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Every diffeomorphism is a total renormalization of a close to identity map

Pierre Berger[*], Nicolaz Gourmelon, Mathieu Helfter[†]

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Abstract

For any $1 \leq r \leq \infty$, we show that every diffeomorphism of a manifold of the form $\mathbb{R}/\mathbb{Z} \times M$ is a total renormalization of a C^r-close to identity map. In other words, for every diffeomorphism f of $\mathbb{R}/\mathbb{Z} \times M$, there exists a map g arbitrarily close to identity such that the first return map of g to a domain is conjugate to f and moreover the orbit of this domain is equal to $\mathbb{R}/\mathbb{Z} \times M$. This enables us to localize nearby the identity the existence of many properties in dynamical systems, such as being Bernoulli for a smooth volume form.

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Introduction

0.1 Statements of the main theorems

Let $\mathbb{B}^n$ be the unit closed ball of $\mathbb{R}^n$.

Definition 0.1. A *renormalization* G of a diffeomorphism $g \in \mathrm{Diff}(\mathbb{B}^n)$ is a rescaling of an iteration of g. In other words, there exists $N \geq 2$ and an embedding $\psi : \mathbb{B}^n \hookrightarrow \mathbb{B}^n$ such that:

$$G = \psi^{-1} \circ g^N \circ \psi.$$

A long standing open problem of dynamical systems theory is:

Problem 0.2 (1971)**.** *Which dynamics can be reached by renormalization of close to identity maps?*

This problem was first studied by Ruelle and Takens in [RT71]. Motivated by the study of turbulence, they proved that for any integer $n \geq 2$, any dynamic on the n-dimensional torus is the renormalization of a C^n-close to identity map. This enabled them to construct perturbations of the identity map of the torus with a strange attractor. Based on this, they conjectured that this appears as well in fluid dynamics and could be used as a mathematical definition of the notion of turbulence [Lor63].

The main mathematical issue with this result is that the regularity is limited by the dimension of the torus. However when considering flows, this problem was solved by Newhouse, Ruelle and Takens in [NRT78]: given any vector filed X equal to a rotation on the torus $\mathbb{T}^n$, $n \geq 3$ and any map $F_0 \in \mathrm{Diff}^\infty(\mathbb{T}^{n-1})$ homotopic to the identity, they perturbed X to $\widetilde{X}$ so that its first return map to a global transverse section is F_0. Yet $\mathrm{Diff}^\infty(\mathbb{T}^n)$ is "much larger" than $\mathrm{Diff}^\infty(\mathbb{T}^{n-1})$ and so the mathematical Problem 0.2 remains unsolved.

A breakthrough was then performed in the seminal work of Turaev who proved that a C^r**-dense subset** of C^r-orientation preserving embeddings of $\mathbb{B}^n$ could be obtained after renormalization of an arbitrarily close to identity map, for every $0 \leq r \leq \infty$.

A first main result is a solution to Problem 0.2, where we improve Turaev's theorem to obtain, via a self-contained and new proof, **any** C^r-orientation preserving map of $\mathbb{B}^n$ (instead of maps among a dense subset):

Theorem A. *For any $1 \leq r \leq \infty$ and any orientation preserving $G \in \mathrm{Diff}^r(\mathbb{B}^n)$, in any neighborhood $\mathcal{N} \subset \mathrm{Diff}^r(\mathbb{B}^n)$ of the identity, there exists $g \in \mathcal{N}$ such that a renormalization of g is **equal** to G. Moreover the rescaling map of this renormalization can be chosen affine.*

A natural open problem is whether g can be obtained conservative or symplectic when G is conservative or symplectic. In this direction let us mention the work of Gonchenko-Shilnikov-Turaev [GST07] who proved that, for every $0 \leq r \leq \infty$, a C^r- dense subset of volume preserving embeddings of $\mathbb{B}^2$ could be obtained after renormalization of an arbitrarily close to identity volume preserving map. Recently Fayad and Saprikina in [FS21] showed that any conservative map of the n-dimensional ball can be realized by renormalized iteration of a conservative C^n-perturbation of the identity.

If all these theorems indicate the richness of the possible dynamical behaviors nearby the identity, one can object the following. In the setting of Definition 0.1, the orbit of $\bigcup_{k=0}^{N-1} \psi(\mathbb{B}^n)$ of the renormalization domain might be extremely small and so experimentally not observable. This objection is lifted completely when the renormalization domain intersects every orbit. This leads us to strengthen the notion of renormalization by the following:

Definition 0.3. Let $r \in \{1, \ldots, \infty\} \cup \{\omega\}$ and let V be a manifold (with boundary). A map $g \in \mathrm{Diff}^r(V)$ is *renormalizable* if there exists a strict submanifold with corners $\Delta \subsetneq V$ such that:

- there exists a bijective, local diffeomorphism $H : \Delta \to V$, called the *rescaling map* of the *renormalization domain* Δ,
- the first return time $\tau : \Delta \to \mathbb{N}^*$ into Δ by g is bounded and the *renormalization* $G = H \circ g^\tau \circ H^{-1}$ belongs to $\mathrm{Diff}^r(V)$.

The map g is *totally renormalizable* if the forward orbit of Δ covers V, i.e. $\bigcup_{n\geq 0} g^n(\Delta) = V$. The map G is then a *total renormalization* of g.

Example 0.4. If $V = \mathbb{B}^n$, any renormalizable map in the sense of Definition 0.1 satisfies Definition 0.3 with $\Delta = \psi(\mathbb{B}^n)$ and $H = \psi^{-1}$. However these maps are not totally renormalizable.

Example 0.5. When V is the circle $\mathbb{T}$, a diffeomorphism g is totally renormalizable iff it does not fix a point. Indeed in this case, take any point $0 \in \mathbb{T}$ and consider the interval $\Delta = [0, g(0))$. Then we glue the two endpoints of Δ using g to obtain a circle and we uniformize it to obtain $\mathbb{T}$. This defines a map H. For this setting one easily shows that the mapping g is renormalizable. This construction was intensively used by Yoccoz [Yoc95b].

Let V be a compact manifold (possibly with corners) and $1 \leq r \leq \infty$. We recall that the *support* $\operatorname{supp} f$ of $f \in \operatorname{Diff}^r(V)$ is the closure of the set of points such that $f(x) \neq x$.

Definition 0.6. Let $\operatorname{Diff}^r_0(V)$ be the component of the identity in $\operatorname{Diff}^r(V)$. Let $\operatorname{Diff}^r_c(V)$ be the subset of $\operatorname{Diff}^r_0(V)$ formed by maps isotopic to id through isotopies $(f_t)_{t\in[0,1]}$ whose support $\bigcup_{t\in[0,1]} \operatorname{supp} f_t$ is a compact subset of $V \setminus \partial V$.

Observe that when V is boundaryless, it holds $\operatorname{Diff}^r_0(V) = \operatorname{Diff}^r_c(V)$. A natural question is:

Question 0.7. *For which manifold V, any map $F \in \operatorname{Diff}^r_c(V)$ is a* **total** *renormalization of a close to identity map?*

So far no example of such a manifold V was known. In this work we give a full class of examples:

Theorem B. *Let $0 \leq r \leq \infty$, let M be a compact manifold of dimension ≥ 1 and put $V := \mathbb{T} \times M$. Let $\mathcal{N} \subset \operatorname{Diff}^r(V)$ be a neighborhood of the identity. Then any $G \in \operatorname{Diff}^r_c(V)$ is a total renormalization of some $g \in \mathcal{N}$.*

If Theorem A implies that every **local** dynamical phenomenon can be found nearby the identity, Theorem B implies that every **global** dynamical phenomenon can be found nearby the identity. A new improvement brought by the latter result is that the renormalization domain is larger than in all of the previous extensions of Ruelle-Takens theorems: its orbit coincides with the whole domain of the dynamic. In Theorem E, we will give a precise formula defining the renormalization domain and rescaling map involved in Theorem B. This will enable new applications such as the proof of existence of maps preserving smooth SRB nearby the identity (see Corollary C) or universal maps whose renormalization domains decrease as slow as we want (see Corollary D).

In Proposition 0.10, we show that Theorem B is wrong when $\mathbb{T} \times M \approx \mathbb{T}$, hence the set of dimensions of the manifold is optimal. On the other hand, a natural open problem communicated to us by Turaev is:

Problem 0.8. *Show that a dense subset of $\operatorname{Diff}^\omega(\mathbb{B}^n)$ is equal to the renormalization of a close to identity map in $\operatorname{Diff}^\omega(\mathbb{B}^n)$?*

Another natural question is:

Question 0.9. *Is Theorem B correct in the area preserving or symplectic categories?*

An extension of Theorem B regards the C^r-families $f_{\mathscr{P}} = (f_p)_{p\in\mathscr{P}}$ of maps $f_p \in \operatorname{Diff}^r(V)$ and indexed by a manifold $\mathscr{P}$. A family $(f_p)_{p\in\mathscr{P}}$ is *of class C^r* if the following is in $\operatorname{Diff}^r(V \times \mathscr{P})$:

$$\widehat{f_{\mathscr{P}}} := (x, p) \mapsto (f_p(x), p) \ . \tag{0.1}$$

We denote by $\operatorname{Diff}^r(V)_{\mathscr{P}}$ the space of such families endowed with the topology induced by $\operatorname{Diff}^r(V \times \mathscr{P})$. Let $\operatorname{Diff}^r_c(V)_{\mathscr{P}}$ be the component of the identity in $\operatorname{Diff}^r(V)_{\mathscr{P}}$ for homotopies $(f_{p,t})_{(p,t)\in\mathscr{P}\times[0,1]} \in \operatorname{Diff}^r(V)_{\mathscr{P}\times[0,1]}$ whose support $\bigcup_{(t,p)\in[0,1]\times\mathscr{P}} \operatorname{supp} f_{p,t} \times \{p\}$ is a compact subset of the interior of $V \times \mathscr{P}$. Observe that $\widehat{f_{\mathscr{P}}} \in \operatorname{Diff}^r_c(V \times \mathscr{P})$.

Theorem B′. *Let $0 \le r \le \infty$, let M and $\mathscr{P}$ be compact manifolds of dim ≥ 1 and set $V := \mathbb{T} \times M$. Let $\mathcal{N} \subset \mathrm{Diff}^r(V)_{\mathscr{P}}$ be a neighborhood of $(id)_{p\in\mathscr{P}}$. Then for any $(G_p)_{p\in\mathscr{P}} \in \mathrm{Diff}^r_c(V)_{\mathscr{P}}$, there exist $(g_p)_{p\in\mathscr{P}} \in \mathcal{N}$ and a rescaling map (independent of $p \in \mathscr{P}$) of a total renormalization domain which renormalizes each g_p to G_p.*

This theorem implies that any bifurcation in $\mathrm{Diff}^r_c(V)$ occurs at small unfolding of the identity. In Section 1.1, we will state the main general Theorem E which implies Theorems A and B and also its parametric counterpart Theorem E′ which implies Theorem B′. In Section 0.2 we will give several applications of them. Now let us discuss the optimality of Theorem B.

Proposition 0.10. *When $r \ge 2$, Theorem B is wrong if $V \approx \mathbb{T}$ (and so $\dim M = 0$).*

Proof. Indeed if F is a renormalization of a close to identity map f for a renormalization domain Δ, then we have necessarily $|\tau(\theta) - \tau(\theta')| \le 1$ for any $\theta, \theta' \in \Delta$. Let $N := \min\{\tau(\theta), \tau(\theta')\}$. We compute the derivative of the $N-1$ first iterates $(\theta_i)_i$ and $(\theta'_i)_i$ of θ and θ':

$$\log\left|\frac{D_\theta F}{D_{\theta'} F}\right| = \log\left|\frac{D_\theta f^N}{D_{\theta'} f^N}\right| + o(1) \quad \text{when } f \to id$$

$$= \sum_{i=0}^{N-1} \log\left|\frac{D_{\theta_i} f}{D_{\theta'_i} f}\right| + o(1) \le \|\log|Df|\|_{C^1} \cdot \sum_{i=0}^{N-1} |\theta_i - \theta'_i| + o(1) = o(1),$$

where the latter inequality uses that $\|\log|Df|\|_{C^1}$ is small while the segment $[\theta_i, \theta'_i]$ are disjoint and so the union of their length is at most 1. Hence this proves that the derivative of F is constant and so that F must be a rotation. □

Also we cannot change $\mathrm{Diff}^r_c(\mathbb{T} \times M)$ by $\mathrm{Diff}^r_0(\mathbb{T} \times M)$ in Theorem B. Indeed the latter proposition applied to the boundary of $[0,1]$ implies immediately:

Corollary 0.11. *There are $G \in \mathrm{Diff}^\infty_0(\mathbb{T}\times[0,1])$ which are not total renormalization of C^2-close to identity map.*

Yet in view of Question 0.7, Theorem B seems to be generalizable for a manifold V on which $\mathbb{T}$ acts properly discontinuously without any fixed point.

0.2 Applications and open problems

Smooth SRB nearby the identity In Section 1.1 we will state Theorem E which will imply together with Katok's theorem [Kat79], an answer to an open question of Thouvenot:

Corollary C. *In any neighborhood $\mathcal{N}$ of $\mathrm{id} \in \mathrm{Diff}^\infty(\mathbb{T}^2)$ there is a map $g \in \mathcal{N}$ which leaves invariant an ergodic smooth volume form and displays positive Lyapunov exponent at Lebesgue a.e. point.*

This corollary will be proved in Section 1.2 from Proposition 1.7, and generalized to higher dimension using [DP02].

Universal mappings A map $f \in \mathrm{Diff}^r(V)$ is said to be *universal* if there exists a dense subset of $\mathrm{Diff}^r_0(V)$ such that each of its element is a renormalization of f. Bonatti and Diaz in [BD03] have shown that universal maps are locally C^1-generic on $\mathbb{B}^3$. Turaev in [Tur15] has shown that universal maps are locally C^∞-generic on $\mathbb{B}^2$. Yet mathematicians wondered whether we can "see" the universality of such mapping. While there are infinitely many renormalization domains, the above proofs lead to a very small volume for the union of their orbits.

Corollary D. *Let $1 \le r \le \infty$ and $n \ge 2$. For any sequence $(F_i)_{i\ge0}$ of maps $F_i \in \mathrm{Diff}^r_0(\mathbb{T}\times[0,1])$ and any sequence of positive numbers $(\ell_i)_{i\ge0}$ s.t. $\sum \ell_i < 1$, there exists a C^r-arbitrary close to identity map $f \in \mathrm{Diff}^r_0(\mathbb{T} \times [0,1])$ which displays a family of renormalization domains $(\Delta_i)_{i\ge0}$ such that:*

1. *a renormalization of f associated to Δ_i is F_i for every $i \geq 0$,*
2. *the orbit $\tilde{\Delta}_i := \bigcup_{n\geq 0} f^n(\Delta_i)$ has volume equal to ℓ_i,*
3. *the sets $\tilde{\Delta}_i$ and $\tilde{\Delta}_j$ are disjoint for $i \neq j$.*

This corollary will be proved in Section 1.2.

The proof of the main theorem is constructive and it seems to us that, in the case where M is boundaryless, the map g of Theorem B depends smoothly on G in a neighborhood of the identity. This leads us to propose:

Conjecture 0.12. *For every compact boundaryless manifold M of dimension ≥ 1, there exists a neighborhood $\mathcal{N}_0$ of $id \in \mathrm{Diff}^\infty(\mathbb{T}\times M)$ such that for every neighborhood $\mathcal{N}$ of $id \in \mathrm{Diff}^\infty(\mathbb{T}\times M)$, there is a smooth (tame) injective map $\mathcal{I} : G \in \mathcal{N}_0 \mapsto g \in \mathcal{N}$ such that G is a total renormalization of $g = \mathcal{I}(G)$ for every $G \in \mathcal{N}_0$.*

Roughly speaking, this conjecture asserts that modulo total renormalization, a fixed neighborhood $\mathcal{N}_0$ of $id \in \mathrm{Diff}^\infty(\mathbb{T}\times M)$ can be smoothly embedded into any smaller neighborhood. This defines infinitely many inverse branches of the renormalization operator with image converging to the identity.

0.3 Sketch of proof

Plugins and pluggable dynamics: The framework of the proof of the main theorem relies on a new object called **plugin** and the notion of **pluggable map**. A plugin is a renormalizable map of a special form, so that it has a canonical renormalization called its **output**. See Def. 1.1 and 1.3 below and Figs. 1 and 2. We will say that a map is **pluggable** if it is the output of an arbitrarily close to identity plugin and likewise for its inverse. In particular a pluggable map is a total renormalization of a close to identity map. Most of this work will be dedicated to show Theorem F stating that:

$$\textit{any map of } \mathrm{Diff}^\infty_c(\mathbb{T}\times M) \textit{ is pluggable.}$$

The finite regularity counterpart of Theorem F is stated as Theorem E in Section 1.1 and will be deduced from Theorem F in Section 2.3. We will deduce Theorems A and B and Corollaries C and D from Theorem E in Section 1.2.

