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► **To cite this version:**

Olivier Debarre, Zhi Jiang, Martí Lahoz, William F Sawin. Rational cohomology tori. *Geometry & Topology Monographs*, 2017, 21 (2), pp.1095-1130. 10.2140/gt.2017.21.1095 . hal-03807432

**HAL Id: hal-03807432**

**<https://hal.science/hal-03807432>**

Submitted on 17 Feb 2024

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## Rational cohomology tori

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APPENDIX BY WILLIAM F SAWIN

We study normal compact varieties in Fujiki's class  $\mathcal{C}$  whose rational cohomology ring is isomorphic to that of a complex torus. We call them rational cohomology tori. We classify, up to dimension three, those with rational singularities. We then give constraints on the degree of the Albanese morphism and the number of simple factors of the Albanese variety for rational cohomology tori of general type (hence projective) with rational singularities. Their properties are related to the birational geometry of smooth projective varieties of general type, maximal Albanese dimension, and with vanishing holomorphic Euler characteristic. We finish with the construction of series of examples.

In an appendix, we show that there are no smooth rational cohomology tori of general type. The key technical ingredient is a result of Popa and Schnell on 1-forms on smooth varieties of general type.

32J27, 32Q15, 32Q55; 14F45, 14E99

### Introduction

Given a compact complex manifold, one fundamental problem is to determine how much information is encoded in its underlying topological space.

Hirzebruch and Kodaira [24] proved that for  $n$  odd, any compact Kähler manifold which is homeomorphic to  $\mathbb{P}^n$  is actually isomorphic to  $\mathbb{P}^n$ ; see also Morrow [32, Theorem 1]. A stronger property is actually conjectured: it should be sufficient to assume that the rings  $H^\bullet(X, \mathbb{Z})$  and  $H^\bullet(\mathbb{P}^n, \mathbb{Z}) = \mathbb{Z}[x]/(x^{n+1})$  are isomorphic and  $c_1(T_X) > 0$  to deduce that  $X$  is isomorphic to  $\mathbb{P}^n$  (this is known in dimensions at most 6; see Fujita [20, Theorem 1], Libgober and Wood [31, Theorem 1] and Debarre [14]).

Catanese [8, Theorem 70] (see also Theorem 1.1) observed that complex tori  $X$  satisfy this stronger property: they can be characterized among compact Kähler manifolds by the fact that there is an isomorphism

$$(1) \quad \bigwedge^\bullet H^1(X, \mathbb{Z}) \xrightarrow{\sim} H^\bullet(X, \mathbb{Z})$$

of graded rings. The Kähler assumption is essential since one can construct non-Kähler compact complex manifolds (hence not biholomorphic to complex tori) that satisfy (1) (see Example 1.6). If we replace (1) with an isomorphism

$$(2) \quad \bigwedge^\bullet H^1(X, \mathbb{Q}) \xrightarrow{\sim} H^\bullet(X, \mathbb{Q})$$

of graded  $\mathbb{Q}$ -algebras, Catanese [8, Conjecture 71] asked whether this property still characterizes complex tori.

We say that a normal compact variety  $X$  in Fujiki's class  $\mathcal{C}$  is a *rational cohomology torus* if it satisfies (2). The main objective of this article is to study the geometry of these varieties: give restrictive properties and construct examples (some smooth) which are not complex tori, thereby answering Catanese's question negatively.

The Albanese morphism of a smooth rational cohomology torus  $X$  is finite (see Catanese [8, Remark 72]). A result of Kawamata (Remark 1.8) then says that there is a morphism  $I_X: X \rightarrow X_1$  which is an Iitaka fibration for  $X$  (see Section 1.1 for the definition) such that  $X_1$  is algebraic and has again a finite morphism to a torus. We prove that  $X_1$  is also a rational cohomology torus, but possibly singular. This leads to the following result, proved in Section 1.

**Theorem A** *Let  $X$  be a normal compact class- $\mathcal{C}$  variety with a finite morphism to a torus. Consider the sequence of Iitaka fibrations*

$$X \xrightarrow{I_X} X_1 \xrightarrow{I_{X_1}} X_2 \rightarrow \cdots \rightarrow X_{k-1} \xrightarrow{I_{X_{k-1}}} X_k,$$

where  $X_1, \dots, X_k$  are normal projective varieties. Then  $X$  is a rational cohomology torus if, and only if,  $X_k$  is a rational cohomology torus. Moreover,

- either  $X_k$  is a point and we say that  $X$  is an Iitaka torus tower;
- or  $X_k$  is of general type (of positive dimension).

It is easy to construct smooth projective surfaces which are Iitaka torus towers but not complex tori (Example 1.11). Since the product of two rational cohomology tori is again a rational cohomology torus, this already gives a negative answer to Catanese's question in any dimension at least 2.

The next question is whether all rational cohomology tori are Iitaka torus towers. By Theorem A, this is the same as asking whether there exist (possibly singular) projective rational cohomology tori of general type. This reduces our problem to the algebraic category; however, the price we have to pay is that we need to deal with singular varieties.

Rational singularities turn out to be the suitable kind of singularities to work with, because they are stable under the construction of the sequence of Iitaka fibrations of Theorem A. Moreover, any desingularization of a projective rational cohomology torus of general type with rational singularities has maximal Albanese dimension and vanishing holomorphic Euler characteristic (Proposition 1.17). These varieties were studied by Chen and Jiang [12] and Chen, Debarre and Jiang [9]. Building on these results, we give a classification, in dimensions up to three, of rational cohomology tori with rational singularities.

**Theorem B** *Let  $X$  be a compact class- $\mathcal{C}$  variety with rational singularities.*

- (1) *If  $X$  is a surface,  $X$  is a rational cohomology torus if, and only if,  $X$  is an Iitaka torus tower.*
- (2) *If  $X$  is a threefold,  $X$  is a rational cohomology torus if, and only if,*
  - *either  $X$  is an Iitaka torus tower;*
  - *or  $X$  has an étale cover which is a Chen–Hacon threefold ( $X$  is then singular, of general type).*
- (3) *Starting from dimension 4, there exist smooth rational cohomology tori that are not Iitaka torus towers.*

This theorem is proved in Section 2. Chen–Hacon threefolds were constructed in [11, Section 4, Example] (see also Example 2.1 and Proposition 2.2). The  $n$ -folds we construct for (3) have Kodaira dimension any number in  $\{3, \dots, n-1\}$ . In Corollary A.2 of the appendix, William Sawin shows that *smooth* rational cohomology tori of general type do not exist (but we construct in Example 4.4 *singular* rational cohomology tori of general type in any dimension at least 3).

After this classification result, we focus on giving restrictions on rational cohomology tori of general type.

**Theorem C** *Let  $X$  be a projective variety of general type with rational singularities. Assume that  $X$  is a rational cohomology torus, with Albanese morphism  $a_X$ . There exists a prime number  $p$  such that  $p^2 \mid \deg(a_X)$ .*

Moreover, if  $\deg(a_X) = p^2$ , the morphism  $a_X$  is a  $(\mathbb{Z}/p\mathbb{Z})^2$ -cover of its image (Corollary 3.8).

We deduce Theorem C and Corollary 3.8 from the analogous restrictions on the degree of the Albanese morphism of a smooth projective variety  $X$  of general type, of maximal Albanese dimension and with  $\chi(X, \omega_X) = 0$  (Theorems 3.6 and 3.7). Theorem 3.6 is probably the deepest result of this article and a key step in its proof is the description

of *minimal primitive varieties of general type with  $\chi = 0$*  (Theorem 3.4), *primitive* in the sense that there exist no proper subvarieties with  $\chi = 0$  through a general point, and *minimal* with respect to the degree of birational factorizations of the Albanese morphism (Definition 3.2). The most difficult part of the proof of Theorem 3.6 is to realize these varieties as Galois quotients of products of lower-dimensional varieties (Lemma 3.5).

Continuing with the idea of giving constraints to the existence of rational cohomology tori of general type, we prove the following condition on the number of simple factors of their Albanese varieties.

**Theorem D** *Let  $X$  be a projective variety of general type with rational singularities. Assume that  $X$  is a rational cohomology torus, with Albanese morphism  $a_X: X \rightarrow A_X$ , and let  $p$  be the smallest prime divisor of  $\deg(a_X)$ . Then  $A_X$  has at least  $p + 1$  simple factors.*

Section 4 is devoted to the construction of examples. First, we construct in Example 4.1, for each prime  $p$ , minimal primitive varieties  $X$  of general type of dimension  $p + 1$  with  $\chi(X, \omega_X) = 0$  whose Albanese morphisms are surjective  $(\mathbb{Z}/p\mathbb{Z})^2$ -covers of a product of  $p + 1$  elliptic curves; they are finite Galois quotients of a product of  $p + 1$  curves. These examples show that the structure of primitive varieties with  $\chi = 0$  is much more complicated than expected by Chen and Jiang [12] (see Remark 4.2).

We then use techniques of Pardini [33] to produce (singular) rational cohomology tori of general type in any dimension at least 3 (Examples 4.3 and 4.4). The first of these examples shows that the lower bound on the degree of the Albanese morphism in Theorem C is optimal, and so is the lower bound on the number of factors of the Albanese variety in Theorem D.

The article ends with a series of examples of nonminimal primitive fourfolds with  $\chi = 0$  whose Albanese variety has 4 simple factors and whose Albanese morphism has degree 8.

**Notation** We work over the complex numbers. A (complex) variety is reduced and integral (and possibly singular; manifolds are smooth). A variety is in the (Fujiki) class  $\mathcal{C}$  if it is compact and bimeromorphic to a compact Kähler manifold (Fujiki [17, Definition 1.2] and Ancona and Gaveau [2, Part I, Section 7.5]).

A projective variety is of general type if a (hence any) desingularization is of general type, ie has maximal Kodaira dimension.

Given a compact Kähler manifold  $X$ , we denote by  $A_X$  its Albanese torus and by  $a_X: X \rightarrow A_X$  its Albanese morphism. Given a complex torus  $A$ , we denote by  $\hat{A}$  its dual.

A proper morphism  $X \rightarrow Y$  between normal varieties is a *fibration* if it is surjective with connected fibers; it is *birationally isotrivial* if the general fibers are all birationally isomorphic to a fixed variety  $F$ . When  $X$  and  $Y$  are algebraic, this is equivalent to saying that after a finite (Galois) base change  $Y' \rightarrow Y$ , the product  $X \times_Y Y'$  is birationally isomorphic, over  $Y'$ , to  $F \times Y'$  (Bogomolov, Böhning and von Bothmer [4]).

**Acknowledgements** We thank Fabrizio Catanese for explaining [8, Conjecture 72] during a talk at the IMJ-PRG–ENS algebraic geometry seminar. This was the starting point of this article. It is also a pleasure to thank Lie Fu, Sándor Kovács, Stefan Schreieder and Claire Voisin for useful conversations and comments, and an anonymous referee for her or his suggestions. Lahoz worked on this article during his stay at IMPA (Brazil) and he is grateful for the support he received on this occasion. Lahoz is partially supported by MTM2015-65361-P.

## 1 Catanese’s theorem and question

Catanese [8] proved the following topological characterization of complex tori.

**Theorem 1.1** [8, Theorem 70] *A compact Kähler manifold  $X$  is biholomorphic to a complex torus if, and only if, there is an isomorphism*

$$(3) \quad \bigwedge^\bullet H^1(X, \mathbb{Z}) \xrightarrow{\sim} H^\bullet(X, \mathbb{Z})$$

*of graded rings.*

Let us recall the proof. The Albanese map  $a_X: X \rightarrow A_X$  induces an isomorphism  $a_X^{*1}: H^1(A_X, \mathbb{Z}) \xrightarrow{\sim} H^1(X, \mathbb{Z})$  and (3) then implies that  $a_X^*: H^\bullet(A_X, \mathbb{Z}) \rightarrow H^\bullet(X, \mathbb{Z})$  is also an isomorphism. Set  $n := \dim(X)$ ; that  $a_X^{*n}: H^{2n}(X, \mathbb{Z}) \xrightarrow{\sim} H^{2n}(A_X, \mathbb{Z})$  is an isomorphism implies that  $a_X$  is birational. Moreover, since we have an isomorphism of the whole cohomology rings and  $X$  is Kähler,  $a_X$  cannot contract any subvariety of  $X$  and is therefore finite. Thus,  $a_X$  is an isomorphism.

If we replace (3) with an isomorphism at the level of rational cohomology, Catanese already observed that the Albanese morphism is still surjective and finite [8, Remark 72].

**Definition 1.2** Let  $X$  be a normal compact class- $\mathcal{C}$  variety. We say that  $X$  is a *rational cohomology torus* if there is an isomorphism

$$(*_X) \quad \bigwedge^\bullet H^1(X, \mathbb{Q}) \xrightarrow{\sim} H^\bullet(X, \mathbb{Q})$$

*of graded  $\mathbb{Q}$ -algebras.*

We also give an a priori slightly different definition.

**Definition 1.3** Let  $X$  be a normal compact class- $\mathcal{C}$  variety and let  $f: X \rightarrow A$  be a morphism to a torus. We say that  $X$  is an  $f$ -rational cohomology torus if  $f$  induces an isomorphism

$$(*_f) \quad f^*: H^\bullet(A, \mathbb{Q}) \xrightarrow{\sim} H^\bullet(X, \mathbb{Q})$$

of graded  $\mathbb{Q}$ -algebras.

Condition  $(*_f)$  certainly implies condition  $(*_X)$ . But because of singularities, the converse is a priori not clear. However, we show that Definitions 1.2 and 1.3 are in fact equivalent.

