



**HAL**  
open science

# L1-Gradient Flow of Convex Functionals

Antonin Chambolle, Matteo Novaga

► **To cite this version:**

Antonin Chambolle, Matteo Novaga. L1-Gradient Flow of Convex Functionals. SIAM Journal on Mathematical Analysis, 2024, 56 (5), pp.5747-5781. 10.1137/22M1527556 . hal-03805962v2

**HAL Id: hal-03805962**

**<https://hal.science/hal-03805962v2>**

Submitted on 12 Oct 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# $L^1$ -GRADIENT FLOW OF CONVEX FUNCTIONALS

ANTONIN CHAMBOLLE AND MATTEO NOVAGA

ABSTRACT. We are interested in the gradient flow of a general first order convex functional with respect to the  $L^1$ -topology. By means of an implicit minimization scheme, we show existence of a global limit solution, which satisfies an energy-dissipation estimate, and solves a non-linear and non-local gradient flow equation, under the assumption of strong convexity of the energy. Under a monotonicity assumption we can also prove uniqueness of the limit solution, even though this remains an open question in full generality. We also consider a geometric evolution corresponding to the  $L^1$ -gradient flow of the anisotropic perimeter. When the initial set is convex, we show that the limit solution is monotone for the inclusion, convex and unique until it reaches the Cheeger set of the initial datum. Eventually, we show with some examples that uniqueness cannot be expected in general in the geometric case.

## CONTENTS

1. Introduction	2
2. Existence of solutions	3
2.1. Minimizing movements	3
2.2. Euler-Lagrange equation	5
2.3. Estimate of the time derivative	6
2.4. Dissipation estimate	8
2.5. Strongly convex case	10
2.6. Minimal surface energy	13
3. Monotone solutions	14
4. The Dirichlet energy	17
4.1. Uniqueness	18
4.2. Energy decay estimate	19
5. Gradient flow of anisotropic perimeters	20
5.1. Outward minimizing case	22
5.2. Convex case	23
Appendix A. Convex functions of gradients	26
A.1. Convex function of measures	26
A.2. Convex functions of gradients	29
A.3. The Dirichlet case	32
References	33

## 1. INTRODUCTION

We consider the functional

$$\Phi(u) := \int_{\Omega} F(Du) \quad u \in BV(\Omega),$$

where  $\Omega$  is a bounded, connected, open subset of  $\mathbb{R}^d$ , and  $F : \mathbb{R}^d \rightarrow [0, +\infty]$  is a convex function with  $F(\xi) \geq c|\xi| - c'$  for some  $c > 0$ ,  $c' \geq 0$  and  $F(0) = 0$ . Here,  $F(Du)$  is understood in the sense of the celebrated paper of Demengel and Temam [16]: when the recession function  $F^\infty$  (see (A.1)) of  $F$  is not infinite,

$$F(Du) = F(\nabla u) dx + F^\infty \left( \frac{D^s u}{|D^s u|} \right) |D^s u|,$$

where  $Du = \nabla u(x) dx + D^s u$  is the Radon-Nikodým decomposition of  $Du$  with respect to the Lebesgue measure. If  $F$  is superlinear and hence  $F^\infty \equiv +\infty$ , then  $\Phi(u) < +\infty$  only if  $u \in W^{1,1}(\Omega)$  and the singular part vanishes in the above formula (see Appendix A for details).

We are interested in the gradient flow of  $\Phi$  with respect to the  $L^1(\Omega)$ -topology, with either homogeneous Neumann, or Dirichlet boundary conditions; in the latter case the functional has to be relaxed, with an appropriate boundary integral, if the function  $F$  has linear growth, see Section A.3.

In order to show existence of a gradient flow, we follow the general approach in [14] (see also the comprehensive reference [4]), which is known as the *minimizing movement scheme* and applies to functionals on metric spaces, under general assumptions. However, most of the theory developed in [4] does not apply to our setting, since the Banach space  $L^1(\Omega)$  does not satisfy the Radon-Nikodým property (see [4, Remark 1.4.6]). In particular, we cannot derive uniqueness of gradient flow solution from general results, and we are able to prove it only in some special cases.

For this reason, there are few results in the literature concerning  $L^1$ -gradient flows. In [13] the author considers the  $L^1$ -gradient flow of a second order functional related to the Willmore energy, and studies in detail rotationally symmetric solutions. We also mention [24] where the authors, motivated by a model of delamination between elastic bodies, study a monotone geometric flow by means of a minimizing movement scheme reminiscent to the one in Section 5. They show existence of a limit solution and discuss some examples.

We recall that the De Giorgi minimizing movements scheme for building gradient flows in a metric space [4] typically builds  $u^{n+1} \approx u((n+1)\tau)$  as a minimizer of

$$\min_u \Phi(u) + \frac{1}{2\tau} \text{dist}(u, u^n)^2.$$

The exponent 2 here is crucial, as it ensures formally that  $\text{dist}(u^{n+1}, u^n) \approx \tau |D\Phi(u^{n+1})|$  (for an appropriate definition of the latter expression), as expected in an (implicit) Euler scheme. In particular, a motion always occurs if the initial point  $u^0$  is not critical for  $\Phi$ , contrarily to what would happen, in the present paper, if we used an exponent 1 and the distance induced by the  $L^1$  norm (then as soon as  $\Phi$  has a subgradient at  $u^0$  which is bounded, one easily obtains that no motion occurs for  $\tau$  small enough).

The plan of the paper is the following: in Section 2 we introduce the minimizing movements and we show convergence of the discrete solutions to a limit solution. We also show a general dissipation estimate from which we derive, under the assumption of strong convexity of  $F$ , a gradient flow equation satisfied by the limit solution.

In Section 3 we analyze the case when the initial datum is a subsolution (see Definition 3.1), in the case where  $F$  is strictly convex with superlinear growth. In such case the limit solution is non-decreasing in time and it is indeed unique.

In Section 4 we consider the simplest possible functional, that is, the Dirichlet energy. In this particular case we can show a stronger uniqueness result, namely that the limit gradient flow equation always admits a unique solution.

Finally, in Section 5 we consider the geometric evolution corresponding to the  $L^1$ -gradient flow of the anisotropic perimeter. Even if we are not able to characterize the limit flow as we do in the case of functions, when the initial set is convex, we can prove that the evolution is unique, monotone for the inclusion, and remains convex until it reaches the Cheeger set of the initial set. In two dimensions we also show that it stays convex until it becomes a Wulff Shape, and then shrinks to a point in finite time. Simple examples show that the geometric evolution is in general non-unique, after reaching the Cheeger set. Appendix A discusses the definition and main properties of the functional  $\Phi$ .

*Acknowledgements.* The second author is member of INDAM-GNAMPA and was supported by the PRIN Project 2019/24. Part of this work was done while he visited CEREMADE, supported by Univ. Paris-Dauphine PSL.

## 2. EXISTENCE OF SOLUTIONS

**2.1. Minimizing movements.** Following [14], we introduce the  $L^1$ -minimizing movement scheme. Given  $u^0 \in L^1(\Omega)$ , we let  $u^n$ , for  $n \geq 1$ , be a minimizer of

$$\min_{u \in L^1(\Omega)} \Phi(u) + \frac{1}{2\tau} \left( \int_{\Omega} |u - u^{n-1}| dx \right)^2. \quad (2.1)$$

If  $F$  has superlinear growth, then  $u^n \in W^{1,1}(\Omega)$ . Assuming in addition that  $F$  is strictly convex, we deduce that if  $u'$  is another solution,  $Du' = Du^n$  a.e., and  $u' - u^n$  is a constant. As a consequence, any other solution is of the form  $u^n + c$  where  $c$  is a minimizer of  $\|u^n - u^{n-1} - c\|_1$ , that is, a median value of  $u^n - u^{n-1}$ . Notice that, by convexity, the set of median values is an interval. If  $u^{n-1} \in W^{1,1}(\Omega)$  (which is true for  $n \geq 2$ , and which will we assume for  $n = 1$ ), then, since  $u^n - u^{n-1} \in W^{1,1}(\Omega)$  and  $\Omega$  is connected, it has a unique median value, hence we have the following result.

**Lemma 2.1.** *Assume that  $F$  is strictly convex with superlinear growth, and that  $\Phi(u^0) < +\infty$ . Then for any  $n \geq 1$ , there is a unique minimizer to (2.1).*

*Remark 2.2.* In case  $F$  is not strictly convex or the growth is not superlinear, the uniqueness is not guaranteed. However, in that case,

- (1) by strong convexity in  $u \mapsto \|u - u^{n-1}\|_1$  of the energy, one easily sees that given any two minimizers  $u, u'$  of (2.1),  $\|u - u^{n-1}\|_1 = \|u' - u^{n-1}\|_1$ . Indeed, one has, for  $\theta \in (0, 1)$ ,

$$\begin{aligned} \|\theta u + (1 - \theta)u' - u^{n-1}\|_1^2 &\leq (\theta\|u - u^{n-1}\|_1 + (1 - \theta)\|u' - u^{n-1}\|_1)^2 \\ &\leq \theta\|u - u^{n-1}\|_1^2 + (1 - \theta)\|u' - u^{n-1}\|_1^2 - \theta(1 - \theta) (\|u - u^{n-1}\|_1 - \|u' - u^{n-1}\|_1)^2, \end{aligned}$$

showing that  $(u + u')/2$  would be otherwise a better minimizer;

- (2) one can easily build measurable selections of the solutions  $\tau \mapsto u^n$  as  $\tau$  varies, as follows. A first observation is that for any  $p \in [1, d/(d - 1)]$ , if the energy of  $u$  in (2.1)

is finite, then  $u \in L^p(\Omega)$ , by Sobolev's embedding and using that  $\Phi(u)$  controls the total variation of  $u$ . Then, given  $p \in (1, d/d - 1)$ , for  $\varepsilon > 0$ , one can consider the unique minimizer  $u_\tau^\varepsilon$  of the strictly convex energy:

$$\Phi(u) + \frac{1}{2\tau} \left( \int_{\Omega} |u - u^{n-1}| dx \right)^2 + \varepsilon \int_{\Omega} |u|^p dx$$

and one easily shows that  $\tau \mapsto u_\tau^\varepsilon$  is continuous (in  $L^1(\Omega)$ , as well as  $L^p(\Omega)$ ). Sending  $\varepsilon \rightarrow 0$  we find that  $u_\tau^\varepsilon \rightarrow u_\tau$ , the solution of (2.1) with minimal  $L^p$  norm. Indeed, if  $u$  is another solution, one can write

$$\begin{aligned} \Phi(u_\tau^\varepsilon) + \frac{1}{2\tau} \left( \int_{\Omega} |u_\tau^\varepsilon - u^{n-1}| dx \right)^2 + \varepsilon \int_{\Omega} |u_\tau^\varepsilon|^p dx \\ \leq \Phi(u) + \frac{1}{2\tau} \left( \int_{\Omega} |u - u^{n-1}| dx \right)^2 + \varepsilon \int_{\Omega} |u|^p dx \\ \leq \Phi(u_\tau^\varepsilon) + \frac{1}{2\tau} \left( \int_{\Omega} |u_\tau^\varepsilon - u^{n-1}| dx \right)^2 + \varepsilon \int_{\Omega} |u|^p dx. \end{aligned}$$

Hence,  $\int_{\Omega} |u_\tau^\varepsilon|^p dx \leq \int_{\Omega} |u|^p dx$  and we conclude thanks to the lower-semicontinuity of the  $p$ -norm. The limit  $u_\tau$  is thus a (Bochner)-measurable selection. We also obtain [21, Thm 8.28] that  $(\tau, x) \mapsto u_\tau(x)$  is measurable.

We can now define  $u_\tau(t) := u^{\lfloor t/\tau \rfloor}$  where  $\lfloor \cdot \rfloor$  is the integer part, and we show the following theorem (whose proof is classical).

**Theorem 2.3.** *Assume that  $\Phi(u^0) < +\infty$ . Then, there exists  $u \in C^0([0, +\infty); L^1(\Omega))$  and a subsequence  $\tau_k \rightarrow 0$  such that  $u_{\tau_k} \rightarrow u$  in  $L^\infty([0, T]; L^1(\Omega))$ , for all  $T > 0$ , and*

$$\|u(s) - u(t)\|_1 \leq \sqrt{2\Phi(u^0)} \sqrt{|t - s|}$$

for any  $t, s \in [0, T]$ .

*Remark 2.4.* If we consider the piecewise affine interpolant  $\hat{u}_\tau$  of  $u^n$  in time, defined as  $u^n + (t/\tau - 1)(u^{n+1} - u^n)$  for  $n\tau \leq t \leq (n+1)\tau$ , rather than the piecewise constant interpolant, then the convergence is also in  $C^0([0, T]; L^1(\Omega))$ .

*Proof.* For any  $0 \leq m < n$ , we have

$$\begin{aligned} \|u_\tau(n\tau) - u_\tau(m\tau)\|_1^2 &\leq \left( \sum_{k=m}^{n-1} \|u_\tau((k+1)\tau) - u_\tau(k\tau)\|_1 \right)^2 \\ &\leq (n-m) \sum_{k=m}^{n-1} \|u_\tau((k+1)\tau) - u_\tau(k\tau)\|_1^2 \\ &\leq 2\tau(n-m) \sum_{k=m}^{n-1} (\Phi(u_\tau(k\tau)) - \Phi(u_\tau((k+1)\tau))) \\ &= 2(\Phi(u^m) - \Phi(u^n))(n\tau - m\tau) \\ &\leq 2\Phi(u^0)(n\tau - m\tau), \end{aligned}$$

where we used the Cauchy-Schwarz inequality and the minimality of  $u_\tau(k\tau)$ , and the fact that the sequence  $(\Phi(u^n))_n$  is non-increasing.

We deduce in addition that  $\Phi(u_\tau(t)) \leq \Phi(u^0)$  for any  $t > 0$ , so that, thanks to the assumptions on  $F$  and together with the bound on  $\|u_\tau(t) - u^0\|_1$ , we find that there is a compact subset of  $L^1(\Omega)$  (even  $L^p(\Omega)$ , for  $p < d/(d-1)$ ) which contains  $u_\tau(t)$  for any  $t > 0$ .

For any  $t, s \geq 0$ , it follows that

$$\|u_\tau(t) - u_\tau(s)\|_1 \leq \sqrt{2\Phi(u^0)} \sqrt{|[t/\tau] - [s/\tau]| \tau} \leq \sqrt{2\Phi(u^0)} \sqrt{\tau + |t - s|}.$$

The compactness and convergence is then deduced by the Ascoli-Arzelà Theorem.  $\square$

By a simple interpolation argument, we can show a slightly improved convergence for the previous theorem.

**Proposition 2.5.** *Let  $p \in [1, d/(d-1))$ . Then the subsequence  $(u_{\tau_k})_k$  in Theorem 2.3 also converges to  $u$  in  $L^\infty([0, T]; L^p(\Omega))$  for any  $T > 0$ , while the piecewise-affine interpolants  $\hat{u}_{\tau_k}$  converge in  $C^0([0, T]; L^p(\Omega))$ .*

*Proof.* By construction, for  $t \in [0, T]$  the norms  $\|u_\tau(t)\|_{d/(d-1)}$  are uniformly bounded and for  $1 < p < d/(d-1)$ , there is a compact set of  $L^p(\Omega)$  such that  $u_\tau(t) \in C_p$ .

For  $0 < \epsilon < 1$ , writing  $|u_\tau(t) - u_\tau(s)|^p = |u_\tau(t) - u_\tau(s)|^{1-\epsilon} |u_\tau(t) - u_\tau(s)|^{p-1+\epsilon}$  and using Hölder's inequality, we have

$$\|u_\tau(t) - u_\tau(s)\|_p^p \leq \|u_\tau(t) - u_\tau(s)\|_1^{1-\epsilon} \left( \int_\Omega |u_\tau(t) - u_\tau(s)|^{\frac{p-1+\epsilon}{\epsilon}} \right)^\epsilon$$

hence if  $(p-1)/\epsilon + 1 \leq d/(d-1)$ , for instance for  $\epsilon = (p-1)(d-1) < 1$  (or any  $\epsilon < 1$  if  $d = 1$ ), we find that

$$\|u_\tau(t) - u_\tau(s)\|_p \leq C \sqrt{\tau + |t - s|}^{\frac{d}{p} - (d-1)}$$

Hence, the convergence is also in  $L^\infty([0, T], L^p(\Omega))$ .  $\square$

**2.2. Euler-Lagrange equation.** The Euler-Lagrange equation for  $u^n$  minimizing (2.1) takes formally the form:

$$\begin{cases} -\operatorname{div} z^n + \frac{\|u^n - u^{n-1}\|_1}{\tau} \operatorname{sign}(u^n - u^{n-1}) \ni 0 \\ z^n \cdot Du^n = F(Du^n) + F^*(z^n) \end{cases} \quad (2.2)$$

(with  $F^*$  the convex conjugate of  $F$ , see Appendix A), where the last statement should be in the sense of [5] if  $F$  has minimal growth 1 ( $Du$  can be a measure), otherwise we just expect  $z^n \in \partial F(Du^n)$  a.e.

This follows from [17, Prop. 5.6], applied in  $V = L^1(\Omega)$  and  $V^* = L^\infty(\Omega)$ . In that case,  $u \mapsto \|u - u^{n-1}\|_1^2/(2\tau)$  is everywhere continuous while  $\Phi$  is lower semicontinuous. Hence,  $\partial(\Phi(\cdot) + \|\cdot - u^{n-1}\|_1^2/(2\tau)) = \partial\Phi + \partial\|\cdot - u^{n-1}\|_1^2/(2\tau)$ , where the subgradients are elements of  $L^\infty(\Omega)$ . So a minimizer  $(u^n)$  is characterized by

$$0 \in \partial\Phi(u^n) + \frac{\|u^n - u^{n-1}\|_1}{\tau} \operatorname{sign}(u^n - u^{n-1}) \quad (2.3)$$

where  $\operatorname{sign}(t) = \{1\}$  for  $t > 0$ ,  $\{-1\}$  for  $t < 0$ , and  $[-1, 1]$  for  $t = 0$ .

