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# On Generalized Homogeneous Leader-Following Consensus

Min Li, Andrey Polyakov and Gang Zheng

**Abstract**—The Multi-agent System (MAS) with agents being linear single input plants is considered. The problem of a design of a leader-following (generalized homogeneous) consensus control protocol is studied under the assumption that a control of the leader is unknown but possibly bounded by a known constant. It is shown that the required control protocol can be obtained as an “upgrade” of the existing linear consensus control protocol.

## I. INTRODUCTION

With the rapid development of intelligent sensing and wireless communication techniques, collective behaviors of the multi-agent system (MAS) have attracted significant interest of the control academy, which has wide potential applications in many industrial scenarios, such as the control and optimization of unmanned aerial systems [1], network security [2], smart grids [3], etc.

Consensus is one of the most typical collective behaviors of MAS, which requires the states of agents reach an agreement. This term is firstly introduced by Olfati-Saber and Murray [4]. After that, recent two decades have witnessed the launch of many classical results on this issue (e.g. [5], [6], [7], [8]).

Homogeneity is a symmetry of an object with respect to a class of transformations called dilation. All linear and many nonlinear systems are homogeneous in a certain sense. It is known [9], [10], [11] that local properties of homogeneous systems, such as local stability, can always be expanded into global sense. The homogeneous system is robust (Input-to-State) stable with respect to a large class of perturbations [12]. Moreover, the homogeneity degree of an asymptotically stable system specifies its convergence rate [13], [14], [15]. For more details about properties of the homogeneous systems we refer the reader to the recent survey [16] and to the book [17].

Early in 2008, homogeneity control strategies have been utilized for consensus control. The pioneer work is carried out by Wang and Hong [18], where the authors designed a homogeneous consensus protocol adapting the well-known controller of Bhat and Bernstein [15] for this purpose. Homogeneous consensus protocols for MAS under directed topologies are developed in [19], [20]. The case of the switching topology is studied in [21], [22]. All mentioned papers deal with agents modeled by double integrators and the weighted homogeneity. Nonlinear control protocols for MAS with high order integrator dynamics are proposed, for example, in [23], [24]. The main difficulty of their applicability is the absence of an efficient procedure for control parameters tuning. The mentioned algorithms (as well as most of algorithms of homogeneous stabilization,

see e.g. [15], [25]) just guarantee the existence of appropriate control parameters but do not explain how to select them.

This paper develops a generalized homogeneous control protocol for MAS, where each agent is modeled by a linear single input plant. We follow the idea of an “upgrade” of an existing linear control to a homogeneous one developed in [17, Chapter 9], [26], [27]. In this paper we showed that such an upgrade is possible under the assumption that MAS has a supervisor, which can observe whole system, but it cannot be utilized as a centralized controller due to communication constraints. The supervisor may just broadcast a small amount of information to all agents simultaneously. A nonlinear consensus protocol is designed for MAS in such a way that the error equation is generalized homogeneous of a desired degree.

The paper is organized as follows. Section II gives some basic knowledge about graph theory and generalized homogeneity. The problem to be studied is formulated in Section III. The basic idea for updating the classical consensus protocol to a homogeneous one is shown in Section IV. The supervisor-based homogeneous control protocol is proposed and convergence results are obtained in Section V. Finally, in Section VI, some simulation examples are presented to illustrate the obtained theoretical results.

*Notation.*  $\mathbb{R}$  is the set of real numbers;  $\mathbb{R}^n$  and  $\mathbb{R}^{n \times n}$  denote the  $n \times 1$  real vector and the  $n \times n$  real matrix, respectively;  $\|\cdot\|$  is a norm in  $\mathbb{R}^n$ ;  $I_N$  is the  $N \times N$  identity matrix;  $e$  is the Euler number;  $\text{diag}\{\sigma_i\}_{i=1}^N$  is the  $N \times N$  diagonal matrix with elements  $\sigma_i$ ;  $\mathbf{1}_N$  is the  $N$ -dimensional vector whose elements are all ones;  $\lambda_{\max}(\cdot)$  and  $\lambda_{\min}(\cdot)$  denote the maximum and minimum eigenvalue of a matrix, respectively;  $P \succ 0$  ( $\prec 0$ ) for  $P \in \mathbb{R}^{n \times n}$  means that the matrix  $P$  is symmetric and positive (negative) definite;  $\otimes$  represents the Kronecker product;  $C(X, Y)$  denotes the space of continuous functions  $X \rightarrow Y$ , where  $X, Y$  are subsets of normed vector spaces;  $C^p(X, Y)$  is the space of functions continuously differentiable at least up to the order  $p$ .

