



HAL
open science

Rational representation of real functions

Wojciech Kucharz, Krzysztof Kurdyka

► **To cite this version:**

Wojciech Kucharz, Krzysztof Kurdyka. Rational representation of real functions. Pure and Applied Mathematics Quarterly, 2021, 17 (1), pp.249-268. hal-03808828

HAL Id: hal-03808828

<https://hal.science/hal-03808828>

Submitted on 10 Oct 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Rational representation of real functions

Wojciech Kucharz and Krzysztof Kurdyka

Abstract. Let X be an irreducible smooth real algebraic variety of dimension at least 2 and let $f: U \rightarrow \mathbb{R}$ be a function defined on a connected open subset $U \subset X(\mathbb{R})$. Assume that for every irreducible smooth real algebraic curve $C \subset X$, for which $C(\mathbb{R})$ is the boundary of a disc embedded in U , the restriction $f|_{C(\mathbb{R})}$ is continuous and has a rational representation. Then f has a rational representation. This is a significant refinement of a recent result of J. Kollár and the authors. The novelty is that existence of rational representation is tested on a much smaller and more rigid class of curves. We also consider the case where U is not necessarily connected and test rationality on subvarieties of dimension greater than 1. For semialgebraic functions our results hold under slightly weaker assumptions.

Key words. Real algebraic variety, rational function, rational representation, semialgebraic function, Nash function.

Mathematics subject classification (2010). 14P05, 14P10, 26C15.

1 Introduction

In this paper we continue the line of research initiated in [9, 10], see also our survey [17], and obtain some new results on rational representation of functions defined on subsets of a real algebraic variety. Here, as in [9, 10], a real algebraic variety is a quasi-projective variety X defined over \mathbb{R} (always reduced but possibly reducible). By a subvariety we mean a closed subvariety. The set of real points is denoted by $X(\mathbb{R})$ and endowed with the Euclidean topology. It is well known that there is an affine open subset $X_0 \subset X$ that contains $X(\mathbb{R})$. Consequently, as in [2], one can always view $X(\mathbb{R})$ as an algebraic subset of \mathbb{R}^n for some n . In particular, $\mathbb{A}^n(\mathbb{R}) = \mathbb{R}^n$. If X is irreducible and smooth, then $X(\mathbb{R})$ is either empty or it is a \mathcal{C}^∞ manifold with $\dim X(\mathbb{R}) = \dim X$.

Let $f: U \rightarrow \mathbb{R}$ be a function defined on some subset $U \subset X(\mathbb{R})$ and let R be a rational function defined on the Zariski closure Y of U in X . We say that R is a *rational representation* of f or that f is *represented* by R if there exists a Zariski open dense subset $Y^0 \subset Y$ with

$$Y^0 \subset Y \setminus \text{Pole}(R) \quad \text{and} \quad f|_{U \cap Y^0} = R|_{U \cap Y^0},$$

where $\text{Pole}(R)$ stands for the polar set of R (no restriction on the values of f on $U \cap (Y \setminus Y^0)$ is imposed). More explicitly, regarding $X(\mathbb{R})$ as an algebraic subset of \mathbb{R}^n , the function f has a rational representation if and only if there exist two polynomials $p, q \in \mathbb{R}[t_1, \dots, t_n]$

W. Kucharz was partially supported by the National Science Center (Poland) under grant number 2014/15/ST1/00046.

K. Kurdyka was partially supported by the ANR project LISA (France).

such that q is not identically equal to 0 on any irreducible component of $Y(\mathbb{R})$ and $f = p/q$ on $U \cap \{q \neq 0\}$. In [9, 10], f is called simply a rational function if it has a rational representation.

According to [9], rationality of f can be tested on algebraic curves, that is, by imposing suitable conditions on the restrictions of f to the intersections of U with real algebraic curves $C \subset X$. In the present paper we give a new criterion for rationality of f , assuming that X is smooth and U is an open subset of $X(\mathbb{R})$. This new criterion involves only irreducible smooth curves C such that, for U connected, $C(\mathbb{R})$ is the boundary of a small disc embedded in U . More generally, along with curves we consider also subvarieties of higher dimension.

Let \mathbb{B}^k denote the closed unit ball in \mathbb{R}^k ,

$$\mathbb{B}^k := \{(x_1, \dots, x_k) \in \mathbb{R}^k : x_1^2 + \dots + x_k^2 \leq 1\},$$

where $k \geq 1$. The boundary $\mathbb{S}^{k-1} = \partial\mathbb{B}^k$ is the unit sphere.

Notation 1.1. Let X be a smooth real algebraic variety of pure dimension at least 2, $U \subset X(\mathbb{R})$ a nonempty open subset, \mathcal{W} an open cover of U , and d an integer satisfying $1 \leq d \leq \dim X - 1$. We denote by $\mathcal{S}_d(X; U, \mathcal{W})$ the collection comprised of all d -dimensional irreducible smooth subvarieties $S \subset X$ satisfying one of the following conditions:

- (i) If U is connected, then $S(\mathbb{R})$ is connected and there exists a \mathcal{C}^∞ embedding $\varphi: \mathbb{B}^{d+1} \rightarrow X(\mathbb{R})$ such that $\varphi(\mathbb{B}^{d+1}) \subset W$ for some $W \in \mathcal{W}$ and $\varphi(\mathbb{S}^d) = S(\mathbb{R})$.
- (ii) If U is disconnected, then $S(\mathbb{R})$ has exactly 2 connected components and is not contained in a single connected component of U . Furthermore, for each connected component M of $S(\mathbb{R})$ there exists a \mathcal{C}^∞ embedding $\varphi_M: \mathbb{B}^{d+1} \rightarrow X(\mathbb{R})$ such that $\varphi_M(\mathbb{B}^{d+1}) \subset W$ for some $W \in \mathcal{W}$ and $\varphi_M(\mathbb{S}^d) = M$.

First we give a criterion for rationality of semialgebraic functions.

Theorem 1.2. *Let X be an irreducible smooth real algebraic variety of dimension at least 2, $f: U \rightarrow \mathbb{R}$ a function defined on a nonempty open subset $U \subset X(\mathbb{R})$, \mathcal{W} an open cover of U , and d an integer satisfying $1 \leq d \leq \dim X - 1$. Assume that U is a semialgebraic set, f is a semialgebraic function, and for every subvariety $S \in \mathcal{S}_d(X; U, \mathcal{W})$ the restriction $f|_{S(\mathbb{R})}$ has a rational representation. Then f has a rational representation.*

For functions that are not necessarily semialgebraic, our criterion takes the following form.

Theorem 1.3. *Let X be an irreducible smooth real algebraic variety of dimension at least 2, $f: U \rightarrow \mathbb{R}$ a function defined on a nonempty open subset $U \subset X(\mathbb{R})$, \mathcal{W} an open cover of U , and d an integer satisfying $1 \leq d \leq \dim X - 1$. Assume that U has finitely many connected components, and for every subvariety $S \in \mathcal{S}_d(X; U, \mathcal{W})$ the restriction $f|_{S(\mathbb{R})}$ is continuous and has a rational representation. Then f has a rational representation. Furthermore, if f is represented by a rational function R on X , then $P := U \cap \text{Pole}(R)$ has codimension at least 2 in U and $f|_{U \setminus P} = R|_{U \setminus P}$.*

In Theorem 1.3, the restrictions $f|_{S(\mathbb{R})}$ are required not only to have rational representations but to be continuous as well. Without the latter condition the function f need not have a rational representation [9, Example 1.8]. Moreover, by [9, Example 2.3], the assumptions in Theorem 1.3 do not imply continuity of f .

