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► **To cite this version:**

Guillaume Landicheck, Tomas Menard, Olivier Gehan, Eric Pigeon. Continuous-Discrete Time Observer Design for Multi-Output Linear Continuous Systems With Aperiodic Asynchronous Measurements. IEEE Control Systems Letters, 2022, 6, pp.3002-3007. 10.1109/LCSYS.2022.3164262. hal-03810598

HAL Id: hal-03810598

<https://hal.science/hal-03810598>

Submitted on 11 Oct 2022

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Continuous-Discrete Time Observer Design for Multi-Output Linear Continuous Systems with Aperiodic Asynchronous Measurements

Guillaume Landicheck, Tomas Ménard, Olivier Gehan and Eric Pigeon

Abstract— In this paper, a new observer is proposed for the state estimation of time-invariant systems with multiple outputs providing non periodic and asynchronous measurements. The state of the proposed hybrid observer is made up of two parts, the first one corresponding to the estimate which is continuous and the second one corresponding to a kind of output error predictor which is updated when new measurements are received. The convergence of the observer is analyzed and ensured with a condition on the maximum value of the outputs sample rates, with the help of a Lyapunov approach and is further recast using a Sum of Squares decomposition approach. Results of various simulations illustrate the efficiency of the observer.

Index Terms— Continuous-discrete time observer, hybrid systems, dwell time, nonuniform and asynchronous sampling.

I. INTRODUCTION

OBSERVERS for continuous time-invariant systems with sampled data have gained a growing attention over the last decades. These observers aim at estimating the state of a system whose outputs are not continuously available with respect to time. Such observers have a lot of applications in a context where controllers and sampled sensors are more and more often used in systems. Applications encompass for example industrial control systems and sensors networks [14], [9], [15].

First developments of observers with sampled data relied on the transformation of the continuous state-space model of the plant into a discrete time model assuming that the measurements are periodically available, as in [21], [17]. Although these observers brought a first solution to this problematic, having a full discrete approach actually brings several drawbacks. Indeed, the use of a full discrete solution usually may prevent from obtaining a global convergence [2] and the assumption that the sampling instants are periodic may not be realistic for instance for network systems. Designing an observer with continuous-time dynamics between the sampling

instants and instantaneous updates when some new measurement is received is more natural. Following this idea, many works have then emerged to propose such observers, denoted continuous-discrete time or hybrid observers in the literature. The case where all the outputs are available at the same instants is the first one that has been considered in the literature and two approaches have emerged. In the first one, the state of the observer is entirely reset when new measurements are available and the observer runs in open loop between two measurements. In [22] a linear system up to input injections with input dependent state is considered following a high-gain approach. A high-gain approach is also used in [1] for a class of uniformly observable system whose nonlinear part Lipschitz constant is input dependent. A class of nonlinear systems is considered in [20] and the convergence is ensured using a parameter dependent LMI over an interval of values. An hybrid approach has also been considered in several works. A class of nonlinear systems whose dynamics is assumed to belong to a polytopic set of matrices is considered in [6]. The paper [11] considers linear systems, and the effect of noise is analyzed. A less conservative approach is proposed in [26] for the same class of linear systems. For the last three papers, a set of LMIs over an interval of values has to be verified to ensure the stability, more tractable conditions are also provided with a finite number of LMI to verify the stability, but which are more complicated. The second approach which provides a continuous estimation has been first proposed in [16]. This observer is based on an output predictor whose state is reset to the value of the plant output at the sampling times. This approach has been further pursued in [7] using a high-gain approach and in [10] with a proof based on LMI conditions. Both continuous and discontinuous approaches have been combined in [12].