## II. PRELIMINARIES: GRAPH THEORY AND GENERALIZED HOMOGENEITY

### A. Graph Theory

The fixed directed graph can be denoted by  $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$ , where  $\mathcal{V} = \{1, 2, \dots, N\}$  is the set of nodes (agents);  $\mathcal{E} = \{(i, j) | i, j \in \mathcal{V}\}$  is the edge set,  $(i, j) \in \mathcal{E}$  if  $i$  can receive the information from  $j$ ;  $n_i$  denotes the number of incoming edges of  $i$ ;  $\mathcal{A} = \{a_{ij}\}$  is the adjacency matrix,  $a_{ij} > 0$  if  $(i, j) \in \mathcal{E}$ , and  $a_{ij} = 0$  otherwise. In this paper, self-loop situation is not taken into consideration, i.e.  $a_{ii} = 0$ . The

element  $l_{ij}$ ,  $i, j \in \mathcal{V}$  of the Laplacian Matrix  $\mathcal{L}$  associated graph  $\mathcal{G}$  is defined as

$$l_{ij} = \begin{cases} -a_{ij}, & i \neq j \\ \sum_{k=1}^N a_{ik}, & i = j, \end{cases}$$

which implies  $\sum_{k=1}^N l_{ik} = 0$ .

### B. Dilation in $\mathbb{R}^n$

Homogeneity is an invariance of an object with respect to a class of transformations called dilations. Choosing a proper dilation group  $\mathbf{d}(s)$ ,  $s \in \mathbb{R}$  is an essential part of the homogeneity-based analysis,  $\mathbf{d}(s)$  is supposed to satisfy the limit property:  $\lim_{s \rightarrow \pm\infty} \|\mathbf{d}(s)x\| = e^{\pm\infty}$  for  $\forall x \neq 0$ . Examples of dilations are as follows:

- Standard dilation (L. Euler 18th century):  $\mathbf{d}(s) = e^s I$ ,  $s \in \mathbb{R}$ ;
- Weighted dilation [9]:  $\mathbf{d}(s) = \text{diag}\{e^{r_1 s}, \dots, e^{r_n s}\} \in \mathbb{R}^{n \times n}$ , where  $r_1, \dots, r_n > 0$ .
- Linear dilation [28]:  $\mathbf{d}(s) = e^{G_{\mathbf{d}} s}$ ,  $s \in \mathbb{R}^n$  where  $G_{\mathbf{d}} \in \mathbb{R}^{n \times n}$  is an anti-Hurwitz matrix known as the generator of dilation.
- Geometric dilation [29], [30]: a flow generated by an unstable  $C^1$  vector field.

### C. Canonical Homogeneous Norm

*Definition 1:* The functional  $\|\cdot\|_{\mathbf{d}} : \mathbb{R}^n \mapsto [0, +\infty)$  defined as  $\|0\|_{\mathbf{d}} = 0$  and

$$\|x\|_{\mathbf{d}} = e^{s_x}, \text{ where } s_x \in \mathbb{R} : \|\mathbf{d}(-s_x)x\| = 1, \quad (1)$$

is called the canonical homogeneous norm in  $\mathbb{R}^n$ , where  $\mathbf{d}$  is a *linear monotone dilation*<sup>1</sup>.

