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# FROM CONTINUOUS RATIONAL TO REGULOUS FUNCTIONS

WOJCIECH KUCHARZ AND KRZYSZTOF KURDYKA

ABSTRACT. Let  $X$  be an algebraic set in  $\mathbb{R}^n$ . Real-valued functions, defined on subsets of  $X$ , that are continuous and admit a rational representation have some remarkable properties and applications. We discuss recently obtained results on such functions, against the backdrop of previously developed theories of arc-symmetric sets, arc-analytic functions, approximation by regular maps, and algebraic vector bundles.

## 1. INTRODUCTION

Our purpose is to report on some new developments in real algebraic geometry, focusing on functions that have a rational representation. Let us initially consider the simplest case. A function  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  of class  $\mathcal{C}^k$ , where  $k$  is a nonnegative integer, is said to have a *rational representation* if there exist two polynomial functions  $p, q$  on  $\mathbb{R}^n$  such that  $q$  is not identically 0 and  $f = p/q$  on  $\{q \neq 0\}$ . A typical example is

(1.1)  $f_k: \mathbb{R}^2 \rightarrow \mathbb{R}$  defined by

$$f_k(x, y) = \frac{x^{3+k}}{x^2 + y^2} \text{ for } (x, y) \neq (0, 0) \text{ and } f(0, 0) = 0.$$

To the best of our knowledge, [46] was the first paper devoted to the systematic study of such functions. This line of research was continued by several mathematicians [10, 32, 33, 42, 43, 48, 50, 51, 53–60, 71, 86], frequently with  $k = 0$ , where the functions admitting a rational representation are only continuous. Let us note that the complex case is quite different. By the Riemann extension theorem, any continuous function from  $\mathbb{C}^n$  into  $\mathbb{C}$  that has a rational representation is a polynomial function.

Henceforth we work with real algebraic sets, which is equivalent to the approach adopted in [13]. By a *real algebraic set* we mean an algebraic subset of  $\mathbb{R}^n$  for some  $n$ . One can realize real projective  $d$ -space  $\mathbb{P}^d(\mathbb{R})$  as a real algebraic set using the embedding

$$(1.2) \quad \mathbb{P}^d(\mathbb{R}) \ni (x_0 : \dots : x_d) \mapsto \left( \frac{x_i x_j}{x_0^2 + \dots + x_d^2} \right) \in \mathbb{R}^{(d+1)^2}.$$

Thus any algebraic subset of  $\mathbb{P}^d(\mathbb{R})$  is an algebraic subset of  $\mathbb{R}^{(d+1)^2}$ . Consequently, many useful constructions can be performed within the class of real algebraic sets, blowing-up being an important example. One can also view any real algebraic set as the set of real points  $V(\mathbb{R})$  of a quasiprojective variety  $V$  defined over  $\mathbb{R}$ .

Unless explicitly stated otherwise, we always assume that real algebraic sets and their subsets are endowed with the Euclidean topology, which is induced by the usual metric on  $\mathbb{R}$ . For a real algebraic set  $X$ , its singular locus  $\text{Sing}(X)$  is an algebraic, Zariski nowhere dense subset of  $X$ . We say that  $X$  is *smooth* if  $\text{Sing}(X)$  is empty. The following examples illustrate some phenomena that do not occur in the complex setting.

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(1.3) The algebraic curve

$$C := (x^4 - 2x^2y - y^3 = 0) \subset \mathbb{R}^2$$

is irreducible and  $\text{Sing}(C) = \{(0, 0)\}$ . Actually,  $C$  is an analytic submanifold of  $\mathbb{R}^2$ .

(1.4) The algebraic curve

$$C := (x^3 - x^2 - y^2 = 0) \subset \mathbb{R}^2$$

is irreducible and  $\text{Sing}(C) = \{(0, 0)\}$ . It has two connected components, the singleton  $\{(0, 0)\}$  and the unbounded branch  $C \setminus \{(0, 0)\}$ .

(1.5) The algebraic curve

$$C := (x^2(x^2 - 1)(x^2 - 4) + y^2 = 0) \subset \mathbb{R}^2$$

is irreducible and  $\text{Sing}(C) = \{(0, 0)\}$ . It has three connected components, the singleton  $\{(0, 0)\}$  and two ovals.

(1.6) The Cartan umbrella

$$S := (x^3 - z(x^2 + y^2) = 0) \subset \mathbb{R}^3$$

is an irreducible algebraic surface with  $\text{Sing}(S) = (z\text{-axis})$ . The surface  $S$  is connected and  $S \setminus \text{Sing}(S)$  is not dense in  $S$ . Furthermore,  $S$  is not coherent when regarded as an analytic subset of  $\mathbb{R}^3$ .

It will be convenient to consider regular functions in a more general setting than usual. Let  $X \subset \mathbb{R}^n$  be an algebraic set and let  $f: W \rightarrow \mathbb{R}$  be a function defined on some subset  $W$  of  $X$ . We say that  $f$  is *regular at a point*  $x \in W$  if there exist two polynomial functions  $p, q$  on  $\mathbb{R}^n$  such that  $q(x) \neq 0$  and  $f = p/q$  on  $W \cap \{q \neq 0\}$ . We say that  $f$  is a *regular function* if it is regular at each point of  $W$ . For any algebraic set  $Y \subset \mathbb{R}^p$ , a map  $\varphi = (\varphi_1, \dots, \varphi_p): W \rightarrow Y$  is *regular* if all the components  $\varphi_i: W \rightarrow \mathbb{R}$  are regular functions. These notions are independent of the algebraic embeddings  $X \subset \mathbb{R}^n$  and  $Y \subset \mathbb{R}^p$ .

Any rational function  $R$  on  $X$  determines a regular function  $R: X \setminus \text{Pole}(R) \rightarrow \mathbb{R}$ , where  $\text{Pole}(R)$  stands for the polar set of  $R$ .

**Contents.** In Section 2 we recall briefly main facts about arc-symmetric sets and arc-analytic functions. These notions, introduced 30 years ago by the second-named author, describe some rigidity phenomena (of an analytic type) of real algebraic sets. They form a background for the subsequent sections in which we present recent developments in the context of rational functions.

Section 3 contains presentation of new results on the geometry defined by regulous functions, that is, continuous functions which admit a strong version of rational representation.

In Section 4 we recall some theorems on approximation of continuous maps with values in spheres by regular maps and give new results in which approximating maps are allowed to be regulous.

In Section 5 we discuss topological, algebraic and regulous vector bundles. Regulous vector bundles have many desirable properties of algebraic vector bundles but are more flexible.

## 2. ARC-SYMMETRIC SETS AND ARC-ANALYTIC FUNCTIONS

Arc-symmetric sets and arc-analytic functions were introduced in [62] by the second-named author. They were further investigated and applied in [1, 7, 9, 31, 41, 45, 63–68, 70, 75–78].

**2.1. Arc-symmetric sets.** We say that a subset  $E \subset \mathbb{R}^n$  is *arc-symmetric* if for every analytic arc  $\gamma: (-1, 1) \rightarrow \mathbb{R}^n$  with  $\gamma((-1, 0)) \subset E$ , we have  $\gamma((-1, 1)) \subset E$ . We are mostly interested in semialgebraic arc-symmetric sets.

Recall that a topological space is called *Noetherian* if every descending chain of its closed subsets is stationary. In particular,  $\mathbb{R}^n$  with the Zariski topology is a Noetherian topological space. In [62], the following is proved.

**Theorem 2.1.** *The semialgebraic arc-symmetric subsets of  $\mathbb{R}^n$  are precisely the closed sets of a certain Noetherian topology on  $\mathbb{R}^n$ .*

Following [62], we call this topology on  $\mathbb{R}^n$  the  $\mathcal{AR}$  topology. Thus a subset of  $\mathbb{R}^n$  is  $\mathcal{AR}$ -closed if and only if it is semialgebraic and arc-symmetric. It follows from the curve selection lemma that each  $\mathcal{AR}$ -closed subset of  $\mathbb{R}^n$  is closed (in the Euclidean topology), cf. [62]. Clearly, any connected component of an  $\mathcal{AR}$ -closed subset of  $\mathbb{R}^n$  is also  $\mathcal{AR}$ -closed. Furthermore, any irreducible analytic component of an algebraic subset of  $\mathbb{R}^n$  is  $\mathcal{AR}$ -closed. However, an  $\mathcal{AR}$ -closed set need not be analytic at every point.

**Example 2.2.** The set

$$E = \{(x, y, z) \in \mathbb{R}^3 : x^3 - z(x^2 + y^2) = 0, x^2 + y^2 \neq 0\} \cup \{(0, 0, 0)\}$$

(the “cloth” of the Cartan umbrella (1.6)) is  $\mathcal{AR}$ -closed, but it is not analytic at the origin of  $\mathbb{R}^3$ .

Given a semialgebraic subset  $E \subset \mathbb{R}^n$ , we say that a point  $x \in E$  is *regular in dimension  $d$*  if for some open neighborhood  $U_x \subset \mathbb{R}^n$  of  $x$ , the intersection  $E \cap U_x$  is a  $d$ -dimensional analytic submanifold of  $U_x$ . We let  $\text{Reg}_d(E)$  denote the locus of regular points of  $E$  in dimension  $d$ . The dimension of  $E$ , written  $\dim E$ , is the maximum  $d$  with  $\text{Reg}_d(E)$  nonempty. If  $V$  is the Zariski closure of  $E$  in  $\mathbb{R}^n$ , then  $\dim E = \dim V$ , cf. [13].

By a *resolution of singularities* of a real algebraic set  $X$  we mean a proper regular map  $\pi: \tilde{X} \rightarrow X$  where  $\tilde{X}$  is a smooth real algebraic set and  $\pi$  is birational.

The following is the key result of [62].