Then, in case  $F$  is 1-homogeneous, (2.2) is deduced from [23, Prop. 3] (in that case, the second equation in (2.2) is to be understood in the sense of [5]). The more general Lipschitz case is studied in [18]. In case both  $F$  and  $F^*$  are superlinear, Lemma A.6 (or Lemma A.7) in Appendix A also shows (2.2). A general case (e.g.,  $F$  neither Lipschitz nor superlinear) remains unclear.

*Remark 2.6.* For varying  $\tau > 0$ , let us denote  $u_\tau$  the minimizer of (2.1) and  $q_\tau \in \partial\Phi(u_\tau)$  the corresponding subgradient in the Euler-Lagrange equation (2.3). Then, as in Remark 2.2-(2), one can build a measurable selection of  $\tau \mapsto q_\tau$ . One first observes that  $q_\tau$  minimizes the dual problem (with  $\Phi^*$  the convex conjugate of  $\Phi$ )

$$\min_q \Phi^*(q) - \int_{\Omega} q(x)u^{n-1}(x) dx + \frac{\tau}{2}\|q\|_{\infty}^2.$$

This is easily deduced from (2.3) and the fact  $u_\tau \in \partial\Phi^*(q_\tau)$ . Then, one perturbs this problem by adding a term  $\varepsilon\|q\|_{p'}^2/2$  for some  $p' \in (1, +\infty)$  (for instance,  $p' = 2$  — using  $p' \geq d$  is less crucial as requiring  $p \leq d/(d-1)$  in the primal problem, since  $q$  has to be bounded anyway). This allows to define a unique minimizer  $q_\tau^\varepsilon$ , which in addition is continuous with respect to  $\tau$  in  $L^{p'}(\Omega)$ .

For each  $\tau$ , as  $\varepsilon \rightarrow 0$ , this minimizer  $q_\tau^\varepsilon$  goes to the solution  $q_\tau$  of the dual problem which is minimal in  $L^{p'}$ -norm, and is thus a Bochner-measurable selection (and measurable as a function of  $(\tau, x)$ ).

**2.3. Estimate of the time derivative.** The class of functionals  $\Phi$  we are considering satisfies the following fundamental estimate: for any  $u, v \in L^1(\Omega)$ ,

$$\Phi(u \wedge v) + \Phi(u \vee v) \leq \Phi(u) + \Phi(v) \quad (2.4)$$

(with equality if  $F$  has superlinear growth), see Lemma A.4. Here for  $x, y \in \mathbb{R}$ ,  $x \vee y = \max\{x, y\}$  and  $x \wedge y = \min\{x, y\}$  and the notation extends to real-valued functions. In this context, we can prove the following:

**Lemma 2.7.** *Let  $v \in L^1(\Omega)$ ,  $q \in L^\infty(\Omega)$  with  $q \in \partial\Phi(v)$ . Let  $u$  be a minimizer of:*

$$\Phi(u) + \frac{1}{2\tau}\|u - v\|_1^2.$$

*Then*

$$\frac{\|u - v\|_1}{\tau} \leq \|q\|_{\infty}.$$

*Proof.* The following remark is crucial: if  $q \in \partial\Phi(v)$ ,  $p \in \partial\Phi(u)$ , then (denoting as usual  $x^+ = x \vee 0$  and  $x^- = (-x)^+$ ):

$$\int_{\Omega} (q - p)(v - u)^+ dx \geq 0 \quad \text{and} \quad \int_{\Omega} (q - p)(v - u)^- dx \leq 0. \quad (2.5)$$

Indeed, one has:

$$\Phi(u \vee v) \geq \Phi(u) + \int_{\Omega} p(u \vee v - u) dx \quad \text{and} \quad \Phi(u \wedge v) \geq \Phi(v) + \int_{\Omega} q(u \wedge v - v) dx. \quad (2.6)$$

Using that  $u \vee v - u = (v - u)^+$  and  $u \wedge v - v = -(v - u)^+$ , the first inequality in (2.5) follows by summing the two previous inequalities and using (2.4). The second is proved similarly.

Since in the Lemma,  $u$  satisfies the equation (cf (2.3)):

$$\exists p \in \partial\Phi(u) \cap -\text{sign}(u - v) \frac{\|u - v\|_1}{\tau},$$

we deduce from (2.5) that (here “sign” is single-valued as the integrand vanishes for  $v \leq u$ ):

$$0 \leq \int_{\Omega} \left( q - \text{sign}(v - u) \frac{\|u - v\|_1}{\tau} \right) (v - u)^+ dx \leq \left( \text{ess sup}_{\Omega} q - \frac{\|u - v\|_1}{\tau} \right) \int_{\Omega} (v - u)^+ dx \quad (2.7)$$

so that if  $\{v > u\}$  has positive measure,  $\frac{\|u-v\|_1}{\tau} \leq \text{ess sup}_\Omega q$ . Similarly (multiplying with  $-(v-u)^-$ ) we show that so that if  $\{v < u\}$  has positive measure,  $\frac{\|u-v\|_1}{\tau} \leq -\text{ess inf}_\Omega q$ . The thesis follows.  $\square$

We deduce immediately the following result, as a consequence of Lemma 2.7 and the Euler-Lagrange equation (2.3).

**Theorem 2.8.** *Let  $u^0 \in L^1(\Omega)$ ,  $\tau > 0$  and  $(u^n)_{n \geq 0}$  defined by the minimizing movement scheme. Then*

i. *for any  $n \geq 1$ ,*

$$\frac{\|u^{n+1} - u^n\|_1}{\tau} \leq \frac{\|u^n - u^{n-1}\|_1}{\tau};$$

ii. *as a result,*

$$\frac{\|u^{n+1} - u^n\|_1}{\tau} \leq \sqrt{\frac{2\Phi(u^0)}{(n+1)\tau}};$$

iii. *if in addition  $\partial\Phi(u^0) \neq \emptyset$ , then for any  $n \geq 0$ ,*

$$\frac{\|u^{n+1} - u^n\|_1}{\tau} \leq \|\partial^0\Phi(u^0)\|_\infty$$

where  $\partial^0\Phi$  denotes the element in the subgradient with minimal norm.

We observe that the set of  $u^0$  such that  $\partial\Phi(u^0)$  contains a bounded element is dense in the domain of  $\Phi$ , see Lemma A.5.

*Proof.* Only point (ii.) still needs to be proven. We write for  $n \geq 0$  (thanks to point (i.):

$$\frac{\|u^{n+1} - u^n\|_1^2}{\tau^2} \leq \frac{2}{(n+1)\tau} \sum_{k=0}^n \frac{\|u^{k+1} - u^k\|_1^2}{2\tau} \leq \frac{2}{(n+1)\tau} \sum_{k=0}^n (\Phi(u^k) - \Phi(u^{k+1})),$$

and the claim follows.  $\square$

**Corollary 2.9.** *Let  $u$  be an evolution provided by Theorem 2.3. Then  $u$  is locally Lipschitz in time. Its time derivative is a bounded measure of the form  $\dot{u}(t) \otimes dt$  which satisfies, for a.e.  $t \geq 0$ ,*

$$|\dot{u}(t)|(\Omega) \leq \sqrt{\frac{2\Phi(u^0)}{t}}. \quad (2.8)$$

*If in addition  $\partial\Phi(u^0) \neq \emptyset$ , then*

$$|\dot{u}(t)|(\Omega) \leq \|\partial^0\Phi(u^0)\|_\infty. \quad (2.9)$$

*Proof.* Indeed, we observe first that given  $\eta$  a smooth function with compact support in  $(t_1, t_2) \times \Omega$ ,  $t_1 < t_2$ , one has for a.e.  $t > 0$ :

$$\begin{aligned} \int_{t_1}^{t_2} \int_{\Omega} u(t, x) \partial_t \eta(t, x) dt dx &= \lim_{\tau \rightarrow 0} \int_{t_1}^{t_2} \int_{\Omega} u(t, x) \frac{\eta(t + \tau, x) - \eta(t, x)}{\tau} dt dx \\ &= \lim_{\tau \rightarrow 0} \int_{t_1}^{t_2} \int_{\Omega} \frac{u(t - \tau, x) - u(t, x)}{\tau} \eta(t, x) dt dx \\ &\leq \min \left\{ \|\partial^0\Phi(u^0)\|_\infty, \sqrt{\frac{2\Phi(u^0)}{t_1}} \right\} \int_{t_1}^{t_2} \|\eta(t, \cdot)\|_\infty dt, \end{aligned}$$



thanks to Theorem 2.8. We deduce that  $\dot{u}$  is a measure, whose marginals are in addition absolutely continuous with respect to the Lebesgue measure  $dt$ . Hence one can disintegrate  $\dot{u}$  as  $\dot{u}(t) \otimes dt$ , and it follows that for a.e.  $t$ , (2.8)-(2.9) hold.  $\square$

**Corollary 2.10.** *Let  $u$  be an evolution provided by Theorem 2.3, assuming as always  $\Phi(u^0) < +\infty$ . Then  $t \mapsto \Phi(u(t))$  is locally Lipschitz. More precisely,*

- i. *If  $\partial\Phi(u^0) \neq \emptyset$ , then  $|\Phi(u(s)) - \Phi(u(t))| \leq \|\partial^0\Phi(u^0)\|_\infty^2 |s - t|$  for any  $s, t \geq 0$ ;*
- ii. *In general, for  $s > t > 0$ ,*

$$|\Phi(u(s)) - \Phi(u(t))| \leq \frac{2\Phi(u^0)}{t} |s - t|.$$

*In particular,  $d\Phi(u(t))/dt$  exists for almost every  $t > 0$ .*

*Proof.* For every  $v \in L^1(\Omega)$  and  $n \geq 1$ , one can write thanks to (2.3):

$$\Phi(v) \geq \Phi(u^n) - \frac{\|u^n - u^{n-1}\|_\tau}{\tau} \int_\Omega \text{sign}(u^n - u^{n-1})(v - u^n) dx \geq \Phi(u^n) - \frac{\|u^n - u^{n-1}\|_1}{\tau} \|v - u^n\|_1.$$

Using Theorem 2.8 (iii.) and letting  $\tau \rightarrow 0$  with  $\tau n \rightarrow t$ , we find:

$$\Phi(v) \geq \Phi(u(t)) - \|\partial^0\Phi(u^0)\|_\infty \|v - u(t)\|_1$$

for all  $t \geq 0$ . Then we conclude choosing  $v = u(s)$ , and observing that Corollary 2.9 yields that  $\|u(t) - u(s)\|_1 \leq \|\partial^0\Phi(u^0)\|_\infty |t - s|$ .

Alternatively, we can also bound  $\frac{\|u^n - u^{n-1}\|_1}{\tau} \|v - u^n\|_1$  using Theorem 2.8 (ii.), and we obtain, letting again  $\tau \rightarrow 0$  with  $n\tau \rightarrow t$ :

$$\Phi(u(t)) \leq \Phi(v) + \sqrt{\frac{2\Phi(u^0)}{t}} \|v - u(t)\|_1.$$

We then choose  $v = u(s)$  (for  $s < t$  and  $s > t$ ), and the thesis follows from Corollary 2.9.  $\square$

**2.4. Dissipation estimate.** We shall prove the following dissipation estimate. This is a variant of [4, Thm. 2.3.3], yet our time derivative is here a measure while we still wish to consider the slopes as elements in  $L^\infty(\Omega)$ .

**Theorem 2.11.** *Let  $u^0$  satisfy  $\Phi(u^0) < +\infty$  and let  $u$  be a limit of minimizing movements given by Theorem 2.3. Then, for any  $t > 0$ ,  $\dot{u}$  is a measure with marginal  $s \mapsto |\dot{u}(s)|(\Omega)$  in  $L^2(0, t)$  and there exists  $q \in L^2((0, t); L^\infty(\Omega))$  with  $q(s) \in -\partial\Phi(u(s))$  for a.e.  $s \geq 0$  such that*

$$\Phi(u(t)) + \frac{1}{2} \int_0^t (|\dot{u}(s)|(\Omega))^2 ds + \frac{1}{2} \int_0^t \|q(s)\|_\infty^2 ds \leq \Phi(u^0). \quad (2.10)$$

*Proof.* We remain in the framework of Theorem 2.3, assuming that  $\Phi(u^0) < +\infty$  and that  $u_\tau$ , defined above converges, up to a subsequence, to a function  $u \in C^{0,1/2}([0, T]; L^1(\Omega))$ .

As usual (see for instance [4, Sec. 3.2]), for  $n\tau < t < (n+1)\tau$ , we let  $\tilde{u}_\tau(t)$  be a minimizer of

$$\min_u \Phi(u) + \frac{1}{2(t - n\tau)} \|u - u^n\|_1^2,$$

which satisfies the Euler-Lagrange equation

$$\partial\Phi(\tilde{u}_\tau(t)) + \frac{\|\tilde{u}_\tau(t) - u^n\|_1}{2(t - n\tau)} \text{sign}(\tilde{u}_\tau(t) - u^n) \ni 0. \quad (2.11)$$

By Remark 2.2-(1), observe that even if the minimizer might be non-unique, the value of  $\|\tilde{u}_\tau(t) - u^n\|_1$  is. In any case, as mentioned in Remark 2.2-(2), we assume that  $t \mapsto \tilde{u}_\tau(t)$  is measurable. We also have that  $\|\tilde{u}_\tau(t) - u_\tau(t)\|_1 \leq \sqrt{2\Phi(u^0)(t - n\tau)} \leq \sqrt{2\Phi(u^0)\tau}$ , so that  $\tilde{u}_\tau$  converges to the same limit as  $u_\tau$ , also uniformly in time.

Now, for  $n \geq 0$ ,  $0 < s < \tau$ , we let  $h(s) = \Phi(\tilde{u}_\tau(n\tau + s)) + \|\tilde{u}_\tau(n\tau + s) - u^n\|_1^2/(2s)$ , hence  $h(\tau) = \Phi(u^{n+1}) + \|u^{n+1} - u^n\|_1^2/(2\tau)$  and  $\lim_{s \rightarrow 0} h(s) = \Phi(u^n)$ . It is standard that:

$$h'(s) \leq -\frac{\|\tilde{u}_\tau(n\tau + s) - u^n\|_1^2}{2s^2},$$

so that (using  $h(\tau) = \lim_{\epsilon \rightarrow 0} h(\epsilon) + \int_\epsilon^\tau h'(s)ds$ )

$$\Phi(u^{n+1}) + \frac{\|u^{n+1} - u^n\|_1^2}{2\tau} \leq \Phi(u^n) - \frac{1}{2} \int_0^\tau \frac{\|\tilde{u}_\tau(n\tau + s) - u^n\|_1^2}{s^2} ds.$$

Thanks to the Euler-Lagrange equation (2.11), we deduce:

$$\Phi(u^{n+1}) + \frac{1}{2} \int_{n\tau}^{(n+1)\tau} \|\dot{u}_\tau(s)\|_1^2 ds + \frac{1}{2} \int_{n\tau}^{(n+1)\tau} \|q_\tau(s)\|_\infty^2 ds \leq \Phi(u^n),$$

where for all  $t$ ,  $q_\tau(t) \in -\partial\Phi(\tilde{u}_\tau(t))$  (and we also assume, reasoning as in Remark 2.6, that  $q_\tau$  is measurable), and  $\hat{u}(t)$  is the piecewise-affine interpolant, which also converges to  $u$  up to a subsequence (in  $C^0([0, T]; L^p(\Omega))$  for  $1 \leq p \leq d/(d-1)$ , see Prop. 2.5). Summing this inequality from  $n = 0$  to  $\lfloor t/\tau \rfloor - 1$ , for  $0 < t \leq T$ , we find:

$$\Phi(u_\tau(t)) + \frac{1}{2} \int_0^{t-\tau} \|\dot{u}_\tau(s)\|_1^2 ds + \frac{1}{2} \int_0^{t-\tau} \|q_\tau(s)\|_\infty^2 ds \leq \Phi(u^0).$$

By lower-semicontinuity of the convex functions appearing in the integrals we claim that (2.10) is deduced, where  $q$  is a weak limit (in  $L^2([0, T]; L^{p'}(\Omega))$ ) of  $q_\tau$ , and  $p'$  the conjugate exponent of some  $p \in (1, d/(d-1))$ .