The canonical homogeneous norm has the following properties [28]

- $\|\mathbf{d}(s)x\|_{\mathbf{d}} = e^s \mathbf{d}(s)$ , for  $x \in \mathbb{R}^n$  and  $s \in \mathbb{R}$ ;
- $\|x\|_{\mathbf{d}} = \|-x\|_{\mathbf{d}}$ ,  $\|x\| = 1 \Leftrightarrow \|x\|_{\mathbf{d}} = 1$ ;
- $\|\cdot\|_{\mathbf{d}} \in C(\mathbb{R}^n)$  is locally Lipschitz continuous on  $\mathbb{R}^n \setminus \{0\}$ ;
- if  $\|\cdot\| \in C^1(\mathbb{R}^n \setminus \{0\})$ , then  $\|\cdot\|_{\mathbf{d}} \in C^1(\mathbb{R}^n \setminus \{0\})$ ;
- if  $\|x\| = \|x\|_P := \sqrt{x^T P x}$ , where  $P$  satisfies

$$P \succ 0, \quad P G_{\mathbf{d}} + G_{\mathbf{d}}^T P \succ 0, \quad (2)$$

then the linear dilation  $\mathbf{d}(s) = e^{s G_{\mathbf{d}}}$  is monotone and

$$\frac{\partial \|x\|_{\mathbf{d}}}{\partial x} = \frac{\|x\|_{\mathbf{d}} x^T \mathbf{d}^T (-\ln \|x\|_{\mathbf{d}}) P \mathbf{d} (-\ln \|x\|_{\mathbf{d}})}{x^T \mathbf{d}^T (-\ln \|x\|_{\mathbf{d}}) (P G_{\mathbf{d}} + G_{\mathbf{d}}^T P) \mathbf{d} (-\ln \|x\|_{\mathbf{d}}) x}, \quad x \neq 0. \quad (3)$$

### D. Homogeneous Systems

*Definition 2:* A vector field  $f : \mathbb{R}^n \mapsto \mathbb{R}^n$  (resp. a function  $h : \mathbb{R}^n \mapsto \mathbb{R}$ ) is said to be  $\mathbf{d}$ -homogeneous if there exist a  $\mu \in \mathbb{R}$  such that

$$f(\mathbf{d}(s)x) = e^{\mu s} \mathbf{d}(s) f(x), \quad \forall s \in \mathbb{R}, \quad \forall x \in \mathbb{R}^n,$$

$$\text{(resp. } h(\mathbf{d}(s)x) = e^{\mu s} h(x), \quad \forall s \in \mathbb{R}, \quad \forall x \in \mathbb{R}^n \text{.)}$$

where  $\mathbf{d}$  is a dilation and the scalar  $\mu \in \mathbb{R}$  is known as the homogeneous degree of  $f$  (resp. of  $h$ ).

<sup>1</sup>A dilation  $\mathbf{d}$  is monotone if the function  $s \mapsto \|\mathbf{d}(s)x\|_{\mathbf{d}}$  is monotone  $\forall x$ .

*Lemma 1:* [17] Let the vector field  $\tilde{f} : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^{n+m}$  given by

$$\tilde{f}(x, q) = \begin{pmatrix} f(x, q) \\ \mathbf{0} \end{pmatrix}, \quad x \in \mathbb{R}^n, \quad q \in \mathbb{R}^m$$

be continuous and  $\mathbf{d}$ -homogeneous of a degree  $\mu$  with respect to a continuous strictly monotone dilation

$$\mathbf{d} = \begin{pmatrix} \mathbf{d}_x & \mathbf{0} \\ \mathbf{0} & \mathbf{d}_q \end{pmatrix}$$

in  $\mathbb{R}^{n+m}$ . If the origin of the system

$$\dot{x}(t) = f(x(t), \mathbf{0}), \quad x(t) \in \mathbb{R}^n$$

is asymptotically stable then the system

$$\dot{x} = f(x, q(t)), \quad q \in L^\infty(\mathbb{R}, \mathbb{R}^m)$$

is Input-to-State Stable (ISS)<sup>2</sup>

*Lemma 2:* [32] Let  $f$  be a continuous  $\mathbf{d}$ -homogeneous vector field of degree  $\mu \in \mathbb{R}$ . The evolution system  $\dot{x} = f(x)$  is globally asymptotically stable if and only if there exists a positive definite  $\mathbf{d}$ -homogeneous function  $V : \mathbb{R}^n \rightarrow [0, +\infty)$  of the degree 1 such that

$$\dot{V}(x) \leq -\rho V^{1+\mu}(x), \quad \forall x \neq 0, \quad \rho > 0.$$

The latter Lemma immediately implies that asymptotically stable homogeneous evolution system  $\dot{x} = f(x)$  is