However, in some cases, the different output measurements are not available at the same instants, these kind of systems are usually denoted multi-rate systems. While interesting in practice, less results are available for the multi-rate case in the literature. On the discontinuous estimate side, a general class of multi rate nonlinear systems has been considered in [19] with a proof of stability based on the nonlinear small gain theorem. A class of non-uniformly observable systems has been considered in [8] with a high-gain approach. The class of multi outputs linear systems whose output measurements are available following a round robin sampling has been

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considered in [27]. On the continuous estimate side, a class of nonlinear uniformly observable systems with a cascade structure has been considered in [28], but the use of a constant gain observer lead to very small practical sampling periods. This class of systems has been further considered in [3] where an observer with continuous estimates and based on an output error predictor and an high-gain approach is proposed. The sampling periods can be taken larger than in [28] in practice, but the maximum values for the sampling periods guaranteed by the convergence analysis may be very low. Furthermore, the observer design relies on the triangular structure of the class of systems and then cannot be applied to the general class of multi-outputs linear systems. Distributed observers for the class of multi outputs linear systems have been considered in [18] following an hybrid approach and stability conditions based on LMIs. Note that, in this case, one observer is used for each sensor and this is a main difference with the structure proposed in the current paper.

The main drawback of the stability results based on LMIs, proposed in the literature is the fact that the LMIs have to be verified for all the values of a parameter lying in an interval which leads to an infinite number of LMIs to check. This drawback can usually be circumvented by considering a polytopic convex hull or a mesh of the interval in consideration. But this may lead to conservativeness or more complex derivations. Instead of using directly LMI conditions, another interesting approach is the Sum Of Squares (SOS) decomposition which can directly deal with LMIs valid for a set of parameters lying in an interval. SOS decompositions have been used to obtain stability conditions for some classes of impulsive systems as in [5], [4] and applied to the stability analysis of sampled control with aperiodic sampling rates.

In this paper, we consider the general class of linear systems with multi outputs sampled at different asynchronous sampling rates. The proposed scheme allows to transform any usual Luenberger continuous-time linear observer into a multi rate one. Furthermore, a convergence condition based on SOS decomposition is proposed which allows to obtain tractable conditions to ensure the convergence of the observer. The SOS conditions can be solved directly using the SOS Matlab toolbox [24] for example.

This paper is organized as follows. In Section II, we first present the class of continuous-time linear systems with sporadic outputs that is considered in the current paper. Section III then presents the proposed observer and the convergence results. Simulation results using the proposed observer design are given in Section IV in order to illustrate the approach. Section V finally concludes this paper.

Notations

For a vector $x \in \mathbb{R}^n$, the distance between x and the set $\mathcal{A} \subset \mathbb{R}^n$ is defined as $|x|_{\mathcal{A}} = \inf_{z \in \mathcal{A}} |x - z|$.

Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix, then the notation $A > 0$ means that A is positive definite.

Let $a_1, \dots, a_n \in \mathbb{R}$, then $diag(a_1, \dots, a_n)$ denotes the diagonal square matrix of dimension $n \times n$ with a_1, \dots, a_n on its diagonal. I_n denotes the identity matrix of dimension n . $0_{n \times m}$

and $1_{n \times m}$ respectively denotes the matrices of size $n \times m$ and with 0 (resp. 1) at each entry.

II. PRESENTATION OF THE PROBLEM

In this paper, we consider the class of continuous time-invariant linear systems with multiple outputs. These systems can be described as follows

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y = Cx \end{cases} \quad (1)$$

where $x \in \mathbb{R}^n$ denotes the state vector, $u \in \mathbb{R}^m$ the known input and $y \in \mathbb{R}^p$ the output of the system. The matrices A, B, C are constant and of appropriate dimensions.

The output y is further composed of p real outputs y^i corresponding to the different sensors and C_i is the vector such that $y^i = C_i x$.

One assumes that each output y^i is only available at some sampling moments $t_k^i, k \in \mathbb{N}$. The sequence of time instants $(t_k^i)_{k \in \mathbb{N}}$ is assumed to be strictly increasing and such that $t_k^i \rightarrow \infty$ when $k \rightarrow \infty$.

Besides, associated to each output $i = 1, \dots, p$, two positive real bounds τ_{\min}^i and τ_{\max}^i are introduced such that:

$$\begin{cases} 0 \leq t_1^i < \tau_{\max}^i \\ (t_{k+1}^i - t_k^i) \in [\tau_{\min}^i, \tau_{\max}^i], \quad \forall k \in \mathbb{N} \\ 0 < \tau_{\min}^i < \tau_{\max}^i. \end{cases} \quad (2)$$

One further denotes $\tau_{\min} = \min\{\tau_{\min}^1, \dots, \tau_{\min}^p\}$ in the following.