In Section 1.3 we precise the topologies of the involved spaces. Also we will show that the following group is formed by pluggable maps, see Proposition 1.16:

$$\mathbb{G}_1 := \{(\theta, y) \mapsto (\theta + \Xi(y), y) : \Xi \in C^\infty_c(M, \mathbb{R})\} .$$

Topological group structure on Plug**.** It is easier to work with pluggable maps rather than directly the set of total renormalizations of close to identity maps. Indeed, we will show in Proposition 2.4 that the set Plug of pluggable maps endowed with the composition rule $\circ$ is a group. To prove this, we will define in Section 2.1 a binary operation $\star$ on compatible plugins g_1, g_2 such that the output of $g_1 \star g_2$ is the composition of the outputs of g_1 and g_2. See Fig. 5. In Section 2.2 we will show that the following group is formed by pluggable maps, see Proposition 2.7:

$$\mathbb{G}_2 := \{(\theta, y) \in \mathbb{T}\times M \mapsto (\theta, F(y)) : F \in \mathrm{Diff}^\infty_c(M)\} .$$

Then in Section 2.3 we will show that Plug is closed in $\mathrm{Diff}^\infty_c(\mathbb{T}\times M)$, see Proposition 2.9. To prove this, we will construct plugins whose dynamics enlarge an iterate of the renormalization domain and will perform a perturbation therein. Note that the elements of $\mathrm{Diff}^\infty_c(\mathbb{T}\times M)$ generated by compositions of elements of $\mathbb{G}_1$ and $\mathbb{G}_2$ have constant derivatives w.r.t. θ. Consequently we will need to construct other diffeomorphisms in Plug to prove Theorem F. To do so we will study the space $\mathfrak{plug}$ of vector fields whose flow is pluggable:

$$\mathfrak{plug} := \{X \in \mathfrak{diff}^\infty_c(\mathbb{T}\times M) : \mathrm{Fl}^t_X \in \mathrm{Plug}, \quad \forall t \in \mathbb{R}\} ,$$

where $\mathfrak{diff}_c^\infty(\mathbb{T}\times M)$ denotes the space of compactly supported smooth vector fields on $\mathbb{T}\times M$ and Fl_X^t denotes the flow of X at time t. Using that Plug is a closed subgroup, we will deduce in Section 2.4 that $\mathfrak{plug}$ is a closed sub-Lie algebra of $\mathfrak{diff}_c^\infty(\mathbb{T}\times M)$, see Proposition 2.16.

Note that the following Lie algebras are formed by fields whose flows are in $\mathbb{G}_1$ or $\mathbb{G}_2$:

$$\mathfrak{g}_1 = \{X_1 : (\theta, y) \in \mathbb{T}\times M \mapsto (\xi(y), 0) \in \mathbb{R}\times TM \ : \ \xi \in C^\infty(M,\mathbb{R})\} \subset \mathfrak{plug}\ ,$$

$$\mathfrak{g}_2 = \{X_2 : (\theta, y) \in \mathbb{T}\times M \mapsto (0, f(y)) \in \mathbb{R}\times TM \ : \ f \in \mathfrak{diff}_c^\infty(M)\} \subset \mathfrak{plug}\ .$$

Construction of pluggable flows. In Section 3.1, using the connectedness of $\mathrm{Diff}_c^\infty(\mathbb{T}\times M)$ and that Plug is a closed group, we will show that, to prove Theorem F, it suffices to show:

$$\mathfrak{plug} = \mathfrak{diff}_c^\infty(\mathbb{T}\times M)\ .$$

This equality will be stated in Proposition 3.3. Its proof relies on two phenomena. The first one is stated in Proposition 3.7 as:

$$\{Y \in \mathfrak{diff}_c^\infty(\mathbb{T}\times M) : \exists X \in \mathfrak{plug} \text{ such that } Y = [X,Y]\} \subset \mathfrak{plug}\ ,$$

where $[\cdot,\cdot]$ denotes the Lie brackets. This will be proved in Section 3.2 by noting that for any $Y = [X,Y]$, it holds $\mathrm{Fl}_Y^t = \mathrm{Fl}_X^s \circ \mathrm{Fl}_{e^{-s}Y}^t \circ \mathrm{Fl}_X^{-s}$. Then by using the same technique as for the proof of the closure of Plug, we will show that arbitrarily close to identity there exists a plugin with output $\mathrm{Fl}_{e^{-s}Y}^t$ for any s large enough. We will conclude the proof by doing two $\star$ products of the latter with plugins with outputs Fl_X^s and Fl_X^{-s}.

The second phenomenon, stated in Proposition 3.9, is that for any vector field $T \in \mathfrak{diff}_c^\infty(M)$, there exist finite families $(X_i)_i, (Y_i)_i, (Z_i)_i$ of vector fields in $\mathfrak{diff}_c^\infty(M)$ such that:

$$T = \sum_i [Y_i, Z_i] \quad \text{and} \quad Y_i = [X_i, Y_i]\ .$$

From this we will deduce the same statement for vector fields in $\mathfrak{g}_2$. To show Proposition 3.9, we will remark that when $M = \mathbb{R}$, $X = 1$ and $Y(\theta, y) = y$, it holds $Y = [X,Y]$ and for any $T \in \mathfrak{diff}_c^\infty(\mathbb{R})$, with $Z = \int_{-\infty}^y T(t)dt$, it also holds $T = [Y,Z]$. Then we will deduce a compactly supported and parametric version of this property which will enable to prove Proposition 3.9 in the case $M = \mathbb{R}^n$. Finally, we will use a partition of unity to deduce the proposition for any manifold.

These phenomena will enable to prove Proposition 3.1 stating that for any $T \in \mathfrak{g}_2$ and $\phi \in C_c^\infty(\mathbb{T}\times M)$ depending only on θ, the field $\phi\cdot T$ is in $\mathfrak{plug}$. Indeed, by the second phenomenon, there are $X_i, Y_i, Z_i \in \mathfrak{g}_2$ s.t.:

$$\phi\cdot T = \sum_i \phi\cdot[Y_i, Z_i] = \sum_i [\phi\cdot Y_i, Z_i] \quad \text{and} \quad Y_i = [X_i, Y_i]\ .$$

Also a simple computation shows that $[\phi\cdot Y_i, X_i] = \phi\cdot Y_i$. As X_i is in $\mathfrak{g}_2 \subset \mathfrak{plug}$, we deduce by the first phenomenon that $\phi\cdot Y_i$ is in $\mathfrak{plug}$. This gives that $\phi\cdot T \in \mathfrak{plug}$ as stated in Proposition 3.1. In Section 3.1, we will use Fourier's Theorem and the closure of $\mathfrak{plug}$ to deduce Proposition 3.1 that any vector field of the form $(0, Y(\theta, y))$ is in $\mathfrak{plug}$. Finally using this with a Lie bracket with an element of $\mathfrak{g}_1$ will enable to obtain that any vector field has a pluggable flow ($\mathfrak{plug} = \mathfrak{diff}_c^\infty(\mathbb{T}\times M)$) as stated by Proposition 3.3.

Parametric counterparts. At the end of each subsection, we will prove a parametric generalization of the aforementioned statements. This will enable to show the parametric counterpart Theorem F′ of Theorem F. It will imply the parametric counterparts Theorem E′ of Theorem E and Theorem B′ of Theorem B.

1 Plugins and Pluggable dynamics

1.1 Plugins

For the rest of this article, we fix $1 \leq r \leq \infty$ and compact connected manifolds M and $\mathscr{P}$ of $\dim \geq 1$.

For $\sigma > 0$, define the rotation:

$$R_\sigma : (\theta, y) \in \mathbb{T} \times M \to (\theta + \sigma, y) \in \mathbb{T} \times M$$

We are now ready to introduce:

Definition 1.1. A *plugin* with *step* $\sigma \in \{2^{-k} : k \geq 1\}$ is a map $g \in \mathrm{Diff}^r(\mathbb{T} \times M)$ satisfying the following assertions:

- (i) g restricted to $\Delta_\sigma := [0, \sigma) \times M$ is equal to R_σ,
- (ii) the first return time in Δ_σ of g is a well defined and bounded function $\tau : \Delta_\sigma \to \mathbb{N}^*$,
- (iii) the union of the iterates $\bigcup_{k \geq 0} g^k(\Delta_\sigma)$ equals $\mathbb{T} \times M$.

Remark 1.2. One can show by compactness of M that, under condition (i), condition (ii) is equivalent to (iii) and that in (ii) the return time is necessarily bounded.

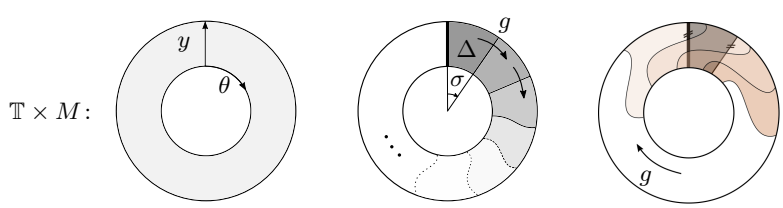


Figure 1: Plugin g of step σ.

Let $H_\sigma := (\theta, y) \in \Delta_\sigma \mapsto (\theta/\sigma, y) \in \mathbb{T} \times M$. It is a bijective local diffeomorphism.

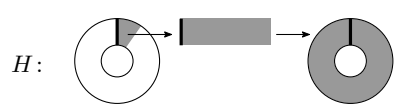


Figure 2: Rescaling map $H_\sigma \colon \Delta_\sigma \to \mathbb{T} \times M$.

Definition 1.3. The *output* of a plugin g of step σ is the following rescaling of the first return map $g^\tau \colon \Delta_\sigma \to \Delta_\sigma$:

$$G := H_\sigma \circ g^\tau \circ H_\sigma^{-1} \colon \mathbb{T} \times M \to \mathbb{T} \times M.$$

Example 1.4. For every $k \geq 0$, the map $g_k : (\theta, y) \mapsto (\theta + 2^{-k}, y)$ is a plugin of step 2^{-k}, iteration 2^k and output the identity.

Actually, we can show that the output of a plugin is always smooth:

Proposition 1.5. *Let $1 \leq r \leq \infty$. The output of a plugin $g \in \mathrm{Diff}^r(\mathbb{T} \times M)$ is in $\mathrm{Diff}^r(\mathbb{T} \times M)$ and depends continuously on g. In particular, the output is a total renormalization of the plugin.*

This proposition is a consequence of a classical, yet beautiful, argument which will be recalled in Appendix A. We are now ready to state the general result:

Theorem E (Main)**.** *Let $1 \leq r \leq \infty$ and a compact manifold M of dimension ≥ 1, and let $\mathcal{N} \subset \mathrm{Diff}^r(\mathbb{T} \times M)$ be a neighborhood of the identity. Then any $G \in \mathrm{Diff}^r_c(\mathbb{T} \times M)$ is the output of some plugin $g_G \in \mathcal{N}$.*

And here is its parametric counterpart:

Theorem E′. *Let $1 \leq r \leq \infty$ and a compact manifold M of dimension ≥ 1, fix a compact manifold $\mathscr{P}$ and a compactly supported family $(G_p)_{p\in\mathscr{P}}$ in $\mathrm{Diff}^r_c(\mathbb{T}\times M)_{\mathscr{P}}$. Let $\mathcal{N} \subset \mathrm{Diff}^r(\mathbb{T}\times M)_{\mathscr{P}}$ be a neighborhood of $(id)_{p\in\mathscr{P}}$. Then there exists a C^r-family of plugins $(g_p)_{p\in\mathscr{P}} \in \mathcal{N}$ such that G_p is the output of g_p for every $p \in \mathscr{P}$.*

1.2 Proof of the corollaries of the main Theorem E

Observe that Theorem E implies immediately Theorem B and that Theorem E′ implies immediately Theorem B′. In this subsection we show that Theorem E implies furthermore Theorem A and Corollaries C and D. To this end, the following will be useful:

Fact 1.6. *If the output of a plugin restricted to $\{0\}\times M$ is the identity, then the return time τ of the plugin is constant on the renormalization domain Δ.*

Proof. As f preserves the orientation it suffices to show that τ is constant on the interior of Δ. As τ is integer valued and M is connected, it suffices to show that τ is continuous on $int\,\Delta$ to conclude the proof. We start by showing that τ is lower semi-continuous on Δ. Suppose that it is not the case. Then there exist $N \geq 2$, a point $x \in \Delta$ and a point $x' \in \Delta$ arbitrarily close to x such that $N = \tau(x') < \tau(x)$. By continuity of g, it holds $g^N(x) \in \mathrm{cl}(\Delta)$. By assumption, we have $g^N(x) \notin \Delta$. Thus $g^N(x) \in \{\sigma\}\times M$ and since g is the translation by σ on Δ it holds then that $g^{N-1}(x) \in \Delta$ and consequently $\tau(x) < N$, which contradicts the assumption.

We now show that τ is upper semi-continuous on $int\,\Delta$. Consider $x \in int\,\Delta$. As g^τ is a bijection of Δ which leaves invariant $\{0\}\times M$, it comes that $g^\tau(x)$ is in the interior of Δ. Then for every x' close to x, the iterate $g^{\tau(x)}(x')$ belongs to Δ and so $\tau(x') \leq \tau(x)$. □

Proof that Theorem E implies Theorem A . Let G be in $\mathrm{Diff}^r(\mathbb{B}^n) = \mathrm{Diff}^r_0(\mathbb{B}^n)$. We observe[1] that G is the restriction to $\mathbb{B}_n$ of a diffeomorphism $\bar{G}$ in $\mathrm{Diff}^r_c(2\cdot\mathbb{B}^n)$. As $2\cdot id$ conjugates $\bar{G}$ to a map in $\mathrm{Diff}^r_c(\mathbb{B}^n)$, without any loss of generality we can assume that G belongs to $\mathrm{Diff}^r_c(\mathbb{B}^n)$.

Let $M := \mathbb{B}^{n-1}$ and embed $i : \mathbb{B}^n \hookrightarrow \mathbb{T}\times M$ so that the embedded ball $i(\mathbb{B}^n)$ does not meet $\{0\}\times M$ nor the boundary of $\mathbb{T}\times M$. Extend then G by id to a diffeomorphism $\tilde{G} \in \mathrm{Diff}^r_0(\mathbb{T}\times M)$. By Theorem E, $\tilde{G}$ is the output of a plugin $\tilde{g} \in \mathrm{Diff}^r(\mathbb{T}\times M)$ arbitrarily close to id. Since $\tilde{G}$ leaves $\{0\}\times M$ invariant it holds by Fact 1.6 that the first return time τ of the plugin $\tilde{g}$ is a constant N. In particular, the restriction $\tilde{g}^N|i(\mathbb{B}^n)$ is conjugate to G.

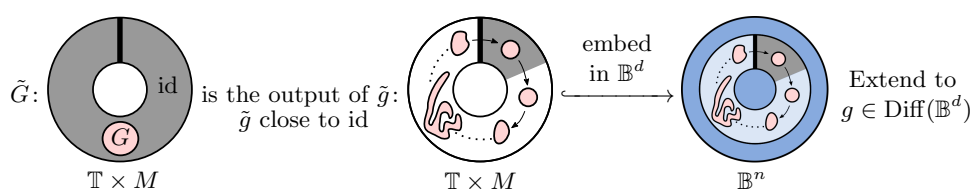


Figure 3: Proof of Theorem A.

To conclude it suffices to embed $\mathbb{T}\times M$ into the interior of $\mathbb{B}^n$ and extend $\tilde{g}$ then the diffeomorphism $\tilde{g}$ to a diffeomorphism g of $\mathbb{B}^n$ that is close to id. □

The following enables to localize nearby the identity the existence of some ergodic properties:

[1] By connectedness, there exists an isotopy $(h_t)_{t\in[0,1]}$ between $G|\partial\mathbb{B}^n$ and $id_{\partial\mathbb{B}^n}$ that can be chosen C^r-smooth, made of diffeomorphisms and and flat at the endpoints. Note that we can extend G on $2\cdot\mathbb{B}^n \setminus \mathbb{B}^n$ by $h_{\|z\|-1}\left(\frac{z}{\|z\|}\right)$ to construct and element of $\mathrm{Diff}^r_0(2\cdot\mathbb{B}^n)$.

Proposition 1.7. *Let g be a plugin of step σ and G its output. Then there is a canonical bijection $\mu_g \to \mu_G$ between G-invariant probability measures and g-invariant probability measures:*

$$\mu_g \mapsto \mu_G := H_{\sigma *}\frac{\mu_g|\Delta}{\mu_g(\Delta)} \quad \textit{and} \quad \mu_G \mapsto \mu_g := \sum_{N\geq 0} g^{*N} H_\sigma^* \mu_G|\{\tau > N\}\ .$$

Moreover:

1. *μ_g is ergodic iff μ_G is ergodic,*
2. *μ_g is hyperbolic iff μ_G is hyperbolic.*
3. *μ_g is a smooth volume form iff μ_G is a smooth volume form,*

Proof. We recall that H_σ conjugates G with the first return time g^τ of g in Δ. Then it is sufficient to prove the analogous properties for the following classical bijection between g^τ-invariant probability measures and g-invariant probability measures:

$$\mu_{g^\tau} \mapsto \mu_g := \sum_{N\geq 0} g^{*N}\mu_{g^\tau}|\{\tau > N\} \quad \text{and} \quad \mu_g \mapsto \mu_{g^\tau} := \frac{\mu_g|\Delta}{\mu_g(\Delta)}$$

Indeed it is well known that μ_{g^τ} is ergodic iff μ_g is ergodic (see for instance [EW13, Lemma 2.43]), this implies 1. Also it well known that the Lyapunov exponent of μ_G are equal to the mean of τ times the Lyapunov exponent of μ_g, from which we deduce 2.