**Proposition 1.4** Let  $X$  be a normal compact class- $\mathcal{C}$  variety. Assume that  $X$  is a rational cohomology torus. There exists a finite morphism  $f: X \rightarrow A$  onto a complex torus such that  $X$  is an  $f$ -rational cohomology torus. In particular, the Hodge structures on  $H^\bullet(X)$  are pure.

**Proof** Set  $n := \dim(X)$ . There are functorial mixed  $\mathbb{Q}$ -Hodge structures on  $H^\bullet(X)$  for which the cup product  $\wedge^{2n} H^1(X) \rightarrow H^{2n}(X)$  is a morphism of mixed Hodge structures [18, Proposition (1.4.1); 2, Part II, Theorem 3.4]. Since  $X$  is compact,  $H^{2n}(X)$  is a 1-dimensional pure Hodge structure, hence  $h^1(X) = 2n$ ,  $W_0 H^1(X) = 0$  and  $H^1(X)$  carries a pure Hodge structure. Therefore,  $H^k(X) \simeq \wedge^k H^1(X)$  has a pure Hodge structure for each  $k \in \{0, \dots, 2n\}$ .

Let  $\mu: X' \rightarrow X$  be a resolution of singularities with  $X'$  Kähler. Since  $H^1(X)$  carries a pure Hodge structure, the pullback map  $\mu^*: H^1(X) \rightarrow H^1(X')$  is injective. Considering the Albanese morphism  $a_{X'}: X' \rightarrow A_{X'}$ , we note that  $a_{X'}^*: H^1(A_{X'}, \mathbb{Q}) \rightarrow H^1(X', \mathbb{Q})$  is an isomorphism. Thus  $H^1(A_{X'})$  has a sub-Hodge structure which is isomorphic to  $H^1(X)$ . Therefore, there exists a quotient  $\pi: A_{X'} \rightarrow A$  of complex tori such that  $g^* H^1(A) = \mu^* H^1(X)$  as sub-Hodge structures of  $H^1(X')$ , where  $g := \pi a_{X'}: X' \rightarrow A$ . We then have  $\mu^* H^k(X, \mathbb{Q}) = g^* H^k(A, \mathbb{Q})$  as subspaces of  $H^k(X', \mathbb{Q})$ .

We claim that  $g$  contracts every fiber of  $\mu$ . Otherwise, let  $F$  be an irreducible closed subvariety contained in a fiber of  $\mu$  and assume  $\dim(g(F)) > 0$ . Let  $F' \rightarrow F$  be a desingularization with  $F'$  Kähler and let  $t: F' \rightarrow F \hookrightarrow X'$  be the composition. Let  $\omega \in g^* H^2(A, \mathbb{C}) \subset H^2(X', \mathbb{C})$  be the pullback of a Kähler form on  $A$ . Since  $\dim(gt(F')) > 0$ , the form  $t^* \omega \in H^2(F', \mathbb{C})$  is nonzero. This is a contradiction, since  $\omega$  is in  $\mu^* H^2(X, \mathbb{C})$  and  $\mu t(F')$  is a point.

Therefore,  $g$  contracts every fiber of  $\mu$ . Moreover,  $\mu$  is birational and  $X$  is normal, hence  $\mu_* \mathcal{O}_{X'} = \mathcal{O}_X$ . Thus the morphism  $g: X' \rightarrow A$  factors through a morphism  $f: X \rightarrow A$  and  $X$  is an  $f$ -rational cohomology torus.  $\square$

**Remark 1.5** If  $X$  is a rational cohomology torus, Proposition 1.4 provides a finite morphism  $f: X \rightarrow A$  which has a universal property (up to translations) for morphisms from  $X$  to complex tori. We will call  $f$  the *Albanese morphism* and  $A$  the *Albanese variety* of  $X$ .

The hypothesis “ $X$  Kähler” in Theorem 1.1 is essential: in dimension at least 3, the topological characterization of tori is not true without this hypothesis, as we show in the following example, whose origins can be traced to [3, page 163] (see also [39, Example 5.1]).

**Example 1.6** (a non-Kähler integral cohomology torus) Let  $E$  be an elliptic curve, let  $L$  be a very ample line bundle on  $E$  and let  $\varphi$  and  $\psi$  be holomorphic sections of  $L$  with no common zeroes on  $E$ . Set

$$J_1 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J_2 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad J_3 := \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \quad J_4 := \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}.$$

Since  $\det(\sum_{i=1}^4 \lambda_i J_i) = \sum_{i=1}^4 \lambda_i^2$ , the group

$$\Gamma := \sum_{i=1}^4 \mathbb{Z} J_i \begin{pmatrix} \varphi \\ \psi \end{pmatrix}$$

is a relative lattice in the total space of the rank-2 vector bundle  $\mathcal{V} := L \oplus L$  over  $E$ . The quotient  $M := \mathcal{V}/\Gamma$  is a complex manifold with a surjective holomorphic map  $\pi: M \rightarrow E$ . By construction,  $\pi$  is smooth, each fiber of  $\pi$  is a complex 2-dimensional torus and its relative canonical bundle  $\omega_{M/E}$  is  $\pi^*L^{-2}$ .

One checks that  $M$  is diffeomorphic to a real torus, hence  $H^\bullet(M, \mathbb{Z}) = \wedge^\bullet H^1(M, \mathbb{Z})$ , but  $M$  is not a complex torus, since it is not Kähler: if it were,  $\pi_*\omega_{M/E}$  would be semipositive [19, Theorem (2.7)].

Answering a question of Ottem, we note that the hypothesis that  $\wedge^\bullet H^1(X, \mathbb{Q}) \simeq H^\bullet(X, \mathbb{Q})$  is a ring isomorphism is crucial to get a finite morphism to an abelian variety: if we only assume that the Hodge numbers of  $X$  are those of a torus, the Albanese morphism is not even necessarily finite as we show in the following example, though strong constraints on these morphisms were found in [13].

**Example 1.7** (a surface with the Hodge numbers of a torus but nonfinite Albanese map) Let  $\rho: D \rightarrow C$  be a double étale cover of smooth projective curves, where  $C$  has genus 2, and let  $\tau$  be the associated involution of  $D$ . Let  $E$  be an elliptic curve and let  $\sigma$  be the involution of  $E$  given by multiplication by  $-1$ , with quotient morphism  $\rho': E \rightarrow \mathbb{P}^1$ . Let  $S := (D \times E)/\langle \tau \times \sigma \rangle$  be the diagonal quotient. The surface  $S$  is



smooth and its Hodge numbers are

$$\begin{matrix} & & 1 & & \\ & & 2 & 2 & \\ & 1 & 4 & 1 & \end{matrix}$$

Indeed, we have

$$\begin{aligned} \rho_* \mathcal{O}_D &= \mathcal{O}_C \oplus L^{-1}, & \text{where } L \in \text{Pic}^0(C) \text{ is a 2-torsion point,} \\ \rho'_* \mathcal{O}_E &= \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2). \end{aligned}$$

If we denote by  $g: S \rightarrow C \times \mathbb{P}^1$  the natural double cover, we have

$$\begin{aligned} g_* \mathcal{O}_S &= \mathcal{O}_{C \times \mathbb{P}^1} \oplus (L^{-1} \boxtimes \mathcal{O}_{\mathbb{P}^1}(-2)), \\ g_* \Omega_S^1 &= (\omega_C \boxtimes \mathcal{O}_{\mathbb{P}^1}) \oplus ((\omega_C \otimes L) \boxtimes \mathcal{O}_{\mathbb{P}^1}(-2)) \oplus (\mathcal{O}_C \boxtimes \omega_{\mathbb{P}^1}) \oplus (L^{-1} \boxtimes \mathcal{O}_{\mathbb{P}^1}), \\ g_* \omega_S &= \omega_{C \times \mathbb{P}^1} \oplus ((\omega_C \otimes L) \boxtimes \mathcal{O}_{\mathbb{P}^1}). \end{aligned}$$

Note that the Albanese variety of  $S$  and the Jacobian  $J(C)$  are isogenous and that the Albanese morphism of  $S$  contracts the elliptic curves  $E$  that are the fibers of  $S \rightarrow C$ .

### 1.1 Iitaka torus towers

By Proposition 1.4, studying rational cohomology tori is equivalent to studying  $f$ -rational cohomology tori. For an  $f$ -rational cohomology torus  $X$ , the property  $(*_f)$  implies, since  $X$  is Kähler, that  $f$  is finite and surjective. A theorem of Kawamata describes Iitaka fibrations for varieties with a finite morphism to a torus.

Recall from [38, Theorem 5.10] that given a normal compact complex variety  $X$  of nonnegative Kodaira dimension  $\kappa(X)$ , there exists a proper modification  $X^* \rightarrow X$  (with  $X^*$  smooth) and a fibration  $I_X: X^* \rightarrow Y^*$  such that  $\dim(Y^*) = \kappa(X)$  and the Kodaira dimension of a general fiber of  $I_X$  is 0. The fibration  $I_X$  is bimeromorphically equivalent to the rational map on  $X$  defined by the sections of  $\omega_X^{\otimes m}$ , for  $m$  sufficiently large and divisible. It is in particular unique up to bimeromorphic equivalence. Any fibration  $X' \rightarrow Y'$  bimeromorphically equivalent to  $I_X$ , with  $X'$  normal but not necessarily smooth, will be called an *Iitaka fibration* of  $X$ .

**Remark 1.8** (reduction to algebraic varieties) Let  $X$  be a normal compact complex variety and let  $f: X \rightarrow A$  be a finite morphism to a torus. By [27, Theorem 23], there are

- an abelian Galois étale cover  $\pi: \tilde{X} \rightarrow X$  with group  $G$ , induced by an étale cover of  $A$ ,
- a subtorus  $K$  of  $A$ ,
- a normal projective variety  $\hat{Y}$  of general type, and

- a commutative diagram

$$(4) \quad \begin{array}{ccccc} \tilde{X} & \xrightarrow{\pi} & X & \xrightarrow[\text{finite}]{f} & A \\ I_{\tilde{X}} \downarrow & & I_X \downarrow & & \downarrow \\ \hat{Y} & \xrightarrow[\text{finite}]{} & Y & \xrightarrow[\text{finite}]{f_Y} & A/K \end{array}$$

with the following properties:

- $I_X$  is the Stein factorization of the composition  $X \rightarrow A \rightarrow A/K$ . It is an Iitaka fibration of  $X$ , with general fiber an étale cover  $\tilde{K}$  of  $K$ .
- $I_{\tilde{X}}$  is the Stein factorization of  $I_X \pi: \tilde{X} \rightarrow Y$ . It is an Iitaka fibration of  $\tilde{X}$  and is an analytic fiber bundle with fiber  $\tilde{K}$ . Hence, there is a natural  $G$ -action on  $\hat{Y}$  which may not be faithful,  $I_{\tilde{X}}$  is  $G$ -equivariant and  $Y = \hat{Y}/G$ .

For any finite group  $G$  acting on a irreducible projective variety  $V$ , we have

$$H^\bullet(V, \mathbb{Q})^G = H^\bullet(V/G, \mathbb{Q});$$

see [6, Chapter III, Theorem 7.2]. Thus, we have isomorphisms

$$\begin{aligned} H^\bullet(X, \mathbb{Q}) &\simeq H^\bullet(\tilde{X}, \mathbb{Q})^G \\ &\simeq (H^\bullet(\hat{Y}, \mathbb{Q}) \otimes H^\bullet(\tilde{K}, \mathbb{Q}))^G \\ &\simeq H^\bullet(\hat{Y}, \mathbb{Q})^G \otimes H^\bullet(K, \mathbb{Q}) \\ &\simeq H^\bullet(Y, \mathbb{Q}) \otimes H^\bullet(K, \mathbb{Q}) \end{aligned}$$

of graded  $\mathbb{Q}$ -algebras, where the second isomorphism holds by the Leray–Hirsch theorem applied to the fiber bundle  $I_{\tilde{X}}$  [5, Theorem 15.11] and the third isomorphism holds because  $G$  acts trivially on  $H^\bullet(\tilde{K}, \mathbb{Q})$ , which is isomorphic to  $H^\bullet(K, \mathbb{Q})$ . Thus,  $(*_f)$  holds if, and only if,  $(*_f_Y)$  holds. In particular, this allows us to reduce the study of property  $(*_f)$  to algebraic varieties.

The following lemma, which implies Theorem A in the introduction, is an easy consequence of the previous remark.

**Lemma 1.9** *Let  $f: X \rightarrow A$  be a finite morphism from a normal compact complex variety to a torus. Let*

$$(5) \quad X \xrightarrow{I_X} X_1 \xrightarrow{I_{X_1}} X_2 \rightarrow \dots \rightarrow X_{k-1} \xrightarrow{I_{X_{k-1}}} X_k$$

*be the tower of Iitaka fibrations as in diagram (4), where the general fibers of  $I_{X_i}$  are complex tori, the  $X_i$  are normal projective varieties with morphisms  $f_i: X_i \rightarrow A_i$*

to quotient tori of  $A$  and  $X_k$  is of general type or a point. Then  $X$  is an  $f$ -rational cohomology torus if, and only if,  $X_k$  is an  $f_k$ -rational cohomology torus. In particular, if  $X_k$  is a point,  $X$  is an  $f$ -rational cohomology torus.