The only difficulty is with the measure term. Given  $\varphi \in C_c^\infty([0, T] \times \Omega)$ , it is not difficult to check that for  $\tau$  small enough:

$$\frac{1}{2} \int_0^{t-\tau} \|\dot{u}_\tau(s)\|_1^2 ds \geq \int_\Omega \varphi(0, x) u^0(x) dx - \int_0^t \int_\Omega \dot{\varphi} \hat{u}_\tau dx ds - \frac{1}{2} \int_0^t \|\varphi(s)\|_\infty^2 ds$$

so that, passing to the limit along an appropriate subsequence,

$$\int_\Omega \varphi(0) u^0 dx - \int_0^t \int_\Omega \dot{\varphi} u dx ds - \frac{1}{2} \int_0^t \|\varphi(s)\|_\infty^2 ds \leq \liminf_{\tau \rightarrow 0} \frac{1}{2} \int_0^{t-\tau} \|\dot{u}_\tau(s)\|_1^2 ds =: \ell.$$

In particular (using also that  $u(t) \rightarrow u^0$  as  $t \rightarrow 0$ ), we deduce immediately that the distribution  $\dot{u}$  is a bounded Radon measure (in  $[0, T] \times \Omega$ ), satisfying for all  $t \leq T$ :

$$\int_{[0, t] \times \Omega} \varphi d\dot{u} - \frac{1}{2} \int_0^t \|\varphi(s)\|_\infty^2 ds \leq \ell.$$

Letting  $n \geq 1$ ,  $0 = t_0 < t_1 < \dots < t_n = t$  and considering  $m_i \geq 0$ ,  $i = 1, \dots, n$ , and the supremum over all functions  $\varphi$  with  $\varphi|_{(t_{i-1}, t_i)} \in C_c^\infty([t_{i-1}, t_i] \times \Omega)$  and  $\|\varphi\|_{L^\infty(t_{i-1}, t_i)} \leq m_i$  we deduce:

$$\sum_{i=1}^n m_i |\dot{u}|([t_{i-1}, t_i] \times \Omega) - (t_i - t_{i-1}) \frac{m_i^2}{2} \leq \ell.$$

By uniform approximation of a smooth function  $\psi \in C_c^\infty([0, t]; \mathbb{R}_+)$  by piecewise constant functions, we deduce that the marginal measure  $|\dot{u}|(\Omega)$  in  $(0, t)$  satisfies:

$$\int_0^t \psi(s) d(|\dot{u}|(\Omega))(s) - \frac{1}{2} \psi^2(s) ds \leq \ell$$

and it follows that  $|\dot{u}|(\Omega)$  is indeed in  $L^2(0, t)$ , with

$$\frac{1}{2} \int_0^t (|\dot{u}|(\Omega))^2 ds \leq \ell.$$

Now, we check that  $q(t) \in \partial\Phi(u(t))$  a.e.: given  $\varphi \in C_c^\infty((0, T) \times \Omega)$ , we have

$$\int_0^T \Phi(\varphi(t)) dt \geq \int_0^T \Phi(\tilde{u}_\tau(t)) dt + \int_0^T \int_\Omega q_\tau(t, x) (\tilde{u}_\tau(t, x) - \varphi(t, x)) dx dt.$$

Since  $\tilde{u}_\tau \rightarrow u$  in  $L^\infty([0, T]; L^p(\Omega))$  (using Prop. 2.5) and  $q_\tau \rightarrow q$  in  $L^2([0, T]; L^{p'}(\Omega))$ , we obtain that

$$\int_0^T \int_\Omega q_\tau(t, x) \tilde{u}_\tau(t, x) dx dt \rightarrow \int_0^T \int_\Omega q(t, x) u(t, x) dx dt.$$

It follows that

$$\int_0^T \Phi(\varphi(t)) dt \geq \int_0^T \Phi(u(t)) dt + \int_0^T \int_\Omega q(t, x) (u(t, x) - \varphi(t, x)) dx dt.$$

We deduce that for a.e.  $t$ ,  $-q(t) \in \partial\Phi(u(t))$ .  $\square$

A dissipation estimate like (2.10) usually implies that the flow  $u(t)$  is a curve of maximal slope in the sense of [4, Def. 1.3.2], satisfying

$$\frac{d\Phi(u(t))}{dt} = - \int_\Omega q(t) \dot{u}(t) dx \quad \text{for a.e. } t \geq 0. \quad (2.12)$$

However, as already observed in the Introduction, the results in [4] fail to apply in the  $(1, \infty)$ -duality, since  $L^1(\Omega)$  does not satisfy the Radon-Nikodým property, and it is not obvious to give a meaning to (2.12) in this context.

We shall rigorously prove (2.12) in the next section, under the additional assumption that  $F$  is strongly convex.

**2.5. Strongly convex case.** In this part, we first assume that in addition there exists  $\gamma > 0$  such that  $F$  is  $\gamma$ -convex:

$$F(\eta) \geq F(\xi) + p \cdot (\eta - \xi) + \frac{\gamma}{2} |\eta - \xi|^2$$

for any  $\eta, \xi \in \mathbb{R}^d$  and  $p \in \partial F(\xi)$ . Then (2.6) becomes, still given  $q \in \partial\Phi(v)$ ,  $p \in \partial\Phi(u)$  (and in particular  $u, v \in \text{dom } \Phi \subseteq H^1(\Omega)$ ):

$$\begin{aligned} \Phi(u \vee v) &\geq \Phi(u) + \int_\Omega p(u \vee v - u) dx + \frac{\gamma}{2} \int_\Omega |D(u \vee v - u)|^2 dx \\ \Phi(u \wedge v) &\geq \Phi(v) + \int_\Omega q(u \wedge v - v) dx + \frac{\gamma}{2} \int_\Omega |D(u \wedge v - v)|^2 dx. \end{aligned}$$

One now deduces, following the arguments in the proof of Lemma 2.7:

$$\begin{aligned} \gamma \int_{\{v>u\}} |Dv - Du|^2 dx &\leq \int_{\Omega} (q - p)(v - u)^+ dx \\ \gamma \int_{\{v<u\}} |Dv - Du|^2 dx &\leq - \int_{\Omega} (q - p)(v - u)^- dx. \end{aligned} \quad (2.13)$$

Summing, we find:

$$\gamma \int_{\Omega} |Dv - Du|^2 dx \leq \int_{\Omega} (q - p)(v - u) dx.$$

Using  $v = u^n$ ,  $u = u^{n+1}$  and (2.3), it follows for all  $n \geq 1$ :

$$\begin{aligned} &\gamma \int_{\Omega} |Du^{n+1} - Du^n|^2 dx \\ &\leq \int_{\Omega} \left( -\text{sign}(u^n - u^{n-1}) \frac{\|u^n - u^{n-1}\|_1}{\tau} + \text{sign}(u^{n+1} - u^n) \frac{\|u^{n+1} - u^n\|_1}{\tau} \right) (u^n - u^{n+1}) dx \\ &\leq -\frac{1}{\tau} \|u^{n+1} - u^n\|_1^2 + \frac{1}{\tau} \|u^n - u^{n-1}\|_1 \|u^{n+1} - u^n\|_1 \end{aligned}$$

(there is an abuse of notation here since “sign” is multivalued, however we use only that  $|\text{sign}| \leq 1$  and  $\text{sign}(u^{n+1} - u^n)(u^n - u^{n+1}) = -|u^{n+1} - u^n|$ ), which we rewrite as:

$$\begin{aligned} \gamma \tau \int_{\Omega} \left| D \frac{u^{n+1} - u^n}{\tau} \right|^2 dx + \frac{1}{2\tau^2} (\|u^{n+1} - u^n\|_1 - \|u^n - u^{n-1}\|_1)^2 + \frac{\|u^{n+1} - u^n\|_1^2}{2\tau^2} \\ \leq \frac{\|u^n - u^{n-1}\|_1^2}{2\tau^2}. \end{aligned} \quad (2.14)$$

Then, summing (2.14), we get the estimate:

$$\gamma \int_0^{n\tau} \|D\dot{u}_\tau(t + \tau)\|_2^2 dt \leq \frac{\|u^1 - u^0\|_1^2}{2\tau^2} \leq \frac{1}{2} \|\partial^0 \Phi(u^0)\|_\infty^2. \quad (2.15)$$

If the initial speed is not bounded we can sum from  $m$  to  $n > m$  and get

$$\gamma \int_{m\tau}^{n\tau} \|D\dot{u}_\tau(t + \tau)\|_2^2 dt \leq \frac{\|u_\tau((m+1)\tau) - u_\tau(m\tau)\|_1^2}{2\tau^2} \leq \frac{\Phi(u^0)}{(m+1)\tau}, \quad (2.16)$$

thanks to Theorem 2.8 (ii). Recalling Theorem 2.8 we are in particular able to deduce the following result:

**Theorem 2.12.** *Assume  $F$  is  $\gamma$ -convex and let  $u$  be given by Theorem 2.3. Then  $\dot{u} \in L^2((t, +\infty); H^1(\Omega))$  for any  $t > 0$ , with*

$$\gamma \int_t^{+\infty} \|D\dot{u}\|_2^2 ds \leq \frac{\Phi(u^0)}{t}.$$

If in addition  $\partial\Phi(u^0) \neq \emptyset$ , then

$$\gamma \int_0^{+\infty} \|D\dot{u}\|_2^2 ds \leq \frac{1}{2} \|\partial^0 \Phi(u^0)\|_\infty^2.$$

*Remark 2.13.* Taking into account (2.13) when deriving (2.7), we can derive slightly more precise estimates which may be useful in case the initial speed  $q^0 \in \partial\Phi(u^0)$  has a sign. Indeed, we obtain for instance that:

- If  $\{u^1 > u^0\}$  has positive measure, then

$$\frac{\|u^1 - u^0\|_1}{\tau} \leq \text{ess sup}(-q^0) - \gamma \frac{\int_{\Omega} |D(u^1 - u^0)^+|^2 dx}{\|(u^1 - u^0)^+\|_1};$$

- If  $\{u^1 < u^0\}$  has positive measure, then

$$\frac{\|u^1 - u^0\|_1}{\tau} \leq \text{ess sup} q^0 - \gamma \frac{\int_{\Omega} |D(u^1 - u^0)^-|^2 dx}{\|(u^1 - u^0)^-\|_1}.$$

In particular, if  $q^0 \leq 0$  a.e., we deduce that  $u^1 \geq u^0$  a.e., but then  $q^1 := -\text{sign}(u^1 - u^0)\|u^1 - u^0\|^2/\tau \in \partial\Phi(u^1)$  is also non-positive and again,  $u^2 \geq u^1$  a.e.: by induction we find that  $u^{n+1} \geq u^n$  for all  $n \geq 0$ .

We now are able to derive rigorously (2.12). First, thanks to Poincaré inequality,  $\dot{u} \in L^2((t, T) \times \Omega)$  for any  $T > t > 0$ . Indeed, in the Dirichlet case, one has  $\int_{\Omega} |\dot{u}(s)|^2 dx \leq c_{\Omega} \int_{\Omega} |D\dot{u}(s)|^2 dx$  for each  $s > 0$  such that the right-hand side integral is finite, with  $c_{\Omega}$  the Poincaré constant of  $H_0^1(\Omega)$ . In the Neumann case, Poincaré-Wirtinger's inequality yields  $\int_{\Omega} |\dot{u}(s) - m(s)|^2 dx \leq c'_{\Omega} \int_{\Omega} |D\dot{u}(s)|^2 dx$  for  $m(s) = (1/|\Omega|) \int_{\Omega} \dot{u}(s) dx$ , which is bounded thanks to Corollary 2.9.

In particular for any  $b > a > 0$ ,

$$\lim_{|s| \rightarrow 0} \int_a^b \int_{\Omega} |\dot{u}(t+s, x) - \dot{u}(t, x)|^2 dx dt = 0$$

and using  $(u(t+s) - u(t))/s = (1/s) \int_0^s \dot{u}(r) dr$  and Jensen's inequality,

$$\lim_{|s| \rightarrow 0} \int_a^b \int_{\Omega} \left| \frac{u(t+s, x) - u(t, x)}{s} - \dot{u}(t, x) \right|^2 dx dt = 0.$$

Hence, we can find a sequence  $s_k \downarrow 0$  such that for a.e.  $t > 0$ ,

$$\lim_{k \rightarrow \infty} \left\| \frac{u(t \pm s_k) - u(t)}{\pm s_k} - \dot{u}(t) \right\|_2 = 0$$

Now we consider  $q(t)$  from Theorem 2.11. For a.e.  $t > 0$ ,  $q(t) \in L^{\infty}(\Omega)$ ,  $\Phi(u(t))$  is differentiable at  $t$ , and one has for  $s$  small (positive or negative):

$$\frac{1}{|s|} \Phi(u(t+s)) - \Phi(u(t)) \geq - \int_{\Omega} q(t) \frac{u(t+s) - u(t)}{|s|} dx.$$

Choosing  $s$  along the sequence  $s_k$  and sending  $k \rightarrow \infty$ , we deduce (2.12).

As a consequence, one has:

$$\Phi(u(0)) - \Phi(u(t)) = \int_0^t \int_{\Omega} q(s) \dot{u}(s) dx ds \leq \frac{1}{2} \int_0^t \|q(s)\|_{\infty}^2 + \|\dot{u}(s)\|_1^2$$

which combined with (2.10), yields that  $q(s) \in \partial\|\cdot\|_1^2(\dot{u}(s))/2$  for a.e.  $s > 0$ .

If  $F$  is  $\gamma$ -convex and  $C^1$ , with full domain, we have additionally that  $q(t) = \text{div} \nabla F(Du(t))$  for a.e.  $t > 0$ , cf Lemmas A.6-A.7. Hence we have:

**Theorem 2.14.** *Assume  $F$  is  $C^1$  and strongly convex, with full domain, and let  $u$  be a limit of minimizing movements given by Theorem 2.3, starting from  $u^0$  with  $\Phi(u^0) < +\infty$ . Then,  $\dot{u} \in L^2((t, +\infty); H^1(\Omega))$  for any  $t > 0$  and satisfies the equations*

$$\begin{cases} |\text{div} \nabla F(Du)| \leq \|\dot{u}\|_1 & \text{a.e. in } (0, +\infty) \times \Omega \\ \dot{u} \text{ div} \nabla F(Du) = |\dot{u}| \|\dot{u}\|_1 & \text{a.e. in } (0, +\infty) \times \Omega. \end{cases} \quad (2.17)$$

**2.6. Minimal surface energy.** The case where  $\Phi(u) = \int_{\Omega} \sqrt{1 + |Du|^2} dx$  is in between the setup of the previous section and that of the last Section 5, where we introduce a geometric version of this gradient flow. In that case, we remark that if we can show that when  $u^0$  is  $L$ -Lipschitz for some constant  $L \geq 0$ ,  $u$  remains  $L$ -Lipschitz, then from Section 2.5 we deduce that the solution satisfies  $\dot{u} \in H^1(\Omega)$  for positive time and that the characterization (2.17) holds. Indeed, in that case, since the gradients are all bounded by  $L$ ,  $F$  is  $\gamma$ -convex, with  $\gamma = (1 + L^2)^{-3/2}$ .

This is the case for instance if we consider the problem in a periodic setting ( $\Omega = \mathbb{R}^d/\mathbb{Z}^d$ ):

**Lemma 2.15.** *Let  $\Omega = \mathbb{R}^d/\mathbb{Z}^d$ ,  $F(p) = \sqrt{1 + |p|^2}$ ,  $v$  a  $L$ -Lipschitz, ( $L \geq 0$ ) function and  $u$  a minimizer of:*

$$\min_u \Phi(u) + \frac{1}{2\tau} \left( \int_{\Omega} |u - v| dx \right)^2. \quad (2.18)$$

*Then  $u$  is  $L$ -Lipschitz, and unique.*

*Proof.* It is enough to show it for the unique solution  $u_p$ ,  $p > 1$ , of:

$$\min_u \Phi(u) + \frac{1}{2\tau} \left( \int_{\Omega} |u - v|^p dx \right)^{2/p} \quad (2.19)$$

since in the limit  $p \rightarrow 1$  one recover a minimizer (hence *the* minimizer) for  $p = 1$ .

We first show a comparison result in a simplified setting: let  $v > v'$ , let  $u$  minimize, for some  $\lambda > 0$ :

$$\min_u \Phi(u) + \frac{\lambda}{p} \int_{\Omega} |u - v|^p dx \quad (2.20)$$

and let  $u'$  solve the same problem with  $v$  replaced with  $v'$ . Then, comparing the energy of  $u$  with the energy of  $u \vee u'$ , and the energy of  $u'$  with the energy of  $u \wedge u'$  and summing both inequalities we end up (using (2.4)) with:

$$\int_{\Omega} |u' - v'|^p dx - \int_{\Omega} |u \wedge u' - v'|^p dx \leq \int_{\Omega} |u \vee u' - v|^p dx - \int_{\Omega} |u - v|^p dx,$$

that is:

$$\int_{\{u < u'\}} |u' - v'|^p - |u - v|^p dx \leq \int_{\{u < u'\}} |u' - v|^p - |u - v|^p dx.$$

One may rewrite this as:

$$\int_{\{u < u'\}} \int_{u(x)}^{u'(x)} p|t - v'(x)|^{p-2} (t - v'(x)) - p|t - v(x)|^{p-2} (t - v(x)) dx \leq 0,$$

which, since  $-v(x) < -v'(x)$ , is not true unless  $u \geq u'$  a.e.

Now, assume  $v$  is  $L$ -Lipschitz and let  $u = u_p$  be the minimizer of (2.19). For  $z \in \mathbb{R}^d$ ,  $\varepsilon > 0$ , let  $v'(x) = v(x - z) - L|z| - \varepsilon < v(x)$  and  $u'(x) = u(x - z) - L|z| - \varepsilon$  be the solution of (2.19) with  $v$  replaced with  $v'$ . The Euler-Lagrange equations for  $v$  and  $v'$  are:

$$\begin{cases} -\partial\Phi(u) + \frac{1}{\tau} \left( \int_{\Omega} |u - v|^p dx \right)^{2/p-1} |u - v|^{p-2} (u - v) = 0, \\ -\partial\Phi(u') + \frac{1}{\tau} \left( \int_{\Omega} |u' - v'|^p dx \right)^{2/p-1} |u' - v'|^{p-2} (u' - v') = 0 \end{cases}$$

hence letting  $\lambda = \|u - v\|^{2-p}/\tau = \|u' - v'\|^{2-p}/\tau$ , we find that  $u$  is a minimizer of (2.20) while  $u'$  is a minimizer of the same problem with  $v$  replaced with  $v'$ . We deduce that  $u' \leq u$ .

Sending  $\varepsilon \rightarrow 0$ , it follows that

$$u(x - z) - L|z| \leq u(x) \quad \forall x \in \Omega, z \in \mathbb{R}^d$$

which shows that  $u$  is  $L$ -Lipschitz. Letting  $p \rightarrow 1$ , we eventually find a  $L$ -Lipschitz solution to (2.18).