- globally uniformly finite-time stable<sup>3</sup> for  $\mu < 0$  with time estimate

$$T(x_0) \leq \frac{V(x_0)^{-\mu}}{(-\mu)\rho}$$

- globally uniformly exponentially stable for  $\mu = 0$ ;
- globally uniformly nearly fixed-time stable<sup>4</sup> for  $\mu > 0$ , and

$$T_r = \frac{1}{\rho \mu r^\mu}.$$

## III. PROBLEM FORMULATIONS

Consider a **MAS** consisting of  $N$  agents, with a topology described by a fixed digraph  $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$  and the dynamics of the agent given by

$$\dot{\xi}_i(t) = A \xi_i(t) + B u_i(t), \quad t > 0, \quad i \in \mathcal{V} = \{1, \dots, N\} \quad (4)$$

where  $\xi_i(t) \in \mathbb{R}^n$  is the state of the  $i$ -th agent,  $u_i \in \mathbb{R}$  is the control input of the  $i$ -th agent,  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^n$ .

*Assumption 1:* The pair  $\{A, B\}$  is assumed to be controllable and the matrix  $A$  is nilpotent.

The condition that the matrix  $A$  is nilpotent, means that the linear vector field  $\xi \mapsto A \xi$  is generalized homogeneous.

<sup>2</sup>The system  $\dot{x} = f(x, q)$ ,  $t > 0$ ,  $x(0) = x_0$  is said to be Input-to-State Stable if [31] there exists  $\beta \in \mathcal{KL}$  and  $\gamma \in \mathcal{K}$  such that

$$\|x(t)\| \leq \beta(\|x_0\|, t) + \gamma(\|q\|_{L^\infty(0, t)})$$

for any  $x_0 \in \mathbb{R}^n$  and for any  $q \in L^\infty$ .

<sup>3</sup>The system  $\dot{x} = f(x)$  is said to be globally uniformly finite-time stable [33] if it is Lyapunov stable and there exists a locally bounded settling-time function  $T(x_0) : \mathbb{R}^n \rightarrow \mathbb{R}_+ \cup \{0\}$ , such that  $\|x(t)\| = 0$  for  $t \geq T(x_0)$ ,  $x_0 \in \mathbb{R}^n$ .

<sup>4</sup>The system  $\dot{x} = f(x)$  is said to be globally uniformly nearly fixed-time stable [16] if it is Lyapunov stable and  $\forall r > 0$ ,  $\exists T_r > 0 : \|x(t)\| < r$ ,  $\forall t \geq T_r$  independently of  $x_0 \in \mathbb{R}^n$ .

This condition is not conservative in our case, since the controllability of  $\{A, B\}$  implies that (see, e.g. [34]) there exists  $K_0 \in \mathbb{R}^{1 \times n}$  such that  $A + BK_0$  is nilpotent, and selecting  $u_i = K_0 \xi_i + u_i^{new}$  we derive that our assumption is fulfilled for the matrix  $A^{new} = A + BK_0$ .

*Assumption 2:* Assume that  $(1, j) \notin \mathcal{E}$ ,  $\forall j = 2, \dots, N$  (i.e., the first agent is a leader) and the control input  $u_1$  of the first agent is unknown,  $u_1 \in L^\infty(\mathbb{R}, \mathbb{R})$ .

In this paper, we assume that **MAS** has an external *supervisor* which can measure the states of all agents and it can broadcast (send) a scalar signal to all agents. A concrete scenario of such a **MAS** is a swarm of drones observed by a cameras of a supervisor. Positions of all drones can be detected/estimated by the supervisor which however cannot realize a centralized control due to communication constraints. Nevertheless, a broadcast of a common signal to all agents (e.g. asking all of them to boost or slow down the motion) is allowable. Mathematically such an assumption can be formalized as follows.

*Assumption 3:* All agents can receive a scalar signal  $u_g(t) \in \mathbb{R}$ , which may depend on the state of the whole **MAS** and can be utilized for the control purpose.

