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Existence and regularity results for some Fully Non Linear singular or degenerate equation

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Abstract

In this article we prove existence, uniqueness and regularity for the singular equation

$$\begin{cases} |\nabla u|^\alpha (F(D^2u) + h(x) \cdot \nabla u) + c(x)|u|^\alpha u + p(x)u^{-\gamma} = 0 & \text{in } \Omega \\ u > 0 & \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega \end{cases}$$

when p is some continuous and positive function, c and h are continuous, $\alpha > -1$ and F is Fully non linear elliptic. Some conditions on the first eigenvalue for the operator $-|\nabla u|^\alpha (F(D^2u) + h(x) \cdot \nabla u) - c(x)|u|^\alpha u$ are required. The results generalize the well known results of Lazer and McKenna.

Key words: Viscosity solutions, Second-order elliptic equations, Nonlinear elliptic equations, Degenerate elliptic equations, Singular problem.

2020 Mathematical subject classification: 35D40, 35J15, 35J60, 35J70, 35J75.

1 Introduction and some useful tools

In this work we study the existence, uniqueness and regularity of solution for the singular equation

$$\begin{cases} |\nabla u|^\alpha (F(D^2u) + h(x) \cdot \nabla u) + c(x)|u|^\alpha u + p(x)u^{-\gamma} = 0 & \text{in } \Omega \\ u > 0 & \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega \end{cases} \quad (1)$$

where Ω is a bounded \mathcal{C}^2 domain, $\alpha > -1$, $p > 0$ is continuous on $\bar{\Omega}$, c and h are continuous on $\bar{\Omega}$. F is Fully Non Linear elliptic, and the solutions are intended in the viscosity sense, this will be precised below.

The results here enclosed generalize the pioneer work of Lazer and McKenna [21], which consider the case where $h = c = \alpha = 0$ and F is the Laplacian. In this simple framework, the solutions can be intended in the variational sense, even if the presence of the singular zero order term $p(x)u^{-\gamma}$ lead the authors in [21] to use some tools usually employed in viscosity framework : Existence of convenient sub- and super-solutions, comparison Theorem, compactness of bounded sequences. In the present work, the difficulties are linked both to the singularity/degeneracy of the term of derivative of order 1, and to the singularity of the zero order term.

When $\alpha = 0$ there is a big amount of articles about the existence, the uniqueness and the regularity of viscosity solutions for the equation $F(D^2u) = f$, as the paper of Caffarelli [12], mainly devoted to the C^1 and higher regularity, the paper of Ishii and Lions [19], the user's guide of Crandall, Ishii and Lions [15], the book of Caffarelli and Cabré [13], and the famous paper of Ishii [18].

The case $\alpha \neq 0$ was introduced by Birindelli and Demengel in [4], [5], which consider the equations

$$|\nabla u|^\alpha (F(D^2u) + h(x) \cdot \nabla u) = f,$$

with f and h continuous and bounded. They provide a definition of viscosity solution which fits the case $\alpha < 0$ (note that in that case, the equation is not well defined on a point where the gradient is zero). In particular this definition can also be used for the $(\alpha + 2)$ -Laplacian (when one works with viscosity solutions in place of classical solutions), or to the infinity Laplacian. It can also be used for the case $\alpha > 0$, note that it is shown in [8] that the solutions are the same as the classical viscosity solutions in that case.

More recently Attouchi and Ruosteenoja in [2] have refined this definition, for equations of the form

$$|\nabla u|^\gamma (\Delta u + (p - 2)\Delta_\infty^N u) = f,$$

with $\gamma > -1$, $p > 1$.

Concerning the optimal regularity expected, the example of $\varphi(r) = r^{\frac{2+\alpha}{1+\alpha}}$ which solves

$$|\nabla \varphi|^\alpha \Delta \varphi = \frac{(2 + \alpha)^{1+\alpha}}{(1 + \alpha)^{2+\alpha}} ((N - 1)\alpha + N)$$

in the ball $B(0, 1)$, shows that, for $\alpha > 0$, this regularity cannot be better than $C^{1, \frac{1}{1+\alpha}}$.

Coming back to the case $\alpha < 0$, a first regularity result is proved in [7] for solutions of the homogeneous Dirichlet problem. When the operator F is concave or convex, the C^2 regularity holds. In the case $\alpha > 0$ and $h = 0$, Imbert and Silvestre proved a $C^{1, \beta}$ interior regularity result in [20]. This result is extended to a local result "up to the boundary" and to the case where $h \neq 0$, [8]. The interior regularity is precised in [1] in the case $\alpha > 0$, which can be $C^{1, \frac{1}{1+\alpha}}$ in the cases where for example F is convex or concave.

Concerning the equation (1) in the variational setting, the work of Crandall Rabinowitz and Tartar [14] extend the results of Lazer McKenna when the Laplacian is replaced by some linear uniformly elliptic operator, positively homogeneous of degree 1, while, for the Fully non linear setting, in [17] the authors consider $F(x, u, Du, D^2u)$ which is homogeneous of degree 1 with respect to all its arguments and Fully Non Linear Elliptic, with some more general singularity than $p(x)u^{-\gamma}$. Always when $\alpha = 0$ and F is replaced by some degenerate Pucci's operators \mathcal{P}_k^\pm , Birindelli and Galise [11] proved existence, uniqueness and some regularity result, result extended to more general singular zero order term in [22].

We now precise the assumptions, and present the main result. We assume that F satisfies the assumptions:

There exist ellipticity constants a and A , $0 < a < A$ so that for any M and N symmetric matrices on \mathbb{R}^N , $N \geq 0$,

$$atr(N) \leq F(M + N) - F(M) \leq Atr(N). \quad (2)$$

We will also assume that F is positively homogenous, i.e

$$F(tM) = tF(M)$$

for all $t > 0$.

A well known example of such operators are the Pucci's operator $\mathcal{M}_{a,A}^\pm$ ($\mathcal{P}_{\lambda,\Lambda}^\pm$ for most of the authors), defined as

$$\mathcal{M}_{a,A}^+(S) = a \sum_{i,\lambda_i>0} \lambda_i(S) + A \sum_{i,\lambda_i<0} \lambda_i(S), \quad \mathcal{M}_{a,A}^-(S) = -\mathcal{M}_{a,A}^+(-S)$$

where the $\lambda_i(S)$ are the eigenvalue of the symmetric matrix S .

In the following Theorem λ_1^c denotes the first eigenvalue for the operator $-|\nabla u|^\alpha(F(D^2u) + h(x) \cdot \nabla u) - \bar{c}(x)|u|^\alpha u$ with the definition precised later.

Theorem 1.1 *Let Ω be a bounded C^2 domain in \mathbb{R}^N . Suppose that F satisfies (2), is positively homogeneous of degree 1, that c, h, p are continuous and bounded, with $p > 0$ on $\bar{\Omega}$. Let us suppose that $\lambda_1^c, \lambda_1^{\frac{(1+\alpha+\gamma)c}{2+\alpha}} > 0$, then there exists a unique solution to (1). In addition $u \in C^{1,\beta}(\Omega)$ for some $\beta > 0$.*

The plan of this paper is as follows : In Section 2 we remind the definition of viscosity solutions adapted to the present context, and recall the maximum and comparison principles, and the regularity results needed in the paper. In Section 3 we prove the existence's result and provide the convenient comparison principle for such equations, which allows to prove the uniqueness of solutions. In Section 4 we study the regularity of the solutions.

2 Background, definitions, and previous existence's and regularity results for singular and degenerate Elliptic equations

$|\nabla u|^\alpha F(D^2u) = f.$

We begin to recall the definition of viscosity solutions adapted to the present context. We denote by \mathcal{S} the space of symmetric matrices on \mathbb{R}^N . Let us define for $f \in C(\Omega)$

$$G(x, u, q, X) = |q|^\alpha (F(X) + h(x) \cdot q) - f \tag{3}$$

where $x \in \mathbb{R}^N, q \in \mathbb{R}^N, u \in \mathbb{R}$ and $X \in \mathcal{S}$.

Definition 2.1 *A function u , upper semicontinuous (USC for short) in Ω is a viscosity sub-solution for (3) (or a solution of $G[u] \geq 0$, a sub-solution of $G[u] = 0$) if whenever $\varphi \in C^2(\Omega)$ and $u - \varphi$ attains a local maximum at $\bar{x} \in \Omega$, then*

1) *Either $\nabla\varphi(\bar{x}) \neq 0$ and*

$$G(\bar{x}, u(\bar{x}), D\varphi(\bar{x}), D^2\varphi(\bar{x})) \geq 0.$$

2) *Or there exists a ball around \bar{x} on which $u(x) = u(\bar{x})$, and*

$$-f(\bar{x}) \geq 0.$$

Similarly, u , lower semicontinuous (LSC for short) in Ω is a viscosity super-solution for (3) (or a solution of $G[u] \leq 0$, a super-solution of $G[u] = 0$) if whenever $\varphi \in C^2(\Omega)$ and $u - \varphi$ attains a local minimum at $\bar{x} \in \Omega$, and $\nabla\varphi(\bar{x}) \neq 0$ then

$$G(\bar{x}, u(\bar{x}), D\varphi(\bar{x}), D^2\varphi(\bar{x})) \leq 0.$$

If u is locally constant around \bar{x}

$$-f(\bar{x}) \leq 0.$$

Of course, $u \in C(\Omega)$ is a viscosity solution of (3) (or a solution of $G[u] = 0$) if u is both a viscosity sub-solution and a viscosity super-solution.

Let $u \in \mathcal{C}(\Omega)$. We define the superjet $J^{2,+}u(x)$ and the subjet $J^{2,-}u(x)$ of the second order.

Definition 2.2

$$J^{2,+}u(x) = \left\{ (p, X) \in \mathbb{R}^N \times \mathbb{S}, \text{ so that } u(x) \leq u(\bar{x}) + \langle p, x - \bar{x} \rangle + \frac{1}{2} \langle X(x - \bar{x}), x - \bar{x} \rangle + o(|x - \bar{x}|^2) \right\}.$$

$$J^{2,-}u(x) = \left\{ (p, X) \in \mathbb{R}^N \times \mathbb{S}, \text{ so that } u(x) \geq u(\bar{x}) + \langle p, x - \bar{x} \rangle + \frac{1}{2} \langle X(x - \bar{x}), x - \bar{x} \rangle + o(|x - \bar{x}|^2) \right\}.$$

More useful are the closed superjet and closed subjet,

$$\overline{J^{2,+}u(\bar{x})} = \{(\bar{p}, \bar{X}), \exists x_n, x_n \rightarrow \bar{x}, \text{ and } (p_n, X_n) \in J^{2,+}u(x_n), (p_n, X_n) \rightarrow (\bar{p}, \bar{X})\}.$$

$\overline{J^{2,-}u(\bar{x})}$ being defined in an obvious symmetric manner.

