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# ON THE CONVERGENCE OF CRITICAL POINTS OF THE AMBROSIO-TORTORELLI FUNCTIONAL

JEAN-FRANÇOIS BABADJIAN, VINCENT MILLOT, AND RÉMY RODIAC

ABSTRACT. This work is devoted to study the asymptotic behavior of critical points  $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0}$  of the Ambrosio-Tortorelli functional. Under a uniform energy bound assumption, the usual  $\Gamma$ -convergence theory ensures that  $(u_\varepsilon, v_\varepsilon)$  converges in the  $L^2$ -sense to some  $(u_*, 1)$  as  $\varepsilon \rightarrow 0$ , where  $u_*$  is a special function of bounded variation. Assuming further the Ambrosio-Tortorelli energy of  $(u_\varepsilon, v_\varepsilon)$  to converge to the Mumford-Shah energy of  $u_*$ , the later is shown to be a critical point with respect to inner variations of the Mumford-Shah functional. As a byproduct, the second inner variation is also shown to pass to the limit. To establish these convergence results, interior  $(\mathcal{C}^\infty)$  regularity and boundary regularity for Dirichlet boundary conditions are first obtained for a fixed parameter  $\varepsilon > 0$ . The asymptotic analysis is then performed by means of varifold theory in the spirit of scalar phase transition problems. Lastly, a complete one-dimensional study allows one to exhibit non-minimizing critical points of the Ambrosio-Tortorelli functional that do satisfy our energy convergence assumption.

## 1. INTRODUCTION

Let  $\Omega \subset \mathbf{R}^N$  be a bounded open set with Lipschitz boundary ( $N \geq 1$ ) and  $g \in H^{\frac{1}{2}}(\partial\Omega)$  be a prescribed Dirichlet boundary data on  $\partial\Omega$ . For infinitesimal parameters  $\varepsilon \rightarrow 0$  and  $\eta_\varepsilon \rightarrow 0$  with  $0 < \eta_\varepsilon \ll \varepsilon$ , we consider the *Ambrosio-Tortorelli functional* defined by

$$AT_\varepsilon(u, v) := \int_{\Omega} (\eta_\varepsilon + v^2) |\nabla u|^2 dx + \int_{\Omega} \left( \varepsilon |\nabla v|^2 + \frac{(v-1)^2}{4\varepsilon} \right) dx, \quad (1.1)$$

for all pairs  $(u, v) \in H^1(\Omega) \times [H^1(\Omega) \cap L^\infty(\Omega)]$  satisfying  $(u, v) = (g, 1)$  on  $\partial\Omega$ . This functional, originally introduced in [3] can be interpreted as a *phase-field regularization* of the Mumford-Shah functional<sup>1</sup>

$$(u, v) \mapsto \begin{cases} MS(u) := \int_{\Omega} |\nabla u|^2 dx + \mathcal{H}^{N-1}(J_u) + \mathcal{H}^{N-1}(\partial\Omega \cap \{u \neq g\}) & \text{if } \begin{cases} u \in SBV^2(\Omega), \\ v = 1 \text{ in } \Omega, \end{cases} \\ +\infty & \text{otherwise.} \end{cases} \quad (1.2)$$

The Mumford-Shah functional is well known as a theoretical tool to approach image segmentation [35, 34, 36]. It is also at the heart of the Francfort-Marigo model in fracture mechanics [14], and the numerical implementation of this model heavily relies on Ambrosio-Tortorelli type functionals [7]. The use of such phase-field approximation in numerics is usually justified through  $\Gamma$ -convergence theory. In terms of the functionals defined above, it states that  $AT_\varepsilon$   $\Gamma$ -converges in the  $[L^2(\Omega)]^2$ -topology as  $\varepsilon \rightarrow 0$  towards the Mumford-Shah functional (see e.g. the seminal paper [4] and Proposition 4.1 below for the proof of the lower bound inequality).

As a consequence, the fundamental theorem of  $\Gamma$ -convergence ensures the convergence of global minimizers  $(u_\varepsilon, v_\varepsilon)$  of  $AT_\varepsilon$  to  $(u, 1)$  as  $\varepsilon \rightarrow 0$  where  $u \in SBV^2(\Omega)$  is a global minimizer of  $MS$ . This result is of course of importance, but it is somehow not fully satisfactory. Beyond the fact that the use of global minimizers in the models mentioned above remains under debate, this convergence result does not really provide a rigorous justification of the numerical simulations based on the Ambrosio-Tortorelli

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<sup>1</sup>We refer to Section 2 for the notation used in the definition of  $MS$ .

functional. One particular feature of  $AT_\varepsilon$  is its lack of convexity due to the nonconvex coupling term  $v^2|\nabla u|^2$  with respect to the pair  $(u, v)$ . This is a high obstacle to reach global minimizers through a numerical method. An idea employed in the context of image segmentation or fracture mechanics consists in performing an alternate minimization algorithm, see [8]. Each iteration of the scheme is well-posed since  $AT_\varepsilon$  is continuous, coercive, and separately strictly convex. Letting the number of steps going to infinity, the sequence of iterates turns out to converge to a critical point of the energy  $AT_\varepsilon$  [7, Theorem 1], but this critical point might fail to be a global minimizer. Consequently, the original target of numerically approximating global minimizers of the Mumford-Shah functional might be lost. These issues motivate the question of convergence as  $\varepsilon \rightarrow 0$  of critical points of the Ambrosio-Tortorelli functional and it constitutes the main goal of this article, continuing a task initiated in [13, 23] in dimension  $N = 1$ . In higher dimension, a fundamental issue in such an analysis is the regularity of critical points of  $AT_\varepsilon$ . It is also of importance for numerics as the efficiency of the numerical methods crucially rests on it. Here, we fully resolve this last question showing smoothness of arbitrary critical points according to the smoothness of  $\partial\Omega$  and the Dirichlet boundary data.

A critical point  $(u_\varepsilon, v_\varepsilon)$  of the Ambrosio-Tortorelli functional is a weak (distributional) solution of the nonlinear elliptic system

$$\begin{cases} -\operatorname{div}((\eta_\varepsilon + v_\varepsilon^2)\nabla u_\varepsilon) = 0 & \text{in } \Omega, \\ -\varepsilon\Delta v_\varepsilon + \frac{v_\varepsilon - 1}{4\varepsilon} + v_\varepsilon|\nabla u_\varepsilon|^2 = 0 & \text{in } \Omega, \\ (u_\varepsilon, v_\varepsilon) = (g, 1) & \text{on } \partial\Omega. \end{cases} \quad (1.3)$$

To be more precise, critical points of  $AT_\varepsilon$  are defined as follows.

**Definition 1.1.** Let  $\Omega \subset \mathbf{R}^N$  be a bounded open set with Lipschitz boundary and  $g \in H^{\frac{1}{2}}(\partial\Omega)$ . A pair

$$(u_\varepsilon, v_\varepsilon) \in \mathcal{A}_g(\Omega) := \left\{ (u, v) \in H^1(\Omega) \times [H^1(\Omega) \cap L^\infty(\Omega)] : (u, v) = (g, 1) \text{ on } \partial\Omega \right\}$$

is a *critical point of the Ambrosio-Tortorelli functional* if

$$\frac{d}{dt} \Big|_{t=0} AT_\varepsilon(u + t\phi, v + t\psi) = 0 \quad \text{for all } (\phi, \psi) \in H_0^1(\Omega) \times [H_0^1(\Omega) \cap L^\infty(\Omega)],$$

that is

$$\int_{\Omega} (\eta_\varepsilon + v_\varepsilon^2) \nabla u_\varepsilon \cdot \nabla \phi \, dx = 0 \quad \text{for all } \phi \in H_0^1(\Omega), \quad (1.4)$$

and

$$\varepsilon \int_{\Omega} \nabla v_\varepsilon \cdot \nabla \psi \, dx + \int_{\Omega} \left( \frac{v_\varepsilon - 1}{4\varepsilon} + v_\varepsilon |\nabla u_\varepsilon|^2 \right) \psi \, dx = 0 \quad \text{for all } \psi \in H_0^1(\Omega) \cap L^\infty(\Omega). \quad (1.5)$$

By density, test functions  $(\phi, \psi)$  in (1.4)-(1.5) can equivalently be chosen in  $[\mathcal{C}_c^\infty(\Omega)]^2$ , and (1.3) holds in the sense of distributions in  $\Omega$ .

One may expect that critical points of  $AT_\varepsilon$  with uniformly bounded energy converge along some subsequence  $\varepsilon \rightarrow 0$  to a limit satisfying some first order criticality conditions for  $MS$ . Unfortunately, the theory of  $\Gamma$ -convergence does not provide convergence of critical points towards critical points of the limiting functional. Even for local minimizers such a result usually fails. We refer to [22, Remark 4.5] and [9, Example 3.5.1] for counter-examples. However, it has been proved in some specific examples that critical points do converge to critical points, possibly under the assumption of convergence of critical values. This is the case for the Allen-Cahn (or Modica-Mortola) functional from phase transitions approximating the  $(N - 1)$ -dimensional area functional [37, 21, 43, 44, 19, 16], the Ginzburg-Landau functional approximating the  $(N - 2)$ -dimensional area functional [2, 5, 28, 42, 38], and the Dirichlet energy of manifold valued stationary harmonic maps [27, 29, 30, 31, 32]. These functionals share many features with  $AT_\varepsilon$ , and we shall take advantage of the existing theory to develop our asymptotic analysis

of critical points of  $AT_\varepsilon$ . In particular, we shall make an essential use of both outer and inner variations of the energy, a common approach in all these studies.

**1.1. Outer and inner variations.** Definition 1.1 is simply saying that the first outer variation of  $AT_\varepsilon$  vanishes at  $(u_\varepsilon, v_\varepsilon) \in \mathcal{A}_g(\Omega)$  in any direction  $(\phi, \psi)$ . In case of a smooth functional like  $AT_\varepsilon$ , outer variations coincide with Gâteaux differentials. For  $(u, v) \in \mathcal{A}_g(\Omega)$  and  $(\phi, \psi) \in H_0^1(\Omega) \times [H_0^1(\Omega) \cap L^\infty(\Omega)]$  as before, we introduce the following notation for the first and second outer variations of  $AT_\varepsilon$  (see Lemma A.9 for explicit formulas)

$$\begin{aligned} dAT_\varepsilon(u, v)[\phi, \psi] &:= \left. \frac{d}{dt} \right|_{t=0} AT_\varepsilon(u + t\phi, v + t\psi), \\ d^2 AT_\varepsilon(u, v)[\phi, \psi] &:= \left. \frac{d^2}{dt^2} \right|_{t=0} AT_\varepsilon(u + t\phi, v + t\psi). \end{aligned}$$

Concerning the Mumford-Shah functional, the notion of critical points requires some definition and notation. Before doing so, let us first comment on the functional  $MS$  in (1.2) we are considering. Contrary to  $AT_\varepsilon$ , the admissible  $u$ 's for  $MS$  are not required to agree with  $g$  on  $\partial\Omega$  in the sense of traces. In turn, the additional term  $\mathcal{H}^{N-1}(\partial\Omega \cap \{u \neq g\})$  in the expression of  $MS(u)$  penalizes “boundary jumps” where the inner trace of  $u$  (still denoted by  $u$ ) differs from  $g$ . The expression  $u \neq g$  on  $\partial\Omega$  is also intended in the sense of traces. In the sequel, we shall often use the following compact notation

$$MS(u) = \int_{\Omega} |\nabla u|^2 dx + \mathcal{H}^{N-1}(\hat{J}_u), \quad u \in SBV^2(\Omega),$$

where  $\hat{J}_u = J_u \cup (\partial\Omega \cap \{u \neq g\})$ , so that

$$\hat{J}_u = J_{\hat{u}} \quad \text{with} \quad \hat{u} := u\mathbf{1}_\Omega + G\mathbf{1}_{\mathbf{R}^N \setminus \Omega} \in SBV^2(\mathbf{R}^N),$$

and  $G \in H^1(\mathbf{R}^N)$  is an arbitrary extension of  $g$ .

Unlike  $AT_\varepsilon$ , the Mumford-Shah functional is not smooth and outer variations must be accordingly defined (see e.g. [1, Section 7.4]). Given  $u, \phi \in SBV^2(\Omega)$  such that  $\hat{J}_\phi \subset \hat{J}_u$ , the first and second outer variations of  $MS$  at  $u$  in the direction  $\phi$  are respectively defined and given by

$$\begin{aligned} dMS(u)[\phi] &:= \left. \frac{d}{dt} \right|_{t=0} MS(u + t\phi) = 2 \int_{\Omega} \nabla u \cdot \nabla \phi dx, \\ d^2 MS(u)[\phi] &:= \left. \frac{d^2}{dt^2} \right|_{t=0} MS(u + t\phi) = 2 \int_{\Omega} |\nabla \phi|^2 dx. \end{aligned}$$

In this definition, the requirement  $\hat{J}_\phi \subset \hat{J}_u$  ensures the differentiability at  $t = 0$  of the function  $t \mapsto MS(u + t\phi)$  since  $\mathcal{H}^{N-1}(\hat{J}_{u+t\phi})$  remains constantly equal to  $\mathcal{H}^{N-1}(\hat{J}_u)$ . As a consequence, these differentials provide only information on the “regular part” of the function  $u$ , and not on the jump set  $\hat{J}_u$ . Note also that the second order condition  $d^2 MS(u)[\phi] \geq 0$  is obviously satisfied at any  $u, \phi$  as above. On the other hand, the condition  $\hat{J}_\phi \subset \hat{J}_u$  also implies that the direction  $\phi$  must agree with  $g$  on  $\partial\Omega \cap \{u = g\}$ , in agreement with the notion of Dirichlet boundary condition.

It is clear that outer variations are not sufficient to define a notion of critical point for  $MS$  since admissible perturbations leave the “singular part”  $\mathcal{H}^{N-1}(\hat{J}_u)$  unchanged. The way to complement outer variations is to consider *inner variations*, i.e., variations under domain deformations. In doing so (up to the boundary), we shall assume that  $\partial\Omega$  is at least of class  $\mathcal{C}^2$ .

Given a vector field  $X \in \mathcal{C}_c^1(\mathbf{R}^N; \mathbf{R}^N)$  satisfying  $X \cdot \nu_\Omega = 0$  on  $\partial\Omega$  (here  $\nu_\Omega$  denotes the outward unit normal field on  $\partial\Omega$ ), we consider its flow map  $\Phi : \mathbf{R} \times \mathbf{R}^N \rightarrow \mathbf{R}^N$ , i.e., for every  $x \in \mathbf{R}^N$ ,  $t \mapsto \Phi(t, x)$

is defined as the unique solution of the system of ODE's

$$\begin{cases} \frac{d\Phi}{dt}(t, x) = X(\Phi(t, x)), \\ \Phi(0, x) = x. \end{cases}$$

According to the standard Cauchy-Lipschitz theory,  $\Phi \in \mathcal{C}^1(\mathbf{R} \times \mathbf{R}^N; \mathbf{R}^N)$  is well-defined, and  $\{\Phi_t\}_{t \in \mathbf{R}}$  with  $\Phi_t := \Phi(t, \cdot)$  is a one-parameter group of  $\mathcal{C}^1$ -diffeomorphisms of  $\mathbf{R}^N$  into itself satisfying  $\Phi_0 = \text{id}$ . Then the requirement  $X \cdot \nu_\Omega = 0$  on  $\partial\Omega$  implies that  $\Phi_t(\partial\Omega) = \partial\Omega$  for every  $t \in \mathbf{R}$ . Hence (the restriction of)  $\Phi_t$  is a  $\mathcal{C}^1$ -diffeomorphism of  $\partial\Omega$  into itself, and a  $\mathcal{C}^1$ -diffeomorphism of  $\Omega$  into itself.

**Definition 1.2.** Let  $u \in SBV^2(\Omega)$ ,  $X \in \mathcal{C}_c^1(\mathbf{R}^N; \mathbf{R}^N)$ , and  $G \in H^1(\mathbf{R}^N)$  satisfying  $X \cdot \nu_\Omega = 0$  and  $G = g$  on  $\partial\Omega$ . Setting  $\{\Phi_t\}_{t \in \mathbf{R}}$  to be the integral flow of  $X$  and

$$u_t := u \circ \Phi_t^{-1} - G \circ \Phi_t^{-1} + G \in SBV^2(\Omega),$$

the first and second inner variations of  $MS$  at  $u$  are defined by

$$\delta MS(u)[X, G] := \left. \frac{d}{dt} \right|_{t=0} MS(u_t), \quad \delta^2 MS(u)[X, G] := \left. \frac{d^2}{dt^2} \right|_{t=0} MS(u_t).$$

It can be checked that, provided  $\partial\Omega$ ,  $g$ , and  $G$  are smooth enough, the above derivatives exist and they can be explicitly computed (see Lemma A.12). Analogously, we define inner variations of the Ambrosio-Tortorelli functional.

**Definition 1.3.** Let  $(u, v) \in \mathcal{A}_g(\Omega)$ ,  $X \in \mathcal{C}_c^1(\mathbf{R}^N; \mathbf{R}^N)$ , and  $G \in H^1(\mathbf{R}^N)$  satisfying  $X \cdot \nu_\Omega = 0$  and  $G = g$  on  $\partial\Omega$ . We set

$$(u_t, v_t) := (u \circ \Phi_t^{-1} - G \circ \Phi_t^{-1} + G, v \circ \Phi_t^{-1}) \in \mathcal{A}_g(\Omega).$$

We define the first and second inner variations of  $AT_\varepsilon$  at  $(u, v)$  by

$$\delta AT_\varepsilon(u, v)[X, G] := \left. \frac{d}{dt} \right|_{t=0} AT_\varepsilon(u_t, v_t), \quad \delta^2 AT_\varepsilon(u, v)[X, G] := \left. \frac{d^2}{dt^2} \right|_{t=0} AT_\varepsilon(u_t, v_t). \quad (1.6)$$

Once again, the limits in (1.6) exist whenever  $\partial\Omega$ ,  $g$ , and  $G$  are sufficiently smooth, and one can compute them explicitly (see Lemmas A.10 & A.11).

We emphasize that we are considering in Definitions 1.2 & 1.3 deformations *up to the boundary*. Compare to the usual deformations involving compactly supported perturbations in  $\Omega$  of the original maps, it requires the additional test function  $G$ . This is of fundamental importance for the  $MS$  functional to recover information at the boundary since the Dirichlet boundary condition is implemented in the functional as a penalization. Of course, the type of deformations we are using includes as a particular case the usual ones defined only through a vector field  $X$  compactly supported in  $\Omega$ , see Remark A.3.

**1.2. First order criticality conditions for  $MS$ .** In view of the discussion above, the non smooth character of  $MS$  forces the appropriate notion of critical point to involve both outer and inner variations. In other words, a critical point of the Mumford-Shah functional is a critical point with respect to both outer and inner variations, a property obviously satisfied by global (and even local) minimizers.

**Definition 1.4.** Let  $\Omega \subset \mathbf{R}^N$  be a bounded open set with boundary of class at least  $\mathcal{C}^2$  and  $g \in \mathcal{C}^2(\partial\Omega)$ . A function  $u_* \in SBV^2(\Omega)$  is a critical point of the Mumford-Shah functional if

$$dMS(u_*)[\phi] = 0 \quad \text{for all } \phi \in SBV^2(\Omega) \text{ with } \widehat{J}_\phi \subset \widehat{J}_{u_*}, \quad (1.7)$$

and

$$\delta MS(u_*)[X, G] = 0 \quad (1.8)$$

for all  $X \in \mathcal{C}_c^1(\mathbf{R}^N; \mathbf{R}^N)$  and  $G \in \mathcal{C}^2(\mathbf{R}^N)$  satisfying  $X \cdot \nu_\Omega = 0$  and  $G = g$  on  $\partial\Omega$ .

From these criticality conditions, one can derive a set of Euler-Lagrange equations which can be written in a strong form if the smoothness of  $u_*$  and  $\widehat{J}_{u_*}$  allows it. Specializing first the condition (1.7) to  $\phi \in \mathcal{C}_c^\infty(\Omega)$  yields

$$\operatorname{div}(\nabla u_*) = 0 \quad \text{in } \mathcal{D}'(\Omega). \quad (1.9)$$

Then, if  $\widehat{J}_{u_*}$  is regular enough, one can choose test functions  $\phi$  in (1.7) with a non trivial jump set but smooth up to  $\widehat{J}_{u_*}$  from both sides. It leads to the following homogeneous Neumann condition

$$\partial_\nu u_* = 0 \quad \text{on } \widehat{J}_{u_*}, \quad (1.10)$$

see [1, formula (7.42)]. In other words, allowing test functions  $\phi$  in (1.7) with  $\widehat{\phi} \subset \widehat{J}_{u_*}$  (and not only in  $\phi \in \mathcal{C}_c^\infty(\Omega)$ ) provides the weak formulation of (1.10) which complements (1.9).

Computing  $\delta MS(u_*)[X, G]$  (see formula (A.38)) and using equation (1.9), the stationarity condition (1.8) appears to be independent of the test function  $G$  and it reduces to

$$\begin{aligned} \int_{\Omega} (|\nabla u_*|^2 \operatorname{Id} - 2\nabla u_* \otimes \nabla u_*) : DX \, dx + \int_{\widehat{J}_{u_*}} \operatorname{div}^{\widehat{J}_{u_*}} X \, d\mathcal{H}^{N-1} = -2 \int_{\partial\Omega} (\nabla u_* \cdot \nu_\Omega)(X_\tau \cdot \nabla_\tau g) \, d\mathcal{H}^{N-1} \\ \text{for all } X \in \mathcal{C}_c^1(\mathbf{R}^N; \mathbf{R}^N) \text{ with } X \cdot \nu_\Omega = 0 \text{ on } \partial\Omega. \end{aligned} \quad (1.11)$$

Here  $\operatorname{div}^{\widehat{J}_{u_*}} X = \operatorname{tr}((\operatorname{Id} - \nu_{u_*} \otimes \nu_{u_*})DX)$  is the tangential divergence of  $X$  on the countably  $\mathcal{H}^{N-1}$ -rectifiable set  $\widehat{J}_{u_*}$  with  $\nu_{u_*}$  the approximate unit normal to that set. The boundary term in the right hand side of (1.11) is interpreted in the sense of duality by (1.9), and  $\nabla_\tau g$  denotes the tangential derivative of  $g$ . If  $J_{u_*}$  and  $u_*$  are regular enough, then (1.11) provides the coupling equation

$$H_{u_*} + [|\nabla u_*|^2]^\pm = 0 \quad \text{on } J_{u_*},$$

where  $H_{u_*}$  denotes the scalar mean curvature of  $J_{u_*}$  with respect to the normal  $\nu_{u_*}$  and  $[|\nabla u_*|^2]^\pm$  the (accordingly oriented) jump of  $|\nabla u_*|^2$  across  $J_{u_*}$  (see [1, Chapter 7, Section 7.4]).

**Remark 1.1 (1D case).** In the one-dimensional case  $N = 1$ , if  $\Omega = (0, L)$  for some  $L > 0$ , we can see that if  $u \in SBV^2(0, L)$  satisfies conditions (1.9)-(1.10), then  $u$  is either piecewise constant with a finite number of jumps or  $u$  is a globally affine function (with no jump). Indeed, the very definition of  $SBV^2(0, L)$  shows that  $u$  has a finite number of jumps. Then, condition (1.9) implies that  $u$  is affine in between to consecutive jump points, and (1.10) implies that the slope of all affine functions must be zero. However, condition (1.11) does not play any role because it only implies that  $|u'|$  is constant in  $(0, L)$ , where  $u'$  is the approximate derivative of  $u$ . From this, we just deduce that  $u$  is a piecewise affine function with equal slopes in absolute value, and it is not sufficient by itself to prove that  $u$  is piecewise constant. It indicates that the use of  $SBV^2$ -test functions in (1.7) can not be relaxed to a class of smooth functions (in any dimension).

**1.3. Main results.** As already mentioned, the main purpose of this article is to investigate the asymptotic behavior of critical points of the Ambrosio-Tortorelli functional as  $\varepsilon \rightarrow 0$ . In view of the  $\Gamma$ -convergence result, one may expect that critical points converge to critical points, possibly under the assumption of convergence of energies. Without fully resolving this question, our analysis provides the first answer in this direction in arbitrary dimensions showing that a limit of critical points of  $AT_\varepsilon$  must at least be a critical point of  $MS$  with respect to inner variations, i.e., a stationary point of  $MS$ . If a critical point  $(u_\varepsilon, v_\varepsilon)$  of  $AT_\varepsilon$  is smooth enough, then it is easy to see that it is also stationary, i.e.,  $\delta AT_\varepsilon(u_\varepsilon, v_\varepsilon) = 0$  (see Lemma A.10). Hence, if regularity of critical points  $AT_\varepsilon$  holds, proving the convergence of the first inner variations implies the announced stationarity of the limit. This is the path we have followed, and the regularity issue is the object of our first main theorem.

**Theorem 1.1.** *Let  $\Omega \subset \mathbf{R}^N$  be a bounded open set with Lipschitz boundary and  $g \in H^{\frac{1}{2}}(\partial\Omega)$ . If  $(u_\varepsilon, v_\varepsilon) \in \mathcal{A}_g(\Omega)$  is a critical point of  $AT_\varepsilon$ , then  $(u_\varepsilon, v_\varepsilon) \in [\mathcal{C}^\infty(\Omega)]^2$  and the following regularity up to the boundary holds.*

- (i) If  $g \in H^{\frac{1}{2}}(\partial\Omega) \cap L^\infty(\partial\Omega)$ , then  $u_\varepsilon \in L^\infty(\Omega)$ .
- (ii) If  $\partial\Omega$  is of class  $\mathcal{C}^{k \vee 2, 1}$  and  $g \in \mathcal{C}^{k, \alpha}(\partial\Omega)$  with  $k \geq 1$  and  $\alpha \in (0, 1)$ , then  $(u_\varepsilon, v_\varepsilon) \in \mathcal{C}^{k, \alpha}(\overline{\Omega}) \times \mathcal{C}^{k \vee 2, \alpha}(\overline{\Omega})$ .

We emphasize that the regularity in Theorem 1.1 is highly non trivial since the second equation in (1.3) is of the form  $\Delta v = f$  with  $f \in L^1$  and standard linear elliptic theory does not directly apply. Instead, we shall rely on arguments borrowed from the regularity theory for harmonic maps into a manifold, or more generally for variational nonlinear elliptic systems, see e.g. [17]. The key issue is to prove Hölder continuity of  $v_\varepsilon$ , that we achieve by proving that it belongs to a suitable Morrey-Campanato space. We treat interior and boundary regularity in a similar way through a reflection argument of independent interest originally devised in [40].

In our second main theorem, we show that, under the assumption of convergence of energies, limits (up to a subsequence) of critical points of  $AT_\varepsilon$  are critical points of  $MS$  for the inner variations.

**Theorem 1.2.** *Assume that  $\Omega \subset \mathbf{R}^N$  is a bounded open set of class  $\mathcal{C}^{2, 1}$  and  $g \in \mathcal{C}^{2, \alpha}(\overline{\Omega})$  for some  $\alpha \in (0, 1)$ . Let  $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon > 0} \subset \mathcal{A}_g(\Omega)$  be a family of critical points of the Ambrosio-Tortorelli functional. Then, the following properties hold:*

- (i) *If the energy bound*

$$\sup_{\varepsilon > 0} AT_\varepsilon(u_\varepsilon, v_\varepsilon) < \infty \quad (1.12)$$

*is satisfied, up to a subsequence,  $u_\varepsilon \rightarrow u_*$  strongly in  $L^2(\Omega)$  as  $\varepsilon \rightarrow 0$  for some  $u_* \in SBV^2(\Omega) \cap L^\infty(\Omega)$  satisfying  $\nabla u_* \cdot \nu_\Omega \in L^2(\partial\Omega)$ , and  $dMS(u_*)[\phi] = 0$  for all  $\phi \in \mathcal{C}_c^\infty(\Omega)$ , i.e.,*

$$\operatorname{div}(\nabla u_*) = 0 \quad \text{in } \mathcal{D}'(\Omega).$$

- (ii) *If, further, the energy convergence*

$$AT_\varepsilon(u_\varepsilon, v_\varepsilon) \rightarrow MS(u_*) \quad (1.13)$$

*is satisfied, then  $\delta MS(u_*) = 0$ , i.e.,*

$$\int_{\Omega} (|\nabla u_*|^2 \operatorname{Id} - 2\nabla u_* \otimes \nabla u_*) : DX \, dx + \int_{\hat{J}_{u_*}} \operatorname{div} \hat{J}_{u_*} X \, d\mathcal{H}^{N-1} = -2 \int_{\partial\Omega} (\nabla u_* \cdot \nu_\Omega)(X_\tau \cdot \nabla_\tau g) \, d\mathcal{H}^{N-1} \quad (1.14)$$

*for all vector field  $X \in \mathcal{C}_c^1(\mathbf{R}^N; \mathbf{R}^N)$  with  $X \cdot \nu_\Omega = 0$  on  $\partial\Omega$ .*

**Remark 1.2.** At this stage, it is still open whether or not  $u_*$  is a critical point of  $MS$  as we do not know if the outer variation  $dMS(u_*)$  also vanishes on arbitrary functions  $\phi \in SBV^2(\Omega)$  satisfying  $\hat{J}_\phi \subset \hat{J}_{u_*}$  (and not only on  $\mathcal{C}_c^\infty(\Omega)$ ). In other words, the weak form of the homogeneous Neumann condition (1.10) on  $\hat{J}_{u_*}$  remains to be established. This is the only missing ingredient to obtain that  $u_*$  is a critical point of  $MS$ .

An assumption of convergence of energies similar to (1.13) has been used in [33, 24, 25, 26] to prove that critical points of the Allen-Cahn functional (from phase transitions) converge towards critical points of the perimeter functional, hence to minimal surfaces. The analysis without this assumption has been first carried out in [21], and it shows that critical points converge (in the sense of inner variations) towards integer multiplicity stationary varifolds, a measure theoretic generalization of minimal surfaces allowing for multiplicities. Interfaces with multiplicities do appear as limits of critical points of the Allen-Cahn energy and cannot be excluded, see e.g. [21, Section 6.3]. In our context, a similar phenomenon may appear, so that assumption (1.13) is probably necessary.

In [24, 25, 26], convergence of energies is also used to pass to the limit in the second inner variation. Following the same path, (1.13) allows us to pass to the limit in the second inner variation of  $AT_\varepsilon$ . It shows that the second inner variations of  $AT_\varepsilon$  do not converge to the second inner variation of  $MS$ , but to the second inner variation plus a residual additional term. As a byproduct, it follows that limits of

stable critical points of  $AT_\varepsilon$  satisfy an “augmented” second order minimality condition. Second order minimality criteria for  $MS$  has been addressed in [11, 6]. We also note that the convergence of the second inner variation for the Allen-Cahn functional without the assumption of convergence of energies has been studied in [15], see also [20]. Convergence of second inner variations is our third and last main result.

**Theorem 1.3.** *Assume that  $\Omega \subset \mathbf{R}^N$  is a bounded open set of class  $\mathcal{C}^{3,1}$  and  $g \in \mathcal{C}^{3,\alpha}(\partial\Omega)$  for some  $\alpha \in (0, 1)$ . Let  $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0} \subset \mathcal{A}_g(\Omega)$  be a family of critical points of the Ambrosio-Tortorelli functional and  $u_* \in SBV^2(\Omega) \cap L^\infty(\Omega)$  be as in Theorem 1.2, satisfying the convergence of energy (1.13). Then,*

(i) *For all  $X \in \mathcal{C}_c^2(\mathbf{R}^N; \mathbf{R}^N)$  and all  $G \in \mathcal{C}^3(\mathbf{R}^N)$  with  $X \cdot \nu_\Omega = 0$  and  $G = g$  on  $\partial\Omega$ ,*

$$\lim_{\varepsilon \rightarrow 0} \delta^2 AT_\varepsilon(u_\varepsilon, v_\varepsilon)[X, G] = \delta^2 MS(u_*)[X, G] + \int_{\hat{J}_{u_*}} |DX : (\nu_{u_*} \otimes \nu_{u_*})|^2 d\mathcal{H}^{N-1}; \quad (1.15)$$

(ii) *If  $(u_\varepsilon, v_\varepsilon)$  is a stable critical point of  $AT_\varepsilon$ , i.e.*

$$d^2 AT_\varepsilon(u_\varepsilon, v_\varepsilon)[\phi, \psi] \geq 0 \quad \text{for all } (\phi, \psi) \in [\mathcal{C}_c^\infty(\Omega)]^2,$$

*then  $u_*$  satisfies the second order inequality*

$$\delta^2 MS(u_*)[X, G] + \int_{\hat{J}_{u_*}} |DX : (\nu_{u_*} \otimes \nu_{u_*})|^2 d\mathcal{H}^{N-1} \geq 0$$

*for all  $X \in \mathcal{C}_c^2(\mathbf{R}^N; \mathbf{R}^N)$  and all  $G \in \mathcal{C}^3(\mathbf{R}^N)$  with  $X \cdot \nu_\Omega = 0$  and  $G = g$  on  $\partial\Omega$ .*

In the one-dimensional case, the asymptotic analysis as  $\varepsilon \rightarrow 0$  of critical points of the Ambrosio-Tortorelli functional has already been carried out in [13, 23], in the slightly different setting where a homogeneous Neumann boundary condition is assumed for the phase field variable  $v$ . The authors proved that if  $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0}$  is a family of critical points of the Ambrosio-Tortorelli functional satisfying (1.12), then, up to a subsequence,  $(u_\varepsilon, v_\varepsilon) \rightarrow (u, 1)$  in  $[L^2(\Omega)]^2$  with  $u \in SBV^2(\Omega)$  that is either globally affine or piecewise constant with a finite number of jumps, see Remark 1.1. This result is extended in [23] to the Ambrosio-Tortorelli functional with a fidelity term. We have also carried out the 1D analysis in our setting, i.e., with the Dirichlet boundary condition on the  $v$  variable. In this case, we establish a convergence result for critical points without assuming the convergence of the energy (1.13), but proving (1.13) as a consequence of the energy bound (1.12), see Theorems 7.1 & 7.2 and Remark 7.2. The analysis is very similar to [13, 23], the only difference being in the boundary condition on  $v$ . As in those references, we observe a selection criterion of all attainable critical points which can be here of two types: either a globally affine function or a piecewise constant function with a single jump at the middle point of the interval  $\Omega = (0, L)$ . As a by-product, this one-dimensional analysis allows us to exhibit non-minimizing critical points of  $AT_\varepsilon$  satisfying our energy convergence assumption (1.13).