Notice that we cannot just select  $u_i = u_g$  and design a purely centralized control protocol, since the obtained system is not controllable. To achieve the consensus, it is necessary to use as well the communication graph and local interactions of the agents. Such a combination, i.e. a common centralized scalar signal and the distributed information of local agents, leads to a "hybrid" consensus protocol, named as quasi-decentralized consensus protocol.

Our *first goal* is to develop the generalized homogeneous consensus control protocol for the considered **MAS**. By definition, this means that, *given  $\mu \in \mathbb{R}$ , we need to design a supervisor's signal  $u_g$  and agent's control laws*

$$u_i = \bar{u}_i(u_g, \xi_i, \xi_{j_1}, \dots, \xi_{j_{n_i}}),$$

where  $j_k \in \mathcal{V} : (i, j_k) \in \mathcal{E}$ ,  $k = 1, \dots, n_i$ ,  $i = 2, \dots, N$  such that

- $\exists f : \mathbb{R}^{(N-1)n} \times \mathbb{R} \rightarrow \mathbb{R}^{(N-1)n}$  such that for

$$e = [(\xi_2 - \xi_1)^\top, \dots, (\xi_N - \xi_1)^\top]^\top$$

the error equation of the closed-loop system has the form

$$\dot{e} = f(e, u_1); \quad (5)$$

- the vector field  $e \mapsto f(e, 0)$  is  $\tilde{\mathbf{d}}$ -homogeneous of degree  $\mu$ ;
- the system (5) is globally asymptotically stable for  $u_1 = 0$  and ISS with respect to  $u_1 \in L^\infty(\mathbb{R}, \mathbb{R})$ .

Notice that the asymptotic stability of the error equation (5) with  $u_1 = 0$  implies its finite-time stability for  $\mu < 0$  and nearly fixed-time stability for  $\mu > 0$ .

Our *second goal* is to design a generalized homogeneous consensus control protocol such that *the error equation (5) is globally uniformly finite-time stable for any*

$$u_1 \in \mathcal{U} = \{u_1 \in L^\infty(\mathbb{R}, \mathbb{R}) : \|u_1\|_{L^\infty} \leq U_{\max}\}$$

provided that  $0 < U_{\max} < +\infty$  is known.

#### IV. BASIC IDEA OF CONTROL PROTOCOL DESIGN

Let us employ the well-known linear consensus control protocol proposed in [35] to demonstrate our basic idea.

*Lemma 3:* Let  $(X, Y) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{1 \times n}$  be a solution of the linear matrix inequality

$$X \succ 0, \quad I_{N-1} \otimes (AX + XA^\top) + (\mathcal{L} \otimes BY + \tilde{\mathcal{L}}^\top \otimes Y^\top B^\top) \prec 0, \quad (6)$$

then the error equation (5) with linear control protocol

$$u_i = K \sum_{j=1}^N a_{ij} (\xi_j - \xi_i), \quad i = 2, \dots, N \quad (7)$$

with  $K = YX^{-1}$  is ISS with respect to  $u_1$ .

The linear control protocol (the error equation) is homogeneous of the zero degree. *An algorithm for upgrading of a linear (centralized) control to a generalized homogeneous one is developed in [36]. It suggests to replace the linear feedback  $Ke$  with the homogeneous one  $\|e\|_{\tilde{\mathbf{d}}}^{1+\mu} K\mathbf{d}(-\ln\|e\|_{\tilde{\mathbf{d}}})e$ , where  $\mathbf{d}$  is a properly defined linear monotone dilation and  $\|\cdot\|_{\tilde{\mathbf{d}}}$  is the canonical homogeneous norm.* Below we follow this idea in order to design a  $\mathbf{d}$ -homogeneous (quasi-decentralized) controller of homogeneous degree  $\mu \neq 0$ .