In the definition of viscosity solutions the test functions can be substituted by the elements of the semi-jets in the sense that in the definition above one can restrict to the functions φ defined by

$$\varphi(x) = u(\bar{x}) + \langle p, x - \bar{x} \rangle + \frac{1}{2} \langle X(x - \bar{x}), x - \bar{x} \rangle$$

with $(p, X) \in J^{2,-}u(\bar{x})$ when u is a super-solution and $(p, X) \in J^{2,+}u(\bar{x})$ when u is a sub-solution. A key tool for the existence's results is the comparison principle, [4].

Theorem 2.3 *Let Ω be a bounded \mathcal{C}^2 domain in \mathbb{R}^N . Let k be continuous with respect to its variables. Suppose that u is a USC solution of*

$$|\nabla u|^\alpha (F(D^2u) + h(x) \cdot \nabla u) - k(x, u) \geq f$$

and v a LSC solution of

$$|\nabla v|^\alpha (F(D^2v) + h(x) \cdot \nabla v) - k(x, v) \leq g$$

with $f, g \in \mathcal{C}(\Omega)$, $f \geq g$ and $u \mapsto k(x, u)$ increasing, or $f > g$ and $u \mapsto k(x, u)$ non decreasing. Then if $u \leq v$ on $\partial\Omega$, $u \leq v$ in Ω .

This comparison theorem, and the construction of sub- and super-solutions which are zero on the boundary, together with a mere adaptation to our context of Perron's method, permit to prove the existence of solutions of the Dirichlet problem, for

$$\begin{cases} |\nabla u|^\alpha (F(D^2u) + h(x) \cdot \nabla u) - k(x, u) = g & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Since we are dealing with positive solutions, we will many times use the strong maximum principle :

Theorem 2.4 *Let Ω be a bounded \mathcal{C}^2 domain in \mathbb{R}^N . Suppose that h and c are continuous and let $u \geq 0$ satisfies*

$$|\nabla u|^\alpha (F(D^2u) + h(x) \cdot \nabla u) + c(x)u^{1+\alpha} \leq 0 \quad \text{in } \Omega.$$

Then one has

$$u > 0 \quad \text{or} \quad u \equiv 0.$$

For some of the results enclosed we will need "Hopf boundary principle", say the fact that near the boundary, the gradient of positive solutions cannot be zero in a neighborhood of the boundary :

Theorem 2.5 Let Ω be a bounded C^2 domain in \mathbb{R}^N . Let $x_0 \in \partial\Omega$ and $\partial\Omega$ satisfy the interior sphere condition at x_0 . Let \vec{n} denotes the inner normal to $\partial\Omega$ at x_0 . If $u > 0$ in Ω and

$$|\nabla u|^\alpha F(D^2u) + h(x) \cdot \nabla u |\nabla u|^\alpha + c(x)u^{1+\alpha} \leq 0 \quad \text{in } \Omega \quad \text{and} \quad u(x_0) = 0$$

then

$$\frac{\partial u}{\partial \vec{n}}(x_0) > 0.$$

in the sense

$$\liminf_{h \rightarrow 0, h > 0} \frac{u(x_0 + h\vec{n}) - u(x_0)}{h} > 0.$$

Remark 1 As an easy consequence, if u is a solution of the equation which is C^1 up to the boundary, we have for some $\kappa > 0$, for any $x_0 \in \partial\Omega$ such that $u(x_0) = 0$,

$$\left| \frac{\partial u}{\partial \vec{n}}(x_0) \right| \geq \kappa \quad \text{and then} \quad |\nabla u| \geq \kappa \quad \text{in a neighborhood of } x_0.$$

We now recall the Lipschitz estimates between sub- and super-solutions (See [19] for example in the case $\alpha = 0$).

Theorem 2.6 1) Let u be USC such that

$$|\nabla u|^\alpha (F(D^2u) + h(x) \cdot \nabla u) \geq f \quad \text{in } B(0, 1)$$

and v is LSC and satisfies

$$|\nabla v|^\alpha (F(D^2v) + h(x) \cdot \nabla v) \leq g \quad \text{in } B(0, 1)$$

with f and g continuous and bounded.

Then $\forall r \in (0, 1)$, there exists $L_r > 0$ such that $\forall (x, y) \in (B(0, r))^2$,

$$u(x) - v(y) \leq \sup(u - v) + L_r |x - y|.$$

Here L_r depends on r , $\sup u - \inf v$, $|f|_\infty$ and $|g|_\infty$.

In particular any solution of $|\nabla v|^\alpha (F(D^2v) + h(x) \cdot \nabla v) = f$, with f continuous and bounded, is locally Lipschitz continuous.

2) Suppose that u is a solution of the Dirichlet problem

$$\begin{cases} |\nabla u|^\alpha (F(D^2u) + h(x) \cdot \nabla u) = f, & f \text{ in } B(0, 1), \\ u = 0 & \text{on } \partial(B(0, 1)). \end{cases}$$

Then u is Lipschitz continuous, with some Lipschitz constant depending on $|f|_\infty$ and $|u|_\infty$.

As a corollary one easily has by using the definition of viscosity sub- and super-solutions

Theorem 2.7 1) Let Ω be a bounded C^2 domain in \mathbb{R}^N . Let u_n be a viscosity solution of

$$|\nabla u_n|^\alpha (F(D^2u_n) + h(x) \cdot \nabla u_n) = f_n$$

in Ω and suppose that f_n converges locally uniformly to f , and u_n is locally uniformly bounded, then one can extract from $(u_n)_n$ a subsequence such that this subsequence converges locally uniformly towards a solution of

$$|\nabla u|^\alpha (F(D^2u) + h(x) \cdot \nabla u) = f.$$

2) If moreover $u_n = 0$ on the boundary and f_n converges uniformly to f , u_n converges uniformly up to a subsequence towards u on $\overline{\Omega}$.

Let us now make precise the $\mathcal{C}^{1,\beta}$ regularity results for the solutions of

$$|\nabla u|^\alpha (F(D^2u) + h(x) \cdot \nabla u) = f,$$

when f is continuous in Ω . The first result obtained in [7] is

Theorem 2.8 *Let Ω be a bounded \mathcal{C}^2 domain in \mathbb{R}^N . Let $\alpha \in]-1, 0]$. Let f and h be continuous on $\overline{\Omega}$. There exists β so that for any u solution of*

$$\begin{cases} |\nabla u|^\alpha (F(D^2u) + h(x) \cdot \nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

$$u \in \mathcal{C}^{1,\beta}(\overline{\Omega}).$$

The case $\alpha > 0$ is treated in [20], for $h = 0$, the precise result is an interior estimate:

Theorem 2.9 *Suppose that $\alpha \geq 0$. There exists $\beta \in]0, 1[$, such that for all $r \in]0, 1[$, there exists C_r so that for any $f \in \mathcal{C}(B(0, 1))$, and for any u solution of*

$$|\nabla u|^\alpha F(D^2u) = f$$

in $B(0, 1)$, one has that u is $\mathcal{C}^{1,\beta}(B(0, r))$, with

$$|u|_{\mathcal{C}^{1,\beta}(B(0,r))} \leq C_r(|u|_\infty + |f|_\infty^{\frac{1}{1+\alpha}}).$$

To complete this interior estimate, a $\mathcal{C}^{1,\beta}(\overline{\Omega})$ regularity result is obtained in [8] :

Theorem 2.10 *Suppose that Ω is a bounded \mathcal{C}^2 domain in \mathbb{R}^N , that f and h are continuous on $\overline{\Omega}$. Let $\varphi \in \mathcal{C}^{1,\beta_0}(\partial\Omega)$. There exists $\beta \leq \inf(\beta_0, \frac{1}{1+\alpha})$, and some constant C so that for any u solution of*

$$\begin{cases} |\nabla u|^\alpha (F(D^2u) + h(x) \cdot \nabla u) = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

one has that $u \in \mathcal{C}^{1,\beta}$ on $\overline{\Omega}$ and

$$|u|_{\mathcal{C}^{1,\beta}(\overline{\Omega})} \leq C(|\varphi|_{\mathcal{C}^{1,\beta_0}(\partial\Omega)} + |f|_\infty^{\frac{1}{1+\alpha}}).$$

Other authors provide precise bounds on β , the interested reader can see [1].

We now precise the definition of the first demi eigenvalues and eigenfunctions, [5]. They are defined on the model of [3] :

$$\lambda_1^{+,c} = \sup\{\lambda, \exists \varphi > 0, |\nabla \varphi|^\alpha (F(D^2\varphi) + h(x) \cdot \nabla \varphi) + (c(x) + \lambda)|\varphi|^\alpha \varphi \leq 0\}.$$

It is clear that $\lambda_1^{+,c}$ exists and $\lambda_1^{+,c} \geq -|c|_\infty$.

Some precise estimates depending on the larger ball contained in Ω , and the smallest ball containing it, can be found in [5]. Even if this article will only need $\lambda_1^{+,c}$, we give the definition of the other demi eigenvalue, say:

$$\lambda_1^{-,c} = \sup\{\lambda, \exists \varphi < 0, |\nabla \varphi|^\alpha (F(D^2\varphi) + h(x) \cdot \nabla \varphi) + (c(x) + \lambda)|\varphi|^\alpha \varphi \geq 0\}.$$

Note that in the case where F is odd, the two eigenvalues coincide.

We have the following maximum and minimum principle "under the first eigenvalues"

Theorem 2.11 *Let Ω be a bounded C^2 domain in \mathbb{R}^N . Under the previous assumptions on F, h, c, α , suppose that $\tau < \lambda_1^{+,c}$ and that u , USC is a sub-solution of*

$$|\nabla u|^\alpha (F(D^2u) + h(x) \cdot \nabla u) + (c(x) + \tau)|u|^\alpha u \geq 0 \text{ in } \Omega$$

and $u \leq 0$ on $\partial\Omega$. Then $u \leq 0$ in Ω .

Suppose that $\tau < \lambda_1^{-,c}$ and that u , LSC is a super-solution of

$$|\nabla u|^\alpha (F(D^2u) + h(x) \cdot \nabla u) + (c(x) + \tau)|u|^\alpha u \leq 0 \text{ in } \Omega$$

and $u \geq 0$ on $\partial\Omega$. Then $u \geq 0$ in Ω .