**Theorem 2.3.** *Let  $X \subset \mathbb{R}^n$  be a  $d$ -dimensional real algebraic set and let  $E \subset \mathbb{R}^n$  be an  $\mathcal{AR}$ -closed irreducible subset with  $E \subset X$  and  $\dim E = d$ . If  $\pi: \tilde{X} \rightarrow X$  is a resolution of singularities of  $X$ , then there exists a unique connected component  $\tilde{E}$  of  $\tilde{X}$  such that  $\pi(\tilde{E})$  is the closure (in the Euclidean topology) of  $\text{Reg}_d(E)$ .*

This is illustrated by an example below.

**Example 2.4.** The real cubic  $C := (x^3 - x - y^2 = 0) \subset \mathbb{R}^2$  is smooth and irreducible. It has two connected components,  $C_1$  which is compact and  $C_2$  which is noncompact. Consider the cone  $X := (x^3 - xz^2 - y^2z = 0) \subset \mathbb{R}^3$  over  $C$ . Note that  $X$  is irreducible and  $\text{Sing}(X) = \{(0, 0, 0)\}$ . Clearly,

$$\pi: \tilde{X} := C \times \mathbb{R} \rightarrow X, \quad (x, y, z) \mapsto (xz, yz, z)$$

is a resolution of singularities of  $X$ . The connected components  $C_1 \times \mathbb{R}$  and  $C_2 \times \mathbb{R}$  of  $\tilde{X}$  correspond via  $\pi$  to the  $\mathcal{AR}$ -irreducible components of  $X$ .

The notion of arc-symmetric set turns out to be related to a notion introduced by Nash in his celebrated paper [72]. We adapt his definition to the case of  $\mathcal{AR}$ -closed sets.

**Definition 2.5.** Let  $E$  be an  $\mathcal{AR}$ -closed subset of  $\mathbb{R}^n$ . We say that a subset  $S \subset E$  is a (Nash) *sheet* of  $E$  if the following conditions are satisfied:

- (i) for any two points  $x_0, x_1$  in  $S$  there exists an analytic arc  $\gamma: [0, 1] \rightarrow \mathbb{R}^n$  with  $\gamma(0) = x_0$ ,  $\gamma(1) = x_1$ , and  $\gamma([0, 1]) \subset S$ ;
- (ii)  $S$  is maximal in the class of subsets satisfying the condition (i);
- (iii) the interior of  $S$  in  $X$  is nonempty.

The following result of [62] gives a positive and precise answer to Nash’s conjecture on sheets of an algebraic set [72].

**Theorem 2.6.** *Let  $X$  be an algebraic subset or more generally an  $\mathcal{AR}$ -closed subset of  $\mathbb{R}^n$ . Then:*

- (i) *There are finitely many sheets in  $X$ .*
- (ii) *Each sheet in  $X$  is semialgebraic and closed (in the Euclidean topology).*
- (iii)  *$X$  is the union of its sheets.*

The proof of this theorem is based on Theorem 2.3 and the notion of *immersed component* of an  $\mathcal{AR}$ -closed set. An immersed component of  $X$  is an  $\mathcal{AR}$ -irreducible subset of  $X$  with nonempty interior in  $X$ . In general,  $X$  may have more immersed components than  $\mathcal{AR}$ -irreducible components. For instance, the Whitney umbrella  $(xy^2 - z^2 = 0) \subset \mathbb{R}^3$  is  $\mathcal{AR}$ -irreducible, but it has two immersed components.

Compact  $\mathcal{AR}$ -closed sets share with compact real algebraic sets all known local and global topological properties. In particular, each compact  $\mathcal{AR}$ -closed set carries the mod 2 fundamental class. It is conjectured that any compact  $\mathcal{AR}$ -closed set is semialgebraically homeomorphic to a real algebraic set.

Recall that a Nash manifold  $X \subset \mathbb{R}^n$  is an analytic submanifold which is a semialgebraic set. Building on Thom's representability theorem [82] and Theorem 2.3, the following was established in [45].

**Theorem 2.7.** *Let  $X \subset \mathbb{R}^n$  be a compact Nash manifold, and  $d$  an integer satisfying  $0 \leq d \leq \dim X$ . Then each homology class in  $H_d(X; \mathbb{Z}/2)$  can be represented by an  $\mathcal{AR}$ -closed subset of  $\mathbb{R}^n$ , contained in  $X$ .*

Now we recall a result of K. Kurdyka and K. Rusek [68] which was motivated by the problem of surjectivity of injective selfmaps.

**Theorem 2.8.** *Let  $X \subset \mathbb{R}^n$  be an  $\mathcal{AR}$ -closed subset of dimension  $d$ , with  $0 \leq d \leq n - 1$ . Then the homotopy group  $\pi_{n-d-1}(\mathbb{R}^n \setminus X)$  is nontrivial.*

As demonstrated in [68], Theorem 2.8 implies the following result of Białynicki-Birula and Rosenlicht [6].

**Theorem 2.9.** *Any injective polynomial map from  $\mathbb{R}^n$  into itself is surjective.*

One should mention that Theorem 2.9, with  $n = 2$ , was established earlier by Newman [73]. In [4], Ax proved that any injective regular map of a complex algebraic variety into itself is surjective. Ax's proof is based on the Lefschetz principle and a reduction to the finite field case. By extending the idea of [6], Borel [28] gave a topological proof of Ax's theorem that works also for injective regular maps of a smooth real algebraic set into itself. Finally, combining Borel's argument with the geometry of  $\mathcal{AR}$ -closed sets, the second-named author proved in [65] the following.

**Theorem 2.10.** *Let  $X$  be a real algebraic set (possibly singular) and let  $f: X \rightarrow X$  be an injective regular map. Then  $f$  is surjective.*

In fact, there is a more general version of Theorem 2.10 due to Parusiński [77], cf. also [66].

**2.2. Arc-analytic maps.** Let  $X \subset \mathbb{R}^n$  and  $Y \subset \mathbb{R}^p$  be some subsets. A map  $f: X \rightarrow Y$  is said to be *arc-analytic* if for every analytic arc  $\gamma: (-1, 1) \rightarrow \mathbb{R}^n$  with  $\gamma((-1, 1)) \subset X$ , the composite  $f \circ \gamma: (-1, 1) \rightarrow \mathbb{R}^p$  is an analytic map. We are mostly interested in the case where  $X$  and  $Y$  are  $\mathcal{AR}$ -closed, and  $f$  is semialgebraic.

The function  $f_k: \mathbb{R}^2 \rightarrow \mathbb{R}$  in (1.1) is arc-analytic and of class  $\mathcal{C}^k$ , but it is not of class  $\mathcal{C}^{k+1}$ . The following fact is recorded in [62].

**Proposition 2.11.** *Let  $X \subset \mathbb{R}^n$  and  $Y \subset \mathbb{R}^p$  be  $\mathcal{AR}$ -closed subsets, and let  $f: X \rightarrow Y$  be a semialgebraic arc-analytic map. Then:*

- (i) *The graph of  $f$  is an  $\mathcal{AR}$ -closed subset of  $\mathbb{R}^n \times \mathbb{R}^p$ .*
- (ii) *If  $Z \subset Y$  is an  $\mathcal{AR}$ -closed set, then so is  $f^{-1}(Z)$ .*
- (iii)  *$f$  is continuous (in the Euclidean topology).*

Arc-analytic functions do not have nice properties without some additional assumptions. For instance, an arc-analytic function on  $\mathbb{R}^n$  need not be subanalytic [63] or continuous [9], and even for  $n = 2$  it may have a nondiscrete singular set [64].

In complex algebraic geometry, the image of an algebraic set by a proper regular map is again an algebraic set. This is trivially false in the real case; consider  $\mathbb{R} \rightarrow \mathbb{R}$ ,  $x \mapsto x^2$ . There is also a more interesting example. Let

$$X := (x - y^2 - 1 = 0) \subset \mathbb{R}^2 \text{ and } Y := (x^3 - x^2 - y^2 = 0) \subset \mathbb{R}^2.$$

Then  $f: X \rightarrow Y$ ,  $(x, y) \mapsto (x, xy)$  is an injective, proper regular map. However,  $f(X)$  is not an algebraic set. Therefore the following embedding theorem of [62] is of interest.

**Theorem 2.12.** *Let  $X \subset \mathbb{R}^n$  be an  $\mathcal{AR}$ -closed subset and let  $f: X \rightarrow \mathbb{R}^p$  be a semialgebraic arc-analytic map that is injective and proper. Then  $f(X) \subset \mathbb{R}^p$  is an  $\mathcal{AR}$ -closed subset.*

Given an  $\mathcal{AR}$ -closed subset  $X \subset \mathbb{R}^n$ , we denote by  $\mathcal{A}_a(X)$  the ring of semialgebraic arc-analytic functions on  $X$ . According to [62], the ring  $\mathcal{A}_a(X)$  is not Noetherian if  $\dim X \geq 2$ . However, any ascending chain of prime ideals of  $\mathcal{A}_a(X)$  is stationary. Furthermore, by [62], there exists a function  $f \in \mathcal{A}_a(\mathbb{R}^n)$  such that  $X \subset f^{-1}(0)$  and  $\dim(f^{-1}(0) \setminus X) < \dim X$ . This latter result has been recently strengthened by Adamus and Sayedinejad [1], who proved that actually  $X = f^{-1}(0)$  for some  $f \in \mathcal{A}_a(\mathbb{R}^n)$ . This enabled them to obtain the Nullstellensatz for the ring  $\mathcal{A}_a(X)$ , generalizing thereby the weak Nullstellensatz of [62].