**1.4. Ideas of the convergence proof.** The proof of Theorem 1.2 relies on the classical compactness argument and the lower bound inequality for the Ambrosio-Tortorelli functional. Indeed, the energy bound for a family  $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0} \subset \mathcal{A}_g(\Omega)$  of critical points for  $AT_\varepsilon$  implies the  $L^2(\Omega)$ -convergence (up to a subsequence) of  $u_\varepsilon$  to a limit  $u_* \in SBV^2(\Omega)$ , together with a  $\Gamma$ -liminf inequality  $MS(u_*) \leq \liminf_\varepsilon AT_\varepsilon(u_\varepsilon, v_\varepsilon)$ . Our energy convergence assumption (1.13) leads to the *equipartition* of phase field energy, as well as the convergence of the bulk energy. Then, as in [21], we associate an  $(N-1)$ -varifold  $V_\varepsilon$  to the phase field variable  $v_\varepsilon$ , which converges (again up to a subsequence) to a limiting varifold  $V_*$ . The energy convergence (1.13) allows us to identify the mass of  $V_*$ , that is  $\|V_*\| = \mathcal{H}^{N-1} \llcorner \hat{J}_{u_*}$ . Next, we use the equations satisfied by  $(u_\varepsilon, v_\varepsilon)$  in their conservative form to pass to the limit, and find an equation satisfied by  $u_*$  and  $V_*$ . The idea is then to employ a blow-up argument similar to [2] to identify (the first moment of)  $V_*$ , and show that it is the rectifiable varifold associated to  $\hat{J}_{u_*}$  with multiplicity one.



To prove Theorem 1.3, we argue as in [24, 25, 26]. We observe that the convergence  $V_\varepsilon \rightarrow V_*$  in the sense of varifolds and the identification of  $V_*$  implies the convergence of quadratic terms  $\varepsilon \nabla v_\varepsilon \otimes \nabla v_\varepsilon \rightarrow \frac{1}{2} \nu_{u_*} \otimes \nu_{u_*} \mathcal{H}^{N-1} \llcorner \hat{J}_{u_*}$  in the sense of measures. This information is precisely what is needed to pass to the limit in the second inner variation of  $AT_\varepsilon$ , and we infer from a stability condition on  $(u_\varepsilon, v_\varepsilon) \in \mathcal{A}_g(\Omega)$  a stability condition on the limit  $u_* \in SBV^2(\Omega)$ .

The paper is organised as follows. Section 2 collects several notation that will be used throughout the paper. In Section 3, we study the regularity theory for critical points of the Ambrosio-Tortorelli functional proving first smoothness in the interior of the domain, and then smoothness at the boundary. In Section 4, we prove compactness of a family  $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0}$  satisfying a uniform energy bound  $\sup_\varepsilon AT_\varepsilon(u_\varepsilon, v_\varepsilon) < \infty$ . The regularity result allows one to derive the conservative form of the equations satisfied by these critical points which itself provides bounds on the normal traces of  $u_\varepsilon$  and  $v_\varepsilon$  on  $\partial\Omega$ . Then, in Section 5, we improve the previous results by assuming the energy convergence  $AT_\varepsilon(u_\varepsilon, v_\varepsilon) \rightarrow MS(u_*)$ . From this assumption we obtain equipartition of the phase field part of the energy. Then, we employ a reformulation in terms of varifolds to pass to the limit in the inner variational equations satisfied by critical points of  $AT_\varepsilon$  to prove that the weak limit  $u_*$  of  $u_\varepsilon$  is a stationary point of the Mumford-Shah energy. The asymptotic behavior of the second inner variations is performed in Section 6. Finally, Section 7 is devoted to the one-dimensional case.

## 2. NOTATION AND PRELIMINARIES

**2.1. Measures.** The Lebesgue measure in  $\mathbf{R}^N$  is denoted by  $\mathcal{L}^N$ , and the  $k$ -dimensional Hausdorff measure by  $\mathcal{H}^k$ . We will sometime write  $\omega_k$  for the  $\mathcal{L}^k$ -measure of the  $k$ -dimensional unit ball in  $\mathbf{R}^k$ .

If  $X \subset \mathbf{R}^N$  is a locally compact set and  $Y$  an Euclidean space, we denote by  $\mathcal{M}(X; Y)$  the space of  $Y$ -valued bounded Radon measures in  $X$  endowed with the norm  $\|\mu\| = |\mu|(X)$ , where  $|\mu|$  is the variation of the measure  $\mu$ . If  $Y = \mathbf{R}$ , we simply write  $\mathcal{M}(X)$  instead of  $\mathcal{M}(X; \mathbf{R})$ . By Riesz representation theorem,  $\mathcal{M}(X; Y)$  can be identified with the topological dual of  $\mathcal{C}_0(X; Y)$ , the space of continuous functions  $f : X \rightarrow Y$  such that  $\{|f| \geq \varepsilon\}$  is compact for all  $\varepsilon > 0$ . The weak\* topology of  $\mathcal{M}(X; Y)$  is defined using this duality.

**2.2. Functional spaces.** We use standard notation for Lebesgue, Sobolev and Hölder spaces. Given a bounded open set  $\Omega \subset \mathbf{R}^N$ , the space of functions of bounded variation is defined by

$$BV(\Omega) = \{u \in L^1(\Omega) : Du \in \mathcal{M}(\Omega; \mathbf{R}^N)\}.$$

We shall also consider the subspace  $SBV(\Omega)$  of special functions of bounded variation made of functions  $u \in BV(\Omega)$  whose distributional derivative can be decomposed as

$$Du = \nabla u \mathcal{L}^N + (u^+ - u^-) \nu_u \mathcal{H}^{N-1} \llcorner J_u.$$

In the previous expression,  $\nabla u$  is the Radon-Nikodým derivative of  $Du$  with respect to  $\mathcal{L}^N$ , and it is called the approximate gradient of  $u$ . The Borel set  $J_u$  is the (approximate) jump set of  $u$ . It is a countably  $\mathcal{H}^{N-1}$ -rectifiable subset of  $\Omega$  oriented by the (approximate) normal direction of jump  $\nu_u : J_u \rightarrow \mathbf{S}^{N-1}$ , and  $u^\pm$  are the one-sided approximate limits of  $u$  on  $J_u$  according to  $\nu_u$ . Finally we define

$$SBV^2(\Omega) = \{u \in SBV(\Omega) : \nabla u \in L^2(\Omega; \mathbf{R}^N) \text{ and } \mathcal{H}^{N-1}(J_u) < \infty\}.$$

We say that a Lebesgue measurable set  $E \subset \Omega$  has finite perimeter in  $\Omega$  if its characteristic function  $1_E \in BV(\Omega)$ . We denote by  $\partial^* E$  its reduced boundary. We refer to [1] for a detailed description of the space  $BV$ .

We recall the following compactness result which can be found in [10, Theorem 16].

**Lemma 2.1.** *Let  $(u_k)_{k \in \mathbf{N}}$  be a sequence in  $H^1(\Omega) \cap L^\infty(\Omega)$  and  $(E_k)_{k \in \mathbf{N}}$  be a sequence of sets with finite perimeter in  $\Omega$ . Assume that*

$$\mathcal{L}^N(E_k) \rightarrow 0, \quad \sup_{k \in \mathbf{N}} \mathcal{H}^{N-1}(\partial^* E_k) < \infty,$$

and that

$$\sup_{k \in \mathbf{N}} \{ \|u_k\|_{L^\infty(\Omega)} + \|\nabla u_k\|_{L^2(\Omega \setminus E_k; \mathbf{R}^N)} \} < \infty.$$

Then, there exist a (not relabled) subsequence and  $u \in SBV^2(\Omega) \cap L^\infty(\Omega)$  such that  $\tilde{u}_k := u_k \mathbf{1}_{\Omega \setminus E_k} \rightarrow u$  strongly in  $L^2(\Omega)$ ,  $\nabla \tilde{u}_k = \nabla u_k \mathbf{1}_{\Omega \setminus E_k} \rightharpoonup \nabla u$  weakly in  $L^2(\Omega; \mathbf{R}^N)$  and

$$\int_{\Omega} |\nabla u|^2 dx \leq \liminf_{k \rightarrow \infty} \int_{\Omega \setminus E_k} |\nabla u_k|^2 dx, \quad 2\mathcal{H}^{N-1}(J_u) \leq \liminf_{k \rightarrow \infty} \mathcal{H}^{N-1}(\partial^* E_k).$$

**2.3. Varifolds.** Let us recall several basic ingredients of the theory of varifolds (see [41] for a detailed description). We denote by  $\mathbf{G}_{N-1}$  the Grassmannian manifold of all  $(N-1)$ -dimensional linear subspaces of  $\mathbf{R}^N$ . The set  $\mathbf{G}_{N-1}$  is as usual identified with the set of all orthogonal projection matrices onto  $(N-1)$ -dimensional linear subspaces of  $\mathbf{R}^N$ , i.e.,  $N \times N$  symmetric matrices  $A$  such that  $A^2 = A$  and  $\text{tr}(A) = N-1$ , in other words, matrices of the form

$$A = \text{Id} - e \otimes e$$

for some  $e \in \mathbf{S}^{N-1}$ .

A  $(N-1)$ -varifold in  $X$  (a locally compact subset of  $\mathbf{R}^N$ ) is a bounded Radon measure on  $X \times \mathbf{G}_{N-1}$ . The class of  $(N-1)$ -varifold in  $X$  is denoted by  $\mathbf{V}_{N-1}(X)$ . The mass of  $V \in \mathbf{V}_{N-1}(X)$  is simply the measure  $\|V\| \in \mathcal{M}(X)$  defined by  $\|V\|(B) = V(B \times \mathbf{G}_{N-1})$  for all Borel sets  $B \subset X$ . We define the first variation of an  $(N-1)$ -varifold in  $V$  in an open set  $U \subset \mathbf{R}^N$  by

$$\delta V(\varphi) = \int_{U \times \mathbf{G}_{N-1}} D\varphi(x) : A dV(x, A) \quad \text{for all } \varphi \in \mathcal{C}_c^1(U; \mathbf{R}^N).$$

We say that an  $(N-1)$ -varifold is stationary in  $U$  if  $\delta V(\varphi) = 0$  for all  $\varphi$  in  $\mathcal{C}_c^1(U; \mathbf{R}^N)$ . We recall that such a varifold satisfies the monotonicity formula

$$\frac{\|V\|(B_\varrho(x_0))}{\varrho^{N-1}} = \frac{\|V\|(B_r(x_0))}{r^{N-1}} + \int_{(B_\varrho(x_0) \setminus B_r(x_0)) \times \mathbf{G}_{N-1}} \frac{|P_{A^\perp}(x - x_0)|^2}{|x - x_0|^{N+1}} dV(x, A)$$

for all  $x_0 \in U$  and  $0 < r < \varrho$  with  $B_\varrho(x_0) \subset U$ , where  $P_{A^\perp}$  is the orthogonal projection onto the one-dimensional space  $A^\perp$  (see [41, paragraph 40]).

**2.4. Tangential divergence.** Let  $\Gamma$  be a countably  $\mathcal{H}^{N-1}$ -rectifiable set and let  $T_x \Gamma$  its approximate tangent space defined for  $\mathcal{H}^{N-1}$ -a.e.  $x \in \Gamma$ . We consider an orthonormal basis  $\{\tau_1(x), \dots, \tau_{N-1}(x)\}$  of  $T_x \Gamma$  and denote by  $\nu(x)$  a normal vector to  $T_x \Gamma$ . If  $\zeta : \mathbf{R}^N \rightarrow \mathbf{R}^N$  is a smooth vector field, we denote by

$$\text{div}^\Gamma \zeta := \sum_{i=1}^{N-1} \tau_i \cdot \partial_{\tau_i} \zeta = (\text{Id} - \nu \otimes \nu) : D\zeta$$

the tangential divergence, and  $(\partial_{\tau_i} \zeta)^\perp = ((\partial_{\tau_i} \zeta) \cdot \nu) \nu = \partial_{\tau_i} \zeta - \sum_{j=1}^{N-1} (\tau_j \cdot \partial_{\tau_i} \zeta) \tau_j$ .

### 3. REGULARITY THEORY FOR CRITICAL POINTS OF THE AMBROSIO-TORTORELLI ENERGY

In this section, we investigate interior and boundary regularity properties of critical points of the Ambrosio-Tortorelli functional  $AT_\varepsilon$  for a parameter  $\varepsilon > 0$  which is kept fixed.

**3.1. Interior regularity.** We first establish interior regularity following ideas used by T. Rivière in [39] to prove the regularity of harmonic maps with values into a revolution torus.

**Theorem 3.1.** *Let  $\Omega \subset \mathbf{R}^N$  be a bounded open set. If  $(u_\varepsilon, v_\varepsilon) \in H^1(\Omega) \times [H^1(\Omega) \cap L^\infty(\Omega)]$  satisfies (1.4)-(1.5), then  $(u_\varepsilon, v_\varepsilon) \in [\mathcal{C}^\infty(\Omega)]^2$ .*

*Proof.* For simplicity, we drop the subscript  $\varepsilon$  in  $(u_\varepsilon, v_\varepsilon)$  and write instead  $(u, v)$ . We also assume  $N \geq 2$  since in the case  $N = 1$ , the regularity of  $(u, v)$  solution of (1.4)-(1.5) is elementary.

By (1.4),  $u$  weakly solves

$$-\operatorname{div}((\eta_\varepsilon + v^2)\nabla u) = 0 \quad \text{in } \Omega. \quad (3.1)$$

Setting  $M := \|v\|_{L^\infty(\Omega)}$ , the matrix field  $(\eta_\varepsilon + v^2)\operatorname{Id}$  has bounded measurable coefficients and it satisfies  $\eta_\varepsilon \operatorname{Id} \leq (\eta_\varepsilon + v^2)\operatorname{Id} \leq (\eta_\varepsilon + M^2)\operatorname{Id}$  a.e. in  $\Omega$  in the sense of quadratic forms. It is therefore uniformly elliptic and the De Giorgi-Nash-Moser regularity theorem applies to equation (3.1). It provides the existence of  $\alpha \in (0, 1)$  such that  $u \in \mathcal{C}_{\text{loc}}^{0,\alpha}(\Omega)$  together with the estimate:

$$K(\omega) := \sup_{x_0 \in \omega, \varrho > 0, B_\varrho(x_0) \subset \omega} \frac{1}{\varrho^{N-2+2\alpha}} \int_{B_\varrho(x_0)} |\nabla u|^2 dx < \infty \quad (3.2)$$

for every open subset  $\omega$  such that  $\bar{\omega} \subset \Omega$  (see e.g. Theorem 8.13 and Eq. (8.18) in [17]).

Now we claim that the function  $v$  belongs to  $\mathcal{C}_{\text{loc}}^{0,\alpha}(\Omega)$ . Before proving this claim, we complete the proof of the theorem. Assuming the claim to be true, we can use the Schauder estimates (see e.g. [17, Theorem 5.19]) to derive from equation (3.1) that  $u \in \mathcal{C}_{\text{loc}}^{1,\alpha}(\Omega)$ . On the other hand, by (1.5),  $v$  weakly solves

$$-\varepsilon \Delta v = \frac{1-v}{4\varepsilon} - |\nabla u|^2 v \quad \text{in } \Omega. \quad (3.3)$$

Since the right-hand-side of (3.3) belongs to  $\mathcal{C}_{\text{loc}}^{0,\alpha}(\Omega)$ , it follows from standard Schauder estimates that  $v \in \mathcal{C}_{\text{loc}}^{2,\alpha}(\Omega)$ . By a classical bootstrap, it now follows from equations (3.1) and (3.3) that both  $u$  and  $v$  are of class  $\mathcal{C}^\infty$  in  $\Omega$ .

Hence, it only remains to show the claim  $v \in \mathcal{C}_{\text{loc}}^{0,\alpha}(\Omega)$ . To this purpose, we fix an arbitrary ball  $\bar{B}_{2R}(x_0) \subset \Omega$ , and we aim to prove that  $v \in \mathcal{C}_{\text{loc}}^{0,\alpha}(B_R(x_0))$ . Consider  $v_1 \in H^1(B_{2R}(x_0))$  to be the unique weak solution of

$$\begin{cases} -\Delta v_1 &= \frac{1-v}{4\varepsilon^2} & \text{in } B_{2R}(x_0), \\ v_1 &= v & \text{on } \partial B_{2R}(x_0). \end{cases} \quad (3.4)$$

Since  $\Delta v_1 \in L^\infty(B_{2R}(x_0))$ , the Calderón-Zygmund estimates yield  $v_1 \in W_{\text{loc}}^{2,p}(B_{2R}(x_0))$  for every  $p < \infty$ . By Sobolev embedding, it follows that  $v_1 \in C_{\text{loc}}^{1,\beta}(B_{2R}(x_0))$  for every  $\beta \in (0, 1)$ . In particular, we have  $v_1 \in L^\infty(B_R(x_0))$ .

Set  $v_2 := v - v_1 \in H_0^1(B_{2R}(x_0))$ . By (3.3) and (3.4),  $v_2$  is a weak solution of

$$-\Delta v_2 = -\frac{1}{\varepsilon} |\nabla u|^2 v \quad \text{in } B_{2R}(x_0). \quad (3.5)$$

To show that  $v_2 \in \mathcal{C}_{\text{loc}}^{0,\alpha}(B_R(x_0))$ , the Morrey-Campanato Theorem (see e.g. [17, Theorem 5.7]) ensures that it suffices to prove the following Morrey type estimate:

$$\sup_{y \in B_R(x_0), \varrho \in (0, R)} \frac{1}{\varrho^{N-2+2\alpha}} \int_{B_\varrho(y)} |\nabla v_2|^2 dx < \infty. \quad (3.6)$$

Let  $y \in B_R(x_0)$  and  $r \in (0, R)$  arbitrary. We denote by  $w \in v_2 + H_0^1(B_r(y))$  the harmonic extension of  $v_2$  in the ball  $B_r(y)$ , i.e., the unique (weak) solution of

$$\begin{cases} -\Delta w &= 0 & \text{in } B_r(y), \\ w &= v_2 & \text{on } \partial B_r(y). \end{cases} \quad (3.7)$$

Since  $v_2 = v - v_1 \in L^\infty(B_R(x_0))$ , we have  $|w| \leq \|v_2\|_{L^\infty(B_R(x_0))}$  on  $\partial B_r(y)$ , and the weak maximum principle implies that  $w \in L^\infty(B_r(y))$  with  $\|w\|_{L^\infty(B_r(y))} \leq \|v_2\|_{L^\infty(B_R(x_0))}$ . Moreover,  $|\nabla w|^2$  being subharmonic in  $B_r(y)$ , we get that for every  $\varrho < r$ ,

$$\int_{B_\varrho(y)} |\nabla w|^2 dx \leq \left(\frac{\varrho}{r}\right)^N \int_{B_r(y)} |\nabla w|^2 dx.$$

Recalling  $w$  also minimizes the Dirichlet integral among all functions agreeing with  $v_2$  on  $\partial B_r(y)$ , we infer that

$$\begin{aligned} \int_{B_\varrho(y)} |\nabla v_2|^2 dx &\leq 2 \int_{B_\varrho(y)} |\nabla w|^2 dx + 2 \int_{B_\varrho(y)} |\nabla(w - v_2)|^2 dx \\ &\leq 2 \left(\frac{\varrho}{r}\right)^N \int_{B_r(y)} |\nabla w|^2 dx + 2 \int_{B_r(y)} |\nabla(w - v_2)|^2 dx \\ &\leq 2 \left(\frac{\varrho}{r}\right)^N \int_{B_r(y)} |\nabla v_2|^2 dx + 2 \int_{B_r(y)} |\nabla(w - v_2)|^2 dx \end{aligned}$$

for every  $\varrho < r$ . Since  $w - v_2 = 0$  on  $\partial B_r(y)$ , (3.5) and (3.7) lead to

$$\int_{B_r(y)} |\nabla(w - v_2)|^2 dx = \frac{1}{\varepsilon} \int_{B_r(y)} |\nabla u|^2 v(w - v_2) dx \leq \frac{2}{\varepsilon} \|v\|_{L^\infty(\Omega)} \|v_2\|_{L^\infty(B_R(x_0))} \int_{B_r(y)} |\nabla u|^2 dx.$$

In view of (3.2), we have thus proved that for every  $y \in B_R(x_0)$  and  $0 < \varrho \leq r < R$ ,

$$\int_{B_\varrho(y)} |\nabla v_2|^2 dx \leq 2 \left(\frac{\varrho}{r}\right)^N \int_{B_r(y)} |\nabla v_2|^2 dx + C_1 r^{N-2+2\alpha}$$

with  $C_1 := \frac{4}{\varepsilon} \|v\|_{L^\infty(\Omega)} \|v_2\|_{L^\infty(B_R(x_0))} K(B_{2R}(x_0))$ . By using a classical iteration lemma (see e.g. [17, Lemma 5.13]), we infer that for every  $y \in B_R(x_0)$  and  $0 < \varrho < R$ ,

$$\int_{B_\varrho(y)} |\nabla v_2|^2 dx \leq C_\alpha \varrho^{N-2+2\alpha} \left( \frac{1}{R^{N-2+2\alpha}} \int_{B_{2R}(x_0)} |\nabla v_2|^2 dx + C_1 \right),$$

for a constant  $C_\alpha$  depending only on  $\alpha$  and  $N$ . Hence  $v_2$  satisfies the Morrey estimate (3.6), and thus  $v_2 \in \mathcal{C}_{\text{loc}}^{0,\alpha}(B_R(x_0))$ . In turn,  $v = v_1 + v_2 \in \mathcal{C}_{\text{loc}}^{0,\alpha}(B_R(x_0))$  and the proof of the claim is complete.  $\square$

**3.2. Maximum principle and boundary regularity.** We first show a (standard) maximum principle which stipulates that  $v_\varepsilon$  takes values between 0 and 1, and that  $u_\varepsilon$  is bounded whenever the boundary condition  $g$  is.

**Lemma 3.1** (Maximum principle). *Let  $\Omega \subset \mathbf{R}^N$  be a bounded open set with Lipschitz boundary and  $(u_\varepsilon, v_\varepsilon) \in H^1(\Omega) \times [H^1(\Omega) \cap L^\infty(\Omega)]$  satisfying (1.4)-(1.5). If  $v_\varepsilon = 1$  on  $\partial\Omega$ , then  $0 \leq v_\varepsilon \leq 1$  a.e. in  $\Omega$ . In addition, if  $u_\varepsilon = g$  on  $\partial\Omega$  for a function  $g \in H^{\frac{1}{2}}(\partial\Omega) \cap L^\infty(\partial\Omega)$ , then  $u_\varepsilon \in L^\infty(\Omega)$  and  $\|u_\varepsilon\|_{L^\infty(\Omega)} \leq \|g\|_{L^\infty(\partial\Omega)}$ .*

*Proof.* For a generic function  $f \in L^1(\Omega)$ , we set  $f^+ := (f + |f|)/2$  and  $f^- := (|f| - f)/2$ . For simplicity, we drop the subscript  $\varepsilon$  in  $(u_\varepsilon, v_\varepsilon)$  and write instead  $(u, v)$ .

Since  $v - 1 \in H_0^1(\Omega) \cap L^\infty(\Omega)$ , it follows that  $(v - 1)^+ \in H_0^1(\Omega) \cap L^\infty(\Omega)$  with  $\nabla(v - 1)^+ = \nabla v \mathbf{1}_{\{v \geq 1\}}$ . Using  $(v - 1)^+$  as a test function in (1.5) leads to

$$\varepsilon \int_{\Omega} |\nabla(v - 1)^+|^2 dx + \int_{\Omega} \left( \frac{v - 1}{4\varepsilon} + v |\nabla u|^2 \right) (v - 1)^+ dx = 0.$$

Since both integrals are non negative, we infer that

$$\int_{\Omega} |(v - 1)^+|^2 dx = 0.$$

Hence  $(v - 1)^+ = 0$  a.e. in  $\Omega$ , which yields  $v \leq 1$  a.e. in  $\Omega$ . Next, since  $v = 1$  on  $\partial\Omega$ , we have  $-v^- \in H_0^1(\Omega) \cap L^\infty(\Omega)$ . The same argument as before with  $-v^-$  as test function in (1.5) shows that  $-v = 0$  a.e. in  $\Omega$ , that is  $v \geq 0$  a.e. in  $\Omega$ .

Now we assume that  $u = g$  on  $\partial\Omega$  with  $g \in H^{\frac{1}{2}}(\partial\Omega) \cap L^\infty(\partial\Omega)$  and we set  $M := \|g\|_{L^\infty(\partial\Omega)}$ . Since  $|g| \leq M$  on  $\partial\Omega$ , we have  $(u - M)^+ \in H_0^1(\Omega)$  with  $\nabla(u - M)^+ = \nabla u \mathbf{1}_{\{u \geq M\}}$ . Using  $(u - M)^+$  as a test function in (1.4) yields

$$0 = \int_{\Omega} (\eta_\varepsilon + v^2) \nabla u \cdot \nabla(u - M)^+ dx = \int_{\Omega} (\eta_\varepsilon + v^2) |\nabla(u - M)^+|^2 dx,$$

Therefore  $\nabla(u - M)^+ = 0$  a.e. in  $\Omega$  which implies that  $(u - M)^+$  is constant. Since  $(u - M)^+ \in H_0^1(\Omega)$ , it follows that  $(u - M)^+ = 0$  a.e. in  $\Omega$ , that is  $u \leq M$  a.e. in  $\Omega$ . The same argument applied to  $(u + M)^- \in H_0^1(\Omega)$  shows that  $u \geq -M$  a.e. in  $\Omega$ , and thus  $\|u\|_{L^\infty(\Omega)} \leq M$ .  $\square$

Next we study the boundary regularity of a critical point  $(u_\varepsilon, v_\varepsilon)$  of the Ambrosio-Tortorelli energy. Our strategy is to use a local reflexion argument to extend  $(u_\varepsilon, v_\varepsilon)$  across the boundary. The extension will then satisfy a modified system of PDEs for which we can apply an interior regularity result (similar to that of Theorem 3.1). The reflexion argument originates in [40] and follows the arguments in [12]. Note that Lemma 3.1 and Theorem 3.2 together with a standard covering argument completes the proof of Theorem 1.1.

**Theorem 3.2.** *Let  $\Omega \subset \mathbf{R}^N$  be a bounded open set and  $(u_\varepsilon, v_\varepsilon) \in H^1(\Omega) \times [H^1(\Omega) \cap L^\infty(\Omega)]$  satisfying (1.4)-(1.5). Assume that in some ball  $B_{4R}(x_0)$  with  $x_0 \in \partial\Omega$ , the boundary portion  $\partial\Omega \cap B_{4R}(x_0)$  is of class  $\mathcal{C}^{k \vee 2, 1}$  and  $(u_\varepsilon, v_\varepsilon) = (g, 1)$  on  $\partial\Omega \cap B_{4R}(x_0)$  for a function  $g \in \mathcal{C}^{k, \alpha}(\partial\Omega \cap B_{4R}(x_0))$  with  $k \geq 1$  and  $\alpha \in (0, 1)$ . Then  $(u_\varepsilon, v_\varepsilon) \in \mathcal{C}^{k, \alpha}(\overline{\Omega} \cap B_{\theta R}(x_0)) \times \mathcal{C}^{k \vee 2, \alpha}(\overline{\Omega} \cap B_{\theta R}(x_0))$  for some constant  $\theta \in (0, 1)$ .*

*Proof.* We start by describing the reflexion method that we use to extend functions across  $\partial\Omega$  in a neighborhood of the point  $x_0$ . We assume that  $x_0 \in \partial\Omega$  and  $R > 0$  are fixed, and that the assumption of the theorem is satisfied. Since  $\partial\Omega \cap B_{4R}(x_0)$  is (at least) of class  $\mathcal{C}^{2, 1}$ , we can find a small  $\delta_0 \in (0, R/2)$  such that the nearest point projection on  $\partial\Omega \cap B_{4R}(x_0)$ , denoted by  $\pi_\Omega$ , is well-defined and (at least) of class  $\mathcal{C}^{1, 1}$  in a tubular neighborhood of size  $2\delta_0$  of  $\partial\Omega \cap B_{4R}(x_0)$  intersected with  $B_{3R}(x_0)$ . For  $\delta \in (0, 2\delta_0]$ , we set

$$\begin{cases} U_\delta := \{x \in \mathbf{R}^N : \text{dist}(x, \partial\Omega) < \delta\} \cap B_{3R}(x_0), \\ U_\delta^{\text{in}} := \Omega \cap U_\delta, \\ U_\delta^{\text{ex}} := U_\delta \setminus \overline{\Omega}. \end{cases}$$

The geodesic reflexion across  $\partial\Omega \cap U_{2\delta_0}$  is denoted by  $\sigma_\Omega : U_{2\delta_0} \rightarrow \mathbf{R}^N$  and it is defined by

$$\sigma_\Omega(x) := 2\pi_\Omega(x) - x \quad \text{for all } x \in U_{2\delta_0}.$$

The mapping  $\sigma_\Omega$  is an involutive  $\mathcal{C}^{1, 1}$ -diffeomorphism (onto its image) which satisfies  $\sigma_\Omega(x) = x$  for all  $x \in \partial\Omega \cap U_{2\delta_0}$ . Reducing the value of  $\delta_0$  if necessary, we have

$$\sigma_\Omega(U_\delta^{\text{ex}} \cap B_R(x_0)) \subset U_\delta^{\text{in}} \cap B_{2R}(x_0) \text{ and } \sigma_\Omega(U_\delta^{\text{in}} \cap B_R(x_0)) \subset U_\delta^{\text{ex}} \cap B_{2R}(x_0) \text{ for } \delta \in (0, 2\delta_0).$$

Next we consider the bounded open set

$$\tilde{\Omega} := (U_{\delta_0} \cap B_R(x_0)) \cup \sigma_\Omega(U_{\delta_0} \cap B_R(x_0)) \subset U_{\delta_0} \cap B_{2R}(x_0). \quad (3.8)$$

The mapping  $\sigma_\Omega$  being involutive, we have

$$\sigma_\Omega(\tilde{\Omega}) = \tilde{\Omega}, \quad \sigma_\Omega(\tilde{\Omega} \cap \Omega) = \tilde{\Omega} \setminus \overline{\Omega} \quad \text{and} \quad \sigma_\Omega(\tilde{\Omega} \setminus \overline{\Omega}) = \tilde{\Omega} \cap \Omega.$$

Differentiating the relation  $\sigma_\Omega(\sigma_\Omega(x)) = x$  yields  $D\sigma_\Omega(x)D\sigma_\Omega(\sigma_\Omega(x)) = \text{Id}$ , and thus

$$D\sigma_\Omega(\sigma_\Omega(x)) = (D\sigma_\Omega(x))^{-1} \quad \text{for every } x \in \tilde{\Omega}. \quad (3.9)$$

For  $x \in \partial\Omega \cap \tilde{\Omega}$ , one has  $(D\sigma_\Omega(x))^T v = 2\mathbf{p}_x(v) - v$  for all  $v \in \mathbf{R}^N$ , where  $\mathbf{p}_x$  is the orthogonal projection from  $\mathbf{R}^N$  onto the tangent space  $T_x(\partial\Omega)$  to  $\partial\Omega$  at  $x$ , i.e.,  $(D\sigma_\Omega(x))^T$  is the reflexion matrix across the hyperplane  $T_x(\partial\Omega)$ . In particular,

$$(D\sigma_\Omega(x))^T D\sigma_\Omega(x) = (D\sigma_\Omega(x))^T (D\sigma_\Omega(x))^T = \text{Id for every } x \in \partial\Omega \cap \tilde{\Omega}. \quad (3.10)$$

Now we define for  $x \in \tilde{\Omega}$ ,

$$j(x) := \begin{cases} 1 & \text{if } x \in \tilde{\Omega} \cap \Omega, \\ |\det D\sigma_\Omega(x)| & \text{if } x \in \tilde{\Omega} \setminus \Omega, \end{cases}$$

and

$$A(x) := \begin{cases} \text{Id} & \text{if } x \in \Omega, \\ j(x) [D\sigma_\Omega(\sigma_\Omega(x))]^T D\sigma_\Omega(\sigma_\Omega(x)) & \text{if } x \in \tilde{\Omega} \setminus \Omega. \end{cases}$$

In view of (3.10),  $j$  and  $A$  are Lipschitz continuous in  $\tilde{\Omega}$  and  $A$  is uniformly elliptic, i.e., there exist two constants  $0 < \lambda_\Omega \leq \Lambda_\Omega$  such that

$$\lambda_\Omega |\xi|^2 \leq A(x) \xi \cdot \xi \leq \Lambda_\Omega |\xi|^2 \quad \text{for every } (x, \xi) \in \tilde{\Omega} \times \mathbf{R}^N.$$

With these geometrical preliminaries, we are now ready to provide the extension of  $(u_\varepsilon, v_\varepsilon)$  to  $\tilde{\Omega}$ . We define for  $x \in \tilde{\Omega}$ ,

$$\hat{u}_\varepsilon(x) := \begin{cases} u_\varepsilon(x) & \text{if } x \in \Omega \\ u_\varepsilon(\sigma_\Omega(x)) & \text{if } x \in \tilde{\Omega} \setminus \Omega \end{cases}, \quad \hat{v}_\varepsilon(x) := \begin{cases} v_\varepsilon(x) & \text{if } x \in \Omega \\ v_\varepsilon(\sigma_\Omega(x)) & \text{if } x \in \tilde{\Omega} \setminus \Omega \end{cases}, \quad (3.11)$$

and

$$\tilde{u}_\varepsilon(x) := \begin{cases} u_\varepsilon(x) & \text{if } x \in \Omega \\ 2g(\pi_\Omega(x)) - u_\varepsilon(\sigma_\Omega(x)) & \text{if } x \in \tilde{\Omega} \setminus \Omega \end{cases}, \quad \tilde{v}_\varepsilon(x) := \begin{cases} v_\varepsilon(x) & \text{if } x \in \Omega \\ 2 - v_\varepsilon(\sigma_\Omega(x)) & \text{if } x \in \tilde{\Omega} \setminus \Omega \end{cases}. \quad (3.12)$$

By the chain rule in Sobolev spaces and the fact that the traces of these functions coincide on both side of  $\partial\Omega \cap \tilde{\Omega}$ , each one of them belongs to  $H^1(\tilde{\Omega})$ . In addition,  $\hat{v}_\varepsilon$  and  $\tilde{v}_\varepsilon$  also belong to  $L^\infty(\tilde{\Omega})$  since  $v_\varepsilon \in L^\infty(\Omega)$ . We finally set

$$\tilde{g} := g \circ \pi_\Omega \in \mathcal{C}^{1,\alpha}(\tilde{\Omega}).$$

Now we show that these extensions satisfy suitable equations in the domain  $\tilde{\Omega}$ .