Let us prove 3. If μ_G is smooth, then $\mu_g|\Delta = \mu_{g^\tau}$ is smooth on Δ and $\mu_g|\Delta \cup g(\Delta) = \mu_{g^\tau} + g^*\mu_{g^\tau}$ is smooth on $\Delta \cup g(\Delta)$. Now observe that for every $x \in \mathbb{T} \times M$ there exists a neighborhood U and $N \geq 0$ such that $g^{-N}(U) \subset \Delta \cup g(\Delta)$. Then by invariance, the density of μ_g on U is as smooth as $\mu_g|\Delta \cup g(\Delta)$. Conversely, if μ_g is smooth, then $\mu_{g^\tau} := \frac{\mu_g|\Delta}{\mu_g(\Delta)}$ must be smooth. Since g is a plugin, the density μ_g at the left hand side of Δ is equal to the translation by $(\sigma, 0)$ of the density of μ_g at the right hand side of $cl(\Delta)$. Thus μ_{g^τ} is pushed forward by H_σ to a smooth volume form μ_G on $\mathbb{T} \times M$. □

Proof of Corollary C. Katok in [Kat79][Thm B] showed the existence of Bernoulli diffeomorphisms homotopic to the identity and preserving a given smooth measure on any surface. Hence there exists $G \in \mathrm{Diff}_0^\infty(\mathbb{T}^2)$ preserving a smooth Bernouilli volume form μ_G. Equivalently by the Pesin Theorem the measure μ_G is a volume form which is ergodic and hyperbolic. By Theorem E, arbitrarily close to the identity, there exists g whose output is G. Then by Proposition 1.7, the volume form μ_G induces an ergodic and hyperbolic volume form μ_g for g. □

We now proceed to:

Proof of Corollary D. We will construct a sequence $(\Delta_i)_{i\geq 1}$ of domains with disjoint closures and of the form $\Delta_i := [0, \sigma_i) \times (s_i, s_i + l_i) \subset \mathbb{T} \times [0, 1]$ with $0 < \sigma_i < 1$ and $0 < s_i < 1 - l_i$. Then we will be able to construct a map f that renormalizes to F_i on each Δ_i and such that the orbit of the renormalization domain Δ_i is exactly $\mathbb{T} \times (s_i, s_i + l_i)$. Moreover we will be able to take f arbitrarily close to identity.

To do so, consider a sequence $(l'_i)_{i\geq 0}$ such that $\sum_i l'_i < 1$ and $l'_i > l_i$ for any $i \geq 0$. For $i \geq 0$ denote $s_i := \sum_{j<i} l'_j$ and $B_i := \mathbb{T} \times [s_i, s_i + l_i]$. Note that B_i has volume l_i and that $\mathrm{cl}(B_i) \cap \mathrm{cl}(B_j) = \emptyset$ for any $i \neq j \geq 0$. Let us define the map $\psi_i : (\theta, y) \in \mathbb{T} \times [0, 1] \mapsto (\theta, s_i + l_i \cdot y)$ that sends $\mathbb{T} \times [0, 1]$ to $\mathrm{cl}(B_i)$. Now by Theorem E, for any $i \geq 0$, there exists a plugin $\tilde{f}_i : \mathbb{T} \times [0, 1] \to \mathbb{T} \times [0, 1]$ arbitrarily close to identity with renormalization domain with output F_i. Let step σ_i be its step. The map $f_i := \psi_i \circ \tilde{f}_i \circ \psi_i^{-1} : B_i \to B_i$ is close to identity. Also a renormalization of f_i associated to $\Delta_i := [0, \sigma_i) \times [s_i, s_i + l_i]$ is F_i. Since the supports $\mathrm{supp}\, f_i = \mathrm{cl}(B_j)$ are disjoint and each f_i is close to identity, there exists a close to identity map $f : \mathbb{T} \times [0, 1] \to \mathbb{T} \times [0, 1]$ that coincides with f_i on B_i. Such a map verifies the conditions of the Corollary D. □

1.3 Pluggable dynamics

We will first work in the C^∞-topology. Indeed, the proof relies on the fact that $\mathrm{Diff}^\infty(V)$ is a Fréchet Lie group, which is not the case of $\mathrm{Diff}^r(V)$ for $r < \infty$. The C^r case will be deduced from the C^∞ case in Section 2.3.

1.3.1 Topologies on spaces of smooth maps and parameter families

We endow V with a Riemannian metric. Let $\mathfrak{diff}^\infty(V)$ be the space of smooth vector fields on V that are tangent to the boundary (if any). We endow $\mathrm{Diff}^\infty(V)$ and the space $\mathfrak{diff}^\infty(V)$ with the following distances:

$$d_{C^\infty}(f,g) = \max_{x\in V} \sum_{k\geq 1} 2^{-k} \min(1, d(D^k_x f, D^k_x g))$$

and

$$d_{C^\infty}(X,Y) = \max_{x\in V} \sum_{k\geq 1} 2^{-k} \min(1, \|D^k_x X - D^k_x Y\|_{C^r}) \ .$$

For these distances $\mathrm{Diff}^\infty(V)$ and $\mathfrak{diff}^\infty(V)$ are complete. Actually $\mathrm{Diff}^\infty(V)$ is a Lie group with algebra the Fréchet space $\mathfrak{diff}^\infty(V)$. We endow the connected component $\mathrm{Diff}^\infty_0(V)$ of the identity with the topology induced by $\mathrm{Diff}^\infty(V)$.

On the other hand, we endow the space $\mathrm{Diff}^\infty_c(V)$ and the space of compactly supported smooth vector fields $\mathfrak{diff}^\infty_c(V)$ with the finer *Whitney topology*. A basis of open sets of these respective topologies is:

$$U_{\eta,f,m} := \{g \in \mathrm{Diff}^\infty_c(V) : d(D^k_x f, D^k_x g) \leq \eta(x), \forall k \leq m\}$$

and

$$U_{\eta,X,m} := \{Y \in \mathfrak{diff}^\infty_c(V) : \|D^k_x X - D^k_x Y\| \leq \eta(x), \forall k \leq m\} \ ,$$

among $m \in \mathbb{N}$, $f \in \mathrm{Diff}^\infty_c(V)$, $X \in \mathfrak{diff}^\infty_c(V)$, and positive continuous functions $\eta : V \setminus \partial V \to (0,\infty)$. A well known theorem asserts that $f_n \to f$ in $\mathrm{Diff}^\infty_c(V)$ if and only if there exists a compact subset $K \subset V \setminus \partial V$ such that the supports of f and f_n are included in K for n and for every $r \geq 1$, we have $f_n \to f$ in the uniform C^r-topology when $n \to \infty$. The analogous property holds true for $\mathfrak{diff}^\infty_c(V)$.

We endow $\mathrm{Diff}^r(V)_{\mathscr{P}}$ and $\mathrm{Diff}^r_c(V)_{\mathscr{P}}$ with the topologies induced by $\mathrm{Diff}^r(V \times \mathscr{P})$ and $\mathrm{Diff}^r_c(V \times \mathscr{P})$ with the inclusions:

$$\mathrm{Diff}^r(V)_{\mathscr{P}} \hookrightarrow \mathrm{Diff}^r(V \times \mathscr{P}) \quad \text{and} \quad \mathrm{Diff}^r_c(V)_{\mathscr{P}} \hookrightarrow \mathrm{Diff}^r_c(V \times \mathscr{P})$$

via $f_{\mathscr{P}} = (f_p)_{p\in\mathscr{P}} \mapsto \widehat{f_{\mathscr{P}}}$ where:

$$\widehat{f_{\mathscr{P}}} := (v,p) \in V \times \mathscr{P} \mapsto (f_p(v), p) \ .$$

The spaces $\mathrm{Diff}^r(V)_{\mathscr{P}}$ and $\mathrm{Diff}^r_c(V)_{\mathscr{P}}$ endowed with the composition law $f_{\mathscr{P}} \circ g_{\mathscr{P}} := (f_p \circ g_p)_{p\in\mathscr{P}}$ are groups.

Note that we have:

Fact 1.8. *The following inclusion is a morphism of topological group:*

$$f_{\mathscr{P}} = (f_p)_{p\in\mathscr{P}} \in \mathrm{Diff}^\infty_0(V)_{\mathscr{P}} \mapsto \widehat{f_{\mathscr{P}}} \in \mathrm{Diff}^\infty_0(V \times \mathscr{P}) \ .$$

The latter fact will be used among some proofs of the parametric counterparts.

From now on we will work with $V = \mathbb{T} \times M$.

1.3.2 The space Plug of Pluggable maps

Every time, but when explicitly stated, we will focus on the case $r = \infty$: plugins will be of class C^∞ and their outputs as well. Recall that we endow $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ with the Euclidean Riemannian metric, M and $\mathscr{P}$ with their Riemannian metric. The product spaces $\mathbb{T} \times M$, $\mathbb{T} \times M \times \mathscr{P}$ and $M \times \mathscr{P}$ are endowed with their product Riemannian metric.

Definition 1.9. A map $G \in \mathrm{Diff}^\infty_c(\mathbb{T} \times M)$ is *semi-pluggable* if there is a sequence $(g_k)_{k\geq 1}$ of plugins $g_k \in \mathrm{Diff}^\infty(\mathbb{T} \times M)$ with step 2^{-k}, so that for every large integer k the output of g_k is G and $g_k \to \mathrm{id}$ for the C^∞-topology when $k \to \infty$. The map G is *pluggable* if G and G^{-1} are semi-pluggable. Let:

$$\mathrm{Plug} := \{G \in \mathrm{Diff}^\infty_c(\mathbb{T} \times M) : G \text{ is pluggable}\} .$$

Example 1.10. The identity id of $\mathbb{T}\times M$ is pluggable with the family $\big((\theta, y) \in \mathbb{T} \times M \mapsto (\theta + 2^{-k}, y)\big)_{k\geq 1}$ as in Example 1.4.

We give in Section 1.3.3 more sophisticated examples of pluggable dynamics. Note that in Definition 1.9, we are only interested by plugins whose output is compactly supported (outside of the boundary). This is because not every mapping of $\mathrm{Diff}^\infty_0(\mathbb{T} \times M)$ is pluggable by Corollary 0.11, while we will show that every map in $\mathrm{Diff}^\infty_c(\mathbb{T} \times M)$ is pluggable:

Theorem F. *We have* $\mathrm{Plug} = \mathrm{Diff}^\infty_c(\mathbb{T} \times M)$.

Observe that the first assertion of Theorem E in the case $r = \infty$ is an immediate consequence of Theorem F.

Parametric counterpart. The proof of Theorem F is (basically) constructive and depends smoothly on the output. For the sake of completeness, we will verify this by giving the parametric counterpart of each statement. Some of the proofs will be designed to be verbatim the same. For a first reading of the proofs, we advise the reader to skip the all parametric counterpart of the arguments. Here is the parametric counterpart of the notion of plugin:

Definition 1.11. A family $g_{\mathscr{P}} = (g_p)_{p\in\mathscr{P}} \in \mathrm{Diff}^\infty(\mathbb{T} \times M)_{\mathscr{P}}$ defines a *$\mathscr{P}$-plugin* if the diffeomorphism:

$$\widehat{g_{\mathscr{P}}} : (z, p) \in M \times \mathscr{P} \mapsto (g_p(x), p)$$

in $\mathrm{Diff}^\infty(\mathbb{T} \times M \times \mathscr{P})$ is a plugin.

Similarly we have the parametric counterpart of the output of a $\mathscr{P}$-plugin:

Definition 1.12. The *output* of a $\mathscr{P}$-plugin $g_{\mathscr{P}} \in \mathrm{Diff}^\infty(\mathbb{T} \times M)_{\mathscr{P}}$ of step σ is the family $G_{\mathscr{P}}$ such that for each $p \in \mathscr{P}$, the map G_p is the rescaling of the return map $g_p^{\tau_p} : \Delta_\sigma \to \Delta_\sigma$:

$$G_p := H_\sigma \circ g_p^{\tau_p} \circ H_\sigma^{-1} : \mathbb{T} \times M \to \mathbb{T} \times M.$$

By Proposition 1.5 the output $\widehat{G_{\mathscr{P}}}$ of the plugin $\widehat{g_{\mathscr{P}}}$ on $\mathbb{T} \times M \times \mathscr{P}$ is smooth and so:

Fact 1.13. *The output of a $\mathscr{P}$-plugin of* $\mathbb{T} \times M$ *lies in* $\mathrm{Diff}^r_0(\mathbb{T} \times V)_{\mathscr{P}}$.

In the parametric setting, Definition 1.9 becomes:

Definition 1.14. A family $G_{\mathscr{P}} \in \mathrm{Diff}^\infty_c(\mathbb{T} \times M)_{\mathscr{P}}$ is *$\mathscr{P}$-semi-pluggable* if there is a sequence $(g_{k\mathscr{P}})_{k\geq 1}$ of $\mathscr{P}$-plugins with step 2^{-k}, so that for every large k, the output of $g_{k\mathscr{P}}$ is $G_{\mathscr{P}}$ and $\widehat{g_{k\mathscr{P}}} \to \mathrm{id}_{\mathbb{T}\times M\times\mathscr{P}}$ for the C^∞-topology. The family of maps $G_{\mathscr{P}}$ is *$\mathscr{P}$-semi-pluggable* if $G_{\mathscr{P}}$ and $G_{\mathscr{P}}^{-1} := (G_p^{-1})_{p\in\mathscr{P}}$ are $\mathscr{P}$-semi-pluggable. Let:

$$\mathrm{Plug}_{\mathscr{P}} := \{G_{\mathscr{P}} \in \mathrm{Diff}^\infty_c(\mathbb{T} \times M)_{\mathscr{P}} : G_{\mathscr{P}} \text{ is } \mathscr{P}\text{-pluggable}\} .$$

To obtain Theorem E′ in the case $r = \infty$, it suffices to prove:

Theorem F′. *We have* $\mathrm{Plug}_{\mathscr{P}} = \mathrm{Diff}^\infty_c(\mathbb{T} \times M)_{\mathscr{P}}$.

1.3.3 Examples of pluggable dynamics

Consider the following subgroup of $\mathrm{Diff}^\infty_c(\mathbb{T}\times M)$:

$$\mathbb{G}_1 := \{(\theta, y) \in \mathbb{T}\times M \mapsto (\theta + \nu(y), y) : \nu \in C^\infty_c(M,\mathbb{R})\} .$$

This gives a first example of a subgroup of pluggable maps:

$$\mathbb{G}_1 \subset \mathfrak{plug} .$$

Proposition 1.15. *The group is included in* Plug.

The subgroup $\mathbb{G}_1$ was first studied in [BT22] into a set of generators of symplectomorphisms.

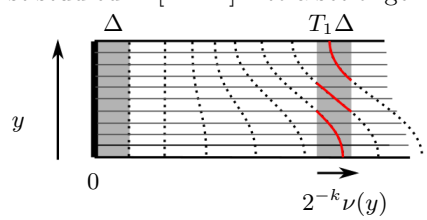


Figure 4: Dynamics of an element of $\mathbb{G}_1$.

Proof. Let $\rho \in C^\infty(\mathbb{T},\mathbb{R}^+)$ be a function with support contained in $[\frac{1}{3},\frac{2}{3}] \subset \mathbb{T}$ and integral 1. For $\epsilon \in [-2^{-k-2}, 2^{-k-2}]$, we define the smooth vector field:

$$X_\epsilon : \theta \in \mathbb{T} \mapsto 2^{-k}/(1+\epsilon\cdot\rho(\theta)) . \tag{1.1}$$

Let ϕ^t_ϵ be the flow of X_ϵ. The time taken to go all around the circle equals to:

$$\tau(\epsilon) = \int_{\mathbb{T}} \frac{1}{X_\epsilon(\theta)} d\theta = \int_{\mathbb{T}} 2^k\cdot(1+\epsilon\cdot\rho(\theta))d\theta = 2^k\cdot(1+\epsilon) . \tag{1.2}$$

If we stop at time 2^k, then the lacking or exceeding time for a complete lap is $2^k\epsilon \in [-\frac{1}{4},\frac{1}{4}]$. As nearby 0 the vector field X_ϵ equal 2^{-k}, this implies that the image of 0 by $\phi^{2^k}_\epsilon$ is equal to ϵ. So for every $\nu \in C^\infty_c(M,\mathbb{R})$, for every k large, the map:

$$g_k := (\theta, y) \mapsto (\phi^1_{2^{-k}\nu(y)}(\theta), y) \tag{1.3}$$

coincides with $R_{2^{-k}}$ on a set which contains the complement of $S = [\frac{1}{4},\frac{3}{4}]\times \operatorname{supp}\nu$. Thus g_k is a plugin with step 2^{-k}. Furthermore, by the above discussion, its output equals to:

$$G : (\theta, y) \mapsto (\theta + \nu(y), y) \text{ with } \nu \in C^\infty_c(M,\mathbb{R}) . \tag{1.4}$$

Finally observe that g_k is C^∞-close to identity. Thus G is semi-pluggable. Hence $\mathbb{G}_1$ is formed by semi-pluggable maps, and as $\mathbb{G}_1$ is a group, it is formed by pluggable maps. □

Parametric counterpart. Let $\mathbb{G}_{1\mathscr{P}}$ be the subset of $\mathrm{Diff}^\infty_c(\mathbb{T}\times M)_{\mathscr{P}}$ formed by families $(G_p)_{p\in\mathscr{P}}$ such that $G_p \in \mathbb{G}_1$ for every $p \in \mathscr{P}$.

We have similarly:

Proposition 1.16. *The group $\mathbb{G}_{1\mathscr{P}}$ is included in* $\mathrm{Plug}_{\mathscr{P}}$.