Our results show that rationality of functions can be tested on very special subvarieties. Such results have not been known heretofore even for X equal to affine space \mathbb{A}^n or projective

space \mathbb{P}^n . The case $d = 1$ is a refined version of [9, Theorem 2.4]; the novelty is that in the present paper we deal with a much more rigid class of curves, which substantially complicates the problem. In order to overcome the difficulties arising in this context we develop some new ideas, involving approximation of \mathcal{C}^∞ manifolds by real algebraic subvarieties (Lemma 2.4) and a nonstandard application of Bertini's theorem (Lemma 2.9).

The description of $\mathcal{S}_d(X; U, \mathcal{W})$ and proofs of Theorems 1.2 and 1.3 are slightly simpler if the set U is connected. However, $X(\mathbb{R})$ can have several connected components (always finitely many) and therefore it is better not to require that U be necessarily connected. Taking $\mathcal{W} = \{U\}$ would not simplify the issue in a significant way.

Theorems 1.2 and 1.3 are proved in Section 2 by making use of suitable embeddings of real algebraic varieties in projective space. In Section 3 we prove an embedding result, not needed in Section 2, which is of independent interest.

Further algebraic and geometric properties of real rational functions are discussed in [4, 5, 9, 10, 15, 19]. Applications of rational functions to topics on the borderline between real algebraic geometry and topology are given in [12, 13, 14, 16] and other papers cited therein.

2 Embeddings and approximation

Let X be a real algebraic variety and let $W \subset X(\mathbb{R})$ be some subset. We say that a function $f: W \rightarrow \mathbb{R}$ is *regular* if for each point $x \in W$ there exist a Zariski open neighborhood $X_x \subset X$ of x and a regular function φ on X_x such that $f|_{W \cap X_x} = \varphi|_{W \cap X_x}$. We say that a map $g = (g_1, \dots, g_k): W \rightarrow \mathbb{R}^k$ is *regular* if all its components g_i are regular functions.

The following fact will be useful.

Lemma 2.1. *Let $X \subset \mathbb{P}^r$ be a real subvariety and let $f: X(\mathbb{R}) \rightarrow \mathbb{R}$ be a regular function. Then there exist two homogeneous polynomials $F, G \in \mathbb{R}[t_0, \dots, t_r]$, which are of the same even degree, such that G has no zero in $\mathbb{R}^{r+1} \setminus \{0\}$ and*

$$f(x_0 : \dots : x_r) = \frac{F(x_0, \dots, x_r)}{G(x_0, \dots, x_r)} \quad \text{for all } (x_0 : \dots : x_r) \in X(\mathbb{R}).$$

Proof. We can find a positive integer k , and for each $i = 1, \dots, k$ two homogeneous polynomials $F_i, G_i \in \mathbb{R}[t_0, \dots, t_r]$, which are of the same degree, such that

$$X(\mathbb{R}) \subset \bigcup_{i=1}^k (\mathbb{P}^r(\mathbb{R}) \setminus V_i) \quad \text{and}$$

$$f(x_0 : \dots : x_r) = \frac{F_i(x_0, \dots, x_r)}{G_i(x_0, \dots, x_r)} \quad \text{for all } (x_0 : \dots : x_r) \in X(\mathbb{R}) \setminus V_i,$$

where $V_i \subset \mathbb{P}^r$ is the hypersurface defined by $G_i = 0$. Multiplying both F_i and G_i by a suitable power of $t_0^2 + \dots + t_r^2$ (depending on i), we may assume that F_i and G_i are of the same degree, say m , for all i . In the case where F_i and G_i are of odd degree we first multiply both of them by G_i .

Setting

$$P := \sum_{i=1}^k F_i G_i \quad \text{and} \quad Q := \sum_{i=1}^k G_i^2,$$

we get

$$f(x_0 : \dots : x_r) = \frac{P(x_0, \dots, x_r)}{Q(x_0, \dots, x_r)} \quad \text{for all } (x_0 : \dots : x_r) \in X(\mathbb{R}).$$

Let $H_1, \dots, H_l \in \mathbb{R}[t_0, \dots, t_r]$ be homogeneous polynomials that generate the homogeneous ideal of the subvariety X of \mathbb{P}^r . If $n_j = \deg H_j$ and $n = \max\{n_1, \dots, n_l\}$, then

$$H := \sum_{j=1}^l (t_0^2 + \dots + t_r^2)^{n-n_j} H_j^2$$

is a homogeneous polynomial of degree $2n$ and

$$X(\mathbb{R}) = \{(x_0 : \dots : x_r) \in \mathbb{P}^r(\mathbb{R}) : H(x_0, \dots, x_r) = 0\}.$$

Setting $q = \max\{m, n\}$, we readily see that the homogeneous polynomials

$$F := (t_0^2 + \dots + t_r^2)^{q-m} P \quad \text{and} \quad G := (t_0^2 + \dots + t_r^2)^{q-m} Q + (t_0^2 + \dots + t_r^2)^{q-n} H$$

satisfy all the requirements. \square

Next we introduce a certain class of subvarieties.

Definition 2.2. Let X be a smooth real algebraic variety. A d -dimensional real subvariety $Z \subset X$ is said to be *admissible* if $Z(\mathbb{R})$ is contained in the nonsingular locus of Z and can be written as $Z(\mathbb{R}) = \varphi(\partial\Omega)$, where Ω is a $(d+1)$ -dimensional compact \mathcal{C}^∞ manifold with boundary $\partial\Omega$, and $\varphi: \Omega \rightarrow X(\mathbb{R})$ is a \mathcal{C}^∞ embedding with trivial normal bundle.

We need the following result on embeddings in projective space.

Proposition 2.3. *Let X be a smooth projective real algebraic variety of pure dimension $n \geq 2$, d an integer satisfying $1 \leq d \leq n-1$, and $Z \subset X$ a d -dimensional admissible subvariety. Then there exist an algebraic embedding $e: X \rightarrow \mathbb{P}^N$ and an $(N-n+d)$ -dimensional plane $L \subset \mathbb{P}^N$ such that $e(X(\mathbb{R}))$ intersects $L(\mathbb{R})$ transversally in $\mathbb{P}^N(\mathbb{R})$ and $e(Z(\mathbb{R})) = e(X(\mathbb{R})) \cap L(\mathbb{R})$.*

Proof. Set $c := n - d$. Since the subvariety $Z \subset X$ is admissible, it follows from [11, Lemma 2.2] that one can find a \mathcal{C}^∞ map $g: X(\mathbb{R}) \rightarrow \mathbb{R}^c$ for which $0 \in \mathbb{R}^c$ is a regular value and $Z(\mathbb{R}) = g^{-1}(0)$. By the relative Weierstrass approximation theorem [2, Lemma 12.5.5], there exists a regular map

$$f = (f_1, \dots, f_c): X(\mathbb{R}) \rightarrow \mathbb{R}^c,$$

arbitrarily close to g in the \mathcal{C}^∞ topology, such that $f(x) = 0$ for all $x \in Z(\mathbb{R})$. It follows that $0 \in \mathbb{R}^c$ is a regular value of f and $Z(\mathbb{R}) = f^{-1}(0)$.