#### V. SUPERVISOR-BASED HOMOGENEOUS CONTROL PROTOCOL

Given  $\mu \neq 0$  and let  $G_{\mathbf{d}} \in \mathbb{R}^{n \times n}$  be an anti-Hurwitz matrix satisfying the identities

$$AG_{\mathbf{d}} = (\mu I_n + G_{\mathbf{d}})A, \quad G_{\mathbf{d}}B = B. \quad (8)$$

In [37] it is shown that such matrix  $G_{\mathbf{d}}$  can always be found provided that  $A$  is a nilpotent matrix.

*Theorem 1:* Let  $(X, Y) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{1 \times n}$  be a solution of the linear matrix inequality

$$I_{N-1} \otimes (AX + XA^\top) + (\mathcal{L} \otimes BY + \tilde{\mathcal{L}}^\top \otimes Y^\top B^\top) \prec 0, \\ G_{\mathbf{d}}X + XG_{\mathbf{d}}^\top \succ 0, \quad X \succ 0, \quad (9)$$

where  $G_{\mathbf{d}}$  satisfies (8). Let  $\|\cdot\|_{\tilde{\mathbf{d}}}$  be the canonical homogeneous norm induced by the norm  $\|e\|_p = \sqrt{e^\top P e}$ ,  $P = I_{N-1} \otimes \{X^{-1}\} \succ 0$ . Then the error equation (5) with  $u_g = \|e\|_{\tilde{\mathbf{d}}}$  and

$$u_i = u_g^{\mu+1} K\mathbf{d}(-\ln u_g) \sum_{j=1}^N a_{ij} (\xi_j - \xi_i), \quad i = 2, \dots, N \quad (10)$$

is

- $\tilde{\mathbf{d}}$ -homogeneous of degree  $\mu$ , where

$$\tilde{\mathbf{d}}(s) = I_{N-1} \otimes \mathbf{d}(s) \in \mathbb{R}^{((N-1) \times n) \times ((N-1) \times n)}$$

and  $\mathbf{d}(s) = e^{sG_{\mathbf{d}}} \in \mathbb{R}^{n \times n}$ ,  $s \in \mathbb{R}$  are linear dilations in  $\mathbb{R}^{(N-1) \times n}$  and  $\mathbb{R}^n$ , respectively;

- globally asymptotically stable for  $u_1 = 0$ ;
- ISS with respect to  $u_1 \in L^\infty(\mathbb{R}, \mathbb{R})$  for  $\mu > -1$ ;
- globally uniformly finite-time stable if  $\mu = -1$  and  $\|u_1\|_{L^\infty} \leq U_{\max}$ , where

$$U_{\max} < \frac{\lambda_{\min}(\Pi_2)\lambda_{\min}(\Pi_1)}{\lambda_{\max}(\Pi_2)\sqrt{(N-1)B^\top X^{-1}B}}, \quad (11)$$

with

$$\begin{aligned}\Pi_1 &= -P^{\frac{1}{2}}(I_{N-1} \otimes (AX + XA^\top) + (\tilde{\mathcal{L}} \otimes BY + \tilde{\mathcal{L}}^\top \otimes Y^\top B^\top))P^{\frac{1}{2}}, \\ \Pi_2 &= P^{\frac{1}{2}}I_{N-1} \otimes (G_d X + XG_d^\top)P^{\frac{1}{2}}.\end{aligned}$$

The proof is based on stability analysis of the closed-loop error equation for  $u_1 = 0$ :

$$\dot{e} = I_{N-1} \otimes Ae + \|e\|_{\tilde{\mathbf{d}}}^{\mu+1} \tilde{\mathcal{L}} \otimes BKd(-\ln \|e\|_{\tilde{\mathbf{d}}})e. \quad (12)$$

It can be shown that, under the condition (9), the canonical homogeneous norm  $\|e\|_{\tilde{\mathbf{d}}}$  is a Lyapunov function of the system. Robustness with respect to the input  $u_1$  can be proven using the results of [12].

The following corollary follows from the asymptotic stability and  $\tilde{\mathbf{d}}$ -homogeneity of the error equation (12).

*Corollary 1:* Let all conditions of Theorem 1 are fulfilled. Let  $u_1 = 0$ , then the error equation (12) is

- globally uniformly finite-time stable for  $\mu < 0$ ;
- globally exponentially stable for  $\mu = 0$ ;
- globally uniformly nearly fixed-time stable for  $\mu > 0$ .