Proof. An element $(G_p)_{p\in\mathscr{P}} \in \mathbb{G}_{1\mathscr{P}}$ is formed by mapping of the form $G_p : (\theta, y) \mapsto (\theta+\nu_p(y), y)$ with $(\nu_p)_{p\in\mathscr{P}}$ smooth. Thus the vector field $Z_p : (\theta, y) \mapsto (X_{2^{-k}\nu_p}(y), 0)$ depends smoothly on p. Hence the time one maps g_p form a family of plugins $(g_p)_{p\in\mathscr{P}}$ (with output G_p and step 2^{-k}) which depends smoothly on $p \in \mathscr{P}$. □

2 Topological group structure on Plug

2.1 Group structure on Plug

In this section we show that Plug endowed with the composition rule is a group. To this end, let $\pi : \bar{\mathbb{T}} \times M := \mathbb{R}/2\mathbb{Z} \times M \to \mathbb{T} \times M = \mathbb{R}/\mathbb{Z} \times M$ be the canonical 2-sheeted covering map and denote by $\psi : (\theta, y) \in \bar{\mathbb{T}} \times M \mapsto (\theta/2, y) \in \mathbb{T} \times M$ the canonical diffeomorphism. The following defines a binary operation $\star$ on the space of plugins:

Definition 2.1. Let g_1 and g_0 be two plugins with same step $\sigma = 2^{-k}$. Let $\bar{g}_1$ and $\bar{g}_0 \in \mathrm{Diff}^\infty(\widehat{\mathbb{T}} \times M)$ be the lifts of g_1 and g_0 such that $\bar{g}_1(0, y) = \bar{g}_0(0, y) = (\sigma, y)$ for every $y \in M$. Let $\overline{g_1 \star g_0}$ be equal to the lift $\bar{g}_0$ on the first half of $\bar{\mathbb{T}} \times M$ and be equal to the lift of $\bar{g}_1$ on second half of $\bar{\mathbb{T}} \times M$:

$$\overline{g_1 \star g_0} : (\theta, y) \in \bar{\mathbb{T}} \times M \mapsto \begin{cases} \bar{g}_0(\theta, y) & \text{if } \theta \in [0, 1) + 2\mathbb{Z}\ , \\ \bar{g}_1(\theta, y), & \text{if } \theta \in [1, 2) + 2\mathbb{Z}\ , \end{cases} \quad \text{and put } g_1 \star g_0 := \psi^{-1} \circ \overline{g_1 \star g_0} \circ \psi\ .$$

Remark 2.2. Given a neighborhood V of id in $\mathrm{Diff}^r_0(\mathbb{T} \times M)$, there exists a neighborhood W of id such that for any pair of plugins $f, g \in W$ of same steps, we have $f \star g \in V$.

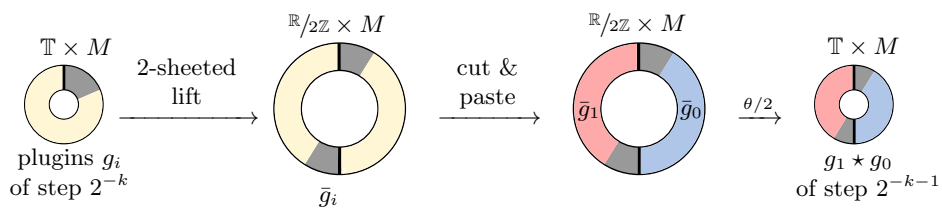


Figure 5: Concatenation of two plugins.

The $\star$-product associates to a pair of plugins of the same step a plugin of half that step and whose output is the composition of the outputs:

Proposition 2.3. *If g_0 and g_1 are plugins with a same step σ and outputs G_0 and G_1 then $g_1 \star g_0$ is a plugin with step $\sigma/2$ and output $G_1 \circ G_0$.*

Proof. The set $\Delta = ([0, \sigma) + \mathbb{Z}) \times M$ lifts into the union of the two sets:

$$\bar{\Delta}_0 = ([0, \sigma) + 2\mathbb{Z}) \times M \quad \text{and} \quad \bar{\Delta}_1 = ([1, 1 + \sigma) + 2\mathbb{Z}) \times M.$$

Let π_i be the restriction of π to each $\bar{\Delta}_i$.

Note that $\bar{g}_0$ and $\bar{g}_1$ coincide with the translation by $(\sigma, 0)$ on $\bar{\Delta}_0 \cup \bar{\Delta}_1$. Hence the map $\overline{g_0 \star g_1}$ is a smooth diffeomorphism that also coincides with the translation by $(\sigma, 0)$ on $\bar{\Delta}_0 \cup \bar{\Delta}_1$. Then $\overline{g_1 \star g_0}^{\tau_1 \circ \pi}$ sends $\bar{\Delta}_0$ onto $\bar{\Delta}_1$ by $\bar{g}_0^{\tau_0 \circ \pi}$ and sends $\bar{\Delta}_1$ onto $\bar{\Delta}_0$ by $\bar{g}_1^{\tau_1 \circ \pi}$. Thus we have:

$$\bar{g}_0^{\tau_0 \circ \pi} | \bar{\Delta}_0 = \pi_1^{-1} \circ H_\sigma^{-1} \circ G_0 \circ H_\sigma \circ \pi_0 \quad \text{and} \quad \bar{g}_1^{\tau_1 \circ \pi} | \bar{\Delta}_1 = \pi_0^{-1} \circ H_\sigma^{-1} \circ G_1 \circ H_\sigma \circ \pi_1\ .$$

Therefore the return time of $\overline{g_1 \star g_0}$ into $\bar{\Delta}_0$ is defined on $\bar{\Delta}_0$ and the first return map is:

$$\pi_0^{-1} \circ H_\sigma^{-1} \circ G_1 \circ G_0 \circ H_\sigma \circ \pi_0.$$

This implies that $g_1 \star g_0$ is a plugin with output $G_1 \circ G_0$ and step $\sigma/2$. □

Proposition 2.4. *The set* Plug *is a subgroup of* $\mathrm{Diff}^\infty_c(\mathbb{T} \times M)$ *endowed with the composition rule.*

Proof. By Example 1.10, Plug contains the identity and by definition it is stable by inversion. Thus it remains to show that semi-pluggability is stable by composition. Let G_1, G_0 be semi-pluggable. Then for every k large, C^∞-close to identity, there are plugins g_1 and g_0 with step 2^{-k} and with outputs G_1 and G_0 respectively. Then by Remark 2.2 , the map $g_1 \star g_0$ is close to identity when k is large. By Proposition 2.3, the output of $g_1 \star g_0$ is $G_1 \circ G_0$ and its step is 2^{-k-1}. Thus $G_1 \circ G_0$ is semi-pluggable. The second assertion is proved similarly. □

Parametric counterpart. The proofs of the two latter propositions imply immediately:

Proposition 2.5. *If $f_{\mathscr{P}}, g_{\mathscr{P}}$ are two $\mathscr{P}$-plugins with same step σ and output $F_{\mathscr{P}}$ and $G_{\mathscr{P}}$, then $(f_p \star g_p)_{p\in\mathscr{P}}$ is a $\mathscr{P}$-plugin with step $\sigma/2$ and output $(F_p \circ G_p)_{p\in\mathscr{P}}$.*

Thus we deduce:

Proposition 2.6. *The set* $\mathrm{Plug}_{\mathscr{P}}$ *is a subgroup of* $\mathrm{Diff}^\infty_c(\mathbb{T}\times M)_{\mathscr{P}}$ *endowed with operation:*

$$(F_p)_{p\in\mathscr{P}} \circ (G_p)_{p\in\mathscr{P}} = (F_p \circ G_p)_{p\in\mathscr{P}} \ .$$

2.2 Another subgroup included in Plug

Consider the following sub-group of $\mathrm{Diff}^\infty_c(\mathbb{T}\times M)$:

$$\mathbb{G}_2 := \{(\theta, y) \in \mathbb{T}\times M \mapsto (\theta, F(y)) : F \in \mathrm{Diff}^\infty_c(M)\}$$

Proposition 2.7. *The group* $\mathbb{G}_2$ *is included in* Plug.

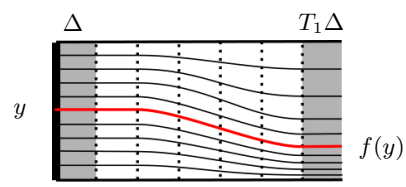


Figure 6: Dynamics of a plugin with output in $\mathbb{G}_2$.

Proof. As $\mathbb{G}_2$ is connected and Plug is a group by Proposition 2.4, it suffices to show that a neighborhood W of id in $\mathbb{G}_2$ is included in Plug. Indeed by [Wil84, Prop 3.18], any such neighborhood W generates $\mathbb{G}_2$. Up to replacing W by $W \cap W^{-1}$, it suffices to show that any element of W is semi-pluggable.

In order to do so, we develop an idea which appears in [NRT78]. Take W sufficiently small so that for every $G : (\theta, y) \mapsto (\theta, F(y)) \in W$, the map F is sufficiently close to id to be isotopic to it via a smooth path. In other words, there exists a C^∞-family $(F_t)_{t\in[0,1]}$ of maps $F_t \in \mathrm{Diff}^\infty_c(M)$ such that $F_0 = \mathrm{id}$ and $F_1 = F$. Such a family can be obtained using the exponential map exp of the Riemannian metric, via the formula $F_t := y \mapsto \exp_y(t \cdot \exp_y^{-1} F(y))$. Define:

$$Y(t, y) := \partial_t F_t \circ F_t^{-1}(y) \ , \tag{2.1}$$

and observe that F is the time one map of the (compactly supported) non-autonomous vector field Y. Let $\tau : \mathbb{T} \to [0, 1]$ be a smooth map which is smooth on $\mathbb{T} \setminus \{0\}$ and such that nearby 0^+ it equals 0 and nearby 0^- is equal 1. For k large, let:

$$X_k : (\theta, y) \in \mathbb{T}\times M \mapsto (2^{-k}, 2^{-k} \cdot \partial_\theta \tau(\theta) \cdot Y(\tau(\theta), y)) \ . \tag{2.2}$$

Let g_k be the time one map of this vector field. Observe that for k large enough, g_k is a plugin with step $\sigma = 2^{-k}$ and return time σ^{-1}. Furthermore, its output is G. Indeed the second coordinate of the output is the time σ^{-1} map of the flow of $\partial_\theta \tau(\theta) \cdot Y(\tau(\theta), y)$ which is the time one map F of the flow of Y. Thus we have:

$$g_k^{1/\sigma} \circ H_\sigma(x) = H_\sigma \circ G(x), \quad \forall x \in \Delta_\sigma \ . \tag{2.3}$$

Furthermore, when k is large, the plugin g_k is close to identity. Thus G is semi-pluggable. □

Parametric counterpart. Let $\mathbb{G}_{2\mathscr{P}}$ be the subset of $\mathrm{Diff}^\infty_c(\mathbb{T}\times M)_{\mathscr{P}}$ formed by families $(G_p)_{p\in\mathscr{P}}$ such that $G_p\in\mathbb{G}_2$ for every $p\in\mathscr{P}$. The following is a counterpart of Proposition 2.7:

Proposition 2.8. *The group $\mathbb{G}_{2\mathscr{P}}$ is included in* $\mathrm{Plug}_{\mathscr{P}}$.

Proof. Let W be as defined in the proof of Proposition 2.7. We define $W_{\mathscr{P}}$ as the subset of $\mathrm{Diff}^\infty_c(\mathbb{T}\times M)_{\mathscr{P}}$ formed by families $(G_p)_{p\in\mathscr{P}}$ such that $G_p\in W$ for every $p\in\mathscr{P}$. For the same reasons it suffices to show that any element of $W_{\mathscr{P}}$ is $\mathscr{P}$-semi-pluggable. Similarly, for any family $(G_p)_{p\in\mathscr{P}}=:(id_{\mathbb{T}}\times F_p)_{p\in\mathscr{P}}\in W_{\mathscr{P}}$, the family $(F_p)_{p\in\mathscr{P}}$ is isotopic to the identity via a smooth path $((F_{pt})_{p\in\mathscr{P}})_{t\in[0,1]}$ where $F_{pt}:=y\mapsto\exp_y(t\cdot\exp_y^{-1}F_p(y))$. We define:

$$Y_p(t,y):=\partial_t F_{pt}\circ F_{pt}^{-1}(y)\ . \tag{2.4}$$

Note that the family of vector fields $(Y_p)_{p\in\mathscr{P}}$ is smooth. Define then the family of vector fields:

$$X_{kp}:(\theta,y)\in\mathbb{T}\times M\mapsto(2^{-k},2^{-k}\cdot\partial_\theta\tau(\theta)\cdot Y_p(\tau(\theta),y))\ , \tag{2.5}$$

where τ is the function defined in Proposition 2.7. For k large enough, the family of time one maps $(g_{kp})_{p\in\mathscr{P}}$ is a $\mathscr{P}$-plugin with output the family $(G_p)_{p\in\mathscr{P}}$. Moreover, when $k\to\infty$, the $\mathscr{P}$-plugin $(g_{kp})_{p\in\mathscr{P}}$ tends to the identity $\mathrm{id}\in\mathrm{Diff}^\infty_c(M)_{\mathscr{P}}$. □

2.3 Closure of the group Plug

In this section we prove that Plug is closed in $\mathrm{Diff}^\infty_c(\mathbb{T}\times M)$.

Proposition 2.9. *The subgroup* $\mathrm{Plug}\subset\mathrm{Diff}^\infty_c(\mathbb{T}\times M)$ *is closed.*

This proposition uses the following lemma proved below:

Lemma 2.10. *For any $1\le r\le\infty$, for any neighborhood $\mathcal{N}$ of $\mathrm{id}\in\mathrm{Diff}^r(\mathbb{T}\times M)$, there exist $N\ge 1$ and a neighborhood $\mathcal{N}_c$ of $\mathrm{id}\in\mathrm{Diff}^r_c(\mathbb{T}\times M)$ such that for all $G\in\mathcal{N}_c$ and $k\ge N$, there is a C^r-plugin $g\in\mathcal{N}$ with output G and step 2^{-k}.*

Note that the latter lemma is redacted for any regularity $1\le r\le\infty$. It will allow to deduce Theorem E from Theorem F.

Proof of Proposition 2.9. It is suffices to show that the set of semi-pluggable maps is closed. Indeed, the continuity of the involution $G\mapsto G^{-1}$ implies that the set of maps with semi-pluggable inverse is closed; and so it comes that the intersection Plug of these two sets is closed.

Let $(G_j)_{j\ge 0}$ be a sequence semi-pluggable maps converging in the C^∞-topology to a diffeomorphism $G\in\mathrm{Diff}^\infty_c(\mathbb{T}\times M)$. Let us show that G is semi-pluggable. In other words, let us show that for every neighborhood $\mathcal{V}$ of $\mathrm{id}\in\mathrm{Diff}^\infty(\mathbb{T}\times M)$, for every k large enough, there exists a plugin $g\in\mathcal{V}$ with output G and step 2^{-k}.

To this end, let us fix a small neighborhood of $\mathcal{N}$ of $\mathrm{id}\in\mathrm{Diff}^\infty(\mathbb{T}\times M)$ and j large so that $G_j^{-1}\circ G$ belongs to the open set $\mathcal{N}_c$ given by Lemma 2.10. Hence for every $k\ge 0$ large enough, there exists a plugin $g_1\in\mathcal{N}$ with output $G_j^{-1}\circ G$ and step 2^{-k}. As G_j is pluggable, for every k large enough, there exists a plugin $g_0\in\mathcal{N}$ with output G_j and step 2^{-k}. Now we merge the plugins g_1 and g_0 to obtain a plugin $g=g_1\star g_0$ of G of step 2^{-k-1}. By Remark 2.2, when $\mathcal{N}$ is small, the map g is close to identity and so in $\mathcal{V}$. □

The idea of the proof of Lemma 2.10 is to find, for each fixed small $\delta>0$, a sequence of close to identity plugins g_k with step 2^{-k} and output the identity such that some iterates of each g_k stretch the 2^{-k}-thin fundamental domain $\Delta_{2^{-k}}$ onto a wider fundamental domain, isometric to $[0,2\delta[\times M$. The iteration by g_k will produce a horizontal zooming effect on $\Delta_{2^{-k}}$. Then we will be able to perturb the plugin on this stretched fundamental domain, to obtain an open set of outputs independent of k.