**Lemma 3.2.** *We have*

$$-\text{div}((\eta_\varepsilon + \hat{v}_\varepsilon^2) A \nabla \tilde{u}_\varepsilon) = -2 \text{div}(\mathbf{1}_{\tilde{\Omega} \setminus \Omega} (\eta_\varepsilon + \hat{v}_\varepsilon^2) A \nabla \tilde{g}) \quad \text{in } \mathcal{D}'(\tilde{\Omega}), \quad (3.13)$$

and

$$-\varepsilon \text{div}(A \nabla \tilde{v}_\varepsilon) = (\mathbf{1}_{\tilde{\Omega} \cap \Omega} - \mathbf{1}_{\tilde{\Omega} \setminus \Omega}) \left( \frac{j}{4\varepsilon} (1 - \hat{v}_\varepsilon) - (A \nabla \hat{u}_\varepsilon \cdot \nabla \hat{u}_\varepsilon) \hat{v}_\varepsilon \right) \quad \text{in } \mathcal{D}'(\tilde{\Omega}). \quad (3.14)$$

*Proof.* Again, for simplicity, we drop the subscript  $\varepsilon$ . We fix an arbitrary test function  $\varphi \in \mathcal{D}(\tilde{\Omega})$ , and we define for  $x \in \tilde{\Omega}$  the *symmetric* and *anti-symmetric* parts of  $\varphi$ ,

$$\varphi^s(x) := \frac{1}{2}(\varphi(x) + \varphi \circ \sigma_\Omega(x)), \quad \varphi^a(x) := \frac{1}{2}(\varphi(x) - \varphi \circ \sigma_\Omega(x)).$$

The functions  $\varphi^s$  and  $\varphi^a$  belong to  $\mathcal{C}^{1,1}(\tilde{\Omega})$  and, by construction,

$$\varphi^s \circ \sigma_\Omega = \varphi^s \quad \text{and} \quad \varphi^a \circ \sigma_\Omega = -\varphi^a.$$

*Step 1: proof of (3.13).* We start with the identity

$$\begin{aligned} \int_{\tilde{\Omega} \setminus \bar{\Omega}} (\eta_\varepsilon + \hat{v}^2) (A \nabla \tilde{u}) \cdot \nabla \varphi^s \, dx &= - \int_{\tilde{\Omega} \setminus \bar{\Omega}} (\eta_\varepsilon + \hat{v}^2) (A \nabla (u \circ \sigma_\Omega)) \cdot \nabla (\varphi^s \circ \sigma_\Omega) \, dx \\ &\quad + 2 \int_{\tilde{\Omega} \setminus \bar{\Omega}} (\eta_\varepsilon + \hat{v}^2) (A \nabla \tilde{g}) \cdot \nabla \varphi^s \, dx. \end{aligned} \quad (3.15)$$

Using relation (3.9) and changing variables yields

$$\begin{aligned} \int_{\tilde{\Omega} \setminus \bar{\Omega}} (\eta_\varepsilon + \hat{v}^2) (A \nabla (u \circ \sigma_\Omega)) \cdot \nabla (\varphi^s \circ \sigma_\Omega) \, dx &= \int_{\tilde{\Omega} \setminus \bar{\Omega}} (\eta_\varepsilon + \hat{v}^2) \nabla u(\sigma_\Omega) \cdot \nabla \varphi^s(\sigma_\Omega) j \, dx \\ &= \int_{\tilde{\Omega} \cap \Omega} (\eta_\varepsilon + \hat{v}^2) (A \nabla \tilde{u}) \cdot \nabla \varphi^s \, dx. \end{aligned} \quad (3.16)$$

Therefore, combining (3.15) and (3.16) yields

$$\int_{\tilde{\Omega}} (\eta_\varepsilon + \hat{v}^2) (A \nabla \tilde{u}) \cdot \nabla \varphi^s \, dx = 2 \int_{\tilde{\Omega} \setminus \bar{\Omega}} (\eta_\varepsilon + \hat{v}^2) (A \nabla \tilde{g}) \cdot \nabla \varphi^s \, dx.$$

In the same way, we have

$$\begin{aligned} \int_{\tilde{\Omega} \setminus \bar{\Omega}} (\eta_\varepsilon + \hat{v}^2) (A \nabla \tilde{u}) \cdot \nabla \varphi^a \, dx &= \int_{\tilde{\Omega} \setminus \bar{\Omega}} (\eta_\varepsilon + \hat{v}^2) (A \nabla (u \circ \sigma_\Omega)) \cdot \nabla (\varphi^a \circ \sigma_\Omega) \, dx \\ &\quad + 2 \int_{\tilde{\Omega} \setminus \bar{\Omega}} (\eta_\varepsilon + \hat{v}^2) (A \nabla \tilde{g}) \cdot \nabla \varphi^a \, dx. \end{aligned}$$

Since  $\varphi^a = 0$  on  $\tilde{\Omega} \cap \partial\Omega$ , we have  $\varphi^a \in H_0^1(\tilde{\Omega} \cap \Omega)$ . Hence, we can use the first equation in (1.3) to infer that

$$\begin{aligned} \int_{\tilde{\Omega} \setminus \bar{\Omega}} (\eta_\varepsilon + \hat{v}^2) (A \nabla (u \circ \sigma_\Omega)) \cdot \nabla (\varphi^a \circ \sigma_\Omega) \, dx &= \int_{\tilde{\Omega} \setminus \bar{\Omega}} (\eta_\varepsilon + \hat{v}^2) \nabla u(\sigma_\Omega) \cdot \nabla \varphi^a(\sigma_\Omega) j \, dx \\ &= \int_{\tilde{\Omega} \cap \Omega} (\eta_\varepsilon + v^2) \nabla u \cdot \nabla \varphi^a \, dx = 0. \end{aligned}$$

Consequently,

$$\int_{\tilde{\Omega}} (\eta_\varepsilon + \hat{v}^2) (A \nabla \tilde{u}) \cdot \nabla \varphi \, dx = 2 \int_{\tilde{\Omega} \setminus \bar{\Omega}} (\eta_\varepsilon + \hat{v}^2) (A \nabla \tilde{g}) \cdot \nabla \varphi \, dx,$$

and (3.13) follows.

*Step 2: proof of (3.14).* We proceed as above, starting with

$$\varepsilon \int_{\tilde{\Omega} \setminus \bar{\Omega}} (A \nabla \tilde{v}) \cdot \nabla \varphi^s \, dx = -\varepsilon \int_{\tilde{\Omega} \setminus \bar{\Omega}} (A \nabla (v \circ \sigma_\Omega)) \cdot \nabla (\varphi^s \circ \sigma_\Omega) \, dx = -\varepsilon \int_{\tilde{\Omega} \cap \Omega} (A \nabla \tilde{v}) \cdot \nabla \varphi^s \, dx,$$

which yields

$$\varepsilon \int_{\tilde{\Omega}} (A \nabla \tilde{v}) \cdot \nabla \varphi^s \, dx = 0. \quad (3.17)$$

On the other hand,

$$\varepsilon \int_{\tilde{\Omega} \setminus \bar{\Omega}} (A \nabla \tilde{v}) \cdot \nabla \varphi^a \, dx = \varepsilon \int_{\tilde{\Omega} \setminus \bar{\Omega}} (A \nabla (v \circ \sigma_\Omega)) \cdot \nabla (\varphi^a \circ \sigma_\Omega) \, dx = \varepsilon \int_{\tilde{\Omega} \cap \Omega} \nabla v \cdot \nabla \varphi^a \, dx. \quad (3.18)$$

Since  $\varphi^a \in H_0^1(\tilde{\Omega} \cap \Omega)$ , we can apply the second equation in (1.3) to deduce that

$$\varepsilon \int_{\tilde{\Omega} \cap \Omega} \nabla v \cdot \nabla \varphi^a \, dx = - \int_{\tilde{\Omega} \cap \Omega} |\nabla u|^2 v \varphi^a \, dx + \frac{1}{4\varepsilon} \int_{\tilde{\Omega} \cap \Omega} (1 - v) \varphi^a \, dx. \quad (3.19)$$

Summing up (3.17), (3.18), and (3.19), and using that  $\sigma_\Omega$  is an involution leads to

$$\begin{aligned} \varepsilon \int_{\tilde{\Omega}} (A \nabla \tilde{v}) \cdot \nabla \varphi \, dx &= - \int_{\tilde{\Omega} \cap \Omega} |\nabla \hat{u}|^2 \hat{v} \varphi \, dx + \frac{1}{4\varepsilon} \int_{\tilde{\Omega} \cap \Omega} (1 - \hat{v}) \varphi \, dx \\ &\quad + \int_{\tilde{\Omega} \cap \Omega} |\nabla \hat{u}|^2 \hat{v} \varphi \circ \sigma_\Omega \, dx - \frac{1}{4\varepsilon} \int_{\tilde{\Omega} \cap \Omega} (1 - \hat{v}) \varphi \circ \sigma_\Omega \, dx. \end{aligned}$$

Changing variables in the two last integrals, we obtain

$$\varepsilon \int_{\tilde{\Omega}} (A \nabla \tilde{v}) \cdot \nabla \varphi \, dx = - \int_{\tilde{\Omega}} (\mathbf{1}_{\tilde{\Omega} \cap \Omega} - \mathbf{1}_{\tilde{\Omega} \setminus \tilde{\Omega}}) ((A \nabla \hat{u}) \cdot \nabla \hat{u}) \hat{v} \varphi \, dx + \frac{1}{4\varepsilon} \int_{\tilde{\Omega}} (\mathbf{1}_{\tilde{\Omega} \cap \Omega} - \mathbf{1}_{\tilde{\Omega} \setminus \tilde{\Omega}}) (1 - \hat{v}) \varphi j \, dx$$

and (3.14) follows.  $\square$

We now provide a general regularity result generalizing the argument used in the proof of the interior regularity.

**Lemma 3.3.** *Let  $A : B_R \rightarrow \mathbf{M}_{\text{sym}}^{N \times N}$  be a Lipschitz field of symmetric  $N \times N$  matrices which is uniformly elliptic (i.e., there exist  $0 < \lambda < \Lambda$  such that  $\lambda|\xi|^2 \leq A(x)\xi \cdot \xi \leq \Lambda|\xi|^2$  for all  $(x, \xi) \in B_R \times \mathbf{R}^N$ ), and  $f \in L^1(B_R)$  satisfying*

$$\sup_{B_\varrho(x_0) \subset B_R} \frac{1}{\varrho^{N-2+\gamma}} \int_{B_\varrho(x_0)} |f| \, dx < \infty, \quad (3.20)$$

for some  $\gamma \in (0, 2]$ . If  $z \in H^1(B_R) \cap L^\infty(B_R)$  solves

$$-\operatorname{div}(A \nabla z) = f \text{ in } \mathcal{D}'(B_R), \quad (3.21)$$

then

$$\sup_{x_0 \in \overline{B}_{R/2}, \varrho \in (0, R/2]} \frac{1}{\varrho^{N-2+2\alpha}} \int_{B_\varrho(x_0)} |\nabla z|^2 \, dx < \infty, \quad (3.22)$$

and  $z \in \mathcal{C}^{0,\alpha}(B_{R/2})$  for every  $\alpha \in (0, \gamma/2)$ .

*Proof.* Throughout the proof, we fix an exponent  $\alpha \in (0, \gamma/2)$  and we set  $\beta := \gamma - 2\alpha > 0$ . We also denote by  $K$  an upper bound for  $\|z\|_{L^\infty(B_R)}$ , and by  $M$  an upper bound for (3.20). Then  $C > 0$  shall stand for a constant (which may vary from line to line) depending only on  $N, \alpha, \gamma, \lambda, \Lambda, K, M$ , and the Lipschitz constant of  $A$ .

Let us fix  $x_0 \in \overline{B}_{R/2}$  and  $\varrho \in (0, R/2]$  arbitrary, and consider  $w \in H^1(B_\varrho(x_0))$  the unique (weak) solution of

$$\begin{cases} -\operatorname{div}(A(x_0) \nabla w) &= 0 & \text{in } B_\varrho(x_0), \\ w &= z & \text{on } \partial B_\varrho(x_0). \end{cases} \quad (3.23)$$

Recalling that

$$\int_{B_\varrho(x_0)} A(x_0) \nabla w \cdot \nabla w \, dx \leq \int_{B_\varrho(x_0)} A(x_0) \nabla \bar{w} \cdot \nabla \bar{w} \, dx \quad \text{for every } \bar{w} \in w + H_0^1(B_\varrho(x_0)),$$

we have

$$\lambda \int_{B_\varrho(x_0)} |\nabla w|^2 \, dx \leq \int_{B_\varrho(x_0)} A(x_0) \nabla w \cdot \nabla w \, dx \leq \int_{B_\varrho(x_0)} A(x_0) \nabla z \cdot \nabla z \, dx \leq \Lambda \int_{B_\varrho(x_0)} |\nabla z|^2 \, dx. \quad (3.24)$$

Moreover, according to the maximum principle,  $\|w\|_{L^\infty(B_\varrho(x_0))} \leq \|z\|_{L^\infty(B_\varrho(x_0))} \leq K$ .



First, we infer from the triangle inequality,

$$\begin{aligned}
\left( \frac{1}{(\varrho/2)^{N-2+2\alpha}} \int_{B_{\frac{\varrho}{2}}(x_0)} A \nabla z \cdot \nabla z \, dx \right)^{\frac{1}{2}} &\leq \left( \frac{1}{(\varrho/2)^{N-2+2\alpha}} \int_{B_{\frac{\varrho}{2}}(x_0)} A \nabla w \cdot \nabla w \, dx \right)^{\frac{1}{2}} \\
&\quad + \left( \frac{1}{(\varrho/2)^{N-2+2\alpha}} \int_{B_{\frac{\varrho}{2}}(x_0)} A (\nabla z - \nabla w) \cdot (\nabla z - \nabla w) \, dx \right)^{\frac{1}{2}} \\
&\leq \left( \frac{1}{(\varrho/2)^{N-2+2\alpha}} \int_{B_{\frac{\varrho}{2}}(x_0)} A \nabla w \cdot \nabla w \, dx \right)^{\frac{1}{2}} \\
&\quad + C \left( \frac{1}{\varrho^{N-2+2\alpha}} \int_{B_{\varrho}(x_0)} A (\nabla z - \nabla w) \cdot (\nabla z - \nabla w) \, dx \right)^{\frac{1}{2}}. \tag{3.25}
\end{aligned}$$

We start by estimating the first term in the right-hand-side of (3.25) using (3.24), and the fact that  $A$  is Lipschitz continuous and uniformly elliptic. It yields

$$\begin{aligned}
\int_{B_{\frac{\varrho}{2}}(x_0)} A \nabla w \cdot \nabla w \, dx &\leq (1 + C\rho) \int_{B_{\varrho}(x_0)} A(x_0) \nabla w \cdot \nabla w \, dx \\
&\leq (1 + C\rho) \int_{B_{\varrho}(x_0)} A(x_0) \nabla z \cdot \nabla z \, dx \leq (1 + C\rho) \int_{B_{\varrho}(x_0)} A \nabla z \cdot \nabla z \, dx. \tag{3.26}
\end{aligned}$$

To estimate the second term in the right-hand-side of (3.25), we make use of equation (3.21) to write

$$\begin{aligned}
\int_{B_{\varrho}(x_0)} A (\nabla z - \nabla w) \cdot (\nabla z - \nabla w) \, dx &= \int_{B_{\varrho}(x_0)} A \nabla z \cdot \nabla (z - w) \, dx - \int_{B_{\varrho}(x_0)} A \nabla w \cdot \nabla (z - w) \, dx \\
&= \int_{B_{\varrho}(x_0)} f(z - w) \, dx - \int_{B_{\varrho}(x_0)} A \nabla w \cdot \nabla (z - w) \, dx.
\end{aligned}$$

Using assumption (3.20) on  $f$ , we infer that

$$\left| \frac{1}{\varrho^{N-2+2\alpha}} \int_{B_{\varrho}(x_0)} f(z - w) \, dx \right| \leq 2KM\varrho^{\beta}. \tag{3.27}$$

On the other hand, Equation (3.23) satisfied by  $w$  implies that

$$\begin{aligned}
\int_{B_{\varrho}(x_0)} A \nabla w \cdot \nabla (z - w) \, dx &= \int_{B_{\varrho}(x_0)} (A - A(x_0)) \nabla w \cdot \nabla (z - w) \, dx \\
&\leq C\varrho \int_{B_{\varrho}(x_0)} [|\nabla w|^2 + |\nabla z|^2] \, dx \\
&\leq C\varrho \int_{B_{\varrho}(x_0)} |\nabla z|^2 \, dx \leq C\varrho \int_{B_{\varrho}(x_0)} A \nabla z \cdot \nabla z \, dx, \tag{3.28}
\end{aligned}$$

where we used again (3.24) together with the ellipticity of  $A$ .

Gathering (3.25), (3.26), (3.27) and (3.28), we get that

$$\left( \frac{1}{(\varrho/2)^{N-2+2\alpha}} \int_{B_{\frac{\varrho}{2}}(x_0)} A \nabla z \cdot \nabla z \, dx \right)^{1/2} \leq (1 + C\sqrt{\varrho}) \left( \frac{1}{\varrho^{N-2+2\alpha}} \int_{B_{\varrho}(x_0)} A \nabla z \cdot \nabla z \, dx \right)^{1/2} + C\varrho^{\beta/2}.$$

We now choose  $\varrho = \varrho_k = 2^{-(k+1)}R$  for  $k \in \mathbf{N}$ , and we obtain

$$\left( \frac{1}{\varrho_{k+1}^{N-2+2\alpha}} \int_{B_{\varrho_{k+1}}(x_0)} A \nabla z \cdot \nabla z \, dx \right)^{1/2} \leq (1 + C\sqrt{R} 2^{-(k+1)/2}) \left( \frac{1}{\varrho_k^{N-2+2\alpha}} \int_{B_{\varrho_k}(x_0)} A \nabla z \cdot \nabla z \, dx \right)^{1/2} + CR^{\beta/2} 2^{-\beta(k+1)/2}.$$

Next, we observe that if  $(\theta_k)_{k \in \mathbf{N}}$ ,  $(\sigma_k)_{k \in \mathbf{N}}$ , and  $(y_k)_{k \in \mathbf{N}}$  are real sequences such that  $\theta_k \in (1, \infty)$ ,  $\theta := \prod_{k=0}^{\infty} \theta_k < \infty$ ,  $\sigma_k \in (0, \infty)$ ,  $\sigma := \sum_{k=0}^{\infty} \sigma_k < \infty$ , and satisfying  $y_{k+1} \leq \theta_k y_k + \sigma_k$  for all  $k \in \mathbf{N}$ , then  $y_k \leq \theta(y_0 + \sigma)$ . Applying this principle with

$$y_k = \left( \frac{1}{\varrho_k^{N-2+2\alpha}} \int_{B_{\varrho_k}(x_0)} A \nabla z \cdot \nabla z \, dx \right)^{1/2}, \quad \theta_k = 1 + C\sqrt{R} 2^{-(k+1)/2}, \quad \sigma_k = CR^{\beta/2} 2^{-\beta(k+1)/2},$$

yields

$$\frac{1}{\varrho_k^{N-2+2\alpha}} \int_{B_{\varrho_k}(x_0)} A \nabla z \cdot \nabla z \, dx \leq Ce^{C\sqrt{R}} \left( \frac{1}{R^{N-2+2\alpha}} \int_{B_R} A \nabla z \cdot \nabla z \, dx + R^{\beta/2} \right) \quad \text{for all } k \in \mathbf{N}$$

(we have also used the elementary estimates  $\theta \leq e^{C\sqrt{R}}$  and  $\sigma \leq CR^{\beta/2}$ ). Since for all  $\varrho \in (0, R/2]$ , there exists a unique  $k \in \mathbf{N}$  such that  $\varrho_{k+1} < \varrho \leq \varrho_k$  and  $\frac{1}{\varrho} < \frac{1}{\varrho_{k+1}} \leq \frac{2}{\varrho_k}$ , we conclude that

$$\frac{1}{\varrho^{N-2+2\alpha}} \int_{B_{\varrho}(x_0)} A \nabla z \cdot \nabla z \, dx \leq Ce^{C\sqrt{R}} \left( \frac{1}{R^{N-2+2\alpha}} \int_{B_R} A \nabla z \cdot \nabla z \, dx + R^{\beta/2} \right) \quad \forall \varrho \in (0, R/2].$$

Finally, by ellipticity of  $A$  and the arbitrariness of  $x_0$ , we conclude that (3.22) holds with

$$\frac{1}{\varrho^{N-2+2\alpha}} \int_{B_{\varrho}(x_0)} |\nabla z|^2 \, dx \leq Ce^{C\sqrt{R}} \left( \frac{1}{R^{N-2+2\alpha}} \int_{B_R} |\nabla z|^2 \, dx + R^{\beta/2} \right) \quad \forall \varrho \in (0, R/2], \quad \forall x_0 \in \overline{B}_{R/2}.$$

By Morrey's Theorem (see e.g. [17, Theorem 5.7]), it then follows that  $v \in \mathcal{C}^{0,\alpha}(B_{R/2})$ .  $\square$

We are now ready to prove the boundary regularity result in Theorem 3.2.

*Proof of Theorem 3.2 completed.* We consider  $(u_\varepsilon, v_\varepsilon) \in H^1(\Omega) \times [H^1(\Omega) \cap L^\infty(\Omega)]$  satisfying (1.4)-(1.5), and we consider the extensions  $\widehat{u}_\varepsilon$ ,  $\widetilde{u}_\varepsilon$ ,  $\widehat{v}_\varepsilon$ , and  $\widetilde{v}_\varepsilon$  to the domain  $\widetilde{\Omega}$  (depending on  $x_0$  and  $R$ ) provided by (3.11)-(3.12) and (3.8). Again, for simplicity, we drop the subscript  $\varepsilon$ .

We first improve the regularity of  $\widetilde{u}$  which satisfies (3.13). We aim to apply the De-Giorgi-Nash-Moser Theorem to infer that  $\widetilde{u}$  is locally Hölder continuous in  $\widetilde{\Omega}$  and that a suitable Morrey estimate holds for  $\nabla \widetilde{u}$ . Since equation (3.13) is linear with respect to  $\widetilde{u}$ , we first observe that

$$\int_{\widetilde{\Omega}} f(x, \nabla \widetilde{u}) \, dx \leq \int_{\widetilde{\Omega}} f(x, \nabla w) \, dx \quad \text{for all } w \in H^1(\Omega) \text{ such that } \text{supp}(w - \widetilde{u}) \subset \widetilde{\Omega},$$

with

$$f(x, \xi) := \frac{1}{2}(\eta_\varepsilon + \widehat{v}_\varepsilon^2(x))A(x)\xi \cdot \xi - h(x) \cdot \xi \quad \text{for a.e. } x \in \widetilde{\Omega} \text{ and all } \xi \in \mathbf{R}^N,$$

and

$$h := 2 \mathbf{1}_{\widetilde{\Omega} \setminus \overline{\Omega}}(\eta_\varepsilon + \widehat{v}^2)A \nabla \widetilde{g} \in L^\infty(\widetilde{\Omega}; \mathbf{R}^N).$$

The function  $f$  is a Carathéodory function, and since  $A$  is uniformly elliptic and the functions  $\widehat{v}$  and  $h$  are essentially bounded, we can find positive constants  $c_1$ ,  $c_2$ , and  $c_3$  such that

$$c_1|\xi|^2 - c_3 \leq f(x, \xi) \leq c_2|\xi|^2 + c_3 \quad \text{for a.e. } x \in \widetilde{\Omega} \text{ and all } \xi \in \mathbf{R}^N.$$

Hence we can apply the De Giorgi-Nash-Moser Theorem (see [18, Theorems 7.5 and 7.6]) to deduce the existence of some  $\beta \in (0, 1)$  such that  $\tilde{u} \in \mathcal{C}_{\text{loc}}^{0,\beta}(\tilde{\Omega})$ . From [18, Theorem 7.7] and [17, Lemma 5.13], we also obtain the Morrey estimate

$$\sup_{B_\varrho(x_*) \subset B_{\delta_0/2}(x_0)} \frac{1}{\varrho^{N-2+2\beta}} \int_{B_\varrho(x_*)} |\nabla \tilde{u}|^2 dx < \infty, \quad (3.29)$$

since  $B_{\delta_0}(x_0) \subset \tilde{\Omega}$ .

Next we consider the equation (3.14) satisfied by  $\tilde{v}$  restricted to  $B_{\delta_0/2}(x_0)$ , that we write

$$-\operatorname{div}(A \nabla \tilde{v}) = H \quad \text{in } \mathcal{D}'(B_{\delta_0/2}(x_0)),$$

with

$$H := \frac{1}{\varepsilon} (\mathbf{1}_{\tilde{\Omega} \cap \Omega} - \mathbf{1}_{\tilde{\Omega} \setminus \bar{\Omega}}) \left( \frac{j}{4\varepsilon} (1 - \tilde{v}) - (A \nabla \hat{u} \cdot \nabla \hat{u}) \tilde{v} \right).$$

Since  $\hat{u} = 2\tilde{g} - \tilde{u}$  and  $\nabla \tilde{g}$ ,  $A$ ,  $j$ , and  $\tilde{v}$  are essentially bounded, we infer from (3.29) that

$$\sup_{B_\varrho(x_*) \subset B_{\delta_0/2}(x_0)} \frac{1}{\varrho^{N-2+2\beta}} \int_{B_\varrho(x_*)} |H| dx < \infty.$$

Applying Lemma 3.3, we deduce that  $\tilde{v} \in \mathcal{C}^{0,\gamma}(B_{\delta_0/4}(x_0))$  for every  $\gamma \in (0, \beta)$ . In particular, we have  $v \in \mathcal{C}^{0,\gamma}(\bar{\Omega} \cap B_{\delta_0/4}(x_0))$  for every  $\gamma \in (0, \beta)$ . Using the equation (1.3) satisfied by  $u$  together with the (up to the boundary) Schauder estimate (see [17, Theorem 5.21]), we obtain that  $u \in \mathcal{C}^{1,\gamma}(\bar{\Omega} \cap B_{\delta_0/5}(x_0))$  for every  $\gamma \in (0, \beta)$ . Then, in view of the equation (1.3) satisfied by  $v$ , and owing to the classical elliptic regularity at the boundary, we obtain  $v \in \mathcal{C}^{2,\gamma}(\bar{\Omega} \cap B_{\delta_0/6}(x_0))$  for every  $\gamma \in (0, \beta)$ . Back to the equation in (1.3) satisfied by  $u$ , elliptic regularity at the boundary now tells us that  $u \in \mathcal{C}^{1,\alpha}(\bar{\Omega} \cap B_{\delta_0/7}(x_0))$  in the case  $g \in \mathcal{C}^{1,\alpha}(\partial\Omega \cap B_{4R}(x_0))$ , and in turn  $v \in \mathcal{C}^{2,\alpha}(\bar{\Omega} \cap B_{\delta_0/8}(x_0))$  still by (1.3). If  $g \in \mathcal{C}^{2,\alpha}(\partial\Omega \cap B_{4R}(x_0))$ , then  $v \in \mathcal{C}^{2,\alpha}(\bar{\Omega} \cap B_{\delta_0/8}(x_0))$  and once again, elliptic boundary regularity implies that  $u \in \mathcal{C}^{2,\alpha}(\bar{\Omega} \cap B_{\delta_0/9}(x_0))$ .

If  $\partial\Omega \cap B_{4R}(x_0)$  is of class  $\mathcal{C}^{k,\alpha}$  and  $g \in \mathcal{C}^{k,\alpha}(\partial\Omega \cap B_{4R}(x_0))$  with  $k \geq 3$ , one can iterate the preceding argument using elliptic boundary regularity to conclude that  $u$  and  $v$  belong to  $\mathcal{C}^{k,\alpha}(\bar{\Omega} \cap B_{\theta R}(x_0))$  for some  $\theta > 0$  small enough.  $\square$

#### 4. COMPACTNESS RESULTS

We start by establishing a weak compactness result, in the spirit of the compactness argument in the  $\Gamma$ -convergence analysis, under the only assumption of uniform energy bound (1.12).

**Proposition 4.1** (Weak compactness). *Let  $\Omega \subset \mathbf{R}^N$  be a bounded open set with Lipschitz boundary,  $g \in H^{\frac{1}{2}}(\partial\Omega) \cap L^\infty(\partial\Omega)$ , and  $\varepsilon_n \rightarrow 0^+$  be an arbitrary sequence. Assume that  $(u_n, v_n) := (u_{\varepsilon_n}, v_{\varepsilon_n}) \in \mathcal{A}_g(\Omega)$  satisfies  $0 \leq v_n \leq 1$  a.e. in  $\Omega$ ,  $\|u_n\|_{L^\infty(\Omega)} \leq \|g\|_{L^\infty(\partial\Omega)}$ , and the uniform energy bound  $\sup_n AT_{\varepsilon_n}(u_n, v_n) < \infty$ . There exist a (not relabelled) subsequence and  $u_* \in SBV^2(\Omega) \cap L^\infty(\Omega)$  such that*

$$\begin{cases} (u_n, v_n) \rightarrow (u_*, 1) & \text{strongly in } [L^2(\Omega)]^2, \\ v_n \nabla u_n \rightharpoonup \nabla u_* & \text{weakly in } L^2(\Omega; \mathbf{R}^N). \end{cases}$$

Moreover,

$$\int_\Omega |\nabla u_*|^2 dx \leq \liminf_{n \rightarrow \infty} \int_\Omega v_n^2 |\nabla u_n|^2 dx \leq \liminf_{n \rightarrow \infty} \int_\Omega (\eta_{\varepsilon_n} + v_n^2) |\nabla u_n|^2 dx, \quad (4.1)$$

and

$$\begin{aligned} \mathcal{H}^{N-1}(J_{u_*} \cup (\partial\Omega \cap \{u_* \neq g\})) &\leq \liminf_{n \rightarrow \infty} \int_{\Omega} (1 - v_n) |\nabla v_n| \, dx \\ &\leq \liminf_{n \rightarrow \infty} \int_{\Omega} \left( \varepsilon_n |\nabla v_n|^2 + \frac{(v_n - 1)^2}{4\varepsilon_n} \right) \, dx. \end{aligned} \quad (4.2)$$

*Proof.* First of all, it is clear from the energy bound that  $v_n \rightarrow 1$  strongly in  $L^2(\Omega)$ . Moreover, up to a subsequence, there exist  $u_* \in L^\infty(\Omega)$  such that  $u_n \rightharpoonup u_*$  weakly\* in  $L^\infty(\Omega)$ .

Let  $\Omega'$  be a bounded open set such that  $\bar{\Omega} \subset \Omega'$ , and  $g' \in H^1(\Omega') \cap L^\infty(\Omega')$  be an arbitrary extension of  $g$  satisfying  $\|g'\|_{L^\infty(\Omega')} \leq \|g\|_{L^\infty(\partial\Omega)}$ . We extend  $(u_n, v_n)$  by  $(g', 1)$  in  $\Omega' \setminus \Omega$  and we still denote by  $(u_n, v_n)$  the extension.

According to Young's inequality, we have

$$\begin{aligned} C &\geq \int_{\Omega'} (\eta_{\varepsilon_n} + v_n^2) |\nabla u_n|^2 \, dx + \int_{\Omega'} \left( \varepsilon_n |\nabla v_n|^2 + \frac{(v_n - 1)^2}{4\varepsilon_n} \right) \, dx \\ &\geq \int_{\Omega'} v_n^2 |\nabla u_n|^2 \, dx + \int_{\Omega'} (1 - v_n) |\nabla v_n| \, dx. \end{aligned} \quad (4.3)$$

Using the coarea formula and the fact that  $0 \leq v_n \leq 1$ , we can reconstruct the second integral in the right hand side by integrating along the level sets of  $v_n$ , i.e.,

$$\int_{\Omega'} (1 - v_n) |\nabla v_n| \, dx = \int_0^1 (1 - t) \mathcal{H}^{N-1}(\partial^* \{v_n < t\}) \, dt \geq \int_a^b \Phi'(t) \mathcal{H}^{N-1}(\partial^* \{v_n < t\}) \, dt,$$

for all  $0 < a < b < 1$ , where  $\Phi(t) := t - t^2/2$ . By a change of variables and the mean value Theorem, we can find some  $t_n \in [a, b]$  such that the set  $E_n := \{v_n < t_n\}$  has finite perimeter in  $\Omega'$  and

$$\int_{\Omega'} (1 - v_n) |\nabla v_n| \, dx \geq [\Phi(b) - \Phi(a)] \mathcal{H}^{N-1}(\partial^* \{v_n < t_n\}). \quad (4.4)$$

Moreover, the previous energy estimate ensures that  $\mathcal{H}^{N-1}(\partial^* E_n) \leq C$ , for some constant  $C > 0$  (possibly depending on  $a$  and  $b$ , but independent of  $n$ ). Then, according to the Chebychev inequality and since  $t_n \leq b$ ,

$$\mathcal{L}^N(E_n) = \mathcal{L}^N(\{1 - v_n > 1 - t_n\}) \leq \frac{1}{(1 - t_n)^2} \int_{\Omega'} (1 - v_n)^2 \, dx \leq \frac{C\varepsilon_n}{1 - b^2} \rightarrow 0.$$

Since  $a \leq t_n \leq v_n$  on  $\Omega' \setminus E_n$ ,

$$\int_{\Omega' \setminus E_n} |\nabla u_n|^2 \, dx \leq \frac{t_n^2}{a^2} \int_{\Omega' \setminus E_n} |\nabla u_n|^2 \, dx \leq \frac{1}{a^2} \int_{\Omega' \setminus E_n} v_n^2 |\nabla u_n|^2 \, dx \leq \frac{C}{a^2}.$$

Applying Lemma 2.1, we can extract a (not relabeled) subsequence and find  $u' \in SBV^2(\Omega') \cap L^\infty(\Omega')$  (at this stage possibly depending on  $a$  and  $b$ ) such that  $\tilde{u}_n := u_n \mathbf{1}_{\Omega' \setminus E_n} \in SBV^2(\Omega') \cap L^\infty(\Omega')$  satisfies

$$\begin{cases} \tilde{u}_n \rightarrow u' & \text{strongly in } L^2(\Omega'), \\ \nabla \tilde{u}_n \rightharpoonup \nabla u' & \text{weakly in } L^2(\Omega'; \mathbf{R}^N), \\ 2\mathcal{H}^{N-1}(J_{u'}) \leq \liminf_n \mathcal{H}^{N-1}(\partial^* E_n). \end{cases}$$

Since  $u_n = g'$  in  $\Omega' \setminus \Omega$  and  $\mathcal{L}^N(E_n) \rightarrow 0$ , we infer that  $u'|_{\Omega' \setminus \Omega} = g'$  and  $\mathcal{H}^{N-1}(J_{u'} \setminus \bar{\Omega}) = 0$  because  $g' \in H^1(\Omega')$ . Moreover, since  $u_n - \tilde{u}_n = u_n \mathbf{1}_{E_n}$ , then  $\|u_n - \tilde{u}_n\|_{L^2(\Omega')} \leq \|g\|_{L^\infty(\partial\Omega)} \mathcal{L}^N(E_n)^{1/2} \rightarrow 0$ , and thus  $u'|_\Omega = u_*$ . We finally get that  $u'$  is independent of  $a$  and  $b$  and that  $J_{u'} = J_{u_*} \cup (\partial\Omega \cap \{u_* \neq g\})$

up to an  $\mathcal{H}^{N-1}$ -negligible set. Consequently,  $u_* \in SBV^2(\Omega) \cap L^\infty(\Omega)$  satisfies

$$\begin{cases} \tilde{u}_n \rightarrow u_*, \quad u_n \rightarrow u_* & \text{strongly in } L^2(\Omega), \\ \nabla \tilde{u}_n \rightharpoonup \nabla u_* & \text{weakly in } L^2(\Omega; \mathbf{R}^N), \\ 2\mathcal{H}^{N-1}(J_{u_*} \cup (\partial\Omega \cap \{u_* \neq g\})) \leq \liminf_n \mathcal{H}^{N-1}(\partial^* E_n). \end{cases}$$

In the sequel, to lighten notation, we shall write

$$\hat{J}_{u_*} := J_{u_*} \cup (\partial\Omega \cap \{u_* \neq g\}).$$

For an arbitrary test function  $\varphi \in \mathcal{C}_c^\infty(\Omega; \mathbf{R}^N)$ , we have

$$\int_{\Omega} (v_n \nabla u_n - \nabla u_*) \cdot \varphi \, dx = \int_{\Omega} (v_n \nabla u_n - \nabla \tilde{u}_n) \cdot \varphi \, dx + \int_{\Omega} (\nabla \tilde{u}_n - \nabla u_*) \cdot \varphi \, dx.$$

The second integral tends to zero as  $n \rightarrow \infty$  because  $\nabla \tilde{u}_n \rightharpoonup \nabla u_*$  weakly in  $L^2(\Omega; \mathbf{R}^N)$ . Concerning the first integral, using that  $\nabla \tilde{u}_n = \mathbf{1}_{\Omega \setminus E_n} \nabla u_n$ , we can estimate it as

$$\begin{aligned} \left| \int_{\Omega} (v_n \nabla u_n - \nabla \tilde{u}_n) \cdot \varphi \, dx \right| &\leq \int_{E_n} v_n |\nabla u_n| |\varphi| \, dx + \int_{\Omega \setminus E_n} (1 - v_n) |\nabla \tilde{u}_n| |\varphi| \, dx \\ &\leq \|v_n \nabla u_n\|_{L^2(\Omega; \mathbf{R}^N)} \|\varphi\|_{L^\infty(\Omega)} \mathcal{L}^N(E_n)^{1/2} \\ &\quad + \|1 - v_n\|_{L^2(\Omega)} \|\nabla \tilde{u}_n\|_{L^2(\Omega; \mathbf{R}^N)} \|\varphi\|_{L^\infty(\Omega)} \rightarrow 0. \end{aligned}$$

Since the sequence  $\{v_n \nabla u_n\}$  is bounded in  $L^2(\Omega; \mathbf{R}^N)$  according to the energy bound, we deduce that  $v_n \nabla u_n \rightharpoonup \nabla u_*$  weakly in  $L^2(\Omega; \mathbf{R}^N)$ . By lower semicontinuity of the norm with respect to weak convergence, we have

$$\int_{\Omega} |\nabla u_*|^2 \, dx \leq \liminf_{n \rightarrow \infty} \int_{\Omega} v_n^2 |\nabla u_n|^2 \, dx \leq \liminf_{n \rightarrow \infty} \int_{\Omega} (\eta_{\varepsilon_n} + v_n^2) |\nabla u_n|^2 \, dx.$$

On the other hand, (4.4) implies that for all  $0 < a < b < 1$ ,

$$\begin{aligned} \liminf_{n \rightarrow \infty} \int_{\Omega} \left( \varepsilon_n |\nabla v_n|^2 + \frac{(v_n - 1)^2}{4\varepsilon_n} \right) \, dx &\geq \liminf_{n \rightarrow \infty} \int_{\Omega} (1 - v_n) |\nabla v_n| \, dx \\ &\geq [\Phi(b) - \Phi(a)] \liminf_{n \rightarrow \infty} \mathcal{H}^{N-1}(\partial^* E_n) \\ &\geq 2[\Phi(b) - \Phi(a)] \mathcal{H}^{N-1}(\hat{J}_{u_*}). \end{aligned}$$

Letting  $a \rightarrow 0$  and  $b \rightarrow 1$  leads to

$$\liminf_{n \rightarrow \infty} \int_{\Omega} \left( \varepsilon_n |\nabla v_n|^2 + \frac{(v_n - 1)^2}{4\varepsilon_n} \right) \, dx \geq \liminf_{n \rightarrow \infty} \int_{\Omega} (1 - v_n) |\nabla v_n| \, dx \geq \mathcal{H}^{N-1}(\hat{J}_{u_*}),$$

since  $\Phi(0) = 0$  and  $\Phi(1) = 1/2$ . □

The regularity of solutions established in Theorem 1.1 allows us to prove that critical points of the Ambrosio-Tortorelli functional satisfies a Noether type conservation law, which is the starting point of their asymptotic analysis.