**2.3. Blow-Nash and blow-analytic functions.** Let  $X$  be a smooth real algebraic set. A *Nash function* on  $X$  is an analytic function which is semialgebraic. A function on  $X$  is said to be *blow-Nash* if it becomes a Nash function after composing with a finite sequence of blowups with smooth nowhere dense centers. It was conjectured by K. Kurdyka (1987) that a function is blow-Nash if and only if it is arc-analytic and semialgebraic. The first proof of this conjecture was published by Bierstone and Milman [7]. They developed techniques which later turned out to be useful in their approach to the resolution of singularities [8]. There is also a second proof due to Parusiński [75]. It is based on the rectilinearization theorem for subanalytic functions [75], which is a prototype of the preparation theorem for subanalytic functions, cf. for example [76].

Less is known on arc-analytic functions which are subanalytic. Any such function is continuous and can be made analytic after composing with finitely many local blowups with smooth centers, cf. [7, 75]. It is not known whether one can use global blowups, that is, whether arc-analytic subanalytic functions coincide with *blow-analytic* functions of Kuo [61], cf. also [34]. In [67] it is proved that the locus of nonanalyticity of an arc-analytic subanalytic function is arc-symmetric and subanalytic. Another result of [67] asserts that in the blow-Nash case, the centers of blowups can be chosen in the locus of nonanalyticity.

**2.4. Some applications.** Recently arc-symmetric sets were used in the construction of new invariants in the singularity theory. These invariants include the virtual Betti numbers of real algebraic sets [70] and arc-symmetric sets [31]. Other invariants, analogous to the zeta function of Denef and Loeser, proved to be useful in the classification of germs of functions with respect to blow-analytic and blow-Nash equivalence, cf. [41] and [34]. Arc-analytic homeomorphisms were recently used in [78] to construct nice trivializations in the stratification theory.

### 3. REGULOUS FUNCTIONS

**3.1. Functions regular on smooth algebraic arcs.** All results presented in this subsection come from our joint paper with J. Kollár [42].

Let  $X$  be a real algebraic set. A subset  $A \subset X$  is called a *smooth algebraic arc* if its Zariski closure  $C$  is an irreducible algebraic curve,  $A \subset C \setminus \text{Sing}(C)$ , and  $A$  is homeomorphic to  $\mathbb{R}$ .

An open subset  $U \subset X$  is said to be *smooth* if it is contained in  $X \setminus \text{Sing}(X)$ .

**Theorem 3.1.** *Let  $X$  be a real algebraic set and let  $f: U \rightarrow \mathbb{R}$  be a function defined on a connected smooth open subset  $U \subset X$ . Assume that the restriction of  $f$  is regular on each smooth algebraic arc contained in  $U$ . Then there exists a rational function  $R$  on  $X$  such that  $P := U \cap \text{Pole}(R)$  has codimension at least 2 and  $f|_{U \setminus P} = R|_{U \setminus P}$ .*

There are two main steps in the proof of Theorem 3.1. Assuming that  $f$  is a semialgebraic function (so  $U$  is a semialgebraic set), one first obtains a local variant of the assertion by means of Bertini's theorem, and then extends it along smooth algebraic arcs. The general case is reduced to the semialgebraic one via some subtle Hartogs-like results on analytic functions due to Blocki [11] and Siciak [80].

A function regular on smooth algebraic arcs need not be continuous.

**Example 3.2.** The function  $f: \mathbb{R}^2 \rightarrow \mathbb{R}$  defined by

$$f(x, y) = \frac{x^8 + y(x^2 - y^3)^2}{x^{10} + (x^2 - y^3)^2} \text{ for } (x, y) \neq (0, 0) \text{ and } f(0, 0) = 0$$

is regular on each smooth algebraic arc in  $\mathbb{R}^2$ , but it is not locally bounded on the curve  $x^2 - y^3 = 0$ .

Let  $X = X_1 \times \cdots \times X_n$  be the product of real algebraic sets and let  $\pi_i: X \rightarrow X_i$  be the projection on the  $i$ th factor. A subset  $K \subset X$  is said to be *parallel to the  $i$ th factor of  $X$*  if  $\pi_j(K)$  consists of one point for each  $j \neq i$ .

**Theorem 3.3.** *Let  $X = X_1 \times \cdots \times X_n$  be the product of real algebraic sets and let  $f: U \rightarrow \mathbb{R}$  be a function defined on a connected smooth open subset  $U \subset X$ . Assume that the restriction of  $f$  is regular on each smooth algebraic arc contained in  $U$  and parallel to one of the factors of  $X$ . Then there exists a rational function  $R$  on  $X$  such that  $P := U \cap \text{Pole}(R)$  has codimension at least 2 and  $f|_{U \setminus P} = R|_{U \setminus P}$ .*

Theorem 3.3, for  $n = 1$ , coincides with Theorem 3.1. The general case is proved by induction on  $n$ , but a detailed argument is fairly long.

As a direct consequence, we get the following.

**Corollary 3.4.** *Let  $f: U \rightarrow \mathbb{R}$  be a function defined on a connected open subset  $U \subset \mathbb{R}^n$ . Assume that the restriction of  $f$  is regular on each open interval contained in  $U$  and parallel to one of the coordinate axes. Then there exists a rational function  $R$  on  $\mathbb{R}^n$  such that  $P := U \cap \text{Pole}(R)$  has codimension at least 2 and  $f|_{U \setminus P} = R|_{U \setminus P}$ .*

Results similar to Corollary 3.4 have been known earlier, but they were obtained under the restrictive assumption that  $f$  is an analytic function on  $U$ , cf. [27].

**3.2. Introducing regulous functions.** Let  $X$  be a real algebraic set,  $f: W \rightarrow \mathbb{R}$  a function defined on some subset  $W \subset X$ , and  $Y$  the Zariski closure of  $W$  in  $X$ .

**Definition 3.5.** A rational function  $R$  on  $Y$  is said to be a *rational representation* of  $f$  if there exists a Zariski open dense subset  $Y^0 \subset Y \setminus \text{Pole}(R)$  such that  $f|_{W \cap Y^0} = R|_{W \cap Y^0}$ .

While the definition makes sense for an arbitrary subset  $W$ , it is sensible only if  $W$  contains a sufficiently large portion of  $Y$ . The key examples of interest are open subsets and semialgebraic subsets, with  $W = X$  being the most important case.

One readily checks that the following conditions are equivalent:

- (3.6) For every algebraic subset  $Z \subset X$  the restriction  $f|_{W \cap Z}$  has a rational representation.
- (3.7) There exists a sequence of algebraic subsets

$$X = X_0 \supset X_1 \supset \cdots \supset X_{m+1} = \emptyset$$

such that the restriction of  $f$  is regular on  $W \cap (X_i \setminus X_{i+1})$  for  $i = 0, \dots, m$ .

- (3.8) There exists a finite stratification  $\mathcal{S}$  of  $X$ , with Zariski locally closed strata, such that the restriction of  $f$  is regular on  $W \cap S$  for every  $S \in \mathcal{S}$ .

**Definition 3.9.** We say that  $f$  is a *regulous function* if it is continuous and the equivalent conditions (3.6), (3.7), (3.8) are satisfied.

In some papers, regulous functions are called *hereditarily rational* [42, 43] or *stratified-regular* [53, 57, 59, 86]. The short name “regulous”, derived from “regular” and “continuous”, was introduced in [32]. A continuous function that has a rational representation is often called simply a *continuous rational function* [42, 43, 46, 48, 50, 51, 54, 56, 58].

Evidently, any regulous function is continuous and has a rational representation. The converse holds in an important special case.

**Proposition 3.10.** *Let  $X$  be a real algebraic set and let  $W \subset X$  be a smooth open subset. For a function  $f: W \rightarrow \mathbb{R}$ , the following conditions are equivalent:*

- (a)  $f$  is regulous.
- (b)  $f$  is continuous and has a rational representation.

The nontrivial implication (b) $\Rightarrow$ (a) is proved in [43]. Suppose that (b) holds, and let  $R$  be a rational representation of  $f$ . Since  $f$  is continuous and  $W$  is smooth, one gets  $f|_{W \setminus P} = R|_{W \setminus P}$ , where  $P = W \cap \text{Pole}(R)$ . Furthermore, it is not hard to see that  $P$  has codimension at least 2. Finally, condition (3.6) can be verified by induction on  $\text{codim } Z$ .

As demonstrated in [43] and recalled below, the smoothness assumption in Proposition 3.10 cannot be dropped.

**Example 3.11.** Consider the algebraic surface

$$S := (x^3 - (1 + z^2)y^3 = 0) \subset \mathbb{R}^3$$

and the function  $f: S \rightarrow \mathbb{R}$  defined by  $f(x, y, z) = (1 + z^2)^{1/3}$ . Note that  $\text{Sing}(S) = z$ -axis and  $f(x, y, z) = x/y$  on  $S \setminus (z\text{-axis})$ . In particular,  $f$  is continuous and has a rational representation. However,  $f$  is not regulous since  $f|_{z\text{-axis}}$  does not have a rational representation. It is also interesting that  $S$  is an analytic submanifold of  $\mathbb{R}^3$ .

The main result of Kollár and Nowak [43] can be stated as follows.

**Theorem 3.12.** *Let  $X$  be a smooth real algebraic set and let  $f: W \rightarrow \mathbb{R}$  be a function defined on an algebraic subset  $W \subset X$ . Then the following conditions are equivalent:*

- (a)  $f$  is regulous.
- (b)  $f = \tilde{f}|_W$ , where  $\tilde{f}: X \rightarrow \mathbb{R}$  is a continuous function that has a rational representation.

The proof of (a) $\Rightarrow$ (b) by induction on  $\dim W$  is tricky. Roughly speaking, one first finds a function on  $X$  that extends  $f$  and has a rational representation. However, such an extension may not be continuous and has to be corrected. This is achieved by analyzing liftings of functions to the blowup of  $X$  at a suitably chosen ideal. The argument relies on a version of the Łojasiewicz inequality given in [13, Theorem 2.6.6].