The following produces the sequence $(g_k)_k$:

Sub-Lemma 2.11. *For every neighborhood $\mathcal{N}$ of* $\mathrm{id} \in \mathrm{Diff}^\infty(\mathbb{T} \times M)$*, for every $\delta > 0$ small, there exists $N \geq 1$ and a sequence $(g_k)_{k \geq N}$ of plugins in $\mathcal{N}$ with step 2^{-k}, output* id *such that:*

- *for all $y \in M$ and $\theta \in [\frac{1}{2}, \frac{2}{3}]$; we have $g_k(\theta, y) = (\theta + \delta, y)$,*
- *g_k is of the form $g_k : (\theta, y) \in \mathbb{T} \times M \mapsto (\phi_k(\theta), y)$.*

Proof. Let $A = [\frac{1}{2}, \frac{2}{3}]$ and $B = [\frac{1}{4}, \frac{1}{3}]$. Let $\psi_A, \psi_B \in C^\infty(\mathbb{T}, [0, 1])$ be two non-negative functions with disjoint supports, vanishing at a neighborhood of 0 and such that:

$$\psi_A | A = 1 \quad \text{and} \quad \psi_B | B = 1 \ . \tag{2.6}$$

For $\beta \geq 0$ we define the following vector field on the circle $\mathbb{T}$:

$$X_{\beta,k} := \delta \cdot \psi_A + \beta \cdot \psi_B + (1 - \psi_A - \psi_B) \cdot 2^{-k}. \tag{2.7}$$

Let $\tau_{\beta,k}$ be the time needed to make one turn around the circle along the flow of X_β. This number is large since δ is small. Note that $\tau_{\beta,k}$ depends smoothly on $\beta > 0$. Also $\tau_{\beta,k} \to \infty$ when $\beta \to 0$ and $\partial_\beta \tau_{\beta,k} < 0$. Let $N \geq 1$ be so that $\delta > 2^{-N}$. Let $k \geq N$. We have $\delta > 2^{-k}$. If $\beta = 2^{-k}$, then the time $\tau_{\beta,k}$ is smaller than 2^k. Thus by the mean value theorem, there exists a unique $\beta = \beta(k, \alpha)$ close to 2^{-k} such that $\tau_{\delta,\beta(k,\delta),k} = 2^k$.

Then the time 1 map g_k of the flow of $(X_{\beta(k),k}, 0)$ satisfies the sought properties. □

We have now the tools to prove the following restricted version of Lemma 2.10:

Sub-Lemma 2.12. *For any $1 \leq r \leq \infty$, for any neighborhood $\mathcal{V}$ of* $\mathrm{id} \in \mathrm{Diff}^r(\mathbb{T} \times M)$*, there exist $N \geq 1$ and a neighborhood $\mathcal{N}_c$ of* $\mathrm{id} \in \mathrm{Diff}^r_c(\mathbb{T} \times M)$ *such that for any $k \geq N$ and every $G \in \mathcal{N}_c$ whose restriction to a neighborhood of $\{0\} \times M$ or a neighborhood of $\{\frac{1}{2}\} \times M$ is the identity, there is a C^r-plugin $g \in \mathcal{V}$ with output G and step 2^{-k}.*

Proof of Sub-Lemma 2.12 . Let $\mathcal{N}$ be a neighborhood of $\mathrm{id} \in \mathrm{Diff}^r(\mathbb{T} \times M)$ such that[2] $\mathcal{N} \Subset \mathcal{V}$. We apply Sub-Lemma 2.11 which provides $\delta > 0$, $N \geq 1$ and a sequence $(g_k)_{k \geq N}$ of plugins $g_k : (\theta, y) \mapsto (\phi_k(\theta), y)$ with output id and step 2^{-k}. By Sub-Lemma 2.11, there exists $n_k > 0$ minimal such that $\theta_k := \phi_k^{n_k}(0) \in [\frac{1}{2}, 1]$. Taking $\mathcal{N}$ small, we have that θ_k is smaller than $\frac{2}{3} - \frac{3}{2}\delta$, and so g_k equals the translation by δ on $(\theta_k, \theta_k + \frac{3}{2}\delta)$. Let $\mathcal{N}_c$ be a small neighborhood of $\mathrm{id} \in \mathrm{Diff}^r_c(\mathbb{T} \times M)$. Let $G \in \mathcal{N}_c$ be equal to the identity nearby $\{0\} \times M$ or $\{\frac{1}{2}\} \times M$. We would like to C^r-perturb g_k so that its output is G.

Case 1: If $G \in \mathcal{N}_c$ coincides with the identity nearby $\{0\} \times M$, then we perform a perturbation of g_k supported by $(\theta_k, \theta_k + \delta) \times M$ and therein equals to:

$$\begin{array}{rcl} \tilde{g}_k : [\theta_k, \theta_k + \delta] \times M & \to & [\theta_k + \delta, \theta_k + 2\delta] \times M \\ (\theta_k + x, y) & \mapsto & (\theta_k + \delta, 0) + \delta \cdot \tilde{G}(\delta^{-1} x, y) \end{array} \tag{2.8}$$

where $\tilde{G} : \mathbb{R} \times M \to \mathbb{R} \times M$ is a lifting of G which fixes $\{0\} \times M$. Note that $\tilde{g}_k$ is a C^r-plugin with output G and step 2^{-k}. Furthermore, if $\mathcal{N}_c$ is small enough (at δ fixed), then for every $k \geq N$, the map $\tilde{g}_k$ is uniformly close to $g_k \in \mathcal{N}$ in the C^r-topology and so $\tilde{g}_k$ belongs to $\mathcal{V}$.

Case 2: If $G \in \mathcal{N}_c$ coincides with the identity nearby $\{\frac{1}{2}\} \times M$, then we perform a perturbation of g_k supported by $(\theta_k + \frac{\delta}{2}, \theta_k + \frac{3}{2}\delta) \times M$ and therein equals to:

$$\begin{array}{rcl} \tilde{g}_k : [\theta_k + \frac{\delta}{2}, \theta_k + \frac{3}{2}\delta] \times M & \to & [\theta_k + \frac{3}{2}\delta, \theta_k + \frac{5}{2}\delta] \times M \\ (\theta_k + \frac{\delta}{2} + x, y) & \mapsto & (\theta_k + \frac{\delta}{2} + \delta, 0) + \delta \cdot \tilde{G}(\delta^{-1} x, y) \ . \end{array} \tag{2.9}$$

Similarly, this is a C^r-plugin of output G and step 2^{-k}, which is in $\mathcal{V}$ for every k provided that $\mathcal{N}_c$ is small enough. □

[2] in the sense that the distance between $\mathcal{N}$ and the complement of $\mathcal{V}$ is positive.

Proof that Sub-Lemma 2.12 implies Lemma 2.10 . First let us 'fragment' any C^r-close to identity map $G \in \mathcal{N}_c$ into the composition of two C^r-maps $G_1 \circ G_0$ such that G_0 coincides with the identity nearby $\{0\} \times M$ and G_1 coincides with the identity nearby $\{\frac{1}{2}\} \times M$.

To this end, we use the exponential map exp associated to the geodesic flow of $\mathbb{T} \times M$ and a function $\rho \in C^\infty(\mathbb{T} \times M, [0,1])$ such that $\rho|\{0\} \times M = 1$ and $\rho|\{\frac{1}{2}\} \times M = 0$. Let $\mathcal{N}_c$ be a sufficiently small neighborhood of $\mathrm{id} \in \mathrm{Diff}^\infty_c(\mathbb{T} \times M)$ such that for every $G \in \mathcal{N}_c$ the following is a smooth diffeomorphism:

$$G_0 := x \mapsto \exp(\rho(x) \cdot \exp_x^{-1} G(x)) \ . \tag{2.10}$$

Let $G_1 := G \circ G_0^{-1}$ and note that $G = G_1 \circ G_0$.

Sub-Lemma 2.12 states that there are C^r-plugins g_1 and g_0 close to identity with step 2^{-k} and outputs G_1 and G_0 for every k large enough. Then by Remark 2.2 and Proposition 2.3, the C^r-plugin $g_1 \star g_0$ of step 2^{-k-1} is close to identity and has output $G_1 \circ G_0 = G$. □

Proof that Theorem F implies Theorem E . When $r = \infty$, the result of Theorem E corresponds to the one Theorem F. Consider now $r < \infty$. Let $G \in \mathrm{Diff}^r_c(\mathbb{T} \times M)$, and $\mathcal{N} \subset \mathcal{V}$ two neighborhoods of $\mathrm{id} \in \mathrm{Diff}^r_c(\mathbb{T} \times M)$. We smooth the map G into a map $\tilde{G} \in \mathrm{Diff}^\infty_c(\mathbb{T} \times M)$ such that the map $\tilde{G}^{-1} \circ G$ belongs to the neighborhood $\mathcal{N}_c$ given by Lemma 2.10. Then for every $k \geq 0$ large enough, there exists a plugin $g_0 \in \mathcal{N}$ with output $\tilde{G}^{-1} \circ G$ and step 2^{-k}. By Theorem F, for k large enough, there exists a plugin $g_1 \in \mathcal{N}$ with output $\tilde{G}$. We merge g_0 and g_1 to get a plugin $g = g_1 \star g_0$ with output G. By Remark 2.2, when $\mathcal{N}$ is small, the map g is in $\mathcal{V}$. □

Parametric counterpart. Here is the parametric counterpart of Proposition 2.9:

Proposition 2.13. *The set* $\mathrm{Plug}_{\mathscr{P}}$ *is closed.*

To show this proposition we will use the following counterpart of Lemma 2.10 proved below:

Lemma 2.14. *For any $1 \leq r \leq \infty$, for any neighborhood $\mathcal{N}$ of $\mathrm{id} \in \mathrm{Diff}^r(\mathbb{T} \times M)$, there exists $N \geq 1$ and a neighborhood $\mathcal{N}_c$ of $\mathrm{id} \in \mathrm{Diff}^r_c(\mathbb{T} \times M)_{\mathscr{P}}$ such that for all $(G_p)_{p \in \mathscr{P}} \in \mathcal{N}_c$ and $k \geq N$, there is a $\mathscr{P}$-C^r-plugin $(g_p)_{p \in \mathscr{P}} \in \mathcal{N}$ with output $(G_p)_{p \in \mathscr{P}}$ and step 2^{-k}.*

Proof of Proposition 2.13. We proceed literally as in the proof Proposition 2.9, by applying Lemma 2.14 instead of Lemma 2.10, and considering families instead of single maps. Note that the continuity of $\star$ given by Remark 2.2 is also valid for families since $(f_p)_p \in \mathrm{Diff}^\infty(\mathbb{T} \times M)_{\mathscr{P}} \mapsto f_{\mathscr{P}} \in \mathrm{Diff}^r(\mathbb{T} \times M \times \mathscr{P})$ is an embedding which commutes with the $\star$-product. □

Proof of Lemma 2.14. The explicit construction of Sub-Lemma 2.12 gives directly a parametric counterpart of this lemma. Indeed note that the maps $\tilde{g}_k$ defined in its proof depends smoothly on G. Moreover the maps G_0 and G_1 obtained in the proof of Lemma 2.10 by the fragmentation formula depends smoothly on the involved diffeomorphism. □

Similarly, we show:

Proof that Theorem F′ implies Theorem E′. This goes exactly the same as for the proof that Theorem F implies Theorem E′. It suffices to replace maps in $\mathrm{Diff}^r_c(\mathbb{T} \times M)$ by families of maps in $\mathrm{Diff}^r_c(\mathbb{T} \times M)_{\mathscr{P}}$, plugins by $\mathscr{P}$-plugins and Lemma 2.10 by Lemma 2.14.

□

2.4 Vector fields whose flow is pluggable

In this section V denotes a manifold. We recall that $\mathrm{Diff}^\infty_c(V)$ endowed with the composition rule $\circ$ is a Lie group. The space $\mathfrak{diff}^\infty_c(V)$ of C^∞-vector fields X on V whose support is a compact subset of $V \setminus \partial V$ is a Lie algebra endowed with the Lie bracket:

$$[X, Y] := DY(X) - DX(Y) \ , \quad \forall X, Y \in \mathfrak{diff}^\infty_c(V).$$

We will work with the Lie algebra counterpart of the considered subgroups. We recall that a subset of $\mathfrak{diff}_c^\infty(V)$ is a *Lie subalgebra* if it is a vector space stable by Lie Bracket. To define the counterpart, we will use the flow $(\mathrm{Fl}_X^t)_t$ of vector fields $X \in \mathfrak{diff}_c^\infty(V)$.

Definition 2.15. We denote $\mathfrak{plug}$ the set of vector fields whose flow is pluggable. In short:

$$\mathfrak{plug} := \{X \in \mathfrak{diff}_c^\infty(\mathbb{T} \times M) : \mathrm{Fl}_X^t \in \mathrm{Plug},\ \forall t \in \mathbb{R}\}\ .$$

Using that Plug is a closed subgroup, the following is an immediate consequence of Proposition B.1 of Appendix B:

Proposition 2.16. *The space* $\mathfrak{plug}$ *is a closed Lie subalgebra of* $\mathfrak{diff}_c^\infty(\mathbb{T} \times M)$.

The following are closed Lie algebras of $\mathfrak{diff}_c^\infty(\mathbb{T} \times M)$:

$$\mathfrak{g}_1 = \mathfrak{g}_1(M) := \{X \in \mathfrak{diff}_c^\infty(\mathbb{T}\times M) : \mathrm{Fl}_X^t \in \mathbb{G}_1,\ \forall t \in \mathbb{R}\} = \{X : (\theta, y) \mapsto (v(y), 0) : v \in C^\infty(M, \mathbb{R})\}\ .$$

$$\mathfrak{g}_2 = \mathfrak{g}_2(M) := \{X \in \mathfrak{diff}_c^\infty(\mathbb{T}\times M) : \mathrm{Fl}_X^t \in \mathbb{G}_2,\ \forall t \in \mathbb{R}\} = \{X : (\theta, y) \mapsto (0, f(y)) : f \in \mathfrak{diff}_c^\infty(M)\}\ .$$

The subgroups $\mathbb{G}_1$ and $\mathbb{G}_2$ are in Plug, so:

$$\mathfrak{g}_1 \subset \mathfrak{plug} \quad \text{and} \quad \mathfrak{g}_2 \subset \mathfrak{plug}\ . \tag{2.11}$$

Parametric counterpart. We denote $\mathfrak{diff}_c^\infty(V)_{\mathscr{P}}$ the subspace of families $X_{\mathscr{P}} = (X_p)_{p\in\mathscr{P}}$ in $\mathfrak{diff}_c^\infty(V)$ such that

$$\widehat{X_{\mathscr{P}}} \colon (x, p) \mapsto (X_p(x), 0) \tag{2.12}$$

is smooth and compactly supported, that is, such that $\widehat{X_{\mathscr{P}}} \in \mathfrak{diff}_c^\infty(V \times \mathscr{P})$. By Fact 1.8, the space $\mathfrak{diff}_c^\infty(V)_{\mathscr{P}}$ is a Lie algebra endowed with the Lie bracket $[X_{\mathscr{P}}, Y_{\mathscr{P}}] := ([X_p, Y_p])_{p\in\mathscr{P}}$.

Definition 2.17. The $\mathscr{P}$-families of vector fields whose flow is $\mathscr{P}$-pluggable is denoted:

$$\mathfrak{plug}_{\mathscr{P}} := \left\{(X_p)_{p\in\mathscr{P}} \in \mathfrak{diff}_c^\infty(\mathbb{T} \times M)_{\mathscr{P}} : (\mathrm{Fl}_{X_p}^t)_{p\in\mathscr{P}} \in \mathrm{Plug}_{\mathscr{P}},\ \forall t \in \mathbb{R}\right\}\ .$$

Using that $\mathrm{Plug}_{\mathscr{P}}$ is a closed subgroup, the following is an immediate consequence of Corollary B.2 of Appendix B:

Proposition 2.18. *The space* $\mathfrak{plug}_{\mathscr{P}}$ *is a closed Lie subalgebra of* $\mathfrak{diff}_c^\infty(\mathbb{T} \times M)_{\mathscr{P}}$.

Also note that the space $\mathfrak{plug}_{\mathscr{P}}$ contains:

$$\mathfrak{g}_{1\mathscr{P}} := \{(X_p)_{p\in\mathscr{P}} \in \mathfrak{diff}_c^\infty(\mathbb{T} \times M)_{\mathscr{P}} : X_p \in \mathfrak{g}_1,\ \forall p \in \mathscr{P}\}\ .$$

and

$$\mathfrak{g}_{2\mathscr{P}} := \{(X_p)_{p\in\mathscr{P}} \in \mathfrak{diff}_c^\infty(\mathbb{T} \times M)_{\mathscr{P}} : X_p \in \mathfrak{g}_2,\ \forall p \in \mathscr{P}\}\ .$$

We define also:

$$\widehat{\mathfrak{g}_{1\mathscr{P}}} := \{\widehat{X_{\mathscr{P}}} : X_{\mathscr{P}} \in \mathfrak{g}_{1\mathscr{P}}\} \quad \text{and} \quad \widehat{\mathfrak{g}_{2\mathscr{P}}} := \{\widehat{X_{\mathscr{P}}} : X_{\mathscr{P}} \in \mathfrak{g}_{2\mathscr{P}}\}\ .$$

Observe that:

$$\widehat{\mathfrak{g}_{1\mathscr{P}}} = \{(\theta, y, p) \mapsto (\nu(y, p), 0, 0) : \nu \in C_c^\infty(M \times \mathscr{P}, \mathbb{R})\}$$

and

$$\widehat{\mathfrak{g}_{2\mathscr{P}}} = \{(\theta, y, p) \mapsto (0, f(y, p), 0) : (f, 0) \in \mathfrak{diff}_c^\infty(M \times \mathscr{P})\}\ .$$

Thus we have $\widehat{\mathfrak{g}_{1\mathscr{P}}} = \mathfrak{g}_1(M \times \mathscr{P})$ and $\widehat{\mathfrak{g}_{2\mathscr{P}}} \subsetneq \mathfrak{g}_2(M \times \mathscr{P})$.

