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Algebraic Thinking

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The Working Group

In CERME12, the Thematic Working Group 3, ‘Algebraic Thinking’, continued its work from previous CERME conferences. We had a total of 27 papers and three posters with a total of 39 people in the group. Participants represented countries from Europe and other continents: Canada, Cyprus, Denmark, Finland, France, Germany, Greece, Hungary, Italy, Norway, Spain, Sweden, The Netherlands, United Kingdom, and the United States of America. The papers loosely centred around six themes. These were: Generalisation and Pattern, Structure, Equations and Variables, Theoretical, Functional Thinking, and Algebraic Thinking. We discuss each of these themes below.

Generalisation and Pattern

Generalisation is a key aspect of algebraic thinking and there were a number of papers which looked at how students might be helped in developing their generalisation skills further. Figural patterns continue to feature as a significant research tool to explore students’ generalization skills, although perhaps less common during this conference than the last. Mazza et al. looked at Grade 10 students as they considered proofs related to figurative patterns. Students were presented with ‘visual proofs’ and asked to justify theorems based upon those pictures. They found that there was a close match between the way in which students described the figural patterns in terms of mathematical properties and their explanations of the theorems. Goñi-Cervera et al. used a well-known growth pattern involving chairs placed around a certain number of tables. Their particular focus was students with autism spectrum disorder (AS) compared with students they describe as ‘typically-developing’ (TD). Although there was more success gained by TD students, it was found that the most frequent strategies for both groups were the same. Lócska and Kovács found that generalisation and reasoning strategies were supported by an intervention with 7th grade Hungarian students. This intervention used numerical tricks based around the array of numbers in a month found in calendars. Reinhardtson and Carlsen’s study involved students approaching introductory algebra through a calculational perspective. They found that teaching norms of emphasising procedures and products remained and that students could evaluate letters but struggled to use them to express generalisation and structure. Kilhamn reported a case study where computer programming was used when working with pattern generalizations. Tinkering with the code sparked students’ curiosity in new ways, and when the computer did all the arithmetic, the students were free to look for pattern and structure.

Structure

A key aspect of algebra which was prominent in the previous CERME was that of structure. The continuum between operational and structural thinking was present in a number of papers. Lenz's study offered situations which included boxes with unknown or indeterminate numbers of marbles in boxes. She looked at kindergarten and elementary school children's approaches to establishing relationships within the continuum of number-orientated and structure-orientated approaches. The need to break away from concrete representation and perceive the variable as a thought object was argued. Unteregge's study involved slightly older children in Grade 4 and looked at the way they justified equalities within the result-orientated and structure-orientated continuum. It was found that students with a clear understanding of equality used different rationales across this range and could easily switch between them. Sencindiver (see Wladis et al. with Sencinder underlined) presented a paper concerning college students doing an algebra course, also using a framework of operational and structural thinking. Students were asked to justify algebraic transformations of expressions; an analysis of their attempts was conducted in terms of the students' understanding of equivalence, substitution, and substitution equivalence. Wladis et al. (paper with Wladis underlined) also used the operation and structural continuum but along with another dimension of extracted vs. stipulated to look at college students' thinking about equivalence. They found that this two-dimensional framework was useful to analyse students' definitions of equivalence. Their finding was that although students noticed 'sameness', they struggled with articulating a more standard definition of equivalence. Vlachos investigated students' understanding of what constituted a set. The sample included students from Grades 6, 9 and 12. Various prevalent misconceptions were identified. Grade 12 students did considerably better but there was no significant difference found between Grade 6 and Grade 9 students.

Equations and Variables

Dealing with equations is a standard part of the algebra curriculum. Roos and Kempen looked at the bar model as a pedagogical tool to assist with solving algebraic equations. Their study involved two cycles with low attaining Grade 10 students followed by Grade 8 students. They found that there was a need for students to develop conceptual understanding of which operations were illustrated within the model, rather than focusing on the numbers involved. López Centella et al. studied Grade 5 students to see whether they could relate a given algebraic equation to five different contextual situations. They found that the students had different forms of justification depending upon the contextual situation, and that the students were able to infer mathematical truths which had not been explicitly taught to them. Korntreff and Prediger's study involved Grade 7 and 8 students. They focused on the variable which appears within certain algebraic activities. Variables can play the role of 'generalizers' or unknowns. Their teaching experiment showed that students could construct both meanings for a variable, but only some students were able to distinguish explicitly the distinction between the two. They also found that the meaning students had of a variable was related to the algebraic activity within which the variable was used. Tondorf and Prediger studied Grade 5 students who were asked to justify the transformation from one arithmetic expression to another, both of which could be represented by the same geometric image. When making sense of the transformation of an expression, many students linked this with the geometric representation. They concluded that the

dynamic transformation of a graphical representation could be used ahead of the dynamic transformation of symbolic expressions.