We now observe that if there is another minimizer  $u' \in BV(\mathbb{R}^d/\mathbb{Z}^d)$  of (2.18), by strict convexity arguments, the absolutely continuous part of the gradient must be the same as  $Du$ , and they can differ only by a singular part. In addition (Remark 2.2-(1)),  $\|u - v\|_1 = \|u' - v\|_1$  so that the energy of  $u'$  is  $\int_{\Omega} F(D^a u') dx + \int_{\Omega} F^{\infty}(D^s u') + \|u' - v\|_1^2/(2\tau) = \int_{\Omega} F(D^a u) dx + \int_{\Omega} F^{\infty}(D^s u') + \|u - v\|_1^2/(2\tau)$  so that  $\int_{\Omega} F^{\infty}(D^s u') = 0$  and  $u = u'$  (up to a possible constant, but then using that  $v$  is Lipschitz and  $\|u - v\|_1 = \|u' - v\|_1$  shows that they cannot differ). Hence there is a unique minimizer of (2.18).  $\square$

Hence, we obtain the following result:

**Theorem 2.16.** *Let  $u^0$  a Lipschitz function over  $\Omega = \mathbb{R}^d/\mathbb{Z}^d$ . Then the discrete motion converges to  $u(t) \in C^0([0, T]; L^p(\Omega))$  for any  $p < d/(d-1)$ , with  $\int_s^{\infty} |D\dot{u}|^2 dt \leq C\Phi(u^0)/s$  for any  $s > 0$ , and  $u$  satisfies*

$$\begin{cases} |\kappa_u(x)| \leq \|\dot{u}\|_1 & \text{a.e. in } \Omega, \\ -\dot{u}\kappa_u(x) = |\dot{u}|\|\dot{u}\|_1 & \text{a.e. in } \Omega \end{cases} \quad \text{for a.e. } t \geq 0,$$

where  $\kappa_u = \operatorname{div} \left( Du / \sqrt{1 + |Du|^2} \right)$  a.e.

*Remark 2.17.* The proof of the existence of a unique  $L$ -Lipschitz solution on the torus when  $u^0$  is  $L$ -Lipschitz only relies on the strict convexity of  $F$ , so that the results in this section are also true for any  $F$  which is strongly convex on bounded subsets of  $\mathbb{R}^d$ .

### 3. MONOTONE SOLUTIONS

In this section we consider the case of Dirichlet boundary conditions ( $\operatorname{dom}(\Phi) = u^0 + H_1^0(\Omega)$ ), and we assume that  $u^0 \in BV(\Omega)$  a *subsolution* in the following sense:

**Definition 3.1.** *We say that  $u^0 \in BV(\Omega)$  is a subsolution if for any  $v \in BV(\Omega)$  with  $\{v \neq u^0\} \subset\subset \Omega$ , we have*

$$v \leq u^0 \Rightarrow \Phi(v) \geq \Phi(u^0).$$

**Lemma 3.2.** *If  $u^0$  is a subsolution then, for any  $v \in BV(\Omega)$  with  $\{v \neq u^0\} \subset\subset \Omega$ , we have*

$$\Phi(\max\{u^0, v\}) \leq \Phi(v).$$

*Proof.* Since  $u_0$  is a subsolution, we know that  $\Phi(\min\{u^0, v\}) \geq \Phi(u^0)$ . Recalling that

$$\Phi(\min\{u^0, v\}) + \Phi(\max\{u^0, v\}) \leq \Phi(v) + \Phi(u^0),$$

it follows that  $\Phi(\max\{u^0, v\}) \leq \Phi(v)$ .  $\square$

Replacing  $u^0$  with  $\max\{u^0, u^1\}$  in the variational problem which defines  $u^1$ , we find that  $u^1 \geq u^0$  a.e. in  $\Omega$ ; in particular, the Euler-Lagrange equation reads:

$$\partial\Phi(u^1) + \frac{\|u^1 - u^0\|_1}{\tau} \varphi = 0, \quad \varphi \in \operatorname{sign}(u^1 - u^0) \quad \text{a.e. in } \Omega.$$

**Proposition 3.3.** *If  $u^0$  is a subsolution then, for any  $n \geq 1$ ,  $u^n \geq u^{n-1}$  and  $u^n$  is also a subsolution.*

*Proof.* This follows the proof of a similar result in [15] for mean-convex sets, see also Sec. 5.1. By Lemma 3.2, for any  $v \in BV(\Omega)$  with  $\{v \neq u^0\} \subset\subset \Omega$ , we have

$$\Phi(\max\{u^0, v\}) \leq \Phi(v).$$

Let  $v \in BV(\Omega)$  with  $\{v \neq u^0\} \subset\subset \Omega$ , and assume  $v \leq u^1$ . We have

$$\begin{aligned} \Phi(v) &\geq \Phi(\max\{u^0, v\}) \geq \Phi(u^1) + \int_{\Omega} -\frac{\|u^1 - u^0\|_1}{\tau} \varphi(\max\{u^0, v\} - u^1) dx \\ &= \Phi(u^1) + \frac{\|u^1 - u^0\|_1}{\tau} \int_{\{u^1 > u^0\}} (u^1 - \max\{u^0, v\}) dx \geq \Phi(u^1) \end{aligned}$$

showing that  $u^1$  is also a subsolution, and the thesis follows by iterating the argument.  $\square$

Let us set now  $\lambda_1 = \|u^1 - u^0\|_1/\tau$ . We observe that for any  $v \geq u^0$  with  $v - u^0$  with compact support, one has

$$\Phi(v) \geq \Phi(u^1) + \lambda_1 \int_{\Omega} -\varphi(v - u^1) dx.$$

Since  $v \geq u^0$ , we get

$$\begin{aligned} \int_{\Omega} -\varphi(v - u^1) dx &= \int_{\{u^1 > u^0\}} u^1 - v dx + \int_{\{u^1 = u^0\}} -\varphi(v - u^0) dx \\ &\geq \int_{\{u^1 > u^0\}} u^1 - v dx - \int_{\{u^1 = u^0\}} v - u^0 dx = \int_{\Omega} u^1 - v dx, \end{aligned}$$

and we deduce that

$$\Phi(v) + \lambda_1 \int_{\Omega} v dx \geq \Phi(u^1) + \lambda_1 \int_{\Omega} u^1 dx.$$

It follows that  $u^1$  is a solution of the obstacle problem (with Dirichlet boundary conditions)

$$\min_{v \geq u^0} \Phi(v) + \lambda_1 \int_{\Omega} v dx.$$

Notice that, if  $F$  has superlinear growth and is strictly convex, the solution is unique.<sup>1</sup> Observe also that, if  $v, v'$  are minimizers of the above obstacle problem for, respectively, two different non-negative parameters  $\lambda$  and  $\lambda'$ , then the inequality

$$\Phi(v) + \lambda \int_{\Omega} v dx + \Phi(v') + \lambda' \int_{\Omega} v' dx \leq \Phi(v \wedge v') + \lambda \int_{\Omega} v \wedge v' dx + \Phi(v \vee v') + \lambda' \int_{\Omega} v \vee v' dx$$

shows that  $(\lambda - \lambda') \int_{\Omega} (v - v')^+ dx \leq 0$ . Hence, if  $\lambda > \lambda'$  one has  $v \leq v'$ .

Let us now introduce, for  $m \geq 0$ , the volume function

$$f(m) := \min \left\{ \Phi(v) : v \geq u^0, v = u^0 \text{ on } \partial\Omega, \int_{\Omega} v - u^0 dx = m \right\}. \quad (3.1)$$

<sup>1</sup>When  $F$  has linear growth such a statement is unclear, we only know that, for all  $\lambda_1$  but a countable number, the solution is unique, otherwise it is trapped in between a minimal and a maximal solution.



From now on we shall assume that  $F$  is strictly convex and superlinear. For any  $\lambda \in \mathbb{R}$ , we define  $v^\lambda$  as the solution of the obstacle problem with parameter  $\lambda$ , that is, the unique minimizer of

$$\min_{v \geq u^0, v = u^0 \text{ on } \partial\Omega} \Phi(v) + \lambda \int_{\Omega} v \, dx. \quad (3.2)$$

Observe that  $\lim_{\lambda \rightarrow +\infty} v^\lambda = u^0$ . Thanks to the uniqueness of the solution and the comparison principle, we observe that the domain  $\mathcal{D} := \{(x, z) : x \in \Omega, u^0(x) < z < \sup_{\lambda \leq 0} v^\lambda(x)\}$  is such that for any  $(x, z) \in \mathcal{D}$ , there is a unique  $\lambda > 0$  such that  $z = v^\lambda(x)$ . Indeed, since both  $\sup_{\lambda' > \lambda} v^{\lambda'}$  and  $\inf_{\lambda' < \lambda} v^{\lambda'}$  are minimizers of (3.2), they must coincide for all  $\lambda$ . In particular, the function

$$\lambda \mapsto \int_{\Omega} v^\lambda - u^0 \, dx =: m^\lambda$$

is continuous and decreasing, going from 0 as  $\lambda \rightarrow +\infty$ , to some maximal value  $\bar{m} \leq +\infty$  as  $\lambda \rightarrow -\infty$ <sup>2</sup>. One can check easily that for any  $m$ , one has  $f(m) = \Phi(v^\lambda)$  for any  $\lambda$  such that  $m = m^\lambda$ . On the other hand, if  $v'$  is another minimizer of (3.1), for  $m = m^\lambda$ , then since  $\Phi(v') + \lambda \int_{\Omega} v' \, dx = \Phi(v^\lambda) + \lambda \int_{\Omega} v^\lambda \, dx$ ,  $v'$  also minimizes (3.2) and by uniqueness  $v' = v^\lambda$ .

In addition, given  $m, m'$  and corresponding  $\lambda, \lambda'$ , we have

$$f(m') + \lambda m' = \Phi(v^{\lambda'}) + \lambda \int_{\Omega} v^{\lambda'} - u^0 \, dx \geq \Phi(v^\lambda) + \lambda \int_{\Omega} v^\lambda - u^0 \, dx = f(m) + \lambda m,$$

showing that

$$f \text{ is convex and } -\lambda \in \partial f(m). \quad (3.3)$$

Observe that, in case  $F$  is not superlinear or not strictly convex, one can still build by approximation an increasing family of minimizers with increasing masses, minimizing the obstacle problem for some non-increasing multipliers, but one might lose uniqueness.

**Proposition 3.4.** *Let  $F$  be strictly convex and superlinear, and let  $u^0$  be a subsolution, then  $u^n = v^{\lambda_n}$  for any  $n \geq 1$ , where  $\lambda_n := \|u^n - u^{n-1}\|_1/\tau$ .*

*Proof.* By the above analysis  $u^n$  is the unique solution of

$$\min_{v \geq u^{n-1}} \Phi(v) + \lambda_n \int_{\Omega} v \, dx$$

with  $\lambda_n = \|u^n - u^{n-1}\|_1/\tau$ . We show by induction that this is also  $v^{\lambda_n}$ , knowing that it is true for  $n = 1$ . Assume it holds for  $u^{n-1}$ , then by comparison principle and the fact  $\lambda_n$  is non-increasing (see Theorem 2.8), one has  $v^{\lambda_n} \geq u^{n-1}$ . Hence we get

$$\Phi(u^n) + \lambda_n \int_{\Omega} u^n \, dx \leq \Phi(v^{\lambda_n}) + \lambda_n \int_{\Omega} v^{\lambda_n} \, dx.$$

But since  $u^n \geq u^0$ , the reverse inequality is also true, hence  $u^n$  and  $v^{\lambda_n}$  are both minimizers of the obstacle problem for  $\lambda_n$ . By uniqueness, we deduce that they coincide.  $\square$

Observe that the value  $\lambda_n$  can be also built as follows: given  $\lambda_{n-1}$ , when  $\lambda$  decreases from  $\lambda_{n-1}$  to 0, then  $\|v^\lambda - u^{n-1}\|_1/\tau$  increases from 0 to  $\|v^0 - v^{\lambda_{n-1}}\|_1/\tau > 0$ , and there is a value in  $(0, \lambda_{n-1})$  for which they coincide. Moreover, letting  $m_n = \int_{\Omega} u^n - u^0 \, dx$ , by Proposition 3.4 and (3.3) we have

$$\frac{m_n - m_{n-1}}{\tau} = \lambda_n \in -\partial f(m_n), \quad (3.4)$$

<sup>2</sup>If domain of  $F$  is not the entire space, the maximal reachable mass  $\bar{m}$  can be finite.

for any  $n \geq 1$ , so that the sequence  $(m_n)_n$  solves the discrete implicit Euler scheme for the gradient flow of the convex function  $f$ .

**Theorem 3.5.** *Let  $F$  be strictly convex and superlinear, and let  $u^0$  be a subsolution with  $\Phi(u^0) < +\infty$ . Then there exists a unique limit solution  $u$  given by Theorem 2.3 with initial datum  $u^0$ . Moreover, the function  $u$  is non-decreasing in  $t$ , and  $u(t) = v^{\lambda(t)}$  for a.e.  $t > 0$ , where  $\lambda(t) \in L^2((0, +\infty))$  is positive and non-increasing.*

*Proof.* The monotonicity of  $u$  in  $t$  follows directly from Proposition 3.3.

Letting  $\lambda_\tau(t) = \lambda_{\lfloor t/\tau \rfloor + 1}$  and  $m_\tau(t) = m_{\lfloor t/\tau \rfloor + 1}$  for  $t \geq 0$ , by Helly's Theorem we may assume that, up to a subsequence,  $\lambda_\tau$  and  $m_\tau$  converge pointwise to functions  $\lambda(t)$  and  $m(t)$  which are respectively non-increasing and non-decreasing. By Proposition 3.4 we then get that  $u_\tau(t) \rightarrow u(t) = v^{\lambda(t)}$  and  $\int_\Omega u_\tau(t) - u^0 dx \rightarrow m(t) = m^{\lambda(t)}$  as  $\tau \rightarrow 0$ , for a.e.  $t > 0$ .

Recalling (3.4) we also have that  $m$  is the unique solution of the gradient flow

$$\dot{m} + \partial f(m) \ni 0$$

with initial value  $m(0) = 0$ , see for instance [8]. It follows that  $u(t)$  is the solution of (3.1) for  $m = m(t)$ , and since the latter is unique, we deduce that also the limit flow  $u(t)$  is unique, and that  $u_\tau \rightarrow u$  as  $\tau \rightarrow 0$ , without passing to a subsequence. The fact that  $\lambda \in L^2((0, +\infty))$  follows by the dissipation estimate (2.10).  $\square$

*Remark 3.6.* Observe that  $\|u^{n\tau} - u^{m\tau}\|_1 = \tau \sum_{l=m+1}^n \lambda_l = \int_{m\tau}^{n\tau} \lambda_\tau(s) ds$ , hence

$$\|u(t_2) - u(t_1)\|_1 = \int_{t_1}^{t_2} \lambda(s) ds \quad \text{for all } 0 \leq t_1 < t_2,$$

which is equivalent to

$$m(t) = \int_\Omega v^{\lambda(t)}(x) - u^0(x) dx = \int_0^t \lambda(s) ds \quad \text{for all } t \geq 0. \quad (3.5)$$

If  $F$  is of class  $C^1$  in  $\mathbb{R}^d$  and superlinear, recalling that the functions  $v^\lambda$  satisfy

$$-\operatorname{div} \nabla F(Dv^\lambda) + \lambda = 0 \quad \text{a.e. in } \{v^\lambda > u^0\}$$

(see Lemma A.7), equation (3.5) implies (2.17).

*Remark 3.7.* If  $F$  is of class  $C^1$  in  $\mathbb{R}^d$  and superlinear one can check that two different values of  $\lambda$  yield different functions (when  $v^\lambda > u^0$ , since in that case one has  $-\operatorname{div} \nabla F(Dv^\lambda) + \lambda = 0$  a.e. in  $\{v^\lambda > u^0\}$ ). Then, using that  $t \mapsto u(t) = v^{\lambda(t)}$  is Hölder continuous in  $L^1(\Omega)$  by Theorem 2.3, it follows that  $\lambda(t)$  is continuous.

#### 4. THE DIRICHLET ENERGY

In this section, we consider the simplest case  $F(\xi) = |\xi|^2/2$ , so that

$$\Phi(u) = \frac{1}{2} \int_\Omega |Du|^2$$

is the Dirichlet energy of  $u$ .

In what follows, we shall consider either the case of Dirichlet boundary conditions ( $\operatorname{dom}(\Phi) = u^0 + H_0^1(\Omega)$ ), or the case of homogeneous Neumann boundary conditions ( $\operatorname{dom}(\Phi) = H^1(\Omega)$ ).

**4.1. Uniqueness.** Assuming that  $\Phi(u^0) < +\infty$ , the limit solution  $u$  provided by Theorem 2.3 satisfies

$$\int_{\Omega} |Du(t, x)|^2 dx + \frac{1}{2} \int_0^t \|\dot{u}(s)\|_1^2 ds + \frac{1}{2} \int_0^t \|\Delta u(s)\|_{\infty}^2 ds \leq \int_{\Omega} |Du^0|^2 dx, \quad (4.1)$$

which is (2.10), and we take into account (cf Theorem 2.12) that  $\dot{u} \in L^{\infty}([t, +\infty]; H^1(\Omega))$  for any  $t > 0$ , and  $\partial\Phi(u(t)) = \{-\Delta u(t)\}$  for a.e.  $t \in (0, +\infty)$ . As usual, this can be rewritten:

$$\int_0^t \left( \int_{\Omega} Du(s, x) \cdot D\dot{u}(s, x) dx + \frac{1}{2} \|\dot{u}(s)\|_1^2 + \frac{1}{2} \|\Delta u(s)\|_{\infty}^2 \right) ds \leq 0,$$

which yields  $\Delta u(s) \in \|\dot{u}\|_1 \text{sign}(\dot{u})$  a.e. in  $\Omega$ , for a.e.  $s \in [0, T]$  (for any  $T > 0$ ), and we obtain the equations

$$\begin{cases} |\Delta u| \leq \|\dot{u}\|_1 & \text{a.e. in } \Omega \\ \dot{u}\Delta u = |\dot{u}|\|\dot{u}\|_1 & \text{a.e. in } \Omega \end{cases} \quad \text{a.e. in } [0, T], \quad (4.2)$$

cf (2.17).