This Theorem allows to prove the existence of a positive eigenfunction for $\lambda_1^{+,c}$ and a negative one for $\lambda_1^{-,c}$:

Theorem 2.12 *Let Ω be a bounded C^2 domain in \mathbb{R}^N . Under the previous assumptions on F, h, c, α , there exists a positive eigenfunction associated to $\lambda_1^{+,c}$, more precisely*

$$\begin{cases} |\nabla \varphi_1^+|^\alpha (F(D^2 \varphi_1^+) + h(x) \cdot \nabla \varphi_1^+) + (c(x) + \lambda_1^{+,c})(\varphi_1^+)^{1+\alpha} = 0 & \text{in } \Omega \\ \varphi_1^+ = 0 & \text{on } \partial\Omega. \end{cases}$$

There exists a negative eigenfunction associated to $\lambda_1^{-,c}$, more precisely

$$\begin{cases} |\nabla \varphi_1^-|^\alpha (F(D^2 \varphi_1^-) + h(x) \cdot \nabla \varphi_1^-) + (c(x) + \lambda_1^{-,c})|\varphi_1^-|^\alpha \varphi_1^- = 0 & \text{in } \Omega \\ \varphi_1^- = 0 & \text{on } \partial\Omega. \end{cases}$$

In the following we will drop the exponent $+$ since we will only use $\lambda_1^{+,c}$.

3 Existence and uniqueness of viscosity solutions

3.1 Existence of viscosity sub- and super-solutions : Proof of Theorem 1.1

Let $\beta_{c,\alpha,\gamma} = \frac{(1+\alpha+\gamma)c}{2+\alpha}$. We begin with some remark about the validity of assumption $\lambda_1^c > 0$, assumed in the existence's Theorem:

Remark 2 *If we take for example $|c|_\infty < \lambda_1^0$, which is known to be > 0 one has $\lambda_1^c > 0$. The same is true for $|\beta_{c,\alpha,\gamma}|_\infty < \lambda_1^0$.*

We begin to exhibit a sub- and a super-solution. Let ϕ_1 be an eigenfunction for $\lambda_1^{\beta_{c,\alpha,\gamma}}$. We first treat the case $\gamma > 1$:

Proposition 1 *Let assume $\gamma > 1$. Let $t = \frac{2+\alpha}{1+\alpha+\gamma}$. There exist b_i , $i = 1, 2$ so that $\psi_i = b_i \phi_1^t$ are respectively sub- and super-solutions.*

Proof. We do the sub-solution case. Let $\psi = b \phi_1^t$. Then $\nabla \psi = bt \phi_1^{t-1} \nabla \phi_1$.

$$D^2 \psi = bt(t-1) \phi_1^{t-2} \nabla \phi_1 \otimes \nabla \phi_1 + bt \phi_1^{t-1} D^2 \phi_1.$$

Then

$$\begin{aligned}
|\nabla\psi|^\alpha(F(D^2\psi)) &+ h(x) \cdot \nabla\psi + c(x)\psi^{1+\alpha} \\
&\geq Ab^{1+\alpha}t^{1+\alpha}(t-1)\phi_1^{t-2+(t-1)\alpha}|\nabla\phi_1|^{2+\alpha} \\
&+ b^{1+\alpha}t^{1+\alpha}\phi_1^{(t-1)(1+\alpha)}\left(|\nabla\phi_1|^\alpha(F(D^2\phi_1) + h(x) \cdot \nabla\phi_1) + \frac{c}{t^{1+\alpha}}\phi_1^{1+\alpha}\right) \\
&= (bt)^{1+\alpha}\phi_1^{(t-1)(1+\alpha)}\left(|\nabla\phi_1|^\alpha(F(D^2\phi_1) + h(x) \cdot \nabla\phi_1) + \left(\frac{c}{t^{1+\alpha}} + \lambda_1^{c,\alpha,\beta}\right)\phi_1^{1+\alpha}\right) \\
&- b^{1+\alpha}t^{1+\alpha}\phi_1^{(t-1)\alpha+t-2}(A(1-t)|\nabla\phi_1|^{2+\alpha} + \lambda_1^{c,\beta,\alpha}\phi_1^{2+\alpha}) \\
&= -b^{1+\alpha}t^{1+\alpha}\phi_1^{t-2+(t-1)\alpha}\left((1-t)A|\nabla\phi_1|^{2+\alpha} + \lambda_1^{\beta c,\alpha,\gamma}\phi_1^{2+\alpha}\right)
\end{aligned}$$

and then if we denote

$$q^A(x, b) = b^{1+\alpha}t^{1+\alpha}\phi_1^{t-2+(t-1)\alpha}\left((1-t)A|\nabla\phi_1|^{2+\alpha} + \lambda_1^{\beta c,\alpha,\gamma}\phi_1^{2+\alpha}\right),$$

one has

$$|\nabla\psi|^\alpha(F(D^2\psi) + h(x) \cdot \nabla\psi) + c(x)\psi^{1+\alpha} + q^A(x, b) \geq 0.$$

Note that using analogous computations, one has

$$|\nabla\psi|^\alpha(F(D^2\psi) + h(x) \cdot \nabla\psi) + c(x)\psi^{1+\alpha} + q^a(x, b) \leq 0$$

where q^a is defined by replacing A by a in the definition of q^A .

Claim: one has the existence of positive constants d_i , $i = 1, 2$ so that

$$d_2 \leq A(1-t)|\nabla\phi_1|^{2+\alpha} + \lambda_1^{\beta c,\alpha,\gamma}\phi_1^{2+\alpha} \leq d_1. \quad (4)$$

Let us admit for a while the claim, and let us take

$$b_1 = \left(\frac{\min p}{d_1 t^{1+\alpha}}\right)^{\frac{1}{1+\alpha+\gamma}},$$

then $q^A(x, b_1) \leq p(x)\psi_1^{-\gamma}$ and

$$\begin{aligned}
|\nabla(b_1\phi^t)|^\alpha(F(D^2(b_1\phi^t))) &+ h(x) \cdot \nabla(b_1\phi^t) \\
&+ c(x)(b_1\phi_1^t)^{1+\alpha} + p(x)(b_1\phi_1^t)^{-\gamma} \\
&\geq |\nabla(b_1\phi_1^t)|^\alpha(F(D^2(b_1\phi_1^t) + h(x) \cdot \nabla(b_1\phi_1^t))) \\
&+ c(x)(b_1\phi_1^t)^{1+\alpha} + q^A(x, b_1).
\end{aligned}$$

Then with that choice of b_1 , $\psi_1 = b_1\phi_1^t$ is a sub-solution of (1).

In the same manner using the left hand side inequality of (4) and taking

$$b_2 = \left(\frac{\max p}{d_2 t^{1+\alpha}}\right)^{\frac{1}{1+\alpha+\gamma}},$$

$\psi_2 = b_2\phi_1^t$ is a super-solution of (1). Note that $b_1 < b_2$ and then $\psi_1 \leq \psi_2$.

We now prove claim (4): Since ϕ_1 is in $C^1(\bar{\Omega})$ the inequality on the right side of (4) is obvious.

To prove the left hand side inequality, near the boundary, using Hopf principle and the fact that ϕ_1 is of class C^1 , $\exists \delta$, $d(x, \partial\Omega) < \delta \Rightarrow |\nabla\phi_1| > m$ (anywhere else one will use $|\nabla\phi_1| \geq 0$).

Now, using $\phi_1 > 0$, $\exists \tilde{m}$ such that $\phi_1 \geq \tilde{m}$ on $d(x, \partial\Omega) \geq \delta$. From all this we derive that

$$c_1|\nabla\phi_1|^{2+\alpha} + c_2\phi_1^{2+\alpha} \geq \min(c_1, c_2) \min(m^{2+\alpha}, \tilde{m}^{2+\alpha}).$$

Taking

$$d_2 = \min(c_1, c_2) \min(m^{2+\alpha}, \tilde{m}^{2+\alpha})$$

we have the left-side inequality of (4). \square

We now treat the case $\gamma < 1$. We begin to introduce a regularized problem depending on some parameter $\delta > 0$,

$$\begin{cases} |\nabla u|^\alpha (F(D^2 u) + h(x) \cdot \nabla u) + c(x)u^{1+\alpha} + p(x)(u + \delta)^{-\gamma} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (5)$$

Lemma 3.1 *Let $\gamma > 0$, and c be so that $\lambda_1^c > 0$, let ψ_1 be some positive eigenfunction for λ_1^c . Then, there exist ε_0 and δ_0 so that for $\varepsilon < \varepsilon_0$ and $\delta \in [0, \delta_0]$, then $u_\star = \varepsilon\psi_1$ is a sub-solution of (5).*

Proof.

Let $u_\star = \varepsilon\psi_1$. Take

$$\delta_0 = \frac{1}{2^{\frac{\gamma}{1+\alpha+\gamma}}} \left(\frac{\min p}{\lambda_1^c} \right)^{\frac{1}{1+\alpha+\gamma}} \quad \text{and} \quad \varepsilon_0 = \frac{1}{2^{\frac{\gamma}{1+\alpha+\gamma}} |\psi_1|_\infty} \left(\frac{\min p}{\lambda_1^c} \right)^{\frac{1}{1+\alpha+\gamma}}.$$

Then, by an easy computation, one has for all $\varepsilon < \varepsilon_0$ and $\delta < \delta_0$,

$$|\nabla u_\star|^\alpha (F(D^2 u_\star) + h(x) \cdot \nabla u_\star) + c(x)u_\star^{1+\alpha} + p(x)(u_\star + \delta)^{-\gamma} \geq 0.$$

\square

Proposition 2 *Suppose that $\gamma < 1$, and $s < 1$ sufficiently close to 1 in order that $\lambda_1^{cs^{-(1+\alpha)}} > 0$. If ψ_2 is an eigenfunction corresponding to $\lambda_1^{cs^{-(1+\alpha)}}$, then there exists d great enough in order that $u^\star = d\psi_2^s$ is a super-solution of (1).*

Remark 3 *The fact that $\lambda_1^{cs^{-(1+\alpha)}} > 0$ when s is sufficiently close to 1 is justified by the continuity result in the Proposition 3 below.*

Proof. of Proposition 2 Let $u^\star = d\psi_2^s$. We have

$$\begin{aligned} |\nabla u^\star|^\alpha (F(D^2 u^\star) + h(x) \cdot \nabla u^\star) + c(u^\star)^{1+\alpha} + p(x)(u^\star)^{-\gamma} \\ \leq d^{1+\alpha} s^{1+\alpha} \psi_2^{s-2+(s-1)\alpha} \left((s-1) |\nabla \psi_2|^{2+\alpha} a \right. \\ \left. + |\nabla \psi_2|^\alpha (F(D^2 \psi_2) + h(x) \cdot \nabla \psi_2) \right) + c(x)(d\psi_2^s)^{1+\alpha} + p(x)d^{-\gamma} \psi_2^{-\gamma s} \\ = (ds)^{1+\alpha} \psi_2^{(s-1)(1+\alpha)} \left(|\nabla \psi_2|^\alpha F(D^2 \psi_2) + h(x) \cdot \nabla \psi_2 + \left(\frac{c}{s^{1+\alpha}} + \lambda_1^{cs^{-(1+\alpha)}} \right) \psi_2^{1+\alpha} \right) \\ - d^{1+\alpha} s^{1+\alpha} \psi_2^{(s-1)\alpha+t-2} \left(a(1-s) |\nabla \psi_2|^{2+\alpha} + \lambda_1^{cs^{-(1+\alpha)}} \psi_2^{2+\alpha} \right) + p(x)d^{-\gamma} \psi_2^{-\gamma s} \\ = -d^{1+\alpha} s^{1+\alpha} \psi_2^{s-2+(s-1)\alpha} \left((1-s) |\nabla \psi_2|^{2+\alpha} a + \lambda_1^{\frac{c}{s^{1+\alpha}}} \psi_2^{2+\alpha} \right) + p(x)d^{-\gamma} \psi_2^{-\gamma s}. \end{aligned}$$

Since $\gamma < 1$ one has $\frac{2+\alpha}{1+\alpha+\gamma} > 1$, and then if $s < 1$, one has $-s(\gamma+1) + (1-s)\alpha + 2 > 0$.