## VI. SIMULATION RESULTS

We consider the MAS (4) with  $N = 4$ . System matrices in (4) are declared as follows:

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The state vector and the consensus error vector are  $\xi = [\xi_1^\top, \xi_2^\top, \xi_3^\top, \xi_4^\top]^\top$  and  $e = [e_1^\top, e_2^\top, e_3^\top]^\top$ , respectively. We denote  $\xi_i^{(k)}$  (resp.  $e_i^{(k)}$ ) as the  $k$ th,  $k = 1, 2$  component of  $\xi_i$  (resp.  $e_i$ ).

The communication topology, as Fig.1 shows, is fixed. The designed parameters for homogeneous control protocol (10) are  $\mu = -1$ ,  $a_{ij} = 1$ ,  $(i, j) \in \mathcal{E}$ ,  $G_d = \text{diag}\{2, 1\}$  according to (8),

$$X = \begin{pmatrix} 4.7799 & -3.0305 \\ -3.0305 & 5.3890 \end{pmatrix}, \quad Y = \begin{pmatrix} 3.0066 & 4.1689 \end{pmatrix}$$

are specified by solving LMI (9) with Matlab tools and  $u_g = \|e\|_{\tilde{\mathbf{d}}}$  is obtained from

$$\|\tilde{\mathbf{d}}(-\ln(\|e\|_{\tilde{\mathbf{d}}}))e\|_P = 1,$$

which is induced by  $\|\cdot\|_P$  with  $P = I_3 \otimes \{X^{-1}\}$ , and  $\tilde{\mathbf{d}}(\cdot)$  is generated by  $G_{\tilde{\mathbf{d}}} = I_3 \otimes G_d$ . In addition,  $\|u_1\|_{L^\infty} < 0.2000$  according to (11), and let  $u_1 = -[1 \ 2]\xi_1 + 0.24\sin(t)$ .

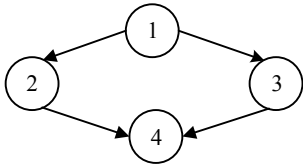


Fig. 1. The topology of the MAS with  $N = 4$ .

Let  $\mu = -1$  and the initial state of the considered MAS be

$$\xi(0) = [0, 0, 0.70, 0.75, 0.55, 0.40, 0.64, 0.15]^\top.$$

We use MATLAB and employ the Euler method with step  $h = 0.001s$  to solve the closed-loop error equation (5), (10), and the agent dynamic equation (4), (10). Meanwhile, we employ the linear control protocol (7) as a reference. The comparison simulation results on  $e$  and  $\xi$  are presented in Fig.2-4. The evolution trajectory of  $u_1$  is as Fig.5. It is clear to see that with a bounded  $u_1$ , the linear control protocol makes consensus errors and followers' trajectories converge to corresponding zones. However, with the homogeneous control protocol, they can converge to their destination in a finite time.

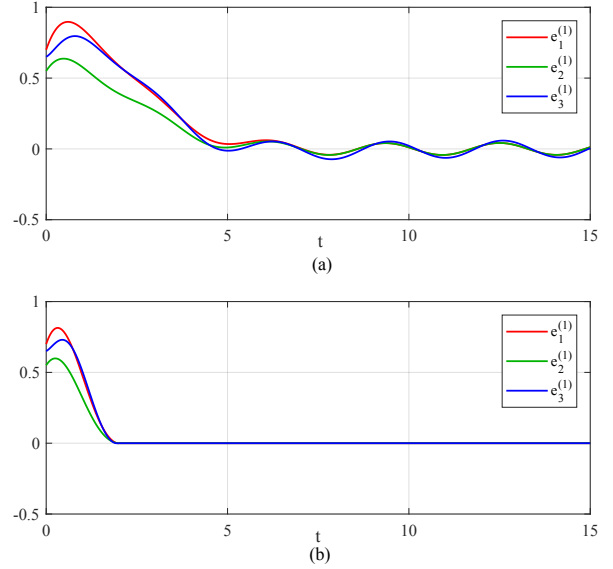


Fig. 2. Evolution of  $e_i^{(1)}$  for  $i = 1, 2, 3$  with (a) linear control protocol (7); (b) homogeneous control protocol (10).