**Definition 1.10** We say that  $X$  is an *Iitaka torus tower* if, in (5),  $X_k$  is a point.

**Example 1.11** (an Iitaka torus tower which is not a torus) Let  $\rho: C \rightarrow E$  be a double cover of smooth projective curves, where  $C$  has genus  $g \geq 2$  and  $E$  is an elliptic curve. Let  $\tau$  be the corresponding involution on  $C$ . Let  $E' \rightarrow E$  be a degree-2 étale cover of elliptic curves and let  $\sigma$  be the corresponding involution on  $E'$ . Let  $X$  be the smooth surface  $(C \times E') / \langle \tau \times \sigma \rangle$ . Then  $X$  is an Iitaka torus tower but has Kodaira dimension 1, hence is not a torus.

The answer to Catanese's original conjecture [8, Conjecture 70] is therefore negative. Nevertheless, we may still ask the following question.

**Question 1.12** Is a compact Kähler manifold which is a rational cohomology torus always an Iitaka torus tower?

To answer (negatively) this question, we study the variety  $X_k$  of Lemma 1.9, which is a possibly singular projective variety.

## 1.2 Rational singularities

Recall the following classical definition.

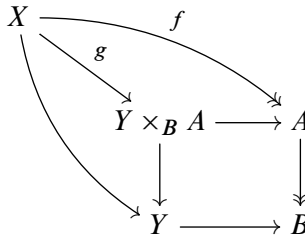
**Definition 1.13** Let  $X$  be a compact complex variety and let  $\mu: X' \rightarrow X$  be a desingularization. We say that  $X$  has *rational singularities* if  $R^i \mu_* \mathcal{O}_{X'} = 0$  for all  $i > 0$  and  $\mu_* \mathcal{O}_{X'} = \mathcal{O}_X$  (equivalently,  $X$  is normal).

The following lemma explains why we work with rational singularities.

**Lemma 1.14** Let  $f: X \rightarrow A$  be a finite and surjective morphism from a projective variety  $X$  with rational singularities to an abelian variety. Consider a quotient  $A \rightarrow B$  of abelian varieties. If the composition  $X \rightarrow A \rightarrow B$  factors through a finite morphism  $Y \rightarrow B$  with  $Y$  normal,  $Y$  has rational singularities.

In particular, the lemma applies when  $X \rightarrow Y \rightarrow B$  is the Stein factorization of  $X \rightarrow B$ .

**Proof** Since  $f: X \rightarrow A$  is finite, so is the induced morphism  $g$  in the commutative diagram



Since  $Y \rightarrow B$  is finite and surjective, so is  $Y \times_B A \rightarrow A$ , hence the image of  $g$  is a component  $\tilde{Y}$  of  $Y \times_B A$ . Since the induced morphism  $X \rightarrow \tilde{Y}$  is finite, the trace operator induces a splitting of the natural morphism  $\mathcal{O}_X \rightarrow Rg_*\mathcal{O}_{\tilde{Y}}$ . Thus, by [29, Theorem 1], since  $X$  has rational singularities, so has  $\tilde{Y}$ . Finally, since  $A \rightarrow B$  is smooth, so is  $\tilde{Y} \hookrightarrow Y \times_B A \rightarrow Y$ . It follows that  $Y$  has rational singularities.  $\square$

It follows from the lemma that if  $X$  has rational singularities, so do all the  $X_i$  in the tower (5). Thus, in order to answer Question 1.12, it suffices to answer the following.

**Question 1.15** Can a projective variety with rational singularities which is a rational cohomology torus be of general type?

To answer (positively) this question, we first prove that any desingularization must satisfy very strong numerical properties.

**Lemma 1.16** Let  $X$  be a projective variety with rational singularities. For each  $k$ , we have an isomorphism

$$H^k(X, \mathcal{O}_X) \simeq \text{Gr}_F^0 H^k(X),$$

where  $F^\bullet$  is the Hodge filtration for Deligne’s mixed Hodge structure on  $H^k(X)$ .

**Proof** By [28, Theorem S], rational singularities are Du Bois. If  $\underline{\Omega}_X^\bullet$  is the Deligne–Du Bois complex of  $X$  [35, Definition 7.34], this means that  $\underline{\Omega}_X^0$  is quasi-isomorphic to  $\mathcal{O}_X$ . By Deligne’s theorem [15, Sections 8.1, 8.2 and 9.3; 30, (4.2.4)], the spectral sequence  $E_1^{p,q} = \mathbb{H}^q(X, \underline{\Omega}_X^p) \Rightarrow H^{p+q}(X, \mathbb{C})$  degenerates at  $E_1$  and abuts to the Hodge filtration of Deligne’s mixed Hodge structure. Thus, we have  $H^k(X, \mathcal{O}_X) \simeq H^k(X, \underline{\Omega}_X^0) = \text{Gr}_F^0 H^k(X)$ .  $\square$

**Proposition 1.17** Let  $X$  be a projective variety with rational singularities. Assume that  $X$  is a rational cohomology torus and let  $\mu: X' \rightarrow X$  be any desingularization.

- (1) We have  $h^k(X', \omega_{X'}) = h^k(X, \omega_X) = \binom{n}{k}$ ; in particular,  $\chi(X', \omega_{X'}) = 0$ .
- (2) The Albanese morphism  $a_{X'}: X' \rightarrow A_{X'}$  factors through  $\mu$  and the induced morphism  $X \rightarrow A_{X'}$  is the Albanese morphism of  $X$  in the sense of Remark 1.5.

**Proof** By Proposition 1.4, there is a finite morphism  $f: X \rightarrow A$  to an abelian variety such that  $X$  is an  $f$ -rational cohomology torus. Hence we have isomorphisms of Hodge structures  $f^*: H^k(A) \xrightarrow{\sim} H^k(X)$  for all  $k$ . In particular,

$$\mathrm{Gr}_F^0 H^k(X) \simeq \mathrm{Gr}_F^0 H^k(A) = H^k(A, \mathcal{O}_A).$$

Since  $X$  has rational singularities, we have  $H^k(X, \mathcal{O}_X) \simeq \mathrm{Gr}_F^0 H^k(X)$  (Lemma 1.16). Hence  $h^k(X, \mathcal{O}_X) = \binom{n}{k}$ .

Let  $\mu: X' \rightarrow X$  be a desingularization. Since  $R\mu_*\omega_{X'} = \omega_X$ , we have

$$h^k(X, \omega_X) = h^k(X', \omega_{X'}) = h^{n-k}(X', \mathcal{O}_{X'}) = h^{n-k}(X, \mathcal{O}_X) = \binom{n}{k}.$$

For (2), note that  $h^1(X', \mathcal{O}_{X'}) = n$ , hence  $\dim(A_{X'}) = \dim(X') = \dim(X) = n$ . By Proposition 1.4, there is a quotient morphism  $A_{X'} \rightarrow A$  with connected fibers, hence  $A_X = A_{X'}$  and  $a_{X'}$  factors through  $\mu$ .  $\square$

This simple but important proposition allows us to use the many known properties of smooth projective varieties  $X'$  of maximal Albanese dimension with  $\chi(X', \omega_{X'}) = 0$  and  $p_g(X') = 1$ .

## 2 Rational cohomology tori in lower dimensions

Thanks to the work of Chen, Debarre and Jiang [9] on smooth varieties of maximal Albanese dimension with  $p_g = 1$ , we can give a classification of rational cohomology tori up to dimension 3. We first recall some important examples.

**Example 2.1** (Ein–Lazarsfeld & Chen–Hacon threefolds) For each  $j \in \{1, 2, 3\}$ , consider an elliptic curve  $E_j$  and a bielliptic curve  $C_j \xrightarrow{2:1} E_j$  of genus  $g_j \geq 2$ , with corresponding involution  $\tau_j$  of  $C_j$ . Set  $A := E_1 \times E_2 \times E_3$  and consider the quotient  $g: C_1 \times C_2 \times C_3 \twoheadrightarrow Z$  by the involution  $\tau_1 \times \tau_2 \times \tau_3$  and the tower of Galois covers

$$C_1 \times C_2 \times C_3 \xrightarrow{g} Z \xrightarrow{f} A$$

of respective degrees 2 and 4. The threefold  $Z$  is of general type with rational singularities and it has  $2^3 \prod_{j=1}^3 (g_j - 1)$  isolated singular points. We call  $Z$  an *Ein–Lazarsfeld threefold* [16, Example 1.13]. If  $X \twoheadrightarrow Z$  is any desingularization, we have  $\chi(X, \omega_X) = \chi(Z, \omega_Z) = 0$ .

A variant of the previous construction gives us varieties with  $p_g = 1$ , as follows. Keeping the same notation, choose points  $\xi_j \in \hat{E}_j$  of order 2 and consider the induced double étale covers  $E'_j \twoheadrightarrow E_j$  and  $C'_j \twoheadrightarrow C_j$ , with associated involution  $\sigma_j$  of  $C'_j$ .

The involution  $\tau_j$  on  $C_j$  pulls back to an involution  $\tau'_j$  on  $C'_j$  (with quotient  $E'_j$ ). Let  $g': C'_1 \times C'_2 \times C'_3 \twoheadrightarrow Z'$  be the quotient by the group (isomorphic to  $(\mathbb{Z}/2\mathbb{Z})^4$ ) of automorphisms generated by  $\text{id}_1 \times \tau'_2 \times \sigma_3$ ,  $\sigma_1 \times \text{id}_2 \times \tau'_3$ ,  $\tau'_1 \times \sigma_2 \times \text{id}_3$  and  $\tau'_1 \times \tau'_2 \times \tau'_3$  and consider the tower

$$C'_1 \times C'_2 \times C'_3 \xrightarrow{g'} Z' \xrightarrow{f'} A$$

of Galois covers of respective degrees  $2^4$  and 4. The threefold  $Z'$  is of general type and has rational singularities. We call  $Z'$  a *Chen–Hacon threefold* [11, Section 4, Example]. For any desingularization  $X' \twoheadrightarrow Z'$ , one has  $p_g(X') = 1$ .

The étale cover  $E'_1 \times E'_2 \times E'_3 \twoheadrightarrow E_1 \times E_2 \times E_3$  pulls back to an étale cover  $Z'' \rightarrow Z'$ , where  $Z''$  is an Ein–Lazarsfeld threefold; in particular,  $Z'$  also has isolated singularities. Moreover, the quotient of  $C'_1 \times C'_2 \times C'_3$  by the group of automorphisms generated by  $\text{id}_1 \times \tau'_2 \times \sigma_3$ ,  $\sigma_1 \times \text{id}_2 \times \tau'_3$  and  $\tau'_1 \times \sigma_2 \times \text{id}_3$  is a smooth double cover of  $Z'$ , since the group acts freely.

This terminology differs from that of [9]: there, Ein–Lazarsfeld and Chen–Hacon threefolds refer to any of their desingularizations. Our singular threefolds can be obtained from their smooth versions by considering the Stein factorizations of their Albanese morphisms.

We can now prove the classification of rational cohomology tori up to dimension 3 stated in the introduction.

**Proof of Theorem B** Let  $X$  be a compact class- $\mathcal{C}$  variety with rational singularities which is a rational cohomology torus. By Lemma 1.9, we may assume that  $X$  is projective. Let  $\mu: X' \rightarrow X$  be a desingularization. By Proposition 1.17, we have  $\chi(X', \omega_{X'}) = 0$  and  $p_g(X') = 1$ .

If  $\dim(X) = 2$  and  $X$  is of general type, we have  $\chi(X', \omega_{X'}) > 0$  by Riemann–Roch, which is a contradiction. If  $\kappa(X) = 0$ , then  $X$  is an abelian variety by [27, Corollary 2]. If  $\kappa(X) = 1$ , in the diagram (4) of Remark 1.8,  $Y$  is an elliptic curve. Hence  $X$  is an Iitaka torus tower. This proves (1).

Assume  $\dim(X) = 3$ . If  $X'$  is of general type, we can apply the structure theorem [9, Theorem 6.3]: there exists an abelian étale cover  $\tilde{A} \twoheadrightarrow A_{X'}$  such that, in the Stein factorization  $X' \times_{A_{X'}} \tilde{A} \rightarrow \tilde{X} \rightarrow \tilde{A}$ , the variety  $\tilde{X}$  is a Chen–Hacon threefold (Example 2.1). As noted in Proposition 1.17(2),  $X$  appears in the Stein factorization of the Albanese morphism of  $X'$ , hence  $\tilde{X}$  is an étale cover of  $X$  and  $X$  is singular. We then apply Lemma 1.9 and part (1) to get the first part of (2).

For the second part of (2), it suffices to show that a Chen–Hacon threefold is a rational cohomology torus. This follows from the more general Proposition 2.2 below.

For (3), we note that there exists a smooth projective threefold  $\widehat{Y}$  with an involution  $\tau$  such that  $Y := \widehat{Y}/\langle\tau\rangle$  is a Chen–Hacon threefold (Example 2.1), hence a rational cohomology torus. Let  $\sigma$  be a translation of order 2 on any nonzero abelian variety  $K$ . The involution  $\tau \times \sigma$  acts freely on  $\widehat{Y} \times K$  and  $X := (\widehat{Y} \times K)/\langle\tau \times \sigma\rangle$  is a smooth projective variety. Moreover, the natural morphism  $X \rightarrow Y$  is the Iitaka fibration in (4). Thus  $X$  is also a rational cohomology torus. Since  $Y$  is of general type,  $X$  is not an Iitaka torus tower.

Moreover, by Example 4.4, there exists a rational cohomology torus  $Y$  with rational singularities and of general type in any dimension at least 3 with a smooth double cover  $\widehat{Y}$ . By the same construction, whenever  $3 \leq m \leq n - 1$ , there exists a smooth rational cohomology torus  $X$  of dimension  $n$  with Kodaira dimension  $m$ .  $\square$

The following proposition, which is further generalized to all abelian covers of abelian varieties in [26], shows in particular that Chen–Hacon threefolds are rational cohomology tori.