It turns out that this defines a *unique* evolution starting from  $u^0 \in H^1(\Omega)$ . Indeed, for different solutions  $u, v$  of (4.2) we have

$$\frac{d}{dt} \int_{\Omega} |Du - Dv|^2 dt = 2 \int_{\Omega} (Du - Dv) \cdot D(\dot{u} - \dot{v}) dx = -2 \int_{\Omega} (\Delta u - \Delta v) \cdot (\dot{u} - \dot{v}) dx \leq 0. \quad (4.3)$$

**Theorem 4.1.** *For any  $u^0 \in H^1(\Omega)$ , there is a unique flow  $u \in C^0([0, +\infty); H^1(\Omega))$  which solves (4.2). In addition, the minimizing movements  $\hat{u}_{\tau}$  converge to  $u$  in  $C^0((0, +\infty); H^1(\Omega))$  (i.e., locally uniformly in time), as  $\tau \rightarrow 0$ . The semi-norm of the speed  $\|D\dot{u}\|_2$  is non-increasing in time.*

*Proof.* The uniqueness follows from (4.3) in the case of Dirichlet boundary conditions. In the case of Neumann conditions, assume we have two different solutions  $u(t)$  and  $u(t) + c(t)$ ,  $c(t) \in \mathbb{R}$  (with  $c \in H^1(\mathbb{R}_+)$  thanks to (4.1)). The equations state, then, that for a.e.  $t$  and a.e.  $x \in \Omega$ , both  $\dot{u}\Delta u = |\dot{u}|\|\dot{u}\|_1$  and  $(\dot{u} + \dot{c})\Delta u = |\dot{u} + \dot{c}|\|\dot{u} + \dot{c}\|_1$ . In case  $\{\Delta u = 0\}$  has positive measure, we deduce that either  $\|\dot{u}\|_1 = 0$  but then  $0 = |\dot{c}|^2|\Omega|$  on a set of positive measure, meaning  $\dot{c} = 0$ , or  $|\dot{u}| = 0$  on a set of positive measure and on the same set,  $|\dot{c}|\|\dot{u} + \dot{c}\|_1 = 0$ . Hence again,  $\dot{c} = 0$  (or we would have  $\dot{u} \equiv -\dot{c} \neq 0$  a.e., a contradiction).

Hence, we assume  $|\{\Delta u = 0\}| = 0$ . Since in addition,  $\int_{\Omega} \Delta u = 0$  (in the case of Neumann conditions),  $\Omega$  is split into two sets  $\Omega^{\pm}$  of positive measure, with  $\Delta u > 0$  a.e. in  $\Omega^+$ , hence  $\dot{u} \geq 0$ , and  $\Delta u < 0$  a.e. in  $\Omega^-$ , hence  $\dot{u} + \dot{c} \leq 0$ . This contradicts  $\dot{u} \in H^1(\Omega)$  if  $\dot{c} > 0$ , since one would have  $|\{-\dot{c} \leq \dot{u} \leq 0\}| = 0$ . Symmetrically, one cannot have  $\dot{c} < 0$ , hence  $\dot{c} = 0$ . We deduce that  $c(t) = 0$  (since  $c$  must be continuous with  $c(0) = 0$ ), and this proves uniqueness in the Neumann case.

The convergence of  $\hat{u}_{\tau}$  to the unique possible limit  $u$  is guaranteed by Theorem 2.3 and Proposition 2.5, at least in  $C^0([0, T]; L^p(\Omega))$  for any  $T > 0$  and  $p < d/(d-1)$ . In addition, it follows from (2.16) by standard arguments that  $\|D\hat{u}_{\tau}(t) - D\hat{u}_{\tau}(s)\|_2^2 \leq 2/\min\{t, s\} \int_{\Omega} |Du^0|^2 dx$ , from which we also deduce the uniform convergence on any interval  $[t, T]$ ,  $T > t > 0$ .

Eventually, the fact that the speed is non-increasing in  $H^1$  follows from the fact that, using (4.3),  $\|Du(t + \varepsilon) - Du(t)\|_2 \leq \|Du(s + \varepsilon) - Du(s)\|_2$  for any  $t > s > 0$  and any  $\varepsilon > 0$ .  $\square$

We observe that the contraction property in the time-continuous setting also has a counterpart for the discrete flow:

**Lemma 4.2.** *Let  $v, v' \in L^1(\Omega)$ ,  $v - v' \in H^1(\Omega)$  (resp.  $H_0^1(\Omega)$  in the case of Dirichlet boundary conditions) and assume  $u$  is a minimizer of*

$$\frac{1}{2\tau} \|u - v\|_1^2 + \Phi(u)$$

and  $u'$  a minimizer of the same problem with  $v$  replaced with  $v'$ . Then:

$$\frac{1}{2} \int_{\Omega} |D(u - u')|^2 dx \leq \frac{1}{2} \int_{\Omega} |D(v - v')|^2 dx - \frac{1}{2} \int_{\Omega} |D(u - u' - v + v')|^2 dx. \quad (4.4)$$

and in particular

$$\|Du - Du'\|_2 \leq \|D(v - v')\|_2.$$

*Proof.* Subtracting the Euler-Lagrange equations for  $u$  and  $u'$ , multiplying by  $(u - v) - (u' - v)$  and integrating by parts, we get:

$$\int_{\Omega} (Du - Du') \cdot (D(u - v - u' + v')) dx \leq 0$$

thanks to the monotonicity of the subgradient of  $\|\cdot\|_1^2/2$ . It follows

$$\int_{\Omega} |D(u - u')|^2 dx \leq \int_{\Omega} D(u - u') \cdot D(v - v') dx$$

from which we deduce (4.4).  $\square$

Specializing (4.4) to the case  $v = u^n$ ,  $v' = u^{n-1}$ , for  $n \geq 1$ , we find:

$$\int_{\Omega} |D(u^{n+1} - u^n)|^2 dx \leq \int_{\Omega} |D(u^n - u^{n-1})|^2 dx - \int_{\Omega} |D(u^{n+1} - 2u^n + u^{n-1})|^2 dx. \quad (4.5)$$

which shows that also for the discrete flow one has that  $\|D\dot{u}_\tau\|_2$  is non-increasing in time.

**4.2. Energy decay estimate.** In the case of homogeneous Neumann or Dirichlet boundary conditions, we expect that  $\lim_{t \rightarrow \infty} \int_{\Omega} |Du(t)|^2 dx = 0$ . Actually, we can even provide the rate of convergence:

**Proposition 4.3.** *Let  $u$  solve (4.2), with  $u^0 \in H^1(\Omega)$  (with Neumann boundary conditions) or  $u^0 \in H_0^1(\Omega)$  (with Dirichlet boundary conditions). Then, for any  $t > 0$  we have*

$$\int_{\Omega} |Du(t)|^2 dx \leq e^{-\frac{t}{c_\Omega}} \int_{\Omega} |Du^0|^2 dx,$$

where  $c_\Omega$  is the constant in the Poincaré-Wirtinger (Neumann) or Poincaré (Dirichlet) inequality.

*Proof.* We consider the minimizing movement scheme, and, given  $\tau > 0$ ,  $n \geq 1$ , we compare the energy of  $u^n$  with the energy of  $u_a := u^{n-1} + a\tau(u^{n-1} - m^{n-1})$ ,  $a \in \mathbb{R}$ , where  $m^{n-1} = \int_{\Omega} u^{n-1} dx / |\Omega|$  is the average of  $u^{n-1}$  for Neumann boundary conditions, and 0 for homogeneous Dirichlet boundary conditions. One has in particular, thanks to the Poincaré(-Wirtinger) inequality:

$$\|u^{n-1} - u_a\|_1^2 = a^2 \tau^2 \|u^{n-1} - m^{n-1}\|_1^2 \leq a^2 \tau^2 c_\Omega \int_{\Omega} |Du^{n-1}|^2.$$

Hence:

$$\begin{aligned} \frac{1}{2} \int_{\Omega} |Du^n|^2 dx + \frac{1}{2\tau} \|u^n - u^{n-1}\|_1^2 \\ \leq (1 + 2a\tau + \tau^2 a^2) \frac{1}{2} \int_{\Omega} |Du^{n-1}|^2 dx + \frac{\tau}{2} a^2 c_{\Omega} \int_{\Omega} |Du^{n-1}|^2. \end{aligned}$$

Choosing  $a = -1/c_{\Omega}$ , we find:

$$\frac{1}{2} \int_{\Omega} |Du^n|^2 dx \leq \frac{1}{2} \int_{\Omega} |Du^{n-1}|^2 \left(1 - \frac{\tau}{c_{\Omega}} + \frac{\tau^2}{c_{\Omega}^2}\right).$$

Hence,

$$\frac{1}{2} \int_{\Omega} |Du^n|^2 dx \leq \frac{1}{2} \int_{\Omega} |Du^0|^2 \left(1 - \frac{\tau}{c_{\Omega}} + \frac{\tau^2}{c_{\Omega}^2}\right)^n.$$

We conclude using that

$$\lim_{n \rightarrow \infty, n\tau \rightarrow t} \left(1 - \frac{\tau}{c_{\Omega}} + \frac{\tau^2}{c_{\Omega}^2}\right)^n = e^{-\frac{t}{c_{\Omega}}}.$$

□

## 5. GRADIENT FLOW OF ANISOTROPIC PERIMETERS

Given a norm  $\varphi$  on  $\mathbb{R}^d$ , we consider the anisotropic perimeter

$$E \mapsto P_{\varphi}(E) := \int_{\partial E} \varphi(\nu) d\mathcal{H}^{d-1}.$$

Letting  $\varphi^{\circ}$  be the dual norm of  $\varphi$ , we recall that the convex set

$$W_{\varphi} := \{x \in \mathbb{R}^d : \varphi^{\circ}(x) \leq 1\},$$

usually called *Wulff Shape*, is the unique volume-constrained minimizer of  $P_{\varphi}$ , up to translations and dilations. We say that  $\varphi$  is smooth (resp. elliptic) if the function  $\varphi^2/2$  is smooth (resp. strongly convex).

We now introduce the geometric  $L^1$ -minimizing movement scheme. Given  $\tau > 0$  and  $E \subset \mathbb{R}^d$ , we consider the minimum problem

$$\min_F P_{\varphi}(F) + \frac{1}{2\tau} |E \Delta F|^2, \tag{5.1}$$

and we let  $T_{\tau}E$  be a (possibly non-unique) minimizer of (5.1).

Given an initial set  $E^0 \subset \mathbb{R}^d$ , for all  $n \in \mathbb{N}$  we let  $E^n := T_{\tau}^n E^0$ . For  $(t, x) \in (0, +\infty) \times \mathbb{R}^d$ , we also let

$$E_{\tau}(t) := E^{\lfloor t/\tau \rfloor} \quad u_{\tau}(t, x) := \chi_{E_{\tau}(t)}(x).$$

The function  $t \mapsto E_{\tau}(t)$  is the discrete  $L^1$ -gradient flow of  $P_{\varphi}$ , with initial datum  $E^0$ .

We point out that the analogous concept for the  $L^2$ -gradient flow of  $P_{\varphi}$ , where (5.1) is replaced by the problem

$$\min_F P_{\varphi}(F) + \frac{1}{2\tau} \int_{E \Delta F} \text{dist}(x, \partial E) dx, \tag{5.2}$$

was originally introduced in [1, 22] as a discrete approximation of the mean curvature flow. It is shown in these references that the “distance term” in (5.2) is indeed a (non symmetric) approximation of the squared 2-distance between the boundaries of  $E$  and  $F$  (a “flat” version

could be something like  $\int_x \int_{t \in [e(x), f(x)]} |t - e(x)| dt dx = \frac{1}{2} \int_x |e(x) - f(x)|^2 dx$ , at least when these boundaries are smooth enough.

From (5.1) it follows that

$$P_\varphi(E^n) \leq P_\varphi(E^{n-1}) \quad \text{and} \quad \frac{1}{2\tau} |E^n \Delta E^{n-1}|^2 \leq P_\varphi(E^{n-1}) - P_\varphi(E^n), \quad (5.3)$$

for all  $n \in \mathbb{N}$ , so that

$$\begin{aligned} |E^n \Delta E^m|^2 &\leq \left( \sum_{k=m+1}^n |E^k \Delta E^{k-1}| \right)^2 \\ &\leq 2\tau(n-m) \sum_{k=m+1}^n \left( P_\varphi(E^{k-1}) - P_\varphi(E^k) \right) \\ &\leq 2\tau(n-m) P_\varphi(E^0), \end{aligned} \quad (5.4)$$

for all  $0 \leq m < n$ .

Reasoning as in the proof of Theorem 2.3, from (5.3) and (5.4) we get the following result.

**Theorem 5.1.** *Assume that  $E^0 \subset \mathbb{R}^d$  is a set of finite perimeter. Then there exist a sequence  $\tau_k \rightarrow \infty$  and a function  $u(x, t) \in L^\infty((0, +\infty), BV(\mathbb{R}^d)) \cap C^{1/2}((0, +\infty), L^1(\mathbb{R}^d))$ , with  $u(x, t) = \chi_{E(t)}(x)$  for some family of sets  $E(t)$ , such that*

$$\lim_{k \rightarrow +\infty} \sup_{t \in [0, T]} \|u(\cdot, t) - u_{\tau_k}(\cdot, t)\|_{L^1(\mathbb{R}^d)} = 0 \quad \forall T > 0.$$

Following [22, Lemma 1.3, Remark 1.4], we show a density estimate for minimizers of (5.1).

**Lemma 5.2.** *There exists  $c > 0$  depending only on  $\varphi$  and the dimension  $d$  such that the following holds: let  $E \subset \mathbb{R}^d$  and  $F$  a minimizer of (5.1), then*

- (1) *for a.e.  $x \in F \setminus E$  and all  $r > 0$  such that  $|B(x, r) \cap E| = 0$ , we have  $|B(x, r) \cap F| \geq cr^d$ ;*
- (2) *for a.e.  $x \notin E$ , and  $r > 0$  such that  $|B(x, r) \cap E| = 0$ , if  $|B(x, r) \cap F| \leq cr^d/2$ , then  $B(x, r/2) \cap F = \emptyset$ .*

*Proof.* Following [22] we compare the energy of  $F$  and  $F \setminus B(x, r)$  in (5.1) and, introducing  $b > a > 0$  s.t.  $a|x| \leq \varphi(x) \leq b|x|$ , we observe that for a.e.  $r > 0$ , if  $|B(x, r) \cap E| = 0$ :

$$P_\varphi(F) + \frac{1}{2\tau} |E \Delta F|^2 \leq P_\varphi(F \setminus B(x, r)) + \frac{1}{2\tau} |(E \Delta F) \setminus B(x, r)|^2$$

implies  $a\mathcal{H}^{d-1}(\partial F \cap B(x, r)) \leq b\mathcal{H}^{d-1}(\partial B(x, r) \cap F)$ . Introducing  $f(r) = |F \cap B(x, r)|$ , this is rewritten as  $(a+b)\mathcal{H}^{d-1}(\partial(F \cap B(x, r))) \leq f'(r)$ , and using the isoperimetric inequality we deduce that for some constant  $\gamma > 0$  depending only on  $d, a, b$ , there holds  $\gamma f(r)^{1-1/d} \leq f'(r)$ . The thesis follows from a version of Gronwall's Lemma.  $\square$

*Remark 5.3.* A symmetric statement holds for points  $x \in E$ , with  $B(x, r) \cap F$  replaced with  $B(x, r) \setminus F$ .

*Remark 5.4.* A similar proof (see [22] again) shows that there exists  $r(\tau) > 0$  such that for  $r < r(\tau)$ , for a.e.  $x \in F$ ,  $|B(x, r) \cap F| \geq cr^d$ , and for a.e.  $x \notin F$ ,  $|B(x, r) \cap F^c| \geq cr^d$ . In particular, the points of Lebesgue density 1 (resp. 0) of  $F$  form an open set, the reduced boundary of  $F$  is  $\mathcal{H}^{d-1}$ -essentially closed, and there is no abuse of notation in denoting it  $\partial F$ .

### 5.1. Outward minimizing case.

**Definition 5.5.** Let  $\Omega \subset \mathbb{R}^d$  be open. We say that a set  $E \subset \Omega$  is outward minimizing if

$$P_\varphi(E) \leq P_\varphi(F) \quad \forall F \supset E, F \subset \Omega.$$

Notice that if  $\varphi$  is smooth and  $E$  is an outer minimizer with boundary of class  $C^2$  then  $E$  is  $\varphi$ -mean convex, that is,  $H_\varphi(x) \geq 0$  for any  $x \in \partial E$ , where  $H_\varphi(x)$  is the  $\varphi$ -mean curvature of  $\partial E$  at  $x$  (see for instance [12] for a precise definition). Conversely, if  $H_\varphi(x) \geq \delta > 0$  for any  $x \in \partial E$  one can build  $\Omega \supset \supset E$  such that  $E$  is outward minimizing in  $\Omega$ . Notice also that a convex set is always outward minimizing.

We recall the following result proved in [15, Lemma 2.5] (see also [12, Section 2.1]).

**Lemma 5.6.**  $E$  is outward minimizing if and only if

$$P_\varphi(E \cap F) \leq P_\varphi(F) \quad \forall F \subset \Omega.$$

From Lemmas 5.2 and 5.6 we obtain the following result.