Then denoting

$$\kappa = \min_{x \in \Omega} \left((1-s) |\nabla \psi_2|^{2+\alpha} a + \lambda_1^{\frac{c}{s^{1+\alpha}}} \psi_2^{2+\alpha} \right)$$

(the existence of κ can be proved in the same manner as the existence of d_2 in (4)) and assuming d large enough in order that

$$d \geq \left(\frac{|p|_\infty (\max \psi_2(x))^{-s(\gamma+1)+(1-s)\alpha+2}}{\kappa s^{1+\alpha}} \right)^{\frac{1}{\alpha+\gamma+1}}.$$

We have

$$|\nabla u^*|^\alpha (F(D^2 u^*) + h(x) \cdot \nabla u^*) + c(u^*)^{1+\alpha} + p(x)(u^*)^{-\gamma} \leq 0.$$

So u^* is a super-solution of equation (1). \square

Furthermore for ε chosen small and d large we have that $u_* < u^*$. Indeed let ψ_1 be a positive eigenfunction for λ_1^c and ψ_2 a positive eigenfunction for $\lambda_1^{c s^{-(1+\alpha)}}$, by the results in [4], there exist positive constants c_i $i = 1, \dots, 4$ so that

$$c_1 d(x, \partial\Omega) \leq \psi_1 \leq c_2 d(x, \partial\Omega), \quad c_3 d(x, \partial\Omega) \leq \psi_2 \leq c_4 d(x, \partial\Omega)$$

and then taking ε small enough and d large enough so that

$$\varepsilon c_2 (\text{diam } \Omega)^{1-s} < d c_3^s$$

one gets that

$$\varepsilon \psi_1 \leq \varepsilon c_2 d(x, \partial\Omega) < d c_3^s d(x, \partial\Omega)^s \leq d \psi_2^s.$$

Proposition 3 *The map $s \mapsto \lambda_1^{c s^{-(1+\alpha)}}$ is continuous. More generally, if c_n converge uniformly to c we have $\lim \lambda_1^{c_n} = \lambda_1^c$.*

Proof. On one hand we have $\limsup \lambda_1^{c_n} \leq \lambda_1^c$. Indeed, let $\lambda = \limsup \lambda_1^{c_n}$, there exists φ_n so that $\varphi_n > 0$, $|\varphi_n|_\infty = 1$ and

$$|\nabla \varphi_n|^\alpha (F(D^2 \varphi_n) + h(x) \cdot \nabla \varphi_n) + (c_n + \lambda_1^{c_n}) \varphi_n = 0.$$

Using the uniform Lipschitz estimate in Theorem 2.6, (φ_n) is uniformly Lipschitz continuous, then one can extract from $(\varphi_n)_n$ a subsequence which converges uniformly on $\overline{\Omega}$ toward some function φ , in particular $(c_n + \lambda_1^{c_n}) \varphi_n$ converges uniformly toward $(c + \lambda) \varphi$, and using Theorem 2.7, one gets that φ satisfies

$$|\nabla \varphi|^\alpha (F(D^2 \varphi) + h(x) \cdot \nabla \varphi) + (c + \lambda) \varphi = 0.$$

By the strong maximum principle, since $|\varphi|_\infty = 1$, $\varphi > 0$ in Ω and then by the definition of λ_1^c ,

$$\lambda_1^c \geq \lambda.$$

On the other hand let $\lambda < \lambda_1^c$, by the existence's result in [4] there exists ψ so that $\psi > 0$ and

$$|\nabla \psi|^\alpha (F(D^2 \psi) + h(x) \cdot \nabla \psi) + (\lambda + c) \psi = -1.$$

Let then N so that for $n > N$ $|c_n - c| |\psi|_\infty < \frac{1}{2}$. Then for such n

$$|\nabla \psi|^\alpha (F(D^2 \psi) + h(x) \cdot \nabla \psi) + (\lambda + c_n) \psi \leq \frac{-1}{2}$$

and then for this range of values of n , $\lambda_1^{c_n} \geq \lambda$. Since λ is arbitrary less than λ_1^c one gets $\liminf \lambda_1^{c_n} \geq \lambda_1^c$. \square

Remark 4 *The function u^* is also a super-solution of the regularized problem (5).*

3.2 Proof of the existence's Theorem

Let us now prove the existence result in Theorem 1.1. We will proceed in two steps.

Step1:

Let u_* and u^* be respectively the sub- and super-solutions (both for the regularized problem) constructed above, with $u_* \leq u^*$. Let k be defined by

$$k > \max \left\{ \frac{\gamma}{1 + \alpha} \frac{|p|_\infty}{\delta^{\alpha+\gamma+1}}; |c|_\infty \right\}.$$

Then the functions

$$f_\delta(x, u) = -k(u + \delta)^{1+\alpha} - p(x)(u + \delta)^{-\gamma}$$

and

$$g_\delta(x, u) = c(x)(u + \delta(\alpha))^{1+\alpha} - k(u + \delta)^{1+\alpha}$$

where

$$\delta(\alpha) = \begin{cases} \delta & \text{if } \alpha < 0 \\ 0 & \text{if not,} \end{cases}$$

are decreasing with respect to $u > 0$.

In the sequel will suppose $\alpha \geq 0$, the case $\alpha < 0$ is left to the reader.

Let us consider the sequence $\{w_n\}$, defined in a recursive way by

$$\begin{cases} |\nabla w_n|^\alpha (F(D^2 w_n) + h(x) \cdot \nabla w_n) + c(x)|w_n|^\alpha w_n - k(w_n + \delta)^{1+\alpha} = f_\delta(x, w_{n-1}) & \text{in } \Omega \\ w_n = 0 & \text{on } \partial\Omega \end{cases},$$

with $w_0 = u_*$.

We will prove that for all $n \in \mathbb{N}$, $u_* \leq w_n \leq w_{n+1} \leq u^*$. Let us show by induction that $\{w_n\}$ is non decreasing.

To prove that $w_1 \geq w_0$ note that :

$$|\nabla w_0|^\alpha (F(D^2 w_0) + h(x) \cdot \nabla w_0) + c(x)|w_0|^\alpha w_0 - k(w_0 + \delta)^{1+\alpha} \geq f_\delta(x, w_0)$$

and

$$|\nabla w_1|^\alpha (F(D^2 w_1) + h(x) \cdot \nabla w_1) + c(x)|w_1|^\alpha w_1 - k(w_1 + \delta)^{1+\alpha} = f_\delta(x, w_0).$$

Using the comparison Theorem 2.3 (with $w_0 = w_1$ on $\partial\Omega$) we have that $w_1 \geq w_0$.

Suppose that $w_n \geq w_{n-1}$ and let us show that $w_{n+1} \geq w_n$:

Since f_δ is decreasing, we have that

$$\begin{aligned} |\nabla w_n|^\alpha (F(D^2 w_n) + h(x) \cdot \nabla w_n) + c(x)|w_n|^\alpha w_n - k(w_n + \delta)^{1+\alpha} \\ = f_\delta(x, w_{n-1}) \geq f_\delta(x, w_n). \end{aligned}$$

This implies that

$$\begin{aligned} |\nabla w_n|^\alpha (F(D^2 w_n) + h(x) \cdot \nabla w_n) + c(x)|w_n|^\alpha w_n - k(w_n + \delta)^{1+\alpha} \\ \geq |\nabla w_{n+1}|^\alpha (F(D^2 w_{n+1}) + h(x) \cdot \nabla w_{n+1}) \\ + c(x)|w_{n+1}|^\alpha w_{n+1} - k(w_{n+1} + \delta)^{1+\alpha}, \end{aligned}$$

and using the comparison Theorem 2.3 (with $w_n = w_{n+1}$ on $\partial\Omega$) one gets $w_{n+1} \geq w_n$.

We have shown that $\{w_n\}$ is non decreasing and since $w_0 > 0$ in Ω one gets $w_n > 0$ in Ω for all $n \geq 0$.

Using the fact that u^* satisfies $|\nabla u^*|^\alpha (F(D^2 u^*) + h(x) \cdot \nabla u^*) + c(x)|u^*|^\alpha u^* - k(u^* + \delta)^{1+\alpha} \leq f_\delta(x, u^*) \leq f_\delta(x, w_n)$ we get at each step that $w_{n+1} \leq u^*$, once we have assumed that $w_n \leq u^*$.

Since the sequence $\{w_n\}$ satisfies the Lipschitz estimates recalled in Theorem 2.6 it converges uniformly to a function Z_δ which satisfies

$$|\nabla Z_\delta|^\alpha (F(D^2 Z_\delta) + h(x) \cdot \nabla Z_\delta) + c(x)|Z_\delta|^\alpha Z_\delta + p(x)(Z_\delta + \delta)^{-\gamma} = 0.$$

Furthermore for any δ one has $u_* \leq Z_\delta \leq u^*$.

Step2: δ tends to 0

Let δ and Z_δ defined by the first step. We note that since $u_* \leq Z_\delta \leq u^*$, the term $p(x)(Z_\delta + \delta)^{-\gamma}$ is uniformly locally bounded independently on δ , and then Z_δ is uniformly locally Lipschitz. It follows, using the uniform Lipschitz estimates in Theorem 2.6, that one can extract from Z_δ a sequence which converges locally uniformly to some function Z , such that $u_* \leq Z \leq u^*$. Passing to the limit with Theorem 2.7, one gets that, since $p(Z_\delta + \delta)^{-\gamma}$ converges locally uniformly (for a subsequence) towards $pZ^{-\gamma}$, Z is a solution of equation (1).