Remark 1: : The lower bound τ_{\min}^i represents the minimum time between two data transmissions from the i -th sensor. The existence of a lower bound is not restrictive since it can be taken as small as desired. This assumption is used for technical reasons in order to avoid Zeno phenomenon. The upper bound τ_{\max}^i represents the maximum time that can elapse between two consecutive measurements from the i -th sensor.

III. OBSERVER DESIGN

A. Proposed Observer

The current objective is to design a general observer providing a continuous estimate \hat{x} of the state of system (1) with the knowledge of the outputs y^i available at their sampling instants t_k^i only. The proposed continuous-discrete time observer is given by:

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) - Kz(t) & \text{if } t \neq t_k^i, \\ \dot{z}_i(t) = -C_i Kz(t) & \forall i = 1, \dots, p, k \in \mathbb{N}, \end{cases} \quad (3a)$$

$$\begin{cases} \hat{x}(t^+) = \hat{x}(t) \\ z_i(t^+) = C_i(\hat{x}(t) - x(t)), & \text{if } t = t_k^i, \forall k \in \mathbb{N} \end{cases} \quad (3b)$$

where $z = [z_1, \dots, z_p]^T$ and $K \in \mathbb{R}^{n \times p}$ is the gain of the observer and is chosen such that the matrix $(A - KC)$ is Hurwitz. This observer evolves following two types of equations:

- the continuous-time equations (3a), between two consecutive measurements, that provides the state estimate,

- the jump equations (3b) corresponding to an update phase at every instant t_k^i when a new data is available from any of the sensors.

The fact that the observer encounters jumps at every sampling time but evolves according to continuous differential equations between those jumps brings the difficulty to assure its convergence. A hybrid approach is then adopted for this convergence analysis.

Remark 2: The convergence of the proposed observer could be obtained by using similar ideas as in [3]. In this case, the convergence is obtained by considering the Lyapunov function corresponding to the continuous observer error system, its derivative is computed and an inequality is obtained with the derivative on the left hand-side and a term depending on the Lyapunov function and its integral on the right-hand side. Finally [3, Lemma 1] is applied. Following similar technical derivations could be possible here but the convergence would be guaranteed only for small sampling periods. The objective of the proposed approach is to derive conditions for larger and hence more realistic sampling periods.

B. Hybrid Error Model

The framework introduced in [13] is used here to express the considered class of systems into a hybrid form. To this aim, a timer variable $\tau \in \mathbb{R}^p$ is introduced to account for the flow of time and the sensors triggering mechanism. Each component τ_i of τ is designed to decrease at the pace of time with a derivative $\dot{\tau}_i = -1$ and triggers a jump every time the condition $\tau_i = 0$ is verified, that is at every sampling instant t_k^i . After each jump, τ_i is reset to a value in $[\tau_{\min}^i, \tau_{\max}^i]$ in regards to (2) in order to keep track of the time before the next reception of data. Note that the new value of τ_i when a jump occurs does not need to be chosen, it simply represents the time until when the next measurement of the output i will be available.

Introducing the error variables $\tilde{x} := \hat{x} - x$ and $\eta := (\hat{y} - y) - z$, let us define the new state $\chi := (\tilde{x}, \eta, \tau)$, $\chi \in \mathcal{X} := \mathbb{R}^n \times \mathbb{R}^p \times \mathcal{T}$ with $\mathcal{T} := [0, \tau_{\max}^1] \times \dots \times [0, \tau_{\max}^p]$. By operating a change of coordinates in (3) to adopt the new state χ as in [10], one can obtain the new hybrid model

$$\mathcal{H} : \begin{cases} \dot{\chi} = f(\chi), & \chi \in \mathcal{C} \\ \chi^+ \in g(\chi), & \chi \in \mathcal{D} \end{cases} \quad (4)$$

where

$$f(\chi) = \begin{bmatrix} (A - KC) & K & 0_{n \times p} \\ CA & 0_{p \times p} & 0_{p \times p} \\ 0_{p \times n} & 0_{p \times p} & 0_{p \times p} \end{bmatrix} \chi + \begin{bmatrix} 0_{n \times 1} \\ 0_{p \times 1} \\ -1_{p \times 1} \end{bmatrix} \quad (5)$$