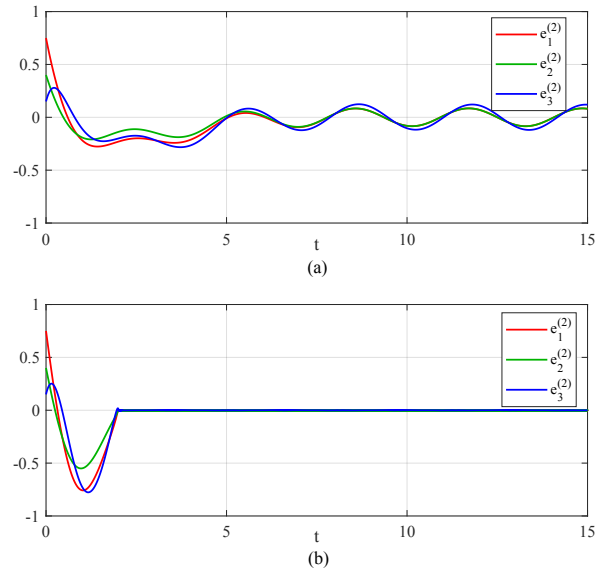


Fig. 3. Evolution of  $e_i^{(2)}$  for  $i = 1, 2, 3$  with (a) linear control protocol (7); (b) homogeneous control protocol (10).

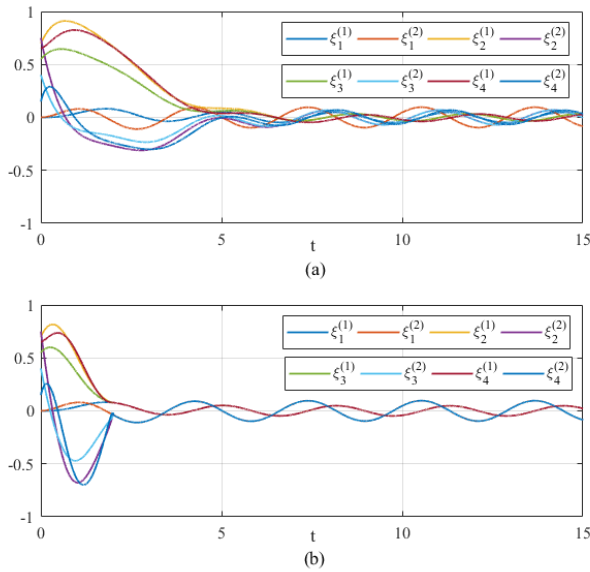


Fig. 4. Evolution of  $\xi_i^{(k)}$  for  $i \in \mathcal{Y}$  with (a) linear control protocol (7); (b) homogeneous control protocol (10).

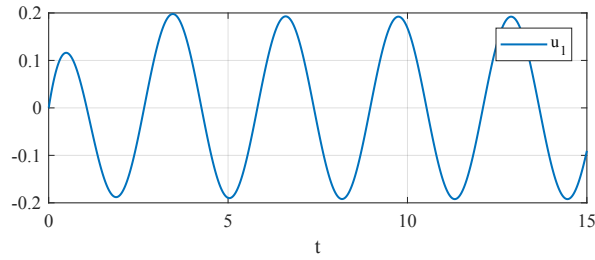


Fig. 5. Evolution of  $u_1$ .

## VII. CONCLUSION

In this paper, a leader-following consensus problem for MAS with agents being linear single input plants is addressed. A novel nonlinear control protocol is proposed by upgrading the classical linear consensus control protocol with the concept of generalized homogeneity. For bounded control input of the leader, ISS and finite-time stability of error equations are guaranteed by non-zero homogeneity degrees. The possibility of such an "upgrade" is proven under assumption that MAS has a super-visor, which may broadcast simultaneously to all agents a small amount of information (a scalar signal). The efficiency of the proposed scheme is demonstrated on numerical simulations. Extension of the obtained results to the multi-input case as well as robustness analysis with respect to measurement noises, delays and sampling effects is the interesting problem for the future research.

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