**Proposition 4.2.** *Let  $\Omega \subset \mathbf{R}^N$  be a bounded open set with boundary of class  $\mathcal{C}^{2,1}$  and  $g \in \mathcal{C}^{2,\alpha}(\partial\Omega)$  for some  $\alpha \in (0, 1)$ . If  $(u_\varepsilon, v_\varepsilon) \in \mathcal{A}_g(\Omega)$  is a critical point of  $AT_\varepsilon$ , then for all  $X \in \mathcal{C}_c^1(\mathbf{R}^N; \mathbf{R}^N)$ ,*

$$\begin{aligned} & \int_{\Omega} (\eta_\varepsilon + v_\varepsilon^2) [2\nabla u_\varepsilon \otimes \nabla u_\varepsilon - |\nabla u_\varepsilon|^2 \text{Id}] : DX \, dx \\ & \quad + \int_{\Omega} \left[ 2\varepsilon \nabla v_\varepsilon \otimes \nabla v_\varepsilon - \left( \varepsilon |\nabla v_\varepsilon|^2 + \frac{(v_\varepsilon - 1)^2}{4\varepsilon} \right) \text{Id} \right] : DX \, dx \\ & = \int_{\partial\Omega} [(\eta_\varepsilon + 1)(\partial_\nu u_\varepsilon)^2 + \varepsilon(\partial_\nu v_\varepsilon)^2 - (\eta_\varepsilon + 1)|\nabla_\tau g|^2](X \cdot \nu_\Omega) \, d\mathcal{H}^{N-1} \\ & \quad + 2(\eta_\varepsilon + 1) \int_{\partial\Omega} (\partial_\nu u_\varepsilon)(X_\tau \cdot \nabla_\tau g) \, d\mathcal{H}^{N-1}, \quad (4.5) \end{aligned}$$

where  $X_\tau := X - (X \cdot \nu_\Omega)\nu_\Omega$  is the tangential part of  $X$ , and  $\nabla_\tau g$  is the tangential gradient of  $g$  on  $\partial\Omega$ .

*Proof.* Let us fix an arbitrary  $X \in \mathcal{C}_c^1(\mathbf{R}^N; \mathbf{R}^N)$ . By Theorem 1.1,  $(u_\varepsilon, v_\varepsilon) \in [\mathcal{C}^{2,\alpha}(\overline{\Omega})]^2$  and (1.3) is satisfied in the classical sense. Multiplying the first equation of (1.3) by  $X \cdot \nabla u_\varepsilon$  (which belongs to  $\mathcal{C}^1(\overline{\Omega})$ ) and integrating by parts yields

$$\begin{aligned} 0 & = \int_{\Omega} (\eta_\varepsilon + v_\varepsilon^2) \nabla u_\varepsilon \cdot \nabla (X \cdot \nabla u_\varepsilon) \, dx - (\eta_\varepsilon + 1) \int_{\partial\Omega} \partial_\nu u_\varepsilon X \cdot \nabla u_\varepsilon \, d\mathcal{H}^{N-1} \\ & = \int_{\Omega} (\eta_\varepsilon + v_\varepsilon^2) (\nabla u_\varepsilon \otimes \nabla u_\varepsilon) : DX \, dx + \frac{1}{2} \int_{\Omega} (\eta_\varepsilon + v_\varepsilon^2) X \cdot \nabla (|\nabla u_\varepsilon|^2) \, dx \\ & \quad - (\eta_\varepsilon + 1) \int_{\partial\Omega} (\partial_\nu u_\varepsilon)^2 (X \cdot \nu_\Omega) \, d\mathcal{H}^{N-1} - (\eta_\varepsilon + 1) \int_{\partial\Omega} (\partial_\nu u_\varepsilon)(X_\tau \cdot \nabla_\tau g) \, d\mathcal{H}^{N-1}, \end{aligned}$$

since  $\nabla_\tau u_\varepsilon = \nabla_\tau g$  on  $\partial\Omega$ . Integrating by parts the second integral in the right hand side, we get

$$\begin{aligned} 0 & = \int_{\Omega} (\eta_\varepsilon + v_\varepsilon^2) (\nabla u_\varepsilon \otimes \nabla u_\varepsilon) : DX \, dx - \frac{1}{2} \int_{\Omega} (\eta_\varepsilon + v_\varepsilon^2) |\nabla u_\varepsilon|^2 \text{div} X \, dx \\ & \quad - \int_{\Omega} v_\varepsilon (X \cdot \nabla v_\varepsilon) |\nabla u_\varepsilon|^2 \, dx + \frac{\eta_\varepsilon + 1}{2} \int_{\partial\Omega} |\nabla u_\varepsilon|^2 (X \cdot \nu_\Omega) \, d\mathcal{H}^{N-1} \\ & \quad - (\eta_\varepsilon + 1) \int_{\partial\Omega} (\partial_\nu u_\varepsilon)^2 (X \cdot \nu_\Omega) \, d\mathcal{H}^{N-1} - (\eta_\varepsilon + 1) \int_{\partial\Omega} (\partial_\nu u_\varepsilon)(X_\tau \cdot \nabla_\tau g) \, d\mathcal{H}^{N-1}. \end{aligned}$$

Rearranging terms leads to

$$\begin{aligned} 0 & = \int_{\Omega} (\eta_\varepsilon + v_\varepsilon^2) \left[ (\nabla u_\varepsilon \otimes \nabla u_\varepsilon) - \frac{1}{2} |\nabla u_\varepsilon|^2 \text{Id} \right] : DX \, dx - \int_{\Omega} v_\varepsilon (X \cdot \nabla v_\varepsilon) |\nabla u_\varepsilon|^2 \, dx \\ & \quad - \frac{\eta_\varepsilon + 1}{2} \int_{\partial\Omega} (\partial_\nu u_\varepsilon)^2 (X \cdot \nu_\Omega) \, d\mathcal{H}^{N-1} + \frac{\eta_\varepsilon + 1}{2} \int_{\partial\Omega} |\nabla_\tau g|^2 (X \cdot \nu_\Omega) \, d\mathcal{H}^{N-1} \\ & \quad - (\eta_\varepsilon + 1) \int_{\partial\Omega} (\partial_\nu u_\varepsilon)(X_\tau \cdot \nabla_\tau g) \, d\mathcal{H}^{N-1}. \quad (4.6) \end{aligned}$$

Similarly, multiplying the second equation in (1.3) by  $X \cdot \nabla v_\varepsilon$  (which belongs to  $\mathcal{C}^1(\overline{\Omega})$ ) and performing similar integration by parts yields

$$\begin{aligned} 0 & = \varepsilon \int_{\Omega} \left[ (\nabla v_\varepsilon \otimes \nabla v_\varepsilon) - \frac{1}{2} \left( |\nabla v_\varepsilon|^2 + \frac{1}{4\varepsilon} (v_\varepsilon - 1)^2 \right) \text{Id} \right] : DX \, dx + \int_{\Omega} v_\varepsilon (X \cdot \nabla v_\varepsilon) |\nabla u_\varepsilon|^2 \, dx \\ & \quad - \frac{\varepsilon}{2} \int_{\partial\Omega} (\partial_\nu v_\varepsilon)^2 (X \cdot \nu_\Omega) \, d\mathcal{H}^{N-1}, \quad (4.7) \end{aligned}$$

since  $v_\varepsilon = 1$  on  $\partial\Omega$ . Then the conclusion follows summing up (4.6) and (4.7).  $\square$

**Remark 4.1.** The fact that critical points  $(u_\varepsilon, v_\varepsilon)$  enjoy the higher regularity  $[\mathcal{C}^{2,\alpha}(\bar{\Omega})]^2$  allows one to obtain a strong form of the conservative equations for  $(u_\varepsilon, v_\varepsilon)$ . In particular, some information on the boundary are recovered since the vector field  $X$  does not need to be tangential on  $\partial\Omega$ . This additional information will be instrumental in Section 5 to characterize the boundary term occurring in the first inner variation of the Mumford-Shah functional. A slightly weaker form can be obtained through inner variations of the energy by using deformations of the form  $u_t := u \circ \Phi_t^{-1} - G \circ \Phi_t^{-1} + G$  and  $v_t := v \circ \Phi_t^{-1}$ , where  $\Phi_t$  is the integral flow of a vector field  $X \in \mathcal{C}_c^1(\mathbf{R}^N; \mathbf{R}^N)$  satisfying  $X \cdot \nu_\Omega = 0$  on  $\partial\Omega$  and  $G$  is an arbitrary extension of  $g$  to the whole domain  $\Omega$ . It leads to the following inner variational equation (see A.11)

$$\begin{aligned} & \int_{\Omega} (\eta_\varepsilon + v_\varepsilon^2) [2\nabla u_\varepsilon \otimes \nabla u_\varepsilon - |\nabla u_\varepsilon|^2 \text{Id}] : DX \, dx \\ & \quad + \int_{\Omega} \left[ 2\varepsilon \nabla v_\varepsilon \otimes \nabla v_\varepsilon - \left( \varepsilon |\nabla v_\varepsilon|^2 + \frac{(v_\varepsilon - 1)^2}{4\varepsilon} \right) \text{Id} \right] : DX \, dx \\ & \quad = 2 \int_{\Omega} (\eta_\varepsilon + v_\varepsilon^2) \nabla u_\varepsilon \cdot \nabla (X \cdot \nabla G) \, dx \end{aligned}$$

for all  $X \in \mathcal{C}_c^1(\mathbf{R}^N; \mathbf{R}^N)$  and  $G \in \mathcal{C}^2(\mathbf{R}^N)$  satisfying  $X \cdot \nu_\Omega = 0$  and  $G = g$  on  $\partial\Omega$ . Integrating by parts the right hand side and using equation (1.3), we arrive at

$$\begin{aligned} & \int_{\Omega} (\eta_\varepsilon + v_\varepsilon^2) [2\nabla u_\varepsilon \otimes \nabla u_\varepsilon - |\nabla u_\varepsilon|^2 \text{Id}] : DX \, dx \\ & \quad + \int_{\Omega} \left[ 2\varepsilon \nabla v_\varepsilon \otimes \nabla v_\varepsilon - \left( \varepsilon |\nabla v_\varepsilon|^2 + \frac{(v_\varepsilon - 1)^2}{4\varepsilon} \right) \text{Id} \right] : DX \, dx \\ & \quad = 2(\eta_\varepsilon + 1) \int_{\partial\Omega} (\partial_\nu u_\varepsilon) (X_\tau \cdot \nabla_\tau g) \, d\mathcal{H}^{N-1}. \end{aligned}$$

Note that we used that  $X \cdot \nabla G = X_\tau \cdot \nabla_\tau g$  on  $\partial\Omega$  since  $X \cdot \nu_\Omega = 0$  on  $\partial\Omega$ .

Owing to the previous results, we get the following property for the weak limit  $u_*$  as  $\varepsilon \rightarrow 0$  of a converging sequence of critical points the Ambrosio-Tortorelli functional.

**Lemma 4.1.** *Let  $\Omega \subset \mathbf{R}^N$  be a bounded open set with boundary of class  $\mathcal{C}^{2,1}$  and  $g \in \mathcal{C}^{2,\alpha}(\partial\Omega)$  for some  $\alpha \in (0, 1)$ . Along a sequence  $\varepsilon \rightarrow 0^+$ , let  $(u_\varepsilon, v_\varepsilon) \in \mathcal{A}_g(\Omega)$  be a critical point of  $AT_\varepsilon$  satisfying the uniform energy bound (1.12) and the conclusion of Proposition 4.1. If  $u_*$  denotes the weak limit of  $u_\varepsilon$  as  $\varepsilon \rightarrow 0$ , then  $\nabla u_* \in L^2(\Omega; \mathbf{R}^N)$  satisfies  $\text{div}(\nabla u_*) = 0$  in  $\mathcal{D}'(\Omega)$ , its normal trace  $\nabla u_* \cdot \nu_\Omega$  belongs to  $L^2(\partial\Omega)$ , and*

$$\partial_\nu u_\varepsilon \rightharpoonup \nabla u_* \cdot \nu_\Omega \quad \text{weakly in } L^2(\partial\Omega) \text{ as } \varepsilon \rightarrow 0.$$

Moreover, up to a subsequence, there exists a nonnegative Radon measure  $\lambda_* \in \mathcal{M}^+(\partial\Omega)$  such that

$$[(\partial_\nu u_\varepsilon)^2 + \varepsilon(\partial_\nu v_\varepsilon)^2] \mathcal{H}^{N-1} \llcorner \partial\Omega \xrightarrow{*} \lambda_* \quad \text{weakly}^* \text{ in } \mathcal{M}(\partial\Omega).$$

*Proof.* We first claim that  $(\eta_\varepsilon + v_\varepsilon^2) \nabla u_\varepsilon \rightharpoonup \nabla u_*$  weakly in  $L^2(\Omega; \mathbf{R}^N)$ . Indeed, on the one hand we have

$$\|\eta_\varepsilon \nabla u_\varepsilon\|_{L^2(\Omega; \mathbf{R}^N)} \leq \sqrt{\eta_\varepsilon} \left\| \sqrt{\eta_\varepsilon + v_\varepsilon^2} \nabla u_\varepsilon \right\|_{L^2(\Omega; \mathbf{R}^N)} \leq C \sqrt{\eta_\varepsilon} \rightarrow 0, \quad (4.8)$$

and on the other hand, for all  $\varphi \in \mathcal{C}_c^\infty(\Omega; \mathbf{R}^N)$ ,

$$\left| \int_{\Omega} v_\varepsilon \nabla u_\varepsilon \cdot \varphi \, dx - \int_{\Omega} v_\varepsilon^2 \nabla u_\varepsilon \cdot \varphi \, dx \right| \leq \|\varphi\|_{L^\infty(\Omega; \mathbf{R}^N)} \|v_\varepsilon \nabla u_\varepsilon\|_{L^2(\Omega; \mathbf{R}^N)} \|v_\varepsilon - 1\|_{L^2(\Omega)} \rightarrow 0.$$

Gathering both information and using Proposition 4.1 leads to  $(\eta_\varepsilon + v_\varepsilon^2) \nabla u_\varepsilon \rightharpoonup \nabla u_*$  weakly\* in  $\mathcal{D}'(\Omega; \mathbf{R}^N)$ . Since the sequence  $\{(\eta_\varepsilon + v_\varepsilon^2) \nabla u_\varepsilon\}$  is bounded in  $L^2(\Omega; \mathbf{R}^N)$ , its weak  $L^2$ -convergence follows. We can thus pass to the limit in (1.4) in the sense of distribution and conclude that  $\text{div}(\nabla u_*) = 0$  in  $\mathcal{D}'(\Omega)$ .

Since  $\nabla u_*$  belongs to  $L^2(\Omega; \mathbf{R}^N)$ , and  $\operatorname{div}(\nabla u_*) = 0$ , the normal trace  $\nabla u_* \cdot \nu_\Omega$  is well defined as an element of  $H^{-\frac{1}{2}}(\partial\Omega)$ . Recalling that  $v_\varepsilon = 1$  on  $\partial\Omega$ , we get that

$$(\eta_\varepsilon + 1)\partial_\nu u_\varepsilon = (\eta_\varepsilon + v_\varepsilon^2)\nabla u_\varepsilon \cdot \nu \rightharpoonup \nabla u_* \cdot \nu_\Omega \quad \text{weakly in } H^{-\frac{1}{2}}(\partial\Omega).$$

We now improve this convergence into a weak convergence in  $L^2(\partial\Omega)$ . For that, let us consider a test function  $X \in \mathcal{C}_c^1(\mathbf{R}^N; \mathbf{R}^n)$  such that  $X = \nu$  on  $\partial\Omega$  in relation (4.5). Using that the left-hand side of (4.5) is clearly controlled by the Ambrosio-Tortorelli energy (see (1.12)), we infer that

$$\sup_{\varepsilon > 0} \int_{\partial\Omega} [(\eta_\varepsilon + 1)(\partial_\nu u_\varepsilon)^2 + \varepsilon(\partial_\nu v_\varepsilon)^2] \, d\mathcal{H}^{N-1} < \infty.$$

On the one hand, we obtain that  $\{\partial_\nu u_\varepsilon\}_{\varepsilon > 0}$  is bounded in  $L^2(\partial\Omega)$ , hence  $\partial_\nu u_\varepsilon \rightharpoonup \nabla u_* \cdot \nu_\Omega$  weakly in  $L^2(\partial\Omega)$ . On the other hand, there exists a nonnegative Radon measure  $\lambda_* \in \mathcal{M}^+(\partial\Omega)$  such that  $[(\partial_\nu u_\varepsilon)^2 + \varepsilon(\partial_\nu v_\varepsilon)^2] \mathcal{H}^{N-1} \llcorner \partial\Omega \xrightarrow{*} \lambda_*$  weakly\* in  $\mathcal{M}(\partial\Omega)$ .  $\square$

**Remark 4.2.** Our choice of Dirichlet boundary conditions for both  $u$  and  $v$  in (1.3) allows one to obtain an  $\varepsilon$ -dependent boundary term which is nonnegative in the boundary integral involving  $X \cdot \nu_\Omega$  in (4.5). This sign information is essential to get a limit boundary term which is a measure  $\lambda_*$  concentrated on  $\partial\Omega$ . If we had chosen a Neumann condition for  $v$  and a Dirichlet condition for  $u$  as in [13], one would have obtained a more involved boundary term which would lead to a first order distribution on  $\partial\Omega$  in the  $\varepsilon \rightarrow 0$  limit. It is not clear in this case how to perform the analysis in Section 5 (in particular Lemma 5.3).

## 5. CONVERGENCE OF CRITICAL POINTS

Our objective is to show that  $u_*$  is a critical point of the Mumford-Shah functional. We now improve the convergence results established at the previous section by additionally assuming the convergence of the energy (1.13), i.e.,

$$AT_\varepsilon(u_\varepsilon, v_\varepsilon) \rightarrow MS(u_*).$$

Under this stronger assumption, we can improve the above established convergences and in particular obtain the equipartition of the phase-field energy.

**Proposition 5.1.** *Let  $\Omega \subset \mathbf{R}^N$  be a bounded open set with Lipschitz boundary and  $g \in H^{\frac{1}{2}}(\partial\Omega) \cap L^\infty(\partial\Omega)$ . Let us consider a critical point  $(u_\varepsilon, v_\varepsilon)$  of the Ambrosio-Tortorelli functional satisfying the energy convergence (1.13) and let  $u_* \in SBV^2(\Omega)$  be given by Proposition 4.1. Then, up to a further subsequence*

$$\sqrt{\eta_\varepsilon + v_\varepsilon^2} \nabla u_\varepsilon \rightarrow \nabla u_*, \quad v_\varepsilon \nabla u_\varepsilon \rightarrow \nabla u_* \quad \text{strongly in } L^2(\Omega; \mathbf{R}^N). \quad (5.1)$$

Moreover, setting  $\Phi(t) := t - t^2/2$ , and

$$w_\varepsilon := \Phi(v_\varepsilon), \quad (5.2)$$

then

$$\begin{cases} \nabla w_\varepsilon \mathcal{L}^N \llcorner \Omega \xrightarrow{*} 0 & \text{weakly* in } \mathcal{M}(\Omega; \mathbf{R}^N), \\ |\nabla w_\varepsilon| \mathcal{L}^N \llcorner \Omega \xrightarrow{*} \mathcal{H}^{N-1} \llcorner \hat{J}_{u_*} & \text{weakly* in } \mathcal{M}(\bar{\Omega}), \end{cases} \quad (5.3)$$

where we recall that  $\hat{J}_{u_*} = J_{u_*} \cup (\partial\Omega \cap \{u_* \neq g\})$ . Finally, there is equipartition of the phase-field energy, i.e.,

$$\lim_{\varepsilon \rightarrow 0} \int_\Omega \left| \varepsilon |\nabla v_\varepsilon|^2 - \frac{1}{4\varepsilon} (1 - v_\varepsilon)^2 \right| dx = \lim_{\varepsilon \rightarrow 0} \int_\Omega |2\varepsilon |\nabla v_\varepsilon|^2 - |\nabla w_\varepsilon|^2| dx = 0. \quad (5.4)$$

*Proof.* According to the convergence of energy assumption (1.13) and the lower semicontinuity properties (4.1)-(4.2) established in Proposition 4.1 (which applies by Lemma 3.1), we deduce that

$$\int_\Omega |\nabla u_*|^2 dx = \lim_{\varepsilon \rightarrow 0} \int_\Omega v_\varepsilon^2 |\nabla u_\varepsilon|^2 dx = \lim_{\varepsilon \rightarrow 0} \int_\Omega (\eta_\varepsilon + v_\varepsilon^2) |\nabla u_\varepsilon|^2 dx \quad (5.5)$$



and

$$\mathcal{H}^{N-1}(\hat{J}_{u_*}) = \lim_{\varepsilon \rightarrow 0} \int_{\Omega} (1 - v_{\varepsilon}) |\nabla v_{\varepsilon}| \, dx = \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \left( \varepsilon |\nabla v_{\varepsilon}|^2 + \frac{(v_{\varepsilon} - 1)^2}{4\varepsilon} \right) \, dx. \quad (5.6)$$

Convergence (5.5) combined with the weak  $L^2$ -convergence of  $v_{\varepsilon} \nabla u_{\varepsilon}$  to  $\nabla u_*$  implies that  $v_{\varepsilon} \nabla u_{\varepsilon} \rightarrow \nabla u_*$  strongly in  $L^2(\Omega; \mathbf{R}^N)$ . Moreover, it follows from (5.5) that  $\sqrt{\eta_{\varepsilon}} \nabla u_{\varepsilon} \rightarrow 0$  strongly in  $L^2(\Omega; \mathbf{R}^N)$ . Hence  $\sqrt{\eta_{\varepsilon} + v_{\varepsilon}^2} \nabla u_{\varepsilon} \rightarrow \nabla u_*$  strongly in  $L^2(\Omega; \mathbf{R}^N)$  as well.

Next, setting  $w_{\varepsilon} = \Phi(v_{\varepsilon})$ , with  $\Phi(t) = t - t^2/2$ , and using that  $v_{\varepsilon} \rightarrow 1$  strongly in  $L^2(\Omega)$  yields  $w_{\varepsilon} \rightarrow 1/2$  strongly in  $L^1(\Omega)$ . Furthermore, owing to the chain rule in Sobolev spaces, we have  $\nabla w_{\varepsilon} = \Phi'(v_{\varepsilon}) \nabla v_{\varepsilon} = (1 - v_{\varepsilon}) \nabla v_{\varepsilon}$ . According to (4.3), we deduce that  $\{\nabla w_{\varepsilon}\}_{\varepsilon > 0}$  is bounded in  $L^1(\Omega; \mathbf{R}^N)$ , and thus  $\nabla w_{\varepsilon} \mathcal{L}^N \llcorner \Omega \xrightarrow{*} 0$  weakly\* in  $\mathcal{M}(\Omega; \mathbf{R}^N)$ . Moreover, reproducing the proof of Proposition 4.1 we can see that for every open set  $U \subset \Omega' \subset \mathbf{R}^N$ , we have

$$\mathcal{H}^{N-1}(\hat{J}_{u_*} \cap U) \leq \liminf_{\varepsilon \rightarrow 0} \int_U |\nabla w_{\varepsilon}| \, dx,$$

and, using (5.6) shows that

$$\int_{\Omega} |\nabla w_{\varepsilon}| \, dx \rightarrow \mathcal{H}^{N-1}(\hat{J}_{u_*}). \quad (5.7)$$

Thus [1, Theorem 1.80] implies that  $|\nabla w_{\varepsilon}| \mathcal{L}^N \llcorner \Omega \xrightarrow{*} \mathcal{H}^{N-1} \llcorner \hat{J}_{u_*}$  weakly\* in  $\mathcal{M}(\bar{\Omega})$ .

The equipartition of energy is obtained by observing that

$$AT_{\varepsilon}(u_{\varepsilon}, v_{\varepsilon}) = \int_{\Omega} (\eta_{\varepsilon} + v_{\varepsilon}^2) |\nabla u_{\varepsilon}|^2 \, dx + \int_{\Omega} |\nabla w_{\varepsilon}| \, dx + \int_{\Omega} \left| \sqrt{\varepsilon} |\nabla v_{\varepsilon}| - \frac{1}{2\sqrt{\varepsilon}} (1 - v_{\varepsilon}) \right|^2 \, dx,$$

and by using (1.13), (5.5) and (5.7). Indeed, we get that

$$\int_{\Omega} \left| \sqrt{\varepsilon} |\nabla v_{\varepsilon}| - \frac{1}{2\sqrt{\varepsilon}} (1 - v_{\varepsilon}) \right|^2 \, dx \rightarrow 0,$$

and then, by the Cauchy-Schwarz inequality,

$$\begin{aligned} & \int_{\Omega} \left| \varepsilon |\nabla v_{\varepsilon}|^2 - \frac{1}{4\varepsilon} (1 - v_{\varepsilon})^2 \right| \, dx \\ & \leq \left( \int_{\Omega} \left| \sqrt{\varepsilon} |\nabla v_{\varepsilon}| - \frac{1}{2\sqrt{\varepsilon}} (1 - v_{\varepsilon}) \right|^2 \, dx \right)^{1/2} \left( \int_{\Omega} \left| \sqrt{\varepsilon} |\nabla v_{\varepsilon}| + \frac{1}{2\sqrt{\varepsilon}} (1 - v_{\varepsilon}) \right|^2 \, dx \right)^{1/2} \\ & \leq C \left( \int_{\Omega} \left| \sqrt{\varepsilon} |\nabla v_{\varepsilon}| - \frac{1}{2\sqrt{\varepsilon}} (1 - v_{\varepsilon}) \right|^2 \, dx \right)^{1/2} \rightarrow 0. \end{aligned}$$

Finally, using again that  $|\nabla w_{\varepsilon}| = (1 - v_{\varepsilon}) |\nabla v_{\varepsilon}|$ , we observe that

$$\begin{aligned} \int_{\Omega} |2\varepsilon |\nabla v_{\varepsilon}|^2 - |\nabla w_{\varepsilon}|| \, dx &= \int_{\Omega} \left| \varepsilon |\nabla v_{\varepsilon}|^2 - \frac{(1 - v_{\varepsilon})^2}{4\varepsilon} + \left( \sqrt{\varepsilon} |\nabla v_{\varepsilon}| - \frac{1}{2\sqrt{\varepsilon}} (1 - v_{\varepsilon}) \right)^2 \right| \, dx \\ &\leq \int_{\Omega} \left| \varepsilon |\nabla v_{\varepsilon}|^2 - \frac{(1 - v_{\varepsilon})^2}{4\varepsilon} \right| \, dx + \int_{\Omega} \left| \sqrt{\varepsilon} |\nabla v_{\varepsilon}| - \frac{1}{2\sqrt{\varepsilon}} (1 - v_{\varepsilon}) \right|^2 \, dx \rightarrow 0. \end{aligned}$$

This implies (5.4).  $\square$

The proof of Theorem 1.2 is based on (geometric) measure theoretic arguments. Let us define the  $(N - 1)$ -varifold  $V_{\varepsilon} \in \mathbf{V}_{N-1}(\bar{\Omega})$  associated to the phase-field variable  $v_{\varepsilon} \in H^1(\Omega)$  by

$$\langle V_{\varepsilon}, \varphi \rangle := \int_{\Omega \cap \{|\nabla w_{\varepsilon}| \neq 0\}} \varphi \left( x, \text{Id} - \frac{\nabla w_{\varepsilon}}{|\nabla w_{\varepsilon}|} \otimes \frac{\nabla w_{\varepsilon}}{|\nabla w_{\varepsilon}|} \right) |\nabla w_{\varepsilon}| \, dx \quad \text{for all } \varphi \in \mathcal{C}(\bar{\Omega} \times \mathbf{G}_{N-1}),$$

where  $w_\varepsilon := \Phi(v_\varepsilon)$ , and  $\Phi(t) = t - t^2/2$  for  $t \in [0, 1]$ . By the coarea formula, this definition is equivalent to the definition of a varifold associated to a function in [21]. By standard compactness of bounded Radon measures, at the expense of extracting a further subsequence, there exists a varifold  $V_* \in \mathbf{V}_{N-1}(\bar{\Omega})$  such that

$$V_\varepsilon \xrightarrow{*} V_* \quad \text{weakly* in } \mathcal{M}(\bar{\Omega} \times \mathbf{G}_{N-1}).$$

Note that  $\|V_\varepsilon\| \xrightarrow{*} \|V_*\|$  weakly\* in  $\mathcal{M}(\bar{\Omega})$  by definition of the mass of a varifold and thanks to the compactness of  $\mathbf{G}_{N-1}$ . Recalling the definition of  $w_\varepsilon$  in (5.2), we observe that  $\|V_\varepsilon\| = |\nabla w_\varepsilon| \mathcal{L}^N \llcorner \Omega \xrightarrow{*} \mathcal{H}^{N-1} \llcorner \hat{J}_{u_*}$  weakly\* in  $\mathcal{M}(\bar{\Omega})$  according to (5.3), and it follows that

$$\|V_*\| = \mathcal{H}^{N-1} \llcorner \hat{J}_{u_*}.$$

According to the disintegration Theorem ([1, Theorem 2.28]), there exists a weak\*  $\mathcal{H}^{N-1}$ -measurable mapping  $x \in \bar{\Omega} \mapsto V_x \in \mathcal{M}(\mathbf{G}_{N-1})$  of probability measures such that  $V_* = (\mathcal{H}^{N-1} \llcorner \hat{J}_{u_*}) \otimes V_x$ , i.e., for all  $\varphi \in \mathcal{C}(\bar{\Omega} \times \mathbf{G}_{N-1})$ ,

$$\int_{\bar{\Omega} \times \mathbf{G}_{N-1}} \varphi(x, A) dV_*(x, A) = \int_{\hat{J}_{u_*}} \left( \int_{\mathbf{G}_{N-1}} \varphi(x, A) dV_x(A) \right) d\mathcal{H}^{N-1}(x). \quad (5.8)$$

For  $\mathcal{H}^{N-1}$  almost every  $x \in \hat{J}_{u_*}$ , we set

$$\bar{A}(x) := \int_{\mathbf{G}_{N-1}} A dV_x(A). \quad (5.9)$$

Owing to our various convergence results, we are now in position to pass to the limit in the inner variation equation (4.5). The limit expression is for now depending on the abstract limit varifold  $V_*$  through its first moment  $\bar{A}$  of  $V_x$ , and the abstract boundary measure  $\lambda_*$  introduced in Lemma 4.1.