We may assume that $X \subset \mathbb{P}^r$ for some positive integer r . Then, according to Lemma 2.1, for each $i = 1, \dots, c$ there exist two homogeneous polynomials $F_i, G_i \in \mathbb{R}[t_0, \dots, t_r]$, which are of the same even degree, such that G_i has no zero in $\mathbb{R}^{r+1} \setminus \{0\}$ and

$$f_i(x_0 : \dots : x_r) = \frac{F_i(x_0, \dots, x_r)}{G_i(x_0, \dots, x_r)} \quad \text{for all } (x_0 : \dots : x_r) \in X(\mathbb{R}).$$

Multiplying F_i and G_i by a suitable power of $t_0^2 + \dots + t_r^2$ (depending on i), we may assume that F_i, G_i are of the same degree, say m , for all i . Let $N := \binom{r+m}{m} - 1$ and let $v_m: \mathbb{P}^r \rightarrow \mathbb{P}^N$ be the Veronese embedding of degree m . Let $L_i \subset \mathbb{P}^N$ be the hyperplane corresponding to F_i . Then $L := L_1 \cap \dots \cap L_c$ is an $(N-n+d)$ -dimensional plane in \mathbb{P}^N . If $e: X \rightarrow \mathbb{P}^N$ is the restriction of v_m , then $e(X(\mathbb{R}))$ intersects $L(\mathbb{R})$ transversally in $\mathbb{P}^N(\mathbb{R})$ and $e(Z(\mathbb{R})) = e(X(\mathbb{R})) \cap L(\mathbb{R})$, as required. \square

Next we describe how to construct admissible subvarieties which have some additional properties.

Lemma 2.4. *Let X be an irreducible smooth projective real algebraic variety of dimension $n \geq 2$, and d an integer satisfying $1 \leq d \leq n - 1$. Let Ω be a $(d + 1)$ -dimensional compact \mathcal{C}^∞ manifold with boundary $\partial\Omega$, $\varphi: \Omega \rightarrow X(\mathbb{R})$ a \mathcal{C}^∞ embedding with trivial normal bundle, and $x_0 \in \varphi(\partial\Omega)$ some point. Then there exists a \mathcal{C}^∞ diffeomorphism $\sigma: X(\mathbb{R}) \rightarrow X(\mathbb{R})$, arbitrarily close to the identity map in the \mathcal{C}^∞ topology, such that $\sigma(x_0) = x_0$ and $\sigma(\varphi(\partial\Omega)) = Z(\mathbb{R})$, where Z is a d -dimensional irreducible smooth admissible subvariety of X . Furthermore, for a suitable algebraic embedding $X \subset \mathbb{P}^N$, the subvariety Z can be written as $Z = X \cap L$, where $L \subset \mathbb{P}^N$ is an $(N - n + d)$ -dimensional plane which intersects X transversally.*

Proof. Set $c = n - d$. In view of [11, Lemma 2.2], there exists a \mathcal{C}^∞ map $g: X(\mathbb{R}) \rightarrow \mathbb{R}^c$ for which $0 \in \mathbb{R}^c$ is a regular value and $g^{-1}(0) = \varphi(\partial\Omega)$. By the relative Weierstrass approximation theorem [2, Lemma 12.5.5], there exists a regular map $f: X(\mathbb{R}) \rightarrow \mathbb{R}^c$, arbitrarily close to g in the \mathcal{C}^∞ topology, such that $f(x_0) = 0$. Then $0 \in \mathbb{R}^c$ is a regular value of f and there exists a \mathcal{C}^∞ diffeomorphism $\tau: X(\mathbb{R}) \rightarrow X(\mathbb{R})$, close to the identity map in the \mathcal{C}^∞ topology, such that

$$\tau(\varphi(\partial\Omega)) = \tau(g^{-1}(0)) = f^{-1}(0) \quad \text{and} \quad \tau(x_0) = x_0.$$

Furthermore, the Zariski closure $V \subset X$ of $f^{-1}(0)$ is an admissible subvariety with

$$V(\mathbb{R}) = f^{-1}(0).$$

Thus, according to Proposition 2.3, we may assume that $X \subset \mathbb{P}^N$ and

$$V(\mathbb{R}) = X(\mathbb{R}) \cap \Lambda(\mathbb{R}),$$

where $\Lambda \subset \mathbb{P}^N$ is an $(N - n + d)$ -dimensional plane such that $\Lambda(\mathbb{R})$ intersects $X(\mathbb{R})$ transversally in $\mathbb{P}^N(\mathbb{R})$.

We express Λ as the intersection $\Lambda = \Lambda_1 \cap \dots \cap \Lambda_c$ of c hyperplanes $\Lambda_i \subset \mathbb{P}^N$. Each Λ_i is defined by $A_i = 0$ for some linear form $A_i \in \mathbb{R}[t_0, \dots, t_N]$. Choose hyperplanes H_1, \dots, H_N in \mathbb{P}^N with $H_1 \cap \dots \cap H_N = \{x_0\}$. Each H_j is defined by $B_j = 0$ for some linear form $B_j \in \mathbb{R}[t_0, \dots, t_N]$. Given a real matrix $\alpha = (\alpha_{ij})$, consider the linear forms

$$A_i^\alpha := A_i + \sum_{j=1}^N \alpha_{ij} B_j, \quad i = 1, \dots, c.$$

For general α_{ij} , close to $0 \in \mathbb{R}$, let $L_i \subset \mathbb{P}^N$ be the hyperplane defined by $A_i^\alpha = 0$. Then $L := L_1 \cap \dots \cap L_c$ is an $(N - n + d)$ -dimensional plane in \mathbb{P}^N which intersects X transversally. Consequently, $Z := X \cap L$ is a d -dimensional smooth subvariety of X . By [18, Theorem 3.3.3], Z is also irreducible. If the α_{ij} are sufficiently close to $0 \in \mathbb{R}$, then there exists a \mathcal{C}^∞ diffeomorphism $\theta: X(\mathbb{R}) \rightarrow X(\mathbb{R})$, close to the identity map in the \mathcal{C}^∞ topology, such that

$$\theta(V(\mathbb{R})) = Z(\mathbb{R}) \quad \text{and} \quad \theta(x_0) = x_0.$$

Setting $\sigma = \theta \circ \tau$, we get

$$\sigma(\varphi(\partial\Omega)) = Z(\mathbb{R}) \quad \text{and} \quad \sigma(x_0) = x_0.$$

In particular, Z is an admissible subvariety. The proof is complete. \square

Lemma 2.4 provides information on the collection of subvarieties $\mathcal{S}_d(X; U, \mathcal{W})$ in Notation 1.1. For the sake of clarity, recall that any vector bundle on \mathbb{B}^{d+1} is trivial.