**Proposition 5.7.** Assume that  $E \subset \subset \Omega$  is outward minimizing in  $\Omega$ . Then, for  $\tau$  small enough (depending only on  $\varphi$ ,  $\text{dist}(E, \partial\Omega)$ , and the dimension) we have that  $T_\tau E \subseteq E$  and  $T_\tau E$  is outward minimizing in  $\Omega$ .

In particular, the limit flow obtained in Theorem 5.1 is non-increasing and outward minimizing in  $\Omega$ .

*Proof.* The first assertion follows from the minimality of  $T_\tau E$  and from the fact that

$$P_\varphi(T_\tau E \cap E) + \frac{1}{2\tau} |(T_\tau E \cap E) \Delta E|^2 \leq P_\varphi(T_\tau E) + \frac{1}{2\tau} |T_\tau E \Delta E|^2,$$

with equality iff  $T_\tau E \subseteq E$ . We use here Lemma 5.6 which holds if we can prove first that  $T_\tau E \subset \Omega$ . Let  $r := \text{dist}(E, \partial\Omega)/2$ . Then for  $\tau$  small enough, we have (comparing the energy of  $T_\tau E$  and  $E$  in (5.1)) that  $|T_\tau E \setminus E| \leq \sqrt{2\tau P_\varphi(E)} \leq cr^d/2$  where  $c$  is the constant in Lemma 5.2. Using point (2) in Lemma 5.2, it follows that  $\{x : r/2 < \text{dist}(x, E) < 3r/2\} \cap T_\tau E = \emptyset$  and we deduce that  $T_\tau E \subset \{x : \text{dist}(x, E) \leq r/2\} \subset \subset \Omega$ .

In order to prove the second assertion, we fix  $F$  such that  $T_\tau E \subset F \subset \Omega$ , and we notice that

$$P_\varphi(T_\tau E) \leq P_\varphi(F \cap E) + \frac{1}{2\tau} |(F \cap E) \Delta E|^2 - \frac{1}{2\tau} |T_\tau E \Delta E|^2 \leq P_\varphi(F \cap E) \leq P_\varphi(F),$$

where the last inequality follows from the outward minimality of  $E$ . □

*Remark 5.8.* From Proposition 5.7 and (5.1) it follows that the set  $T_\tau E$  solves the minimum problem

$$\min_{F \subset E} P_\varphi(F) - \frac{1}{\tau} |E| |F| + \frac{1}{2\tau} |F|^2, \quad (5.5)$$

hence  $T_\tau E$  is also a solution of the volume-constrained isoperimetric problem (see also (5.12) later on)

$$\min_{F \subset E, |F|=|T_\tau E|} P_\varphi(F). \quad (5.6)$$

If  $\varphi$  is smooth and elliptic (that is,  $\varphi^2/2$  is smooth and strongly convex), from (5.5) it follows that  $T_\tau E \cap \text{int}(E)$  is smooth and satisfies the Euler-Lagrange equation

$$H_\varphi(x) = \frac{|E \setminus T_\tau E|}{\tau} \quad \text{for } x \in \partial T_\tau E \cap \text{int}(E). \quad (5.7)$$

If in addition  $\partial E$  is of class  $C^{1,1}$ , by classical regularity results for the obstacle problem [9, 11]  $\partial T_\tau E$  is also of class  $C^{1,1}$  outside a closed singular set of Hausdorff dimension  $d - 2$ , and satisfies the Euler-Lagrange inequality

$$0 \leq H_\varphi(x) \leq \frac{|E \setminus T_\tau E|}{\tau} \quad \text{for a.e. } x \in \partial T_\tau E. \quad (5.8)$$

Passing to the limit in (5.7) and (5.8) as  $\tau \rightarrow 0$ , and reasoning as in Theorem 2.14, we may expect that the limit flow  $E(t)$  satisfies the equations

$$\begin{cases} 0 \leq H_\varphi \leq -\frac{d}{dt}|E(t)| & \text{a.e. on } \partial E(t) \\ H_\varphi = -\frac{d}{dt}|E(t)| & \text{a.e. on } \partial E(t) \cap \text{int}(E^0), \end{cases} \quad (5.9)$$

for a.e.  $t > 0$ .

*Remark 5.9.* We cannot expect that there always exists a constant  $\lambda > 0$  such that  $T_\tau E$  is a solution of

$$\min_{F \subset E} P_\varphi(F) - \lambda|F|, \quad (5.10)$$

as it happens in the case of functions (see Section 3). In the sequel we shall see that, in the case  $E$  is convex, this is true only if  $E$  does not coincide with its Cheeger set, and  $\tau$  is small enough so that  $\lambda$  is greater than the Cheeger constant of  $E$ .

**5.2. Convex case.** We now consider the special case of a convex initial set.

**Proposition 5.10.** *Let  $d = 2$  and assume that  $\varphi$  is smooth and elliptic. Assume also that  $E^0$  is a bounded convex set. Then the limit flow  $E(t)$  obtained in Theorem 5.1 is given by a decreasing family of convex subsets of  $E^0$ .*

*Proof.* As in [24, Section 4.1] one can easily show that  $T_\tau E$  is a convex subset of  $E$  with boundary of class  $C^{1,1}$ , satisfying (5.7) and (5.8). As in [6, Section 9] (see also [20, Theorem 2.3]), it follows that each connected component of  $\partial T_\tau E \cap \text{int}(E)$  is a graph and it is contained in  $r\partial W_\varphi$ , with  $r = \tau/|E \setminus T_\tau E|$ . As a consequence, if  $r$  is greater than the inradius of  $E$ , then

$$T_\tau E = E_r^- := \bigcup_{x+rW_\varphi: (x+rW_\varphi) \subset E} (x+rW_\varphi),$$

otherwise  $T_\tau E = rW_\varphi + s$  for some segment  $s \subset E$ , and  $s$  is a point if  $r$  is smaller than the inradius of  $E$ .

By iterating the previous argument, and taking the limit as  $\tau \rightarrow 0$ , get the thesis. □

*Remark 5.11.* By the argument above we get that

$$E(t) = E_{r(t)}^- \quad t \in [0, T],$$

where  $r(t)$  is continuous, increasing, and  $T \geq 0$  is the first time such that  $r(T)$  equals the inradius of  $E$ . In particular, the limit flow is unique on  $[0, T]$ .

*Remark 5.12.* By approximating a general norm  $\varphi$  with a sequence of smooth and elliptic norms, following the proof of Proposition 5.10, we obtain that there exists  $r > 0$  such that  $T_\tau E = E_r^-$  or  $T_\tau E = rW_\varphi + s$  for some segment  $s \subset E$ .



As a consequence, also in the case of a general norm, there exists at least one limit flow  $E(t)$  given by a decreasing family of convex subsets of  $E^0$ . We point out that, in the general case, we do not prove uniqueness of the limit flow.

In [10] it has been proved that, in any dimension  $d \geq 2$ , a volume-constrained minimizer of  $P_\varphi$  inside a convex set  $E$  is unique and convex if its volume is greater or equal than the volume of the Cheeger set of  $E$ . We recall that the Cheeger set of  $E$  is the minimizer  $F^*$  of the variational problem

$$\min_{F \subset E} \frac{P_\varphi(F)}{|F|} =: \lambda^*$$

( $\lambda^*$  is called the Cheeger constant of  $E$ ). This Cheeger set is unique when  $E$  is convex [2,11,19]. The Cheeger set is also characterized as the largest minimizer ( $\emptyset$  being the smallest one) of the problem  $\min_{F \subset E} P_\varphi(F) - \lambda^*|F|$ , which has value 0.

For  $\lambda > \lambda^*$ , there is a unique minimizer  $F^\lambda$  to (5.10), which is convex, and coincides with the above volume-constrained minimizer (and is continuous with respect to  $\lambda$ , see [10]). Moreover, if  $\varphi$  is smooth and elliptic,  $\lambda$  coincides with the mean curvature  $H_\varphi$  of  $\partial F^\lambda \cap \text{int} E$  (otherwise it can be thought of as a variational mean curvature).

It follows that, as long as  $|E^n| \geq |F^*|$ , where  $F^*$  is the Cheeger set of  $E^0$ , we can define a non-increasing sequence  $\lambda_n \geq \lambda^*$  such that  $E^n = F^{\lambda_n}$  and which satisfies, for  $n \geq 1$ ,

$$\frac{|E^{n-1}| - |E^n|}{\tau} = \lambda_n,$$

or equivalently for all  $n \geq 1$ ,

$$|E^0| - |E^n| = \tau \sum_{k=0}^n \lambda_k.$$

In the limit  $\tau \rightarrow 0$ , similarly to Section 3, up to a subsequence the non-increasing function  $\lambda_{\lfloor t/\tau \rfloor + 1}$  converges pointwise to a non-increasing function  $\lambda(t)$ , while  $E_\tau(t)$  converges to  $F^{\lambda(t)}$ . In particular, in the limit we find that:

$$|E^0| - |E(t)| = \int_0^t \lambda(s) ds \tag{5.11}$$

for all  $0 \leq t \leq T^*$ , where  $\lambda(T^*) = \lambda^*$ .

If  $\varphi$  is smooth and elliptic, since the sets  $F^\lambda$  are all different, the function  $t \mapsto \lambda(t)$  is continuous on  $[0, T^*]$ , so that the function  $t \mapsto |E(t)|$  is of class  $C^1$  by (5.11). We deduce that (5.9) holds for all  $t \in (0, T^*)$ .

We then obtain a partial extension of Proposition 5.10 to arbitrary dimensions and for a general norm  $\varphi$ .

**Proposition 5.13.** *Assume that  $E^0$  is a bounded convex set not coinciding with its Cheeger set. Then there exists  $T^* > 0$  such that limit flow  $E(t)$  is given by a decreasing family of convex subsets of  $E^0$  for  $t \in [0, T^*]$ . Moreover, each set  $E(t)$  is a volume-constrained minimizer of  $P_\varphi$  inside  $E^0$ , and  $E(T^*) = F^*$  is the Cheeger set of  $E^0$ . In particular, for  $t \in (0, T^*]$   $E(t)$  is the unique minimizer  $F^{\lambda(t)}$  of (5.10) for some  $\lambda(t) > \lambda^*$  which solves (5.11).*

For  $m \in [0, |E^0|]$  we let

$$f(m) := \min \{ P_\varphi(F) : F \subset E^0, |E^0 \setminus F| = m \}. \tag{5.12}$$

Reasoning as in Section 3 we have that, for any  $m \in [0, |E^0 \setminus F^*|]$  there exists a unique  $\lambda^m \geq \lambda^*$  such that  $|E^0 \setminus F^{\lambda^m}| = m$  and  $P_\varphi(F^{\lambda^m}) = f(m)$ . Moreover the function  $m \rightarrow \lambda^m$  is non-increasing in this interval, and

$$-\lambda^m \in \partial f(m), \tag{5.13}$$

which implies that  $f$  is convex on  $[0, |E^0 \setminus F^*|]$ . With almost the same proof as Theorem 3.5, we can show the following uniqueness result for the limit flow  $E(t)$ .

**Proposition 5.14.** *Assume that  $E^0$  is a bounded convex set not coinciding with its Cheeger set. Then the flow  $E(t)$  given by Proposition 5.13 is unique and satisfies*

$$\frac{d|E(t)|}{dt} = -\lambda(t) \tag{5.14}$$

for all  $t \in (0, T^*)$ , where  $\lambda(t)$  coincides with the mean curvature of  $\partial E(t)$  inside  $E^0$  and  $E(T^*)$  is the Cheeger set of  $E^0$ .

*Remark 5.15.* If  $\varphi$  is smooth and elliptic, from (5.14) it follows that  $E(t)$  satisfies (5.9).

Recalling the proof of Proposition 5.10, when  $d = 2$  the minimizer  $E^m$  in (5.12) is uniquely characterized and coincides with the set  $E_{r^m}^-$  as long as  $m \geq |E_{r^0}^-|$ , where  $r^0$  is the inradius of  $E^0$  and  $r^m \geq r^0$  is such that  $|E_{r^m}^-| = m$ . When  $m < |E_{r^0}^-|$  the minimizer  $E^m$  is only unique up to translations.

If in addition  $\varphi(x) = |x|$ , it has been proved in [20] that the function  $f$  is convex on  $[0, m^0]$ , where  $m^0 = |E^0| - |B_{r^0}|$  and  $E^m$  is a solution of (5.10) with  $\lambda = 1/r_m$ , among sets of volume greater or equal to  $|B_{r_m}|$ . Observing also that  $f(m) = 2\sqrt{\pi}(|E^0| - m)$  for  $m \in [m_0, |E^0|]$ , reasoning as above we get that  $f$  satisfies (5.13) for all  $m \in (0, |E^0|)$ , so that we can partly extend the result in Proposition 5.14.

**Proposition 5.16.** *Let  $d = 2$ ,  $\varphi(x) = |x|$ , and assume that  $E^0$  is a bounded convex set. Then the flow  $E(t)$  is defined on a maximal time interval  $[0, T_{\max})$ , with*

$$\lim_{t \rightarrow T_{\max}} |E(t)| = 0,$$

it is unique up to translations, and satisfies (5.14) for all  $t \in (0, T_{\max})$ , where  $\lambda(t)$  coincides with the curvature of  $\partial E(t)$  inside  $E^0$ . Moreover,  $E(t)$  is unique as long as  $|E(t)| \geq |E_{r^0}^-|$ .

We show with two simple examples that uniqueness of the flow cannot be expected for  $t > T^*$ . In the following we fix  $d = 2$  and  $\varphi(x) = |x|$ .

*Example 1.* Let  $E^0 = B_R(x_0)$  for some  $R > 0$  and  $x_0 \in \mathbb{R}^2$ . Then, by the isoperimetric inequality,  $T_\tau E^0$  is a ball contained in  $E^0$  of radius  $r$  minimizing the function

$$r \mapsto 2\pi r + \frac{\pi^2}{2\tau}(R^2 - r^2)^2, \tag{5.15}$$

that is,  $r = R - \tau/(2\pi R^2) + o(\tau)$  as  $\tau \rightarrow 0$ . By iteration, it follows that the discrete evolutions  $E_\tau(t)$  converge, up to a subsequence as  $\tau \rightarrow 0$ , to  $E(t) = B_{R(t)}(x(t))$ , with

$$R(t) := \left( R^3 - \frac{3t}{2\pi} \right)^{\frac{1}{3}} \quad \text{for } t \in \left[ 0, \frac{2}{3}\pi R^3 \right)$$

and  $x(t)$  is a Lipschitz function such that  $|\dot{x}(t)| \leq |\dot{R}(t)|$  for a.e.  $t \in [0, \frac{2}{3}\pi R^3)$ .

Notice that in this case the limit evolution is non-unique.

*Example 2.* Let  $E^0 = B_{R_1}(x_1) \cup B_{R_2}(x_2)$  for some  $R_1 \geq R_2 > 0$  and  $x_1, x_2 \in \mathbb{R}^2$  such that  $|x_1 - x_2| > R_1 + R_2$ . As in the previous example, we have that  $T_\tau E^0 = B_{r_1}(\tilde{x}_1) \cup B_{r_2}(\tilde{x}_2)$ , with  $B_{r_i}(\tilde{x}_i) \subseteq B_{R_i}(x_i)$  for  $i \in \{1, 2\}$ , and the radii  $r_i$  minimize the function

$$(r_1, r_2) \mapsto 2\pi(r_1 + r_2) + \frac{\pi^2}{2\tau} (R_1^2 - r_1^2 + R_2^2 - r_2^2)^2. \quad (5.16)$$

If  $R_1 > R_2$ , by an easy computation it follows that  $r_1 = R_1$  and  $r_2$  minimize the function in (5.15) with  $R$  replaced by  $R_2$ , that is,  $r_2 = R_2 - \tau/(2\pi R_2^2) + o(\tau)$  as  $\tau \rightarrow 0$ . In particular, in the limit as  $\tau \rightarrow 0$ , we obtain the evolution  $E(t) = B_{R_1}(x_1) \cup B_{R_2(t)}(x_2(t))$ , with  $R_2(t)$  and  $x_2(t)$  as in the previous case of a single ball.

On the other hand, If  $R_1 = R_2 = R$  then either  $r_1 = R$  and  $r_2$  minimize the function in (5.15), or viceversa  $r_2 = R$  and  $r_1$  minimize the function in (5.15). This implies that, in the limit as  $\tau \rightarrow 0$ , only one of the two balls start shrinking, whereas the other does not move until the first ball disappears. As above the limit evolution is non-unique.

## APPENDIX A. CONVEX FUNCTIONS OF GRADIENTS

**A.1. Convex function of measures.** We give here an alternative proof of (a simpler variant of) the main result of [16], with less hypotheses on  $F$ . We consider  $F : \mathbb{R}^m \rightarrow [0, +\infty]$ ,  $m \geq 1$  a convex, lower semicontinuous function.

We start by assuming that  $F(0) = 0$  so that 0 is a minimizer of  $F$ . In particular, denoting  $F^*$  the convex conjugate of  $F$ , one has  $F^*(q) = \sup_p p \cdot q - F(p) \geq 0$  (choosing  $p = 0$ ), and  $F^*(0) = -\min_p F(p) = 0$ , hence 0 is also a minimizer of  $F^*$ .

Under these assumptions, one has the following variant of [16, Thm 2.1]:

**Theorem A.1.** *For any vectorial Borel (or Radon) measure  $\mu$ ,  $A \subset \Omega$  open,*

$$\begin{aligned} \sup \left\{ \int_\Omega \varphi \cdot \mu - \int_\Omega F^*(\varphi(x)) dx : \varphi \in C_c^\infty(A; \mathbb{R}^m) \right\} \\ = \int_A F(\mu^a(x)) dx + \int_A F^\infty \left( \frac{\mu^s}{|\mu^s|}(x) \right) |\mu^s|(x) \end{aligned}$$

(possibly infinite).