3.3 Comparison principle and Uniqueness result

We begin to prove some Lipschitz estimate between sub- and super-solutions of equation (1)

Theorem 3.2 *Suppose that u is a positive, bounded by above, solution of*

$$\begin{cases} |\nabla u|^\alpha (F(D^2 u) + h(x) \cdot \nabla u) + p(x)u^{-\gamma} \geq f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

and v is a positive solution of

$$\begin{cases} |\nabla v|^\alpha (F(D^2 v) + h(x) \cdot \nabla v) + p(x)v^{-\gamma} \leq g & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

with f, g and h continuous and bounded, and $p > 0$ is Hölder continuous of exponent τ_p . Then :

- If $\sup_{\overline{\Omega}}(u - v) > 0$, there exists c depending on $\Omega, |u|_\infty, |f|_\infty, |g|_\infty, |h|_\infty, |p|_\infty$ so that for all (x, y) in $\overline{\Omega}^2$

$$u(x) - v(y) \leq \sup(u - v) + C|x - y|$$

- If $\sup_{\overline{\Omega}}(u - v) = 0$, and if there exist $\tau_1 \leq \inf(1, \frac{2}{1+\gamma})$, and $C_1 > 0$ so that

$$u(x) \leq C_1 d(x, \partial\Omega)^{\tau_1},$$

then, there exists some constant C depending on $\Omega, |u|_\infty, |f|_\infty, |g|_\infty, |h|_\infty, \tau_p$, and C_1 so that for all (x, y) in $\overline{\Omega}^2$

$$u(x) - v(y) \leq C|x - y|^\tau,$$

with $\tau = \inf(\tau_1, \frac{2+\alpha+\tau_p}{1+\alpha+\gamma})$.

Remark 5 *In the sequel, we will apply this result with $f = -cu^{1+\alpha}$ and $g = -cv^{1+\alpha}$, mainly to prove the uniqueness. On the other hand, we get Hölder regularity of this solution, by the second part of the Theorem above, by recalling that from the first sections, one has an exponent τ_1 which can be taken arbitrarily close to 1 in the case $\gamma < 1$ and is equal to $\frac{2}{1+\gamma}$ if $\gamma > 1$.*

Proof

Let ω be defined on \mathbb{R}^+ by

$$\omega(s) = s - \frac{s^{1+\varepsilon}}{2(1+\varepsilon)}$$

where $\varepsilon \in]0, 1[$. Let us introduce

$$\psi(x, y) = u(x) - v(y) - \sup(u - v) - M\omega(|x - y|)$$

where M will be chosen large enough later.

It is clear that it is sufficient to prove that for $|x - y| < \frac{1}{2}$,

$$\psi(x, y) \leq 0$$

Indeed, if $|x - y| > \frac{1}{2}$ and if we assume that $\frac{M}{2} > \sup u - \inf v$ the required result holds.

We argue by contradiction and suppose that $\sup_{(x,y) \in \overline{\Omega}^2} \psi(x, y) > 0$. Then by the upper-semicontinuity of ψ , it is achieved on some pair $(\bar{x}, \bar{y}) \in \overline{\Omega}^2$. In the following $\delta > 0$ is a positive parameter, take

$$M = \frac{2(\sup u - \inf v)}{\delta},$$

then from the definition of \bar{x} and \bar{y} , $|\bar{x} - \bar{y}| \leq \delta$. So saying that M is large is equivalent to say that δ is small.

We first remark that neither \bar{x} nor \bar{y} belongs to the boundary. Indeed, \bar{x} cannot be on the boundary by the positivity of v , and if $\bar{y} \in \partial\Omega$ then we would have

$$u(\bar{x}) \geq \sup(u - v) + \frac{M}{2}|\bar{x} - \bar{y}|. \quad (6)$$

This is contradicted by the continuity of u , since there exists δ_1 so that for $d(x, \partial\Omega) < \delta_1$ one has

$$u(x) \leq \frac{\sup(u - v)}{2}$$

while if $d(x, \partial\Omega) > \delta$

$$\frac{M}{2}d(\bar{x}, \partial\Omega) + \sup(u - v) \geq \frac{M\delta}{2} > \sup u$$

which also contradicts (6).

We have obtained that $(\bar{x}, \bar{y}) \in \Omega^2$. Furthermore $\bar{x} \neq \bar{y}$.

By Ishii's lemma (Lemma 9) in [18] (see also [6]) for all $\zeta > 0$ there exist X_ζ and Y_ζ so that

$$(q, X_\zeta) \in \overline{\mathcal{J}}^{2,+}u(\bar{x}), \quad (q, -Y_\zeta) \in \overline{\mathcal{J}}^{2,-}v(\bar{y})$$

with $q = M\omega'(|\bar{x} - \bar{y}|)$, and

$$\begin{pmatrix} X_\zeta & 0 \\ 0 & Y_\zeta \end{pmatrix} \leq M \begin{pmatrix} B & -B \\ -B & B \end{pmatrix} + \zeta M^2 \begin{pmatrix} B^2 & -B^2 \\ -B^2 & B^2 \end{pmatrix}$$

with $B = D^2(\omega(|\cdot|))(\bar{x} - \bar{y})$. One has

$$B = \left(\omega'' - \frac{\omega'}{r} \right) \frac{x \otimes x}{|x|^2} + \frac{\omega'}{r} I$$

and then

$$B^2 = \left((\omega'')^2 - \frac{(\omega')^2(r)}{r^2} \right) \frac{x \otimes x}{|x|^2} + \frac{(\omega')^2(r)}{r^2} I.$$

So taking $\zeta = \frac{1}{M(1+2|\omega''(r)|+\frac{\omega'(r)}{r})}$, $B + \zeta MB^2$ has the eigenvalues $\omega'' + \zeta M(\omega'')^2 \leq \frac{\omega''}{2}$ and $\frac{\omega'(r)}{r} + \zeta \frac{(\omega')^2}{r^2} \leq 2\frac{\omega'}{r}$. With that choice of ζ , dropping the index ζ for X_ζ and Y_ζ , one has $X + Y \leq 0$ and for any x

$${}^t(x, -x) \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \begin{pmatrix} x \\ -x \end{pmatrix} \leq 4M^t x(B + \varepsilon B^2)x.$$

In particular there exist at least one eigenvalue which is less or equal to $4\frac{\omega''}{2} = 2\omega''$. Using $X + Y \leq 0$, one has $\text{tr}(X + Y) \leq 2M\omega''$.

We have then

$$\text{tr}(X + Y) \leq -CM|\bar{x} - \bar{y}|^{\varepsilon-1}$$

while always by Ishii's lemma

$$|X| + |Y| \leq CM|B|_\infty \leq CM|\bar{x} - \bar{y}|.$$

On the other hand

$$\frac{M}{2} \leq |q| \leq M.$$

Then, using in the following lines $F(X) - F(-Y) \leq \text{atr}(X + Y)$,

$$\begin{aligned} -|p|_\infty(\sup(u - v))^{-\gamma} + f(\bar{x}) &\leq -p(\bar{x})(u(\bar{x}))^{-\gamma} + f(\bar{x}) \\ &\leq |q|^\alpha F(X) + h(\bar{x}) \cdot q|q|^\alpha \\ &\leq |q|^\alpha F(-Y) + h(\bar{y}) \cdot q|q|^\alpha + 2|h|_\infty M^{1+\alpha} \\ &\quad + M^{1+\alpha} \left(\frac{1}{2}\right)^{|\alpha|} \text{atr}(X + Y) \\ &\leq -p(\bar{y})v(\bar{y})^{-\gamma} + g(\bar{y}) + 2|h|_\infty M^{1+\alpha} \\ &\quad - CM^{1+\alpha}|\bar{x} - \bar{y}|^{\varepsilon-1} \\ &\leq 2|h|_\infty M^{1+\alpha} - CM^{1+\alpha}|\bar{x} - \bar{y}|^{\varepsilon-1} + g(\bar{y}) \end{aligned}$$

from this we get for some positive constant C , and for δ small (so that $|h|_\infty \delta^{1-\varepsilon} < \frac{C}{2}$),

$$\frac{C}{2}M^{1+\alpha}|\bar{x} - \bar{y}|^{\varepsilon-1} \leq |p|_\infty(\sup(u - v))^{-\gamma} + |f|_\infty + |g|_\infty + 2|h|_\infty M^{1+\alpha},$$

clearly a contradiction as soon as M is large enough, since $\varepsilon < 1$. We have then obtained that $\psi(x, y) \leq 0$ for all $x, y \in \bar{\Omega}^2$.

We now do the case where $\sup(u - v) = 0$. We take the function

$$\psi(x, y) = u(x) - v(y) - M|x - y|^\tau$$

where $M = \frac{2(\sup u - \inf v)}{\delta^\tau}$, this forces \bar{x} and \bar{y} to satisfy $|\bar{x} - \bar{y}| \leq \delta$. We take $\tau = \inf(\tau_1, \frac{2+\alpha+\tau_p}{1+\alpha+\gamma})$. We argue by contradiction and suppose that the supremum of ψ is > 0 . Then it is achieved on some pair $(\bar{x}, \bar{y}) \in \bar{\Omega}^2$. By taking M larger than C where C is so that

$$u(\bar{x}) \leq Cd(\bar{x}, \partial\Omega)^{\tau_1}$$

one obtains that \bar{y} cannot belong to the boundary. On the other hand, the positivity of v implies that \bar{x} cannot be on the boundary.

Note for further purpose that

$$-p(\bar{x})u(\bar{x})^{-\gamma} \geq -p(\bar{y})u(\bar{x})^{-\gamma} - C_p|\bar{x} - \bar{y}|^{\tau_p}(M|\bar{x} - \bar{y}|^\tau)^{-\gamma} \geq -p(\bar{y})v(\bar{y})^{-\gamma} - CM^{-\gamma}|\bar{x} - \bar{y}|^{\tau_p - \tau\gamma}.$$

We have then by Ishii's lemma, for all ζ the existence of X_ζ and Y_ζ in \mathcal{S} so that with $q = M\tau|\bar{x} - \bar{y}|^{\tau-1}$

$$(q, X_\zeta) \in \overline{\mathcal{J}}^{2,+}u(\bar{x}), (q, -Y_\zeta) \in \overline{\mathcal{J}}^{2,-}v(\bar{y})$$

$$\begin{pmatrix} X_\zeta & 0 \\ 0 & Y_\zeta \end{pmatrix} \leq M \begin{pmatrix} B + \zeta MB^2 & -B - \zeta MB^2 \\ -B + \zeta MB^2 & B + \zeta MB^2 \end{pmatrix}$$

with $B = D^2(|\cdot|^\tau)(\bar{x} - \bar{y})$. By the choice of ζ as the first part of the proof, and dropping the index ζ , one has $X + Y \leq 0$, and $\text{tr}(X + Y) \leq -CM|\bar{x} - \bar{y}|^{\tau-2}$.