Lemma 2.5. *Let X be an irreducible smooth projective real algebraic variety of dimension $n \geq 2$, $U \subset X(\mathbb{R})$ a nonempty open subset, \mathcal{W} an open cover of U , and d an integer satisfying $1 \leq d \leq n - 1$. Let $U_0 \subset U$ be a connected open subset, $U_1 \subset U_0$ a nonempty open subset, and $x_0 \in U_0$ some point. Assume that $U_0 \subset W_0$ for some $W_0 \in \mathcal{W}$. Then there exists a subvariety $S \in \mathcal{S}_d(X; U, \mathcal{W})$ such that $S(\mathbb{R}) \cap U_0$ is a connected component of $S(\mathbb{R})$, $x_0 \in S(\mathbb{R})$, and $S(\mathbb{R}) \cap U_1 \neq \emptyset$. Furthermore, for a suitable algebraic embedding $X \subset \mathbb{P}^N$, the subvariety S can be written as $S = X \cap L$, where $L \subset \mathbb{P}^N$ is an $(N - n + d)$ -dimensional plane which intersects X transversally.*

Proof. Suppose that the set U is connected. We can find a \mathcal{C}^∞ embedding $\varphi: \mathbb{B}^{d+1} \rightarrow X(\mathbb{R})$ such that $\varphi(\mathbb{B}^{d+1}) \subset U_0$, $x_0 \in \varphi(\mathbb{S}^d)$, and $\varphi(\mathbb{S}^d) \cap U_1 \neq \emptyset$. In view of Lemma 2.4, we obtain S with the required properties.

Now suppose that the set U is disconnected, and let U' be a connected component of U which is disjoint from U_0 . Choose $W_1 \in \mathcal{W}$ with $U' \cap W_1 \neq \emptyset$. Setting

$$\Omega := (\{0\} \times \mathbb{B}^{d+1}) \cup (\{1\} \times \mathbb{B}^{d+1}),$$

we can find a \mathcal{C}^∞ embedding $\psi: \Omega \rightarrow X(\mathbb{R})$ such that $\psi(\{0\} \times \mathbb{B}^{d+1}) \subset U_0$, $x_0 \in \psi(\{0\} \times \mathbb{S}^d)$, $\psi(\{0\} \times \mathbb{S}^d) \cap U_1 \neq \emptyset$, and $\psi(\{1\} \times \mathbb{B}^{d+1}) \subset U' \cap W_1$. Making use of Lemma 2.4, we obtain S with the required properties. \square

Henceforth we will frequently appeal to Bertini's theorem [18, Theorems 3.3.1 and 3.3.3] to produce irreducible smooth subvarieties. Given integers k and N with $1 \leq k \leq N$, we denote by $\text{Gr}(k, N)$ the Grassmann variety of k -dimensional planes in \mathbb{P}^N .

Lemma 2.6. *Let X be an irreducible smooth projective real algebraic variety of dimension at least 2, $f: U \rightarrow \mathbb{R}$ a function defined on a nonempty open subset $U \subset X(\mathbb{R})$, \mathcal{W} an open cover of U , and d an integer satisfying $1 \leq d \leq \dim X - 1$. Assume that U is a semialgebraic set, f is a semialgebraic function, and for every subvariety $S \in \mathcal{S}_d(X; U, \mathcal{W})$ the restriction $f|_{S(\mathbb{R})}$ has a rational representation. Then, for each nonempty open subset $U' \subset U$, there exists a connected open subset $U_0 \subset U'$ such that the restriction $f|_{U_0}$ has a rational representation.*

Proof. Let $U' \subset U$ be a nonempty open subset. By definition of semialgebraic function, there exist a connected open subset $U_0 \subset U'$ and an irreducible hypersurface $Y \subset X \times \mathbb{A}^1$ such that the graph of $f|_{U_0}$ is contained in Y . Then $f|_{U_0}$ has a rational representation if and only if the first projection $\pi_1: Y \rightarrow X$ is birational.

Suppose to the contrary that π_1 is not birational, that is, π_1 has degree $m > 1$. According to Lemma 2.5, we may assume that $X \subset \mathbb{P}^N$ and there exists a k -dimensional plane $L_0 \subset \mathbb{P}^N$, where $k := N - \dim X + d$, such that L_0 intersects X transversally, the subvariety $S_0 := X \cap L_0$ belongs to $\mathcal{S}_d(X; U, \mathcal{W})$, and $S_0(\mathbb{R}) \cap U_0$ is a connected component of $S_0(\mathbb{R})$. Let $\mathcal{B} \subset \text{Gr}(k, N)$ be the collection of all planes L for which $X \cap L$ and $\pi_1^{-1}(L)$ are d -dimensional irreducible smooth subvarieties. For each $L \in \mathcal{B}$ the restriction $\pi_1^{-1}(X \cap L) \rightarrow X \cap L$ of π_1 has degree m . By Bertini's theorem, \mathcal{B} is open dense in the Zariski topology. Thus, we can choose an $L \in \mathcal{B}$ such that $S := X \cap L$ is in $\mathcal{S}_d(X; U, \mathcal{W})$ and $S(\mathbb{R}) \cap U_0$ is a connected component of $S(\mathbb{R})$. The graph of $f|_{S(\mathbb{R}) \cap U_0}$ lies on $\pi_1^{-1}(S) = \pi_1^{-1}(X \cap L)$, hence the restriction $f|_{S(\mathbb{R}) \cap U_0}$ has no rational representation. This is a contradiction since $f|_{S(\mathbb{R})}$ has a rational representation by assumption. \square

In the proofs of Theorems 1.2 and 1.3 we will also make use of the following.

Proposition 2.7. *Let X be an irreducible smooth real algebraic variety of dimension at least 2, $f: U \rightarrow \mathbb{R}$ a function defined on a nonempty open subset $U \subset X(\mathbb{R})$, \mathcal{W} an open cover of U , and d an integer satisfying $1 \leq d \leq \dim X - 1$. Assume that the following hold:*

- (a) U is disconnected and the collection $\{U_i\}_{i \in I}$ of all its connected components is finite;
- (b) the restriction $f|_{U_i}$ is represented by a rational function R_i on X for every $i \in I$;
- (c) for every subvariety $S \in \mathcal{S}_d(X; U, \mathcal{W})$ the restriction $f|_{S(\mathbb{R})}$ has a rational representation.

Then f has a rational representation. Moreover, R_i is a rational representation of f , and the rational functions R_i and R_j coincide for $i, j \in I$.

Proof. By the resolution of singularities theorem [7, 8], we may assume that the variety X is irreducible smooth and projective. In view of (a) and (b), we can find a Zariski open dense subset $X^0 \subset X$ such that

$$(2.1) \quad X^0 \subset X \setminus \text{Pole}(R_i) \quad \text{and} \quad f|_{U_i \cap X^0} = R_i|_{U_i \cap X^0} \quad \text{for all } i \in I.$$

It suffices to prove that $R_i = R_j$, as rational functions, for all $i, j \in I$.