In this statement,  $F^\infty$  is the recession function of  $F$ , given by

$$F^\infty(p) = \lim_{t \rightarrow +\infty} \frac{1}{t} F(tp + \hat{p}) = \sup_{t > 0} \frac{1}{t} F(tp + \hat{p}) = \sup_{q: F^*(q) < +\infty} q \cdot p \quad (A.1)$$

(where  $\hat{p}$  is any point in the relative interior of the domain of  $F$ ), see [16, §1];  $\mu = \mu^a(x)dx + \mu^s$  is the Radon-Nikodým decomposition of  $\mu$  with respect to the Lebesgue measure;  $\frac{\mu^s}{|\mu^s|}$  is the Radon-Besicovitch derivative of the singular part  $\mu^s$  with respect to its variation  $|\mu^s|$ .

*Proof.* We define a measure as

$$\lambda(A) = \sup \left\{ \int_\Omega \varphi \cdot \mu - \int_\Omega F^*(\varphi(x)) dx : \varphi \in C_c^\infty(A; \mathbb{R}^m) \right\}$$

for  $A$  open.

Step 1. We claim that

- for any open sets  $A, B \subset \Omega$ ,  $\lambda(A \cup B) \leq \lambda(A) + \lambda(B)$ ,
- with equality if  $A \cap B = \emptyset$ , and
- for any open set  $A$ ,  $\lambda(A) = \sup\{\lambda(B) : B \subset \Omega \text{ open}, \bar{B} \subset A\}$ .

Then thanks to De Giorgi-Letta's theorem [3, Thm 1.53], the extension

$$\lambda(B) = \inf\{\lambda(A) : B \subset A \subset \Omega, A \text{ open}\}$$

defines a metric outer measure on  $\Omega$ , and in particular a Borel positive measure. As the two last points follow quite obviously from the definition of  $\lambda$ , the only point to check is the first one: we consider  $A, B$  open sets, possibly intersecting, and given  $\varepsilon > 0$  we choose  $\varphi \in C_c^\infty(A \cup B; \mathbb{R}^m)$  such that

$$\lambda(A \cup B) \leq \int_\Omega \varphi \cdot \mu - \int_\Omega F^*(\varphi(x)) dx + \varepsilon.$$

(If  $\lambda(A \cup B) = +\infty$ , we require rather than the integrals are larger than  $1/\varepsilon$ , the rest of the proof is modified accordingly.) We consider a smooth partition of the unity  $\eta_A, \eta_B$  subject to the sets  $A, B$ . Then, using that  $F^*(\eta_A \varphi) \leq \eta_A F^*(\varphi) + (1 - \eta_A) F^*(0) \leq F^*(\varphi)$  (as 0 is a minimizer of  $F^*$ ), denoting  $\varphi_A = \varphi \eta_A$ ,  $\varphi_B = \varphi \eta_B$ , so that in particular  $\varphi = \varphi_A + \varphi_B$ , one has:

$$\begin{aligned} & \int_\Omega \varphi \cdot \mu - \int_\Omega F^*(\varphi(x)) dx \\ & \leq \int_\Omega \varphi_A \cdot \mu - \int_\Omega F^*(\varphi_A(x)) dx + \int_\Omega \varphi_B \cdot \mu - \int_\Omega F^*(\varphi_B(x)) dx \\ & \quad + \int_{\{\eta_A \eta_B > 0\}} F^*(\varphi(x)) dx. \end{aligned}$$

As it is possible to chose  $\eta_A, \eta_B$  such that the intersection of their support  $\{\eta_A \eta_B > 0\}$  is arbitrarily small, one can assume that  $\int_{\{\eta_A \eta_B > 0\}} F^*(\varphi(x)) dx \leq \varepsilon$ . We deduce that

$$\lambda(A \cup B) \leq \lambda(A) + \lambda(B) + 2\varepsilon,$$

and as  $\varepsilon$  is arbitrary, the claim follows.

Step 2. Quite obviously, for any  $\varphi \in C_c^\infty(A; \mathbb{R}^m)$  such that  $\varphi(x) \in \text{dom } F^*$  a.e.,

$$\begin{aligned} \int_A F(\mu^a) dx + \int_A F^\infty\left(\frac{\mu^s}{|\mu^s|}\right) |\mu^s| & \geq \int_A \varphi \cdot \mu^a - F^*(\varphi) dx + \int_A \varphi \cdot \frac{\mu^s}{|\mu^s|} |\mu^s| \\ & = \int_A \varphi \cdot \mu - \int_A F^*(\varphi) dx, \end{aligned}$$

so that one needs only to show that in the sense of measures,  $\lambda \geq F(\mu^a) dx + F^\infty\left(\frac{\mu^s}{|\mu^s|}\right) |\mu^s|$ .

Let  $x \in \Omega$ ,  $r > 0$  with  $B_r(x) \subset \Omega$ , and choose any  $q \in \text{dom } F^*$ . Let  $\eta \in C_c^\infty(B_r(x); [0, 1])$  be a smooth cutoff. Then,  $\lambda(B_r(x)) \geq \int_{B_r(x)} \eta q \cdot \mu - \int_{B_r(x)} F^*(\eta q) dz$ . As before, as 0 is a minimizer of  $F^*$  and  $\eta(x) \in [0, 1]$ , one has  $F^*(\eta q) \leq \eta F^*(q) + (1 - \eta) F^*(0) \leq F^*(q)$ , so that:

$$\lambda(B_r(x)) \geq q \cdot \mu(B_r(x)) - |B_r(x)| F^*(q) - |q| \int_{B_r(x)} (1 - \eta) |\mu|.$$

Now, we may send  $\eta$  to  $\chi_{B_r(x)}$ , and this sends the last term to 0. It follows that

$$\lambda(B_r(x)) \geq q \cdot \mu(B_r(x)) - |B_r(x)| F^*(q). \quad (\text{A.2})$$

Next, we consider a point  $x$  where both the derivatives of  $\mu$  and  $\lambda$  with respect to the Lebesgue measure exist. Then, one has:

$$\mu^a(x) = \lim_{r \rightarrow 0} \frac{\mu(B_r(x))}{|B_r(x)|}.$$

We rewrite (A.2) as

$$\frac{\lambda(B_r(x))}{|B_r(x)|} \geq q \cdot \frac{\mu(B_r(x))}{|B_r(x)|} - F^*(q),$$

and let  $r \rightarrow 0$ . We find that  $\lambda^a(x) \geq q \cdot \mu^a(x) - F^*(q)$ , and since this holds for any  $q$  with  $F^*(q) < +\infty$ , we deduce  $\lambda^a(x) \geq F^a(\mu^a(x))$ .

We now consider the singular part  $\mu^s$ . We recall that the Radon-Nikodým derivation theorem (or the more general Besicovitch's derivation theorem [3, Thm. 2.22]) states that  $\mu^s = \mu \llcorner E$  where:

$$E = \left\{ x \in \Omega : \lim_{r \rightarrow 0} \frac{|\mu|(B_r(x))}{|B_r(x)|} = +\infty \right\}.$$

In addition, the same theorem ensures that one can further restrict  $E$  to the points  $x$  where  $|\mu^s|(B_r(x)) > 0$  for all  $r > 0$  and such that

$$\lim_{r \rightarrow 0} \frac{\mu^s(B_r(x))}{|\mu^s|(B_r(x))} = \frac{\mu^s}{|\mu^s|}(x)$$

exists and has norm 1; then  $\mu^s = \frac{\mu^s}{|\mu^s|} |\mu^s|$ . We remark also that  $|\mu|$ -a.e. in  $E$ , this limit is also:

$$\lim_{r \rightarrow 0} \frac{\mu(B_r(x))}{|\mu|(B_r(x))} = \frac{\mu}{|\mu|}(x)$$

since  $\mu = \frac{\mu}{|\mu|} |\mu|$  so that  $\mu^s = \frac{\mu}{|\mu|} |\mu| \llcorner E$ , which may hold only if  $|\mu^s| = |\mu| \llcorner E$  and  $\frac{\mu}{|\mu|} = \frac{\mu^s}{|\mu^s|}$   $|\mu|$ -a.e. in  $E$ . We further restrict  $E$  to the points where this holds.

For  $x \in E$ , we then consider:

$$\ell(x) := \limsup_{r \rightarrow 0} \frac{\lambda(B_r(x))}{|\mu|(B_r(x))} \in [0, +\infty],$$

then thanks to [3, Prop. 2.21], for any  $t \geq 0$  and any Borel set  $F \subset \{x \in E : \ell(x) > t\}$ ,

$$\lambda(F) \geq t |\mu|(F).$$

In particular for  $F \subset \{x \in E : \ell(x) = +\infty\}$ , either  $|\mu|(F) = 0$ , or  $\lambda(F) = +\infty$ .

Let  $x \in E' := \{x \in E : \ell(x) < +\infty\}$ . Using (A.2) again, we find that for any  $q \in \text{dom } F^*$ ,

$$\frac{\lambda(B_r(x))}{|\mu|(B_r(x))} \geq q \cdot \frac{\mu(B_r(x))}{|\mu|(B_r(x))} - F^*(q) \frac{|B_r(x)|}{|\mu|(B_r(x))}.$$

By definition of  $E$ ,  $|B_r(x)|/(|\mu|(B_r(x))) \rightarrow 0$  as  $r \rightarrow 0$ , hence taking the limsup, we find that  $\ell(x) \geq q \cdot \frac{\mu^s}{|\mu^s|}(x)$ . Since this is true for any  $q$  with  $F^*(q) < +\infty$ , it follows  $\ell(x) \geq F^\infty \left( \frac{\mu^s}{|\mu^s|}(x) \right)$ .

We then proceed as in the proof of [3, Thm 2.22] to deduce that  $\lambda \llcorner E' \geq \ell |\mu| \llcorner E'$  and it follows

$$\lambda \llcorner E \geq F^\infty \left( \frac{\mu^s}{|\mu^s|}(x) \right) |\mu^s|.$$

□

*Remark A.2.* The measure  $\lambda$  is usually denoted  $F(\mu)$ .

**Corollary A.3.** *Let  $F : \mathbb{R}^n \rightarrow [0, +\infty]$  be convex, lsc, and assume there is  $\hat{p} \in \text{dom } F$  a minimizer of  $F$ . Then, for any  $A$  open and bounded,*

$$\begin{aligned} \sup \left\{ \int_A \varphi \cdot \mu - \int_A F^*(\varphi(x)) dx : \varphi \in C_c^\infty(A; \mathbb{R}^m) \right\} \\ = \int_A F(\mu^a(x)) dx + \int_A F^\infty \left( \frac{\mu^s}{|\mu^s|}(x) \right) |\mu^s|(x) \end{aligned}$$

*Proof.* Let  $\tilde{F}(p) = F(\hat{p} + p) - F(\hat{p})$ . Then  $\tilde{F}$  satisfies the assumptions of Theorem A.1. In addition, observe that

$$\tilde{F}^*(q) = \sup_p q \cdot p - F(\hat{p} + p) + F(\hat{p}) = F(\hat{p}) - q \cdot \hat{p} + F^*(q),$$

and  $\tilde{F}^\infty = F^\infty$ . Hence, for any  $\mathbb{R}^n$ -valued Radon measure  $\mu$ ,

$$\begin{aligned} \sup \left\{ \int_A \varphi \cdot \mu - \int_A F^*(\varphi(x)) dx + \int_A \varphi(x) \cdot \hat{p} dx - |A|F(\hat{p}) : \varphi \in C_c^\infty(A; \mathbb{R}^m) \right\} \\ = \int_A F(\hat{p} + \mu^a(x)) dx - |A|F(\hat{p}) + \int_A F^\infty \left( \frac{\mu^s}{|\mu^s|}(x) \right) |\mu^s|(x) \end{aligned}$$

Writing the above equality for the shifted measure  $\mu - \hat{p} dx$  shows the claim.  $\square$

**A.2. Convex functions of gradients.** Now, we consider  $F : \mathbb{R}^d \rightarrow [0, +\infty]$  convex, lsc, with  $F(0) = 0$ , and in addition, we assume there exist  $a > 0$ ,  $b \geq 0$  such that

$$F(p) \geq a|p| - b$$

for all  $p \in \mathbb{R}^d$ . In particular,  $F^*(q) \leq b + \delta_{B_a(0)}$  (the characteristic of the ball of radius  $a$ ), and for  $u \in L^1(\Omega)$ ,

$$\sup \left\{ - \int_\Omega u \operatorname{div} \varphi dx - \int_\Omega F^*(\varphi(x)) dx : \varphi \in C_c^\infty(A; \mathbb{R}^m) \right\} \quad (\text{A.3})$$

can be bounded only if  $a|Du|(A) < +\infty$ , that is if  $u \in BV(A)$ . (Of course, depending on  $F$ , higher integrability on the gradient might also be implied.) In that case, we define as before the measure  $F(Du) := F(D^a u) dx + F^\infty(D^s u)$ , with  $F^\infty(D^s u) = F^\infty(D^s u / |D^s u|) |D^s u|$ , and Theorem A.1 yields that the sup in (A.3) is nothing but  $F(Du)(A) = \int_A F(Du)$ .

We therefore can define a convex, lsc. functional for  $u \in L^1(A)$  (or  $L^1_{\text{loc}}(\Omega)$ ):

$$\Phi(u; A) := F(Du)(A) = \int_A F(Du)$$

when  $u \in BV(A)$ , and  $\Phi(u; A) = +\infty$  else; we denote  $\Phi(u) = \Phi(u; \Omega)$ . We prove here a series of useful lemmas.

**Lemma A.4.** *For any  $u, v \in L^1(\Omega)$ ,*

$$\Phi(u \wedge v) + \Phi(u \vee v) \leq \Phi(u) + \Phi(v) \quad (2.4)$$

*Proof.* Let  $\rho$  be a symmetric mollifier with support in the unit ball, and  $\rho_\varepsilon(x) = \varepsilon^{-d}\rho(x/\varepsilon)$ . Let also  $\Omega_\varepsilon = \{x \in \Omega : \text{dist}(x, \partial\Omega) > \varepsilon\}$ . Then,

$$\begin{aligned} & \int_{\Omega_\varepsilon} F(D(u * \rho_\varepsilon)) dx \\ &= \sup \left\{ - \int_{\Omega} u * \rho_\varepsilon \operatorname{div} \varphi \, dx - \int_{\Omega} F^*(\varphi(x)) \, dx : \varphi \in C_c^\infty(\Omega_\varepsilon; \mathbb{R}^d) \right\} \\ &= \sup \left\{ - \int_{\Omega} u \operatorname{div}(\rho_\varepsilon * \varphi) \, dx - \int_{\Omega} F^*(\varphi(x)) \, dx : \varphi \in C_c^\infty(\Omega_\varepsilon; \mathbb{R}^d) \right\} \\ &\leq \sup \left\{ - \int_{\Omega} u \operatorname{div}(\rho_\varepsilon * \varphi) \, dx - \int_{\Omega} F^*(\rho_\varepsilon * \varphi(x)) \, dx : \varphi \in C_c^\infty(\Omega_\varepsilon; \mathbb{R}^d) \right\} \leq \Phi(u), \end{aligned}$$

where we have used Jensen's inequality:

$$F^* \left( \int_{B_1(0)} \rho(z) \varphi(x - \varepsilon z) \, dz \right) \leq \int_{B_1(0)} \rho(z) F^*(\varphi(x - \varepsilon z)) \, dz.$$

On the other hand, for  $A \subset\subset \Omega$ ,

$$\Phi(u; A) \leq \liminf_{\varepsilon \rightarrow 0} \int_A F(D(u * \rho_\varepsilon)) dx \leq \liminf_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} F(D(u * \rho_\varepsilon)) dx$$

by lower-semicontinuity of  $\Phi(\cdot; A)$ . Since  $\Phi(u) = \sup_{A \subset\subset \Omega} \Phi(u; A)$ , we deduce that

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} F(D(u * \rho_\varepsilon)) dx = \Phi(u).$$

Given  $u, v \in L^1(\Omega)$ , one has (quite obviously, and even if it is not finite):

$$\int_{\Omega_\varepsilon} F(D(u * \rho_\varepsilon \wedge v * \rho_\varepsilon)) dx + \int_{\Omega_\varepsilon} F(D(u * \rho_\varepsilon \vee v * \rho_\varepsilon)) dx = \int_{\Omega_\varepsilon} F(D(u * \rho_\varepsilon)) dx + \int_{\Omega_\varepsilon} F(D(v * \rho_\varepsilon)) dx.$$

Passing to the limit, and using the lower-semicontinuity of  $\Phi$  again, we deduce (2.4).  $\square$

**Lemma A.5.** *Let  $D_\infty := \{u \in L^1(\Omega) \cap \operatorname{dom} \Phi : \partial\Phi(u) \cap L^\infty(\Omega) \neq \emptyset\}$ . Then  $\overline{D_\infty} \supseteq \operatorname{dom} \Phi$ .*

*Proof.* Let  $u \in \operatorname{dom} \Phi$  and for  $k \geq 1$ ,  $u_k = (-k) \vee (u \wedge k)$ . Then  $u_k \rightarrow u$  (in  $L^1(\Omega)$ , or  $L^p(\Omega)$  if  $u \in L^p(\Omega)$ ,  $p \in [1, +\infty]$ ; in particular using  $\Phi(u) < +\infty$  this holds for any  $p \leq d/(d-1)$ ). We claim that  $\Phi(u_k) \leq \Phi(u)$ , in fact, Lemma A.4 shows that  $\Phi(u \wedge k) + \Phi(u \vee k) \leq \Phi(u) + \Phi(k) = \Phi(u)$ , the claim follows. We deduce (thanks to the lower semicontinuity of  $\Phi$ )  $\lim_{k \rightarrow +\infty} \Phi(u_k) = \Phi(u)$ .