and

$$g(\chi) = [\tilde{x}^T g_1^\eta(\chi) \dots g_p^\eta(\chi) g_1^\tau(\chi) \dots g_p^\tau(\chi)]^T \quad (6)$$

with

$$g_i^\eta(\chi) = \begin{cases} 0, & \text{if } \chi \in \mathcal{D}_i \\ \eta_i, & \text{else} \end{cases}, \quad i = 1, \dots, p \quad (7)$$

$$g_i^\tau(\chi) = \begin{cases} [\tau_{\min}^i, \tau_{\max}^i], & \text{if } \chi \in \mathcal{D}_i \\ \tau_i, & \text{else} \end{cases}, \quad i = 1, \dots, p \quad (8)$$

and the flow set \mathcal{C} and the jump set \mathcal{D} are respectively defined as $\mathcal{C} := \mathcal{X}$ and $\mathcal{D} := \bigcup_{i=1}^p \mathcal{D}_i$, $\mathcal{D}_i := \{\chi \in \mathcal{X}, \tau_i = 0\}$.

Given the hybrid framework used here, the solutions $\phi(t, j)$ of system (4) are now parameterized by both the time t and the number of jumps j . The hybrid time domain of ϕ is denoted $\text{dom } \phi$ (see [13] for more details).

Lemma 1: For each $\xi \in \mathcal{C} \cup \mathcal{D}$, there exists a non trivial solution ϕ of system \mathcal{H} such that $\phi(0, 0) = \xi$. Furthermore, every maximal solution ϕ of the hybrid system (4) is complete. Considering the structure of the considered hybrid system, the proof is straightforward by using proposition 2.10 in [13] and hence is then not reported here.

Remark 3: Contrary to [18] where the hybrid system fulfills the usual hybrid basic condition, the hybrid system (4) considered here does not satisfy the hybrid basic conditions (see [13] for more details) due to the jump map not being outer semicontinuous. A modification of (4) could allow to satisfy the hybrid basic condition, however, the results presented here do not rely on the satisfaction of these conditions.

C. Stability Analysis

According to the new coordinates, the convergence condition translates to \tilde{x} converging to 0. Given the structure of the proposed observer, convergence on \tilde{x} also means that η also converges to 0. Therefore, we are going to look for sufficient conditions ensuring that the set

$$\mathcal{A} := \{0_n\} \times \{0_p\} \times \mathcal{T} \quad (9)$$

is globally exponentially stable for system \mathcal{H} according to the following definition.

Definition 1: [30] For a hybrid system \mathcal{H} , the closed set \mathcal{A} is said to be exponentially stable if there exists strictly positive real numbers ω, ϱ such that each maximal solution ϕ is complete and satisfies for all $(t, j) \in \text{dom } \phi$

$$|\phi(t, j)|_{\mathcal{A}} \leq \omega \exp(-\varrho(t + j)) |\phi(0, 0)|_{\mathcal{A}} \quad (10)$$

One considers the following Lyapunov candidate function:

$$V(\chi) := \tilde{x}^T P \tilde{x} + \eta^T Q(\tau) \eta, \quad \forall \chi \in \mathcal{X} \quad (11)$$

where P is a symmetric positive definite matrix and $Q(\tau)$ is chosen more specifically as

$$Q(\tau) = \text{diag}\{Q_1(\tau_1), \dots, Q_p(\tau_p)\} \quad (12)$$

where the functions $Q_i : \mathbb{R} \rightarrow \mathbb{R}$ are such that $Q_i(\tau_i) > 0$, for all $\tau_i \in [0, \tau_{\max}^i]$, $i = 1, \dots, p$.

Theorem 1: Consider system (1) with sampling times verifying conditions (2) and consider observer (3a)-(3b) with K such that $(A - KC)$ is Hurwitz. If there exist a symmetric positive definite matrix $P \in \mathbb{R}^{n \times n}$ and some real functions Q_i such that $Q_i(\tau_i) > 0, \forall \tau_i \in [0, \tau_{\max}^i]$ and

$$M(\tau) < 0, \quad \forall \tau \in \mathcal{T} \quad (13)$$

where the symmetric matrix $M(\tau)$ is defined as

$$M(\tau) \triangleq \begin{bmatrix} (A - KC)^T P + P(A - KC) & \star \\ K^T P + Q(\tau)CA & -\tilde{Q}(\tau) \end{bmatrix} \quad (14)$$

with $\tilde{Q}(\tau) = \text{diag} \{d(Q_1(\tau_1))/d\tau_1, \dots, d(Q_p(\tau_p))/d\tau_p\}$ then the set \mathcal{A} defined by (9) is globally exponentially stable for the hybrid system (4).