Suppose to the contrary that $R_i \neq R_j$ for some fixed i, j . Then we get

$$R_i|_{U_i \cap X^0} \neq R_j|_{U_i \cap X^0},$$

hence

$$U_i^* = \{x \in U_i \cap X^0 : R_i(x) \neq R_j(x)\}$$

is a nonempty open subset of U_i . Pick sets $W_i, W_j \in \mathcal{W}$ such that $W_i \cap U_i^* \neq \emptyset$ and $W_j \cap U_j \neq \emptyset$. Set

$$\Omega := (\{i\} \times \mathbb{B}^{d+1}) \cup (\{j\} \times \mathbb{B}^{d+1}),$$

and let $\varphi: \Omega \rightarrow X(\mathbb{R})$ be a \mathcal{C}^∞ embedding with

$$\varphi(\{i\} \times \mathbb{B}^{d+1}) \subset W_i \cap U_i^* \quad \text{and} \quad \varphi(\{j\} \times \mathbb{B}^{d+1}) \subset W_j \cap U_j \cap X^0.$$

Denote by S the Zariski closure of $\varphi(\partial\Omega)$ in X . By Lemma 2.4, φ can be chosen so that $S \in \mathcal{S}_d(X; U, \mathcal{W})$. Since the restriction $f|_{S(\mathbb{R})}$ has a rational representation, there exist a Zariski open dense subset $S^0 \subset S$ and a rational function R_S on S such that

$$(2.2) \quad S^0 \subset S \setminus \text{Pole}(R_S) \quad \text{and} \quad f|_{S(\mathbb{R}) \cap S^0} = R_S|_{S(\mathbb{R}) \cap S^0}.$$

By comparing (2.1) and (2.2), we get

$$R_i|_{U_i \cap S(\mathbb{R}) \cap S^0 \cap X^0} = R_S|_{U_i \cap S(\mathbb{R}) \cap S^0 \cap X^0},$$

which in turn implies

$$(2.3) \quad R_i|_{S(\mathbb{R}) \cap S^0 \cap X^0} = R_S|_{S(\mathbb{R}) \cap S^0 \cap X^0},$$

since S is irreducible. On the other hand, (2.1) and (2.2) also yield

$$R_j|_{U_j \cap S(\mathbb{R}) \cap S^0 \cap X^0} = R_S|_{U_j \cap S(\mathbb{R}) \cap S^0 \cap X^0},$$

hence

$$(2.4) \quad R_j|_{S(\mathbb{R}) \cap S^0 \cap X^0} = R_S|_{S(\mathbb{R}) \cap S^0 \cap X^0}.$$

In view of (2.3) and (2.4), we get

$$R_i|_{S(\mathbb{R}) \cap S^0 \cap X^0} = R_j|_{S(\mathbb{R}) \cap S^0 \cap X^0}.$$

The last equality leads to a contradiction since $U_j^* \cap S(\mathbb{R}) \cap S^0 \cap X^0 \neq \emptyset$. The proof is complete. \square

Proof of Theorem 1.2. By the resolution of singularities theorem [7, 8], we may assume that the variety X is irreducible smooth and projective. Since the set U is semialgebraic, it has finitely many connected components. In view of Proposition 2.7, it suffices to prove that the restriction of f to any connected component of U has a rational representation.

Let U' be a connected component of U . Pick a set $W_0 \in \mathcal{W}$ with $U' \cap W_0 \neq \emptyset$. By Lemma 2.6, there exist a connected open subset $U_0 \subset U' \cap W_0$ and a rational function R on X such that

$$(2.5) \quad U_0 \subset X \setminus \text{Pole}(R) \quad \text{and} \quad f|_{U_0} = R|_{U_0}.$$

Since f is a semialgebraic function, there exists a Zariski nowhere dense real subvariety $Z \subset X$ such that the restriction $f|_{U \setminus Z}$ is a continuous function. Set $P := U \cap (Z \cup \text{Pole}(R))$. It remains to prove that

$$(2.6) \quad f|_{U' \setminus P} = R|_{U' \setminus P}.$$

For the proof fix a point $p_0 \in U_0$ and let $p \in U' \setminus P$ be an arbitrary point. Let $\gamma: [0, 1] \rightarrow U'$ be a continuous path with $\gamma(0) = p_0$ and $\gamma(1) = p$. We can cover the compact subset $\gamma([0, 1]) \subset U'$ by a finite collection of connected open subsets U_0, U_1, \dots, U_r such that each U_k is contained in some $W_k \in \mathcal{W}$, and the intersection $U_l \cap U_{l+1}$ is nonempty for every $l = 0, \dots, r-1$. We use induction on i to show that

$$(2.7) \quad f|_{U_i \setminus P} = R|_{U_i \setminus P} \quad \text{for } i = 0, \dots, r.$$

This is clear for $i = 0$ by (2.5). Suppose that (2.7) holds for $i = j$, where $0 \leq j < r$, and $f(x_0) \neq R(x_0)$ for some point $x_0 \in U_{j+1} \setminus P$. Then

$$U_{j+1}^* := \{x \in U_{j+1} \setminus P : f(x) \neq R(x)\}$$

is an open neighborhood of x_0 in U_{j+1} . By Lemma 2.5, there exists a subvariety $S \in \mathcal{S}_d(X; U, \mathcal{W})$ such that $x_0 \in S(\mathbb{R})$, $S(\mathbb{R}) \cap (U_j \setminus P) \neq \emptyset$, and $S(\mathbb{R}) \cap U_{j+1}$ is a connected component of $S(\mathbb{R})$. Since the restriction $f|_{S(\mathbb{R})}$ has a rational representation, there exist a Zariski open dense subset $S^0 \subset S$ and a rational function R_S on S such that

$$(2.8) \quad S^0 \subset S \setminus \text{Pole}(R_S) \quad \text{and} \quad f|_{S(\mathbb{R}) \cap S^0} = R_S|_{S(\mathbb{R}) \cap S^0}.$$

By shrinking S^0 if necessary, we obtain

$$(2.9) \quad S(\mathbb{R}) \cap S^0 \subset U \setminus P.$$

In view of (2.8), (2.9) and the induction hypothesis, we get

$$f = R \quad \text{on } U_j \cap S(\mathbb{R}) \cap S^0,$$

hence

$$(2.10) \quad f|_{S(\mathbb{R}) \cap S^0} = R|_{S(\mathbb{R}) \cap S^0}$$

since both restrictions in (2.10) are regular functions. This leads to a contradiction, the intersection $U_{j+1}^* \cap S(\mathbb{R}) \cap S^0$ being nonempty. Thus (2.7) holds, which in turn implies $f(p) = R(p)$. Consequently, (2.6) follows and the proof is complete. \square

The proof of Theorem 1.3 is more involved and requires additional preparation. Let X be a real algebraic variety, and $U \subset X(\mathbb{R})$ a semialgebraic open subset which is contained in the nonsingular locus of X . Recall that a function $f: U \rightarrow \mathbb{R}$ is called a *Nash function* (or in older texts an *algebraic function*) if it is analytic and semialgebraic [2].

As a consequence of some subtle Hartogs-like results on real analytic functions due to Błocki [1] and Siciak [20], in conjunction with a classical theorem presented in Bochner–Martin [3], the following fact was noted in [9, Corollary 2.10].

Proposition 2.8. *Let $f: U \rightarrow \mathbb{R}$ be a function defined on a nonempty open subset $U \subset \mathbb{R}^n$. Assume that the restriction of f is a Nash function on any open interval contained in U and parallel to one of the coordinate axes. Then there exists a nonempty semialgebraic open subset $U_0 \subset U$ such that the restriction $f|_{U_0}$ is a Nash function.*

Proposition 2.8 allows us to prove the following variant of Lemma 2.6.