Now, we consider  $v_{k,l}$  the minimizer of

$$\frac{l}{2} \int_{\Omega} (v - u_k)^2 dx + \Phi(v).$$

We have that  $\|v_{k,l}\|_\infty \leq k$ , indeed otherwise letting  $v' = (-k) \vee (v_{k,l} \wedge k)$ , one would have  $\int_{\Omega} (v' - u_k)^2 dx < \int_{\Omega} (v_{k,l} - u_k)^2 dx$ , and  $\Phi(v') \leq \Phi(v_{k,l})$  (as before). The Euler-Lagrange equation for this problem can be written:

$$q_{k,l} := l(u_k - v_{k,l}) \in \partial\Phi(v_{k,l}),$$

and since  $\|q_{k,l}\|_\infty \leq 2kl$ , this shows that  $\partial\Phi(v_{k,l}) \cap L^\infty(\Omega) \neq \emptyset$ . When  $l \rightarrow \infty$ ,  $v_{k,l} \rightarrow u_k$  (in  $L^2(\Omega)$ , and then also  $L^1(\Omega)$  or  $L^p(\Omega)$  for any  $p < +\infty$ ), which shows the Lemma.  $\square$

We state a last result which describes the subgradient of  $\Phi$  in the simpler case where  $F$  and  $F^*$  are superlinear, so that, in particular, the domain of  $\Phi$  is a subset of  $W^{1,1}(\Omega)$ . The case where  $F$  is one-homogeneous is discussed for instance in [23], and has been recently generalized to the Lipschitz case in [18]. The fully general case is a difficult issue. In the lemma below, one considers  $\Phi$  as a functional in  $L^p(\Omega)$ ,  $1 \leq p \leq +\infty$ , and  $\Phi^*$  is defined in  $L^{p'}(\Omega)$  with  $1/p + 1/p' = 1$ . In the non-reflexive cases  $p \in \{1, +\infty\}$ , the convergence  $-\operatorname{div} \varphi_n$  to  $w$  below has to be understood in the weak or weak-\* sense.

**Lemma A.6.** *Assume  $F, F^*$  have full domain (equivalently,  $F, F^*$  are superlinear). If  $w \in \partial\Phi(u)$ , there exists  $z \in L^1(\Omega; \mathbb{R}^d)$  such that  $z(x) \in \partial F(Du(x))$  a.e. and  $w = -\operatorname{div} z$  in the distributional sense (with  $z \cdot \nu_\Omega = 0$  in the weak sense on  $\partial\Omega$ , that is,  $\int z \cdot Dv \, dx = \int wv \, dx$  for any  $v \in W^{1,\infty}(\Omega)$ ). In particular, if  $F$  is  $C^1$ , one has  $w = -\operatorname{div} DF(Du)$ .*

*Proof.* We first give in a first step a partial description of the conjugate of  $\Phi$  (see [7] for a description a general setting); then in Step 2, we characterize the subgradients.

Step 1 - Description of the conjugate. We introduce the convex function:

$$H(w) := \begin{cases} \min \left\{ \int_\Omega F^*(\varphi) \, dx : \varphi \in C_c^\infty(\Omega; \mathbb{R}^d), -\operatorname{div} \varphi = w \right\} & \text{if this set is nonempty;} \\ +\infty & \text{else.} \end{cases}$$

Then one has (Theorem A.1)  $\Phi(u) = \int_\Omega F(Du) = \sup_q \int_\Omega wu \, dx - H(w) = H^*(u)$ . Hence,  $\Phi^*(w) = H^{**}(w)$  is the lsc. envelope of  $H$  and one has:

$$\Phi^*(w) = \inf \left\{ \liminf_n \int_\Omega F^*(\varphi_n) \, dx : \varphi_n \in C_c^\infty(\Omega; \mathbb{R}^d), -\operatorname{div} \varphi_n \rightarrow w \text{ in } L^{p'}(\Omega) \right\}.$$

Now, since  $F^*$  is superlinear, if  $(\varphi_n)_n$  is a minimizing sequence in the above infimum, and if the latter is finite,  $(\varphi_n)_n$  has (up to a subsequence) a weak limit  $z$  in  $L^1(\Omega; \mathbb{R}^d)$ . For any  $v \in W^{1,\infty}(\Omega)$ , one has

$$\int_\Omega \varphi_n \cdot Dv \, dx = - \int_\Omega \operatorname{div} \varphi_n v \, dx$$

so that in the limit,

$$\forall v \in W^{1,\infty}(\Omega), \quad \int_\Omega z \cdot Dv \, dx = \int_\Omega wv \, dx. \tag{A.4}$$

(In particular, of course,  $-\operatorname{div} z = w$  in the sense of distributions.)

In addition, by lower-semicontinuity, one has

$$\int_\Omega F^*(z) \, dx \leq \liminf_n \int_\Omega F^*(\varphi_n) \, dx = \Phi^*(w).$$

Step 2 - Subgradient. Let now  $w \in \partial\Phi(u)$ . Equivalently  $u \in \partial\Phi^*(w)$ , and one has

$$\Phi(u) + \Phi^*(w) = \int_\Omega uw \, dx$$

Since  $\Phi^*(w) < +\infty$ , we may consider  $(\varphi_n)_n$  and  $z$  as built in Step 1, with  $\varphi_n \rightharpoonup z$  in  $L^1(\Omega)$ .

One has  $u \in W^{1,1}(\Omega)$  and  $A_n := F(Du) + F^*(\varphi_n) - \varphi_n \cdot Du \geq 0$  a.e., while

$$\int_\Omega A_n \, dx = \Phi(u) + \int_\Omega F^*(\varphi_n) \, dx + \int_\Omega u \operatorname{div} \varphi_n \, dx \rightarrow \Phi(u) + \Phi^*(w) - \int_\Omega wu \, dx = 0$$



since  $-\operatorname{div} \varphi_n \rightarrow w$  in  $L^{p'}(\Omega)$  and  $u \in L^p(\Omega)$ . Hence,  $A_n \rightarrow 0$  in  $L^1(\Omega)$ . In particular, for any  $M \geq 0$ , letting  $E_M := \{x : |Du(x)| \leq M\}$ , one has  $\int_{E_M} A_n dx \rightarrow 0$ ,  $\int_{\Omega} \varphi_n \cdot (\chi_{E_M} Du) dx \rightarrow \int_{E_M} z \cdot Du dx$ , and  $\int_{E_M} F^*(z) dx \leq \liminf_n \int_{E_M} F^*(\varphi_n)$  and we deduce

$$\int_{E_M} F^*(z) + F(Du) - z \cdot Du dx \leq 0.$$

Since  $M$  is arbitrary, it follows that  $F(Du) + F^*(z) = z \cdot Du$  (equivalently  $z \in \partial F(Du)$ ) a.e. in  $\Omega$ .  $\square$

**A.3. The Dirichlet case.** We describe here how the previous results should be adapted to consider Dirichlet boundary constraints. In this section,  $\Omega$  is a bounded, open, Lipschitz-regular set in  $\mathbb{R}^d$ . We consider also  $u^0 \in L^1(\partial\Omega)$  and we assume that there exists an extension  $u^0 \in W^{1,1}(\Omega)$  and  $t > 1$  with

$$\int_{\Omega} F(tDu^0(x)) dx < +\infty. \quad (\text{A.5})$$

(If  $F$  is Lipschitz this is a standard result, if  $F$  has a higher growth this imposes an additional constraint on  $u^0|_{\partial\Omega}$ .)

Let us introduce the functional:

$$\Psi_0(u) = \begin{cases} \Phi(u) \in [0, +\infty] & \text{if } u \in W^{1,1}(\Omega), u = u^0 \text{ on } \partial\Omega, \\ +\infty & \text{else.} \end{cases}$$

In case  $F$  is superlinear, it is well known that  $\Psi = \Psi_0$  is lower-semicontinuous, and can be recovered by duality as:

$$\Psi(u) = \sup \left\{ \int_{\partial\Omega} u^0 \varphi \cdot \nu_{\Omega} d\mathcal{H}^{d-1} - \int_{\Omega} (u \operatorname{div} \varphi + F^*(\varphi)) dx : \varphi \in C^{\infty}(\bar{\Omega}; \mathbb{R}^d) \right\} \quad (\text{A.6})$$

(where  $\nu_{\Omega}$  is the outer normal to  $\partial\Omega$ ) if  $u = u^0$  on  $\partial\Omega$ . Here we have simply used Green's formula and the boundary condition to transform the term  $\int_{\Omega} \varphi \cdot Du dx$ .

In the general case, one checks that the functional  $\Psi$  given by (A.6) is also given, for a general  $u \in L^1(\Omega)$  by:

$$\Psi(u) = \begin{cases} \int_{\Omega} F(Du) + \int_{\partial\Omega} F^{\infty}((u^0 - u)\nu_{\Omega}) d\mathcal{H}^{d-1} \in [0, +\infty] & \text{if } u \in BV(\Omega), \\ +\infty & \text{else.} \end{cases} \quad (\text{A.7})$$

This is easily obtained applying Theorem A.1 to the measure  $Du \llcorner \Omega + (u - u^0)\nu_{\Omega} \mathcal{H}^{d-1} \llcorner \partial\Omega$  (in a larger domain), when  $u$  is  $BV$ , so that in particular its trace is well defined on the boundary.

The fact  $\Psi$  is then the lower-semicontinuous envelope of  $\Psi_0$  is well known in case  $F$  is superlinear (since then  $\Psi = \Psi_0$ ) and in case  $F$  is Lipschitz (approximating  $u$  with  $u + \chi_{\{\operatorname{dist}(\cdot, \partial\Omega) < 1/n\}}(u^0 - u)$ ). The situation where  $F$  is neither Lipschitz nor superlinear remains unclear, we choose in that case to use (A.6)–(A.7) as a definition for the Dirichlet problem.

We turn ourself to the characterization of the subgradient of  $\Psi$ . As before, we restrict ourselves to the Lipschitz case, for which we refer to [18]: in this case, one can show that  $w \in \partial\Psi(u)$  if and only if  $w = -\operatorname{div} z$  with  $z \cdot Du = F(Du) + F^*(z)$  in the sense of [5], and  $z \cdot \nu_{\Omega} = F(\operatorname{sign}(u - u^0)\nu_{\Omega})$  a.e. on the boundary. In the superlinear case, the following holds:

**Lemma A.7.** *Assume  $F, F^*$  have full domain (equivalently,  $F, F^*$  are superlinear). If  $w \in \partial\Psi(u)$ , there exists  $z \in L^1(\Omega; \mathbb{R}^d)$  such that  $z(x) \in \partial F(Du(x))$  a.e. and  $w = -\operatorname{div} z$  in the distributional sense. In particular, if  $F$  is  $C^1$ , one has  $w = -\operatorname{div} DF(Du)$ .*

*Proof.* The proof is almost the same as for Lemma A.6 and we only sketch it. We introduce again:

$$H(w) := \begin{cases} \min \left\{ \int_{\Omega} F^*(\varphi) dx - \int_{\partial\Omega} u^0 \varphi \cdot \nu_{\Omega} d\mathcal{H}^{d-1} : \varphi \in C_c^{\infty}(\bar{\Omega}; \mathbb{R}^d), -\operatorname{div} \varphi = w \right\} \\ \quad \text{if this set is nonempty;} \\ +\infty \text{ else.} \end{cases}$$

and we find that since  $\Psi = H^*$ ,  $\Psi^*$  is the convex, l.s.c. envelope of  $H$ . Hence if  $w \in \partial\Psi(u)$  (where  $u = u^0$  on  $\partial\Omega$ , since  $F^{\infty} \equiv +\infty$ ), there is  $\varphi_n$  with  $-\operatorname{div} \varphi_n \rightarrow w$ ,

$$\lim_n \int_{\Omega} F^*(\varphi_n) dx - \int_{\partial\Omega} u^0 \varphi_n \cdot \nu_{\Omega} d\mathcal{H}^{d-1} = \Psi^*(w)$$

and

$$\Psi^*(w) + \Psi(u) = \int_{\Omega} wu \, dx.$$

Using

$$\int_{\partial\Omega} u^0 \varphi_n \cdot \nu_{\Omega} d\mathcal{H}^{d-1} = \int_{\Omega} u^0 \operatorname{div} \varphi_n + \varphi_n \cdot Du^0 \, dx \leq C + \frac{1}{t} \int_{\Omega} F(tDu^0) + F^*(\varphi_n) dx,$$

where  $t > 1$  is from (A.5), we deduce that  $\int F^*(\varphi_n) dx$  is bounded and as in the Neumann case we may assume, up to a subsequence, that  $\varphi_n$  converges weakly to some vector field  $z \in L^1(\Omega; \mathbb{R}^n)$  with  $-\operatorname{div} z = w$ . We let again  $A_n = F(Du) + F^*(\varphi_n) - \varphi_n \cdot Du \geq 0$ , and use Green's formula to obtain:

$$\begin{aligned} \int_{\Omega} A_n \, dx &= \int_{\Omega} F(Du) + F^*(\varphi_n) dx + \int_{\Omega} u \operatorname{div} \varphi_n \, dx - \int_{\partial\Omega} u_0 \varphi_n \cdot \nu_{\Omega} d\mathcal{H}^{d-1} \\ &= \Psi(u) + \int_{\Omega} F^*(\varphi_n) dx - \int_{\partial\Omega} u_0 \varphi_n \cdot \nu_{\Omega} d\mathcal{H}^{d-1} + \int_{\Omega} u \operatorname{div} \varphi_n \, dx \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

We conclude as in the proof of Lemma A.6.  $\square$

## REFERENCES

- [1] Fred Almgren, Jean E. Taylor, and Lihe Wang. Curvature-driven flows: a variational approach. *SIAM J. Control Optim.*, 31(2):387–438, 1993.
- [2] François Alter and Vicent Caselles. Uniqueness of the Cheeger set of a convex body. *Nonlinear Anal.*, 70(1):32–44, 2009.
- [3] Luigi Ambrosio, Nicola Fusco, and Diego Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.
- [4] Luigi Ambrosio, Nicola Gigli, and Giuseppe Savaré. *Gradient flows in metric spaces and in the space of probability measures*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, second edition, 2008.
- [5] Gabriele Anzellotti. Pairings between measures and bounded functions and compensated compactness. *Ann. Mat. Pura Appl. (4)*, 135:293–318, 1983.
- [6] Giovanni Bellettini, Matteo Novaga, and Maurizio Paolini. Characterization of facet breaking for non-smooth mean curvature flow in the convex case. *Interfaces Free Bound.*, 3(4):415–446, 2001.
- [7] Guy Bouchitte. Convex analysis and duality, 2020.

- [8] Haïm Brézis. *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*. North-Holland Mathematics Studies, No. 5. North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973.
- [9] Luis Caffarelli. The obstacle problem revisited. *J. Fourier Anal. Appl.*, 4(4-5):383–402, 1998.
- [10] Vicent Caselles, Antonin Chambolle, Salvador Moll, and Matteo Novaga. A characterization of convex calibrable sets in  $\mathbb{R}^N$  with respect to anisotropic norms. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 25(4):803–832, 2008.
- [11] Vicent Caselles, Antonin Chambolle, and Matteo Novaga. Some remarks on uniqueness and regularity of Cheeger sets. *Rend. Semin. Mat. Univ. Padova*, 123:191–201, 2010.
- [12] Antonin Chambolle and Matteo Novaga. Anisotropic and crystalline mean curvature flow of mean-convex sets. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 23(2):623–643, 2022.
- [13] François Dayrens. The  $L^1$  gradient flow of a generalized scale invariant Willmore energy for radially non-increasing functions. *Adv. Calc. Var.*, 10(4):331–355, 2017.
- [14] Ennio De Giorgi. New problems on minimizing movements. In *Boundary value problems for partial differential equations and applications*, volume 29 of *RMA Res. Notes Appl. Math.*, pages 81–98. Masson, Paris, 1993.
- [15] Guido De Philippis and Tim Laux. Implicit time discretization for the mean curvature flow of mean convex sets. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 21:911–930, 2020.
- [16] Françoise Demengel and Roger Temam. Convex function of a measure: the unbounded case. In *FERMAT days 85: mathematics for optimization (Toulouse, 1985)*, volume 129 of *North-Holland Math. Stud.*, pages 103–134. North-Holland, Amsterdam, 1986.
- [17] Ivar Ekeland and Roger Témam. *Convex analysis and variational problems*, volume 28 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1999.
- [18] Wojciech Górny and José M. Mazón. A duality-based approach to gradient flows of linear growth functionals, 2022. Preprint arXiv:2212.08725.
- [19] Bernd Kawohl and Thomas Lachand-Robert. Characterization of Cheeger sets for convex subsets of the plane. *Pacific J. Math.*, 225(1):103–118, 2006.
- [20] Gian Paolo Leonardi and Giorgio Saracco. The isoperimetric problem in 2d domains without necks. *Calc. Var. Partial Differential Equations*, 61(2):Paper No. 56, 23, 2022.
- [21] Giovanni Leoni. *A first course in Sobolev spaces*, volume 181 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2017.
- [22] Stephan Luckhaus and Thomas Sturzenhecker. Implicit time discretization for the mean curvature flow equation. *Calc. Var. Partial Differential Equations*, 3(2):253–271, 1995.
- [23] J. S. Moll. The anisotropic total variation flow. *Math. Ann.*, 332(1):177–218, 2005.
- [24] Riccarda Rossi, Ulisse Stefanelli, and Marita Thomas. Rate-independent evolution of sets. *Discrete Contin. Dyn. Syst. Ser. S*, 14(1):89–119, 2021.

CEREMADE, CNRS AND UNIVERSITÉ PARIS DAUPHINE, PARIS, FRANCE

*Email address:* `chambolle@ceremade.dauphine.fr`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PISA, PISA, ITALY

*Email address:* `matteo.novaga@unipi.it`