Remark 4: Condition (13) implies that $(A - KC)^T P + P(A - KC) < 0$, which means that, with the proposed approach, the gain K always corresponds to the gain of an equivalent Luenberger continuous time observer.

Proof: Consider the Lyapunov candidate $V : \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}$ given by (11). Let us first show that V is such that $V(\chi) = 0$ for all $\chi \in \mathcal{A}$ and $V(\chi) > 0$ for all $\chi \notin \mathcal{A}$. Since \mathcal{T} is a compact set, and since P and Q are symmetric matrices taken such that $P > 0$ and $Q(\tau) > 0, \forall \tau \in \mathcal{T}$, there exist $a_1, a_2 > 0$ such that

$$a_1(\|\tilde{x}\|^2 + \|\eta\|^2) \leq V(\chi) \leq a_2(\|\tilde{x}\|^2 + \|\eta\|^2) \quad (15)$$

Let $t > 0$ and consider a solution ϕ of \mathcal{H} whose hybrid time domain is given by $\cup_{l=0}^{j-1} [t_l, t_{l+1}] \times \{l\}$ with $t_0 = 0$ and $t_j = t$. The number j thus corresponds to the number of sampling instants before t .

Two cases have to be considered for system (4), either $\chi \in \mathcal{D}$ or $\chi \in \mathcal{C}$.

In the first case, there exists at least one $k \in \{1, \dots, p\}$ such that $\chi \in \mathcal{D}_k$, then for any $\chi = [\tilde{x}^T, \eta^T, \tau^T]^T \in \mathcal{X}$, one has

$$V(\chi^+) = V(g(\chi)) = \tilde{x}^T P \tilde{x} + g^\eta(\chi) Q(\tau) g^\eta(\chi) \quad (16)$$

$$= \tilde{x}^T P \tilde{x} + \sum_{i=1}^p (g_i^\eta(\chi))^2 Q_i(\tau) \quad (17)$$

$$\leq \tilde{x}^T P \tilde{x} + \sum_{i=1}^p (\eta_i)^2 Q_i(\tau) = V(\chi) \quad (18)$$

since $g_i^\eta(\chi)$ is either equal to 0 or η_i .

In the second case, the solution follows the differential equation $\dot{\chi} = f(\chi)$, then

$$\dot{V}(\chi) = \dot{\tilde{x}}^T P \tilde{x} + \tilde{x}^T P \dot{\tilde{x}} \quad (19)$$

$$+ \dot{\eta}^T Q(\tau) \eta + \eta^T Q(\tau) \dot{\eta} + \eta^T \left(\frac{d}{dt} (Q(\tau)) \right) \eta$$

$$= \tilde{x}^T ((A - KC)^T P + P(A - KC)) \tilde{x} \quad (20)$$

$$+ \tilde{x}^T (PK + A^T C^T Q(\tau)) \eta$$

$$+ \eta^T (K^T P + Q(\tau) C A) \tilde{x} - \eta^T \left(\tilde{Q}(\tau) \right) \eta$$

$$= \begin{bmatrix} \tilde{x} \\ \eta \end{bmatrix}^T M(\tau) \begin{bmatrix} \tilde{x} \\ \eta \end{bmatrix} \quad (21)$$

where $M(\tau)$ is defined by (14) and $\tilde{Q}(\tau) = \text{diag} \{d(Q_1(\tau_1))/d\tau_1, \dots, d(Q_p(\tau_p))/d\tau_p\}$. Given the assumption on M and inequality (15), there exists $b_1 > 0$ such that