**Proposition 2.2** *For each  $j \in \{1, \dots, n\}$ , let  $\rho_j: C_j \rightarrow E_j$  be an abelian Galois cover with group  $G_j$ , where  $C_j$  is a smooth projective curve and  $E_j$  an elliptic curve. Take a subgroup  $G$  of  $G_1 \times \dots \times G_n$  and set  $X := (C_1 \times \dots \times C_n)/G$ . Assume  $h^0(X, \omega_X) = 1$ ; then  $X$  is a rational cohomology torus with rational singularities.*

**Proof** Set  $V := C_1 \times \dots \times C_n$  and  $A := E_1 \times \dots \times E_n$ , and let

$$\rho: V \xrightarrow{f} X \xrightarrow{g} A$$

be the quotient morphisms. The variety  $X$  has finite quotient singularities, which are rational singularities. In particular,  $H^k(X)$  has a pure Hodge structure for all  $k \in \{0, \dots, n\}$  [35, Theorem 2.43]. More precisely, if  $\iota: X_{\text{reg}} \hookrightarrow X$  is the smooth locus of  $X$  and we set  $\Omega_X^{[p]} := \iota_*(\Omega_{X_{\text{reg}}}^p)$ , we have  $\Omega_X^{[p]} = (f_*\Omega_V^p)^G$  and  $\Omega_X^{[\bullet]}$  is a resolution of the constant sheaf  $\mathbb{C}_X$  [35, Lemma 2.46]. Thus,  $H^q(X, \Omega_X^{[p]}) = \text{Gr}_F^q H^{p+q}(X, \mathbb{C})$  [35, proof of Theorem 2.43].

We may assume that each projection  $G \rightarrow G_j$  is surjective. Indeed, if we denote by  $H_j$  the image of this projection, there are natural morphisms  $X \rightarrow C_j/H_j \rightarrow E_j$ . Since  $X$  has maximal Albanese dimension, the condition  $h^0(X, \omega_X) = 1$  implies  $h^0(C_j/H_j, \omega_{C_j/H_j}) = 1$  [25, Lemma 2.3], so  $C_j/H_j$  is also an elliptic curve. Then we simply replace  $G_j$  with  $H_j$  and  $E_j$  with  $C_j/H_j$ .

Let  $j \in \{1, \dots, n\}$ . Since  $\rho_j$  is an abelian Galois cover, we may write

$$(6) \quad \rho_{j*}\omega_{C_j} = \mathcal{O}_{E_j} \oplus \bigoplus_{1 \neq \chi_j \in G_j^\vee} L_{\chi_j}, \quad \rho_{j*}\mathcal{O}_{C_j} = \mathcal{O}_{E_j} \oplus \bigoplus_{1 \neq \chi_j \in G_j^\vee} L_{\chi_j}^{-1},$$

where  $G_j^\vee$  is the character group of  $G_j$  and  $L_{\chi_j}$  is the line bundle on  $E_j$  associated with the character  $\chi_j \in G_j^\vee$ . Since  $G \rightarrow G_j$  is surjective, its dual  $G_j^\vee \rightarrow G^\vee$  is injective.

Since  $\rho_{j*}\omega_{C_j}$  is a nef vector bundle on  $E_j$  [19, Theorem (3.1)], each line bundle  $L_{\chi_j}$  is nef and, for  $\chi_j \neq 1$ , it is either ample or nontrivial torsion in  $\widehat{E}_j$ . Moreover, if  $L_{\chi_j}$  is a torsion line bundle, so is  $L_{\chi_j^m} = L_{\chi_j}^{\otimes m}$  for each  $m \in \mathbb{Z}$ . Thus,  $L_{\chi_j}$  is nontrivial torsion if, and only if,  $L_{\chi_j^{-1}} = L_{\chi_j}^{-1}$  is nontrivial torsion.

We compute

$$\begin{aligned} g_*\omega_X &= g_*((f_*\omega_V)^G) = (\rho_*\omega_V)^G \\ &= \left( \bigoplus_{\substack{\chi_j \in G_j^\vee \\ 1 \leq j \leq n}} (L_{\chi_1} \boxtimes \cdots \boxtimes L_{\chi_n}) \right)^G \\ &= \bigoplus_{\substack{\chi_j \in G_j^\vee \\ \chi_1 \cdots \chi_n = 1 \in G^\vee}} (L_{\chi_1} \boxtimes \cdots \boxtimes L_{\chi_n}). \end{aligned}$$

Since  $\mathcal{O}_A$  is a direct summand of  $g_*\omega_X$  and  $h^0(X, \omega_X) = 1$ , we conclude that, for any  $(\chi_1, \dots, \chi_n) \in G_1^\vee \times \cdots \times G_n^\vee$  not all trivial such that  $\chi_1 \cdots \chi_n = 1 \in G^\vee$ , at least one of the corresponding line bundles  $L_{\chi_j}$  is nontrivial torsion.

For any subset  $J = \{j_1, \dots, j_p\}$  of  $T := \{1, \dots, n\}$ , we set  $V_J := C_{j_1} \times \cdots \times C_{j_p}$  and we let  $p_J: V \rightarrow V_J$  be the projection. We also denote by  $q_j: A \rightarrow E_j$  the projections. Then

$$\begin{aligned} g_*\Omega_X^{[p]} &= g_*((f_*\Omega_V^p)^G) = (\rho_*\Omega_V^p)^G \\ &= \left( \rho_* \left( \bigoplus_{\substack{J \subset T \\ |J|=p}} p_J^* \omega_{V_J} \right) \right)^G \\ &= \left( \bigoplus_{\substack{J \subset T \\ |J|=p}} \left( \bigoplus_{\substack{\chi_j \in G_j^\vee \\ \text{for all } j \in J}} \left( \bigotimes_{j \in J} q_j^* L_{\chi_j} \right) \right) \otimes \left( \bigoplus_{\substack{\chi_k \in G_k^\vee \\ \text{for all } k \in J^c}} \left( \bigotimes_{k \in J^c} q_k^* L_{\chi_k}^{-1} \right) \right) \right)^G \\ &= \bigoplus_{\substack{J \subset T \\ |J|=p}} \bigoplus_{\substack{\chi_j \in G_j^\vee, \chi_k \in G_k^\vee \\ \prod_{j \in J} \chi_j \prod_{k \in J^c} \chi_k^{-1} = 1 \in G^\vee}} \left( \bigotimes_{\substack{\chi_j \in G_j^\vee \\ j \in J}} q_j^* L_{\chi_j} \right) \otimes \left( \bigotimes_{\substack{\chi_k \in G_k^\vee \\ k \in J^c}} q_k^* L_{\chi_k}^{-1} \right). \end{aligned}$$

For example, for  $J = \{1, \dots, p\}$ , the fourth equality reads

$$\rho_* p_J^* \omega_{V_J} = \rho_{1*} \omega_{C_1} \boxtimes \cdots \boxtimes \rho_{p*} \omega_{C_p} \boxtimes \rho_{p+1*} \mathcal{O}_{C_{p+1}} \boxtimes \cdots \boxtimes \rho_{n*} \mathcal{O}_{C_n}.$$



For any nontrivial solution of  $\prod_{j \in J} \chi_j \prod_{k \in J^c} \chi_k^{-1} = 1 \in G^\vee$ , we have already seen that the condition  $h^0(X, \omega_X) = 1$  implies that either there exists  $j \in J$  such that  $L_{\chi_j}$  is nontrivial torsion or there exists  $k \in J^c$  such that  $L_{\chi_k^{-1}}$  is nontrivial torsion, in which case  $L_{\chi_k^{-1}}$  is also nontrivial torsion. Thus, by the Künneth formula, only the trivial direct summands of  $g_*\Omega_X^p$  have nontrivial cohomology groups, since all the others contain a nontrivial torsion line bundle. Therefore,

$$H^q(X, \Omega_X^{[p]}) = H^q(A, g_*\Omega_X^{[p]}) = H^q(A, \Omega_A^p).$$

Since  $H^q(X, \Omega_X^{[p]}) = \text{Gr}_F^q H^{p+q}(X, \mathbb{C})$ , we conclude that  $X$  is a rational cohomology torus. □

### 3 Constraints on the Albanese morphism

In this section, we study how the condition  $\chi(X, \omega_X) = 0$  gives restrictions on the degree of the Albanese morphism. Note that  $\chi(X, \omega_X)$  is a birational invariant. It would be interesting to study the nonbirational conditions  $\chi(X, \Omega_X^p) = 0$  for  $0 < p < \dim(X)$  to get further restrictions on the structure of rational cohomology tori.

We first recall some facts about projective varieties  $X$  of general type, of maximal Albanese dimension and with  $\chi(X, \omega_X) = 0$  (see [9; 12] for more details).

Let  $X$  be a smooth projective variety and let  $f: X \rightarrow A$  be a generically finite morphism to an abelian variety. We set

$$V^i(f_*\omega_X) := \{[P] \in \widehat{A} \mid H^i(A, f_*\omega_X \otimes P) \neq 0\}.$$

By [21; 22; 37; 23],  $V^i(f_*\omega_X)$  is a union of torsion translates of abelian subvarieties of  $\widehat{A}$  of codimension  $\geq i$ . The set

$$(7) \quad S_f := \{T \subset \widehat{A} \mid \exists i \geq 1 \text{ } T \text{ is a component of } V^i(f_*\omega_X) \text{ with } \text{codim}_{\widehat{A}}(T) = i\}$$

controls the positivity of the sheaf  $f_*\omega_X$  [12; 25, Section 3].

We use the following notation: for any abelian subvariety  $\widehat{B} \subset \widehat{A}$ , we let

$$(8) \quad X \twoheadrightarrow X_B \xrightarrow{f_B} B$$

be the Stein factorization of the composition  $X \xrightarrow{f} A \twoheadrightarrow B$ . After birational modifications, we may assume that  $X_B$  is also smooth. Note that when  $f$  is the Albanese morphism of  $X$  and  $\widehat{B} \in S_f$ , the map  $f_B$  is the Albanese morphism of  $X_B$ .

**Lemma 3.1** [21; 16; 9] *Let  $X$  be a smooth projective variety of general type with a generically finite morphism  $f: X \rightarrow A$  to an abelian variety.*

- (1) *We have  $\chi(X, \omega_X) = 0$  if, and only if,  $V^0(f_*\omega_X)$  is a proper subset of  $\widehat{A}$ . If these properties hold, the abelian variety  $A$  has at least 3 simple factors.*

- (2) If  $\chi(X, \omega_X) = 0$  and  $T$  is an irreducible component of  $V^0(f_*\omega_X)$ , we have  $T \in S_f$ . More precisely,  $T$  is an irreducible component of  $V^i(f_*\omega_X)$ , where  $i = \text{codim}_{\hat{A}}(T)$ .
- (3) For any abelian variety  $\hat{B} \in S_f$ , the variety  $X_B$  is of general type and we have  $\chi(X_B, \omega_{X_B}) > 0$ .

**Proof** The equivalence in (1) follows from generic vanishing [21, Theorem 1; 16, Remark 1.6, Theorem 1.2] and the other statement is [9, Corollary 3.4]. For (2), see [16, Claim (1.10)]. For (3), see [9, Theorem 3.1]. □

We introduce the notion of *primitive* and *minimal primitive* varieties. In Section 3.1, we study the structure of minimal primitive varieties and prove Theorem C. We provide examples in Section 4.

**Definition 3.2** [12, Definition 6.1] Let  $X$  be a smooth projective variety of general type, of maximal Albanese dimension and with  $\chi(X, \omega_X) = 0$ . We say that  $X$  is *primitive* if there exist no proper smooth subvarieties  $F$  through a general point of  $X$  such that  $\chi(F, \omega_F) = 0$ .

We say that  $X$  is *minimal primitive* if it is primitive and, for any rational factorization  $X \xrightarrow{a} Y \rightarrow A_X$  of the Albanese morphism of  $X$  through a smooth projective variety  $Y$  of general type, the map  $a$  is birational.

This definition of primitive is different from [7, Definition 1.24] but is equivalent to [12, Definition 6.1].

We will use the following results about primitive varieties.

**Lemma 3.3** [9; 12] *Let  $X$  be a smooth projective variety of general type, of maximal Albanese dimension and with  $\chi(X, \omega_X) = 0$ .*

- (1) *If  $f: X \rightarrow A$  is a morphism to an abelian variety, then there exists a quotient  $A \twoheadrightarrow B$  of abelian varieties such that the general fiber  $F_B$  of the induced morphism  $X \twoheadrightarrow X_B$  is primitive with  $\chi(F_B, \omega_{F_B}) = 0$ .*

*Assume now that  $X$  is primitive.*

- (2) *The Albanese morphism  $a_X: X \rightarrow A_X$  is surjective.*
- (3) *For any quotient  $A_X \twoheadrightarrow B$  to a simple abelian variety, with connected fibers, the composition  $X \xrightarrow{a_X} A_X \twoheadrightarrow B$  is a fibration.*
- (4) *If the abelian variety  $A$  has  $m$  simple factors, then  $V^0(f_*\omega_X)$  has at least  $m$  irreducible components; each component is a torsion translate of an abelian variety with  $m - 1$  simple factors and the intersection of these components has dimension 0.*

**Proof** For the first assertion in (1), see [9, Theorem 3.1]. The second follows from [12, Proposition 6.2]. For (2), see [9, Lemma 4.6] and the definition of primitive. For (3), see [12, Lemma 6.4]. Statement (4) also follows from [12, Lemma 6.4], since for any quotient  $\widehat{A}_X \twoheadrightarrow \widehat{K}$  of abelian varieties, the composition  $V^0(f_*\omega_X) \hookrightarrow A_X \twoheadrightarrow \widehat{K}$  is surjective.  $\square$

### 3.1 The structure of minimal primitive varieties

We describe the structure of minimal primitive varieties  $X$ : we prove that each simple factor  $K_j$  of the Albanese variety  $A_X$  has a birational model  $K'_j$  that admits a Galois cover  $F_j \rightarrow K'_j$  with finite Galois group  $G_j$  such that  $X$  is a quotient of the product of the  $F_j$  by a subgroup of the product of the  $G_j$ . When the  $F_j$  are curves, these quotient varieties already played an important role in Proposition 2.2.