We have then by using the fact that u and v are respectively sub- and super-solutions

$$\begin{aligned} -p(\bar{y})v(\bar{y})^{-\gamma} & - CM^{-\gamma}|\bar{x} - \bar{y}|^{\tau p - \tau\gamma} + f(\bar{x}) \\ & \leq |q|^\alpha F(X) + M^{1+\alpha}(h(\bar{x}) \cdot \bar{x} - \bar{y})|\bar{x} - \bar{y}|^{(\tau-1)\alpha-1} \\ & \leq |q|^\alpha F(-Y) + M^{1+\alpha}|h|_\infty|\bar{x} - \bar{y}|^{(\tau-1)\alpha} - CM^{1+\alpha}|\bar{x} - \bar{y}|^{(\tau-1)\alpha + \tau - 2} \\ & \leq -p(\bar{y})(v(\bar{y})^{-\gamma} + g(\bar{y}) + M^{1+\alpha}|h|_\infty|\bar{x} - \bar{y}|^{(\tau-1)\alpha} - CM^{1+\alpha}|\bar{x} - \bar{y}|^{(\tau-1)\alpha + \tau - 2}). \end{aligned}$$

From this one derives that for some constants

$$CM^{1+\alpha}|\bar{x} - \bar{y}|^{(\tau-1)\alpha + \tau - 2} \leq CM^{1+\alpha}|h|_\infty|\bar{x} - \bar{y}|^{(\tau-1)\alpha} + CM^{-\gamma}|\bar{x} - \bar{y}|^{\tau p - \tau\gamma} + |f|_\infty + |g|_\infty$$

which is a contradiction as soon as δ is small enough, by the assumption on τ . We have obtained that for all x, y in $\overline{\Omega}$

$$u(x) - v(y) \leq C|x - y|^\tau.$$

□

Corollary 1 *The solutions constructed in the proof of the previous section are Hölder continuous up to the boundary, with an exponent τ arbitrary close to 1 when $\gamma < 1$, and $\tau \leq \frac{\tau_p + \alpha + 2}{1 + \alpha + \gamma}$, when $\gamma > 1$. They are in both cases Lipschitz continuous inside Ω .*

Proof The Lipschitz interior continuity is immediate by using the results of [5], remarking that on a compact set of Ω , by the strong maximum principle and the continuity of u , $pu^{-\gamma}$ is bounded. □

Using Theorem 3.2 we have the following comparison result between sub- and super solutions.

Theorem 3.3 *Suppose that $u, v > 0$ are respectively sub- and super-solutions of*

$$|\nabla u|^\alpha (F(D^2u) + h(x) \cdot \nabla u) + c(x)u^{1+\alpha} + p(x)u^{-\gamma} = 0 \text{ in } \Omega$$

and are zero on the boundary. Then $u \leq v$ in Ω .

Proof. Note that for $y < \varepsilon := \left(\frac{\gamma \min p}{(1+\alpha)|c|_\infty}\right)^{\frac{1}{1+\alpha+\gamma}}$ the function $y \mapsto c(x)y^{1+\alpha} + p(x)y^{-\gamma}$ is decreasing. By upper-semicontinuity, since u is zero on the boundary, for all ε there exists δ so that for $d < \delta$ one has $u(x) \leq \varepsilon$.

Suppose by contradiction that $u > v$ somewhere. We suppose first that the supremum of $u - v$ is achieved inside $d(x, \partial\Omega) \leq \delta$. Let \bar{x} be some point in this set where the supremum is achieved. Then one has $0 < v(\bar{x}) < u(\bar{x}) \leq \varepsilon$. In particular $c(\bar{x})u(\bar{x})^{1+\alpha} + p(\bar{x})(u(\bar{x}))^{-\gamma} < c(\bar{x})v(\bar{x})^{1+\alpha} + p(\bar{x})(v(\bar{x}))^{-\gamma}$.

Using the usual doubling of variables, say defining in the case $\alpha > 0$

$$\psi_j(x, y) = u(x) - v(y) - \frac{j}{2}|x_j - y_j|^2$$

(the case $\alpha < 0$ requires the changes provided at the end of the proof), there exist x_j and y_j in $\{x \in \Omega : d(x, \partial\Omega) \leq \delta\}$ and (X_j, Y_j) in \mathcal{S}^2 so that

$$(j(x_j - y_j), X_j) \in \overline{\mathcal{J}}^{2,+}u(x_j), (j(x_j - y_j), -Y_j) \in \overline{\mathcal{J}}^{2,-}v(y_j)$$

and

$$\begin{pmatrix} X_j & 0 \\ 0 & Y_j \end{pmatrix} \leq 2j \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

Furthermore by the boundary conditions, neither x_j , nor y_j belong to $\partial\Omega$.

From the Lipschitz estimates between sub and super-solutions in Theorem 3.2, one has

$$j|x_j - y_j|^2 + \sup(u - v) \leq u(x_j) - v(y_j) \leq \sup(u - v) + |x_j - y_j|.$$

From what we derive that $j|x_j - y_j|$ is bounded. In particular, using h continuous, one has $|h(x_j) - h(y_j)|(j|x_j - y_j|)^{1+\alpha} = o(1)$.

Using $X_j + Y_j \leq 0$, and, since F satisfies (2), one can write

$$\begin{aligned} -c(x_j)u(x_j)^{1+\alpha} - p(x_j)u(x_j)^{-\gamma} &\leq |j(x_j - y_j)|^\alpha (F(X_j) + h(x_j) \cdot j(x_j - y_j)) \\ &\leq |j(x_j - y_j)|^\alpha (F(-Y_j) + h(y_j) \cdot j(x_j - y_j)) + o(1) \\ &\leq -c(y_j)v(y_j)^{1+\alpha} - p(y_j)(v(y_j))^{-\gamma} + o(1), \end{aligned}$$

and using the continuity of c , h and p and passing to the limit when j goes to infinity, one gets a contradiction.

Suppose now that \bar{x} is not in $\{x \in \Omega : d(x, \partial\Omega) \leq \delta\}$. Then there exists $0 < \kappa < M$ so that for all $x \in \Omega$ such that $d(x) < \delta$, $u(x) \leq v(x) + \sup(u - v) - \kappa := v(x) + M - \kappa$. We then use the change of function $U = \log u$ and $V = \log(v + M - \kappa)$ in the set $\Omega_\delta = \{x \in \Omega, d(x, \partial\Omega) > \delta\}$. One has $U \leq V$ on the boundary, and U and V are respectively sub- and super-solutions of the equations

$$|\nabla U|^\alpha (F(D^2U + \nabla U \otimes \nabla U) + h(x) \cdot \nabla U) + c(x) + e^{-U(1+\alpha+\gamma)}p(x) \geq 0$$

and

$$|\nabla V|^\alpha (F(D^2V + \nabla V \otimes \nabla V) + h(x) \cdot \nabla V) + c(x) + e^{-V(1+\alpha+\gamma)}p(x) \leq 0.$$

Using the comparison principle for these type of equations, (see for example [10], Theorem 5.1), remarking that $y \mapsto e^{-y(1+\alpha+\gamma)}p(x)$ is decreasing one gets that $U \leq V$ everywhere in Ω_δ . This implies that $u \leq v + M - \kappa$ everywhere in Ω , a contradiction with the definition of the supremum.

We briefly give the changes to bring in the case $\alpha < 0$. In that case in the first step, the function ψ_j must be replaced by

$$\psi_j(x, y) = u(x) - v(y) - \frac{j}{q}|x_j - y_j|^q$$

where $q > \frac{\alpha+2}{1+\alpha}$. We next follow the lines in the comparison Theorem 3.3 in [9], (A key point consists in observing that $x_j \neq y_j$).

□

Corollary 2 *There is uniqueness of solution for the equation (1).*

4 Regularity of the unique viscosity solution of (1).

4.1 Interior regularity of the viscosity solution

Interior regularity of the unique viscosity solution of the problem (1) is easily obtained by the results about regularity of viscosity solutions of the following equation

$$|\nabla u|^\alpha F(D^2u) + h(x) \cdot \nabla u |\nabla u|^\alpha = f \tag{7}$$

which are recalled in the introduction, ([20], [9], [8]). Indeed, let u be a solution of equation (1), let $f = -c(x)u^{1+\alpha} - p(x)u^{-\gamma}$, and let us consider the equation

$$|\nabla v|^\alpha (F(D^2v) + h(x) \cdot \nabla v) = -c(x)u^{1+\alpha} - p(x)u^{-\gamma} \text{ in } \Omega.$$

It is immediate to check that u is a viscosity solution of this equation, hence, using the classical regularity results recalled below, $u \in \mathcal{C}^{1,\beta}(\Omega)$.

4.2 "Regularity" up to the boundary

We begin to remark that when $\gamma > 1$, even the Lipschitz regularity up to the boundary does not hold. Next we will see some cases in which we can ensure the \mathcal{C}^1 regularity for $\gamma < 1$. We note that if $x_0 \in \partial\Omega$, \vec{n} denotes the inner normal to $\partial\Omega$ at x_0 , and ϕ_1 is some eigenfunction for $\lambda_1^{\beta,c,\alpha,\gamma}$, then by Hopf boundary principle

$$\lim_{s \rightarrow 0^+} \frac{\phi_1(x_0 + s\vec{n})}{s} = \lim_{s \rightarrow 0^+} \frac{\phi_1(x_0 + s\vec{n}) - \phi_1(x_0)}{s} = \nabla\phi_1(x_0) \cdot \vec{n}.$$

If $\gamma > 1$, let us recall that for some convenient $b_1 > 0$ and for $t = \frac{2+\alpha}{1+\alpha+\gamma} < 1$, $b_1\phi_1(x)^t$ is a sub-solution, and then by the comparison principle, $u(x) \geq b_1\phi_1(x)^t$. It follows that, for $s > 0$,

$$\frac{u(x_0 + s\vec{n}) - u(x_0)}{s} \geq b_1\phi_1(x_0 + s\vec{n})^{t-1} \frac{\phi_1(x_0 + s\vec{n})}{s}.$$

Therefore since $t < 1$, $\phi_1 = 0$ on the boundary, using the existence of a positive constant C such that $\frac{\phi_1(x_0 + s\vec{n})}{s} > C$ given by Hopf principle, then

$$\lim_{s \rightarrow 0^+} \frac{u(x_0 + s\vec{n}) - u(x_0)}{s} = +\infty,$$

so u cannot be Lipschitz continuous on $\bar{\Omega}$.

4.3 Regularity up to the boundary when $N = 1$

Suppose that $N = 1$, $h = c = 0$ and $p \equiv 1$. We prove below the \mathcal{C}^1 regularity up to the boundary in the case $\gamma < 1$, while in the case $\gamma > 1$ the solution cannot be Lipschitz continuous up to the boundary. We can suppose without loss of generality that $\Omega =]0, 1[$. Let us consider the equation

$$|u'|^\alpha u'' + u^{-\gamma} = 0,$$

$$u(0) = u(1) = 0,$$

multiplying by u' and integrating one gets

$$\frac{|u'|^{2+\alpha}}{2+\alpha} + \frac{u^{1-\gamma}}{1-\gamma} = C,$$

where C is a constant, and then when $\gamma > 1$, $\lim_{x \rightarrow (0,1)} \frac{|u'|^{2+\alpha}}{2+\alpha} = +\infty$.