Lemma 2.9. *Let X be an irreducible smooth projective real algebraic variety of dimension at least 2, $f: U \rightarrow \mathbb{R}$ a function defined on a nonempty open subset $U \subset X(\mathbb{R})$, \mathcal{W} an open cover of U , and d an integer satisfying $1 \leq d \leq \dim X - 1$. Assume that for every subvariety $S \in \mathcal{S}_d(X; U, \mathcal{W})$ the restriction $f|_{S(\mathbb{R})}$ is continuous and has a rational representation. Then, for each nonempty open subset $U' \subset U$, there exists a connected open subset $U_0 \subset U'$ such that the restriction $f|_{U_0}$ is a Nash function.*

Proof. It suffices to consider a connected open subset $U' \subset U$ which is contained in some set that belongs to \mathcal{W} . Set $n = \dim X$. According to Lemma 2.5, we may assume that $X \subset \mathbb{P}^N$ and there exists an $(N - n + d)$ -dimensional plane $L_0 \subset \mathbb{P}^N$ such that L_0 intersects X transversally, the subvariety $S_0 := X \cap L_0$ is in $\mathcal{S}_d(X; U, \mathcal{W})$, and $S_0(\mathbb{R}) \cap U'$ is a connected component of $S_0(\mathbb{R})$.

Let $E \subset \mathbb{P}^N$ be an $(N - n - 1)$ -dimensional plane that is disjoint from X and contained in L_0 . We choose homogeneous coordinates $(x_0 : \dots : x_N)$ in \mathbb{P}^N so that E is defined by $x_0 = 0, \dots, x_n = 0$, and L_0 is defined by $x_{d+1} = 0, \dots, x_n = 0$. Then the projection $\pi_E: \mathbb{P}^N \setminus E \rightarrow \mathbb{P}^n$ with center E is given by $\pi_E(x_0 : \dots : x_N) = (x_0 : \dots : x_n)$, and $\Lambda_0 \subset \mathbb{P}^n$ defined by $x_{d+1} = 0, \dots, x_n = 0$ is a unique d -dimensional plane such that L_0 is the Zariski closure of $\pi_E^{-1}(\Lambda_0)$ in \mathbb{P}^N . Furthermore, the restriction $\pi := (\pi_E)|_X: X \rightarrow \mathbb{P}^n$ is a finite morphism with $S_0 = \pi^{-1}(\Lambda_0)$.

By Bertini's theorem, the subset $\mathcal{B} \subset \text{Gr}(d, n)$ comprised of all planes Λ for which $\pi^{-1}(\Lambda)$ is a d -dimensional irreducible smooth subvariety of X is Zariski open dense. Let \mathcal{B}_0 be the set of all planes $\Lambda \in \mathcal{B}$ such that the subvariety $S := \pi^{-1}(\Lambda)$ belongs to $\mathcal{S}_d(X; U, \mathcal{W})$, and $S(\mathbb{R}) \cap U'$ is a connected component of $S(\mathbb{R})$. By construction, $\Lambda_0 \in \mathcal{B}_0$.

Pick a point $p \in U' \cap L_0 = U' \cap S_0(\mathbb{R})$ and affine coordinates (z_1, \dots, z_n) in $\mathbb{A}^n \subset \mathbb{P}^n$ with origin at $\pi(p)$. Set $c = n - d$. Given a c -tuple of integers (i_1, \dots, i_c) with $1 \leq i_1 < \dots < i_c \leq n$, let $A(i_1, \dots, i_c) \subset \mathbb{A}^n$ be the d -dimensional coordinate plane defined by $z_{i_1} = 0, \dots, z_{i_c} = 0$, and let $\Lambda(i_1, \dots, i_c) \subset \mathbb{P}^n$ be its Zariski closure. Let $\mathcal{P}(i_1, \dots, i_c)$ denote the set of all planes $\Lambda \in \text{Gr}(d, n)$ with $\mathbb{A}^n \cap \Lambda$ parallel to $A(i_1, \dots, i_c)$. By shrinking U' , we may assume

that the coordinates (z_1, \dots, z_n) are chosen so that $\Lambda_0 = \Lambda(d+1, \dots, n)$, and every plane $\Lambda \in \mathcal{P}(i_1, \dots, i_c)$ with $\pi(U') \cap \Lambda \neq \emptyset$ belongs to \mathcal{B}_0 (we have to ensure that each plane $\Lambda \in \mathcal{P}(i_1, \dots, i_c)$ with $\pi(U') \cap \Lambda \neq \emptyset$ is sufficiently close to Λ_0). After further shrinking U' , we may also assume that the restriction $\sigma: U' \rightarrow \sigma(U') = \pi(U')$ of π is a Nash diffeomorphism and

$$\sigma(U') \subset \mathbb{R}^n = \mathbb{A}^n(\mathbb{R}) \subset \mathbb{P}^n(\mathbb{R}).$$

Let $I \subset \sigma(U')$ be an open segment that is parallel to one of the coordinate axes. Set $A := \sigma^{-1}(I)$, and let C be the Zariski closure of A in X . Obviously, C is an irreducible real algebraic curve whose nonsingular locus contains A . We have $I \subset \sigma(U') \cap \Lambda$, where $\Lambda \in \mathcal{P}(i_1, \dots, i_c)$ for some c -tuple (i_1, \dots, i_c) . For $S := \pi^{-1}(\Lambda)$, we get $C \subset S$. Since $S \in \mathcal{S}_d(X; U, \mathcal{W})$, the restriction $f|_{S(\mathbb{R})}$ is continuous and has a rational representation. Hence, according to [10, Proposition 8], $f|_{C(\mathbb{R})}$ has a rational representation. Since A is contained in the nonsingular locus of C , we conclude that $f|_A$ is a regular function. Consequently, $(f \circ \sigma^{-1})|_I$ is a Nash function. In view of Proposition 2.8, the restriction of $f \circ \sigma^{-1}: \sigma(U') \rightarrow \mathbb{R}$ to some connected open subset of $\sigma(U')$ is a Nash function. This completes the proof. \square

Proof of Theorem 1.3. By the resolution of singularities theorem [7, 8], we may assume that the variety X is irreducible smooth and projective.

Let U' be a connected component of U . Pick a set $W_0 \in \mathcal{W}$ with $U' \cap W_0 \neq \emptyset$. By Lemmas 2.6 and 2.9, there exist a connected open subset $U_0 \subset U' \cap W_0$ and a rational function R on X such that

$$(2.11) \quad U_0 \subset X \setminus \text{Pole}(R) \quad \text{and} \quad f|_{U_0} = R|_{U_0}.$$

Set $P := U \cap \text{Pole}(R)$. We assert that

$$(2.12) \quad f|_{U' \setminus P} = R|_{U' \setminus P}.$$

For the proof of (2.12) fix a point $p_0 \in U_0$ and let $p \in U' \setminus P$ be an arbitrary point. Let $\gamma: [0, 1] \rightarrow U'$ be a continuous path with $\gamma(0) = p_0$ and $\gamma(1) = p$. We can cover the compact subset $\gamma([0, 1]) \subset U'$ by a finite collection of connected open subsets U_0, \dots, U_r such that each U_k is contained in some $W_k \in \mathcal{W}$, and the intersection $U_l \cap U_{l+1}$ is nonempty for every $l = 0, \dots, r-1$. We use induction on i to show that

$$(2.13) \quad f|_{U_i \setminus P} = R|_{U_i \setminus P} \quad \text{for } i = 0, \dots, r.$$