The implication (b) $\Rightarrow$ (a) follows from Proposition 3.10.

As it was noted on various occasions (see for example [46, p. 528] or [32, Théorème 3.11]), Hironaka’s theorem on resolution of indeterminacy points [38] implies immediately the following.

**Proposition 3.13.** *Let  $X$  be a smooth real algebraic set. For a function  $f: X \rightarrow \mathbb{R}$ , the following conditions are equivalent:*

- (a)  $f$  is continuous and has a rational representation.
- (b) There exists a regular map  $\pi: X' \rightarrow X$ , which is the composite of a finite sequence of blowups with smooth Zariski nowhere dense centers, such that  $f \circ \pi: X' \rightarrow \mathbb{R}$  is a regular function.

Fefferman and Kollár [30] study the following problem. Consider a linear equation

$$f_1 y_1 + \cdots + f_r y_r = g,$$

where  $g$  and the  $f_i$  are regular (or polynomial) functions on  $\mathbb{R}^n$ . Assume that it admits a solution where the  $y_i$  are continuous functions on  $\mathbb{R}^n$ . Then, according to [30, Section 2], it has also a continuous semialgebraic solution. One could hope to prove that it has a regulous



solution. This is indeed the case for  $n = 2$  [58, Corollary 1.7], but fails for any  $n \geq 3$  [43, Example 6]. It would be interesting to decide which linear equations have regulous solutions. Of course, the problem can be considered in a more general setting, replacing  $\mathbb{R}^n$  by a real algebraic set.

**3.3. Curve-regulous and arc-regulous functions.** All results discussed in this subsection come from our joint paper with J. Kollár [42], where it is proved that regulous functions can be characterized by restrictions to algebraic curves or algebraic arcs.

**Definition 3.14.** Let  $X$  be a real algebraic set and let  $f: W \rightarrow \mathbb{R}$  be a function defined on some subset  $W \subset X$ .

We say that  $f$  is *regulous on algebraic curves* or *curve-regulous* for short if for every irreducible algebraic curve  $C \subset X$  the restriction  $f|_{W \cap C}$  is regulous (equivalently,  $f|_{W \cap C}$  is continuous and has a rational representation).

Furthermore, we say that  $f$  is *regulous on algebraic arcs* or *arc-regulous* for short if for every irreducible algebraic curve  $C \subset X$  and every point  $x \in W \cap C$  there exists an open neighborhood  $U_x \subset W$  of  $x$  such that the restriction  $f|_{U_x \cap C}$  is regulous (equivalently,  $f|_{U_x \cap C}$  is continuous and has a rational representation).

Obviously, any curve-regulous function is arc-regulous. The converse does not hold for a rather obvious reason. For instance, consider the hyperbola  $H \subset \mathbb{R}^2$  defined by  $xy - 1 = 0$ . Any function on  $H$  that is constant on each connected component of  $H$  is arc-regulous, but it must be constant to be regulous.

In [42], curve-regulous (resp. arc-regulous) functions are called *curve-rational* (resp. *arc-rational*).

Our main result on curve-regulous functions is the following.

**Theorem 3.15.** *Let  $X$  be a real algebraic set and let  $W \subset X$  be a subset that is either open or semialgebraic. For a function  $f: W \rightarrow \mathbb{R}$ , the following conditions are equivalent:*

- (a)  $f$  is regulous.
- (b)  $f$  is curve-regulous.

The corresponding result for arc-regulous functions takes the following form.

**Theorem 3.16.** *Let  $X$  be a real algebraic set and let  $W \subset X$  be a connected smooth open subset. For a function  $f: W \rightarrow \mathbb{R}$ , the following conditions are equivalent:*

- (a)  $f$  is regulous.
- (b)  $f$  is arc-regulous.

The crucial ingredient in the proofs of Theorems 3.15 and 3.16 is Theorem 3.1. In both cases, only the implication (b) $\Rightarrow$ (a) is not obvious. It is essential that testing curves and arcs are allowed to have singularities.

The main properties of arc-regulous functions on semialgebraic sets can be summarized as follows.

**Theorem 3.17.** *Let  $X$  be a real algebraic set and let  $f: W \rightarrow \mathbb{R}$  be an arc-regulous function defined on a semialgebraic subset  $W \subset X$ . Then  $f$  is continuous and there exists a sequence of semialgebraic sets*

$$W = W_0 \supset W_1 \supset \cdots \supset W_{m+1} = \emptyset,$$

*which are closed in  $W$ , such that the restriction of  $f$  is a regular function on each connected component of  $W_i \setminus W_{i+1}$  for  $i = 0, \dots, m$ . In particular,  $f$  is a semialgebraic function.*

We also establish a connection between arc-regulous functions and, discussed in Section 2, arc-analytic functions.

**Theorem 3.18.** *Let  $X$  be a real algebraic set and let  $f: W \rightarrow \mathbb{R}$  be an arc-regulous function defined on an open subset  $W \subset X$ . Then  $f$  is continuous and arc-analytic.*

In [42] there are several other related results.

**3.4. Constructible topology and  $k$ -regulous functions.** We consider regulous functions of class  $\mathcal{C}^k$ .

**Definition 3.19.** Let  $X$  be a smooth real algebraic set and let  $f: U \rightarrow \mathbb{R}$  be a function defined on an open subset  $U \subset X$ .

We say that  $f$  is a  $k$ -regulous function, where  $k$  is a nonnegative integer, if it is of class  $\mathcal{C}^k$  and regulous; or equivalently, by Proposition 3.10, if it is of class  $\mathcal{C}^k$  and has a rational representation.

The set  $\mathcal{R}^k(U)$  of all  $k$ -regulous functions on  $U$  forms a ring. An example of a  $k$ -regulous function on  $\mathbb{R}^2$  is provided by (1.1).

A function on  $U$  which is of class  $\mathcal{C}^\infty$  and regulous is actually regular, cf. [46]. Therefore one gains no new insight by considering such functions.

All results discussed in the remainder of this subsection come from the paper of Fichou, Huisman, Mangolte and Monnier [32], where they are stated for functions defined on  $\mathbb{R}^n$ . The ring  $\mathcal{R}^k(\mathbb{R}^n)$  is not Noetherian if  $n \geq 2$ . Nevertheless it has some remarkable properties.

Given a collection  $F$  of real-valued functions on  $\mathbb{R}^n$ , we set

$$Z(F) := \{x \in \mathbb{R}^n : f(x) = 0 \text{ for all } f \in F\}$$

and write  $Z(f)$  for  $Z(F)$  if  $F = \{f\}$ .

The following is a variant of the classical Nullstellensatz for the ring  $\mathcal{R}^k(\mathbb{R}^n)$ .

**Theorem 3.20.** *Let  $I$  be an ideal of the ring  $\mathcal{R}^k(\mathbb{R}^n)$ . If a function  $f$  in  $\mathcal{R}^k(\mathbb{R}^n)$  vanishes on  $Z(I)$ , then  $f^m$  belongs to  $I$  for some positive integer  $m$ .*

Recall that the Nullstellensatz for the ring of polynomial or regular functions on  $\mathbb{R}^n$  requires an entirely different formulation, cf. [13].

The subsets of  $\mathbb{R}^n$  of the form  $Z(I)$  for some ideal  $I$  of  $\mathcal{R}^k(\mathbb{R}^n)$  can be characterized in terms of constructible sets. A subset of  $\mathbb{R}^n$  is said to be *constructible* if it belongs to the Boolean algebra generated by the algebraic subsets of  $\mathbb{R}^n$ ; or equivalently if it is a finite union of Zariski locally closed subsets of  $\mathbb{R}^n$ .

**Theorem 3.21.** *For a subset  $E \subset \mathbb{R}^n$ , the following conditions are equivalent:*

- (a)  $E = Z(I)$  for some ideal  $I$  of  $\mathcal{R}^k(\mathbb{R}^n)$ .
- (b)  $E = Z(f)$  for some function  $f$  in  $\mathcal{R}^k(\mathbb{R}^n)$ .
- (c)  $E$  is closed and constructible.

Theorem 3.21 can be illustrated as follows.

**Example 3.22.** Consider the Cartan umbrella  $S \subset \mathbb{R}^3$  defined in (1.6), and let  $E$  be the closure of  $S \setminus (z\text{-axis})$ . It is clear that  $E$  is a closed constructible set. Moreover,  $E = Z(f)$ , where  $f: \mathbb{R}^3 \rightarrow \mathbb{R}$  is the regulous function defined by

$$f(x, y, z) = z - \frac{x^3}{x^2 + y^2} \text{ on } \mathbb{R}^3 \setminus (z\text{-axis}) \text{ and } f(x, y, z) = z \text{ on the } z\text{-axis.}$$

The collection of all closed constructible subsets of  $\mathbb{R}^n$  forms the family of closed subsets for a Noetherian topology on  $\mathbb{R}^n$ , called the *constructible topology*. Any constructible-closed subset of  $\mathbb{R}^n$  is  $\mathcal{AR}$ -closed. The converse does not hold if  $n \geq 2$ .

In what follows we consider  $\mathbb{R}^n$  endowed with the constructible topology. The assignment  $\mathcal{R}^k: U \mapsto \mathcal{R}^k(U)$ , where  $U$  runs through the open subsets of  $\mathbb{R}^n$ , is a sheaf of rings on  $\mathbb{R}^n$ , and  $(\mathbb{R}^n, \mathcal{R}^k)$  is a locally ringed space. Sheaves of  $\mathcal{R}^k$ -modules on  $\mathbb{R}^n$  are called  *$k$ -regulous sheaves*.

It follows from Theorem 3.20 that the ringed space  $(\mathbb{R}^n, \mathcal{R}^k)$  carries essentially the same information as the affine scheme  $\text{Spec}(\mathcal{R}^k(\mathbb{R}^n))$ . In particular, Cartan's theorems A and B are available for  $k$ -regulous sheaves.