**Lemma 5.1.** *Let  $\Omega \subset \mathbf{R}^N$  is a bounded open set of class  $\mathcal{C}^{2,1}$  and  $g \in \mathcal{C}^{2,\alpha}(\partial\Omega)$  for some  $\alpha \in (0, 1)$ . Let  $u_* \in SBV^2(\Omega)$  be a limit of critical points of the Ambrosio-Tortorelli functional as in Proposition 5.1. For all  $X \in \mathcal{C}_c^1(\mathbf{R}^N; \mathbf{R}^N)$ , we have*

$$\begin{aligned} & \int_{\Omega} (|\nabla u_*|^2 \text{Id} - 2\nabla u_* \otimes \nabla u_*) : DX \, dx + \int_{\hat{J}_{u_*}} \bar{A} : DX \, d\mathcal{H}^{N-1} \\ &= - \int_{\partial\Omega} (X \cdot \nu_\Omega) d\lambda_* + \int_{\partial\Omega} |\nabla_\tau g|^2 (X \cdot \nu_\Omega) d\mathcal{H}^{N-1} - 2 \int_{\partial\Omega} (\nabla u_* \cdot \nu_\Omega) (X_\tau \cdot \nabla_\tau g) d\mathcal{H}^{N-1}. \end{aligned} \quad (5.10)$$

*Proof.* Using the strong convergence (5.1) established before, it is easy to pass to the limit in the first integral and in the left hand side of (4.5). We get for all  $X \in \mathcal{C}_c^1(\mathbf{R}^N; \mathbf{R}^N)$ ,

$$\begin{aligned} & \int_{\Omega} \left( 2(\sqrt{\eta_\varepsilon + v_\varepsilon^2} \nabla u_\varepsilon) \otimes (\sqrt{\eta_\varepsilon + v_\varepsilon^2} \nabla u_\varepsilon) - (\eta_\varepsilon + v_\varepsilon^2) |\nabla u_\varepsilon|^2 \text{Id} \right) : DX \, dx \\ & \xrightarrow{\varepsilon \rightarrow 0} \int_{\Omega} (2\nabla u_* \otimes \nabla u_* - |\nabla u_*|^2 \text{Id}) : DX \, dx. \end{aligned} \quad (5.11)$$

According to Lemma 4.1 we can also pass to the limit in the boundary integrals in the right hand side of (4.5), we get that

$$\begin{aligned} & \int_{\partial\Omega} [(\eta_\varepsilon + 1)(\partial_\nu u_\varepsilon)^2 + \varepsilon(\partial_\nu v_\varepsilon)^2] X \cdot \nu \, d\mathcal{H}^{N-1} \\ & - (1 + \eta_\varepsilon) \int_{\partial\Omega} X \cdot \nu |\nabla_\tau g|^2 \, d\mathcal{H}^{N-1} + 2(\eta_\varepsilon + 1) \int_{\partial\Omega} (\partial_\nu u_\varepsilon) (X_\tau \cdot \nabla_\tau g) \, d\mathcal{H}^{N-1} \\ & \xrightarrow{\varepsilon \rightarrow 0} \int_{\partial\Omega} (X \cdot \nu_\Omega) d\lambda_* - \int_{\partial\Omega} |\nabla_\tau g|^2 (X \cdot \nu_\Omega) d\mathcal{H}^{N-1} + 2 \int_{\partial\Omega} (\nabla u_* \cdot \nu_\Omega) (X_\tau \cdot \nabla_\tau g) \, d\mathcal{H}^{N-1}. \end{aligned} \quad (5.12)$$

It remains to pass to the limit in the second integral in the left hand side of (4.5). Using the chain rule, we have  $\nabla w_\varepsilon = \Phi'(v_\varepsilon)\nabla v_\varepsilon$ . The equipartition of energy (5.4) thus implies that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \left( 2\varepsilon \nabla v_\varepsilon \otimes \nabla v_\varepsilon - \varepsilon |\nabla v_\varepsilon|^2 \text{Id} - \frac{(v_\varepsilon - 1)^2}{4\varepsilon} \text{Id} \right) : DX \, dx \\ = \lim_{\varepsilon \rightarrow 0} \int_{\Omega \cap \{\nabla v_\varepsilon \neq 0\}} 2\varepsilon |\nabla v_\varepsilon|^2 \left( \frac{\nabla v_\varepsilon}{|\nabla v_\varepsilon|} \otimes \frac{\nabla v_\varepsilon}{|\nabla v_\varepsilon|} - \text{Id} \right) : DX \, dx \\ = \lim_{\varepsilon \rightarrow 0} \int_{\Omega \cap \{\nabla w_\varepsilon \neq 0\}} |\nabla w_\varepsilon| \left( \frac{\nabla w_\varepsilon}{|\nabla w_\varepsilon|} \otimes \frac{\nabla w_\varepsilon}{|\nabla w_\varepsilon|} - \text{Id} \right) : DX \, dx \\ = - \int_{\bar{\Omega} \times \mathbf{G}_{N-1}} A : DX(x) \, dV_*(x, A) = - \int_{\hat{J}_{u_*}} \bar{A} : DX \, d\mathcal{H}^{N-1}. \quad (5.13) \end{aligned}$$

Gathering (5.11), (5.12) and (5.13), we infer that (5.10) holds.  $\square$

Let us now identify the first moment  $\bar{A}$  of the measure  $V_x$ . We first establish some algebraic properties of this matrix.

**Lemma 5.2.** *For  $\mathcal{H}^{N-1}$ -almost every  $x \in \hat{J}_{u_*}$ , the matrix  $\bar{A}(x)$  satisfies*

$$\bar{A}(x) \geq 0, \quad \text{tr}(\bar{A}(x)) = N - 1, \quad \rho(\bar{A}(x)) = 1,$$

where  $\rho$  denotes the spectral radius.

*Proof.* To simplify notation, we set

$$A_\varepsilon := \left( \text{Id} - \frac{\nabla w_\varepsilon}{|\nabla w_\varepsilon|} \otimes \frac{\nabla w_\varepsilon}{|\nabla w_\varepsilon|} \right).$$

The matrix  $A_\varepsilon$  is well-defined on the set  $\Omega \cap \{\nabla w_\varepsilon \neq 0\}$ , it is a symmetric matrix corresponding to the orthogonal projection on  $\{\nabla w_\varepsilon\}^\perp$ . It satisfies

$$A_\varepsilon \geq 0, \quad \text{tr}(A_\varepsilon) = N - 1 \quad \text{and} \quad \rho(A_\varepsilon) = 1 \quad \text{in } \Omega \cap \{\nabla w_\varepsilon \neq 0\}.$$

For all  $\varphi \in \mathcal{C}^0(\bar{\Omega})$ , we have

$$\begin{aligned} \int_{\hat{J}_{u_*}} \text{tr}(\bar{A}) \varphi \, d\mathcal{H}^{N-1} &= \int_{\bar{\Omega} \times \mathbf{G}_{N-1}} \text{tr}(A) \varphi(x) \, dV_*(x, A) \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega \times \mathbf{G}_{N-1}} \text{tr}(A) \varphi(x) \, dV_\varepsilon(x, A) \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega \cap \{\nabla w_\varepsilon \neq 0\}} \text{tr}(A_\varepsilon) \varphi |\nabla w_\varepsilon| \, dx \\ &= (N - 1) \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \varphi |\nabla w_\varepsilon| \, dx \\ &= (N - 1) \int_{\hat{J}_{u_*}} \varphi \, d\mathcal{H}^{N-1}, \end{aligned}$$

which shows that  $\text{tr}(\bar{A}) = (N - 1) \mathcal{H}^{N-1}$ -a.e. on  $\hat{J}_{u_*}$ . If further  $\varphi \geq 0$  and  $z \in \mathbf{R}^N$ , then

$$\begin{aligned} \int_{\hat{J}_{u_*}} (\bar{A}z \cdot z) \varphi \, d\mathcal{H}^{N-1} &= \int_{\bar{\Omega} \times \mathbf{G}_{N-1}} (Az \cdot z) \varphi(x) \, dV_*(x, A) \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega \times \mathbf{G}_{N-1}} (Az \cdot z) \varphi(x) \, dV_\varepsilon(x, A) \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega \cap \{\nabla w_\varepsilon \neq 0\}} (A_\varepsilon z \cdot z) \varphi |\nabla w_\varepsilon| \, dx \geq 0. \end{aligned}$$

As a consequence, for all  $z \in \mathbf{R}^N$ , we have  $\bar{A}z \cdot z \geq 0$   $\mathcal{H}^{N-1}$ -a.e. on  $\hat{J}_{u_*}$ , from which we deduce that  $\bar{A}$  is a nonnegative matrix  $\mathcal{H}^{N-1}$ -a.e. on  $\hat{J}_{u_*}$ .

Since for all  $\varphi \in \mathcal{C}^0(\bar{\Omega})$  we have

$$\int_{\Omega \cap \{\nabla w_\varepsilon \neq 0\}} |\nabla w_\varepsilon| \left( \text{Id} - \frac{\nabla w_\varepsilon}{|\nabla w_\varepsilon|} \otimes \frac{\nabla w_\varepsilon}{|\nabla w_\varepsilon|} \right) \varphi \, dx \rightarrow \int_{\bar{\Omega} \times \mathbf{G}_{N-1}} A\varphi(x) \, dV_*(x, A) = \int_{\hat{J}_{u_*}} \bar{A}\varphi \, d\mathcal{H}^{N-1},$$

we deduce that

$$A_\varepsilon |\nabla w_\varepsilon| \mathcal{L}^N \llcorner \Omega \xrightarrow{*} \bar{A} \mathcal{H}^{N-1} \llcorner \hat{J}_{u_*} \quad \text{weakly}^* \text{ in } \mathcal{M}(\bar{\Omega}; \mathbf{M}^{N \times N}). \quad (5.14)$$

Using that the spectral radius  $\rho$  is a convex, continuous, and positively 1-homogeneous function on the set of symmetric matrices<sup>2</sup>, it follows from Reshetnyak continuity Theorem (see [1, Theorem 2.39]) that for all  $\varphi \in \mathcal{C}^0(\bar{\Omega})$ ,

$$\int_{\hat{J}_{u_*}} \varphi \, d\mathcal{H}^{N-1} = \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \varphi |\nabla w_\varepsilon| \, dx = \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \varphi \rho(A_\varepsilon) |\nabla w_\varepsilon| \, dx = \int_{\hat{J}_{u_*}} \varphi \rho(\bar{A}) \, d\mathcal{H}^{N-1},$$

hence  $\rho(\bar{A}) = 1$   $\mathcal{H}^{N-1}$ -a.e. on  $\hat{J}_{u_*}$ .  $\square$

We now focus on the interior structure of the varifold  $V_*$ .

**Lemma 5.3.** *For  $\mathcal{H}^{N-1}$ -a.e.  $x$  in  $J_{u_*} = \hat{J}_{u_*} \cap \Omega$ ,*

$$\bar{A}(x) = \text{Id} - \nu_{u_*}(x) \otimes \nu_{u_*}(x),$$

where  $\nu_{u_*}$  is the approximate normal to the countably  $\mathcal{H}^{N-1}$ -rectifiable set  $J_{u_*}$ .

*Proof. Step 1: Let us show that for  $\mathcal{H}^{N-1}$ -a.e.  $x$  in  $J_{u_*}$ ,  $\bar{A}(x)$  is a projection matrix onto a  $(N-1)$ -dimensional hyperplane.*

To this aim, we perform a blow-up argument on the first variation equation (5.10). Let  $x_0 \in J_{u_*}$  be such that

- (1)  $x_0$  is a Lebesgue point of  $\bar{A}$  with respect to  $\mathcal{H}^{N-1} \llcorner J_{u_*}$ ;
- (2)  $J_{u_*}$  admits an approximate tangent space at  $x_0$  given by  $T_{x_0} = \{\nu_{u_*}(x_0)\}^\perp$ ;
- (3)

$$\lim_{\varrho \rightarrow 0} \frac{\mathcal{H}^{N-1}(J_{u_*} \cap B_\varrho(x_0))}{\omega_{N-1} \varrho^{N-1}} = 1; \quad (5.15)$$

(4)

$$\lim_{\varrho \rightarrow 0} \frac{1}{\varrho^{N-1}} \int_{B_\varrho(x_0)} |\nabla u_*|^2 \, dx = 0.$$

It turns out that  $\mathcal{H}^{N-1}$ -almost every point  $x_0 \in J_{u_*}$  satisfies these properties. Indeed, (1) is a consequence of the Besicovitch differentiation theorem, (2) and (3) are consequences of the rectifiability of  $J_{u_*}$  (see Theorems 2.63 and 2.83 in [1]), while condition (4) is a consequence of (3) together with the fact that the measure  $|\nabla u_*|^2 \mathcal{L}^N \llcorner \Omega$  is singular with respect to  $\mathcal{H}^{N-1} \llcorner J_{u_*}$ .

Let  $x_0 \in J_{u_*}$  be such a point and let  $\varrho > 0$  be such that  $\overline{B_\varrho(x_0)} \subset \Omega$ . For  $\phi \in \mathcal{C}_c^\infty(\mathbf{R}^N; \mathbf{R}^N)$  such that  $\text{Supp}(\phi) \subset B_1$ , we set

$$\phi_{x_0, \varrho}(x) := \phi \left( \frac{x - x_0}{\varrho} \right) \quad \text{for all } x \in \mathbf{R}^N,$$

so that  $\phi_{x_0, \varrho} \in \mathcal{C}_c^\infty(\mathbf{R}^N; \mathbf{R}^N)$  and  $\text{Supp}(\phi_{x_0, \varrho}) \subset B_\varrho(x_0)$ . Taking  $\phi_{x_0, \varrho}$  as test vector field in (5.10) (note that  $\phi_{x_0, \varrho} = 0$  in a neighbourhood of  $\partial\Omega$ ) yields

$$\int_{J_{u_*} \cap B_\varrho(x_0)} \bar{A} : D\phi_{x_0, \varrho} \, d\mathcal{H}^{N-1} = - \int_{B_\varrho(x_0)} (|\nabla u_*|^2 \text{Id} - 2\nabla u_* \otimes \nabla u_*) : D\phi_{x_0, \varrho} \, dx.$$

<sup>2</sup>For a symmetric matrix we can write  $\rho(A) = \max_{|x|=1} Ax \cdot x$ , hence  $\rho$  is convex as a supremum of affine functions.

Dividing this identity by  $\varrho^{N-2}$  yields

$$\begin{aligned} \frac{1}{\varrho^{N-1}} \int_{J_{u_*} \cap B_\varrho(x_0)} \bar{A} : D\phi \left( \frac{\cdot - x_0}{\varrho} \right) d\mathcal{H}^{N-1} \\ = \frac{-1}{\varrho^{N-1}} \int_{B_\varrho(x_0)} (|\nabla u_*|^2 \text{Id} - 2\nabla u_* \otimes \nabla u_*) : D\phi \left( \frac{\cdot - x_0}{\varrho} \right) dx. \end{aligned}$$

We first show that the left hand side of the previous equality tends to zero as  $\varrho \rightarrow 0$ . Indeed, thanks to our choice of  $x_0$ , we have

$$\begin{aligned} \left| \frac{1}{\varrho^{N-1}} \int_{B_\varrho(x_0)} (|\nabla u_*|^2 \text{Id} - 2\nabla u_* \otimes \nabla u_*) : D\phi \left( \frac{x - x_0}{\varrho} \right) dx \right| \\ \leq C \frac{\|D\phi\|_{L^\infty(B_1; \mathbf{M}^{N \times N})}}{\varrho^{N-1}} \int_{B_\varrho(x_0)} |\nabla u_*|^2 dx \rightarrow 0, \end{aligned}$$

for some constant  $C > 0$ . For what concerns the right hand side, using first that  $x_0$  is a Lebesgue point of  $\bar{A}$  and (5.15), we get that

$$\begin{aligned} \left| \frac{1}{\varrho^{N-1}} \int_{J_{u_*} \cap B_\varrho(x_0)} (\bar{A} - \bar{A}(x_0)) : D\phi \left( \frac{\cdot - x_0}{\varrho} \right) d\mathcal{H}^{N-1} \right| \\ \leq \frac{\|D\phi\|_{L^\infty(B_1; \mathbf{M}^{N \times N})}}{\varrho^{N-1}} \int_{J_{u_*} \cap B_\varrho(x_0)} |\bar{A} - \bar{A}(x_0)| d\mathcal{H}^{N-1} \rightarrow 0, \end{aligned}$$

so that

$$\begin{aligned} \lim_{\varrho \rightarrow 0} \frac{1}{\varrho^{N-1}} \int_{J_{u_*} \cap B_\varrho(x_0)} \bar{A} : D\phi \left( \frac{\cdot - x_0}{\varrho} \right) d\mathcal{H}^{N-1} \\ = \bar{A}(x_0) : \lim_{\varrho \rightarrow 0} \frac{1}{\varrho^{N-1}} \int_{J_{u_*} \cap B_\varrho(x_0)} D\phi \left( \frac{\cdot - x_0}{\varrho} \right) d\mathcal{H}^{N-1}. \end{aligned}$$

Using next that  $J_{u_*}$  admits an approximate tangent space that we denote by  $T_{x_0}$  at  $x_0$ , we obtain that

$$\lim_{\varrho \rightarrow 0} \frac{1}{\varrho^{N-1}} \int_{J_{u_*} \cap B_\varrho(x_0)} D\phi \left( \frac{\cdot - x_0}{\varrho} \right) d\mathcal{H}^{N-1} = \int_{T_{x_0} \cap B_1} D\phi d\mathcal{H}^{N-1}.$$

Hence,

$$\int_{T_{x_0} \cap B_1} \bar{A}(x_0) : D\phi d\mathcal{H}^{N-1} = 0 \quad \text{for all } \phi \in \mathcal{C}_c^\infty(B_1; \mathbf{R}^N). \quad (5.16)$$

Let  $t \in (0, 1)$  be such that  $t < \sqrt{N}^{-N}$ , the measure  $\nu := \frac{1}{\omega_{N-1}} \mathcal{H}^{N-1} \llcorner T_{x_0}$  satisfies

$$t^{N-1} \leq \nu(\bar{B}_t) \leq \nu(B_1) \leq 1.$$

According to [2, Lemma 3.9] with  $\beta = s = N - 1$ , we get that the matrix  $\bar{A}(x_0)$  has at most  $N - 1$  nonzero eigenvalues. Recalling that  $\text{tr}(\bar{A}(x_0)) = N - 1$  and that all eigenvalues of  $\bar{A}(x_0)$  belong to  $[0, 1]$ , this implies that  $\bar{A}(x_0)$  has exactly  $N - 1$  eigenvalues which are equal to 1, and one eigenvalue which is zero. Hence, there exists  $e \in \mathbf{S}^{N-1}$  such that  $A = \text{Id} - e \otimes e$ .

*Step 2: Let us show that  $e = \pm \nu_{u_*}(x_0)$ .* Let us consider the varifold

$$\tilde{V} := \mathcal{H}^{N-1} \llcorner T_{x_0} \otimes \delta_{\bar{A}(x_0)} \in \mathcal{M}(B_1 \times \mathbf{G}_{N-1}),$$

whose action is given by

$$\int_{B_1 \times \mathbf{G}_{N-1}} \varphi(x, A) d\tilde{V}(x, A) = \int_{B_1 \cap T_{x_0}} \varphi(x, \bar{A}(x_0)) d\mathcal{H}^{N-1}(x) \quad \text{for all } \varphi \in \mathcal{C}_c^0(B_1 \times \mathbf{G}_{N-1}).$$

Since  $\bar{A}(x_0)$  is a projection matrix onto the hyperplane  $e^\perp$ , it follows that  $\tilde{V} \in \mathbf{V}_{N-1}(B_1)$  is a  $(N-1)$ -varifold in  $B_1$  with  $\|\tilde{V}\| = \mathcal{H}^{N-1} \llcorner T_{x_0}$ . Moreover, condition (5.16) shows that  $\tilde{V}$  is a stationary varifold, cf. Section 2.3. It follows from the monotonicity formula (see e.g. formula (40.3) page 236 in [41]) that for all  $x \in T_{x_0} \cap B_1$  and all  $\varrho > 0$  such that  $B_\varrho(x) \subset B_1$ ,

$$\frac{\mathcal{H}^{N-1}(T_{x_0} \cap B_\varrho(x))}{\varrho^{N-1}} = \frac{\mathcal{H}^{N-1}(T_{x_0} \cap B_r(x))}{r^{N-1}} + \int_{T_{x_0} \cap B_\varrho(x) \setminus B_r(x)} \frac{|e \cdot (y-x)|^2}{|y-x|^{N+1}} d\mathcal{H}^{N-1}(y)$$

for all  $0 < r < \varrho$ . Since

$$\frac{\mathcal{H}^{N-1}(T_{x_0} \cap B_r(x))}{r^{N-1}} = \frac{\mathcal{H}^{N-1}(T_{x_0} \cap B_\varrho(x))}{\varrho^{N-1}} = \omega_{N-1},$$

we deduce that

$$\int_{T_{x_0} \cap B_\varrho(x) \setminus B_r(x)} \frac{|e \cdot (y-x)|^2}{|y-x|^{N+1}} d\mathcal{H}^{N-1}(y) = 0.$$

Choosing  $x = 0$ ,  $\varrho = 1$ , and letting  $r \rightarrow 0^+$ , we infer that  $y \cdot e = 0$  for  $\mathcal{H}^{N-1}$ -a.e.  $y \in T_{x_0} \cap B_1$  which implies that  $T_{x_0} = e^\perp$ , hence  $e = \pm \nu_{u_*}(x_0)$ .  $\square$

Next we focus on boundary points.

**Lemma 5.4.** *For  $\mathcal{H}^{N-1}$ -a.e.  $x \in \hat{J}_{u_*} \cap \partial\Omega$ , we have*

$$\bar{A}(x) = \text{Id} - \nu_\Omega(x) \otimes \nu_\Omega(x),$$

where  $\nu_\Omega$  is the outward unit normal to  $\partial\Omega$ .

*Proof.* We perform again a blow-up argument, this time at boundary points. Let  $x_0 \in \hat{J}_{u_*} \cap \partial\Omega$  be such that:

- (1)  $x_0$  is a Lebesgue point of  $\bar{A}$  with respect to  $\mathcal{H}^{N-1} \llcorner \hat{J}_{u_*}$ ;
- (2)  $\hat{J}_{u_*}$  admits an approximate tangent space at  $x_0$  which coincides with the (usual) tangent space of  $\partial\Omega$  at  $x_0$  (this in particular implies that  $\nu_{u_*}(x_0) = \pm \nu_\Omega(x_0)$ );
- (3)

$$(4) \quad \lim_{\varrho \rightarrow 0} \frac{\mathcal{H}^{N-1}(\hat{J}_{u_*} \cap B_\varrho(x_0))}{\omega_{N-1} \varrho^{N-1}} = 1;$$

$$(5) \quad \lim_{\varrho \rightarrow 0} \frac{1}{\varrho^{N-1}} \int_{B_\varrho(x_0) \cap \Omega} |\nabla u_*|^2 dx = 0;$$

$$\lim_{\varrho \rightarrow 0} \frac{\lambda_*(B_\varrho(x_0))}{\varrho^{N-2}} = 0, \quad \lim_{\varrho \rightarrow 0} \frac{1}{\varrho^{N-2}} \int_{\partial\Omega \cap B_\varrho(x_0)} |\nabla u_* \cdot \nu| d\mathcal{H}^{N-1} = 0.$$

It turns out that  $\mathcal{H}^{N-1}$  almost every point  $x_0 \in \hat{J}_{u_*} \cap \partial\Omega$  satisfies these properties. Indeed, (1) is a consequence of the Besicovitch differentiation Theorem while (2) comes from the rectifiability of  $\hat{J}_{u_*}$  (see [1, Theorems 2.83]) together with the locality of approximate tangent spaces (see [1, Proposition 2.85]). Condition (3) is again a consequence of the rectifiability of  $\hat{J}_{u_*}$  and the Besicovitch-Marstrand-Mattila theorem (see [1, Proposition 2.63]). Condition (4) is a consequence of (3) together with the fact that the measure  $|\nabla u_*|^2 \mathcal{L}^N \llcorner \Omega$  is singular with respect to  $\mathcal{H}^{N-1} \llcorner \hat{J}_{u_*}$ . To justify (5), we define for  $x \in \partial\Omega$ ,

$$\Theta(x) := \limsup_{\varrho \rightarrow 0} \frac{\lambda_*(B_\varrho(x))}{\varrho^{N-2}}.$$

According to standard density arguments (see e.g. [1, Theorem 2.56]), we have

$$t\mathcal{H}^{N-2}(\{\Theta \geq t\}) \leq \lambda_*(\{\Theta \geq t\}) < \infty \quad \text{for all } t > 0.$$

Hence  $\mathcal{H}^{N-1}(\{\Theta \geq t\}) = 0$  for all  $t > 0$ . As a consequence,  $\mathcal{H}^{N-1}(\{\Theta > 0\}) = 0$ . The second property of (5) can be obtained similarly replacing  $\lambda_*$  by  $|\nabla u_* \cdot \nu| \mathcal{H}^{N-1} \llcorner \partial\Omega$ .

We choose such a point  $x_0 \in \partial\Omega \cap \hat{J}_{u_*}$  and we take  $\varrho > 0$ .

*Step 1: We first prove that  $\nu_\Omega(x_0)$  is an eigenvector of  $\bar{A}(x_0)$ .* Consider first a test vector field  $\phi$  of the form

$$\phi(x) := \varphi \left( \frac{x - x_0}{\varrho} \right) \tilde{\tau}(x) \quad \text{for all } x \in B_\varrho(x_0),$$

where  $\varphi \in \mathcal{C}_c^\infty(B_1)$  and  $\tilde{\tau} \in \mathcal{C}_c^1(\mathbf{R}^N; \mathbf{R}^N)$  is such that  $\tilde{\tau} \cdot \nu_\Omega = 0$  on  $\partial\Omega$ . Plugging  $\phi$  in (5.10) and using estimates similar to the proof of Lemma 5.3, we obtain

$$\int_{T_{x_0} \cap B_1} \bar{A}(x_0) : (\tilde{\tau}(x_0) \otimes \nabla \varphi) d\mathcal{H}^{N-1} = 0 \quad \text{for all } \varphi \in \mathcal{C}_c^\infty(B_1). \quad (5.17)$$

Note that to get (5.17), the boundary term is cancelled thanks to the second property of (5). Let  $\{\tau_1, \dots, \tau_{N-1}\}$  be an orthonormal basis of  $T_{x_0}$ , and  $\nu := \nu_\Omega(x_0)$  be the outward unit normal to  $\Omega$  at  $x_0$  (i.e.  $\nu$  is a normal vector to  $T_{x_0}$ ). We choose the vector field  $\tilde{\tau}$  in such a way that  $\tilde{\tau}(x_0) = \tau_i$ , and we decompose  $\nabla \varphi$  along the orthonormal basis  $\{\tau_1, \dots, \tau_{N-1}, \nu\}$  of  $\mathbf{R}^N$  as

$$\nabla \varphi = \sum_{j=1}^{N-1} (\partial_j \varphi) \tau_j + (\partial_\nu \varphi) \nu.$$

Since  $\int_{T_{x_0} \cap B_1} \partial_j \varphi d\mathcal{H}^{N-1} = 0$  for all  $1 \leq j \leq N-1$ , we infer from (5.17) that

$$\left( (\bar{A}(x_0) \tau_i) \cdot \nu \right) \int_{T_{x_0} \cap B_1} \partial_\nu \varphi d\mathcal{H}^{N-1} = 0.$$

From the arbitrariness of  $\varphi$ , it follows that  $(\bar{A}(x_0) \tau_i) \cdot \nu = 0$  for all  $1 \leq i \leq N-1$ . Since  $\bar{A}(x_0)$  is symmetric, we deduce that  $\bar{A}(x_0) \nu \in T_{x_0}^\perp$ , that is  $\bar{A}(x_0) \nu = c \nu$  for some  $c \in [0, 1]$  (recall that all eigenvalues of  $\bar{A}(x_0)$  belong to  $[0, 1]$  by Lemma 5.2). Thus  $\nu$  is an eigenvector of  $\bar{A}(x_0)$ , and by the spectral theorem, we can also assume without loss of generality that  $\tau_1, \dots, \tau_{N-1}$  are also eigenvectors of  $\bar{A}(x_0)$ .

*Step 2: We next show that  $\bar{A}(x_0)$  is the projection matrix onto the tangent space to  $\partial\Omega$  at  $x_0$ .*

We now consider a test vector field  $\phi$  of the form

$$\phi(x) := \tilde{\nu}(x) \varphi \left( \frac{x - x_0}{\varrho} \right) \quad \text{for all } x \in B_\varrho(x_0),$$

where  $\varphi \in \mathcal{C}_c^\infty(B_1)$  and  $\tilde{\nu} \in \mathcal{C}_c^1(\mathbf{R}^N; \mathbf{R}^N)$  is such that  $\tilde{\nu}(x)$  is a normal vector to  $\partial\Omega$  at  $x \in \partial\Omega$  satisfying  $\tilde{\nu}(x_0) = \nu_\Omega(x_0)$ . Using again estimates in a similar way to the proof of Lemma 5.3 (this time, the boundary term is cancelled thanks to the first property of (5)), we obtain that

$$\int_{T_{x_0} \cap B_1} \bar{A}(x_0) : (\nu \otimes \nabla \varphi) d\mathcal{H}^{N-1} = 0 \quad \text{for all } \varphi \in \mathcal{C}_c^\infty(B_1),$$

and thus, by Step 1,

$$c \int_{T_{x_0} \cap B_1} \partial_\nu \varphi d\mathcal{H}^{N-1} = 0.$$

By arbitrariness of  $\varphi$ , this last equality shows that  $c = 0$ . As a consequence, there exist real numbers  $c_1, \dots, c_{N-1} \in [0, 1]$  (the eigenvalues of  $\bar{A}(x_0)$  associated to the eigenvectors  $\tau_1, \dots, \tau_{N-1}$ ) such that

$$\bar{A}(x_0) = \sum_{i=1}^{N-1} c_i \tau_i \otimes \tau_i.$$

According to Lemma 5.2,  $\text{tr}(\bar{A}(x_0)) = c_1 + \dots + c_{N-1} = N-1$ , and we deduce that  $c_1 = \dots = c_{N-1} = 1$ . Hence,

$$\bar{A}(x_0) = \sum_{i=1}^{N-1} \tau_i \otimes \tau_i = \text{Id} - \nu \otimes \nu,$$

as announced.  $\square$

We can now complete the proof of our second main theorem.

*Proof of Theorem 1.2.* i) This point is a consequence of Lemma 4.1.

ii) Using that  $\nu_{u_*} = \pm \nu_\Omega$   $\mathcal{H}^{N-1}$ -a.e. in  $\partial\Omega \cap \hat{J}_{u_*}$  and gathering Lemmas 5.3 and 5.4 yields

$$\bar{A} = \text{Id} - \nu_{u_*} \otimes \nu_{u_*} \quad \mathcal{H}^{N-1}\text{-a.e. in } \hat{J}_{u_*}.$$

Thus according to (5.8) and (5.9), we get that

$$\int_{\hat{J}_{u_*}} \bar{A} : DX \, d\mathcal{H}^{N-1} = \int_{\hat{J}_{u_*}} (\text{Id} - \nu_{u_*} \otimes \nu_{u_*}) : DX \, d\mathcal{H}^{N-1}.$$

Then Lemma 5.1 implies that

$$\begin{aligned} & \int_{\Omega} (|\nabla u_*|^2 \text{Id} - 2\nabla u_* \otimes \nabla u_*) : DX \, dx + \int_{\hat{J}_{u_*}} (\text{Id} - \nu_{u_*} \otimes \nu_{u_*}) : DX \, d\mathcal{H}^{N-1} \\ &= - \int_{\partial\Omega} (X \cdot \nu_\Omega) \, d\lambda_* + \int_{\partial\Omega} |\nabla_\tau g|^2 (X \cdot \nu_\Omega) \, d\mathcal{H}^{N-1} - 2 \int_{\partial\Omega} (\nabla u_* \cdot \nu_\Omega) (X_\tau \cdot \nabla_\tau g) \, d\mathcal{H}^{N-1} \end{aligned}$$

for all  $X \in \mathcal{C}_c^1(\mathbf{R}^N; \mathbf{R}^N)$ . Specifying this identity to vector fields  $X \in \mathcal{C}_c^1(\mathbf{R}^N; \mathbf{R}^N)$  satisfying  $X \cdot \nu_\Omega = 0$  on  $\partial\Omega$  leads to

$$\begin{aligned} & \int_{\Omega} (|\nabla u_*|^2 \text{Id} - 2\nabla u_* \otimes \nabla u_*) : DX \, dx + \int_{\hat{J}_{u_*}} (\text{Id} - \nu_{u_*} \otimes \nu_{u_*}) : DX \, d\mathcal{H}^{N-1} \\ &= -2 \int_{\partial\Omega} (\nabla u_* \cdot \nu_\Omega) (X_\tau \cdot \nabla_\tau g) \, d\mathcal{H}^{N-1}, \end{aligned}$$

and (1.14) follows from the definition of the tangential divergence of  $X$  on the countably  $\mathcal{H}^{N-1}$ -rectifiable set  $\hat{J}_{u_*}$ .  $\square$

The results of this section also give the following convergences that will be used in Section 6.

**Corollary 5.1.** *Let  $(u_\varepsilon, v_\varepsilon) \in \mathcal{A}_g(\Omega)$ ,  $w_\varepsilon$  given by (5.2), and  $u_* \in SBV^2(\Omega) \cap L^\infty(\Omega)$  be as in Theorem 1.2. Then,*

$$\frac{\nabla w_\varepsilon}{|\nabla w_\varepsilon|} \otimes \frac{\nabla w_\varepsilon}{|\nabla w_\varepsilon|} |\nabla w_\varepsilon| \mathcal{L}^N \llcorner \Omega \xrightarrow{*} \nu_{u_*} \otimes \nu_{u_*} \mathcal{H}^{N-1} \llcorner \hat{J}_{u_*} \quad \text{weakly}^* \text{ in } \mathcal{M}(\bar{\Omega}; \mathbf{M}^{N \times N}), \quad (5.18)$$

$$\varepsilon \nabla v_\varepsilon \otimes \nabla v_\varepsilon \mathcal{L}^N \llcorner \Omega \xrightarrow{*} \frac{1}{2} \nu_{u_*} \otimes \nu_{u_*} \mathcal{H}^{N-1} \llcorner \hat{J}_{u_*} \quad \text{weakly}^* \text{ in } \mathcal{M}(\bar{\Omega}; \mathbf{M}^{N \times N}). \quad (5.19)$$

*Proof.* The first point (5.18) follows from (5.14) together with Lemmas 5.3 and 5.4 by observing that  $\nu_{u_*} = \pm \nu_\Omega$   $\mathcal{H}^{N-1}$ -a.e. in  $\partial\Omega \cap \hat{J}_{u_*}$ . The second point (5.19) follows from the first one, the equipartition of energy (5.4) and the fact that  $\frac{\nabla v_\varepsilon}{|\nabla v_\varepsilon|} = \frac{\nabla w_\varepsilon}{|\nabla w_\varepsilon|}$ .  $\square$



## 6. PASSING TO THE LIMIT IN THE SECOND INNER VARIATION

The aim of this section is to complement Theorem 1.2 proving also the convergence of the second inner variation of  $AT_\varepsilon$ . As a consequence, we shall deduce that if the limit  $u_*$  comes from stable critical points of  $AT_\varepsilon$ , then  $u_*$  satisfies a certain stability condition for  $MS$ . Our analysis and result parallels completely the ones in [24, 25, 26] for the Allen-Cahn type energies arising in phase transitions problems.

*Proof of Theorem 1.3.* Assume that  $\partial\Omega$  is of class  $\mathcal{C}^{3,1}$  and  $g \in \mathcal{C}^{3,\alpha}(\partial\Omega)$  for some  $\alpha \in (0, 1)$ . By Theorem 3.2, if  $(u_\varepsilon, v_\varepsilon)$  is a critical points of the Ambrosio-Tortorelli functional then it belongs to  $[\mathcal{C}^{3,\alpha}(\bar{\Omega})]^2$ .