**Theorem 3.4** *Let  $X$  be a smooth projective variety of general type, of maximal Albanese dimension and with  $\chi(X, \omega_X) = 0$ . We assume that  $X$  is minimal primitive. For some  $m \geq 3$ , there exist*

- smooth projective varieties  $F_1, \dots, F_m$  of general type,
- nontrivial finite groups  $G_j$  acting faithfully on  $F_j$  such that the quotient  $F_j/G_j$  is birationally isomorphic to a simple (nonzero) abelian variety  $K_j$ ,
- an isogeny  $K_1 \times \dots \times K_m \rightarrow A_X$  which induces an étale cover  $\widetilde{X} \rightarrow X$ ,
- a subgroup  $G$  of  $G_1 \times \dots \times G_m$ ,

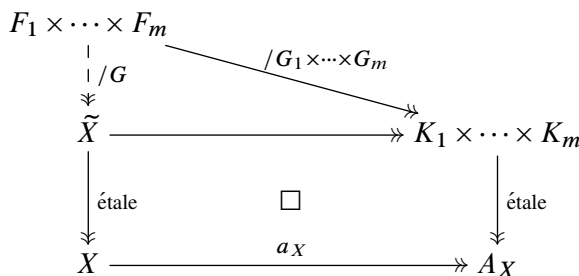
such that  $\widetilde{X}$  is birationally isomorphic to  $(F_1 \times \dots \times F_m)/G$ .

Furthermore, we can assume that the projections

$$\pi_{ij}: G \rightarrow G_1 \times \dots \times G_{i-1} \times G_{i+1} \times \dots \times G_{j-1} \times G_{j+1} \times \dots \times G_m$$

are injective and the projections  $G \twoheadrightarrow G_i$  are surjective whenever  $1 \leq i < j \leq m$ .

We summarize part of the conclusions of the theorem in a commutative diagram:



Minimal primitive varieties with  $\chi = 0$  will be constructed in Examples 4.1 and 4.3.

**Proof** The proof is divided into four steps.

**Step 1** (reduction via étale covers) Taking if necessary an étale cover of  $A_X$  (which induces an étale cover of  $X$ ), we may assume that each element of  $S_{a_X}$  (see (7)), which by Lemma 3.1(1)–(2) is nonempty, contains the origin  $0_{\widehat{A}_X}$ .

Assume that  $A_X$  has  $m$  simple factors. Since we are assuming that  $X$  is primitive, there exist by Lemma 3.3(4) irreducible components  $\widehat{A}_1, \dots, \widehat{A}_m$  of  $V^0(a_{X*}\omega_X)$  such that each  $\widehat{A}_j$  has  $m - 1$  simple factors and

$$(9) \quad \dim\left(\bigcap_{1 \leq j \leq m} \widehat{A}_j\right) = 0.$$

The quotient  $\widehat{K}_j := \widehat{A}_X / \widehat{A}_j$  is a simple abelian variety and we consider the dual injective morphism  $K_j \hookrightarrow A_X$ . By (9), the sum morphism

$$\pi: A' := K_1 \times \dots \times K_m \rightarrow A_X$$

is an isogeny.

If  $X' \twoheadrightarrow X$  is the étale cover induced by  $\pi$ , then  $\pi^*(\widehat{A}_j) = \widehat{K}_1 \times \dots \times \{0_{\widehat{K}_j}\} \times \dots \times \widehat{K}_m$ . Thus  $V^0(a_{X'*}\omega_{X'})$  contains at least the  $m$  components  $\widehat{K}_1 \times \dots \times \{0_{\widehat{K}_j}\} \times \dots \times \widehat{K}_m$ . Moreover,  $A_{X'} = K_1 \times \dots \times K_m$ .

Thus, we have constructed the following elements from the statement of the theorem: the simple abelian varieties  $K_i$  and the isogeny  $K_1 \times \dots \times K_m \rightarrow A_X$ . We still need to identify the fibers  $F_j$  and the groups  $G_j$  and  $G$ .

**Step 2** (a special property of fiber products) By Step 1, we can suppose that  $\widehat{A}_j := \widehat{K}_1 \times \dots \times \{0_{\widehat{K}_j}\} \times \dots \times \widehat{K}_m$  is a component of  $V^0(a_{X*}\omega_X)$  and  $A_X = K_1 \times \dots \times K_m$ .

For each  $1 \leq i < j \leq m$ , set  $\widehat{A}_{ij} := \widehat{A}_i \cap \widehat{A}_j$ . Using the notation (8), we have a commutative diagram

$$(10) \quad \begin{array}{ccccc} & & X & & \\ & & \downarrow f_{ij} & & \\ & & Y_{ij} & & \\ & f_i \swarrow & & \searrow f_j & \\ X_{A_i} & & & & X_{A_j} \\ & \swarrow g_{ij} & & \nwarrow g_{ji} & \\ & & X_{A_{ij}} & & \end{array}$$

where  $Y_{ij}$  is a desingularization of the main component of  $X_{A_i} \times_{X_{A_{ij}}} X_{A_j}$ . Since  $A_X = K_1 \times \cdots \times K_m$ , we have  $A_i \times_{A_{ij}} A_j \simeq A_X$ . Hence, the Albanese morphism factors as

$$(11) \quad a_X: X \xrightarrow{f_{ij}} Y_{ij} \rightarrow A_X.$$

Since  $X_{A_i}$  and  $X_{A_j}$  are of general type by Lemma 3.1(3),  $Y_{ij}$  is of general type by Viehweg’s subadditivity theorem [40, Corollary IV]. Moreover, the assumption that  $X$  is *minimal* implies that  $f_{ij}$  is birational.

In other words,  $X$  is birationally isomorphic to the fiber product of any two of the fibrations induced by the simple factors of the Albanese variety.

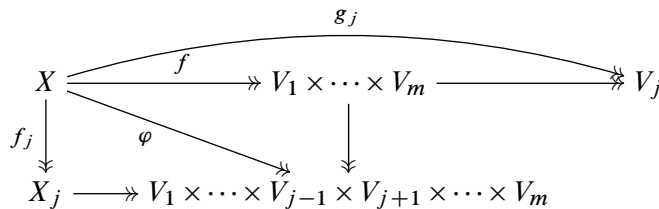
**Step 3** (the  $f_j$  are all birationally isotrivial fibrations) We use the notation of (10). Since  $f_{12}$  is birational, for a general point  $x \in X_{A_1}$  the fiber  $F_x$  of  $f_1$  is birationally isomorphic to  $g_{21}^{-1}(g_{12}(x))$ . Hence  $F_x$  is birationally isomorphic to  $F_y$  for  $y \in g_{12}^{-1}(g_{12}(x))$  general. Similarly,  $F_x$  is birationally isomorphic to  $F_y$  for  $y \in g_{1j}^{-1}(g_{1j}(x))$  general for all  $j \in \{2, \dots, m\}$ . Any two points of  $X_{A_1}$  can be connected by a chain of fibers of  $g_{12}, g_{13}, \dots$  or  $g_{1m}$ . For general points  $x$  and  $y$  of  $X_{A_1}$ , the fiber  $F_x$  is therefore birationally isomorphic to  $F_y$  and  $f_1$  is a birationally isotrivial fibration.

By the same argument, we see that  $f_j$  is a birationally isotrivial fibration for each  $j \in \{1, \dots, m\}$ . We denote by  $F_j$  its general fiber; since  $X$  is of general type, so is  $F_j$ .

We have now constructed the varieties  $F_j$  in the statement of the theorem. It remains to see that there are finite groups  $G_j$  acting faithfully on  $F_j$  such that  $F_j/G_j$  is birationally isomorphic to  $K_j$  and  $X$  is birationally isomorphic to the quotient  $(F_1 \times \cdots \times F_m)/G$  for some subgroup  $G$  of  $G_1 \times \cdots \times G_m$ .

**Step 4** (the finite groups  $G_1, \dots, G_m$  and the subgroup  $G$  of  $G_1 \times \cdots \times G_m$ ) The following lemma allows us to characterize varieties which are finite group quotients of a product of varieties and finishes the proof of Theorem 3.4.

**Lemma 3.5** *Let  $f: X \rightarrow V_1 \times \cdots \times V_m$  be a generically finite and surjective morphism between normal projective varieties. Assume that  $X$  is of general type and that, for each  $j \in \{1, \dots, m\}$ , there is a commutative diagram*



where  $X \xrightarrow{f_j} X_j$  is the Stein factorization of  $\varphi$ , the variety  $X_j$  is of general type,  $f_j$  is a birationally isotrivial fibration with general fiber  $F_j$  and  $g_j$  is a fibration.

Then there exists a finite group  $G_j$  acting faithfully on  $F_j$  such that  $F_j/G_j$  is birationally isomorphic to  $V_j$ . Moreover,  $X$  is birationally isomorphic to a quotient  $(F_1 \times \dots \times F_m)/G$ , where  $G$  is a subgroup of  $G_1 \times \dots \times G_m$  with surjective projections  $G \twoheadrightarrow G_j$ .

Before giving the proof of the lemma, note that it applies to our situation

$$f: X \rightarrow K_1 \times \dots \times K_m,$$

thanks to Step 3 and Lemma 3.3(3), which ensures that the  $g_j: X \rightarrow K_j$  are fibrations.

**Proof** Let  $\tilde{X}_1$  be a general fiber of  $g_1: X \rightarrow V_1$  and let  $f|_{\tilde{X}_1}: \tilde{X}_1 \rightarrow V_2 \times \dots \times V_m$  be the induced generically finite morphism. For  $j \in \{2, \dots, m\}$ , denote by  $\tilde{X}_1 \twoheadrightarrow V'_j \twoheadrightarrow V_j$  the Stein factorization of the natural morphism  $\tilde{X}_1 \twoheadrightarrow V_j$ . The induced morphism  $f': \tilde{X}_1 \rightarrow V'_2 \times \dots \times V'_m$  is generically finite and surjective. For each  $j \in \{2, \dots, m\}$ , let

$$\tilde{X}_1 \xrightarrow{f'_j} Y_j \twoheadrightarrow V'_2 \times \dots \times V'_{j-1} \times V'_{j+1} \times \dots \times V'_m$$

be the Stein factorization of the natural morphism. We summarize these constructions in the commutative diagram

$$\begin{array}{ccccccc}
 \tilde{X}_1 & \xrightarrow{f'_j} & Y_j & \twoheadrightarrow & V'_2 \times \dots \times V'_{j-1} \times V'_{j+1} \times \dots \times V'_m & \twoheadrightarrow & \{*\} \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 X & \xrightarrow{f_j} & X_j & \twoheadrightarrow & V_1 \times \dots \times V_{j-1} \times V_{j+1} \times \dots \times V_m & \twoheadrightarrow & V_1 \\
 & & & \searrow & & & \\
 & & & & & & g_1
 \end{array}$$

where the second and third vertical arrows are finite. Since the images of the  $Y_j$  in  $X_j$  cover  $X_j$ , and  $X_j$  is of general type by hypothesis,  $Y_j$  is also of general type; similarly,  $\tilde{X}_1$  is also of general type. Moreover,  $f'_j$  is also a birationally isotrivial fibration with general fiber  $F_j$ .

The morphism  $f': \tilde{X}_1 \rightarrow V'_2 \times \dots \times V'_m$  satisfies again the hypotheses of the lemma. Thus, by induction on  $m$ , we obtain that  $\tilde{X}_1$  is birationally isomorphic to a quotient of  $F_2 \times \dots \times F_m$ . Since a fixed variety can only dominate finitely many birational classes of varieties of general type,  $g_1: X \rightarrow V_1$  is birationally isotrivial.

Thus, after a suitable finite Galois base change  $F'_1 \rightarrow V_1$  with Galois group  $G_1$ , where  $F'_1$  is normal, we have a birational isomorphism  $F'_1 \times \tilde{X}_1 \xrightarrow{\sim} F'_1 \times_{V_1} X$ . Since  $\tilde{X}_1$  is

of general type, its birational automorphism group is finite. The action of  $G_1 \times \text{id}$  on  $F'_1 \times_{V_1} X$  therefore induces a birational action of  $G_1$  on  $\tilde{X}_1$  such that  $X$  is birationally isomorphic to the quotient of  $F'_1 \times \tilde{X}_1$  by the diagonal action of  $G_1$ .