3 Construction of Pluggable flows

In this section we show Theorem F by proving that any vector field has a pluggable flow. In order to do so we will show that the following subspace of $\mathfrak{diff}_c^\infty(\mathbb{T}\times M)$ is in $\mathfrak{plug}$:

$$\mathfrak{g}_3 := \mathfrak{g}_3(M) = \{X \in: \mathfrak{diff}_c^\infty(\mathbb{T}\times M) : X \text{ has null } \mathbb{T}\text{-coordinate}\} .$$

The non-trivial remaining part of the proof is to show the following:

Proposition 3.1. *Any vector field of $\mathfrak{g}_3$ has pluggable flows:*

$$\mathfrak{g}_3 \subset \mathfrak{plug} .$$

The proof of this proposition will occupy Sections 3.2 and 3.3. In the next section we show that it implies main Theorem F. The proof of several propositions will involve the following notation. If $\mathfrak{g}$ and $\mathfrak{h}$ are two sub-Lie algebras, we denote $[\mathfrak{g},\mathfrak{h}]$ the vector space spanned by Lie brackets of elements of $\mathfrak{g}$ and $\mathfrak{h}$:

$$[\mathfrak{g},\mathfrak{h}] := \left\{ \sum_{\text{i:finite}} [X_i, Y_i] : X_i \in \mathfrak{g},\ Y_i \in \mathfrak{h} \right\} .$$

Parametric counterpart. Similarly we set:

$$\mathfrak{g}_{3\mathscr{P}} := \{(X_p)_{p\in\mathscr{P}} \in \mathfrak{diff}_c^\infty(\mathbb{T}\times M)_{\mathscr{P}} : X_p \in \mathfrak{g}_3,\ \forall p \in \mathscr{P}\} ,$$

and

$$\widehat{\mathfrak{g}_{3\mathscr{P}}} := \{\widehat{X_{\mathscr{P}}} : X_{\mathscr{P}} \in \mathfrak{g}_{3\mathscr{P}}\} \subsetneq \mathfrak{g}_3(M\times\mathscr{P}) .$$

We will prove the following parametric counterpart of Proposition 3.1 in Sections 3.2 and 3.3:

Proposition 3.2. *Every smooth family of vector fields in $\mathfrak{g}_3$ has a $\mathscr{P}$-pluggable flow:*

$$\mathfrak{g}_{3\mathscr{P}} \subset \mathfrak{plug}_{\mathscr{P}} .$$

3.1 Proof of main Theorem F and Theorem F′

The first step of the proof is the following:

Proposition 3.3. *Any vector field has a pluggable flow:*

$$\mathfrak{plug} = \mathfrak{diff}_c^\infty(\mathbb{T}\times M) .$$

Proof. As $\mathfrak{plug}$ is a Lie algebra by Proposition 2.16 and since $\mathfrak{g}_1$ and $\mathfrak{g}_3$ are in $\mathfrak{plug}$ by Eq. (2.11) and Proposition 3.1, it suffices to show that the Lie algebra generated by $\mathfrak{g}_1$ and $\mathfrak{g}_3$ equals $\mathfrak{diff}_c^\infty(\mathbb{T}\times M)$. We first prove the statement of the proposition for $M=\mathbb{R}^n$. Let $X \in \mathfrak{diff}_c^\infty(\mathbb{T}\times M)$ and $f \in C_c^\infty(\mathbb{T}\times\mathbb{R}^n,\mathbb{R})$ be its $\mathbb{T}$-coordinate. Let $W \in \mathfrak{g}_3$ be:

$$W := (0, f, 0, \dots, 0) . \tag{3.1}$$

Let $\rho \in C^\infty(\mathbb{T}\times M,\mathbb{R})$ be a compactly supported function which is equal to 1 nearby the support of f. Let $Y \in \mathfrak{g}_1$ be:

$$Y(\theta, y_1, \dots, y_n) = (\rho(\theta, y)\cdot y_1, 0, \dots, 0) . \tag{3.2}$$

A computation gives a Lie bracket of the form:

$$[W, Y](\theta, y) = (f(\theta, y), -\partial_\theta f(\theta, y)\cdot y_1, 0, \dots, 0) . \tag{3.3}$$

Thus $X - [W, Y]$ is in $\mathfrak{g}3$ which gives the sought result for $M=\mathbb{R}^n$.

When M is another manifold, we fix a locally finite covering $(U_i)_i$ by balls U_i. Using a partition of the unity, every $X \in \mathfrak{diff}_c^\infty(\mathbb{T} \times M)$ can be written as a sum:

$$X = \sum_i X_i \ . \tag{3.4}$$

where each X_i is supported by U_i. As X is compactly supported, the X_i are almost all null and thus the above sum is finite. As each U_i is diffeomorphic to $\mathbb{R}^n$, we can apply the case $M = \mathbb{R}^n$ which gives $Y_i, Z_i \in \mathfrak{g}_3$ and $W_i \in \mathfrak{g}_1$ all supported by $\mathbb{T} \times U_i$ such that $[W_i, Y_i] + Z_i = X_i$. We conclude by summing over i. □

Remark 3.4. We proved that $[\mathfrak{g}_3, \mathfrak{g}_1] + \mathfrak{g}_3 = \mathfrak{diff}_c^\infty(\mathbb{T} \times M)$.

We now have the tools to show:

Proof of Theorem F. Let $G \in \mathrm{Diff}_c^\infty(\mathbb{T}\times M)$ and let $(G_t)_{t\in[0,1]}$ be a compactly supported smooth path from $G_0 = \mathrm{id}$ to $G_1 = G$ in $\mathrm{Diff}_c^\infty(\mathbb{T} \times M)$. Derivatives $X_t := \partial_t G_t$ define a smooth family $X = (X_t)_t$ of vector fields all supported in a compact subset $K \subset V \setminus \partial V$. In particular, the time $1/N$-map F_i of the vector field $X_{i/N}$ is supported by K, and likewise for $F = F_{N-1} \circ \cdots \circ F_0$. By definition of the Whitney topology, it suffices to show that for every $r \geq 1$, when N is large, the map F is C^r-close to G to obtain that F is close to G in $\mathrm{Diff}_c^\infty(\mathbb{T} \times M)$.

Note that indeed each F_i is $O_{C^r}(1/N^2)$-close to $\tilde{F}_i := G_{(i+1)/N} \circ G_{i/N}^{-1}$ and so it holds:

$$G = \tilde{F}_{N-1} \circ \cdots \circ \tilde{F}_0 = F_{N-1} \circ \cdots \circ F_0 + O_{C^r}(1/N) = F + O_{C^r}(1/N) \ . \tag{3.5}$$

As each F_i belongs to Plug, and since Plug is a closed group by Proposition 2.13, it comes that F belongs to Plug and its limit G when $N \to \infty$ as well. □

Parametric counterpart. To prove Theorem F′ we will use the following parametric counterpart of Proposition 3.3:

Proposition 3.5. *The flow of every $\mathscr{P}$-family of vector fields is $\mathscr{P}$-pluggable:*

$$\mathfrak{plug}_{\mathscr{P}} = \mathfrak{diff}_c^\infty(\mathbb{T} \times M)_{\mathscr{P}} \ . \tag{3.6}$$

Proof. As in the proof of Proposition 3.3, it suffices to show that the Lie algebra generated by $\mathfrak{g}_{1\mathscr{P}}$ and $\mathfrak{g}_{3\mathscr{P}}$ equals $\mathfrak{plug}_{\mathscr{P}}$. We start with the case $M = \mathbb{R}^n$ and $\mathscr{P} = \mathbb{R}^d$. Let $X_{\mathscr{P}} = (X_p)_{p\in\mathscr{P}}$ be a $\mathscr{P}$-family in $\mathfrak{diff}_c^\infty(\mathbb{T} \times M)_{\mathscr{P}}$ and $f_p \in \mathfrak{diff}_c(\mathbb{T} \times M)$ be the $\mathbb{T}$-coordinate of X_p for each $p \in \mathscr{P}$. Let $W_{\mathscr{P}} \in \mathfrak{g}_{3\mathscr{P}}$ be:

$$W_{\mathscr{P}} := ((0, f_p, 0, \ldots, 0))_{p\in\mathscr{P}} \ .$$

With Y as defined in Eq. (3.2), it holds by Eq. (3.3):

$$[W_p, Y](\theta, y) = (f_p(\theta, y), -\partial_\theta f_p(\theta, y) \cdot y_1, 0, \ldots, 0))_{p\in\mathscr{P}} \ .$$

We set $Y_{\mathscr{P}} := (Y)_{p\in\mathscr{P}}\mathfrak{g}_{1\mathscr{P}}$. The latter computation gives:

$$[W_{\mathscr{P}}, Y_{\mathscr{P}}](\theta, y) = ((f_p(\theta, y), -\partial_\theta f_p(\theta, y) \cdot y_1, 0, \ldots, 0))_{p\in\mathscr{P}} \ . \tag{3.7}$$

Thus $X_{\mathscr{P}} - [W_{\mathscr{P}}, Y_{\mathscr{P}}]$ is in $\mathfrak{g}_{3\mathscr{P}}$. Now for the general case we conclude by using a partition of unity as before. □

It allows to conclude:

Proof of Theorem F′. Consider $G_{\mathscr{P}} \in \mathfrak{diff}_c^\infty(\mathbb{T} \times M)_{\mathscr{P}}$. As a corollary of Theorem F, for any $r \geq 0$, the map $\widehat{G_{\mathscr{P}}} \in \mathrm{Diff}_c^\infty(\mathbb{T} \times M \times \mathscr{P})$ can be approximated by a composition of flows of vector fields with a zero $\mathscr{P}$-coordinate. Thus the family $G_{\mathscr{P}}$ can be approximated arbitrarily close by a composition of compactly supported families of flows of vector fields in the C^r-norm for any $r \geq 0$. By Proposition 3.3, each of the families of flows belongs to $\mathrm{Plug}_{\mathscr{P}}$. Since $\mathrm{Plug}_{\mathscr{P}}$ is a group by Proposition 2.13 the above composition belongs to $\mathrm{Plug}_{\mathscr{P}}$ as well and it comes that $(G_p)_{p\in\mathscr{P}}$ is in $\mathrm{Plug}_{\mathscr{P}}$ by closure of $\mathrm{Plug}_{\mathscr{P}}$. □

3.2 Eigenvectors of the adjoint representation

Observe that the Lie algebra generated by $\mathfrak{g}_1$ and $\mathfrak{g}_2$ contains the vector fields that do not depend on the $\mathbb{T}$-coordinate. To obtain more elements in $\mathfrak{plug}$ we study the eigenvectors of operators of the form:

$$\mathrm{ad}_X : Y \in \mathfrak{diff}^\infty_c(\mathbb{T} \times M) \mapsto [X, Y] \in \mathfrak{diff}^\infty_c(\mathbb{T} \times M) \ ,$$

for $X \in \mathfrak{diff}^\infty_c(\mathbb{T} \times M)$.

In this subsection, we show that if Y is an eigenvector of ad_X for $X \in \mathfrak{plug}$ then Y is in $\mathfrak{plug}$ (see Proposition 3.7). This will enable to prove Proposition 3.1 in Section 3.3.

Definition 3.6. Let V be a manifold and let $\mathfrak{g}$ be a subspace of $\mathfrak{diff}^\infty_c(V)$. We denote:

$$\mathfrak{Eig}(\mathrm{ad}_{\mathfrak{g}}) := \{Y \in \mathfrak{diff}^\infty_c(V) : \exists X \in \mathfrak{g} \text{ such that } Y = [X, Y]\} \ .$$

The following is the key proposition enabling to construct new examples of pluggable flows:

Proposition 3.7. *If $Y \in \mathfrak{diff}^\infty_c(\mathbb{T} \times M)$ satisfies $\mathrm{ad}_X Y = Y$ for $X \in \mathfrak{plug}$ then Y is in $\mathfrak{plug}$:*

$$\mathfrak{Eig}(\mathrm{ad}_{\mathfrak{plug}}) \subset \mathfrak{plug} \ .$$

Proof. We recall that the *adjoint* of X by $f \in \mathrm{Diff}^\infty_c(V)$ is $\mathrm{Ad}_f X := Df \circ X \circ f^{-1}$. The following contains a key idea for the proof of the main theorem:

Lemma 3.8. *If Y satisfies $[X, Y] = Y$, then for every $s, t \in \mathbb{R}$ it holds:*

$$\mathrm{Fl}^t_Y = \mathrm{Fl}^{-s}_X \circ \mathrm{Fl}^{t \cdot e^{-s}}_Y \circ \mathrm{Fl}^s_X \ . \tag{3.8}$$

Proof. Let $Y_s := \mathrm{Ad}_{\mathrm{Fl}^s_X}(Y) := D\mathrm{Fl}^s_X \circ Y \circ \mathrm{Fl}^{-s}_X$. Observe that $X = \mathrm{Ad}_{\mathrm{Fl}^s_X}(X)$. Thus since $Y = [X, Y]$, it holds:

$$[X, Y_s] = [\mathrm{Ad}_{\mathrm{Fl}^s_X}(X), \mathrm{Ad}_{\mathrm{Fl}^s_X}(Y)] = \mathrm{Ad}_{\mathrm{Fl}^s_X}([X, Y]) = \mathrm{Ad}_{\mathrm{Fl}^s_X}(Y) = Y_s \ . \tag{3.9}$$

Also for every $s \in \mathbb{R}$, we have:

$$\partial_s Y_s = \partial_t (Y_{s+t})_{|t=0} = \partial_t \mathrm{Ad}_{\mathrm{Fl}^t_X}(Y_s)_{|t=0} = \partial_t (D\mathrm{Fl}^t_X \circ Y_s \circ \mathrm{Fl}^{-t}_X)_{|t=0} \tag{3.10}$$

and thus:

$$\partial_s Y_s = DX \circ Y_s - DY_s \circ X = -[X, Y_s] = -Y_s \ . \tag{3.11}$$

Consequently $\partial_s Y_s = -Y_s$ and thus $Y_s = e^{-s} \cdot Y$. It follows that $e^s \cdot Y = \mathrm{Ad}_{\mathrm{Fl}^s_X}(Y)$. After integration between 0 and $e^{-s} \cdot t$, we obtain the sought equality. □

We can now prove Proposition 3.7. Fix a neighborhood $\mathcal{N}$ of $\mathrm{id} \in \mathrm{Diff}^\infty(\mathbb{T} \times M)$ and let us show the existence of a plugin in $\mathcal{N}$ whose output is Fl^t_Y. Let $\mathcal{N}_c$ be the neighborhood of $\mathrm{id} \in \mathrm{Diff}^\infty_c(\mathbb{T} \times M)$ given by Lemma 2.10. Let $s \in \mathbb{R}$ be sufficiently large such that $\mathrm{Fl}^{e^{-s} \cdot t}_Y$ belongs to the neighborhood $\mathcal{N}_c$ of the identity. Then for any $k \geq 1$ large enough $\mathrm{Fl}^{e^{-s} \cdot t}_Y$ is the output of a plugin $g \in \mathcal{N}$ of step 2^{-k}. Since $X \in \mathfrak{plug}$, it holds $\mathrm{Fl}^s_X \in \mathrm{Plug}$ so for any $k \geq 1$ large enough Fl^s_X and Fl^{-s}_X are the outputs of plugins h and f in $\mathcal{N}$ of respective steps 2^{-k} and 2^{-k-1}. So for any large k the map Fl^t_Y is the output of the plugin $(h \star g) \star f$ with step 2^{-k-2}. Since this holds for any neighbourhood $\mathcal{N}$ of the identity the plugin $(h \star g) \star f$ can be taken arbitrary close to identity. Therefore for every t, there exists a plugin with output Fl^t_X arbitrarily close to identity for any small enough step. Hence X is in $\mathfrak{plug}$. □

A second main ingredient of the proof of the main theorem is the following of independent interest:

Proposition 3.9. *For every $T \in \mathfrak{diff}_c^\infty(V)$, there exist finite families $(X_i)_i, (Y_i)_i, (Z_i)_i$ of vector fields in $\mathfrak{diff}_c^\infty(V)$ such that:*

$$T = \sum_i [Y_i, Z_i] \quad \textit{and} \quad Y_i = [X_i, Y_i] \ .$$

The proof of this proposition and its parametric counterpart will occupy the full Section 3.3. Now note that the following is an isomorphism of Lie algebras:

$$\mathfrak{i}_2 : X \in \mathfrak{diff}_c^\infty(M) \mapsto (0, X) \in \mathfrak{g}_2 \ .$$

Thus, by applying this isomorphism to the image of sets involved in the statement of Proposition 3.9 for every $T \in \mathfrak{g}_2$, there exist finite families $(X_i)_i, (Y_i)_i, (Z_i)_i$ of vector fields in $\mathfrak{g}_2$ such that:

$$T = \sum_i [Y_i, Z_i] \quad \text{and} \quad Y_i = [X_i, Y_i] \ .$$

In other words, we proved:

Corollary 3.10. *Any element of $\mathfrak{g}_2$ can be written as a sum of Lie brackets of elements of $\mathfrak{g}_2$ with elements in $\mathfrak{Eig}(\mathrm{ad}_{\mathfrak{g}_2}) \cap \mathfrak{g}_2$.*

This corollary allows to deduce:

Proof of Proposition 3.1. Let $p_\theta : \mathbb{T} \times M \to \mathbb{T}$ be the projection on the $\mathbb{T}$-coordinate.