To prove the convergence of the second inner variation, we use Lemma A.11 and formula (A.25). From Proposition 5.1, we know that

$$\begin{cases} \sqrt{\eta_\varepsilon + v_\varepsilon^2} \nabla u_\varepsilon \rightarrow \nabla u_* & \text{strongly in } L^2(\Omega; \mathbf{R}^N), \\ \varepsilon |\nabla v_\varepsilon|^2 \mathcal{L}^N \llcorner \Omega \xrightarrow{*} \frac{1}{2} \mathcal{H}^{N-1} \llcorner \hat{J}_{u_*} & \text{weakly* in } \mathcal{M}(\bar{\Omega}), \\ \frac{(v_\varepsilon - 1)^2}{4\varepsilon} \mathcal{L}^N \llcorner \Omega \xrightarrow{*} \frac{1}{2} \mathcal{H}^{N-1} \llcorner \hat{J}_{u_*} & \text{weakly* in } \mathcal{M}(\bar{\Omega}). \end{cases}$$

On the other hand, Corollary 5.1 ensures that

$$\varepsilon \nabla v_\varepsilon \otimes \nabla v_\varepsilon \mathcal{L}^N \llcorner \Omega \xrightarrow{*} \frac{1}{2} \nu_{u_*} \otimes \nu_{u_*} \mathcal{H}^{N-1} \llcorner \hat{J}_{u_*} \text{ weakly* in } \mathcal{M}(\bar{\Omega}; \mathbf{M}^{N \times N}).$$

Let  $X \in \mathcal{C}_c^2(\mathbf{R}^N; \mathbf{R}^N)$  and  $G \in \mathcal{C}^2(\mathbf{R}^N)$  be such that  $X \cdot \nu_\Omega = 0$  and  $G = g$  on  $\partial\Omega$ , and set  $Y := (DX)X$ . Observing that  $|DX^T \nabla v_\varepsilon|^2 = (DX(DX)^T) : (\nabla v_\varepsilon \otimes \nabla v_\varepsilon)$ , we can pass to the limit in all the terms of  $\delta^2 AT_\varepsilon(u_\varepsilon, v_\varepsilon)[X, G]$  in (A.25) to find that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \delta^2 AT_\varepsilon(u_\varepsilon, v_\varepsilon)[X, G] &= \int_{\Omega} (|\nabla u_*|^2 \text{Id} - 2(\nabla u_* \otimes \nabla u_*)) : DY \, dx + \int_{\hat{J}_{u_*}} \text{div}^{\hat{J}_{u_*}} Y \, d\mathcal{H}^{N-1} \\ &+ \int_{\Omega} |\nabla u_*|^2 ((\text{div} X)^2 - \text{tr}(DX)^2) - 4((\nabla u_* \otimes \nabla u_*) : DX) \text{div} X \, dx \\ &+ \int_{\Omega} [4(\nabla u_* \otimes \nabla u_*) : (DX)^2 + 2|DX^T \nabla u_*|^2] \, dx \\ &+ 4 \int_{\Omega} [\nabla u_* \cdot \nabla(X \cdot \nabla G) \text{div} X \\ &\quad - (\nabla u_* \otimes \nabla(X \cdot \nabla G)) : (DX + (DX)^T)] \, dx \\ &+ 2 \int_{\Omega} \nabla u_* \cdot \nabla(X \cdot \nabla(X \cdot G)) \, dx + 2 \int_{\Omega} |\nabla(X \cdot \nabla G)|^2 \, dx \\ &+ \int_{\hat{J}_{u_*}} [(\text{div} X)^2 - \text{tr}(DX)^2 - 2((\nu_{u_*} \otimes \nu_{u_*}) : DX) \text{div} X] \, d\mathcal{H}^{N-1} \\ &+ 2 \int_{\hat{J}_{u_*}} [(\nu_{u_*} \otimes \nu_{u_*}) : (DX)^2 + |DX^T \nu_{u_*}|^2] \, d\mathcal{H}^{N-1}. \end{aligned} \quad (6.1)$$

Using the geometric formulas stated in the proof of Theorem 1.1 p. 1851–1852 in [24], we infer that

$$\begin{aligned} &(\text{div} X)^2 - \text{tr}[(DX)^2] - 2((\nu_{u_*} \otimes \nu_{u_*}) : DX) \text{div} X + 2(\nu_{u_*} \otimes \nu_{u_*}) : (DX)^2 + |DX^T \nu_{u_*}|^2 \\ &= (\text{div}^{J_{u_*}} X)^2 + \sum_{i=1}^{N-1} |(\partial_{\tau_i} X)^\perp|^2 - \sum_{i,j=1}^{N-1} (\tau_i \cdot \partial_{\tau_j} X) (\tau_j \cdot \partial_{\tau_i} X) + ((\nu_{u_*} \otimes \nu_{u_*}) : DX)^2. \end{aligned} \quad (6.2)$$

According to the expression of the inner second variation of the Mumford-Shah energy stated in Lemma A.12, (6.1) and (6.2), we infer that

$$\lim_{\varepsilon \rightarrow 0} \delta^2 AT_\varepsilon(u_\varepsilon, v_\varepsilon)[X, G] = \delta^2 MS(u_*)[X, G] + \int_{\hat{J}_{u_*}} ((\nu_{u_*} \otimes \nu_{u_*}) : DX)^2 d\mathcal{H}^{N-1}.$$

Now assume that  $(u_\varepsilon, v_\varepsilon) \in \mathcal{A}_g(\Omega)$  is a stable critical point of  $AT_\varepsilon$ , i.e.

$$d^2 AT_\varepsilon(u_\varepsilon, v_\varepsilon)[\phi, \psi] \geq 0 \quad \text{for all } (\phi, \psi) \in H_0^1(\Omega) \times [H_0^1(\Omega) \cap L^\infty(\Omega)], \quad (6.3)$$

where  $d^2 AT_\varepsilon(u_\varepsilon, v_\varepsilon)$  is the second outer variation of  $AT_\varepsilon$  at  $(u_\varepsilon, v_\varepsilon)$  given by formula (A.19).

Let us fix an arbitrary vector field  $X \in \mathcal{C}_c^2(\mathbf{R}^N; \mathbf{R}^N)$  and an arbitrary function  $G \in \mathcal{C}^3(\mathbf{R}^N)$  satisfying  $X \cdot \nu_\Omega = 0$  and  $G = g$  on  $\partial\Omega$ . According to Lemma A.10, we have

$$\begin{aligned} \delta^2 AT_\varepsilon(u_\varepsilon, v_\varepsilon)[X, G] &= d^2 AT_\varepsilon(u_\varepsilon, v_\varepsilon)[X \cdot \nabla(u_\varepsilon - G), X \cdot \nabla v_\varepsilon] \\ &\quad + dAT_\varepsilon(u_\varepsilon, v_\varepsilon)[X \cdot \nabla(X \cdot \nabla(u_\varepsilon - G)), X \cdot \nabla(X \cdot \nabla v_\varepsilon)]. \end{aligned}$$

Since  $(u_\varepsilon, v_\varepsilon) = (g, 1)$  and  $X \cdot \nu_\Omega = 0$  on  $\partial\Omega$ , we have  $X \cdot \nabla(u_\varepsilon - G) = X \cdot \nabla v_\varepsilon = 0$  on  $\partial\Omega$ . As a consequence, the functions  $X \cdot \nabla(u_\varepsilon - G)$  and  $X \cdot \nabla v_\varepsilon$  belong to  $\mathcal{C}^2(\bar{\Omega})$  and also vanish on  $\partial\Omega$ . Therefore,

$$X \cdot \nabla(X \cdot \nabla(u_\varepsilon - G)) = X \cdot \nabla(X \cdot \nabla v_\varepsilon) = 0 \quad \text{on } \partial\Omega.$$

Next,  $(u_\varepsilon, v_\varepsilon)$  being a critical point of  $AT_\varepsilon$ , it follows that

$$dAT_\varepsilon(u_\varepsilon, v_\varepsilon)[X \cdot \nabla(X \cdot \nabla(u_\varepsilon - G)), X \cdot \nabla(X \cdot \nabla v_\varepsilon)] = 0.$$

Back to (6.3), it follows that

$$\delta^2 AT_\varepsilon(u_\varepsilon, v_\varepsilon)[X, G] = d^2 AT_\varepsilon(u_\varepsilon, v_\varepsilon)[X \cdot \nabla(u_\varepsilon - G), X \cdot \nabla v_\varepsilon] \geq 0.$$

Passing now to the limit in the second inner variation yields

$$\delta^2 MS(u_*)[X, G] + \int_{\hat{J}_{u_*}} ((\nu_{u_*} \otimes \nu_{u_*}) : DX)^2 d\mathcal{H}^{N-1} \geq 0,$$

and the proof of Theorem 1.3 is now complete.  $\square$

**Remark 6.1.** In [11, 6], the authors explore second order minimality conditions for the Mumford-Shah functional in the case where the jump set is regular enough. Such conditions could be derived in our context, taking care of the Dirichlet boundary data and thus of the fact that the jump set can charge the boundary. We do not develop this point here and refer to [11, Theorem 3.6] where the authors provide another expression for  $\delta^2 MS(u)$  defined for smooth vector fields  $X$  compactly supported in  $\Omega$  (see Remark A.3). But we indicate that, as a consequence of Theorem 1.3, it can be seen that, if  $(u_\varepsilon, v_\varepsilon) \in \mathcal{A}_g(\Omega)$  is a stable critical point of  $AT_\varepsilon$  such that, up to a subsequence,  $u_\varepsilon \rightarrow u_*$  in  $L^2(\Omega)$  and (1.13) hold, then  $u_*$  satisfies the second order minimality condition for the Mumford-Shah functional derived in [11, 6]) provided  $\hat{J}_{u_*}$  is sufficiently smooth. This follows by choosing  $X \in \mathcal{C}_c^\infty(\mathbf{R}^N; \mathbf{R}^N)$  of the form  $X = \varphi \nu_{u_*} \circ \Pi_{\hat{J}_{u_*}}$  in a neighborhood of  $\hat{J}_{u_*}$  and satisfying  $\nu_{u_*} \cdot (DX \nu_{u_*}) = 0$  on  $\hat{J}_{u_*}$ , where  $\Pi_{\hat{J}_{u_*}}$  denotes the nearest point projection onto  $\hat{J}_{u_*}$  and  $\varphi$  is an arbitrary smooth scalar function.

## 7. THE ONE-DIMENSIONAL CASE

The goal of this section is to investigate the one-dimensional case ( $N = 1$ ) and show the existence of non globally minimizing critical points  $(u_\varepsilon, v_\varepsilon)$  of the Ambrosio-Tororelli energy satisfying the energy convergence (1.13). Most of the arguments are borrowed from [13] (see also [23]). However, we cannot directly use their results because of our different boundary condition for the phase-field variable.

Let  $N = 1$ ,  $\Omega = (0, L)$  with  $L > 0$  and a Dirichlet boundary data  $g(0) = 0$  and  $g(L) = a > 0$ . As explained in Remark 1.1, in this setting, a function  $u \in SBV^2(0, L)$  is a critical point of the Mumford-Shah functional in  $SBV^2(0, L)$  if and only if  $u(x) = \frac{ax}{L}$  for  $x \in [0, L]$  or  $u$  is piecewise constant with

a finite number of jumps  $\widehat{J}_u = \{x_1, \dots, x_m\}$  with  $x_i \in [0, L]$  for all  $i = 1, \dots, m$ . In the first case the energy of  $u$  is

$$MS(u) = \int_0^L \left| \frac{a}{L} \right|^2 dx = \frac{a^2}{L},$$

while in the second case

$$MS(u) = \#(\widehat{J}_u \cap [0, L]) = m.$$

Thus, if  $a^2 < L$  then  $u(x) = ax/L$  is the global minimizer whereas if  $a^2 > L$  then a constant function with exactly one jump anywhere in the closed interval  $[0, L]$  is a global minimizer (if  $a^2 = L$  then all previous functions are global minimizers). In particular, we have

$$\min_{SBV^2(0, L)} MS = \min \left\{ \frac{a^2}{L}, 1 \right\}. \quad (7.1)$$

We now consider a family of critical points of  $AT_\varepsilon$ , i.e., a family  $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon > 0}$  in  $[H^1(0, L)]^2$  satisfying

$$\begin{cases} [(\eta_\varepsilon + v_\varepsilon^2)u'_\varepsilon]' = 0 & \text{in } (0, L), \\ -\varepsilon v''_\varepsilon + v_\varepsilon |u'_\varepsilon|^2 + \frac{v_\varepsilon - 1}{4\varepsilon} = 0 & \text{in } (0, L), \\ u_\varepsilon(0) = 0, \quad u_\varepsilon(L) = a, \\ v_\varepsilon(0) = v_\varepsilon(L) = 1. \end{cases} \quad (7.2)$$

We also assume the uniform energy bound

$$AT_\varepsilon(u_\varepsilon, v_\varepsilon) = \int_0^L (\eta_\varepsilon + v_\varepsilon^2) |u'_\varepsilon|^2 dx + \int_0^L \left( \varepsilon |v'_\varepsilon|^2 + \frac{(v_\varepsilon - 1)^2}{4\varepsilon} \right) dx \leq C. \quad (7.3)$$

The following result extends [13, Theorem 2.2] to the case of Dirichlet boundary conditions for the phase-field variable  $v$ . It also improves Theorem 1.2 in the one-dimensional case since assumption (1.13) on the convergence of energies is no longer required.

**Theorem 7.1.** *Let  $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon > 0} \subset [H^1(0, L)]^2$  be a family satisfying (7.2) and (7.3). Then, up to a subsequence  $\varepsilon \rightarrow 0$ ,*

(i)  $(u_\varepsilon, v_\varepsilon) \rightarrow (u_*, 1)$  in  $[L^2(0, L)]^2$  where  $u_* \in \{u_{\text{jump}}, u_{\text{aff}}\}$  with

$$u_{\text{jump}} = a \mathbf{1}_{[L/2, L]}, \quad u_{\text{aff}}(x) = \frac{ax}{L} \quad \text{for all } x \in [0, L]. \quad (7.4)$$

(ii)  $(\eta_\varepsilon + v_\varepsilon^2) |u'_\varepsilon|^2 \mathcal{L}^1 \llcorner (0, L) \xrightarrow{*} |u'_*|^2 \mathcal{L}^1 \llcorner (0, L)$  weakly\* in  $\mathcal{M}([0, L])$ ,

(iii)  $\left| \varepsilon |v'_\varepsilon|^2 - \frac{(1 - v_\varepsilon)^2}{4\varepsilon} \right| \rightarrow 0$  strongly in  $L^1(0, L)$ ,

(iv)  $\varepsilon |v'_\varepsilon|^2 \mathcal{L}^1 \llcorner (0, L) \xrightarrow{*} \alpha \delta_{\frac{L}{2}}$  weakly\* in  $\mathcal{M}([0, L])$ , with  $\alpha = 0$  or  $\alpha = 1/2$ . Moreover, if  $u_* = u_{\text{jump}}$ , then  $\alpha = 1/2$ .

**Remark 7.1.** We emphasize that we must have  $\alpha = 1/2$  for  $u_* = u_{\text{jump}}$ . However we are not able to prove that  $\alpha = 0$  for  $u_* = u_{\text{aff}}$ . Indeed  $\alpha = 1/2$  provided  $v_\varepsilon$  has a  $v$ -jump at  $L/2$  in the terminology of [13], i.e., as soon as  $v_\varepsilon(L/2) \leq \sqrt{C_* \varepsilon}$  for some constant  $C_* > 0$ . However it could happen that this  $v$ -jump disappears in the limit and does not create a discontinuity for  $u_*$ .

As in the case with Neumann boundary conditions we are in presence of a selection phenomenon. Critical points of  $AT_\varepsilon$  cannot approximate any critical points of  $MS$  but only specific ones. Here the selection phenomenon is much stronger in the sense that only two critical points  $u_{\text{jump}}$  and  $u_{\text{aff}}$  of  $MS$  can be reached as limits of critical points of  $AT_\varepsilon$ .

As in [13], we also show the existence of a family of critical points of  $AT_\varepsilon$  approximating  $u_{\text{jump}}$ .

**Theorem 7.2.** *There exists a family  $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0} \subset [H^1(0, L)]^2$  satisfying (7.2)-(7.3) and such that  $(u_\varepsilon, v_\varepsilon) \rightarrow (u_{\text{jump}}, 1)$  strongly in  $[L^2(0, L)]^2$  as  $\varepsilon \rightarrow 0$ .*

**Remark 7.2.** According to Theorem 7.1, we obtain that the family in Theorem 7.2 satisfies the energy convergence (1.13) i.e.  $AT_\varepsilon(u_\varepsilon, v_\varepsilon) \rightarrow MS(u_{\text{jump}})$ . Furthermore, if  $a^2 < L$ , then (7.1) ensures that  $u_{\text{jump}}$  is not a global minimizer of  $MS$ . This shows the existence of non minimizing critical points of  $AT_\varepsilon$  satisfying the assumption of convergence of energy in Theorem 1.2. This example can be made  $N$ -dimensional adding artificially  $(N - 1)$ -variables and considering periodic boundary conditions in those extra variables.

The rest of this section is devoted to the proofs of Theorem 7.1 and Theorem 7.2, reproducing some of the arguments in [13] for completeness.

**7.1. Preliminary estimates.** We start by using the first equation in (7.2) to find a constant  $c_\varepsilon \in \mathbf{R}$  such that

$$(\eta_\varepsilon + v_\varepsilon^2)u'_\varepsilon = c_\varepsilon, \quad (7.5)$$

which implies that  $u'_\varepsilon$  has a constant sign. Since we assume  $u_\varepsilon(0) = 0$  and  $u_\varepsilon(L) = a > 0$ , we deduce that  $u'_\varepsilon \geq 0$  and  $c_\varepsilon \geq 0$ . Then the second equation in (7.2) can be rewritten as

$$-\varepsilon v''_\varepsilon + \frac{c_\varepsilon^2 v_\varepsilon}{(\eta_\varepsilon + v_\varepsilon^2)^2} + \frac{v_\varepsilon - 1}{4\varepsilon} = 0. \quad (7.6)$$

We observe that, thanks to the energy bound (7.3),

$$ac_\varepsilon = \int_0^L u'_\varepsilon c_\varepsilon dx = \int_0^L (\eta_\varepsilon + v_\varepsilon^2)|u'_\varepsilon|^2 dx \leq AT_\varepsilon(u_\varepsilon, v_\varepsilon) \leq C,$$

hence  $\{c_\varepsilon\}_{\varepsilon>0}$  is bounded and, up to a subsequence, we can assume that

$$c_\varepsilon \rightarrow c_0. \quad (7.7)$$

As in [13, Lemma 3.2] (and using Proposition 4.1), we have the following result.

**Lemma 7.1.** *Let  $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0} \subset [H^1(0, L)]^2$  satisfying (7.2) and (7.3). Then, up to a subsequence,  $(u_\varepsilon, v_\varepsilon) \rightarrow (u_*, 1)$  strongly in  $[L^2(0, L)]^2$  with  $u_* \in SBV^2(0, L)$ . Furthermore,  $u'_\varepsilon \rightarrow c_0$  a.e. in  $(0, L)$ ,  $|Du_*|(0, L) \leq a$  and  $c_0 \leq a/L$ .*

In the one-dimensional setting, the Noether type conservation law of Proposition 4.2 reads as

$$\left( \frac{(1 - v_\varepsilon)^2}{4\varepsilon} - \varepsilon|v'_\varepsilon|^2 - (\eta_\varepsilon + v_\varepsilon^2)|u'_\varepsilon|^2 \right)' = 0 \text{ in } (0, L),$$

and it implies the existence of a constant  $d_\varepsilon \in \mathbf{R}$ , sometimes called discrepancy, such that

$$\frac{(1 - v_\varepsilon)^2}{4\varepsilon} - \varepsilon|v'_\varepsilon|^2 - (\eta_\varepsilon + v_\varepsilon^2)|u'_\varepsilon|^2 = d_\varepsilon. \quad (7.8)$$

Thanks to the energy bound (7.3), it is easy to see that  $\{d_\varepsilon\}_{\varepsilon>0}$  is a bounded sequence, and thus (up to a further subsequence)

$$d_\varepsilon \rightarrow d_0. \quad (7.9)$$

It also ensures the following uniform bounds (see [13, Lemma 3.4]).

**Lemma 7.2.** *For  $\varepsilon > 0$  small enough,*

$$\|u'_\varepsilon\|_{L^\infty(0, L)} \leq \frac{2}{\sqrt{\varepsilon\eta_\varepsilon}}, \quad \|v'_\varepsilon\|_{L^\infty(0, L)} \leq \frac{2}{\varepsilon}.$$

Moreover, if  $c_0 > 0$ , then the following refined estimates hold

$$\|u'_\varepsilon\|_{L^\infty(0, L)} \leq \frac{2}{c_0\varepsilon}, \quad \min_{[0, L]} v_\varepsilon \geq \frac{c_0\sqrt{\varepsilon}}{2}.$$

We next show the following strong maximum principle.

**Lemma 7.3.** *Let  $(u_\varepsilon, v_\varepsilon) \in [H^1(0, L)]^2$  satisfying (7.2), then  $0 < v_\varepsilon < 1$  in  $(0, L)$ .*

*Proof.* Let  $x_0$  be a minimum point of  $v_\varepsilon$  in  $[0, L]$ . If  $v_\varepsilon(x_0) = 1$ , using that  $0 \leq v_\varepsilon \leq 1$ , we deduce that  $v_\varepsilon \equiv 1$  in  $[0, L]$ . Inserting into the second equation of (7.2), we find that  $u_\varepsilon$  is a constant function in  $[0, L]$  which is in contradiction with  $u_\varepsilon(0) = 0$  and  $u_\varepsilon(L) = a > 0$ . As a consequence of the Dirichlet boundary condition for  $v_\varepsilon$ , we have  $x_0 \in (0, L)$  and thus  $v_\varepsilon''(x_0) \geq 0$ . If  $v_\varepsilon(x_0) = 0$ , using again the second equation in (7.2) we find that  $-\varepsilon v_\varepsilon''(x_0) = \frac{1}{4\varepsilon} > 0$  which is a contradiction. Therefore,  $v_\varepsilon \geq v_\varepsilon(x_0) > 0$  in  $(0, L)$ .

Likewise, let  $x_1$  be a maximum point of  $v_\varepsilon$  in  $[0, L]$ . If  $x_1 \in (0, L)$  and  $v_\varepsilon(x_1) = 1$ , then we use that  $v_\varepsilon''(x_1) \leq 0$  together with (7.6) to obtain that  $c_\varepsilon = 0$ . This implies by (7.5) that  $u'_\varepsilon = 0$  which is a contradiction since  $u_\varepsilon(0) = 0$  and  $u_\varepsilon(L) = a > 0$ . It shows again that  $v_\varepsilon < 1$  in  $(0, L)$ .  $\square$

The selection phenomenon already observed in [13] is due to the following symmetry property which is similar to [13, Lemma 4.1].

**Proposition 7.1.** *Let  $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0}$  be a family in  $[H^1(0, L)]^2$  satisfying (7.2) and (7.3). Then  $v_\varepsilon$  possesses a unique critical point in  $(0, L)$  located at  $L/2$ , which is a minimum of  $v_\varepsilon$  on  $[0, L]$ . Moreover,  $v_\varepsilon$  is decreasing in  $(0, L/2)$ , increasing in  $(L/2, L)$  and the graph of  $v_\varepsilon$  is symmetric with respect to the vertical line  $x = L/2$ .*

*Proof.* From Lemma 7.3,  $v_\varepsilon$  cannot be identically constant equal to 1. Thus by Rolle's theorem,  $v_\varepsilon$  admits critical points in  $(0, L)$ .

Let  $x_0 \in (0, L)$  be an arbitrary critical point of  $v_\varepsilon$  in  $(0, L)$ . If  $x_0 \in (0, L/2)$  then the function

$$\tilde{v}_\varepsilon(x) = \begin{cases} v_\varepsilon(x) & \text{if } x \in (0, x_0] \\ v_\varepsilon(2x_0 - x) & \text{if } x \in (x_0, 2x_0) \end{cases}$$

is a solution of (7.6) in the interval  $(0, 2x_0)$ . In particular,  $v_\varepsilon$  and  $\tilde{v}_\varepsilon$  are two solutions of an ODE of the form  $v_\varepsilon'' = f_\varepsilon(x, v_\varepsilon)$  in  $(x_0, 2x_0)$  for some function  $f_\varepsilon$  of class  $\mathcal{C}^2$  with  $v_\varepsilon(x_0) = \tilde{v}_\varepsilon(x_0)$  and  $v_\varepsilon'(x_0) = \tilde{v}_\varepsilon'(x_0) = 0$ . Cauchy-Lipschitz Theorem yields in turn that  $v_\varepsilon = \tilde{v}_\varepsilon$  in  $(x_0, 2x_0)$ . In particular,  $v_\varepsilon(2x_0) = v_\varepsilon(0) = 1$  which contradicts Lemma 7.3 since  $2x_0 \in (0, L)$ . Thus  $x_0 \in [L/2, L)$  and a symmetric argument shows that  $x_0 \in (0, L/2]$ . Finally, the only possibility left is  $x_0 = L/2$ .

In particular,  $v_\varepsilon$  admits a unique critical point in  $(0, L)$  at the point  $L/2$ , which must be a minimum of  $v_\varepsilon$  on  $[0, L]$ . Moreover, the graph of  $v_\varepsilon$  is symmetric with respect to the vertical line  $\{x = L/2\}$ . Since  $v_\varepsilon$  is a smooth function satisfying  $v_\varepsilon(0) = 1$ ,  $v_\varepsilon(L/2) < 1$  and  $v'_\varepsilon \neq 0$  in  $(0, L/2)$ , we deduce that  $v_\varepsilon$  is decreasing in  $(0, L/2)$ . By symmetry  $v_\varepsilon$  is increasing in  $(L/2, L)$ .  $\square$

A crucial step in the proof of Theorem 7.1 is the following characterization of possible limiting slopes  $c_0$  in (7.7), which strongly rests on the symmetry property of  $v_\varepsilon$ . We refer to [13, Lemma 4.4] for the proof.

**Lemma 7.4.** *The limiting slope  $c_0$  in (7.7) satisfies that either  $c_0 = 0$  or  $c_0 = a/L$ .*

Using that  $u'_\varepsilon = \frac{c_\varepsilon}{\eta_\varepsilon + v_\varepsilon^2}$  in (7.8), we find that

$$\frac{(1 - v_\varepsilon)^2}{4\varepsilon} - \varepsilon |v'_\varepsilon|^2 - \frac{c_\varepsilon^2}{\eta_\varepsilon + v_\varepsilon^2} = d_\varepsilon.$$

Thus, since  $v_\varepsilon \leq 1$  and  $c_\varepsilon$  and  $d_\varepsilon$  are bounded, we obtain that, for some constant  $C_* > 0$  independent of  $\varepsilon$ ,

$$v_\varepsilon^2(L/2) \left(1 - v_\varepsilon(L/2)\right)^2 \leq \left(\eta_\varepsilon + v_\varepsilon^2(L/2)\right) \left(1 - v_\varepsilon(L/2)\right)^2 \leq C_* \varepsilon.$$

This implies, thanks to the study of the function  $X \mapsto X^2(1-X)^2$  on  $[0, 1]$ , that

$$\text{either } v_\varepsilon\left(\frac{L}{2}\right) \geq 1 - \sqrt{C_*\varepsilon} \quad \text{or} \quad v_\varepsilon\left(\frac{L}{2}\right) \leq \sqrt{C_*\varepsilon}.$$

In the latter case,  $L/2$  corresponds to a so-called  $v$ -jump according to the terminology of [13]. The previous dichotomy implies that either  $v_\varepsilon$  converges uniformly to 1 or there exists exactly one  $v$ -jump which is a minimum of  $v_\varepsilon$  located at  $L/2$ .

**7.2. Proof of Theorem 7.1.** We are now ready to prove items i) and ii) of Theorem 7.1, i.e., the selection principle for limit of critical points of  $AT_\varepsilon$  and the convergence of the bulk energy.

*Proof of i) and ii) in Theorem 7.1. Step 1.* Assume first that  $v_\varepsilon(L/2) \geq 1 - \sqrt{C_*\varepsilon}$ . Then we have that  $0 \leq 1 - v_\varepsilon \leq \sqrt{C_*\varepsilon} \rightarrow 0$  uniformly in  $[0, L]$ . For  $\varepsilon$  small enough we also have that  $v_\varepsilon \geq 1/2$  so that the energy bound (7.3) yields

$$\frac{1}{2} \int_0^L |u'_\varepsilon|^2 dx \leq \int_0^L (\eta_\varepsilon + v_\varepsilon^2) |u'_\varepsilon|^2 dx \leq C.$$

Since  $\{u_\varepsilon\}_{\varepsilon>0}$  is bounded in  $H^1(0, L)$  up to a subsequence we have that  $u_\varepsilon \rightharpoonup u_*$  weakly in  $H^1(0, L)$  with  $u_* \in H^1(0, L)$ . We can pass to the limit in the first equation of (7.5) using (7.7) to obtain that  $u'_* = c_0$  in  $(0, L)$ . Since  $u_*(0) = 0$  and  $u_*(L) = a$  we find that  $u_* = u_{\text{aff}}$  and  $c_0 = a/L$ . Moreover, thanks to the uniform convergence of  $v_\varepsilon$ ,

$$\lim_{\varepsilon \rightarrow 0} \int_0^L |u'_\varepsilon|^2 dx = \lim_{\varepsilon \rightarrow 0} \int_0^L (\eta_\varepsilon + v_\varepsilon^2) |u'_\varepsilon|^2 dx = \lim_{\varepsilon \rightarrow 0} c_\varepsilon \int_0^L u'_\varepsilon dx = c_0 \int_0^L u'_{\text{aff}} dx = \int_0^L |u'_{\text{aff}}|^2 dx.$$

It yields  $u_\varepsilon \rightarrow u_{\text{aff}}$  strongly in  $H^1(0, L)$  and, in particular  $(\eta_\varepsilon + v_\varepsilon^2) |u'_\varepsilon|^2 \rightarrow |u'_{\text{aff}}|^2$  strongly in  $L^1(0, L)$ , hence also weakly\* in  $\mathcal{M}([0, L])$ .

*Step 2.* Assume now that  $v_\varepsilon(L/2) < \sqrt{C_*\varepsilon}$ . We first notice that  $u_\varepsilon(L/2) = a/2$ . Indeed, thanks to the symmetry property of  $v_\varepsilon$  and a change of variables, we find that

$$a = \int_0^L u'_\varepsilon dx = \int_0^L \frac{c_\varepsilon}{\eta_\varepsilon + v_\varepsilon^2} dx = 2 \int_0^{L/2} \frac{c_\varepsilon}{\eta_\varepsilon + v_\varepsilon^2} dx = 2u_\varepsilon\left(\frac{L}{2}\right). \quad (7.10)$$

We next claim that for each  $0 < \delta < L/2$ , the function  $v_\varepsilon \rightarrow 1$  uniformly on  $[0, \delta]$ . To this purpose, define  $A_\varepsilon := \{x \in (0, L) : v_\varepsilon(x) \leq 1 - \varepsilon^{1/4}\}$ . By the monotonicity properties of  $v_\varepsilon$ , the set  $A_\varepsilon$  is a closed interval centered in  $L/2$ . Thanks to the energy bound (7.3),

$$C \geq \int_0^L \frac{(1 - v_\varepsilon)^2}{\varepsilon} dx \geq \frac{\mathcal{L}^1(A_\varepsilon)}{\varepsilon^{1/2}},$$

which implies that

$$\text{diam}(A_\varepsilon) \leq C\varepsilon^{1/2} < \frac{L}{2} - \delta$$

for  $\varepsilon$  small enough. Hence  $A_\varepsilon \cap [0, \delta] = \emptyset$  for  $\varepsilon$  small. In particular  $v_\varepsilon \rightarrow 1$  uniformly on  $[0, \delta]$ , and then  $u'_\varepsilon = \frac{c_\varepsilon}{\eta_\varepsilon + v_\varepsilon^2} \rightarrow c_0$  uniformly on  $[0, \delta]$ . We deduce that

$$u_\varepsilon(x) = \int_0^x u'_\varepsilon(t) dt \rightarrow c_0 x \text{ uniformly with respect to } x \in [0, \delta].$$

Thus  $u_\varepsilon(x) \rightarrow c_0 x$  for a.e.  $x \in (0, \frac{L}{2})$ , and we prove in the same way that  $u_\varepsilon(x) \rightarrow a - c_0(L - x)$  for a.e.  $x \in (\frac{L}{2}, L)$ . Since  $c_0 = 0$  or  $c_0 = a/L$  by Lemma 7.4, then we find that either  $u_* = u_{\text{jump}}$  or  $u_* = u_{\text{aff}}$  (see (7.4)). Observe that the case  $u_* = u_{\text{jump}}$  only occurs in the case  $v_\varepsilon(L/2) \leq \sqrt{C_*\varepsilon}$ .

We finally show the convergence of the bulk energy. From the first equation in (7.2) we can write that  $(\eta_\varepsilon + v_\varepsilon^2)|u'_\varepsilon|^2 = c_\varepsilon u'_\varepsilon$ . Thus, for all  $\varphi \in \mathcal{C}_c^\infty(\mathbf{R})$ ,

$$\int_0^L (\eta_\varepsilon + v_\varepsilon^2)|u'_\varepsilon|^2 \varphi \, dx = c_\varepsilon \int_0^L u'_\varepsilon \varphi \, dx = -c_\varepsilon \int_0^L u_\varepsilon \varphi' \, dx + c_\varepsilon a \varphi(L) \rightarrow -c_0 \int_0^L u_* \varphi' \, dx + c_0 a \varphi(L).$$

If  $c_0 = 0$ , then  $u_* = u_{\text{jump}}$  and  $u'_{\text{jump}} = 0$ . We thus get in that case,

$$-c_0 \int_0^L u_{\text{jump}} \varphi' \, dx + c_0 a \varphi(L) = 0 = \int_0^L |u'_{\text{jump}}|^2 \varphi \, dx.$$

If  $c_0 = a/L$ , then  $u_* = u_{\text{aff}}$  and thus

$$-c_0 \int_0^L u_{\text{aff}} \varphi' \, dx + c_0 a \varphi(L) = \frac{a}{L} \int_0^L u'_{\text{aff}} \varphi \, dx = \int_0^L |u'_{\text{aff}}|^2 \varphi \, dx.$$

In any case, we obtain

$$\int_0^L (\eta_\varepsilon + v_\varepsilon^2)|u'_\varepsilon|^2 \varphi \, dx \rightarrow \int_0^L |u'_*|^2 \varphi \, dx,$$

which proves the announced items i) and ii).  $\square$

From now on, the function  $u$  stands for either  $u_{\text{aff}}$  or  $u_{\text{jump}}$ . The argument in the previous proof actually shows that  $u = u_{\text{aff}}$  if  $c_0 = a/L$ , while  $u = u_{\text{jump}}$  if  $c_0 = 0$ .

At this stage, it remains to show the equipartition of energy and the convergence of the diffuse surface energy (points iii) and iv) in Theorem 7.1). The key argument is the following result stating that there is very few diffuse surface energy far way from  $L/2$ , the only possible limit jump point. The proof is an adaptation of [13, Lemma 6.1].