For the special case where $\gamma = 1$, multiplying by u' and integrating one gets the equation

$$\frac{|u'|^{2+\alpha}}{2+\alpha} + \log u = C,$$

Note that the equation is invariant by the change $x \mapsto 1-x$, u is concave and then $\frac{1}{2}$ is a maximum point for u , so $u'(\frac{1}{2}) = 0$. Then, for some positive constant C defined by $u(\frac{1}{2}) = e^C$,

$$u'(x) = \begin{cases} ((2+\alpha)(C - \log u))^{\frac{1}{2+\alpha}} & \text{if } x < \frac{1}{2} \\ -((2+\alpha)(C - \log u))^{\frac{1}{2+\alpha}} & \text{if } x > \frac{1}{2}. \end{cases} \quad (8)$$

Consequently,

$$\lim_{x \rightarrow 0} u' = \infty.$$

If $\gamma < 1$ the solutions are given, for some positive constant C defined by $u(\frac{1}{2}) = (C(1-\gamma))^{\frac{1}{1-\gamma}}$

$$u'(x) = \begin{cases} \left((2+\alpha) \left(C - \frac{u^{1-\gamma}}{1-\gamma} \right) \right)^{\frac{1}{2+\alpha}} & \text{if } x < \frac{1}{2} \\ - \left((2+\alpha) \left(C - \frac{u^{1-\gamma}}{1-\gamma} \right) \right)^{\frac{1}{2+\alpha}} & \text{if } x > \frac{1}{2}. \end{cases} \quad (9)$$

From equation (9), u' is continuous up to the boundary.

4.4 Existence of radial solution when $N > 1$ and regularity up to the boundary when $\gamma < 1$ and Ω is a ball.

In all this sub-section we still suppose that $h = c = 0$, $p \equiv 1$ and that $N \geq 2$.

We begin to prove the existence of a radial solution in the particular case where $F = tr$. We will do the general case later. We begin to construct in a neighborhood of 0 a solution by using a fixed point argument. First of all suppose $v(0) = 1$, $v'(0) = 0$ (necessary in the radial case). Note that we follow the method employed in particular in [16]. Let us consider the map $v \mapsto T(v)$ where

$$T(v)(r) = 1 - \int_0^r \left(\frac{1}{s^{(1+\alpha)(N-1)}} \int_0^s \lambda^{(1+\alpha)(N-1)} (1+\alpha) v^{-\gamma}(\lambda) d\lambda \right)^{\frac{1}{1+\alpha}} ds.$$

We prove that for r_o small enough, T possesses a fixed point defined in $[0, r_o[$. Let us define

$$r_o = \left(\frac{((N-1)(1+\alpha) + 1)^{\frac{1}{1+\alpha}} (2+\alpha)}{2^{1 + \frac{1+\alpha}{1+\alpha} \gamma} (\max(\gamma, 1+\alpha))^{\frac{1}{1+\alpha}} (1+\alpha)} \right)^{\frac{1+\alpha}{2+\alpha}},$$

and let us consider the ball

$$B = \left\{ v \in \mathcal{C}(B(0, r_o)), |v(x) - 1|_{\infty} < \frac{1}{2} \right\}.$$

Then for $r < r_o$, T maps B into itself. Indeed :

$$\begin{aligned}
|T(v) - 1|_\infty &\leq \int_0^r \left(\frac{1}{s^{(1+\alpha)(N-1)}} \int_0^s \lambda^{(1+\alpha)(N-1)} (1+\alpha) 2^\gamma d\lambda \right)^{\frac{1}{1+\alpha}} ds \\
&= r^{\frac{2+\alpha}{1+\alpha}} \frac{(2^\gamma(1+\alpha))^{\frac{1}{1+\alpha}} (1+\alpha)}{((N-1)(1+\alpha)+1)^{\frac{1}{1+\alpha}} (2+\alpha)} \\
&\leq \frac{1}{2}.
\end{aligned}$$

In order to check that T is a contracting mapping on B , we denote for v, w in B

$$X(s) = \frac{1}{s^{(1+\alpha)(N-1)}} \int_0^s \lambda^{(1+\alpha)(N-1)} (1+\alpha) v^{-\gamma} d\lambda$$

and

$$Y(s) = \frac{1}{s^{(1+\alpha)(N-1)}} \int_0^s \lambda^{(1+\alpha)(N-1)} (1+\alpha) w^{-\gamma} d\lambda.$$

Note that

$$\begin{aligned}
\text{i)} \quad &|v^{-\gamma} - w^{-\gamma}| \leq 2^{\gamma+1} \gamma |v - w|, \\
\text{ii)} \quad &\frac{s 2^{-\gamma} (1+\alpha)}{(1+\alpha)(N-1)+1} \leq X(s) \leq \frac{s 2^\gamma (1+\alpha)}{(1+\alpha)(N-1)+1} \\
&\quad \text{and} \quad \frac{s 2^{-\gamma} (1+\alpha)}{(1+\alpha)(N-1)+1} \leq Y(s) \leq \frac{s 2^\gamma (1+\alpha)}{(1+\alpha)(N-1)+1}, \\
\text{iii)} \quad &|X(s) - Y(s)| \leq \frac{1+\alpha}{(1+\alpha)(N-1)+1} |v^{-\gamma} - w^{-\gamma}| s.
\end{aligned}$$

Then using i), ii), iii) and the mean value's Theorem, for some $\theta \in]0, 1[$

$$\begin{aligned}
|X(s)^{\frac{1}{1+\alpha}} - Y(s)^{\frac{1}{1+\alpha}}| &\leq \frac{1}{1+\alpha} |X(s) - Y(s)| |X(s) + \theta(Y(s) - X(s))|^{-\frac{\alpha}{1+\alpha}} \\
&\leq 2^{1+\frac{|\alpha|+1}{1+\alpha}} \gamma \frac{1}{1+\alpha} \left(\frac{1+\alpha}{(1+\alpha)(N-1)+1} \right)^{\frac{1}{1+\alpha}} s^{\frac{1}{1+\alpha}} |v - w|.
\end{aligned}$$

As a consequence

$$\begin{aligned}
|T(v) - T(w)| &\leq \int_0^r |X(s)^{\frac{1}{1+\alpha}} - Y(s)^{\frac{1}{1+\alpha}}| ds \\
&\leq 2^{1+\frac{1+|\alpha|}{1+\alpha}} \gamma \frac{1}{1+\alpha} \left(\frac{1+\alpha}{(1+\alpha)(N-1)+1} \right)^{\frac{1}{1+\alpha}} \left(\frac{1+\alpha}{2+\alpha} \right) r^{\frac{2+\alpha}{1+\alpha}} |v - w|,
\end{aligned}$$

and then T is a contraction mapping.

This gives the local existence and uniqueness of a fixed point, denoted u , around 0. We can suppose, up to replace r_o by some smaller number, that $u' < 0$ and $u > 0$ in the whole interval $]0, r_o[$. Now if $r_1 > 0$ is so that $u'(r_1) < 0$, Cauchy Lipschitz Theorem gives the local existence and uniqueness of a solution. For that it is sufficient to consider the ordinary differential equation :

$$\begin{pmatrix} v' \\ w' \end{pmatrix} := \varphi(v, w) = \begin{pmatrix} |w|^{-\frac{\alpha}{\alpha+1}} w \\ -v^{-\gamma} - \frac{N-1}{r} w \end{pmatrix},$$

with $v(r_1) = u(r_1)$, $w(r_1) = u'(r_1)$, and to observe that φ is a Lipschitz function of (v, w) as long as neither w , nor v takes the value 0. We denote by u the fixed point for T in $[0, r_1]$ extended by the unique local solution of the previous ODE, as long as it is defined.

Note that u satisfies on $[r_1, r[$

$$u(r) = u(r_1) - \int_{r_1}^r \left(\frac{1}{s^{(1+\alpha)(N-1)}} \int_0^s \lambda^{(1+\alpha)(N-1)} (1+\alpha) u^{-\gamma}(\lambda) d\lambda \right)^{\frac{1}{1+\alpha}} ds.$$

Then as long as $u > 0$ one has

$$u(r) = 1 - \int_0^r \left(\frac{1}{s^{(1+\alpha)(N-1)}} \int_0^s \lambda^{(1+\alpha)(N-1)} (1+\alpha) u^{-\gamma}(\lambda) d\lambda \right)^{\frac{1}{1+\alpha}} ds.$$

Since u has values in $]0, 1[$, and u is not identically equal to 1

$$\int_0^r \left(\frac{1}{s^{(1+\alpha)(N-1)}} \int_0^s \lambda^{(1+\alpha)(N-1)} (1+\alpha) u^{-\gamma}(\lambda) d\lambda \right)^{\frac{1}{1+\alpha}} ds \geq r^{\frac{2+\alpha}{1+\alpha}} \frac{(1+\alpha)^{\frac{2+\alpha}{1+\alpha}}}{((N-1)(1+\alpha)+1)^{\frac{1}{1+\alpha}} (2+\alpha)}$$

and then taking R so that $R^{\frac{2+\alpha}{1+\alpha}} \frac{(1+\alpha)^{\frac{2+\alpha}{1+\alpha}}}{((N-1)(1+\alpha)+1)^{\frac{1}{1+\alpha}} (2+\alpha)} > 1$, one obtains that there exists $\bar{r} < R$, so that $u(\bar{r}) = 0$.

We then consider \tilde{u} defined as :

$$\tilde{u}(r) = Cu(\bar{r} r), \text{ with } C := \bar{r}^{-\frac{2+\alpha}{\gamma+\alpha+1}}.$$

Then \tilde{u} solves the equation

$$|\tilde{u}'|^\alpha (\tilde{u}'' + \frac{N-1}{r} \tilde{u}') + \tilde{u}^{-\gamma} = 0$$

and $\tilde{u}(1) = 0$.

The computations above can easily be generalized to the case where p is a radial function which satisfies the assumptions of the article.

We now observe that in the radial case, when $\gamma < 1$ the solution \tilde{u} above is \mathcal{C}^1 . Indeed, multiplying

$$|\tilde{u}'|^\alpha \tilde{u}'' + \frac{N-1}{r} |\tilde{u}'|^\alpha \tilde{u}' + \tilde{u}^{-\gamma} = 0$$

by $\tilde{u}' r^{(N-1)(2+\alpha)}$ and integrating, one has

$$\frac{d}{dr} \left(\frac{|\tilde{u}'|^{2+\alpha}}{2+\alpha} r^{(N-1)(2+\alpha)} + \frac{\tilde{u}^{1-\gamma}}{1-\gamma} r^{(N-1)(2+\alpha)} \right) = (N-1)(2+\alpha) r^{(N-1)(2+\alpha)-1} \frac{\tilde{u}^{1-\gamma}}{1-\gamma},$$

and then, integrating between $\frac{1}{2}$ and r one gets that

$$\frac{|\tilde{u}'|^{2+\alpha}}{2+\alpha} r^{(N-1)(2+\alpha)} + \frac{\tilde{u}^{1-\gamma}}{1-\gamma} r^{(N-1)(2+\alpha)} - C = \frac{1}{1-\gamma} \int_{\frac{1}{2}}^r (N-1)(2+\alpha) s^{(N-1)(2+\alpha)-1} \tilde{u}^{1-\gamma}(s) ds,$$

where $C = \left(\frac{|\tilde{u}'|^{2+\alpha}}{2+\alpha} r^{(N-1)(2+\alpha)} + \frac{\tilde{u}^{1-\gamma}}{1-\gamma} r^{(N-1)(2+\alpha)} \right) \left(\frac{1}{2} \right)$, which proves that $|\tilde{u}'|^{2+\alpha}$ has a finite limit when r goes to 1.