First note that by Fourier's theorem, the space $\mathfrak{g}_3$ is the closure of the vector space spanned by elements of the form:

$$\phi \circ p_\theta \cdot Y \quad \text{for } \phi \in C^\infty(\mathbb{T}, \mathbb{R}) \quad \text{and} \quad Y \in \mathfrak{g}_2 \ .$$

Since $\mathfrak{plug}$ is a closed vector space by Proposition 2.16, it suffices to show any such $\phi \circ p_\theta \cdot Y$ is in $\mathfrak{plug}$. To do so, we first start with the case where there exists $X \in \mathfrak{g}_2$ such that $Y = [X, Y]$, i.e. we assume that $Y \in \mathfrak{Eig}(\mathrm{ad}_{\mathfrak{g}_2})$. Since the $\mathbb{T}$-coordinate $p_\theta \circ X$ of X is zero, it follows:

$$[X, \phi \circ p_\theta \cdot Y] = \phi \circ p_\theta \cdot DY(X) - DX(\phi \circ p_\theta \cdot Y) = \phi \circ p_\theta \cdot [X, Y] = \phi \circ p_\theta \cdot Y \ . \tag{3.12}$$

Thus by Proposition 3.7 we have $\phi \circ p_\theta \cdot Y \in \mathfrak{plug}$ since $X \in \mathfrak{g}_2 \subset \mathfrak{plug}$. Now in the general case, for every $Y \in \mathfrak{g}_2$, by Corollary 3.10, there exist an integer $N \geq 1$, $Z_i \in \mathfrak{g}_2$ and $Y_i \in \mathfrak{Eig}(\mathrm{ad}_{\mathfrak{g}_2}) \cap \mathfrak{g}_2$ for any $1 \leq i \leq N$, such that:

$$Y = \sum_{1 \leq N} [Y_i, Z_i] \ . \tag{3.13}$$

By the first case for any $1 \leq i \leq N$ we have $\phi \circ p_\theta \cdot Y_i \in \mathfrak{plug}$. Since $\mathfrak{plug}$ is a Lie-algebra and each Z_i is in $\mathfrak{g}_2 \subset \mathfrak{plug}$, it immediately follows:

$$\phi \circ p_\theta \cdot Y = \sum_{1 \leq i \leq N} \phi \circ p_\theta \cdot [Y_i, Z_i] = \sum_{1 \leq i \leq N} [\phi \circ p_\theta \cdot Y_i, Z_i] \in \mathfrak{plug} \ . \tag{3.14}$$

□

Parametric counterpart. The following is the parametric counterpart of Proposition 3.7.

Proposition 3.11. *If $Y_{\mathscr{P}} \in \mathfrak{diff}_c^\infty(\mathbb{T} \times M)_{\mathscr{P}}$ satisfies $[X_{\mathscr{P}}, Y_{\mathscr{P}}] = Y_{\mathscr{P}}$ for $X_{\mathscr{P}} \in \mathfrak{plug}_{\mathscr{P}}$, then $Y_{\mathscr{P}} \in \mathfrak{plug}_{\mathscr{P}}$.*

Proof. Let $X_{\mathscr{P}} =: (X_p)_{p\in\mathscr{P}}$ and $Y_{\mathscr{P}} =: (Y_p)_{p\in\mathscr{P}}$. By Lemma 3.8, it holds:

$$\mathrm{Fl}^t_{Y_p} = \mathrm{Fl}^{-s}_{X_p} \circ \mathrm{Fl}^{t\cdot e^{-s}}_{Y_p} \circ \mathrm{Fl}^s_{X_p} \ , \tag{3.15}$$

for any $t, s \in \mathbb{R}$. For any neighborhood $\mathcal{N}$ of $\mathrm{id} \in \mathrm{Diff}^\infty(\mathbb{T}\times M)_{\mathscr{P}}$, we denote $\mathcal{N}_c$ the neighborhood of $\mathrm{id} \in \mathrm{Diff}^\infty_c(\mathbb{T}\times M)_{\mathscr{P}}$ given by Lemma 2.14. For all $t \in \mathbb{R}$ and for s large enough $(\mathrm{Fl}^{t\cdot e^{-s}}_{Y_p})_{p\in\mathscr{P}} \subset \mathcal{N}_c$ and thus it is the output of a $\mathscr{P}$-plugin in $\mathcal{N}$ of any small step. We now regard the $\star$ product of the latter plugins with $\mathscr{P}$-plugins of $(\mathrm{Fl}^s_{X_p})_{p\in\mathscr{P}}$ and $(\mathrm{Fl}^{-s}_{X_p})_{p\in\mathscr{P}}$ to obtain a $\mathscr{P}$-plugin with output $(\mathrm{Fl}^t_{Y_p})_{p\in\mathscr{P}}$ of any small step. Moreover, by construction, this $\mathscr{P}$-plugin can be taken arbitrarily close to identity family, which ends the proof. □

For $\mathfrak{g}_{\mathscr{P}} \subset \mathfrak{diff}^\infty_c(V)_{\mathscr{P}}$, we denote:

$$\mathfrak{Eig}(\mathrm{ad}_{\mathfrak{g}_{\mathscr{P}}}) := \{Y_{\mathscr{P}} \in \mathfrak{diff}^\infty_c(V)_{\mathscr{P}} : \exists X_{\mathscr{P}} \in \mathfrak{g}_{\mathscr{P}} \text{ such that } Y_{\mathscr{P}} = [X_{\mathscr{P}}, Y_{\mathscr{P}}]\} \ .$$

The following parametric counterpart of Proposition 3.9 holds:

Proposition 3.12. *For every $T_{\mathscr{P}} \in \mathfrak{diff}^\infty_c(V)_{\mathscr{P}}$, there exist finite families $(X_{i\mathscr{P}})_i, (Y_{i\mathscr{P}})_i, (Z_{i\mathscr{P}})_i$ of vector fields in $\mathfrak{diff}^\infty_c(V)_{\mathscr{P}}$ such that:*

$$T_{\mathscr{P}} = \sum_i [Y_{i\mathscr{P}}, Z_{i\mathscr{P}}] \quad \textit{and} \quad Y_{i\mathscr{P}} = [X_{i\mathscr{P}}, Y_{i\mathscr{P}}] \ .$$

Using the isomorphism:

$$X_{\mathscr{P}} \in \mathfrak{diff}^\infty_c(M)_{\mathscr{P}} \mapsto (0, X_p)_{p\in\mathscr{P}} \in \mathfrak{g}_{2\mathscr{P}}$$

leads as before to:

Corollary 3.13. *Any element of $\mathfrak{g}_{2\mathscr{P}}$ can be written as a sum of Lie brackets of elements of $\mathfrak{g}_{2\mathscr{P}}$ with elements in $\mathfrak{Eig}(\mathrm{ad}_{\mathfrak{g}_{2\mathscr{P}}}) \cap \mathfrak{g}_{2\mathscr{P}}$.*

The latter allows to deduce:

Proof of Proposition 3.2. By the Fourier theorem, the space $\mathfrak{g}_{3\mathscr{P}}$ is the closure in $\mathfrak{diff}^\infty_c(\mathbb{T}\times M)$ of the vector space spanned by vector fields of the form $(\phi\circ p_\theta \cdot Y_p)_{p\in\mathscr{P}}$ with $\phi \in C^\infty(\mathbb{T},\mathbb{R})$ and $(Y_p)_{p\in\mathscr{P}} \in \mathfrak{g}_{2\mathscr{P}}$. Since $\mathfrak{plug}_{\mathscr{P}}$ is a closed vector space by Proposition 2.18, it suffices then to show that $(\phi\circ p_\theta \cdot Y_p)_{p\in\mathscr{P}}$ is in $\mathfrak{plug}_{\mathscr{P}}$. To do so, we first start with the case where there exists $X_{\mathscr{P}} \in \mathfrak{g}_{2\mathscr{P}}$ such that $Y_{\mathscr{P}} = [X_{\mathscr{P}}, Y_{\mathscr{P}}]$. By Eqs. (3.12) and (3.14) of the proof of Proposition 3.1, for each $p \in \mathscr{P}$, it holds $\phi\circ p_\theta \cdot Y_p = [X_p, \phi\circ p_\theta \cdot Y_p]$. And thus the family $(\phi\circ p_\theta \cdot Y_p)_{p\in\mathscr{P}}$ is in $\mathfrak{plug}_{\mathscr{P}}$ by Proposition 3.11.

Now in the general, by the latter Corollary 3.13 we can decompose:

$$Y_{\mathscr{P}} = \sum [Y_{i\mathscr{P}}, Z_{i\mathscr{P}}] \ , \tag{3.16}$$

with $Y_{i\mathscr{P}} \in \mathfrak{Eig}(\mathfrak{g}_{2\mathscr{P}}) \cap \mathfrak{g}_{2\mathscr{P}}$ and $Z_{i\mathscr{P}} \in \mathfrak{g}_{2\mathscr{P}}$. Thus Eq. (3.14) applied for each $p \in \mathscr{P}$ leads to:

$$(\phi\circ p_\theta \cdot Y_p)_{p\in\mathscr{P}} = \sum [(\phi\circ p_\theta \cdot Y_{ip})_{p\in\mathscr{P}}, Z_{i\mathscr{P}}] \ , \tag{3.17}$$

where $(Y_{ip})_{p\in\mathscr{P}} = Y_{i\mathscr{P}}$. Since $Z_{i\mathscr{P}} \in \mathfrak{g}_{2\mathscr{P}} \subset \mathfrak{plug}_{\mathscr{P}}$ and $(\phi\circ p_\theta \cdot Y_{ip})_{p\in\mathscr{P}} \in \mathfrak{plug}_{\mathscr{P}}$ by the first case, it follows that $(\phi\circ p_\theta \cdot Y_p)_{p\in\mathscr{P}}$ lies in the sub Lie algebra $\mathfrak{plug}_{\mathscr{P}}$ as well. □

3.3 Decomposition of vector fields

To prove the Theorem F, it remains only to show Proposition 3.9. We first prove this proposition in the case $M = \mathbb{R}^n$, which we will deduce from the case $M = \mathbb{R}$ and a parametric version thereof. This whole section is dedicated to this proof. For a manifold M, let us denote:

$$\mathfrak{Eig}(M) := \{Y \in \mathfrak{diff}_c^\infty(M) : \exists X \in \mathfrak{diff}_c^\infty(M) \text{ such that } \mathrm{ad}_X Y = [X, Y] = Y\} .$$

We can rephrase Proposition 3.9 as:

Proposition 3.14. *Every vector field in $T \in \mathfrak{diff}_c^\infty(V)$ is a finite sum of vector fields of the form $[Y_i, Z_i]$ with $Y_i \in \mathfrak{Eig}(M)$. In other words:*

$$[\mathfrak{Eig}(M), \mathfrak{diff}_c^\infty(M)] = \mathfrak{diff}_c^\infty(M).$$

Two key ingredients of the proof of this proposition are the following observations:

Fact 3.15. *The vector field $Y : y \mapsto 1$ on $\mathbb{R}$ satisfies $[X, Y] = Y$ with $X : y \in \mathbb{R} \mapsto -y$.*

Fact 3.16. *For any diffeomorphism $\psi\colon W \to V$ between manifolds and any vector fields $X, Y \in \mathfrak{diff}_c^\infty(V)$ satisfying $[X, Y] = Y$, it holds $[\psi^* X, \psi^* Y] = \psi^* Y$.*

An important consequence of the latter fact is:

Fact 3.17. *For any diffeomorphism $\psi\colon W \to V$ from a manifold W into V, if $X \in [\mathfrak{Eig}(V), \mathfrak{diff}_c^\infty(V)]$, then $\psi^* X \in [\mathfrak{Eig}(W), \mathfrak{diff}_c^\infty(W)]$.*

Let $\mathcal{Q}$ be a manifold. We define:

$$\mathfrak{Eig}(\mathbb{R})_{\mathcal{Q}} = \Big\{Y_{\mathcal{Q}} \in \mathfrak{diff}_c^\infty(\mathbb{R})_{\mathcal{Q}} : \text{ there exists a family } X_{\mathcal{Q}} \in \mathfrak{diff}_c^\infty(\mathbb{R})_{\mathcal{Q}} \text{ such that } [X_{\mathcal{Q}}, Y_{\mathcal{Q}}] = Y_{\mathcal{Q}}\Big\}.$$

We can prove Proposition 3.14 in the case $M = \mathbb{R}$:

Lemma 3.18. *We have:*

$$[\mathfrak{Eig}(\mathbb{R}), \mathfrak{diff}_c^\infty(\mathbb{R})] = \mathfrak{diff}_c^\infty(\mathbb{R}) .$$

Moreover for any manifold $\mathcal{Q}$, every $T_{\mathcal{Q}} \in \mathfrak{diff}_c^\infty(\mathbb{R})_{\mathcal{Q}}$ satisfies $T_{\mathcal{Q}} = [Y_{\mathcal{Q}}, Z_{\mathcal{Q}}]$ for some $Y_{\mathcal{Q}} \in \mathfrak{Eig}(\mathbb{R})_{\mathcal{Q}}$ and $Z_{\mathcal{Q}} \in \mathfrak{diff}_c^\infty(\mathbb{R})_{\mathcal{Q}}$.

Proof. We first give an intuitive idea of the proof using Fact 3.15. First note that for every $T \in \mathfrak{diff}_c^\infty(\mathbb{R})$, there exist $X, Y, Z \in \mathfrak{diff}^\infty(\mathbb{R})$ such that $Y = [X, Y]$ and $T = [Y, Z]$. Indeed it suffices to take $X\colon y \mapsto -y$, $Y\colon y \mapsto 1$ and $Z\colon y \mapsto \int_{-\infty}^y T(t)\,dt$. This almost proves the first assertion of the Lemma. To have exactly the sought result we shall modify X, Y and Z to make them compactly supported. Let $\widetilde{T} \in \mathfrak{diff}_c^\infty(\mathbb{R})$. Take intervals $[-A, A] \subset (-a, a)$ containing its support. Consider the map $\psi\colon y \mapsto y \cdot e^{e^{(a^2-y^2)^{-1}}}$ from $(-a, a)$ to $\mathbb{R}$. We compute:

$$\partial_y \psi(y) = \phi(y) \cdot e^{e^{(a^2-y^2)^{-1}}} , \quad \text{with } \phi(y) = 1 + 2y^2 \cdot (a^2 - y^2)^{-2} e^{(a^2-y^2)^{-1}} . \tag{3.18}$$

Thus the map ψ is a diffeomorphism. This allows to consider the push-forward:

$$T = \psi_* \widetilde{T} = D\psi \circ \widetilde{T} \circ \psi^{-1} . \tag{3.19}$$

Now we define as above:

$$X\colon y \mapsto -y , \quad Y\colon y \mapsto 1 \quad \text{and} \quad Z\colon y \mapsto \int_{-\infty}^y T(t)\,dt . \tag{3.20}$$

Note that $\tilde{T} = \psi^* T$ and put $\widetilde{X} = \psi^* X$ and $\widetilde{Y} = \psi^* Y$. Then by Fact 3.16, it holds:

$$\widetilde{Y} = [\widetilde{X}, \widetilde{Y}] \quad \text{and} \quad [\widetilde{Y}, \widetilde{Z}] = \widetilde{T} . \tag{3.21}$$

Let us show that these pulled-back vector fields on $(-a,a)$ extend smoothly by 0 to $\mathbb{R}$. For any vector field $S \in \mathfrak{diff}_c^\infty(\mathbb{R})$, we have $\psi^* S = (\partial_y \psi)^{-1} \cdot S \circ \psi = \frac{S \circ \psi}{\partial_y \psi}$. It follows:

$$\widetilde{X}(y) = \psi^* X(y) = \tfrac{-\psi(y)}{\partial_y \psi(y)} = \tfrac{-y}{\phi(y)} \quad \text{and} \quad \widetilde{Y}(y) = \psi^* Y(y) = \tfrac{1}{\partial_y \psi(y)}, \tag{3.22}$$

Observe that $\phi(y)$ grows exponentially fast to $+\infty$ when $|y| \to a$. Thus $\widetilde{X}$ and $\widetilde{Y}$ have all their derivatives tending to 0 as y tends to $\pm a$. So they extend smoothly by 0 to vector fields in $\mathfrak{diff}_c^\infty(\mathbb{R})$. Also note that $Z(y) = 0$ for $y \leq \psi(-A)$ and $Z(y) = \int_{\mathbb{R}} T(t)\, dt$ for $y \geq \psi(A)$. Thus:

$$\widetilde{Z}(y) = \psi^* Z(y) = 0 \quad \text{for } -a < y \leq -A \quad \text{and} \quad \widetilde{Z}(y) = \frac{1}{\partial_y \psi(y)} \cdot \int_{\mathbb{R}} T(t) \cdot dt \quad \text{for } A \leq y < a. \tag{3.23}$$

Likewise $\widetilde{Z}$ extends smoothly by 0 to form a vector field in $\mathfrak{diff}_c^\infty(\mathbb{R})$. As $\widetilde{Y} = [\widetilde{X}, \widetilde{Y}]$ and $[\widetilde{Y}, \widetilde{Z}] = \widetilde{T}$, this proves the first assertion of the lemma.

For the parametric assertion, observe that ψ, $\widetilde{X}$ and $\widetilde{Y}$ depends only on the segment $[-a,a]$. Hence given $\widetilde{T}_{\mathcal{Q}} = (\widetilde{T}_q)_{q \in \mathcal{Q}} \in \mathfrak{diff}_c^\infty(\mathbb{R})_{\mathcal{Q}}$, we set $[-A,A] \subset (-a,a)$ containing the supports of all $\widetilde{T}_q$ and define ψ, $\widetilde{X}$ and $\widetilde{Y}$ as above. Then we observe that $\widetilde{Z}_q = \psi^* \int T_q$ depends smoothly on q and define a family $\widetilde{Z}_{\mathcal{Q}} \in \mathfrak{diff}_c^\infty(\mathbb{R})_{\mathcal{Q}}$ which satisfies the sough equalities with $\widetilde{T}_{\mathcal{Q}}$ and the constant families $(\widetilde{X})_{q \in \mathcal{Q}}$ and $(\widetilde{Y})_{q \in \mathcal{Q}}$. □

We are going to use the parametric assertion of the latter lemma to obtain:

Lemma 3.19. *We have* $[\mathfrak{Eig}(\mathbb{R}^n), \mathfrak{diff}_c^\infty(\mathbb{R}^n)] = \mathfrak{diff}_c^\infty(\mathbb{R}^n)$.