**Lemma 7.5.** *For every compact set  $K \subset [0, L] \setminus \{L/2\}$ , there exists a constant  $C_K > 0$  such that*

$$\int_K \left( \varepsilon |v'_\varepsilon|^2 + \frac{(1 - v_\varepsilon)^2}{4\varepsilon} \right) dx \leq C_K \varepsilon^{1/4}.$$

*Proof.* We already know that the set  $A_\varepsilon := \{x \in (0, L) : v_\varepsilon(x) \leq 1 - \varepsilon^{1/4}\}$  is a closed interval centred at  $L/2$  with  $\text{diam}(A_\varepsilon) \leq C\sqrt{\varepsilon}$ . Let  $\delta < \frac{1}{2} \text{dist}(L/2, K)$ . If  $\varepsilon > 0$  is small enough then  $A_\varepsilon \subset [L/2 - \delta, L/2 + \delta]$ , hence  $K \cap A_\varepsilon = \emptyset$ . Since  $K \subset V_\delta := [0, L] \setminus [L/2 - \delta, L/2 + \delta]$ , it suffices to show that

$$\int_{V_\delta} \left( \varepsilon |v'_\varepsilon|^2 + \frac{(1 - v_\varepsilon)^2}{4\varepsilon} \right) dx \leq C_K \varepsilon^{1/4}.$$

We multiply the second equation in (7.2) by  $v_\varepsilon - 1 \in H_0^1(0, L)$  and we integrate by parts to obtain

$$\begin{aligned} \int_{V_\delta} \left( \varepsilon |v'_\varepsilon|^2 + \frac{(v_\varepsilon - 1)^2}{4\varepsilon} \right) dx &= \varepsilon v'_\varepsilon(L/2 - \delta)(v_\varepsilon(L/2 - \delta) - 1) \\ &\quad - \varepsilon v'_\varepsilon(L/2 + \delta)(v_\varepsilon(L/2 + \delta) - 1) + \int_{V_\delta} \frac{c_\varepsilon^2 v_\varepsilon (1 - v_\varepsilon)}{(\eta_\varepsilon + v_\varepsilon^2)^2} dx. \end{aligned}$$

By definition of  $A_\varepsilon$  and  $V_\delta$ , we have  $|1 - v_\varepsilon| \leq \varepsilon^{1/4}$  on  $V_\delta$ . Using further the gradient bound for  $v_\varepsilon$  in Lemma 7.2, we find that

$$\int_{V_\delta} \left( \varepsilon |v'_\varepsilon|^2 + \frac{(v_\varepsilon - 1)^2}{4\varepsilon} \right) dx \leq C_K \varepsilon^{1/4},$$

which completes the proof of the lemma.  $\square$

Arguing as in [13, Lemma 6.3], we also have the following result which relates the limit slope  $c_0$  to the limit  $d_0$  (respectively defined in (7.7) and (7.9)).

**Lemma 7.6.** *The real numbers  $c_0$  and  $d_0$  satisfy  $d_0 + c_0^2 = 0$ .*

We are now in position to complete the proof of Theorem 7.1.

*Proof of iii) and iv) in Theorem 7.1. Step 1.* Let us consider the function

$$f_\varepsilon := \frac{(v_\varepsilon - 1)^2}{4\varepsilon} - \varepsilon|v'_\varepsilon|^2 = d_\varepsilon + (\eta_\varepsilon + v_\varepsilon^2)|u'_\varepsilon|^2 = d_\varepsilon + c_\varepsilon u'_\varepsilon = d_\varepsilon + \frac{c_\varepsilon^2}{\eta_\varepsilon + v_\varepsilon^2}.$$

If  $u_* = u_{\text{jump}}$ , then

$$\int_0^L (\eta_\varepsilon + v_\varepsilon^2)|u'_\varepsilon|^2 dx \rightarrow 0,$$

hence  $c_\varepsilon a = c_\varepsilon \int_0^L u'_\varepsilon dx = \int_0^L (\eta_\varepsilon + v_\varepsilon^2)|u'_\varepsilon|^2 dx \rightarrow 0$ . It shows that  $c_0 = 0$  and thus  $d_0 = 0$  owing to Lemma 7.6. We thus infer that

$$\int_0^L \left| \frac{(v_\varepsilon - 1)^2}{4\varepsilon} - \varepsilon|v'_\varepsilon|^2 \right| dx \rightarrow 0.$$

Assume next that  $u_* = u_{\text{aff}}$ . In that case, we have

$$\int_0^L (\eta_\varepsilon + v_\varepsilon^2)|u'_\varepsilon|^2 dx \not\rightarrow 0,$$

and the same argument as before shows that  $c_0 \neq 0$ . Then  $d_0 \neq 0$  by Lemma 7.6. In particular  $d_\varepsilon \neq 0$  for  $\varepsilon > 0$  small enough. The function  $f_\varepsilon$  reaches its maximum when  $v_\varepsilon$  is minimal, i.e., at the point  $L/2$ . Since  $L/2$  is a critical point of  $v_\varepsilon$ , we have

$$\max_{[0,L]} f_\varepsilon = f_\varepsilon(L/2) = \frac{(v_\varepsilon(L/2) - 1)^2}{4\varepsilon} \geq 0.$$

Similarly,  $f_\varepsilon$  reaches its minimum when  $v_\varepsilon$  attains its maximum. Since  $v_\varepsilon$  is maximal on the boundary with  $v_\varepsilon(0) = v_\varepsilon(L) = 1$  we find that

$$\min_{[0,L]} f_\varepsilon = f_\varepsilon(0) = -\varepsilon|v'_\varepsilon(0)|^2 \leq 0.$$

As a consequence, there exists  $s_\varepsilon \in (0, L/2)$  such that  $f_\varepsilon(s_\varepsilon) = 0$ . From Lemma 7.6, it follows that

$$v_\varepsilon^2(s_\varepsilon) = -\eta_\varepsilon - \frac{c_\varepsilon^2}{d_\varepsilon} \rightarrow -\frac{c_0^2}{d_0} = 1.$$

Up to a subsequence, there exists  $s_0 \in [0, L/2]$  such that  $s_\varepsilon \rightarrow s_0$ . By monotonicity of  $v_\varepsilon$ , we get that  $v_\varepsilon \mathbf{1}_{[0,s_\varepsilon]} \rightarrow \mathbf{1}_{[0,s_0]}$  for a.e.  $s \in (0, L/2)$ . Hence using again Lemma 7.6,

$$d_\varepsilon s_\varepsilon + c_\varepsilon u_\varepsilon(s_\varepsilon) = \int_0^{s_\varepsilon} \left( d_\varepsilon + \frac{c_\varepsilon^2}{\eta_\varepsilon + v_\varepsilon^2} \right) dx \rightarrow \int_0^{s_0} (d_0 + c_0^2) dx = 0. \quad (7.11)$$

Using the symmetry of  $v_\varepsilon$ , (7.10) and (7.11), we compute

$$\begin{aligned} \int_0^L \left| \frac{(v_\varepsilon - 1)^2}{4\varepsilon} - \varepsilon|v'_\varepsilon|^2 \right| dx &= \int_0^L |f_\varepsilon| dx = 2 \int_0^{L/2} |f_\varepsilon| dx \\ &= -2 \int_0^{s_\varepsilon} f_\varepsilon dx + 2 \int_{s_\varepsilon}^{L/2} f_\varepsilon dx \\ &= 2d_\varepsilon \left( \frac{L}{2} - 2s_\varepsilon \right) + 2c_\varepsilon \left( u_\varepsilon \left( \frac{L}{2} \right) - 2u_\varepsilon(s_\varepsilon) \right) \\ &\rightarrow Ld_0 + c_0 a = 0. \end{aligned}$$

It completes the proof of the equipartition of energy.



*Step 2.* We finally show the convergence of the diffuse surface energy. According to Lemma 7.5, we have

$$\left( \varepsilon |v'_\varepsilon|^2 + \frac{(v_\varepsilon - 1)^2}{4\varepsilon} \right) \mathcal{L}^1 \llcorner (0, L) \xrightarrow{*} \mu \quad \text{weakly* in } \mathcal{M}([0, L]),$$

for some nonnegative measure  $\mu \in \mathcal{M}([0, L])$  supported on  $\{L/2\}$ , and thus of the form  $\mu = c\delta_{L/2}$  with  $c \geq 0$ . On the one hand, since  $\mu$  is concentrated at  $L/2$ , we have

$$c = \mu((0, L)) = \lim_{\varepsilon \rightarrow 0} \int_0^L \left( \varepsilon |v'_\varepsilon|^2 + \frac{(v_\varepsilon - 1)^2}{4\varepsilon} \right) dx.$$

On the other hand, the equipartition of energy ensures that

$$\begin{aligned} \int_0^L \left( \varepsilon |v'_\varepsilon|^2 + \frac{(v_\varepsilon - 1)^2}{4\varepsilon} - |v'_\varepsilon|(1 - v_\varepsilon) \right) dx &= \int_0^L \left( \sqrt{\varepsilon} |v'_\varepsilon| - \frac{1 - v_\varepsilon}{2\sqrt{\varepsilon}} \right)^2 dx \\ &\leq \int_0^L \left| \varepsilon |v'_\varepsilon|^2 - \frac{|1 - v_\varepsilon|^2}{4\varepsilon} \right| dx \rightarrow 0. \end{aligned}$$

Since  $v'_\varepsilon \leq 0$  on  $(0, L/2)$  and  $v'_\varepsilon \geq 0$  on  $(L/2, L)$ , by symmetry of  $v_\varepsilon$  with respect to the vertical axis  $\{x = L/2\}$ , we have

$$\int_0^L |v'_\varepsilon|(1 - v_\varepsilon) dx = -2 \int_0^{L/2} v'_\varepsilon(1 - v_\varepsilon) dx = (1 - v_\varepsilon(L/2))^2.$$

If  $v_\varepsilon(L/2) \leq \sqrt{C_*\varepsilon}$ , then  $c = 1$  and  $\mu = \delta_{L/2}$ , while if  $v_\varepsilon(L/2) \geq 1 - \sqrt{C_*\varepsilon}$ , then  $c = 0$  and  $\mu = 0$ . Using again the equipartition of energy, we infer that

$$\varepsilon |v'_\varepsilon|^2 \mathcal{L}^1 \llcorner (0, L) \xrightarrow{*} \frac{c}{2} \delta_{L/2} \quad \text{weakly* in } \mathcal{M}([0, L]),$$

so that the desired convergence holds with  $\alpha = \frac{c}{2} \in \{0, \frac{1}{2}\}$ . If  $u_* = u_{\text{jump}}$ , then we must have  $v_\varepsilon(L/2) \leq \sqrt{C_*\varepsilon}$ , and it follows that  $\alpha = 1/2$  in that case.  $\square$

**7.3. Proof of Theorem 7.2.** This section is devoted to prove Theorem 7.2, following again arguments similar to those of [13, Section 5]. By the symmetry properties of Theorem 7.1 it suffices to construct a critical point  $(u_\varepsilon, v_\varepsilon)$  of  $AT_\varepsilon$  in  $(0, L/2)$  such that  $v_\varepsilon(0) = 1$ ,  $v'_\varepsilon(L/2) = 0$ ,  $u_\varepsilon(0) = 0$  and  $u_\varepsilon(L/2) = a/2$ .

To this aim, let  $\alpha \in (0, 1)$  independent of  $\varepsilon$ , and set

$$\mathcal{V} := \left\{ v \in H^1(0, L/2) : v(0) = 1, v\left(\frac{L}{2}\right) \leq \alpha \right\},$$

$$\mathcal{B} := \left\{ (u, v) \in H^1(0, L/2) \times \mathcal{V} : u(0) = 0, u\left(\frac{L}{2}\right) = \frac{a}{2} \right\}.$$

For  $(u, v) \in \mathcal{B}$  and  $0 \leq r \leq s \leq L/2$ , we define the localized bulk and diffuse surface energies by

$$\mathcal{E}_\varepsilon(u, v; r, s) = \int_r^s (\eta_\varepsilon + v^2) |u'|^2 dx, \quad \mathcal{F}_\varepsilon(v; r, s) := \int_r^s \left( \varepsilon |v'|^2 + \frac{(1 - v)^2}{4\varepsilon} \right) dx,$$

and the Ambrosio-Tortorelli energy localized on  $(0, L/2)$  by

$$\widetilde{AT}_\varepsilon(u, v) := \mathcal{E}_\varepsilon(u, v; 0, L/2) + \mathcal{F}_\varepsilon(v; 0, L/2).$$

The following result has been established in [13, Section 5].

**Lemma 7.7.** *For all  $x_1, x_2$  and  $x_3 \in (0, L/2)$  we have*

$$\mathcal{F}_\varepsilon(v; x_1, x_3) \geq |\Phi(v(x_1)) + \Phi(v(x_3)) - 2\Phi(v(x_2))|,$$

with  $\Phi(t) = t - t^2/2$ .

Using the monotone increasing character of  $\Phi$  on  $[0, 1]$  and choosing  $x_1 = x_2 = 0$  and  $x_3 = L/2$ , we get for all  $v \in \mathcal{V}$

$$\mathcal{F}_\varepsilon \left( v; 0, \frac{L}{2} \right) \geq \Phi(1) - \Phi \left( v \left( \frac{L}{2} \right) \right) \geq \frac{1}{2} - \Phi(\alpha). \quad (7.12)$$

Moreover, arguing as in [13, Section 5], we can show the existence of a minimizer  $(u_\varepsilon, v_\varepsilon)$  over  $\mathcal{B}$  of  $\widetilde{AT}_\varepsilon$  such that

$$\limsup_{\varepsilon \rightarrow 0} \widetilde{AT}_\varepsilon(u_\varepsilon, v_\varepsilon) \leq \frac{1}{2}. \quad (7.13)$$

We will show that, for a convenient choice of  $\alpha \in (0, 1)$ , such a minimizer is a critical point of  $\widetilde{AT}_\varepsilon$  with the desired boundary conditions. The proof of the following result is similar to that of [13, Lemma 5.1 & Proposition 5.2].

**Lemma 7.8.** *There exists  $\alpha \in (0, 1)$  independent of  $\varepsilon > 0$  such that if  $(u_\varepsilon, v_\varepsilon)$  is a minimizer over  $\mathcal{B}$  of  $\widetilde{AT}_\varepsilon$ , then it is a critical point of  $\widetilde{AT}_\varepsilon$  with  $v'_\varepsilon(L/2) = 0$  for  $\varepsilon$  small enough.*

*Proof.* It is sufficient to show the existence of  $\alpha \in (0, 1)$ , independent of  $\varepsilon$ , such that if  $(u_\varepsilon, v_\varepsilon)$  is a minimizer over  $\mathcal{B}$  of  $\widetilde{AT}_\varepsilon$ , then  $v_\varepsilon(L/2) < \alpha$ . Indeed, in that case the minimizer  $(u_\varepsilon, v_\varepsilon)$  belongs to the interior of  $\mathcal{B}$  and variations of the form  $(u_\varepsilon + t\phi, v_\varepsilon + t\psi)$  with  $\phi \in \mathcal{C}_c^\infty((0, L/2))$  and  $\psi \in \mathcal{C}_c^\infty((0, L/2])$  are allowed. Let  $\alpha \in (0, 1)$  small enough so that

$$\frac{a^2}{L} - 2\Phi(\alpha) > 0. \quad (7.14)$$

Assume by contradiction that there exists  $\varepsilon_j \rightarrow 0$  such that  $v_{\varepsilon_j}(L/2) = \alpha$  for all  $j \in \mathbf{N}$ . Then consider the sequence

$$\alpha_j^* := \min_{x \in [0, L/2]} v_{\varepsilon_j}(x) \leq \alpha.$$

Applying Lemma 7.7 with  $x_1 = 0, x_3 = L/2$  and  $x_2 = y_j$  where  $v_{\varepsilon_j}(y_j) = \alpha_j^*$  leads to

$$\widetilde{AT}_{\varepsilon_j}(u_{\varepsilon_j}, v_{\varepsilon_j}) \geq \frac{1}{2} + \Phi(\alpha) - 2\Phi(\alpha_j^*).$$

We claim that for all  $j \in \mathbf{N}$ ,

$$\Phi(\alpha) - 2\Phi(\alpha_j^*) \geq \frac{\Phi(\alpha)}{2} > 0. \quad (7.15)$$

Provided the claim is proved, we infer from (7.13) that

$$\frac{1}{2} \geq \limsup_{j \rightarrow \infty} \widetilde{AT}_{\varepsilon_j}(u_{\varepsilon_j}, v_{\varepsilon_j}) \geq \frac{1}{2} + \frac{\Phi(\alpha)}{2},$$

which is a contradiction since  $\Phi(\alpha) > 0$ . We are now reduced to show (7.15).

*Proof of the Claim.* Let  $\alpha_1 \in (0, \alpha)$  be such that  $\Phi(\alpha) \geq 4\Phi(\alpha_1)$  and assume by contradiction that  $v_{\varepsilon_j}(x) \geq \alpha_1$  for all  $x \in [0, L/2]$ . Using variations with compact support in  $(0, L/2)$ , we get that  $(u_{\varepsilon_j}, v_{\varepsilon_j})$  solves

$$\begin{cases} \left( (\eta_{\varepsilon_j} + v_{\varepsilon_j}^2) u'_{\varepsilon_j} \right)' = 0 & \text{in } (0, L/2), \\ -\varepsilon_j v''_{\varepsilon_j} + v_{\varepsilon_j} |u'_{\varepsilon_j}|^2 + \frac{v_{\varepsilon_j} - 1}{\varepsilon_j} = 0 & \text{in } (0, L/2). \end{cases}$$

From the first equation we obtain that  $u'_{\varepsilon_j}(\eta_{\varepsilon_j} + v_{\varepsilon_j}^2) = c_j$  a.e. in  $(0, L/2)$ , for some constant  $c_j \in \mathbf{R}$ . The upper bound (7.13) shows that  $v_{\varepsilon_j} \rightarrow 1$  in  $L^2(0, L/2)$  and

$$\widetilde{AT}_{\varepsilon_j}(u_{\varepsilon_j}, v_{\varepsilon_j}) \geq \int_0^{L/2} (\eta_{\varepsilon_j} + v_{\varepsilon_j}^2) |u'_{\varepsilon_j}|^2 dx = \int_0^{L/2} c_j u'_{\varepsilon_j} dx = \frac{ac_j}{2}.$$

It implies that  $\{c_j\}_{j \in \mathbf{N}}$  is bounded so that, up to a subsequence,  $c_j \rightarrow c_0$  for some  $c_0 \in \mathbf{R}$ . Since  $v_{\varepsilon_j} \geq \alpha_1$ , we deduce that  $\{u'_{\varepsilon_j}\}_{j \in \mathbf{N}}$  is bounded in  $L^\infty(0, L/2)$ . Then Lebesgue's dominated convergence yields

$$\frac{a}{2} = \int_0^{L/2} u'_{\varepsilon_j} dx = \int_0^{L/2} \frac{c_j}{\eta_{\varepsilon_j} + v_{\varepsilon_j}^2} dx \rightarrow \int_0^{L/2} c_0 dx,$$

so that  $c_0 = a/L$ . Now,

$$\liminf_{j \rightarrow \infty} \int_0^{L/2} (\eta_{\varepsilon_j} + v_{\varepsilon_j}^2) |u'_{\varepsilon_j}|^2 dx = \liminf_{j \rightarrow \infty} \int_0^{L/2} c_j u'_{\varepsilon_j} dx = c_0 a = \frac{a^2}{L},$$

and thus, by (7.12) and (7.13),

$$\frac{1}{2} \geq \limsup_{j \rightarrow \infty} \widetilde{AT}_{\varepsilon_j}(u_{\varepsilon_j}, v_{\varepsilon_j}) \geq \frac{a^2}{L} + \frac{1}{2} - \Phi(\alpha),$$

which is in contradiction with (7.14).

We have thus proved by contradiction that  $\min_{[0, L/2]} v_{\varepsilon_j} = \alpha_j^* \leq \alpha_1$ . Since we assumed that  $\Phi(\alpha) \geq 4\Phi(\alpha_1) \geq 4\Phi(\alpha_j^*)$ , we infer that (7.15) is satisfied.  $\square$

We can now conclude the proof of Theorem 7.2.

*Proof of Theorem 7.2.* Let  $(u_\varepsilon, v_\varepsilon) \in \mathcal{B}$  and  $\alpha \in (0, 1)$  given by Lemma 7.8 (see (7.14)) so that  $(u_\varepsilon, v_\varepsilon)$  satisfies

$$\begin{cases} \left( (\eta_\varepsilon + v_\varepsilon^2) u'_\varepsilon \right)' = 0 & \text{in } (0, L/2), \\ -\varepsilon v''_\varepsilon + v_\varepsilon |u'_\varepsilon|^2 + \frac{v_\varepsilon - 1}{4\varepsilon} = 0 & \text{in } (0, L/2), \\ u_\varepsilon(0) = 0, \quad u_\varepsilon(L/2) = a/2, \\ v_\varepsilon(0) = 0, \quad v'_\varepsilon(L/2) = 0. \end{cases}$$

By the first equation, there exists  $c_\varepsilon \in \mathbf{R}$  such that  $u'_\varepsilon = \frac{c_\varepsilon}{\eta_\varepsilon + v_\varepsilon^2}$ . Extending  $v_\varepsilon$  to  $(0, L)$  by symmetry with respect to the vertical axis  $\{x = L/2\}$ , we obtain a function (still denoted by  $v_\varepsilon$ ) which belongs to  $H^1(0, L)$  with  $v_\varepsilon(0) = v_\varepsilon(L) = 1$  (this reflexion argument is possible since  $v'_\varepsilon(L/2) = 0$ ). Note that the boundary conditions satisfied by  $u_\varepsilon$  implies

$$c_\varepsilon = \frac{a}{2} \left( \int_0^{L/2} \frac{dx}{\eta_\varepsilon + v_\varepsilon^2} \right)^{-1} = a \left( \int_0^L \frac{dx}{\eta_\varepsilon + v_\varepsilon^2} \right)^{-1},$$

where the last equality holds because  $v_\varepsilon$  is symmetric with respect to the vertical axis  $\{x = L/2\}$ . The function  $u_\varepsilon$  is extended to  $(0, L)$  (into a function still denoted by  $u_\varepsilon$ ) by setting

$$u_\varepsilon(x) = \int_0^x \frac{c_\varepsilon}{\eta_\varepsilon + v_\varepsilon^2(t)} dt.$$

By construction,  $(u_\varepsilon, v_\varepsilon)$  solves

$$\begin{cases} \left( (\eta_\varepsilon + v_\varepsilon^2) u'_\varepsilon \right)' = 0 & \text{in } (0, L), \\ -\varepsilon v''_\varepsilon + v_\varepsilon |u'_\varepsilon|^2 + \frac{v_\varepsilon - 1}{4\varepsilon} = 0 & \text{in } (0, L), \\ u_\varepsilon(0) = 0, \quad u_\varepsilon(L) = a, \\ v_\varepsilon(0) = v_\varepsilon(L) = 0. \end{cases}$$

Moreover, the symmetry properties of  $u_\varepsilon$  and  $v_\varepsilon$  together with a change of variable yield

$$\limsup_{\varepsilon \rightarrow 0} AT_\varepsilon(u_\varepsilon, v_\varepsilon) = 2 \limsup_{\varepsilon \rightarrow 0} \widetilde{AT}_\varepsilon(u_\varepsilon, v_\varepsilon) \leq 1. \quad (7.16)$$

By Theorem 7.1, up to a subsequence,  $(u_\varepsilon, v_\varepsilon) \rightarrow (u_*, 1)$  in  $[L^2(0, L)]^2$  where  $u_* \in \{u_{\text{jump}}, u_{\text{aff}}\}$ . Assume by contraction that  $u_* = u_{\text{aff}}$ . According to Proposition 4.1, we have

$$\frac{a^2}{L} \leq \liminf_{\varepsilon \rightarrow 0} \int_0^L (\eta_\varepsilon + v_\varepsilon^2) |u'_\varepsilon|^2 dx. \quad (7.17)$$

By (7.12) together with a change of variable and the symmetry property of  $v_\varepsilon$ , we obtain

$$\liminf_{\varepsilon \rightarrow 0} \int_0^L \left( \varepsilon |v'_\varepsilon|^2 + \frac{(1 - v_\varepsilon)^2}{4\varepsilon} \right) dx = 2 \liminf_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon \left( v_\varepsilon; 0, \frac{L}{2} \right) \geq 1 - 2\Phi(\alpha). \quad (7.18)$$

Combining (7.16), (7.17) and (7.18) leads to  $a^2/L - \Phi(\alpha) \leq 0$ , which is in contradiction with our choice of  $\alpha$  in (7.14). Therefore  $u_* = u_{\text{jump}}$ , and the proof is complete.  $\square$

#### APPENDIX: FIRST AND SECOND VARIATIONS

In this appendix we derive explicit expressions for outer and inner variations of the Ambrosio-Tortorelli and Mumford-Shah functionals.

First, we recall the expression of the first and second outer variations of  $AT_\varepsilon$  defined by

$$dAT_\varepsilon(u, v)[\phi, \psi] := \frac{d}{dt} \Big|_{t=0} AT_\varepsilon(u + t\phi, v + t\psi),$$

and

$$d^2 AT_\varepsilon(u, v)[\phi, \psi] := \frac{d^2}{dt^2} \Big|_{t=0} AT_\varepsilon(u + t\phi, v + t\psi).$$

**Lemma A.9.** *Let  $\Omega \subset \mathbf{R}^N$  be a bounded open set with Lipschitz boundary and  $g \in H^{\frac{1}{2}}(\partial\Omega)$ . For all  $(u, v) \in \mathcal{A}_g(\Omega)$  and all  $(\phi, \psi) \in H_0^1(\Omega) \times [H_0^1(\Omega) \cap L^\infty(\Omega)]$ ,*

$$\begin{aligned} dAT_\varepsilon(u, v)[\phi, \psi] &= 2 \int_\Omega \left[ (\eta_\varepsilon + v^2) \nabla u \cdot \nabla \phi + \varepsilon \nabla v \cdot \nabla \psi + |\nabla u|^2 v \psi + \frac{(v-1)\psi}{4\varepsilon} \right] dx, \\ d^2 AT_\varepsilon(u, v)[\phi, \psi] &= \int_\Omega \left[ (\eta_\varepsilon + v^2) |\nabla \phi|^2 + 4v\psi \nabla u \cdot \nabla \phi + \varepsilon |\nabla \psi|^2 + |\nabla u|^2 \psi^2 + \frac{\psi^2}{4\varepsilon} \right] dx. \end{aligned} \quad (\text{A.19})$$

The computations of inner variations rely on one-parameter groups of diffeomorphisms over  $\bar{\Omega}$ , or equivalently on their infinitesimal generators. More precisely, assuming that  $\partial\Omega$  is of class  $\mathcal{C}^{k+1}$  with  $k \geq 1$ , and given a vector field  $X \in \mathcal{C}_c^k(\mathbf{R}^N; \mathbf{R}^N)$  satisfying  $X \cdot \nu_\Omega = 0$  on  $\partial\Omega$ , we consider the integral flow  $\{\Phi_t\}_{t \in \mathbf{R}}$  of  $X$  defined through the resolution of the ODE for every  $x \in \mathbf{R}^N$ ,

$$\begin{cases} \frac{d}{dt} \Phi_t(x) = X(\Phi_t(x)), \\ \Phi_0(x) = x. \end{cases} \quad (\text{A.20})$$

Then  $\Phi_0 = \text{id}$  and the flow rule asserts that  $\Phi_{t+s} = \Phi_t \circ \Phi_s$ . Since  $X \cdot \nu_\Omega = 0$  on  $\partial\Omega$ ,  $\{\Phi_t\}_{t \in \mathbf{R}}$  is a one-parameter group of  $\mathcal{C}^k$ -diffeomorphism from  $\Omega$  into itself, and from  $\partial\Omega$  into itself.

Given a (sufficiently smooth) boundary data  $g$ , we consider an arbitrary (smooth) extension  $G$  of  $g$  to  $\bar{\Omega}$  to define a one-parameter family of deformations  $\{(u_t, v_t)\}_{t \in \mathbf{R}} \subset \mathcal{A}_g(\Omega)$  satisfying  $(u_0, v_0) = (u, v)$  for a given pair  $(u, v) \in \mathcal{A}_g(\Omega)$  by setting

$$u_t := u \circ \Phi_{-t} - G \circ \Phi_{-t} + G \quad \text{and} \quad v_t := v \circ \Phi_{-t}.$$

The first and second inner variations of  $AT_\varepsilon$  at  $(u, v)$  are then defined by

$$\delta AT_\varepsilon(u, v)[X, G] := \frac{d}{dt} \Big|_{t=0} AT_\varepsilon(u_t, v_t),$$

and

$$\delta^2 AT_\varepsilon(u, v)[X, G] := \frac{d^2}{dt^2} \Big|_{t=0} AT_\varepsilon(u_t, v_t).$$

**Remark A.3.** We emphasize that  $\delta AT_\varepsilon(u, v)[X, G]$  and  $\delta^2 AT_\varepsilon(u, v)[X, G]$  depend on both the vector field  $X$  and the extension  $G$  of the boundary data  $g$ , because the family of deformations  $\{(u_t, v_t)\}_{t \in \mathbf{R}}$  depends on  $X$  and  $G$ . It allows one to perform inner variations of the energy *up to the boundary*. This type of deformations includes the more usual variation  $\{(u \circ \Phi_{-t}, v \circ \Phi_{-t})\}_{t \in \mathbf{R}}$  with  $X$  compactly supported in  $\Omega$ . Indeed, in this case we may choose an extension  $G$  supported in a small neighborhood of  $\partial\Omega$  in such a way that  $\text{supp } G \cap \text{supp } X = \emptyset$ . Then  $G \circ \Phi_{-t} = G$ , and thus  $u_t = u \circ \Phi_{-t}$ .

If the pair  $(u, v)$  and  $\partial\Omega$  are smooth enough, one can compute the first and second inner variations of  $AT_\varepsilon$  at  $(u, v)$  using the Taylor expansion of  $(u_t, v_t)$  with respect to the parameter  $t$ . One may for instance follow the general setting of [26, Lemma 2.2 and Corollary 2.3].

**Lemma A.10.** *Let  $\Omega \subset \mathbf{R}^N$  be a bounded open set with boundary of class  $\mathcal{C}^2$ ,  $g \in \mathcal{C}^2(\partial\Omega)$  and  $(u, v) \in \mathcal{A}_g(\Omega) \cap [\mathcal{C}^2(\overline{\Omega})]^2$ .*

- (i) *Then for every vector field  $X \in \mathcal{C}_c^1(\mathbf{R}^N; \mathbf{R}^N)$  and every extension  $G \in \mathcal{C}^2(\mathbf{R}^N)$  satisfying  $X \cdot \nu_\Omega = 0$  and  $G = g$  on  $\partial\Omega$ ,*

$$\delta AT_\varepsilon(u, v)[X, G] = -dAT_\varepsilon(u, v)[X \cdot \nabla(u - G), X \cdot \nabla v].$$

- (ii) *If further  $\partial\Omega$  is of class  $\mathcal{C}^3$ ,  $g \in \mathcal{C}^3(\partial\Omega)$ , and  $(u, v) \in \mathcal{A}_g(\Omega) \cap [\mathcal{C}^3(\overline{\Omega})]^2$ , then for every vector field  $X \in \mathcal{C}_c^2(\mathbf{R}^N; \mathbf{R}^N)$  and every extension  $G \in \mathcal{C}^3(\mathbf{R}^N)$  satisfying  $X \cdot \nu_\Omega = 0$  and  $G = g$  on  $\partial\Omega$ ,*

$$\begin{aligned} \delta^2 AT_\varepsilon(u, v)[X, G] &= d^2 AT_\varepsilon(u, v)[X \cdot \nabla(u - G), X \cdot \nabla v] \\ &\quad + dAT_\varepsilon(u, v)[X \cdot \nabla(X \cdot \nabla(u - G)), X \cdot \nabla(X \cdot \nabla v)]. \end{aligned}$$

*Proof.* Define  $\bar{u}_t := u \circ \Phi_{-t} - G \circ \Phi_{-t}$ . Since  $(u, v)$  and  $G$  belong to  $\mathcal{C}^2(\overline{\Omega})$ , we can differentiate  $(u_t, v_t)$  with respect to  $t$  and use (A.20) with the flow rule  $\Phi_{t+s} = \Phi_t \circ \Phi_s$  to find

$$\dot{u}_t := \frac{d}{dt} u_t = \left. \frac{d}{ds} \right|_{s=0} \bar{u}_{t+s} = -X \cdot \nabla \bar{u}_t \in \mathcal{C}^1(\overline{\Omega}),$$

and

$$\dot{v}_t := \frac{d}{dt} v_t = \left. \frac{d}{ds} \right|_{s=0} v_{t+s} = -X \cdot \nabla v_t \in \mathcal{C}^1(\overline{\Omega}).$$

In particular, we have

$$(\dot{u}_0, \dot{v}_0) = (-X \cdot \nabla(u - G), -X \cdot \nabla v). \quad (\text{A.21})$$

If  $\partial\Omega$  is of class  $\mathcal{C}^3$ ,  $g \in \mathcal{C}^3(\partial\Omega)$ , and  $(u, v) \in \mathcal{A}_g(\Omega) \cap [\mathcal{C}^3(\overline{\Omega})]^2$ , then we can differentiate  $(\dot{u}_t, \dot{v}_t)$  with respect to  $t$  to obtain

$$\ddot{u}_t := \frac{d}{dt} \dot{u}_t = -X \cdot \nabla \dot{u}_t = X \cdot \nabla(X \cdot \nabla \bar{u}_t)$$

and

$$\ddot{v}_t := \frac{d}{dt} \dot{v}_t = -X \cdot \nabla \dot{v}_t = X \cdot \nabla(X \cdot \nabla v_t).$$

Hence,

$$\ddot{u}_0 = X \cdot \nabla(X \cdot \nabla(u - G)) \quad \text{and} \quad \ddot{v}_0 = X \cdot \nabla(X \cdot \nabla v). \quad (\text{A.22})$$

Next elementary computations yield

$$\frac{d}{dt} AT_\varepsilon(u_t, v_t) = dAT_\varepsilon(u_t, v_t)[\dot{u}_t, \dot{v}_t],$$

and

$$\frac{d^2}{dt^2} AT_\varepsilon(u_t, v_t) = d^2 AT_\varepsilon(u_t, v_t)[\dot{u}_t, \dot{v}_t] + dAT_\varepsilon(u_t, v_t)[\ddot{u}_t, \ddot{v}_t],$$

so that the conclusion follows from (A.21)-(A.22) evaluating those derivatives at  $t = 0$ .  $\square$

In case the pair  $(u, v)$  only belongs to the energy space  $\mathcal{A}_g(\Omega)$ , we can compute the first and second inner variations of  $AT_\varepsilon$  by making the change variables  $y = \Phi_t(x)$  in the integrals defining  $AT_\varepsilon(u_t, v_t)$ . Then one expands the result with respect to  $t$  using a Taylor expansion of  $\Phi_t$ . If  $X \in \mathcal{C}_c^2(\mathbf{R}^N; \mathbf{R}^N)$ , the second order Taylor expansion near  $t = 0$  of the flow map  $\Phi_t$  induced by  $X$  is given by

$$\Phi_t = \text{id} + tX + \frac{t^2}{2}Y + o(t^2), \quad (\text{A.23})$$

where  $Y \in \mathcal{C}_c^1(\mathbf{R}^N; \mathbf{R}^N)$  denotes the vector field

$$Y := (DX)X,$$

$DX$  being the Jacobian matrix of  $X$  (i.e.,  $(DX)_{ij} = \partial_j X_i$  with  $i$  the row index and  $j$  the column index), and  $o(s)$  denotes a quantity satisfying  $o(s)/s \rightarrow 0$  as  $s \rightarrow 0$  uniformly with respect to  $x \in \mathbf{R}^N$ .