This is clear for $i = 0$ by (2.11). Suppose that (2.13) holds for $i = j$, where $0 \leq j < r$, and consider a point $x_0 \in U_{j+1} \setminus P$. By Lemma 2.5, there exists a subvariety $S \in \mathcal{S}_d(X; U, \mathcal{W})$ such that $x_0 \in S(\mathbb{R})$, $S(\mathbb{R}) \cap (U_j \setminus P) \neq \emptyset$, and $S(\mathbb{R}) \cap U_{j+1}$ is a connected component of $S(\mathbb{R})$. Since the restriction $f|_{S(\mathbb{R})}$ has a rational representation, there exist a Zariski open dense subset $S^0 \subset S$ and a rational function R_S on S such that

$$(2.14) \quad S^0 \subset S \setminus \text{Pole}(R_S) \quad \text{and} \quad f|_{S(\mathbb{R}) \cap S^0} = R_S|_{S(\mathbb{R}) \cap S^0}.$$

By shrinking S^0 if necessary, we obtain

$$(2.15) \quad S(\mathbb{R}) \cap S^0 \subset U \setminus P.$$

In view of (2.14), (2.15) and the induction hypothesis, we get

$$f = R \quad \text{on } U_j \cap S(\mathbb{R}) \cap S^0,$$

hence

$$(2.16) \quad f|_{S(\mathbb{R}) \cap S^0} = R|_{S(\mathbb{R}) \cap S^0}$$

since both restrictions in (2.16) are regular functions. Moreover, the functions $f|_{S(\mathbb{R})}$ and $R|_{U \setminus P}$ are continuous, $x_0 \in S(\mathbb{R}) \cap U'$, and $S(\mathbb{R}) \cap S^0$ is dense in $S(\mathbb{R})$. Therefore (2.16) yields $f(x_0) = R(x_0)$, which completes the proof of (2.13). Consequently, $f(p) = R(p)$, and hence (2.12) holds.

In view of (2.12) and Proposition 2.7, R is a rational representation of f and

$$(2.17) \quad f|_{U \setminus P} = R|_{U \setminus P}.$$

It remains to prove

$$(2.18) \quad \text{codim}_U P \geq 2.$$

Suppose to the contrary that $\text{codim}_U P = 1$. Then one can find a codimension 1 irreducible component Y of $\text{Pole}(R)$ such that the intersection P_0 of U with the nonsingular locus of Y is nonempty. Obviously, $P_0 \subset U$ is a \mathcal{C}^∞ submanifold of codimension 1. Making use of Lemma 2.4 and arguing as in the proof of Lemma 2.5, we obtain a subvariety $T \in \mathcal{S}_d(X; U, \mathcal{W})$ with $T(\mathbb{R})$ transverse to P_0 and $T(\mathbb{R}) \cap P_0 \neq \emptyset$. Applying Lemma 2.4 once again, we get an irreducible smooth real algebraic curve $C \subset T$ with $C(\mathbb{R})$ transverse to P_0 and $C(\mathbb{R}) \cap P_0 \neq \emptyset$. By assumption, the restriction $f|_{T(\mathbb{R})}$ is continuous and has a rational representation. Hence, in view of [10, Proposition 8], the restriction $f|_{C(\mathbb{R})}$ has a rational representation. Actually, the function $f|_{C(\mathbb{R})}$ is regular since it is continuous and the curve C is smooth. According to (2.17), we have

$$(f|_{C(\mathbb{R})})|_{C(\mathbb{R}) \setminus P} = f|_{C(\mathbb{R}) \setminus P} = R|_{C(\mathbb{R}) \setminus P},$$

which means that R cannot have a pole along $C(\mathbb{R})$, a contradiction. Thus (2.18) holds, as required. \square

3 Appendix

The embedding result proved in this section is not used elsewhere in our paper. However, it is related to Proposition 2.3 and may be of independent interest.

Proposition 3.1. *Let X be a smooth projective real algebraic variety of pure dimension at least 1. Let $Y \subset X$ be a real subvariety of pure codimension 1, whose nonsingular locus contains $Y(\mathbb{R})$. Then there exist an algebraic embedding $e: X \rightarrow \mathbb{P}^N$ and a hyperplane $H \subset \mathbb{P}^N$ such that $e(X(\mathbb{R}))$ intersects $H(\mathbb{R})$ transversally in $\mathbb{P}^N(\mathbb{R})$ and $e(Y(\mathbb{R})) = e(X(\mathbb{R})) \cap H(\mathbb{R})$.*

Proof. It will be convenient to identify isomorphic sheaves. Furthermore, we will make no distinction between invertible sheaves and line bundles on real algebraic varieties.

For any positive integer n , let $\mathcal{O}_n(1)$ denote the twisting line bundle on \mathbb{P}^n . Recall that $\mathcal{O}_n(q) = \mathcal{O}_n(1)^{\otimes q}$ for every positive integer q .

Given a homogeneous polynomial $F \in \mathbb{R}[t_0, \dots, t_n]$, we set

$$V(F) := (F(t_0, \dots, t_n) = 0) \subset \mathbb{P}^n \quad \text{and} \quad \mathbb{P}_F^n := \mathbb{P}^n \setminus V(F).$$

We also set

$$\mathbb{P}_0^n := \mathbb{P}^n \setminus V(t_0^2 + \dots + t_n^2).$$

Claim 1. The restriction of the line bundle $\mathcal{O}_n(2)$ to \mathbb{P}_0^n is trivial.

This is the case since the section of $\mathcal{O}_n(2)$ corresponding to the homogeneous polynomial $t_0^2 + \dots + t_n^2$ does not vanish anywhere on \mathbb{P}_0^n .

Claim 2. For any line bundle \mathcal{L} on X , there exists an algebraic embedding $\varepsilon: X \rightarrow \mathbb{P}^n$ such that $\mathcal{L}|_{X_0} = (\varepsilon^*\mathcal{O}_n(1))|_{X_0}$ for some Zariski open neighborhood $X_0 \subset X$ of $X(\mathbb{R})$.

We may assume that $X \subset \mathbb{P}^k$ for some k . Then $X_0 := X \cap \mathbb{P}_0^k$ is a Zariski open neighborhood of $X(\mathbb{R})$ in X .

Set $\mathcal{O}_X(q) := \mathcal{O}_k(q)|_X$. By Serre's theorem [6, p. 121, Theorem 5.17], there exists a positive integer q_0 such that the line bundle $\mathcal{L} \otimes \mathcal{O}_X(q)$ is generated by global sections for every $q \geq q_0$. Furthermore, in view of [6, p. 169, Exercise 7.5 (d)], the line bundle

$$\mathcal{L} \otimes \mathcal{O}_X(q+1) = (\mathcal{L} \otimes \mathcal{O}_X(q)) \otimes \mathcal{O}_X(1)$$

is very ample for $q \geq q_0$. Fix $q \geq q_0 + 1$ and consider the very ample line bundle $\mathcal{L} \otimes \mathcal{O}_X(2q)$. There exists an algebraic embedding $\varepsilon: X \rightarrow \mathbb{P}^n$ such that

$$\mathcal{L} \otimes \mathcal{O}_X(2q) = \varepsilon^*\mathcal{O}_n(1).$$

According to Claim 1, the restriction $\mathcal{O}_X(2)|_{X_0}$ is a trivial line bundle. Since $\mathcal{O}_X(2q) = \mathcal{O}_X(2)^{\otimes q}$, we have

$$(\varepsilon^*\mathcal{O}_n(1))|_{X_0} = (\mathcal{L}|_{X_0}) \otimes (\mathcal{O}_X(2)|_{X_0})^{\otimes q} = \mathcal{L}|_{X_0},$$

which completes the proof of Claim 2.