Note that  $G_1$  acts on the canonical models of  $\tilde{X}_1$  and  $F'_1$ . After an equivariant resolution of singularities [1, Theorem 0.1], we may assume that  $\tilde{X}_1$  and  $F'_1$  are smooth, still with  $G_1$ -actions, and that  $G_1$  acts faithfully on  $F'_1$ . We have the commutative diagram

$$\begin{array}{ccccc}
 F'_1 \times \tilde{X}_1 & \twoheadrightarrow & (F'_1 \times \tilde{X}_1)/G_1 & \xrightarrow{\sim} & X & \xrightarrow{f} & V_1 \times \cdots \times V_m \\
 & \searrow \pi & & & \downarrow f_1 & & \downarrow \\
 & & & & X_1 & \twoheadrightarrow & V_2 \times \cdots \times V_m
 \end{array}$$

Let  $x \in F'_1$  be a general point; there is a dominant rational map  $f_{1x} = f_1 \circ \pi|_{\{x\} \times \tilde{X}_1} : \tilde{X}_1 \dashrightarrow X_1$ . Since any family of dominant maps between varieties of general type is locally constant, we have  $f_{1x} = f_{1y}$  for  $x$  and  $y$  general points of  $F'_1$ . Thus,  $f_1$  contracts the image of  $F'_1$  in  $X$  and  $F'_1 \dashrightarrow F_1$ , hence  $F_1 \rightarrow V_1$  is a birational Galois cover with Galois group  $G_1$ .

Similarly, for each  $j \in \{1, \dots, m\}$ , the map  $F_j \rightarrow V_j$  is a birational Galois cover with Galois group  $G_j$  and there are dominant maps

$$\begin{array}{ccc}
 & G_1 \times \cdots \times G_m & \\
 & \text{Galois} & \\
 F_1 \times \cdots \times F_m & \dashrightarrow & X \longrightarrow V_1 \times \cdots \times V_m
 \end{array}$$

Thus, there is a subgroup  $G$  of  $G_1 \times \cdots \times G_m$  such that  $X$  is birationally isomorphic to  $(F_1 \times \cdots \times F_m)/G$ . Since  $g_j : X \rightarrow V_j$  is a fibration, the projection  $G \rightarrow G_j$  is surjective for each  $j \in \{1, \dots, m\}$ . □

To finish the proof of Theorem 3.4, it only remains to prove the injectivity assertion of the projection to the product with two factors missing. This follows from the minimality assumption: the morphisms  $f_{ij} : X \rightarrow Y_{ij}$  are birational, thus a general fiber of  $X \rightarrow X_{A_{ij}}$  is birationally isomorphic to  $F_i \times F_j$ . This implies that the projections  $\pi_{ij}$  are injective. □

### 3.2 Divisibility properties of the degree of the Albanese morphism

The main result of this section is the following theorem.

**Theorem 3.6** *Let  $X$  be a smooth projective variety of general type and maximal Albanese dimension and let  $a_X$  be its Albanese morphism. If  $\chi(X, \omega_X) = 0$ , there exists a prime number  $p$  such that  $p^2$  divides the degree of  $a_X$  onto its image.*

By Proposition 1.17, if  $X$  is a rational cohomology torus of general type with rational singularities, we have  $\chi(X, \omega_X) = \chi(X', \omega_{X'}) = 0$  for any desingularization  $X' \rightarrow X$ . Thus Theorem C in the introduction directly follows from Theorem 3.6.

**Proof** We first reduce to the case of minimal primitive varieties (see Definition 3.2) using induction on the dimension. Then, we apply Theorem 3.4 and study the numerical properties of the degree of the Albanese morphism.

**Step 1** (reduction to minimal primitive varieties) We may assume that  $X$  is primitive with  $\chi(X, \omega_X) = 0$ . Otherwise, by Lemma 3.3(1) (see also (8)), there exists a quotient  $A_X \twoheadrightarrow B := A_X/K$  such that the general fiber  $F$  of the induced fibration  $X \twoheadrightarrow X_B$  is primitive with  $\chi(F, \omega_F) = 0$ . The restriction  $a_X|_F: F \rightarrow K$  factors through the (surjective) Albanese morphism of  $F$  and we can argue by induction on the dimension.

Moreover, if  $a_X$  factors as  $X \dashrightarrow Y \rightarrow A_X$  through a variety of general type  $Y$ , after birational modifications, we may assume that we have morphisms between smooth projective varieties

$$a_X: X \twoheadrightarrow Y \xrightarrow{a_Y} A_X = A_Y.$$

Therefore, we can replace  $X$  with  $Y$  and study the structure of  $a_Y$ . Note that  $Y$  may not be primitive and we need to reapply induction on the dimension as before. Finally, we get an  $X$  which is a minimal primitive variety of general type.

The structure of  $a_X$  remains the same after taking abelian étale covers of  $X$ . Thus, using Theorem 3.4, we can assume that  $X$  is birationally isomorphic to

$$(F_1 \times \cdots \times F_m)/G,$$

where  $F_j$  is a smooth projective variety acted on faithfully by the finite group  $G_j$  such that  $F_j/G_j$  is birationally isomorphic to a simple abelian variety  $K_j$ , the group  $G$  is a subgroup of  $G_1 \times \cdots \times G_m$  and

$$A_X = K_1 \times \cdots \times K_m.$$

Furthermore, we can assume that, for all  $1 \leq i < j \leq m$ , the projection

$$G \rightarrow G_1 \times \cdots \times G_{j-1} \times G_{j+1} \times \cdots \times G_{j-1} \times G_{j+1} \times \cdots \times G_m$$

is injective and the projection  $G \twoheadrightarrow G_j$  is surjective.

**Step 2** (computation of  $\deg(a_X)$ ) Set  $g := |G|$ . For each  $j \in \{1, \dots, m\}$ , set  $g_j := |G_j|$ . Since  $A_X = K_1 \times \cdots \times K_m$  and  $K_j$  is birationally isomorphic to  $F_j/G_j$ , we have

$$\deg(a_X) = \frac{1}{g} \prod_{1 \leq j \leq m} g_j.$$



Moreover, since the projection  $G \twoheadrightarrow G_j$  is surjective, we have  $g_j \mid g$ , hence

$$(12) \quad \deg(a_X) \mid \prod_{2 \leq j \leq m} g_j.$$

Finally, since the projections  $G \rightarrow G_1 \times \cdots \times G_{j-1} \times G_{j+1} \times \cdots \times G_{k-1} \times G_{k+1} \times \cdots \times G_m$  are injective, we have

$$(13) \quad g_j g_k \mid \deg(a_X) \quad \text{for all } 1 \leq j < k \leq m.$$

Let now  $p$  be a prime factor of  $g_1$ . Then  $p \mid \deg(a_X)$  by (13), hence  $p \mid g_j$  for some  $j \in \{2, \dots, m\}$  by (12), and  $p^2 \mid g_1 g_j \mid \deg(a_X)$  by (13) again. This finishes the proof of Theorem 3.6.  $\square$

Given a smooth projective variety  $X$  of general type, of maximal Albanese dimension and primitive with  $\chi(X, \omega_X) = 0$ , we know by Theorem 3.6 that  $p^2 \mid \deg(a_X)$  for some prime number  $p$ . Using the proofs of Theorems 3.6 and 3.4, we study the extremal case  $\deg(a_X) = p^2$ .

**Theorem 3.7** *Let  $X$  be a smooth projective variety of general type, of maximal Albanese dimension and with  $\chi(X, \omega_X) = 0$ . If  $\deg(a_X) = p^2$  for some prime number  $p$ , the morphism  $a_X$  is birationally a  $(\mathbb{Z}/p\mathbb{Z})^2$ -cover of its image.*

**Proof** We use the same notation as in Theorem 3.4.

There exists by Lemma 3.3(1) a quotient  $A_X \twoheadrightarrow B = A_X/K$  such that the general fiber  $F$  of the induced fibration  $X \twoheadrightarrow X_B$  is primitive with  $\chi(F, \omega_F) = 0$ . There is a factorization

$$a_X|_F: F \xrightarrow{a_F} A_F \xrightarrow{h} K \hookrightarrow A_X.$$

On the other hand, we have

$$p^2 = \deg(a_X) = \deg(h a_F) \deg(X_B \xrightarrow{a_{X_B}} B).$$

By Theorem 3.6,  $\deg(a_F)$  is divisible by the square of a prime number. It follows that this prime number must be  $p$  and that  $a_{X_B}$  is birational onto its image.

Let  $\eta \in X_B$  be the generic point. The geometric generic fiber  $X_{\bar{\eta}}$  of  $X \twoheadrightarrow X_B$  is then primitive and satisfies  $\chi(X_{\bar{\eta}}, \omega_{X_{\bar{\eta}}}) = 0$  and  $\deg(X_{\bar{\eta}} \rightarrow A_{\bar{\eta}}) = p^2$ . We are therefore reduced to the case where  $X$  is primitive.

Note that  $a_X: X \rightarrow A_X$  is *minimal* (see Definition 3.2). We can therefore apply Theorem 3.4. Keeping its notation, we see  $G$  has index  $\deg(a_X) = p^2$  in  $G_1 \times \cdots \times G_m$ ; since  $\pi_{ij}$  is injective whenever  $1 \leq i < j \leq m$ , we obtain that each  $G_j$  is isomorphic to  $\mathbb{Z}/p\mathbb{Z}$  and that  $a_X: X \rightarrow A_X$  is a  $(\mathbb{Z}/p\mathbb{Z})^2$ -cover.  $\square$

**Corollary 3.8** *Let  $X$  be a projective variety of general type with rational singularities. If  $X$  is a rational cohomology torus and  $\deg(a_X) = p^2$  for some prime number  $p$ , then  $a_X$  is a  $(\mathbb{Z}/p\mathbb{Z})^2$ -cover.*

**Proof** Let  $\mu: X' \rightarrow X$  be a desingularization. By Proposition 1.17,  $A_X = A_{X'}$  and  $X$  is the Stein factorization of the Albanese morphism  $X' \rightarrow A_X$ . Hence  $a_X$  is a  $(\mathbb{Z}/p\mathbb{Z})^2$ -cover. □

### 3.3 Simple factors of the Albanese variety

Using the description of minimal primitive varieties of general type with  $\chi = 0$ , we obtain restrictions on the number of simple factors of their Albanese varieties (which we already know is at least 3 by Lemma 3.1(1)).

**Proposition 3.9** *Let  $X$  be a smooth projective variety of general type, of maximal Albanese dimension, with  $\chi(X, \omega_X) = 0$ . If  $p$  is the smallest prime divisor of the degree of the Albanese map  $a_X$ , the Albanese variety  $A_X$  has at least  $p + 1$  simple factors.*

**Proof** We argue by induction on  $\dim(X)$ . As in Step 1 of the proof of Theorem 3.6, we can assume that  $X$  is minimal primitive (in the sense of Definition 3.2). Then, applying Theorem 3.4, we may assume, after taking an étale cover of  $X$ , that  $X$  is birationally isomorphic to a quotient  $(F_1 \times \cdots \times F_m)/G$ , the abelian variety  $A_X$  has  $m$  simple factors,  $V^0(a_{X*}\omega_X)$  has  $m$  irreducible components

$$\begin{aligned} \widehat{B}_1 &:= \{0_{\widehat{K}_1}\} \times \widehat{K}_2 \times \cdots \times \widehat{K}_m, \\ \widehat{B}_2 &:= \widehat{K}_1 \times \{0_{\widehat{K}_2}\} \times \cdots \times \widehat{K}_m, \\ &\vdots \\ \widehat{B}_m &:= \widehat{K}_1 \times \cdots \times \widehat{K}_{m-1} \times \{0_{\widehat{K}_m}\} \end{aligned}$$

and all elements of  $S_{a_X}$  contain the origin  $0_{\widehat{A}_X}$ .

By the decomposition theorem [12, Theorems 1.1 and 3.5], we have

$$a_{X*}\omega_X = \bigoplus_{\widehat{B} \in S_{a_X}} p_B^* \mathcal{F}_B,$$

where  $p_B: A_X \rightarrow B$  is the natural quotient and  $\mathcal{F}_B$  is a coherent sheaf supported on  $B$ .

On the other hand, by [12, Lemma 3.7], we have, for each  $j \in \{1, \dots, m\}$ ,

$$a_{X_{B_j}} * \omega_{X_{B_j}} = \bigoplus_{\substack{\widehat{B} \in S_{a_X} \\ \widehat{B} \subset \widehat{B}_j}} p_{B,j}^* \mathcal{F}_B,$$

where  $p_{B,j}: B_j \twoheadrightarrow B$  is the natural quotient, hence

$$\deg(a_{X_{B_j}}) = 1 + \sum_{\substack{\widehat{B} \in S_{a_X} \\ \widehat{B} \subset \widehat{B}_j}} \text{rank}(\mathcal{F}_B).$$

Since all elements of  $S_{a_X}$  are contained in  $\bigcup_{1 \leq j \leq m} \widehat{B}_j$ , we have

$$(14) \quad \deg(a_X) - 1 \leq \sum_{1 \leq j \leq m} (\deg(a_{X_{B_j}}) - 1).$$

With the notation of the proof of Theorem 3.4 ( $g = |G|$  and  $g_j = |G_j|$ ), we get

$$\deg(a_X) = \frac{1}{g} \prod_{1 \leq j \leq m} g_j \quad \text{and} \quad \deg(a_{X_{B_k}}) = \frac{1}{g} \prod_{j \neq k} g_j.$$

We may assume that  $\deg(a_{X_{B_1}})$  is maximal among all  $\deg(a_{X_{B_k}})$ . Using (14), we then obtain  $m \deg(a_{X_{B_1}}) \geq \deg(a_X) + m - 1 > g_1 \deg(a_{X_{B_1}})$ . Hence  $m \geq g_1 + 1 \geq p + 1$ .  $\square$

By Proposition 1.17, we obtain Theorem D in the introduction as a direct corollary.