**Lemma A.11.** *Let  $\Omega \subset \mathbf{R}^N$  be a bounded open set with boundary of class  $\mathcal{C}^2$ ,  $g \in \mathcal{C}^2(\partial\Omega)$  and  $(u, v) \in \mathcal{A}_g(\Omega)$ .*

- (i) *Then for every vector field  $X \in \mathcal{C}_c^1(\mathbf{R}^N; \mathbf{R}^N)$  and every extension  $G \in \mathcal{C}^2(\mathbf{R}^N)$  satisfying  $X \cdot \nu_\Omega = 0$  and  $G = g$  on  $\partial\Omega$ ,*

$$\begin{aligned} \delta AT_\varepsilon(u, v)[X, G] &= \int_\Omega (\eta_\varepsilon + v^2) \left[ |\nabla u|^2 \text{Id} - 2\nabla u \otimes \nabla u \right] : DX \, dx \\ &+ \int_\Omega \left[ \left( \varepsilon |\nabla v|^2 + \frac{(v-1)^2}{4\varepsilon} \right) \text{Id} - 2\varepsilon \nabla v \otimes \nabla v \right] : DX \, dx + 2 \int_\Omega (\eta_\varepsilon + v^2) \nabla u \cdot \nabla (X \cdot \nabla G) \, dx. \end{aligned} \quad (\text{A.24})$$

- (ii) *If further  $\partial\Omega$  is of class  $\mathcal{C}^3$ ,  $g \in \mathcal{C}^3(\partial\Omega)$ , then for every vector field  $X \in \mathcal{C}_c^2(\mathbf{R}^N; \mathbf{R}^N)$  and every extension  $G \in \mathcal{C}^3(\mathbf{R}^N)$  satisfying  $X \cdot \nu_\Omega = 0$  and  $G = g$  on  $\partial\Omega$ ,*

$$\begin{aligned} \delta^2 AT_\varepsilon(u, v)[X, G] &= \int_\Omega (\eta_\varepsilon + v^2) \left( |\nabla u|^2 \text{Id} - 2(\nabla u \otimes \nabla u) \right) : DY \, dx \\ &+ 4 \int_\Omega (\eta_\varepsilon + v^2) \left[ (\nabla u \cdot \nabla (X \cdot \nabla G)) \text{div} X \right. \\ &\quad \left. - (\nabla u \otimes \nabla (X \cdot \nabla G)) : (DX + (DX)^T) \right] \, dx \\ &+ 2 \int_\Omega (\eta_\varepsilon + v^2) \nabla u \cdot \nabla (X \cdot \nabla (X \cdot \nabla G)) \, dx + 2 \int_\Omega (\eta_\varepsilon + v^2) |\nabla (X \cdot \nabla G)|^2 \, dx \\ &+ \int_\Omega \left[ \left( \varepsilon |\nabla v|^2 + \frac{(v-1)^2}{4\varepsilon} \right) \text{Id} - 2\varepsilon \nabla v \otimes \nabla v \right] : DY \, dx \\ &\int_\Omega (\eta_\varepsilon + v^2) \left\{ |\nabla u|^2 ((\text{div} X)^2 - \text{tr}((DX)^2)) - 4((\nabla u \otimes \nabla u) : DX) \text{div} X \right. \\ &\quad \left. + 4(\nabla u \otimes \nabla u) : (DX)^2 + 2|DX^T \nabla u|^2 \right\} \, dx \\ &+ \int_\Omega \left( \varepsilon |\nabla v|^2 + \frac{(v-1)^2}{4\varepsilon} \right) ((\text{div} X)^2 - \text{tr}((DX)^2)) \, dx \\ &- 4 \int_\Omega \varepsilon ((\nabla v \otimes \nabla v) : DX) \text{div} X \, dx + 4 \int_\Omega \varepsilon (\nabla v \otimes \nabla v) : (DX)^2 \, dx \\ &+ 2 \int_\Omega \varepsilon |DX^T \nabla v|^2 \, dx. \end{aligned} \quad (\text{A.25})$$

**Remark A.4.** From (A.24) we see that if  $(u, v) \in \mathcal{A}_g(\Omega)$  is a critical point of  $AT_\varepsilon$  in the sense that  $dAT_\varepsilon(u, v)[\phi, \psi] = 0$  for all admissible  $(\phi, \psi)$ , then  $\delta AT_\varepsilon(u, v)[X, G]$  is independent of the extension  $G$ .

Indeed, an integration by parts shows that, in this case,

$$\begin{aligned} \int_{\Omega} (\eta_{\varepsilon} + v^2) \nabla u \cdot \nabla (X \cdot \nabla G) \, dx &= - \int_{\Omega} \operatorname{div}((\eta_{\varepsilon} + v^2) \nabla u) (X \cdot \nabla G) \, dx \\ &\quad + (\eta_{\varepsilon} + 1) \int_{\partial\Omega} \partial_{\nu} u (X \cdot \nabla G) \, d\mathcal{H}^{N-1}. \end{aligned}$$

The first term in the right-hand side vanishes thanks to the first equation in (1.3) and, since  $X \cdot \nu_{\Omega} = 0$  on  $\partial\Omega$ , we can write  $X \cdot \nabla G = X_{\tau} \cdot \nabla_{\tau} G = X_{\tau} \cdot \nabla_{\tau} g$  where  $X_{\tau} = X - (X \cdot \nu_{\Omega}) \nu_{\Omega}$  and  $\nabla_{\tau}$  is the tangential gradient on  $\partial\Omega$ .

For the second inner variation (A.25), even if  $(u, v) \in \mathcal{A}_g(\Omega)$  satisfies  $dAT_{\varepsilon}(u, v)[\phi, \psi] = 0$  for  $(\phi, \psi) \in [\mathcal{C}_c^{\infty}(\Omega)]^2$ , the expression  $\delta^2 AT_{\varepsilon}(u, v)[X, G]$  does depend on the extension  $G$  and not only on the boundary data  $g$  because of the terms

$$\int_{\Omega} (\eta_{\varepsilon} + v^2) [(\nabla u \cdot \nabla (X \cdot \nabla G)) \operatorname{Id} - 2 \nabla u \otimes \nabla (X \cdot \nabla G)] : DX \, dx + \int_{\Omega} (\eta_{\varepsilon} + v^2) |\nabla (X \cdot \nabla G)|^2 \, dx.$$

If we take  $X \in \mathcal{C}_c^{\infty}(\Omega; \mathbf{R}^N)$  and  $G \in \mathcal{C}^2(\mathbf{R}^N)$  an extension of  $g$  such that  $\operatorname{supp} G \cap \operatorname{supp} X = \emptyset$ , and if we assume  $(u, v) \in \mathcal{A}_g(\Omega)$  to be a critical of  $AT_{\varepsilon}$ , then the expression of the second-inner variation (A.25) simplifies. Indeed the terms that contain  $Y = (DX)X$  disappear, since by regularity we have  $\delta AT_{\varepsilon}(u, v)[Y, G] = 0$ , and all terms containing  $G$  disappear. In this case, we end up with

$$\begin{aligned} \delta^2 AT_{\varepsilon}(u, v)[X, G] &= \int_{\Omega} (\eta_{\varepsilon} + v^2) \left\{ ((\operatorname{div} X)^2 - \operatorname{tr}((DX)^2)) - 4((\nabla u \otimes \nabla u) : DX) \operatorname{div} X \right. \\ &\quad \left. + 4(\nabla u \otimes \nabla u) : (DX)^2 + 2|DX^T \nabla u|^2 \right\} \, dx \\ &\quad + \int_{\Omega} \left( \varepsilon |\nabla v|^2 + \frac{(v-1)^2}{4\varepsilon} \right) ((\operatorname{div} X)^2 - \operatorname{tr}((DX)^2)) \, dx \\ &\quad - 4 \int_{\Omega} \varepsilon ((\nabla v \otimes \nabla v) : DX) \operatorname{div} X \, dx + 4 \int_{\Omega} \varepsilon (\nabla v \otimes \nabla v) : (DX)^2 \, dx \\ &\quad + 2 \int_{\Omega} \varepsilon |DX^T \nabla v|^2 \, dx. \end{aligned}$$

*Proof of Lemma A.11.* For simplicity, we assume that  $\partial\Omega$  is of class  $\mathcal{C}^3$ ,  $g \in \mathcal{C}^3(\partial\Omega)$ , and we observe that the computation of  $\delta AT_{\varepsilon}$  below only requires  $\mathcal{C}^2$  regularity. We fix  $X \in \mathcal{C}_c^2(\mathbf{R}^N; \mathbf{R}^N)$  and  $G \in \mathcal{C}^3(\overline{\Omega})$  satisfying  $X \cdot \nu = 0$  and  $G = g$  on  $\partial\Omega$ . We set

$$\widehat{u}_t := u \circ \Phi_{-t} \quad \text{and} \quad G_t := G \circ \Phi_{-t}.$$

Since  $u_t = \widehat{u}_t - (G_t - G)$ , we have

$$\begin{aligned} AT_{\varepsilon}(u_t, v_t) &= AT_{\varepsilon}(\widehat{u}_t, v_t) - 2 \int_{\Omega} (\eta_{\varepsilon} + v_t^2) \nabla \widehat{u}_t \cdot \nabla (G_t - G) \, dx + \int_{\Omega} (\eta_{\varepsilon} + v_t^2) |\nabla (G_t - G)|^2 \, dx \\ &=: \mathcal{A}(t) + \mathcal{B}(t) + \mathcal{C}(t). \end{aligned} \tag{A.26}$$

We aim to compute the first and second derivatives at  $t = 0$  of  $\mathcal{A}$ ,  $\mathcal{B}$ , and  $\mathcal{C}$ , starting with  $\mathcal{A}$ .

By the chain rule in Sobolev spaces, we have

$$\nabla \widehat{u}_t = [D\Phi_{-t}]^T \nabla u(\Phi_{-t}) = [D\Phi_t(\Phi_{-t})]^{-T} \nabla u(\Phi_{-t}). \tag{A.27}$$

On the other hand, in view of (A.23), we have

$$[D\Phi_t]^{-1} = I - tDX - \frac{t^2}{2}Y + t^2(DX)^2 + o(t^2), \tag{A.28}$$

and

$$\det(D\Phi_t) = 1 + t \operatorname{div} X + \frac{t^2}{2} [\operatorname{div} Y + (\operatorname{div} X)^2 - \operatorname{tr}((DX)^2)] + o(t^2). \tag{A.29}$$

Using the change of variables  $x = \Phi_t(y)$ , we obtain

$$\int_{\Omega} (\eta_{\varepsilon} + v_t^2) |\nabla \hat{u}_t|^2 \, dx = \int_{\Omega} (\eta_{\varepsilon} + v^2) |[D\Phi_t]^{-T} \nabla u|^2 \det(D\Phi_t) \, dy.$$

Hence,

$$\begin{aligned} \int_{\Omega} (\eta_{\varepsilon} + v_t^2) |\nabla \hat{u}_t|^2 \, dx &= \int_{\Omega} (\eta_{\varepsilon} + v^2) |\nabla u|^2 \, dx + t \int_{\Omega} (\eta_{\varepsilon} + v^2) [|\nabla u|^2 \text{Id} - 2\nabla u \otimes \nabla u] : DX \, dx \\ &+ \frac{t^2}{2} \left[ \int_{\Omega} (\eta_{\varepsilon} + v^2) |\nabla u|^2 (\text{div} Y + (\text{div} X)^2 - \text{tr}((DX)^2)) \, dx - 4 \int_{\Omega} (\eta_{\varepsilon} + v^2) ((\nabla u \otimes \nabla u) : DX) \text{div} X \, dx \right. \\ &\quad \left. - 2 \int_{\Omega} (\eta_{\varepsilon} + v^2) (\nabla u \otimes \nabla u) : (DY - 2(DX)^2) \, dx + 2 \int_{\Omega} (\eta_{\varepsilon} + v^2) |DX^T \nabla u|^2 \, dx \right] + o(t^2). \quad (\text{A.30}) \end{aligned}$$

Similarly,

$$\begin{aligned} \int_{\Omega} |\nabla v_t|^2 \, dx &= \int_{\Omega} |\nabla v|^2 \, dx + t \int_{\Omega} [|\nabla v|^2 \text{Id} - 2\nabla v \otimes \nabla v] : DX \, dx \\ &+ \frac{t^2}{2} \left[ \int_{\Omega} |\nabla v|^2 (\text{div} Y + (\text{div} X)^2 - \text{tr}((DX)^2)) \, dx - 4 \int_{\Omega} ((\nabla v \otimes \nabla v) : DX) \text{div} X \, dx \right. \\ &\quad \left. - 2 \int_{\Omega} (\nabla v \otimes \nabla v) : (DY - 2(DX)^2) \, dx + 2 \int_{\Omega} |DX^T \nabla v|^2 \, dx \right] + o(t^2), \quad (\text{A.31}) \end{aligned}$$

and

$$\begin{aligned} \int_{\Omega} (v_t - 1)^2 \, dx &= \int_{\Omega} (v - 1)^2 \, dx + t \int_{\Omega} (v - 1)^2 \text{div} X \, dx \\ &\quad + \frac{t^2}{2} \int_{\Omega} (v - 1)^2 [\text{div} Y + (\text{div} X)^2 - \text{tr}((DX)^2)] \, dx + o(t^2). \quad (\text{A.32}) \end{aligned}$$

Gathering (A.30)-(A.31)-(A.32) leads to

$$\begin{aligned} \mathcal{A}'(0) &= \int_{\Omega} (\eta_{\varepsilon} + v^2) [|\nabla u|^2 \text{Id} - 2\nabla u \otimes \nabla u] : DX \, dx \\ &\quad + \int_{\Omega} \left[ \left( \varepsilon |\nabla v|^2 + \frac{(v-1)^2}{4\varepsilon} \right) \text{Id} - 2\varepsilon \nabla v \otimes \nabla v \right] : DX \, dx, \quad (\text{A.33}) \end{aligned}$$

and

$$\begin{aligned} \mathcal{A}''(0) &= \int_{\Omega} (\eta_{\varepsilon} + v^2) \left\{ |\nabla u|^2 (\text{div} Y + (\text{div} X)^2 - \text{tr}((DX)^2)) - 4((\nabla u \otimes \nabla u) : DX) \text{div} X \right. \\ &\quad \left. - 2(\nabla u \otimes \nabla u) : (DY - 2(DX)^2) + 2|DX^T \nabla u|^2 \right\} \, dx \\ &\quad + \int_{\Omega} \left( \varepsilon |\nabla v|^2 + \frac{(v-1)^2}{4\varepsilon} \right) (\text{div} Y + (\text{div} X)^2 - \text{tr}((DX)^2)) - 4\varepsilon ((\nabla v \otimes \nabla v) : DX) \text{div} X \, dx \\ &\quad - 2 \int_{\Omega} \varepsilon (\nabla v \otimes \nabla v) : (DY - 2(DX)^2) \, dx + 2 \int_{\Omega} \varepsilon |DX^T \nabla v|^2 \, dx. \quad (\text{A.34}) \end{aligned}$$

Next we compute the derivatives of  $\mathcal{B}$ . To this purpose, we observe that

$$\nabla \hat{u}_{t+s} = [D\Phi_s(\Phi_{-s})]^{-T} \nabla \hat{u}_t(\Phi_{-s}), \quad \nabla G_{t+s} = [D\Phi_s(\Phi_{-s})]^{-T} \nabla G_t(\Phi_{-s}),$$

and by (A.28),

$$[D\Phi_s]^{-T} \nabla G_t - \nabla G(\Phi_s) = \nabla(G_t - G) - s(DX)^T \nabla(G_t - G) - s \nabla(X \cdot \nabla G) + o(s).$$



Using the change of variables  $x = \Phi_s(y)$  and (A.28)–(A.29) again, we obtain

$$\begin{aligned}
& \int_{\Omega} (\eta_{\varepsilon} + v_{t+s}^2) \nabla \hat{u}_{t+s} \cdot \nabla (G_{t+s} - G) \, dx \\
&= \int_{\Omega} (\eta_{\varepsilon} + v_t^2) ([D\Phi_s]^{-T} \nabla \hat{u}_t) \cdot ([D\Phi_s]^{-T} \nabla G_t - \nabla G(\Phi_s)) \det(D\Phi_s) \, dy \\
&= \int_{\Omega} (\eta_{\varepsilon} + v_t^2) \nabla \hat{u}_t \cdot \nabla (G_t - G) \, dx - s \int_{\Omega} (\eta_{\varepsilon} + v^2) \nabla \hat{u}_t \cdot \nabla (X \cdot \nabla G) \, dx \\
&\quad + s \int_{\Omega} (\eta_{\varepsilon} + v_t^2) [(\nabla \hat{u}_t \cdot \nabla (G_t - G)) \operatorname{div} X - (\nabla \hat{u}_t \otimes \nabla (G_t - G)) : (DX + (DX)^T)] \, dx + o(s).
\end{aligned}$$

Consequently,

$$\begin{aligned}
\mathcal{B}'(t) &= 2 \int_{\Omega} (\eta_{\varepsilon} + v_t^2) \nabla \hat{u}_t \cdot \nabla (X \cdot \nabla G) \, dx \\
&\quad - 2 \int_{\Omega} (\eta_{\varepsilon} + v_t^2) [(\nabla \hat{u}_t \cdot \nabla (G_t - G)) \operatorname{div} X - (\nabla \hat{u}_t \otimes \nabla (G_t - G)) : (DX + (DX)^T)] \, dx,
\end{aligned}$$

and in particular,

$$\mathcal{B}'(0) = 2 \int_{\Omega} (\eta_{\varepsilon} + v^2) \nabla u \cdot \nabla (X \cdot \nabla G) \, dx. \quad (\text{A.35})$$

To compute  $\mathcal{B}''(0)$ , we write  $\mathcal{B}'(t) =: I(t) - II(t)$ , and we set for simplicity

$$H := X \cdot \nabla G.$$

Since

$$\nabla H(\Phi_t) = \nabla H + t \nabla (X \cdot \nabla H) - t (DX)^T \nabla H + o(t),$$

we can change variables  $y = \Phi_t(x)$  and use again (A.27)–(A.28)–(A.29) to find

$$\begin{aligned}
I(t) &= 2 \int_{\Omega} (\eta_{\varepsilon} + v^2) \nabla u \cdot \nabla H \, dx + 2t \int_{\Omega} (\eta_{\varepsilon} + v^2) \nabla u \cdot \nabla (X \cdot \nabla H) \, dx \\
&\quad + 2t \int_{\Omega} (\eta_{\varepsilon} + v^2) [(\nabla u \cdot \nabla H) \operatorname{div} X - (\nabla u \otimes \nabla H) : (DX + (DX)^T)] \, dx + o(t).
\end{aligned}$$

Since  $G_t - G = -tH + o(t)$ , we easily infer that

$$II(t) = -2t \int_{\Omega} (\eta_{\varepsilon} + v^2) [(\nabla u \cdot \nabla H) \operatorname{div} X - (\nabla u \otimes \nabla H) : (DX + (DX)^T)] \, dx + o(t),$$

and consequently,

$$\begin{aligned}
\mathcal{B}''(0) &= 2 \int_{\Omega} (\eta_{\varepsilon} + v^2) \nabla u \cdot \nabla (X \cdot \nabla H) \, dx \\
&\quad + 4 \int_{\Omega} (\eta_{\varepsilon} + v^2) [(\nabla u \cdot \nabla H) \operatorname{div} X - (\nabla u \otimes \nabla H) : (DX + (DX)^T)] \, dx. \quad (\text{A.36})
\end{aligned}$$

Similarly,

$$\mathcal{C}(t) = t^2 \int_{\Omega} (\eta_{\varepsilon} + v^2) |\nabla H|^2 \, dx + o(t^2),$$

so that

$$\mathcal{C}'(0) = 0 \quad \text{and} \quad \mathcal{C}''(0) = 2 \int_{\Omega} (\eta_{\varepsilon} + v^2) |\nabla H|^2 \, dx. \quad (\text{A.37})$$

In view of (A.26), gathering (A.33)–(A.35)–(A.37) or (A.34)–(A.36)–(A.37) leads to the announced formula for  $\delta AT_{\varepsilon}(u, v)[X, G]$  and  $\delta^2 AT_{\varepsilon}(u, v)[X, G]$  respectively.  $\square$

Similar computations together with the well known first and second inner variations of a countably  $\mathcal{H}^{N-1}$ -rectifiable set (see e.g. [41, Chapter 2, Section 9]) lead to explicit expressions for the first and second inner variations of the Mumford-Shah functional.

**Lemma A.12.** *Let  $\Omega \subset \mathbf{R}^N$  be a bounded open set of class  $\mathcal{C}^2$ ,  $g \in \mathcal{C}^2(\partial\Omega)$  and  $u \in SBV^2(\Omega)$ .*

- (i) *Then for all vector field  $X \in \mathcal{C}_c^1(\mathbf{R}^N; \mathbf{R}^N)$  and every extension  $G \in \mathcal{C}^2(\mathbf{R}^N)$  with  $X \cdot \nu_\Omega = 0$  and  $G = g$  on  $\partial\Omega$ , we have*

$$\begin{aligned} \delta MS(u)[X, G] &= \int_{\Omega} (|\nabla u|^2 \text{Id} - 2\nabla u \otimes \nabla u) : DX \, dx \\ &\quad + \int_{\hat{J}_u} \text{div}^{\hat{J}_u} X \, d\mathcal{H}^{N-1} + 2 \int_{\Omega} \nabla u \cdot \nabla (X \cdot \nabla G) \, dx. \end{aligned} \quad (\text{A.38})$$

- ii) *If further  $\Omega$  is of class  $\mathcal{C}^3$  and  $g \in \mathcal{C}^3(\partial\Omega)$ , then for all vector field  $X \in \mathcal{C}_c^2(\mathbf{R}^N; \mathbf{R}^N)$  and every extension  $G \in \mathcal{C}^3(\mathbf{R}^N)$  with  $X \cdot \nu_\Omega = 0$  and  $G = g$  on  $\partial\Omega$ , we have*

$$\begin{aligned} \delta^2 MS(u)[X, G] &= \int_{\Omega} (|\nabla u|^2 \text{Id} - 2\nabla u \otimes \nabla u) : DY \, dx + \int_{\hat{J}_u} \text{div}^{\hat{J}_u} Y \, d\mathcal{H}^{N-1} \\ &\quad + \int_{\Omega} [|\nabla u|^2 ((\text{div} X)^2 - \text{tr}((DX)^2)) - 4((\nabla u \otimes \nabla u) : DX) \text{div} X] \, dx \\ &\quad + \int_{\Omega} [4(\nabla u \otimes \nabla u) : (DX)^2 + 2|DX^T \nabla u|^2] \, dx \\ &\quad + \int_{\hat{J}_u} \left[ (\text{div}^{\hat{J}_u} X)^2 + \sum_{i=1}^{N-1} |(\partial_{\tau_i} X)^\perp|^2 - \sum_{i,j=1}^{N-1} (\tau_i \cdot \partial_{\tau_j} X) (\tau_j \cdot \partial_{\tau_i} X) \right] d\mathcal{H}^{N-1} \\ &\quad + 2 \int_{\Omega} \nabla u \cdot \nabla (X \cdot \nabla (X \cdot G)) \, dx + 2 \int_{\Omega} |\nabla (X \cdot \nabla G)|^2 \, dx \\ &\quad + 4 \int_{\Omega} [(\nabla u \cdot \nabla (X \cdot \nabla G)) \text{div} X - (\nabla u \otimes \nabla (X \cdot \nabla G)) : (DX + (DX)^T)] \, dx, \end{aligned}$$

where  $(\tau_1, \dots, \tau_{N-1})$  is a basis of the tangent plane to  $\hat{J}_u$  at a given point  $x \in \hat{J}_u$ .

*Proof.* The second inner variation of the part  $\int_{\Omega} |\nabla u|^2 dx$  is computed exactly as in the proof of Lemma A.11, recalling that the chain rule still holds for the approximate gradient  $\nabla(u \circ \Phi_{-t}) = [D\Phi_{-t}]^T \nabla u(\Phi_{-t})$ . For the singular part of the energy, we use that  $\mathcal{H}^{N-1}(\hat{J}_{u_t}) = \mathcal{H}^{N-1}(J_{\hat{u}_t})$  where  $\hat{u}_t = \hat{u} \circ \Phi_{-t}$  and  $\hat{u} = u \mathbf{1}_\Omega + G \mathbf{1}_{\mathbf{R}^N \setminus \Omega}$ . The second variation of such a functional is computed with the area formula as in [41, Chapter 2] together with the following geometric formulas

$$\begin{aligned} (\text{div} X)^2 - \text{tr}[(\nabla X)^2] &- 2((\nu_u \otimes \nu_u) : DX) \text{div} X + 2(\nu_u \otimes \nu_u) : (DX)^2 + |DX^T \nu_u|^2 \\ &= (\text{div}^{J_u} X)^2 + \sum_{i=1}^{N-1} |(\partial_{\tau_i} X)^\perp|^2 - \sum_{i,j=1}^{N-1} (\tau_i \cdot \partial_{\tau_j} X) (\tau_j \cdot \partial_{\tau_i} X) + ((\nu_u \otimes \nu_u) : \nabla X)^2 \end{aligned}$$

stated in the proof of Theorem 1.1 p. 1851–1852 in [24].  $\square$

**Remark A.5.** As in Remark A.4,  $\delta MS(u)[X, G]$  is independent of the extension  $G$  when  $u$  is a critical point for the outer variations of  $MS$ , while  $\delta^2 MS(u)[X, G]$  does depend on the extension  $G$  in general. If  $u$  satisfies  $\delta MS(u)[X, G] = 0$ , then for all  $X \in \mathcal{C}_c^\infty(\Omega, \mathbf{R}^N)$  with  $\text{supp } G \cap \text{supp } X = \emptyset$ , then the

formula of the second inner variation reduces to

$$\begin{aligned} \delta^2 MS(u)[X, G] &= \int_{\Omega} |\nabla u|^2 ((\operatorname{div} X)^2 - \operatorname{tr}((DX)^2)) - 4((\nabla u \otimes \nabla u) : DX) \operatorname{div} X \, dx \\ &+ \int_{\Omega} 4 \nabla u \otimes \nabla u : (DX)^2 + 2 |DX^T \nabla u|^2 \, dx \\ &+ \int_{\hat{\mathcal{J}}_u} \left[ (\operatorname{div} \hat{\mathcal{J}}_u X)^2 + \sum_{i=1}^{N-1} |(\partial_{\tau_i} X)^\perp|^2 - \sum_{i,j=1}^{N-1} (\tau_i \cdot \partial_{\tau_j} X) (\tau_j \cdot \partial_{\tau_i} X) \right] d\mathcal{H}^{N-1}. \end{aligned}$$

## REFERENCES

- [1] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.
- [2] L. Ambrosio and H. M. Soner. A measure-theoretic approach to higher codimension mean curvature flows. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 25(1-2):27–49, 1997.
- [3] L. Ambrosio and V. M. Tortorelli. Approximation of functionals depending on jumps by elliptic functionals via  $\Gamma$ -convergence. *Comm. Pure Appl. Math.*, 43(8):999–1036, 1990.
- [4] L. Ambrosio and V. M. Tortorelli. On the approximation of free discontinuity problems. *Boll. Un. Mat. Ital. B (7)*, 6(1):105–123, 1992.
- [5] F. Bethuel, H. Brezis, and G. Orlandi. Asymptotics for the Ginzburg-Landau equation in arbitrary dimensions. *J. Funct. Anal.*, 186(2):432–520, 2001.
- [6] M. Bonacini and M. Morini. Stable regular critical points of the Mumford-Shah functional are local minimizers. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 32(3):533–570, 2015.
- [7] B. Bourdin. Numerical implementation of the variational formulation for quasi-static brittle fracture. *Interfaces Free Bound.*, 9(3):411–430, 2007.
- [8] B. Bourdin, G. A. Francfort, and J.-J. Marigo. *The variational approach to fracture*. Springer, New York, 2008.
- [9] A. Braides. *Local minimization, variational evolution and  $\Gamma$ -convergence*, volume 2094 of *Lecture Notes in Mathematics*. Springer, Cham, 2014.
- [10] A. Braides, A. Chambolle, and M. Solci. A relaxation result for energies defined on pairs set-function and applications. *ESAIM Control Optim. Calc. Var.*, 13(4):717–734, 2007.
- [11] F. Cagnetti, M. G. Mora, and M. Morini. A second order minimality condition for the Mumford-Shah functional. *Calc. Var. Partial Differential Equations*, 33(1):37–74, 2008.
- [12] F. Dipasquale, V. Millot, and A. Pisante. Torus-like solutions for the Landau-de Gennes model. Part I: The Lyuksyutov regime. *Arch. Ration. Mech. Anal.*, 239(2):599–678, 2021.
- [13] G. A. Francfort, N. Q. Le, and S. Serfaty. Critical points of Ambrosio-Tortorelli converge to critical points of Mumford-Shah in the one-dimensional Dirichlet case. *ESAIM Control Optim. Calc. Var.*, 15(3):576–598, 2009.
- [14] G. A. Francfort and J.-J. Marigo. Revisiting brittle fracture as an energy minimization problem. *J. Mech. Phys. Solids*, 46(8):1319–1342, 1998.
- [15] P. Gaspar. The second inner variation of energy and the Morse index of limit interfaces *J. Geom. Anal.*, 30(1):69–85, 2020.
- [16] P. Gaspar and M. A. M. Guaraco. The Allen-Cahn equation on closed manifolds. *Calc. Var. Partial Differential Equations*, 57(4):Paper No. 101, 42, 2018.
- [17] M. Giaquinta and L. Martinazzi. *An introduction to the regularity theory for elliptic systems, harmonic maps and minimal graphs*, volume 11 of *Appunti. Scuola Normale Superiore di Pisa (Nuova Serie) [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)]*. Edizioni della Normale, Pisa, second edition, 2012.
- [18] E. Giusti. *Direct methods in the calculus of variations*. World Scientific Publishing Co., Inc., River Edge, NJ, 2003.
- [19] M. A. M. Guaraco. Min-max for phase transitions and the existence of embedded minimal hypersurfaces. *J. Differential Geom.*, 108(1):91–133, 2018.
- [20] F. Hiesmayr. Spectrum and index of two-sided Allen-Cahn minimal hypersurfaces. *Comm. Partial Differential Equations*, 43(11):1541–1565, 2018.
- [21] J. E. Hutchinson and Y. Tonegawa. Convergence of phase interfaces in the van der Waals-Cahn-Hilliard theory. *Calc. Var. Partial Differential Equations*, 10(1):49–84, 2000.
- [22] R. L. Jerrard and P. Sternberg. Critical points via  $\Gamma$ -convergence: general theory and applications. *J. Eur. Math. Soc. (JEMS)*, 11(4):705–753, 2009.
- [23] N. Q. Le. Convergence results for critical points of the one-dimensional Ambrosio-Tortorelli functional with fidelity term. *Adv. Differential Equations*, 15(3-4):255–282, 2010.

- [24] N. Q. Le. On the second inner variation of the Allen-Cahn functional and its applications. *Indiana Univ. Math. J.*, 60(6):1843–1856, 2011.
- [25] N. Q. Le. On the second inner variations of Allen-Cahn type energies and applications to local minimizers. *J. Math. Pures Appl. (9)*, 103(6):1317–1345, 2015.
- [26] N. Q. Le and P. J. Sternberg. Asymptotic behavior of Allen-Cahn-type energies and Neumann eigenvalues via inner variations. *Ann. Mat. Pura Appl. (4)*, 198(4):1257–1293, 2019.
- [27] F. H. Lin. Gradient estimates and blow-up analysis for stationary harmonic maps. *Annals of Math.*, 132:785–829, 1999.
- [28] F.-H. Lin and T. Rivière. A quantization property for static Ginzburg-Landau vortices. *Comm. Pure Appl. Math.*, 54(2):206–228, 2001.
- [29] F.-H. Lin and T. Rivière. Energy quantization for harmonic maps. *Duke Math. J.*, 111:177–193, 2002.
- [30] F. H. Lin and C. Y. Wang. Harmonic and quasi-harmonic spheres. *Comm. Anal. Geom.*, 7:397–429, 1999.
- [31] F. H. Lin and C. Y. Wang. Harmonic and quasi-harmonic spheres. II. *Comm. Anal. Geom.*, 10:341–475, 2002.
- [32] F. H. Lin and C. Y. Wang. Harmonic and quasi-harmonic spheres. III. Rectifiability of the parabolic defect measure and generalized varifold flows. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 19:209–259, 2002.
- [33] S. Luckhaus and L. Modica. The Gibbs-Thompson relation within the gradient theory of phase transitions. *Arch. Rational Mech. Anal.*, 107(1):71–83, 1989.
- [34] J. Marroquin, S. Mitter, and T. Poggio. Probabilistic solution of ill-posed problems in computational vision. *Journal of the american statistical association*, 82(397):76–89, 1987.
- [35] D. Mumford and J. Shah. Boundary detection by minimizing functionals. In *IEEE Conference on Computer Vision and Pattern Recognition*, volume 17, pages 137–154. San Francisco, 1985.
- [36] D. Mumford and J. Shah. Optimal approximations by piecewise smooth functions and associated variational problems. *Comm. Pure Appl. Math.*, 42(5):577–685, 1989.
- [37] P. Padilla and Y. Tonegawa. On the convergence of stable phase transitions. *Comm. Pure Appl. Math.*, 51(6):551–579, 1998.
- [38] A. Pigati and D. Stern. Minimal submanifolds from the abelian Higgs model. *Invent. Math.*, 223(3):1027–1095, 2021.
- [39] T. Rivière. Harmonic maps with values into torii of revolution. Thèse de l’université Paris 6, 1993.
- [40] C. Scheven. Partial regularity for stationary harmonic maps at a free boundary. *Math. Z.*, 253(1):135–157, 2006.
- [41] L. Simon. *Lectures on geometric measure theory*, volume 3 of *Proceedings of the Centre for Mathematical Analysis, Australian National University*. Australian National University, Centre for Mathematical Analysis, Canberra, 1983.
- [42] D. Stern. Existence and limiting behavior of min-max solutions of the Ginzburg-Landau equations on compact manifolds. *J. Differential Geom.*, 118(2):335–371, 2021.
- [43] Y. Tonegawa. Phase field model with a variable chemical potential. *Proc. Roy. Soc. Edinburgh Sect. A*, 132(4):993–1019, 2002.
- [44] Y. Tonegawa. A diffused interface whose chemical potential lies in a Sobolev space. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 4(3), 2005.

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