**Theorem 3.23.** *For any quasi-coherent  $k$ -regulous sheaf  $\mathcal{F}$  on  $\mathbb{R}^n$ , the following hold:*

- (A)  $\mathcal{F}$  is generated by global sections.
- (B)  $H^i(\mathbb{R}^n, \mathcal{F}) = 0$  for  $i \geq 1$ .

As is well-known, Cartan's theorems A and B fail for coherent algebraic sheaves on  $\mathbb{R}^n$ .

Let  $V \subset \mathbb{R}^n$  be a constructible-closed subset. The sheaf  $\mathcal{J}_V \subset \mathcal{R}^k$  of ideals of  $k$ -regulous functions vanishing on  $V$  is a quasi-coherent  $k$ -regulous sheaf on  $\mathbb{R}^n$ , and the quotient sheaf  $\mathcal{R}^k/\mathcal{J}_V$  has support  $V$ . Endow  $V$  with the induced (constructible) topology, and let  $\mathcal{R}_V^k$  be the restriction of the sheaf  $\mathcal{R}^k/\mathcal{J}_V$  to  $V$ . The locally ringed space  $(V, \mathcal{R}_V^k)$  is called a *closed  $k$ -regulous subvariety of  $(\mathbb{R}^n, \mathcal{R}^k)$* . One can consider  $k$ -regulous sheaves on  $V$ , which are just sheaves of  $\mathcal{R}_V^k$ -modules. By a standard argument, Theorem 3.23 implies that Cartan's theorems A and B hold also for quasi-coherent  $k$ -regulous sheaves on  $V$ .

An *affine  $k$ -regulous variety* is a locally ringed space isomorphic to a closed  $k$ -regulous subvariety of  $\mathbb{R}^n$  for some  $n$ . An *abstract  $k$ -regulous variety* can be defined in the standard way. The geometry of  $k$ -regulous varieties is to be developed.

#### 4. HOMOTOPY AND APPROXIMATION

In this section we discuss some homotopy and approximation results in the framework of real algebraic geometry, focusing on maps with values in the unit  $p$ -sphere

$$\mathbb{S}^p = (u_0^2 + \cdots + u_p^2 - 1 = 0) \subset \mathbb{R}^{p+1}.$$

Approximation of continuous maps means approximation in the compact-open topology.

**4.1. Approximation by regular maps.** The theory of regular maps between real algebraic sets was developed by J. Bochnak and the first-named author [14–17, 20, 23, 24, 25] who joined forces with R. Silhol working on [26]. Regular maps are studied also in [36, 37, 40, 47, 49, 52, 69, 74, 79, 84, 85]. We make no attempt to survey this theory, but give instead a sample of results that motivated later work described in the next subsection.

**Problem 4.1.** Let  $X$  be a compact real algebraic set. For a continuous map  $f: X \rightarrow \mathbb{S}^p$ , consider the following questions:

- (i) Is  $f$  homotopic to a regular map?
- (ii) Can  $f$  be approximated by regular maps?

It is expected that questions (i) and (ii) are equivalent, however, the proof is available only for special values of  $p$ , cf. [14].

**Theorem 4.2.** *Let  $X$  be a compact real algebraic set. For a continuous map  $f: X \rightarrow \mathbb{S}^p$ , where  $p \in \{1, 2, 4\}$ , the following conditions are equivalent:*

- (a)  $f$  is homotopic to a regular map.
- (b)  $f$  can be approximated by regular maps.

Basic topological properties of regular maps between unit spheres still remain mysterious.

**Conjecture 4.3.** *For any pair  $(n, p)$  of positive integers, the following assertions hold:*

- (i) Each continuous map from  $\mathbb{S}^n$  into  $\mathbb{S}^p$  is homotopic to a regular map.
- (ii) Each continuous map from  $\mathbb{S}^n$  into  $\mathbb{S}^p$  can be approximated by regular maps.

Conjecture 4.3 (i) is known to be true in several cases [13, 14, 15, 79, 84, 85]; for example if  $n = p$  or  $(n, p) = (2q + 14, 2q + 1)$  with  $q \geq 7$ .

Conjecture 4.3 (ii) holds if either  $n < p$  (trivial) or  $p \in \{1, 2, 4\}$  [14]. Nothing is known for other pairs  $(n, p)$ .

However, a complete solution to Problem 4.1 is known in several cases. The simplest one, noted in [14], is the following.

**Proposition 4.4.** *Let  $X$  be a compact smooth real algebraic curve. Then each continuous map from  $X$  into  $\mathbb{S}^1$  can be approximated by regular maps.*

Going beyond curves is a lot harder. Nevertheless, it can happen also in higher dimension that the behavior of regular maps is determined entirely by the topology of the real algebraic sets involved.

Consider a compact  $C^\infty$  manifold  $M$ . A smooth real algebraic set diffeomorphic to  $M$  is called an *algebraic model* of  $M$ . By the Nash–Tognoli theorem [72, 83],  $M$  has algebraic models. Actually, according to [21], there exists an uncountable family of pairwise birationally nonequivalent algebraic models of  $M$ , provided that  $\dim M \geq 1$ . If  $\dim M \leq 2$ , the existence of algebraic models of  $M$  follows easily from the well-known classification of such manifolds. As  $X$  runs through the class of all algebraic models of  $M$ , the topological properties of regular maps from  $X$  into  $\mathbb{S}^p$  may vary; this phenomenon is extensively investigated in [15, 16, 17, 19, 23, 47].

A detailed study of regular maps into  $\mathbb{S}^1$  is contained in [17] where in particular the following result is proved.

**Theorem 4.5.** *Let  $M$  be a compact  $C^\infty$  manifold. Then there exists an algebraic model  $X$  of  $M$  such that each continuous map from  $X$  into  $\mathbb{S}^1$  can be approximated by regular maps.*

For simplicity, we state the next result of [17] only for surfaces.

**Theorem 4.6.** *Let  $M$  be a connected, compact  $C^\infty$  surface. Then the following conditions are equivalent:*

- (a) *For any algebraic model  $X$  of  $M$ , each continuous map from  $X$  into  $\mathbb{S}^1$  can be approximated by regular maps.*
- (b)  *$M$  is homeomorphic to the unit 2-sphere or the real projective plane or the Klein bottle.*

In [15], one finds the following.

**Theorem 4.7.** *Let  $M$  be a compact  $C^\infty$  manifold of dimension  $p$ . Then there exists an algebraic model  $X$  of  $M$  such that each continuous map from  $X$  into  $\mathbb{S}^p$  is homotopic to a regular map.*

Theorem 4.7, for  $p = 1$ , is of course weaker than Proposition 4.4. The cases  $p = 2$  and  $p = 4$  are of particular interest in view of Theorem 4.2.

Numerous results on algebraic models and regular maps into even-dimensional spheres are included in [16, 19, 23, 47]. The following comes from [16].

**Theorem 4.8.** *Let  $M$  be a connected, compact  $C^\infty$  surface. Then the following conditions are equivalent:*

- (a) *For any algebraic model  $X$  of  $M$ , each continuous map from  $X$  into  $\mathbb{S}^2$  can be approximated by regular maps.*
- (b)  *$M$  is nonorientable of odd genus.*

The true complexity of Problem 4.1 becomes apparent for surfaces of other types.

Consider smooth cubic curves in  $\mathbb{P}^2(\mathbb{R})$ . Each such cubic is either connected or has two connected components, and its Zariski closure in  $\mathbb{P}^2(\mathbb{C})$  is also smooth. If  $C_1$  and  $C_2$  are smooth cubic curves in  $\mathbb{P}^2(\mathbb{R})$ , then  $C_1 \times C_2$  can be oriented in such a way that for each regular map  $\varphi: C_1 \times C_2 \rightarrow \mathbb{S}^2$ , the topological degree  $\deg(\varphi|_A)$  of the restriction of  $\varphi$  to a connected component  $A$  of  $C_1 \times C_2$  does not depend on the choice of  $A$ . Moreover, the set

$$\text{Deg}_{\mathcal{R}}(C_1, C_2) := \{m \in \mathbb{Z} : m = \deg(\psi|_A) \text{ for some regular map } \psi: C_1 \times C_2 \rightarrow \mathbb{S}^2\}$$

is a subgroup of  $\mathbb{Z}$ . These assertions are proved in [23, Theorem 3.1]. Define  $b(C_1, C_2)$  to be the unique nonnegative integer satisfying

$$\text{Deg}_{\mathcal{R}}(C_1, C_2) = b(C_1, C_2)\mathbb{Z}.$$

According to Hopf's theorem and Theorem 4.2, a continuous map  $f: C_1 \times C_2 \rightarrow \mathbb{S}^2$  is homotopic to a regular map (or equivalently can be approximated by regular maps) if and only if for every connected component  $A$  of  $C_1 \times C_2$ , one has  $\deg(f|_A) = b(C_1, C_2)r$  for some integer  $r$

independent of  $A$ . Thus, in this context, Problem 4.1 is reduced to the computation of the numerical invariant  $b(C_1, C_2)$ .