$$V(\phi(t_{l+1}, l)) \leq e^{-b_1(t_{l+1}-t_l)} V(\phi(t_l, l)), \quad (22)$$

for all $l = 0, \dots, j-1$. Further using inequality (18) gives

$$V(\phi(t_{l+1}, l+1)) \leq V(\phi(t_{l+1}, l))$$

$$\leq e^{-b_1(t_{l+1}-t_l)} V(\phi(t_l, l)) \quad (23)$$

for all $l = 0, \dots, j-1$. Using iteratively inequality (23) then yields

$$V(\phi(t, j)) = V(\phi(t_j, j)) \leq e^{-b_1 t} V(\phi(0, 0)) \quad (24)$$

Since at most p timers can reset in a time window of length τ_{\min} , one can show that $t \geq \tau_{\min} \left(\frac{j}{p} - 1 \right)$. Using this inequality together with inequality (24) yields

$$V(\phi(t, j)) \leq e^{-b_1 t} V(\phi(0, 0)) \leq e^{-b_1 \frac{t}{2} - b_1 \frac{t}{2}} V(\phi(0, 0))$$

$$\leq e^{-b_1 \frac{t}{2} - \frac{b_1}{2} \tau_{\min} \left(\frac{j}{p} - 1 \right)} V(\phi(0, 0))$$

The exponential stability finally follows by taking $\omega = \exp\left(\frac{b_1}{2}\right)$ and $\varrho = \min\left(\frac{b_1}{2}, \frac{b_1 \tau_{\min}}{2p}\right)$. ■

Remark 5: It should be noted that while the proposed observer is convergent as long as condition (2) holds, the convergence still holds if condition (2) is violated finitely many times over $[0, \infty)$. The proof of this fact is direct since the observer cannot escape in finite-time due to its linear flow and the structure of the jump map. This makes the proposed observer inherently robust to measurement dropouts as long as there is only a finite number of occurrences of them. Indeed dropouts can be seen as a violation of condition (2) since the observer continues to run as long as no new measurements are received.

Remark 6: While the time sequences considered here have to follow the classical strict dwell time condition, it should be noted that wider classes of time sequences such as average dwell time with eventually uniformly bounded impulse frequency may be considered in order to relax the sufficient convergence conditions proposed here.

D. Sum of Squares Conditions

The convergence condition (13) given in Theorem 1 has to be valid for all $\tau \in \mathcal{T}$. This means that an infinite number of LMIs have to be checked and hence classical LMI algorithms may not be adapted. To cope with this difficulty, we adopt here a method based on sum-of-squares (SOS) decomposition which can naturally deal with these kind of conditions and provide a computationally tractable solution to the considered problem.

The definition of a SOS polynomial is given next.

Definition 2: For $x \in \mathbb{R}^n$, a multivariate polynomial $p(x)$ is a sum of squares if there exists some polynomials $f_i(x), i = 1, \dots, M$ such that $p(x) = \sum_{i=1}^M f_i^2(x)$.

As it can be seen, if a polynomial is SOS, then it is positive. The contrary is not true except for some specific cases. The advantage of requiring the polynomial to be SOS is that this property can be easily checked by using a computer and solutions can be given by using a dedicated toolbox such as SOSTOOLS [23] with a semidefinite solver such as SeDuMi [29].

Remark 7: The SOS decomposition can be readily extended to symmetric matrices. Indeed, a symmetric matrix $A(x) \in \mathbb{R}^{n \times n}$ is said to be SOS if the polynomial $p(x, y) = y^T A(x) y$ is SOS, where $y \in \mathbb{R}^n$ is a new set of indeterminate variables. The convergence conditions for the observer are now recast using an SOS framework in the next proposition. The proposed result is based on the Positivstellensatz as the one given in [25].

Proposition 1: Consider system (1) with sampling times verifying conditions (2) and consider observer (3a)-(3b) with

K such that $(A - KC)$ is Hurwitz. Then, if there exist SOS polynomials $\gamma_i(\tau_i), Q_i(\tau_i)$, SOS matrices P and $\Gamma_i(\tau)$, $i = 1, \dots, p$, and $\epsilon > 0$ such that

$$Q_i(\tau_i) - \gamma_i(\tau_i)\tau_i(\tau_{\max}^i - \tau_i) - \epsilon \text{ is SOS} \quad (25a)$$

$$-M(\tau) - \epsilon I_{n+2p} - \sum_{i=1}^p \Gamma_i(\tau)\tau_i(\tau_{\max}^i - \tau_i) \text{ is SOS} \quad (25b)$$

then the set \mathcal{A} defined by (9) is globally exponentially stable for the hybrid system (4).