### 4 Construction of examples

We show that the varieties in Theorem 3.7 and Corollary 3.8 actually exist. The lower bounds on the degree of the Albanese morphisms in Theorem C, Theorem 3.6 and Proposition 3.9 are therefore optimal.

We first construct, for every prime  $p$ , a series of examples of (smooth) minimal primitive varieties  $X$  of general type with  $\chi(X, \omega_X) = 0$ , such that  $X$  is a finite quotient of a product of  $p + 1$  curves and the Albanese morphism  $a_X$  is a  $(\mathbb{Z}/p\mathbb{Z})^2$ -cover. Then we show that a slight modification of this construction leads to rational cohomology tori.

**Example 4.1** ((smooth) minimal primitive varieties of general type with  $\chi = 0$  whose Albanese morphisms are  $(\mathbb{Z}/p\mathbb{Z})^2$ -covers of a product of  $p + 1$  elliptic curves) Let  $p$  be a prime number. For each  $j \in \{1, \dots, p + 1\}$ , let  $\rho_j: C_j \rightarrow E_j$  be a  $(\mathbb{Z}/p\mathbb{Z})$ -cover,

where  $C_j$  is a smooth projective curve of genus  $g_j \geq 2$  and  $E_j$  is an elliptic curve. For each  $j \in \{1, \dots, p + 1\}$ , write, as in (6),

$$\rho_{j*}\omega_{C_j} = \mathcal{O}_{E_j} \oplus \bigoplus_{1 \leq m \leq p-1} L_{\chi_j^m},$$

where  $\chi_j \in (\mathbb{Z}/p\mathbb{Z})^\vee$  is a generator and  $L_{\chi_j^m} = L_{\chi_j}^{\otimes m}$  is an ample line bundle on  $E_j$  for each  $m \in \{1, \dots, p - 1\}$ . Let  $\rho: V = C_1 \times \dots \times C_{p+1} \rightarrow A = E_1 \times \dots \times E_{p+1}$  be the corresponding  $G$ -cover, where  $G := (\mathbb{Z}/p\mathbb{Z})^{p+1}$ .

We now construct a subgroup  $H$  of  $G$  isomorphic to  $(\mathbb{Z}/p\mathbb{Z})^{p-1}$ . Actually, we will describe dually the quotient morphism of character groups. Let  $p_j: G \rightarrow \mathbb{Z}/p\mathbb{Z}$  be the projection to the  $j^{\text{th}}$  factor, let  $\chi \in (\mathbb{Z}/p\mathbb{Z})^\vee$  be a generator and set  $\chi_j := \chi \circ p_j: G \rightarrow \mathbb{C}^*$ . Then  $(\chi_1, \dots, \chi_{p+1})$  is a generating family of  $G^\vee \simeq (\mathbb{Z}/p\mathbb{Z})^{p+1}$ . On the other hand, let  $(e_1, \dots, e_{p-1})$  be the canonical basis for  $(\mathbb{Z}/p\mathbb{Z})^{p-1}$ . Define the quotient morphism  $G^\vee \twoheadrightarrow H^\vee$  by

$$\pi: G^\vee \rightarrow (\mathbb{Z}/p\mathbb{Z})^{p-1}, \quad \chi_j \mapsto \begin{cases} e_j & \text{for } j \leq p-1, \\ \sum_{1 \leq j \leq p-1} e_j & \text{for } j = p, \\ \sum_{1 \leq j \leq p-1} j e_j & \text{for } j = p+1. \end{cases}$$

Let  $H$  be the corresponding subgroup of  $G$  and set  $X := V/H$ . We consider

$$\rho: V \xrightarrow{f} X \xrightarrow{g} A = E_1 \times \dots \times E_{p+1}$$

and compute

$$\rho_*\omega_V = \bigoplus_{\tau \in G^\vee} L_\tau = \bigoplus_{(m_1, \dots, m_{p+1}) \in (\mathbb{Z}/p\mathbb{Z})^{p+1}} (L_{\chi_1^{m_1}} \boxtimes \dots \boxtimes L_{\chi_{p+1}^{m_{p+1}}}).$$

Moreover, as in the proof of Proposition 2.2, we have

$$\begin{aligned} g_*\omega_X &= (\rho_*\omega_V)^H = \bigoplus_{\substack{\tau \in G^\vee \\ \pi(\tau) = 1 \in H^\vee}} L_\tau \\ &= \bigoplus_{\substack{(m_1, \dots, m_{p+1}) \in (\mathbb{Z}/p\mathbb{Z})^{p+1} \\ m_j + m_p + j m_{p+1} = 0 \text{ for all } j \in \{1, \dots, p-1\}}} (L_{\chi_1^{m_1}} \boxtimes \dots \boxtimes L_{\chi_{p+1}^{m_{p+1}}}) \\ &= \bigoplus_{(a,b) \in (\mathbb{Z}/p\mathbb{Z})^2} (L_{\chi_1^{-a-b}} \boxtimes L_{\chi_2^{-a-2b}} \boxtimes \dots \boxtimes L_{\chi_{p-1}^{-a-(p-1)b}} \boxtimes L_{\chi_p^a} \boxtimes L_{\chi_{p+1}^b}). \end{aligned}$$

For  $a$  and  $b$  both nontrivial, there exists a unique  $j \in \{1, \dots, p - 1\}$  such that  $a + jb = 0 \in \mathbb{Z}/p\mathbb{Z}$ . Thus,  $\chi(X, \omega_X) = \chi(A, g_*\omega_X) = 0$ .

Moreover, we see that

$$(15) \quad V^0(g_*\omega_X) = \bigcup_{1 \leq j \leq p+1} \hat{E}_1 \times \cdots \times \{0_{\hat{E}_j}\} \times \cdots \times \hat{E}_{p+1}.$$

By [10, Theorem 1],  $X$  is of general type.

By studying the fibrations induced by the components of  $V^0(g_*\omega_X)$ , we deduce that  $X$  is primitive by Lemma 3.3(1). Since  $\deg(a_X) = p^2$ , Theorem 3.6 implies that  $X$  is minimal.

**Remark 4.2** We saw that  $X$  is primitive in the sense of Definition 3.2 (see also [12, Section 6]). However,  $g: X \rightarrow A$  is a  $(\mathbb{Z}/p\mathbb{Z})^2$ -cover. This shows that [12, Conjecture 6.6] is false: the structure of primitive varieties with  $\chi = 0$  is more complicated than expected.

**Example 4.3** (rational cohomology tori of general type with finite quotient singularities whose Albanese morphisms are  $(\mathbb{Z}/p\mathbb{Z})^2$ -covers of a product of  $p + 1$  elliptic curves) We first recall some constructions of Pardini. Let  $X$  be a normal projective variety, let  $A$  be a smooth variety, let  $G$  be a finite abelian group and let  $g: X \rightarrow A$  be a  $G$ -cover. We write, as in (6),

$$g_*\mathcal{O}_X = \bigoplus_{\tau \in G^\vee} L_\tau^{-1},$$

where the  $L_\tau$  are line bundles on  $A$ . The algebra structure on  $g_*\mathcal{O}_X$  gives rise to effective divisors  $(D_{\tau,\tau'})_{(\tau,\tau') \in G^\vee \times G^\vee}$  such that  $L_\tau \otimes L_{\tau'} \simeq L_{\tau,\tau'}(D_{\tau,\tau'})$ . Conversely, the data  $(L_\tau)_{\tau \in G^\vee}$  and  $(D_{\tau,\tau'})_{(\tau,\tau') \in G^\vee \times G^\vee}$  define a  $G$ -cover as above; see [33, Theorem 2.1].

Let  $p$  be a prime number, set  $G := (\mathbb{Z}/p\mathbb{Z})^2$  and consider the  $G$ -cover  $g: X \rightarrow A$  from Example 4.1 with its associated data  $(L_\tau)_{\tau \in G^\vee}$  and  $(D_{\tau,\tau'})_{(\tau,\tau') \in G^\vee \times G^\vee}$ .

Let  $P_j$  and  $P'_j$  be  $p$ -torsion line bundles on  $E_j$  such that their classes  $[P_j]$  and  $[P'_j]$  generate the group  $\hat{E}_j[p] \simeq (\mathbb{Z}/p\mathbb{Z})^2$  of  $p$ -torsion line bundles on  $E_j$ . Set  $P := P_1 \boxtimes \cdots \boxtimes P_{p+1}$  and  $P' := P'_1 \boxtimes \cdots \boxtimes P'_{p+1}$ .

We pick generators  $\tau_1$  and  $\tau_2$  of  $G^\vee \simeq (\mathbb{Z}/p\mathbb{Z})^2$  and, for  $(a, b) \in (\mathbb{Z}/p\mathbb{Z})^2$ , we set

$$L'_{\tau_1^a \cdot \tau_2^b} := L_{\tau_1^a \cdot \tau_2^b} \otimes P^{\otimes a} \otimes P'^{\otimes b}.$$

The relations  $L'_\tau \otimes L'_{\tau'} \simeq L'_{\tau,\tau'}(D_{\tau,\tau'})$  hold for all  $(\tau, \tau') \in G^\vee \times G^\vee$ . Thus, by [33, Theorem 2.1], we get a  $G$ -cover  $g': X' \rightarrow A$  such that

$$g'_*\mathcal{O}_{X'} = \bigoplus_{\tau \in G^\vee} L'_\tau^{-1}.$$

Both covers have the same branch divisors  $D_{\tau, \tau'}$ , so  $X'$  has the same singularities as  $X$ . Therefore,  $X'$  has finite quotient singularities and we have by duality (as in (6))

$$g'_* \omega_{X'} = \bigoplus_{\tau \in G^\vee} L'_\tau.$$

Hence,  $V^0(g'_* \omega_{X'})$  still generates  $A$ , so  $X'$  is also of general type by [10, Theorem 1].

We saw in Example 4.1 that for any  $\tau$  nontrivial,  $L_\tau$  is trivial when restricted to some  $E_j$ . However, since  $P_j$  and  $P'_j$  generate  $\widehat{E}_j[p] \simeq (\mathbb{Z}/p\mathbb{Z})^2$ ,  $L'_\tau$  restricted to  $E_j$  is a nontrivial torsion line bundle. Therefore,  $H^k(A, L'_\tau) = 0$  for any  $k \in \mathbb{Z}$  and  $\tau$  nontrivial. This implies  $h^0(X', \omega_{X'}) = 1$ .

Let  $A' \rightarrow A$  be the abelian cover induced by  $P$  and  $P'$ . Since they have the same building data, the Galois covers  $X \times_A A' \rightarrow A'$  and  $X' \times_A A' \rightarrow A'$  are isomorphic [33, Theorem 2.1]. This shows that  $X' \times_A A'$  is a quotient of a product of curves as in Proposition 2.2, hence so is  $X'$ . It follows from Proposition 2.2 that  $X'$  is a rational cohomology torus.

Now we show that there exist rational cohomology tori of general type with mild singularities in any dimension at least 3.

**Example 4.4** (rational cohomology tori of general type with finite quotient singularities of any dimension at least 3) For each  $j \in \{1, 2, 3\}$ , consider a nonzero abelian variety  $A_j$ , an ample line bundle  $L_j$  on  $A_j$ , a smooth divisor  $D_j \in |2L_j|$  and a nontrivial line bundle  $P_j \in \widehat{A}_j$  of order two. Let  $X_j \rightarrow A_j$  be the double cover associated with the data  $L_j$  and  $D_j$ , with involution  $\tau_j$ , and let  $X'_j \rightarrow X_j$  be the étale cover defined by  $P_j$ , with involution  $\sigma_j$ . Moreover, let  $\tau'_j$  be a lifting of  $\tau_j$  to  $X'_j$ .

Set  $G := (\mathbb{Z}/2\mathbb{Z})^2$ . Let  $H_1, H_2$  and  $H_3$  be the nontrivial cyclic subgroups of  $G$  and let  $\chi_1, \chi_2$  and  $\chi_3$  be the nontrivial characters of  $G$ , with  $\text{Ker}(\chi_j) = H_j$ . We now define data in order to construct a  $G$ -cover of  $A := A_1 \times A_2 \times A_3$ . Let  $p_j: A \rightarrow A_j$  be the projections. We define

$$\begin{aligned} L_{\chi_1} &:= P_1 \boxtimes L_2 \boxtimes (L_3 \otimes P_3), & D_{H_1} &:= p_1^* D_1, \\ L_{\chi_2} &:= (L_1 \otimes P_1) \boxtimes P_2 \boxtimes L_3, & D_{H_2} &:= p_2^* D_2, \\ L_{\chi_3} &:= L_1 \boxtimes (L_2 \otimes P_2) \boxtimes P_3, & D_{H_3} &:= p_3^* D_3. \end{aligned}$$

For  $\{i, j, k\} = \{1, 2, 3\}$ , we have  $L_{\chi_i} \otimes L_{\chi_i} \simeq \mathcal{O}_A(D_{H_j} + D_{H_k})$  and  $L_{\chi_i} \otimes L_{\chi_j} \simeq L_{\chi_k}(D_k)$ . By [33, Theorem 2.1], there exists a  $G$ -cover  $f: X \rightarrow A$  such that

$$f_* \mathcal{O}_X = \mathcal{O}_A \oplus L_{\chi_1}^{-1} \oplus L_{\chi_2}^{-1} \oplus L_{\chi_3}^{-1},$$

with branch locus  $D = D_{H_1} + D_{H_2} + D_{H_3}$ . Since  $D$  is a normal crossing divisor, a local computation shows that  $X$  has finite group quotient singularities. In particular,  $X$  has rational singularities. Actually,  $X$  is isomorphic to the quotient of  $X'_1 \times X'_2 \times X'_3$  by the automorphism group generated by  $\text{id}_1 \times \tau'_2 \times \sigma_3$ ,  $\sigma_1 \times \text{id}_2 \times \tau'_3$ ,  $\tau'_1 \times \sigma_2 \times \text{id}_3$  and  $\tau'_1 \times \tau'_2 \times \tau'_3$ .