Since Y is of pure codimension 1, there exists a line bundle \mathcal{E} on X and a global section s of \mathcal{E} with

$$Y = Z(s),$$

where $Z(s)$ stands for the scheme of zeros of s . By Claim 2, we may assume that

$$X \subset \mathbb{P}^n \quad \text{and} \quad \mathcal{E}|_{X_1} = \mathcal{O}_n(1)|_{X_1},$$

where $X_1 \subset X$ is a Zariski open neighborhood of $X(\mathbb{R})$.

Claim 3. There exists a homogeneous polynomial $F \in \mathbb{R}[t_0, \dots, t_n]$ such that

$$X(\mathbb{R}) \subset X \cap \mathbb{P}_F^n \subset X_1.$$

Indeed, we have

$$X_1 = X \cap (\mathbb{P}^n \setminus (V(F_1) \cap \dots \cap V(F_r)))$$

for some homogeneous polynomials F_1, \dots, F_r in $\mathbb{R}[t_0, \dots, t_n]$. Set $m_i = \deg F_i$ and $m = \max\{m_1, \dots, m_r\}$. Then the homogeneous polynomial

$$F := \sum_{i=1}^r (t_0^2 + \dots + t_n^2)^{m-m_i} F_i^2$$

has the properties required in Claim 3.

Thus we have

$$X \subset \mathbb{P}^n \quad \text{and} \quad \mathcal{E}|_{X_F} = \mathcal{O}_n(1)|_{X_F},$$

where F is as in Claim 3 and $X_F := X \cap \mathbb{P}_F^n$.

Claim 4. The section $s|_{X_F}$ of $\mathcal{O}_n(1)|_{X_F}$ can be extended to a section u of the sheaf $\mathcal{F} := \mathcal{O}_n(1)|_{\mathbb{P}_F^n}$ on \mathbb{P}_F^n .

Let \mathcal{I} be the ideal sheaf of X_F in \mathbb{P}_F^n . Since \mathbb{P}_F^n is an affine variety, the obvious exact sequence of sheaves

$$0 \longrightarrow \mathcal{I}\mathcal{F} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}/\mathcal{I}\mathcal{F} \longrightarrow 0$$

gives rise to an exact sequence of the corresponding modules of global sections [6, p. 215, Theorem 3.5]. Claim 4 follows because $s|_{X_F}$ induces a global section of the quotient sheaf $\mathcal{F}/\mathcal{I}\mathcal{F}$.

One can complete the proof of Proposition 3.1 as follows. Let $Z \subset \mathbb{P}^n$ be the Zariski closure of $Z(u)$. Then Z is of pure codimension 1, $Z(\mathbb{R})$ is contained in the nonsingular locus of Z , $Z(\mathbb{R})$ intersects $X(\mathbb{R})$ transversally in $\mathbb{P}^n(\mathbb{R})$, and $Y(\mathbb{R}) = X(\mathbb{R}) \cap Z(\mathbb{R})$. Let d be the degree of Z and $N := \binom{n+d}{d} - 1$. Let $v_d: \mathbb{P}^n \rightarrow \mathbb{P}^N$ be the Veronese embedding of degree d and let $H \subset \mathbb{P}^N$ be the hyperplane corresponding to Z . Define $e: X \rightarrow \mathbb{P}^N$ to be the restriction of v_d . Then e and H satisfy all the requirements. \square

References

- [1] Z. Błocki, Singular sets of separately analytic functions, *Ann. Polon. Math.* 56 (2) (1992), 219–225.
- [2] J. Bochnak, M. Coste and M.-F. Roy, *Real Algebraic Geometry*, *Ergeb. Math. Grenzgeb.* Vol. 36, Springer, 1998.
- [3] S. Bochner and W. Martin, *Several Complex Variables*, Princeton University Press, Princeton, 1948.
- [4] G. Fichou, J. Huisman, F. Mangolte and J.-P. Monnier, Fonctions régulières, *J. Reine Angew. Math.* 718 (2016), 103–151.
- [5] G. Fichou, J.-P. Monnier and R. Quarez, Continuous functions on the plane regular after one blowing up, *Math. Z.* 285 (2017), 287–323.
- [6] R. Hartshorne, *Algebraic Geometry*, GTM 52, Springer, 1977.
- [7] H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero, *Ann. of Math.* 79 (1964), 109–326.
- [8] J. Kollár, *Lectures on Resolution of Singularities*, *Ann. of Math. Studies*, vol. 166, Princeton University Press, Princeton, 2007.
- [9] J. Kollár, W. Kucharz and K. Kurdyka, Curve-rational functions, *Math. Ann.* 370 (2018), 39–69.
- [10] J. Kollár and K. Nowak, Continuous rational functions on real and p -adic varieties, *Math. Z.* 279 (2015), 85–97.
- [11] W. Kucharz, On homology of real algebraic sets, *Invent. Math.* 82 (1985), 19–25.
- [12] W. Kucharz, Approximation by continuous rational maps into spheres, *J. Eur. Math. Soc.* 16 (2014), 1555–1569.
- [13] W. Kucharz, Continuous rational maps into spheres, *Math. Z.* 283 (2016), 1201–1215.

- [14] W. Kucharz, Piecewise-regular maps, *Math. Ann.* (online 2017), DOI 10.1007/s00208-017-1607-2.
- [15] W. Kucharz and K. Kurdyka, Linear equations on real algebraic surfaces, *Manuscripta Math.* 154 (2017), 285–296.
- [16] W. Kucharz and K. Kurdyka, Stratified-algebraic vector bundles, *J. Reine Angew. Math.* (online 2016), DOI 10.1515/crelle-2015-0105.
- [17] W. Kucharz and K. Kurdyka, From continuous rational to regulous functions, *Proc. ICM Congress Rio de Janeiro*, August 2018, to appear.
- [18] R. Lazarsfeld, *Positivity in Algebraic Geometry. I. Classical Setting: Line Bundles and Linear Series*, *Ergeb. Math. Grenzgeb. Vol. 48*, Springer, 2004.
- [19] K.J. Nowak, Some results on algebraic geometry over Henselian rank one valued fields, *Sel. Math. New Ser.* 23 (2017), 455–495.
- [20] J. Siciak, Singular sets of separately analytic functions, *Colloq. Math.* 60/61 (1990), 281–290.

Wojciech Kucharz
Institute of Mathematics
Polish Academy of Sciences
ul. Śniadeckich 8
00-656 Warszawa
Poland
and
Institute of Mathematics
Faculty of Mathematics and Computer Science
Jagiellonian University
ul. Łojasiewicza 6
30-348 Kraków
Poland
E-mail address: Wojciech.Kucharz@im.uj.edu.pl

Krzysztof Kurdyka
Laboratoire de Mathématiques
UMR 5127 de CNRS
Université Savoie Mont Blanc
Campus Scientifique
73 376 Le Bourget-du-Lac Cedex
France
E-mail address: krzysztof.kurdyka@univ-smb.fr