Remark 8: It should be noted that $\epsilon > 0$ is only used to guarantee positive definiteness and can be taken as small as desired. The unknown variables P, Q_i, γ_i and Γ_i are found with the help of a computer, for example with MATLAB and the toolbox SOSTOOLS as this is done on an example in section IV.

Proof: One wants to show here that $Q_i(\tau_i) > 0$ for all $\tau_i \in [0, \tau_{\max}^i]$ and $M(\tau) < 0$ for all $\tau \in \mathcal{T}$. Conditions (25a) and (25b) yields $Q_i(\tau_i) \geq \gamma_i(\tau_i)\tau_i(\tau_{\max}^i - \tau_i) \geq \epsilon$ and $-M(\tau) - \sum_{i=1}^p \Gamma_i(\tau)\tau_i(\tau_{\max}^i - \tau_i) \geq \epsilon I_{n+p}$ for all $\tau \in \mathcal{T}$. Since the functions $\tau_i(\tau_{\max}^i - \tau_i)$ are positive for all $\tau_i \in [0, \tau_{\max}^i]$, then, since $\gamma_i(\tau_i)$ and $\Gamma_i(\tau)$ are SOS, they are positive and one has $Q_i(\tau_i) \geq \epsilon, \forall \tau_i \in [0, \tau_{\max}^i]$ and $M(\tau) \leq -\epsilon I_{n+p}, \forall \tau \in \mathcal{T}$. One can then apply Theorem 1 which ends the proof. ■

IV. EXAMPLE

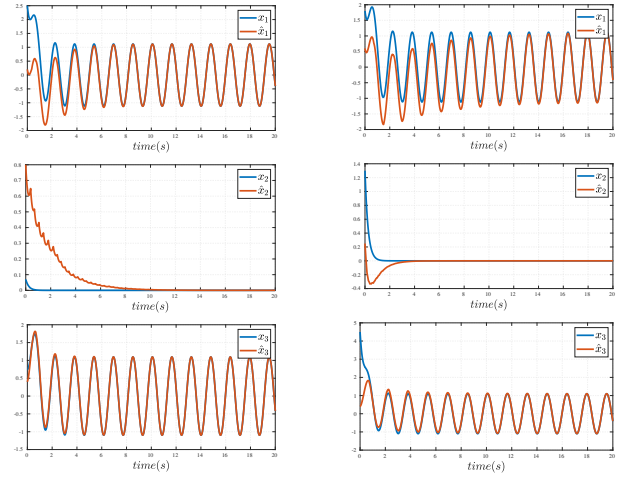
The performances of the designed continuous-discrete time observer are now illustrated through MATLAB simulations. The following system of dimension $n = 3$ with $p = 2$ output sensors is considered:

$$\begin{cases} \dot{x} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -4 & 0 \\ 5/6 & 1/6 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x \end{cases} \quad (26)$$

In order to implement observer (3), the gain K is chosen such that the eigenvalues of $(A - KC)$ are equal to $\{-1, -2, -3\}$. For all the simulations, the sampling periods of the two outputs are chosen such that they follow a uniform probabilistic law on $[\tau_{\min}^1, \tau_{\max}^1]$ and $[\tau_{\min}^2, \tau_{\max}^2]$, the input $u(t)$ is taken as $u(t) = 5 \sin(4t)$, the system is initialized with random values $x_i(0) \in [0, 1]$ and the observer with random values $\hat{x}_i(0) \in [0, 5], i = 1, 2, 3$.

In order to guarantee the convergence of the proposed observer, the MATLAB toolbox SOSTOOLS v3.03 has been used together with the toolbox SeDuMi v1.3. These frameworks were used to find Q_i, P, γ_i and Γ_i such that equations (25a) and (25b) are verified. Besides the matrices A, B, C and K , the only design parameter that has to be set is ϵ which has been chosen here as $\epsilon = 10^{-5}$.