As in Proposition 2.2, we set  $\Omega_X^{[s]} = \iota_*(\Omega_{X_{\text{reg}}}^s)$ , where  $\iota: X_{\text{reg}} \hookrightarrow X$  is the open subset of smooth points. By [33, Theorem 4.1] or [34, Proposition 1.2], we have

$$\begin{aligned} f_*\Omega_X^{[s]} &= \Omega_A^s \oplus (\Omega_A^s(\log(D_{H_2} + D_{H_3})) \otimes L_{\chi_1}^{-1}) \\ &\quad \oplus (\Omega_A^s(\log(D_{H_3} + D_{H_1})) \otimes L_{\chi_2}^{-1}) \\ &\quad \oplus (\Omega_A^s(\log(D_{H_1} + D_{H_2})) \otimes L_{\chi_3}^{-1}), \end{aligned}$$

hence

$$H^t(X, \Omega_X^{[s]}) \simeq H^t(A, \Omega_A^s) \quad \text{for all } s, t \geq 0,$$

and  $X$  is a rational cohomology torus of general type with finite quotient singularities. Moreover, let  $Y$  be the quotient of  $X'_1 \times X'_2 \times X'_3$  by the automorphism group generated by  $\text{id}_1 \times \tau'_2 \times \sigma_3$ ,  $\sigma_1 \times \text{id}_2 \times \tau'_3$  and  $\tau'_1 \times \sigma_2 \times \text{id}_3$ . Then  $Y \rightarrow X$  is a double cover and  $Y$  is smooth.

The following example exhibits (smooth) primitive fourfolds with  $\chi = 0$  whose Albanese varieties have 4 simple factors and whose Albanese morphisms have degree 8, which is not a square. Thus, we have constructed two essentially different series of examples of (smooth) primitive varieties with  $\chi = 0$  whose Albanese varieties have 4 simple factors: the minimal varieties provided by Example 4.1 taking  $p = 3$  and the following nonminimal primitive varieties.

**Example 4.5** (nonminimal (smooth) primitive fourfolds of general type with  $\chi = 0$  whose Albanese morphisms are  $(\mathbb{Z}/2\mathbb{Z})^3$ -covers of a product of 4 elliptic curves) Let  $\rho_1: C_1 \rightarrow E_1$  be a  $(\mathbb{Z}/2\mathbb{Z})^2$ -cover, where  $C_1$  is a smooth projective curve of genus  $g_1 \geq 2$  and  $E_1$  is an elliptic curve. Let  $\sigma$  and  $\tau$  be generators of the Galois group and let  $\sigma^\vee$  and  $\tau^\vee$  be the dual characters. For  $j \in \{2, 3, 4\}$ , let  $\rho_j: C_j \rightarrow E_j$  be a  $(\mathbb{Z}/2\mathbb{Z})$ -cover with associated involution  $\iota_j$ , where  $C_j$  is a smooth projective curve of genus at least 2 and  $E_j$  an elliptic curve.

Thus, we are considering the case where  $G_1 = (\mathbb{Z}/2\mathbb{Z})^2$  and  $G_2 = G_3 = G_4 = \mathbb{Z}/2\mathbb{Z}$  in Proposition 2.2 or Theorem 3.4. We have

$$\begin{aligned} \rho_{1*}\omega_{C_1} &= \mathcal{O}_{E_1} \oplus L_{\sigma^\vee} \oplus L_{\tau^\vee} \oplus L_{\sigma^\vee\tau^\vee}, \\ \rho_{j*}\omega_{C_j} &= \mathcal{O}_{E_j} \oplus L_{\iota_j^\vee} \quad \text{for } j \in \{2, 3, 4\}, \end{aligned}$$

where  $L_{\chi^\vee}$  is an ample line bundle corresponding to the character  $\chi^\vee$ .

We then set  $X := (C_1 \times C_2 \times C_3 \times C_4) / \langle \sigma \times \iota_1 \times \iota_2 \times \text{id}_{F_3}, \tau \times \iota_1 \times \text{id}_{F_2} \times \iota_3 \rangle$  and consider the  $(\mathbb{Z}/2\mathbb{Z})^2$ -quotient

$$f: C_1 \times C_2 \times C_3 \times C_4 \rightarrow X.$$

With the notation of Theorem 3.4 or Proposition 2.2, we have  $G = (\mathbb{Z}/2\mathbb{Z})^2$ .

Let  $g: X \rightarrow A = E_1 \times E_2 \times E_3 \times E_4$  be the morphism such that the composition

$$\rho: Z = C_1 \times C_2 \times C_3 \times C_4 \xrightarrow[4:1]{f} X \xrightarrow[8:1]{g} A$$

is the quotient by  $G_1 \times G_2 \times G_3 \times G_4$ .

Abusing the notation, we can describe the quotient  $(G_1 \times G_2 \times G_3 \times G_4)^\vee \rightarrow G^\vee$  by

$$\begin{aligned} \sigma^\vee &\mapsto (1, 0), & \iota_1^\vee &\mapsto (1, 0), \\ \tau^\vee &\mapsto (0, 1), & \iota_2^\vee &\mapsto (0, 1), \\ & & \iota_3^\vee &\mapsto (1, 1). \end{aligned}$$

One checks that

$$\begin{aligned} g_*\omega_X &\simeq (\rho_*\omega_Z)^G \\ &\simeq \mathcal{O}_A \oplus (\mathcal{O}_{E_1} \boxtimes L_{\iota_1^\vee} \boxtimes L_{\iota_2^\vee} \boxtimes L_{\iota_3^\vee}) \\ &\quad \oplus (L_{\sigma^\vee} \boxtimes L_{\iota_1^\vee} \boxtimes \mathcal{O}_{E_2} \boxtimes \mathcal{O}_{E_3}) \oplus (L_{\sigma^\vee} \boxtimes \mathcal{O}_{E_1} \boxtimes L_{\iota_2^\vee} \boxtimes L_{\iota_3^\vee}) \\ &\quad \oplus (L_{\tau^\vee} \boxtimes \mathcal{O}_{E_1} \boxtimes L_{\iota_2^\vee} \boxtimes \mathcal{O}_{E_3}) \oplus (L_{\tau^\vee} \boxtimes L_{\iota_1^\vee} \boxtimes \mathcal{O}_{E_2} \boxtimes L_{\iota_3^\vee}) \\ &\quad \oplus (L_{\sigma^\vee\tau^\vee} \boxtimes \mathcal{O}_{E_1} \boxtimes \mathcal{O}_{E_2} \boxtimes L_{\iota_3^\vee}) \oplus (L_{\sigma^\vee\tau^\vee} \boxtimes L_{\iota_1^\vee} \boxtimes L_{\iota_2^\vee} \boxtimes \mathcal{O}_{E_3}). \end{aligned}$$

Thus,  $\chi(X, \omega_X) = \chi(A, g_*\omega_X) = 0$ . Moreover, we obtain

$$V^0(g_*\omega_X) = \bigcup_{1 \leq j \leq 4} \hat{E}_1 \times \cdots \times \{0_{\hat{E}_j}\} \times \cdots \times \hat{E}_4.$$

By [10, Theorem 1],  $X$  is of general type and, by studying the fibrations induced by the components of  $V^0(g_*\omega_X)$ , we obtain that  $X$  is primitive by Lemma 3.3(1).

Note that  $X$  is not minimal primitive: if we consider the quotient

$$(G_1 \times G_2 \times G_3 \times G_4)^\vee \rightarrow H^\vee \simeq (\mathbb{Z}/2\mathbb{Z})^3$$

defined by

$$\begin{aligned} \sigma^\vee &\mapsto (1, 0, 0), & \iota_1^\vee &\mapsto (1, 0, 1), \\ \tau^\vee &\mapsto (0, 1, 0), & \iota_2^\vee &\mapsto (0, 1, 1), \\ & & \iota_3^\vee &\mapsto (1, 1, 1), \end{aligned}$$



and the varieties  $Z := C_1 \times C_2 \times C_3 \times C_4$  and  $Y := Z/H$ , we have a factorization

$$\rho: Z \xrightarrow[4:1]{f} X \xrightarrow[2:1]{} Y \xrightarrow[4:1]{h} A.$$

One checks that

$$\begin{aligned} h_*\omega_Y = (\rho_*\omega_Z)^H &= \mathcal{O}_A \oplus (L_{\sigma^\vee} \boxtimes \mathcal{O}_{E_1} \boxtimes L_{i_2^\vee} \boxtimes L_{i_3^\vee}) \\ &\oplus (L_{\tau^\vee} \boxtimes L_{i_1^\vee} \boxtimes \mathcal{O}_{E_2} \boxtimes L_{i_3^\vee}) \\ &\oplus (L_{\sigma^\vee\tau^\vee} \boxtimes L_{i_1^\vee} \boxtimes L_{i_2^\vee} \boxtimes \mathcal{O}_{E_3}). \end{aligned}$$

Thus,  $\chi(Y, \omega_Y) = \chi(A, h_*\omega_Y) = 0$ . Moreover, we have  $V^0(g_*\omega_X) = V^0(h_*\omega_Y)$ , so by [10, Theorem 1],  $Y$  is of general type.

## Appendix: Nonexistence of smooth rational cohomology tori of general type

by William F Sawin

**Theorem A.1** *Let  $f: X \rightarrow A$  be a finite morphism from a smooth projective variety of general type  $X$  to an abelian variety  $A$ , all over  $\mathbb{C}$ . Let  $n$  be the dimension of  $X$ . Then*

$$(-1)^n \chi_{\text{top}}(X) > 0.$$

**Proof** Recall that  $(-1)^n \chi_{\text{top}}(X)$  is the top Chern class of the cotangent bundle, or, equivalently, the intersection number of a section of the cotangent bundle and the zero section. We will compute this by taking a generic 1-form of  $A$  and pulling it back to  $X$ . We will show that its vanishing locus is 0-dimensional and nonempty, which implies that the intersection number is positive.

First we will show that the vanishing locus is 0-dimensional. Let

$$Z \subseteq X \times H^0(A, \Omega_A^1)$$

be the locus of pairs of a point  $x \in X$  and a 1-form  $\omega$  on  $A$  such that  $f^*\omega$  vanishes at  $x$ . Let  $m$  be the dimension of  $A$ . Then the dimension of  $Z$  is at most  $m$ : because it is a closed subset, it is sufficient to check that for each subvariety  $Y \subseteq X$  of dimension  $k$  with generic point  $\eta$ , the fiber  $Z_\eta$  has dimension at most  $m - k$ . The map  $f$  remains finite when restricted to  $Y$  and finite morphisms in characteristic 0 are generically unramified, so the map

$$H^0(A, \Omega_A^1) \otimes_{\mathbb{C}} \mathbb{C}(\eta) = \Omega_{A, f(\eta)}^1 \otimes_{\mathbb{C}(f(\eta))} \mathbb{C}(\eta) \rightarrow \Omega_{Y, \eta}^1$$

from the cotangent space of  $f(\eta)$  to the cotangent space of  $\eta$  is surjective, hence its kernel has dimension  $m - k$ . Then the kernel of the natural map

$$H^0(A, \Omega_A^1) \otimes_{\mathbb{C}} \mathbb{C}(\eta) \rightarrow \Omega_{X,\eta}^1$$

has dimension at most  $m - k$ , because it is contained in the previous kernel. But  $Z_\eta$  is precisely the affine space corresponding to this kernel, viewed as a vector space over  $\mathbb{C}(\eta)$ . So the dimension of  $Z_\eta$  equals the dimension of the kernel and is at most  $m - k$ , and thus the dimension of  $Z$  is at most  $m$ , as desired. Hence the vanishing locus of a generic 1-form from  $A$  is 0-dimensional.

By a result of Popa and Schnell [36, Conjecture 1], any 1-form on  $X$  vanishes at some point. So the vanishing locus is nonempty. Now the Chern number  $c_n(\Omega_X^1)$  is the intersection number of the zero section with this generic 1-form. Because the intersection consists of finitely many points, the intersection number is a sum of contributions at those points, which is 1 if they are transverse but is always positive in general, so the total intersection number is positive. Thus

$$(-1)^n \chi_{\text{top}}(X) = c_n(\Omega_X^1) > 0. \quad \square$$

**Corollary A.2** *Let  $X$  be a smooth projective variety of general type. Then  $X$  is not a rational cohomology torus.*

**Proof** If it is, then by a remark of Catanese [8, Remark 72], its Albanese morphism  $X \rightarrow A_X$  is finite. So by Theorem A.1, its topological Euler characteristic is nonzero. But because its rational cohomology is the same as that of an abelian variety, its Euler characteristic must be the same as that of an abelian variety, which is zero. This is a contradiction, so  $X$  is not a rational cohomology torus.  $\square$

**Acknowledgements** Sawin was supported by NSF Grant No. DGE-1148900 when writing this appendix and would like to thank the organizers of the 2015 AMS Summer Institute on Algebraic Geometry for providing the venue where he worked on this problem.

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Received: 14 September 2015  
Revised: 11 April 2016