For any real number  $\alpha$  in  $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$ , set

$$\tau_\alpha = \frac{1}{2}(1 + \alpha\sqrt{-1}) \text{ if } \alpha > 0 \text{ and } \tau_\alpha = \alpha\sqrt{-1} \text{ if } \alpha < 0.$$

The lattice  $\Lambda_\alpha := \mathbb{Z} + \mathbb{Z}\tau_\alpha$  in  $\mathbb{C}$  is stable under complex conjugation. Hence the numbers

$$g_2(\tau_\alpha) = 60 \sum_{\omega} \omega^{-4}, \quad g_3(\tau_\alpha) = 140 \sum_{\omega} \omega^{-6}$$

(summation over  $\omega \in \Lambda_\alpha \setminus \{0\}$ ) are real and

$$D_\alpha := \{(x : y : z) \in \mathbb{P}^2(\mathbb{R}) : y^2 z = 4x^3 - g_2(\tau_\alpha)xz^2 - g_3(\tau_\alpha)z\}$$

is a smooth cubic curve in  $\mathbb{P}^2(\mathbb{R})$ . Each smooth cubic curve in  $\mathbb{P}^2(\mathbb{R})$  is biregularly isomorphic to exactly one cubic  $D_\alpha$ . Thus  $\mathbb{R}^*$  can be regarded as a moduli space of smooth cubic curves in  $\mathbb{P}^2(\mathbb{R})$ . For  $\alpha > 0$  (resp.  $\alpha < 0$ ) the cubic  $D_\alpha$  is connected (resp. has two connected components).

The invariant  $b(D_{\alpha_1}, D_{\alpha_2})$  is explicitly computed in [23] for all pairs  $(\alpha_1, \alpha_2)$ . In particular, it can take any nonnegative integer value. We recall only two cases.

First we deal with generic pairs  $(\alpha_1, \alpha_2)$ .

**Theorem 4.9.** *For  $(\alpha_1, \alpha_2)$  in  $\mathbb{R}^* \times \mathbb{R}^*$ , the following conditions are equivalent:*

- (a) *Each regular map from  $D_{\alpha_1} \times D_{\alpha_2}$  into  $\mathbb{S}^2$  is null homotopic.*
- (b)  *$b(D_{\alpha_1}, D_{\alpha_2}) = 0$ .*
- (c) *The product  $\alpha_1\alpha_2$  is an irrational number.*

From the viewpoint of approximation, the following case is of greatest interest.

**Theorem 4.10.** *For  $(\alpha_1, \alpha_2)$  in  $\mathbb{R}^* \times \mathbb{R}^*$ , the following conditions are equivalent:*

- (a) *Each continuous map from  $D_{\alpha_1} \times D_{\alpha_2}$  into  $\mathbb{S}^2$  can be approximated by regular maps.*
- (b)  *$\alpha_1 > 0$ ,  $\alpha_2 > 0$ , and  $b(D_{\alpha_1}, D_{\alpha_2}) = 1$ .*
- (c)  *$\alpha_i = (p_i/q_i)\sqrt{d}$  for  $i = 1, 2$ , where  $p_i, q_i, d$  are positive integers,  $p_i$  and  $q_i$  are relatively prime,  $d$  is square free,  $d \equiv 3 \pmod{4}$ ,  $p_1p_2q_1q_2 \equiv 1 \pmod{2}$ , and  $p_1p_2d$  is divisible by  $q_1q_2$ .*

Theorems 4.9 and 4.10 show that a small perturbation of  $(\alpha_1, \alpha_2)$  can drastically change topological properties of regular maps from  $D_{\alpha_1} \times D_{\alpha_2}$  into  $\mathbb{S}^2$ . Thus, in general, one cannot hope to find a comprehensive solution to Problem 4.1, even for  $X$  smooth with  $\dim X = p$ . It is therefore desirable to introduce maps which have good features of regular maps but are more flexible.

**4.2. Approximation by regulous maps.** Let  $X$  and  $Y \subset \mathbb{R}^q$  be smooth real algebraic sets, and let  $k$  be a nonnegative integer. A map  $f = (f_1, \dots, f_q): X \rightarrow Y$  is said to be  $k$ -regulous if its components  $f_i: X \rightarrow \mathbb{R}$  are  $k$ -regulous functions; 0-regulous maps are called *regulous*.

If  $f$  is a regulous map, we denote by  $P(f)$  the smallest algebraic subset of  $X$  such that  $f|_{X \setminus P(f)}: X \setminus P(f) \rightarrow Y$  is a regular map. Obviously,  $P(f)$  is Zariski nowhere dense in  $X$ . We say that  $f$  is *nice* if  $f(P(f)) \neq Y$ .

We state the next result for  $C^\infty$  maps. This is convenient since such maps have regular values by Sard's theorem.

**Theorem 4.11.** *Let  $X$  be a compact smooth real algebraic set,  $f: X \rightarrow \mathbb{S}^p$  a  $C^\infty$  map with  $p \geq 1$ , and  $y \in \mathbb{S}^p$  a regular value of  $f$ . Assume that the  $C^\infty$  submanifold  $f^{-1}(y)$  of  $X$  is isotopic to a smooth Zariski locally closed subset of  $X$ . Then:*

- (i)  *$f$  is homotopic to a nice  $k$ -regulous map, where  $k$  is an arbitrary nonnegative integer.*
- (ii)  *$f$  can be approximated by nice regulous maps.*

Theorem 4.11 is due to the first-named author. Part (i) is a simplified version of [46, Theorem 2.4]. The proof is based on the Pontryagin construction (framed cobordism), Łojasiewicz inequality, and Hironaka's resolution of singularities. In turn (ii) follows from [51, Theorem 1.2] since, by [44, Theorem 2.1],  $f^{-1}(y)$  can be approximated by smooth Zariski locally closed subsets of  $X$ .

Theorem 4.11 provides information also on continuous maps since they can be approximated by  $C^\infty$  maps.

According to [2, Theorem A], any compact  $C^\infty$  submanifold of  $\mathbb{R}^n$  (resp.  $\mathbb{S}^n$ ) is isotopic to a smooth Zariski locally closed subset of  $\mathbb{R}^n$  (resp.  $\mathbb{S}^n$ ). In particular, Theorem 4.11 yields the following.

**Corollary 4.12.** *Let  $(n, p)$  be a pair of positive integers. Then:*

- (i) *Each continuous map from  $\mathbb{S}^n$  into  $\mathbb{S}^p$  is homotopic to a nice  $k$ -regulous map, where  $k$  is an arbitrary nonnegative integer.*
- (ii) *Each continuous map from  $\mathbb{S}^n$  into  $\mathbb{S}^p$  can be approximated by nice regulous maps.*

With notation as in Theorem 4.11 we have  $\dim f^{-1}(y) = \dim X - p$ . Hence we get immediately the following.

**Corollary 4.13.** *Let  $X$  be a compact smooth real algebraic set of dimension  $p$ . Then:*

- (i) *Each continuous map from  $X$  into  $\mathbb{S}^p$  is homotopic to a nice  $k$ -regulous map, where  $k$  is an arbitrary nonnegative integer.*
- (ii) *Each continuous map from  $X$  into  $\mathbb{S}^p$  can be approximated by nice regulous maps.*

Comparing Theorems 4.8, 4.9 and 4.10 with Corollary 4.13 we see that  $k$ -regulous maps are indeed more flexible than regular ones. However, for each integer  $p \geq 1$  there exist a compact smooth real algebraic set  $Y$  and a continuous map  $g: Y \rightarrow \mathbb{S}^p$  such that  $\dim Y = p + 1$  and  $g$  is not homotopic to any regulous map, cf. [57, Theorem 7.9]. In particular, Theorem 4.11 does not hold without some assumption on the  $C^\infty$  submanifold  $f^{-1}(y) \subset X$ . It would be very useful to formulate an appropriate assumption in terms of bordism. This is related to a certain conjecture, which has nothing to do with regulous maps and originates from the celebrated paper of Nash [72] and the subsequent developments due to Tognoli [83], Akbulut and King [2], and other mathematicians.

For a real algebraic set  $X$ , a bordism class in the unoriented bordism group  $\mathfrak{N}_*(X)$  is said to be *algebraic* if it can be represented by a regular map from a compact smooth real algebraic set into  $X$ .

**Conjecture 4.14.** *For any smooth real algebraic set  $X$ , the following holds: If  $M$  is a compact  $C^\infty$  submanifold of  $X$  and the unoriented bordism class of the inclusion map  $M \hookrightarrow X$  is algebraic, then  $M$  is  $\varepsilon$ -isotopic to a smooth Zariski locally closed subset of  $X$ .*

Here “ $\varepsilon$ -isotopic” means isotopic via a  $C^\infty$  isotopy that can be chosen arbitrarily close, in the  $C^\infty$  topology, to the inclusion map. A slightly weaker assertion than the one in Conjecture 4.14 is known to be true: If the unoriented bordism class of the inclusion map  $M \hookrightarrow X$  is algebraic, then the  $C^\infty$  submanifold  $M \times \{0\}$  of  $X \times \mathbb{R}$  is  $\varepsilon$ -isotopic to a smooth Zariski locally closed subset of  $X \times \mathbb{R}$ , cf. [2, Theorem F].

**Remark 4.15.** Let  $X$  be a compact smooth real algebraic set and let  $f: X \rightarrow \mathbb{S}^p$  be a continuous map. According to [56, Proposition 1.4], if Conjecture 4.14 holds, then the following conditions are equivalent:

- (a)  $f$  is homotopic to a nice regulous map.
- (b)  $f$  can be approximated by nice regulous maps.

Using a method independent of Conjecture 4.14, the first-named author proved in [54] the following weaker result.

**Theorem 4.16.** *Let  $X$  be a compact smooth real algebraic set and let  $p$  be an integer satisfying  $\dim X + 3 \leq 2p$ . For a continuous map  $f: X \rightarrow \mathbb{S}^p$ , the following conditions are equivalent:*

- (a)  $f$  is homotopic to a nice regulous map.
- (b)  $f$  can be approximated by nice regulous maps.

Other results on topological properties of regulous maps can be found in [48, 50, 54, 57, 86].