Two different situations have been considered. In the first case, one chooses $\tau_{\max} = \tau_{\max}^1 = \tau_{\max}^2$ for different values. It can be seen on table I that the maximum theoretically guaranteed admissible τ_{\max} can be increased by taking a larger order for the polynomials $Q_i, i = 1, 2$. The computation times for the different cases are also reported on table I and



(a) $\tau_{\min}^1 = 0.3, \tau_{\max}^1 = 0.4, \tau_{\min}^2 = 0.3, \tau_{\max}^2 = 0.4$ (b) $\tau_{\min}^1 = 0.7, \tau_{\max}^1 = 0.8, \tau_{\min}^2 = 0.1, \tau_{\max}^2 = 0.2$

Fig. 1: Simulation of each component of the system and its estimation for a sinusoidal input.

shows that it increases as the order of polynomials increases, however it should be noted that the computations are done off-line to guarantee the convergence beforehand, this fact is therefore not restrictive. The computer used for the simulations runs with an Intel Core i7-6600U with 4 cores cadenced at 2.60GHz and 32Go of RAM. The maximum value from which the convergence is obtained in simulation when taking $\tau_{\min} = \tau_{\max}$ is equal to 0.5s which shows that the theoretically guaranteed values are not very conservative. A particular case is depicted on figures 1-(a), 2-(a) and 3-(a) and shows the well behavior of the proposed observer.

A second case is considered with a smaller fixed value for the maximum sampling periods of the second output, that is $\tau_{\max}^2 = 0.2s$ and different values for τ_{\max}^1 . The maximum theoretically guaranteed admissible value for τ_{\max}^1 in this case is equal to 0.8. This shows the interest to have an approach with different sampling bounds for the different outputs since in this case one can consider larger possible sampling periods for the first output by taking smaller sampling periods for the second output. For $\tau_{\min}^2 = \tau_{\max}^2$, the maximum possible value of $\tau_{\min}^1 = \tau_{\max}^1$ obtained in simulation and for which the observer is convergent is equal to 1s, which shows again that the proposed approach is not very conservative. A particular case is depicted on figures 1-(b), 2-(b) and 3-(b).

deg(Q_i)	2	4	6	8	10
Admissible τ_{\max}	0.19s	0.32s	0.38s	0.41s	0.41s
Computation time	140s	351s	638s	1318s	3125s

TABLE I: Evolution of the maximum sampling rate depending on the degree of Q_i

V. CONCLUSION

In this paper a continuous-discrete time observer has been proposed for linear systems with multiple non periodic and

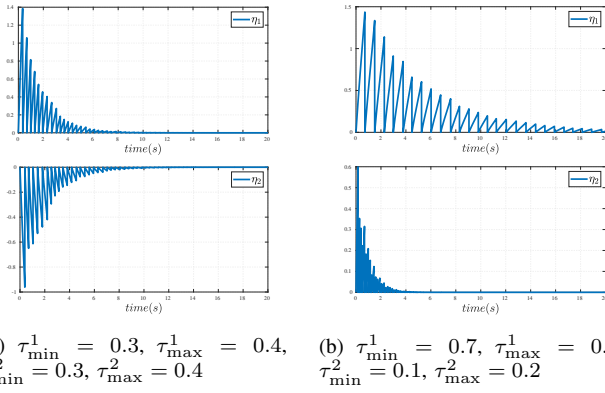


Fig. 2: Evolution of the components of η for a sinusoidal input

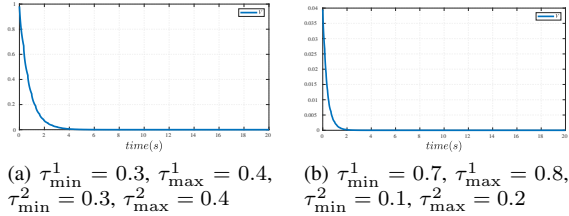


Fig. 3: Evolution of the value of the Lyapunov function (11) associated to the hybrid system (4)

asynchronous outputs. This observer provides a continuous-time estimation of the state of the system from the discrete-time sampled outputs. Theoretical conditions for exponential convergence have been derived based on the use of a hybrid formulation. Furthermore, the conditions have been recast using an SOS decomposition approach which gives computationally tractable theoretically guaranteed conditions. The performances of the proposed observer have been highlighted with different simulations. Future works will focus on expanding these results to include nonlinearities in the system dynamics.

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