Proof. Lemma 3.18 corresponds to the case $n = 1$. For $n \geq 2$, given $T \in \mathfrak{diff}_c^\infty(\mathbb{R}^n)$, we write its components as $T = (T_1, \dots, T_n)$. By linearity of the condition, it suffices to show that each vector field $(0, \dots, 0, T_i, 0, \dots, 0)$ is in $[\mathfrak{Eig}(\mathbb{R}^n), \mathfrak{diff}_c^\infty(\mathbb{R}^n)]$. Using an adjunction by a permutation of the coordinates and Fact 3.17, its suffices to show that each $(T_i, 0, \dots, 0)$ is in $[\mathfrak{Eig}(\mathbb{R}^n), \mathfrak{diff}_c^\infty(\mathbb{R}^n)]$. In other words, it suffices to prove that the following subalgebra $\mathfrak{h}(\mathbb{R}^n)$ of $\mathfrak{diff}_c^\infty(\mathbb{R}^n)$ is included in $[\mathfrak{Eig}(\mathbb{R}^n), \mathfrak{diff}_c^\infty(\mathbb{R}^n)]$:

$$\mathfrak{h}(\mathbb{R}^n) := \{y \in \mathbb{R}^n \mapsto (h(y), 0, \dots, 0) : h \in C_c^\infty(\mathbb{R}^n, \mathbb{R})\} .$$

To this end, note that $X_{\mathbb{R}^{n-1}} \in \mathfrak{diff}_c^\infty(\mathbb{R})_{\mathbb{R}^{n-1}} \mapsto \widehat{X_{\mathbb{R}^{n-1}}} \in \mathfrak{h}(\mathbb{R}^n)$ is an isomorphism of Lie algebras. By Lemma 3.18 with $\mathcal{Q} = \mathbb{R}^{n-1}$, we have:

$$\mathfrak{h}(\mathbb{R}^n) = [\widehat{\mathfrak{Eig}(\mathbb{R})}_{\mathcal{Q}}, \mathfrak{h}(\mathbb{R}^n)] .$$

Finally we note that $\widehat{\mathfrak{Eig}(\mathbb{R})}_{\mathcal{Q}}$ is formed by vector fields of the form $\widehat{Y_{\mathcal{Q}}}$ such that $Y_{\mathcal{Q}} = [X_{\mathcal{Q}}, Y_{\mathcal{Q}}]$ for $X_{\mathcal{Q}} \in \mathfrak{diff}_c^\infty(\mathbb{R})_{\mathcal{Q}}$. Thus $\widehat{Y_{\mathcal{Q}}} = [\widehat{X_{\mathcal{Q}}}, \widehat{Y_{\mathcal{Q}}}]$, this proves that $\mathfrak{h}(\mathbb{R}^n) \subset [\mathfrak{Eig}(\mathbb{R}^n), \mathfrak{h}(\mathbb{R}^n)]$.

□

We can now treat the general case:

Proof of Proposition 3.14. Let $T \in \mathfrak{diff}_c^\infty(M)$. Then it decomposes in a finite sum $T = \sum_i T_i$ where each T_i is compactly supported in an open set U_i which is diffeomorphic to $\mathbb{R}^n$ via a map $\psi_i \colon U_i \to \mathbb{R}^n$. Consider the push forward $\psi_{i*} T_i \in \mathfrak{diff}_c^\infty(\mathbb{R}^n)$. By Lemma 3.19 the field $\psi_{i*} T_i | U_i$ belongs to $[\mathfrak{Eig}(\mathbb{R}^n), \mathfrak{diff}_c^\infty(\mathbb{R}^n)]$. Thus by Lemma 3.18, it holds $T_i | U_i \in [\mathfrak{Eig}(U_i), \mathfrak{diff}_c^\infty(U_i)]$. This means that $T_i = \sum_{\text{Finite}} [Y_j, Z_j]$ with $Y_j = [X_j, Y_j]$ for some $X_j, Y_j, Z_j \in \mathfrak{diff}_c^\infty(\mathbb{R}^n)$. Extending all these vector fields by 0, we obtain that T_i belongs to $[\mathfrak{Eig}(M), \mathfrak{diff}_c^\infty(M)]$. So does $T = \sum_i T_i$. □

Parametric counterpart. We set:

$$\mathfrak{Eig}(M)_{\mathscr{P}} := \{Y_{\mathscr{P}} \in \mathfrak{diff}_c^\infty(M)_{\mathscr{P}} : Y_{\mathscr{P}} = [X_{\mathscr{P}}, Y_{\mathscr{P}}] \text{ with } X_{\mathscr{P}} \in \mathfrak{diff}_c^\infty(M)_{\mathscr{P}}\} .$$

We shall prove Proposition 3.12 that we rephrase as:

Proposition 3.20. *It holds:*

$$[\mathfrak{Eig}(M)_{\mathscr{P}}, \mathfrak{diff}_c^\infty(M)_{\mathscr{P}}] = \mathfrak{diff}_c^\infty(M)_{\mathscr{P}}.$$

Similarly to the proof of Proposition 3.14, Proposition 3.20 is an easy consequence of the following:

Lemma 3.21. *We have* $[\mathfrak{Eig}(\mathbb{R}^n)_{\mathscr{P}}, \mathfrak{diff}_c^\infty(\mathbb{R}^n)_{\mathscr{P}}] = \mathfrak{diff}_c^\infty(\mathbb{R}^n)_{\mathscr{P}}$.

Proof. As in the proof of Proposition 3.14, we only need to prove that the following Lie subalgebra $\mathfrak{h}(\mathbb{R}^n)_{\mathscr{P}}$ of $\mathfrak{diff}_c^\infty(\mathbb{R}^n)_{\mathscr{P}}$ is included in $[\mathfrak{Eig}(\mathbb{R}^n)_{\mathscr{P}}, \mathfrak{diff}_c^\infty(\mathbb{R}^n)_{\mathscr{P}}]$:

$$\mathfrak{h}(\mathbb{R}^n)_{\mathscr{P}} := \{((h_p, 0, \dots, 0))_{p\in\mathscr{P}} : h_{\mathscr{P}} = (h_p)_{p\in\mathscr{P}} \in C_c^\infty(\mathbb{R}^n, \mathbb{R})_{\mathscr{P}}\} .$$

To this end we proceed as in Lemma 3.19, by using the isomorphism of Lie algebra:

$$X_{\mathbb{R}^{n-1}\times\mathscr{P}} \in \mathfrak{diff}_c^\infty(\mathbb{R})_{\mathbb{R}^{n-1}\times\mathscr{P}} \mapsto Y_{\mathscr{P}} \in \mathfrak{h}(\mathbb{R}^n)_{\mathscr{P}} \text{ such that } \widehat{X_{\mathbb{R}^{n-1}\times\mathscr{P}}} = \widehat{Y_{\mathscr{P}}} ,$$

and using Lemma 3.18. □

A Smoothness of outputs

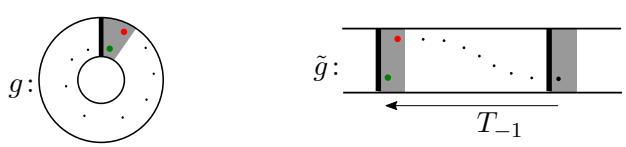
The proof of Proposition 1.5 stating that the output of a plugin is necessarily smooth is similar to the classical renormalization performed by Douady-Ghys [Dou87], Yoccoz [Yoc95a] and Shilnikov-Turaev [ST00].

Proof of Proposition 1.5. Let g be a plugin with step σ. Let $\pi : \tilde{\mathbb{T}} \times M := \mathbb{R} \times M \to \mathbb{T} \times M =$ be the canonical cover. Let $\tilde{g}$ be a lifting of g such that $\tilde{g}(0, y) = (\sigma, y)$ for every $y \in M$.

Fact A.1. *The action* $\phi : (k, z) \in \mathbb{Z}\times\tilde{\mathbb{T}}\times M \mapsto \tilde{g}^k(z) \in \tilde{\mathbb{T}}\times M$ *is free, proper and discontinuous.*

Proof. The action is free since no point of $\Delta+\mathbb{Z}$ is fixed by $\tilde{g}$ nor in its complement (every point must come back to $\Delta + \mathbb{Z}$). It is discontinuous since any $x \in \mathbb{R} \times M$ has its orbit which equals the one of a certain $z \in [k, k + \sigma) \times M$ for some $k \in \mathbb{Z}$ by Definition 1.1.(iii), and the orbit of z is discrete by Definition 1.1.(i). Finally the action is proper since τ is bounded by some $N \geq 1$ by Definition 1.1.(ii), and so any $\tilde{g}^N(\theta, y)$ has its $\mathbb{R}$-coordinate greater than $\theta + 1$. □

Thus the quotient $C := \tilde{\mathbb{T}} \times M/\phi$ is a manifold. As $\tilde{g}$ sends the left hand side of Δ to its right hand side, the image of Δ by the group action is both open and close, hence equal to the connected set C. Therefore, $\Delta_\sigma = [0, \sigma) \times M$ is a fundamental domain of this group action. Also the rescaling map $H_\sigma : \Delta_\sigma \to \mathbb{T} \times M$ induces a diffeomorphism between C and $\mathbb{T} \times M$.



Now observe that $\tilde{T} : (\theta, y) \mapsto (\theta + 1, y)$ and $\tilde{g}$ commutes: $\tilde{T} \circ \tilde{g} = \tilde{g} \circ \tilde{T}$. Thus $\tilde{T}$ defines a smooth diffeomorphism T on C. To determine T, we (abusively) identify Δ_σ to a subset of both $\tilde{\mathbb{T}} \times M$ and $\mathbb{T} \times M$. Given $x \in \Delta_\sigma$, the point $\tilde{T}(x) \in [1, 1 + \sigma) \times M$ is equivalent to the point $y \in \Delta_\sigma$ such that there exists $k \geq 0$ satisfying $\tilde{g}^k(y) = \tilde{T}(x)$. Note that $k = \tau(y)$ and

so $\tilde{T}(x) = \tilde{g}^\tau(y)$. Composing by π, we obtain that $x = g^\tau(y)$. Hence T is equal to the inverse of the action of g^τ on C. Therefore G is a diffeomorphism. Observe that this construction depends continuously on g and so the output depends continuously on the plugin. As the space of plugins is connected, it comes that the space of output is connected to id by Example 1.4. $\square$

B Lie algebras associated to closed subgroups

This section is dedicated to show Proposition 2.16 which states that $\mathfrak{plug}$ is a closed Lie algebra. We prove this using general arguments on closed subgroups of $\mathrm{Diff}^\infty_c(V)$ where V is a manifold.

Proposition B.1. *For every closed subgroup $\mathbb{G} \subset \mathrm{Diff}^\infty_c(V)$, the following is a closed Lie subalgebra of $\mathfrak{diff}^\infty_c(V)$:*

$$\mathfrak{g} := \{X \in \mathfrak{diff}^\infty_c(V) : \mathrm{Fl}^t_X \in \mathbb{G},\ \forall t \in \mathbb{R}\}.$$

We immediately deduce the result of Proposition 2.16 by applying the latter proposition with $V = \mathbb{T} \times M$ and $\mathbb{G} = \mathrm{Plug}$.

Proof of Proposition B.1. $\underline{\mathfrak{g} \text{ is a vector space.}}$ First note that if $X \in \mathbb{G}$ and $\lambda \in \mathbb{R}$, it holds $\lambda \cdot X \in \mathfrak{g}$. Now for $X, Y \in \mathfrak{g}$, for any large integer N and $r \geq 1$, observe that $\mathrm{Fl}^{1/N}_{X+Y} = \mathrm{Fl}^{1/N}_X \circ \mathrm{Fl}^{1/N}_Y + O(N^{-2})$ for the C^r-norm. Thus we have:

$$Fl^1_{X+Y} = \left(Fl^{1/N}_X \circ \mathrm{Fl}^{1/N}_Y\right)^N + O_{C^r}(N^{-1}),$$

and the supports of $\left(Fl^{1/N}_X \circ \mathrm{Fl}^{1/N}_Y\right)^N$ are included in the union of those of X and Y. Thus Fl^1_{X+Y} is the limit of $\left(Fl^{1/N}_X \circ \mathrm{Fl}^{1/N}_Y\right)^N$ when $N \to \infty$ in the topology of $\mathrm{Diff}^\infty_c(V)$. As $\mathbb{G}$ is a group, the map $(Fl^{1/N}_X \circ \mathrm{Fl}^{1/N}_Y)^N$ belongs to $\mathbb{G}$, and since $\mathbb{G}$ is closed the map Fl^1_{X+Y} also belongs to $\mathbb{G}$. Also for every $t \in \mathbb{R}$, by replacing X, Y by (tX, tY), we obtain that $Fl^t_{X+Y} = Fl^1_{tX+tY}$ belongs to $\mathbb{G}$.

$\underline{\mathfrak{g} \text{ is a Lie algebra.}}$ For $X, Y \in \mathfrak{g}$ and $r \geq 1$, we have for the C^r-norm:

$$\mathrm{Fl}^{\tau^2}_{[X,Y]} = [\mathrm{Fl}^\tau_X, \mathrm{Fl}^\tau_Y] + O(\tau^3)$$

Thus by taking $\tau^2 = 1/N$ small we have:

$$\mathrm{Fl}^1_{[X,Y]} = \left([\mathrm{Fl}^{\sqrt{N}}_X, \mathrm{Fl}^{\sqrt{N}}_Y]\right)^N + O(\sqrt{N}^{-1})\ .$$

So $\mathrm{Fl}^1_{[X,Y]}$ is in $\mathbb{G}$. Similarly, we have for any t that $\mathrm{Fl}^1_{[tX,Y]} \in \mathbb{G}$, and so $\mathrm{Fl}^t_{[X,Y]} = \mathrm{Fl}^1_{[tX,Y]} \in \mathbb{G}$.

$\underline{\mathfrak{g} \text{ is closed.}}$ As for every $t \geq 0$, the map $\mathrm{Fl}^t : X \in \mathrm{Diff}^\infty_c(V) \mapsto \mathrm{Fl}^t_X$ is continuous and $\mathbb{G}$ is closed, the set $\{X \in \mathfrak{diff}^\infty_c(V) : \mathrm{Fl}^t_X \in \mathbb{G}\}$ is closed. Thus the intersection $\mathfrak{g}$ of the latter sets for all t is closed. $\square$

To state the parameteric counterpart of the latter proposition, given a manifold $\mathscr{P}$, we define:

$$\mathrm{Fl}^t : X_{\mathscr{P}} = (X_p)_{p \in \mathscr{P}} \in \mathfrak{diff}^\infty_c(V)_{\mathscr{P}} \mapsto (\mathrm{Fl}^t_{X_p})_{p \in \mathscr{P}} \in \mathrm{Diff}^\infty_c(V)_{\mathscr{P}}\ .$$

Note that the following diagram commutes:

$$\begin{array}{ccccc} & & \mathfrak{inc} & & \\ \mathrm{Fl}^t & \mathfrak{diff}^\infty_c(V)_{\mathscr{P}} & \hookrightarrow & \mathfrak{diff}^\infty_c(V \times \mathscr{P}) & \\ & \downarrow & & \downarrow & \mathrm{Fl}^t \\ & \mathrm{Diff}^\infty_c(V)_{\mathscr{P}} & \hookrightarrow & \mathrm{Diff}^\infty_c(V \times \mathscr{P}) & \\ & & inc & & \end{array}$$

where $\mathfrak{inc}(X_{\mathscr{P}}) := \widehat{X_{\mathscr{P}}}$ and $inc(f_{\mathscr{P}}) := \widehat{f_{\mathscr{P}}}$.

Corollary B.2. *For every closed subgroup $\mathbb{G}_{\mathscr{P}} \subset \mathrm{Diff}_c^\infty(V)_{\mathscr{P}}$, the following is a closed Lie subalgebra of $\mathfrak{diff}_c^\infty(V)_{\mathscr{P}}$:*

$$\mathfrak{g}_{\mathscr{P}} := \{X_{\mathscr{P}} \in \mathfrak{diff}_c^\infty(V)_{\mathscr{P}} : \mathrm{Fl}_X^t \in \mathbb{G},\ \forall t \in \mathbb{R}\}.$$

Proof. First note that $inc(\mathbb{G}_{\mathscr{P}})$ is a closed Lie sub-group of $\mathrm{Diff}_c^\infty(V \times \mathscr{P})$. Hence it define via Proposition B.1 a closed Lie algebra $\widehat{\mathfrak{g}_{\mathscr{P}}}$. By commutativity of the diagram, we have:

$$\mathfrak{inc}(\mathfrak{g}_{\mathscr{P}}) := (\widehat{\mathfrak{g}_{\mathscr{P}}})\ .$$

Hence $\mathfrak{g}_{\mathscr{P}}$ is a closed Lie subalgebra of $\mathfrak{diff}_c^\infty(V)_{\mathscr{P}}$. □

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