## 5. VECTOR BUNDLES

Let  $\mathbb{F}$  stand for  $\mathbb{R}$ ,  $\mathbb{C}$  or  $\mathbb{H}$  (the quaternions). We consider only left  $\mathbb{F}$ -vector spaces. When convenient,  $\mathbb{F}$  will be identified with  $\mathbb{R}^{d(\mathbb{F})}$ , where  $d(\mathbb{F}) = \dim_{\mathbb{R}} \mathbb{F}$ .

**5.1. Algebraic versus topological vector bundles.** Let  $X$  be a real algebraic set. For any nonnegative integer  $n$ , let

$$\varepsilon_X^n(\mathbb{F}) = (X \times \mathbb{F}^n, p, X)$$

denote the product  $\mathbb{F}$ -vector bundle of rank  $n$  on  $X$ , where  $X \times \mathbb{F}^n$  is regarded as a real algebraic set and  $p: X \times \mathbb{F}^n \rightarrow X$  is the canonical projection.

An *algebraic  $\mathbb{F}$ -vector subbundle on  $X$*  is an algebraic  $\mathbb{F}$ -vector subbundle of  $\varepsilon_X^n(\mathbb{F})$  for some  $n$  (cf. [13] for various characterizations of algebraic  $\mathbb{F}$ -vector bundles). The category of algebraic  $\mathbb{F}$ -vector bundles on  $X$  is equivalent to the category of finitely generated projective left  $\mathcal{R}(X, \mathbb{F})$ -modules, where  $\mathcal{R}(X, \mathbb{F})$  is the ring of  $\mathbb{F}$ -valued regular functions on  $X$ .

Any algebraic  $\mathbb{F}$ -vector bundle on  $X$  can be regarded also as a topological  $\mathbb{F}$ -vector bundle. A topological  $\mathbb{F}$ -vector bundle is said to *admit an algebraic structure* if it is topologically isomorphic to an algebraic  $\mathbb{F}$ -vector bundle.

**Problem 5.1.** Which topological  $\mathbb{F}$ -vector bundles on  $X$  admit an algebraic structure?

It is convenient to bring into play the reduced Grothendieck group  $\tilde{K}_{\mathbb{F}}(X)$  of topological  $\mathbb{F}$ -vector bundles on  $X$ . Since  $X$  has the homotopy type of a compact polyhedron [13], the Abelian group  $\tilde{K}_{\mathbb{F}}(X)$  is finitely generated [3, 29]. We let  $\tilde{K}_{\mathbb{F}\text{-alg}}(X)$  denote the subgroup of  $\tilde{K}_{\mathbb{F}}(X)$  generated by the classes of all topological  $\mathbb{F}$ -vector bundles on  $X$  that admit an algebraic structure.

If  $X$  is compact, then Problem 5.1 is equivalent to providing a description of  $\tilde{K}_{\mathbb{F}\text{-alg}}(X)$ . More precisely, the following holds.

**Theorem 5.2.** *Let  $X$  be a compact real algebraic set. Then:*

- (i) *Two algebraic  $\mathbb{F}$ -vector bundles on  $X$  are algebraically isomorphic if and only if they are topologically isomorphic.*
- (ii) *A topological  $\mathbb{F}$ -vector bundle on  $X$  admits an algebraic structure if and only if its class in  $\tilde{K}_{\mathbb{F}}(X)$  belongs to  $\tilde{K}_{\mathbb{F}\text{-alg}}(X)$ .*

Theorem 5.2 follows from [81, Theorem 2.2], and a geometric proof is given in [13]. Note that  $\tilde{K}_{\mathbb{F}\text{-alg}}(X) = 0$  if and only if each algebraic  $\mathbb{F}$ -vector bundle on  $X$  is algebraically stably trivial. In turn,  $\tilde{K}_{\mathbb{F}\text{-alg}}(X) = \tilde{K}_{\mathbb{F}}(X)$  if and only if each topological  $\mathbb{F}$ -vector bundle on  $X$  admits an algebraic structure.

According to Fossum [35] and Swan [81], we have the following.

**Theorem 5.3.** *For the unit  $n$ -sphere  $\mathbb{S}^n$ , the equality  $\tilde{K}_{\mathbb{F}\text{-alg}}(\mathbb{S}^n) = \tilde{K}_{\mathbb{F}}(\mathbb{S}^n)$  holds.*

Benedetti and Tognoli [5] proved that algebraization of topological vector bundles on a compact  $\mathcal{C}^\infty$  manifold is always possible.

**Theorem 5.4.** *Let  $M$  be a compact  $\mathcal{C}^\infty$  manifold. Then there exists an algebraic model  $X$  of  $M$  such that  $\tilde{K}_{\mathbb{F}\text{-alg}}(X) = \tilde{K}_{\mathbb{F}}(X)$  for  $\mathbb{F} = \mathbb{R}$ ,  $\mathbb{F} = \mathbb{C}$  and  $\mathbb{F} = \mathbb{H}$ .*

The groups  $\tilde{K}_{\mathbb{F}\text{-alg}}(-)$  have been extensively investigated by J. Bochnak and the first-named author [18, 19, 22], who worked jointly with M. Buchner on [12]. In many cases,  $\tilde{K}_{\mathbb{F}\text{-alg}}(-)$  are “small” subgroups of  $\tilde{K}_{\mathbb{F}}(-)$ . The following is a simplified version of [12, Theorem 7.1].

**Theorem 5.5.** *Let  $M$  be a compact  $C^\infty$  submanifold of  $\mathbb{R}^{n+1}$ , with  $\dim M = n \geq 1$ . Then  $M$  is  $\varepsilon$ -isotopic to a smooth algebraic subset  $X$  of  $\mathbb{R}^{n+1}$  such that the group  $\tilde{K}_{\mathbb{F}\text{-alg}}(X)$  is finite for  $\mathbb{F} = \mathbb{R}$ ,  $\mathbb{F} = \mathbb{C}$  and  $\mathbb{F} = \mathbb{H}$ .*

The conclusion of Theorem 5.5 can be strengthened in some cases.

**Example 5.6.** Recall that  $\tilde{K}_{\mathbb{F}}(\mathbb{S}^{4d}) = \mathbb{Z}$  for every positive integer  $d$ , cf. [39]. Hence, by Theorem 5.5,  $\mathbb{S}^{4d}$  is  $\varepsilon$ -isotopic in  $\mathbb{R}^{4d+1}$  to a smooth algebraic subset  $\Sigma^{4d}$  such that

$$\tilde{K}_{\mathbb{F}\text{-alg}}(\Sigma^{4d}) = 0 \text{ and } \tilde{K}_{\mathbb{F}}(\Sigma^{4d}) = \mathbb{Z}$$

for  $\mathbb{F} = \mathbb{R}$ ,  $\mathbb{F} = \mathbb{C}$  and  $\mathbb{F} = \mathbb{H}$ .

One readily sees that each topological  $\mathbb{R}$ -vector bundle on a smooth real algebraic curve admits an algebraic structure. Next we discuss some results on vector bundles on a product of real algebraic curves.

**Example 5.7.** Let  $X = C_1 \times \cdots \times C_n$ , where  $C_1, \dots, C_n$  are connected, compact smooth real algebraic curves. Then  $\tilde{K}_{\mathbb{R}\text{-alg}}(X) = \tilde{K}_{\mathbb{R}}(X)$  if  $n = 2$  or  $n = 3$ . This assertion is a special case of [18, Theorem 1.6].

In Example 5.7, one cannot take  $n \geq 4$ .

**Example 5.8.** Let  $\mathbb{T}^n := \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$  be the  $n$ -fold product of  $\mathbb{S}^1$ . According to [15],  $\tilde{K}_{\mathbb{C}\text{-alg}}(\mathbb{T}^n) = 0$ . Furthermore, by [57, Example 1.11], we have  $\tilde{K}_{\mathbb{R}\text{-alg}}(\mathbb{T}^n) \neq \tilde{K}_{\mathbb{R}}(\mathbb{T}^n)$  and  $\tilde{K}_{\mathbb{H}\text{-alg}}(\mathbb{T}^n) \neq \tilde{K}_{\mathbb{H}}(\mathbb{T}^n)$  for  $n \geq 4$ .

Let  $\mathbb{K}$  be a subfield of  $\mathbb{F}$ , where  $\mathbb{K}$  (as  $\mathbb{F}$ ) stands for  $\mathbb{R}$ ,  $\mathbb{C}$  or  $\mathbb{H}$ . Any  $\mathbb{F}$ -vector bundle  $\xi$  can be regarded as a  $\mathbb{K}$ -vector bundle, which is indicated by  $\xi_{\mathbb{K}}$ .

**Example 5.9.** Let  $\lambda$  be a nontrivial topological  $\mathbb{C}$ -line bundle on  $\mathbb{T}^2$ . By Example 5.7, the  $\mathbb{R}$ -vector bundle  $\lambda_{\mathbb{R}}$  on  $\mathbb{T}^2$  admits an algebraic structure. However, in view of Example 5.8,  $\lambda$  does not admit an algebraic structure. The  $r$ th tensor power  $\lambda^{\otimes r}$  is a nontrivial  $\mathbb{C}$ -line bundle, hence it does not admit an algebraic structure.

The next two theorems come from [22]. We use smooth real cubic curves  $D_\alpha \subset \mathbb{P}^2(\mathbb{R})$ , with  $\alpha \in \mathbb{R}^*$ , introduced in Subsection 4.1.

**Theorem 5.10.** *Let  $X = D_\alpha \times \cdots \times D_\alpha$  be the  $n$ -fold product of  $D_\alpha$ , where  $\alpha$  is in  $\mathbb{R}^*$  and  $n \geq 2$ . Then the following conditions are equivalent:*

- (a)  $\tilde{K}_{\mathbb{C}\text{-alg}}(X) = 0$ .
- (b) *The number  $\alpha^2$  is irrational.*

In this context, the equality  $\tilde{K}_{\mathbb{C}\text{-alg}}(-) = \tilde{K}_{\mathbb{C}}(-)$  is characterized as follows.