Theoretical

There were a number of papers which were based around theoretical arguments or developing models related to the teaching and learning of mathematics. Eriksson carried out a literature review on algebraic thinking related to students of 5-12 years old. She considered three perspectives in respect to teaching approaches which were based upon whether arithmetic thinking or algebraic thinking was developed first or whether they were developed at the same time. It was found that more research is needed on the perspective of developing algebraic thinking before arithmetic thinking. Weigand et al. came from the German tradition of *Grundvorstellungen*, which they translated as Basic Mental Models. They compared four different models for an equation and explored the relationship between these models and solving equations, particularly in relation to the use of technology. Three other papers used a similar framework to each other in their studies. Strømskag and Chevillard analysed textbooks from different countries, along with other publications from influential authors. They found that the evolution of curricula related to the notion of functions has seen a decline in algebra being taught as a modelling tool, and the reduced inclusion of parameters in algebraic equations. Hällback et al. analysed the algebra content of two Swedish upper secondary programmes and found that the programme for vocational education and training was more focused on *know-how* aspects rather than *know-why*. In contrast, the programme for higher educational preparation was more evenly balanced between these two aspects. Finally, Tonnesen constructed a praxeological reference model to develop a diagnostic test tool. This was used to examine students' technical and theoretical knowledge of basic algebra.

Functional thinking

Functional thinking is a significant aspect of algebraic thinking. Frey et al. interviewed 35 educational experts across five countries about their views of what constitutes functional thinking. In their preliminary results from two German interviewees and one interviewee from the Netherlands, it was found that there were different views across those experts. Sterner's study involved contextual growth patterns being presented to Grade 1 students and found that graphical representation, along with well-thought-through terminology, was significant in developing students' reasoning about recursive and also covariational relationships. Pittalis et al. focused more widely on algebraic thinking. They proposed and empirically validated a framework describing algebraic thinking abilities of Grade 3 students. Functional thinking, along with two other abilities, were found to form an index of those students' capacity to respond to algebraic tasks.

Algebraic thinking

Algebraic thinking is, of course, the focus of our Thematic Working Group. In that sense this is a theme which pervades all the papers. There are papers, though, where this is a more explicit focus. Radford presented his own conception of what constitutes algebraic thinking before then going on to analyse Grade 3 students' engagement in tasks presented in concrete and iconic semiotic systems. He found that the students generated two key algebraic ideas: that of 'removing' (from both sides) and 'separating' (effectively reducing the coefficient of the unknown to 1). Bräuer's study also avoided

use of symbolic language by using the image of balance scales to support learning of linear equation systems, involving multiple unknowns, with Grade 3 and 4 students. It was found that the students could use algebraic strategies in an informal way without the need for an explicit intervention programme. Akinwunmi and Steinweg made a distinction between focusing on patterns and focusing on structure in their study with Grade 3 and 4 students. They argued that awareness of structure was important for students to engage in reasoning about patterns and made a case for the importance of material representations to help develop more structural arguments. Chimoni (Pitta-Pantazi et al.) presented a paper where they also made the case that seeing structure is important in early algebraic thinking. They found that students who solved arithmetical tasks using algebraic strategies were also able to solve algebraic tasks. They concluded that generalised arithmetic abilities underpin manipulation of algebraic expressions. Lastly, Fred et al. saw algebraic thinking as a resource or tool for action. They argued that explicit connections should be made between algebraic thinking and ways in which algebra plays a role in addressing challenges such as sustainability and climate change. In their analysis of two major research review books, they found that most of the ‘powerful algebraic ideas’ had a logical or psychological focus.

Discussions and further directions for TWG3

There was much discussion about the relationship between arithmetic thinking and algebraic thinking. Sometimes these were mentioned as a transition from one to the other. If that were to be the case, then is there a type of continuum between the two or is there an abrupt shift from one to the other? What constitutes the beginning of algebraic thinking? Alternatively, are these seen as running in parallel with each other, or are they actually more separate than sometimes imagined? For example, algebraic thinking can exist within a non-arithmetic context. We feel the link between these two is an important area for future work, perhaps considering the idea of *advanced arithmetic thinking* (which looks forward from arithmetic, rather than *pre-algebra*, which looks back from algebra). We also noted that terms, such as arithmetic thinking, functional thinking, relational thinking, and algebraic thinking were used without them always being defined. Different ways in which such terms are used can result in quite different analysis and conclusions being drawn from studies. The use of representations/models was a feature in a number of papers, and we felt that more work could be focused on how the abstraction process might involve gradual moving away from the use of representations/models. Algebra was often presented as a desired endpoint within some empirical studies, and we felt that more research could be done around the way in which algebra can be used, such as a modelling tool. We were aware of some important areas which were not represented so much in the papers and posters at this CERME. This included the use of technology. Also, there was no reference to the aesthetic aspect of algebraic activity. This is far from a dry area of mathematics and can bring many insights and Aha! moments which have an affective impact. We are aware that some papers may be presented within other TWGs in relation to both these aspects, but we feel that research around both these would be a very welcome addition to the next CERME in this thematic working group. Lastly, we noted the relative lack of discussion about the role of teacher interactions with students whilst involved with algebraic activities. Often focus was on the tasks presented to students but the way in which tasks are introduced and the nature of the teacher-student interactions which follow can be just as, or even more, significant as the tasks themselves.