We now do the general case . We begin with the case of one of the Pucci's operator. We will deduce the general case by using the fact that the operator F is sandwiched between the two Pucci's operators, $\mathcal{M}_{a,A}^+$ and $\mathcal{M}_{a,A}^-$.

Suppose that $F = \mathcal{M}_{a,A}^+$. We begin to prove local existence and uniqueness of a solution near 0. For that aim we argue as in the case of the Laplacian , say we observe that if r_o is replaced by $r_o a^{\frac{1}{2+\alpha}}$

and B is defined as in the Laplacian case, we have a fixed point, denoted u_o , defined on $[0, r_o[$ for the operator T in B :

$$T(v) = 1 - \int_0^r \left(\frac{1}{s^{(N-1)(1+\alpha)}} \int_0^s \lambda^{(N-1)(1+\alpha)} \frac{(1+\alpha)}{a} v^{-\gamma} d\lambda \right)^{\frac{1}{1+\alpha}} ds.$$

Up to replace r_o by some smaller number, one can assume that for $r < r_o$, $\frac{u^{-\gamma}}{a} + \frac{(N-1)u'}{r} < 0$. Let r_1 be so that $r_1 \in]0, r_o[$. We consider for $r > r_o$, the ordinary differential equation

$$\begin{pmatrix} v' \\ w' \end{pmatrix} := \varphi(v, w) = \begin{pmatrix} |w|^{-\frac{\alpha}{\alpha+1}} w \\ f_{a,A}(-v^{-\gamma} - \frac{N-1}{r} aw) \end{pmatrix}$$

where

$$f_{a,A}(x) = \frac{x^+}{A} - \frac{x^-}{a},$$

with $v(r_o) = u(r_o) \neq 0$, $w(r_o) = u'(r_o) \neq 0$. The function φ is Lipschitz continuous as long as $v \neq 0$ and $w \neq 0$. Then Cauchy Lipschitz Theorem ensures local existence and uniqueness of solution for $r > r_o$. Let $r_1 > r_o$ and u_1 defined on $]r_o, r_1[$ which solves this ordinary differential equation. Let

$$u = \begin{cases} u_o & \text{if } r < r_o \\ u_1 & \text{if } r \in]r_o, r_1[\end{cases}$$

We observe that $u' < 0$ as long as $u(r) > 0$. Indeed, suppose that $r_2 > r_o$ is so that $u'(r) \leq 0$, $u(r) > 0$ for $r < r_2$, and $u'(r_2) = 0$. One would have, since $(|u'|^\alpha u')$ is continuous, $(|u'|^\alpha u)'(r_2) \geq 0$, while by the equation this quantity is < 0 on r_2 , a contradiction. As a consequence, u is a solution of the equation related to $\mathcal{M}_{a,A}^+$ on $[0, R[$ where $R \leq \infty$ is so that $u(r) > 0$, $u'(r) < 0$ for $r < R$ and $\lim_{r \rightarrow R} \inf(u, u')(r) = 0$. As a conclusion we have obtained a solution u on some interval $[0, R[$, with R so that $u(r) > 0$, $u'(r) < 0$ for $r < R$, and $\lim_{r \rightarrow R} u(r) = 0$. In the following lines we prove that $R < \infty$.

Since $u' \neq 0$ on $]0, R[$, u'' is continuous and then the sets $\{r > 0, u''(r) < 0\}$ and $\{r > 0, u''(r) > 0\}$ are open. Then each of these sets is a countable union of intervals.

So there exist a numerable set of r_i so that on $]0, r_1[$, and on $]r_{2i}, r_{2i+1}[$, $u'' < 0$, on $]r_{2i+1}, r_{2i+2}[$, $u'' > 0$. Then on $]r_{2i}, r_{2i+1}[$, u satisfies

$$\begin{aligned} -(-u')^{1+\alpha}(s) &= -(-u')^{1+\alpha}(r_{2i}) \left(\frac{r_{2i}}{s} \right)^{(N-1)(1+\alpha)} \\ &- \frac{1}{s^{(N-1)(1+\alpha)}} \int_{r_{2i}}^s \frac{(1+\alpha)}{a} u^{-\gamma}(\lambda) \lambda^{(N-1)(1+\alpha)} d\lambda \\ &:= h_{2i}(s) (\leq 0), \end{aligned}$$

and then

$$u(r) = u(r_{2i}) - \int_{r_{2i}}^r (-h_{2i}(s))^{\frac{1}{1+\alpha}} ds.$$

On $]r_{2i+1}, r_{2i+2}[$ we have the analogous formula with

$$\begin{aligned} h_{2i+1}(s) &= -(-u')^{1+\alpha}(r_{2i+1}) \left(\frac{r_{2i+1}}{s} \right)^{\frac{(N-1)(1+\alpha)a}{A}} \\ &- \frac{1}{s^{\frac{(N-1)(1+\alpha)a}{A}}} \int_{r_{2i+1}}^s \frac{(1+\alpha)}{A} u^{-\gamma}(\lambda) \lambda^{\frac{(N-1)(1+\alpha)a}{A}} d\lambda. \end{aligned}$$

We observe next that there exists \bar{r} so that $u(\bar{r}) = 0$. For that aim we write for $r \in]r_{2k-1}, r_{2k}[$

$$u(r) = u(0) + (u(r_0) - u(0)) + \sum_0^{k-1} (u(r_{2i+1}) - u(r_{2i})) + u(r) - u(r_{2k-1}).$$

Note that $u(r_{2i+1}) - u(r_{2i}) < 0$, and since u is bounded by $u(0)$,

$$\begin{aligned} u(r) - u(r_{2k-1}) &\leq -(u(0))^{\frac{-\gamma}{1+\alpha}} \int_{r_{2k-1}}^r \left(\frac{1}{s^{\frac{(N-1)(1+\alpha)a}{A}}} \int_{r_{2i+1}}^s \frac{(1+\alpha)}{A} \lambda^{\frac{(N-1)(1+\alpha)a}{A}} d\lambda \right)^{\frac{1}{1+\alpha}} ds \\ &\leq -C(u(0))^{\frac{-\gamma}{1+\alpha}} (r^{\frac{2+\alpha}{1+\alpha}} - r_{2k-1}^{\frac{2+\alpha}{1+\alpha}}). \end{aligned}$$

where C is a positive constant. An analogous formula holds for $r \in]r_{2k}, r_{2k+1}[$.

Suppose that $r_{2i} > 1$, then for all $k \geq i$, on $[r_{2k}, r_{2k+1}[$, $-\lambda^{(N-1)(1+\alpha)} \leq -\lambda^{\frac{(N-1)(1+\alpha)a}{A}}$ so in the previous inequality we do "as if" the same formula holds for the case $u'' \geq 0$ or ≤ 0 as soon as r_{2i} is greater than 1, and then for $r \in [r_{2i}, r_{2i+1}[$

$$u(r) \leq u(r_{2i}) - \int_{r_{2i}}^r \left(\frac{1}{s^{(N-1)(1+\alpha)}} \int_{r_{2i}}^s \frac{(1+\alpha)}{a} u^{-\gamma}(\lambda) \lambda^{\frac{(N-1)(1+\alpha)a}{A}} d\lambda \right)^{\frac{1}{1+\alpha}} ds.$$

Finally one has for any r , $r > 1$, denoting by i_o the first i so that $r_{2(i_o+1)} > 1$, if it exists,

$$\begin{aligned} u(r) &\leq u(0) + (u(r_0) - u(0)) + \sum_{i, i \leq i_o-1} (u(r_{2i+1}) - u(r_{2i})) + u(r) - u(r_{2i_o}) \\ &\leq u(0) - \int_{r_{2i_o}}^r \left(\frac{1}{s^{(N-1)(1+\alpha)}} \int_{r_{2i}}^s \frac{(1+\alpha)}{a} u^{-\gamma}(\lambda) \lambda^{\frac{(N-1)(1+\alpha)a}{A}} d\lambda \right)^{\frac{1}{1+\alpha}} ds \\ &\leq u(0) - C u(0)^{\frac{-\gamma}{1+\alpha}} (r^{\frac{2+\alpha}{1+\alpha}} - r_{2i_o}^{\frac{2+\alpha}{1+\alpha}}) \\ &\leq u(0) + C u(0)^{\frac{-\gamma}{1+\alpha}} - C u(0)^{\frac{-\gamma}{1+\alpha}} r^{\frac{2+\alpha}{1+\alpha}}. \end{aligned}$$

And then for r large enough this quantity becomes negative. If for all i , $r_{2i} \leq 1$, since r_{2i} is increasing let l its limit, then one can write for $r > l$

$$u(r) \leq u(0) - C u(0)^{\frac{-\gamma}{1+\alpha}} (r^{\frac{2+\alpha}{1+\alpha}} - l^{\frac{2+\alpha}{1+\alpha}}) \leq u(0) - C u(0)^{\frac{-\gamma}{1+\alpha}} (r^{\frac{2+\alpha}{1+\alpha}} - 1)$$

and the same conclusion follows. Let then \bar{r} be so that $u(\bar{r}) = 0$. We end the proof as in the case where $\mathcal{M}_{a,A}^+$ is replaced by the Laplacian, and we have obtained a radial solution of the equation related to $\mathcal{M}_{a,A}^+$ in the ball $B(0, 1)$.

For the general case we observe that by the previous computations, there exists \bar{u} a radial solution for

$$|\bar{u}'|^\alpha \mathcal{M}_{a,A}^+(D^2 \bar{u}) = -\bar{u}^{-\gamma}, \bar{u}(1) = 0.$$

Then it provides a super-solution for the equation, while, by obvious changes in the analysis above, there exists \underline{u} a radial solution for

$$|\underline{u}'|^\alpha \mathcal{M}_{a,A}^-(D^2 \underline{u}) = -\underline{u}^\gamma, \underline{u}(1) = 0$$

which provides a sub-solution. By the comparison principle (in the uniqueness part), $\underline{u} \leq \bar{u}$. Using Perron's method adapted to the present context (see [4]), we obtain the existence of a radial solution of (1) which lies between \bar{u} and \underline{u} .

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