**Theorem 5.11.** *Let  $X = D_{\alpha_1} \times \cdots \times D_{\alpha_n}$ , where  $\alpha_1, \dots, \alpha_n$  are in  $\mathbb{R}^*$  and  $n \geq 2$ . Then the following conditions are equivalent:*

- (a)  $\tilde{K}_{\mathbb{C}\text{-alg}}(X) = \tilde{K}_{\mathbb{C}}(X)$ .
- (b)  $\alpha_i > 0$  for all  $i$ , and  $b(D_{\alpha_i}, D_{\alpha_j}) = 1$  for  $i \neq j$ .

The pairs  $(\alpha_i, \alpha_j)$  with  $b(D_{\alpha_i}, D_{\alpha_j}) = 1$  are explicitly described in Theorem 4.10.

In the next subsection we deal with vector bundles of a new type, which occupy an intermediate position between algebraic and topological vector bundles.

**5.2. Regulous versus topological vector bundles.** Let  $X$  be a real algebraic set. As in (3.8), by a stratification of  $X$  we mean a finite collection  $\mathcal{S}$  of pairwise disjoint Zariski locally closed subsets whose union is  $X$ . An  $\mathcal{S}$ -algebraic  $\mathbb{F}$ -vector bundle on  $X$  is a topological  $\mathbb{F}$ -vector subbundle of  $\varepsilon_X^n(\mathbb{F})$ , for some  $n$ , such that the restriction  $\xi|_S$  of  $\xi$  to each stratum  $S \in \mathcal{S}$  is an algebraic  $\mathbb{F}$ -vector subbundle of  $\varepsilon_S^n(\mathbb{F})$ . If  $\xi$  and  $\eta$  are  $\mathcal{S}$ -algebraic  $\mathbb{F}$ -vector bundles on  $X$ , then an  $\mathcal{S}$ -algebraic morphism  $\varphi: \xi \rightarrow \eta$  is a morphism of topological  $\mathbb{F}$ -vector bundles which induces a morphism of algebraic  $\mathbb{F}$ -vector bundles  $\varphi_S: \xi|_S \rightarrow \eta|_S$  for each stratum  $S \in \mathcal{S}$ .



**Definition 5.12.** A *regulous  $\mathbb{F}$ -vector bundle on  $X$*  is an  $\mathcal{S}$ -algebraic  $\mathbb{F}$ -vector bundle for some stratification  $\mathcal{S}$  of  $X$ . If  $\xi$  and  $\eta$  are regulous  $\mathbb{F}$ -vector bundles on  $X$ , then a *regulous morphism*  $\varphi: \xi \rightarrow \eta$  is an  $\mathcal{S}$ -algebraic morphism for some stratification  $\mathcal{S}$  of  $X$  such that both  $\xi$  and  $\eta$  are  $\mathcal{S}$ -algebraic  $\mathbb{F}$ -vector bundles.

In our joint paper [57], we introduced and investigated regulous (= stratified-algebraic) vector bundles. The main focus of [57] and the subsequent papers [53, 55, 59, 60] is on comparison of algebraic, regulous, and topological vector spaces.

Regulous  $\mathbb{F}$ -vector bundles on  $X$  (together with regulous morphisms) form a category, which is equivalent to the category of finitely generated projective left  $\mathcal{R}^0(X, \mathbb{F})$ -modules, where  $\mathcal{R}^0(X, \mathbb{F})$  is the ring of  $\mathbb{F}$ -valued regulous functions on  $X$ , cf. [57, Theorem 3.9].

A topological  $\mathbb{F}$ -vector bundle on  $X$  is said to *admit a regulous structure* if it is topologically isomorphic to a regulous  $\mathbb{F}$ -vector bundle. We let  $\tilde{K}_{\mathbb{F}\text{-reg}}(X)$  denote the subgroup of  $\tilde{K}_{\mathbb{F}}(X)$  generated by the classes of all topological  $\mathbb{F}$ -vector bundles on  $X$  that admit a regulous structure.

We have the following counterpart of Theorem 5.2, cf. [57].

**Theorem 5.13.** *Let  $X$  be a compact real algebraic set. Then:*

- (i) *Two regulous  $\mathbb{F}$ -vector bundles on  $X$  are regulously isomorphic if and only if they are topologically isomorphic.*
- (ii) *A topological  $\mathbb{F}$ -vector bundle on  $X$  admits a regulous structure if and only if its class in  $\tilde{K}_{\mathbb{F}}(X)$  belongs to  $\tilde{K}_{\mathbb{F}\text{-reg}}(X)$ .*

Hence  $\tilde{K}_{\mathbb{F}\text{-reg}}(X) = \tilde{K}_{\mathbb{F}}(X)$  if and only if each topological  $\mathbb{F}$ -vector bundle on  $X$  admits a regulous structure.

The following result of [57] should be compared with Theorem 5.3 and Example 5.6.

**Theorem 5.14.** *Let  $X$  be a compact real algebraic set that is homotopically equivalent to the unit  $n$ -sphere  $\mathbb{S}^n$ . Then  $\tilde{K}_{\mathbb{F}\text{-reg}}(X) = \tilde{K}_{\mathbb{F}}(X)$ .*

In contrast to Example 5.8 and Theorems 5.10 and 5.11, we proved in [57] the following.

**Theorem 5.15.** *Let  $X = X_1 \times \cdots \times X_n$ , where  $X_i$  is a compact real algebraic set that is homotopically equivalent to the unit  $d_i$ -sphere  $\mathbb{S}^{d_i}$  for  $1 \leq i \leq n$ . Then*

$$2\tilde{K}_{\mathbb{R}}(X) \subset \tilde{K}_{\mathbb{R}\text{-reg}}(X), \quad \tilde{K}_{\mathbb{C}\text{-reg}}(X) = \tilde{K}_{\mathbb{C}}(X) \quad \text{and} \quad \tilde{K}_{\mathbb{H}\text{-reg}}(X) = \tilde{K}_{\mathbb{H}}(X).$$

It is possible that  $\tilde{K}_{\mathbb{R}\text{-reg}}(X) = \tilde{K}_{\mathbb{R}}(X)$  in Theorem 5.15, but no proof is available even for  $X = \mathbb{T}^n$  with  $n \geq 4$ .

Our next result comes from [59].

**Theorem 5.16.** *Let  $X$  be a compact real algebraic set that is homotopically equivalent to  $\mathbb{S}^{d_1} \times \cdots \times \mathbb{S}^{d_n}$ . Then the quotient group  $\tilde{K}_{\mathbb{F}}(X)/\tilde{K}_{\mathbb{F}\text{-reg}}(X)$  is finite.*

If  $n \geq 5$ , then there exists an algebraic model  $X$  of  $\mathbb{T}^n$  with  $\tilde{K}_{\mathbb{F}}(X)/\tilde{K}_{\mathbb{F}\text{-reg}}(X) \neq 0$  for  $\mathbb{F} = \mathbb{R}$ ,  $\mathbb{F} = \mathbb{C}$  and  $\mathbb{F} = \mathbb{H}$ , cf. [57, Example 7.10].

For any real algebraic set  $X$ , we let  $\tilde{K}_{\mathbb{F}}^{(\text{crk})}(X)$  denote the subgroup of  $\tilde{K}_{\mathbb{F}}(X)$  generated by the classes of all topological  $\mathbb{F}$ -vector bundles of constant rank. Define  $\Gamma_{\mathbb{F}}(X)$  to be the quotient group

$$\Gamma_{\mathbb{F}}(X) := \tilde{K}_{\mathbb{F}}^{(\text{crk})}(X) / (\tilde{K}_{\mathbb{F}\text{-reg}}(X) \cap \tilde{K}_{\mathbb{F}}^{(\text{crk})}(X))$$

(cf. [59] for an equivalent description). Evidently,  $\Gamma_{\mathbb{F}}(X) = \tilde{K}_{\mathbb{F}}(X)/\tilde{K}_{\mathbb{F}\text{-reg}}(X)$  if  $X$  is connected. Note, however, that for the real algebraic curve  $C$  of (1.5), one has  $\Gamma_{\mathbb{R}}(C) = 0$ , while the group  $\tilde{K}_{\mathbb{R}}(C)/\tilde{K}_{\mathbb{R}\text{-reg}}(C)$  is infinite.

**Conjecture 5.17.** *For any compact real algebraic set  $X$ , the group  $\Gamma_{\mathbb{F}}(X)$  is finite.*

In [59], we proved that Conjecture 5.17 holds in low dimensions.

**Theorem 5.18.** *If  $X$  is a compact real algebraic set of dimension at most 8, then the group  $\Gamma_{\mathbb{F}}(X)$  is finite.*

For  $\mathbb{C}$ -line bundles, the following is expected.

**Conjecture 5.19.** *For any compact real algebraic set  $X$  and any topological  $\mathbb{C}$ -line bundle  $\lambda$  on  $X$ , the  $\mathbb{C}$ -line bundle  $\lambda^{\otimes 2}$  admits a regulous structure.*

According to [53], Conjecture 4.14 implies Conjecture 5.19. If  $\dim X \leq 8$ , then for some positive integer  $r$ , the  $\mathbb{C}$ -line bundle  $\lambda^{\otimes r}$  admits a regulous structure, cf. [59]. This should be compared with Example 5.9.

The key role in the proofs of the results presented in this subsection plays the following theorem of [57].

**Theorem 5.20.** *Let  $X$  be a compact real algebraic set. A topological  $\mathbb{F}$ -vector bundle  $\xi$  on  $X$  admits a regulous structure if and only if the  $\mathbb{R}$ -vector bundle  $\xi_{\mathbb{R}}$  admits a regulous structure.*

By Example 5.9, one cannot substitute “algebraic” for “regulous” in Theorem 5.20.

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