A note on polarized varieties with high nef value

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Abstract. We study the classification problem for polarized varieties with high nefvalue. We give a complete list of isomorphism classes for normal polarized varieties with high nefvalue. This generalizes classical work on the smooth case by Fujita, Beltrametti and Sommese. As a consequence we obtain that polarized varieties with slc singularities and high nefvalue, are birationally equivalent to projective bundles over nodal curves.

Contents

1. Introduction

A projective variety X together with an ample line bundle L on X is called a polarized variety and is denoted by (X, L). A classical result on polarized varieties is the Kobayashi-Ochiai theorem:

Theorem 1.1 (Generalized Kobayashi-Ochiai Theorem, *cf.* [BS95, Theorem 3.1.6])**.** *Let X be an n-dimensional connected normal projective scheme and L an ample line bundle on X. Then we have*

- $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ *if and only if* $K_X + (n+1)L \equiv_{\mathrm{num}} \mathcal{O}_X$*;*
- $(X, L) \cong (Q, \mathcal{O}_Q(1))$ *where* $Q \subset \mathbb{P}^{n+1}$ *is a hyperquadric in* $\mathbb{P}^{n+1}$ *if and only if* $K_X + nL \equiv_{\mathrm{num}} \mathcal{O}_X$*.*

To study polarized varieties, Fujita introduced the Δ-genus $\Delta(X, L) := n + L^n - h^0(X, L)$ of polarized varieties, which encodes the dimension of the variety X and L^n, and develops classification theories for polarized varieties with small Δ-genus under certain assumptions on the singularities of X and positivity on L. For Fujita's work, we refer to [Fuj90, Chapter 1].

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When a foliation $\mathcal{F}$ is algebraically integrable, one can define naturally general log leaves of $\mathcal{F}$ (*cf.* [AD14, Definition 3.11]). A *general log leaf* $(\tilde{F}, \tilde{\Delta})$ comprises a normalization of the closure of a general leaf F of $\mathcal{F}$ and an effective Weil $\mathbb{Q}$-divisor $\tilde{\Delta}$. Let $e : \tilde{F} \to F$ be the normalization map. Then $\tilde{\Delta}$ is given by $K_{\tilde{F}} + \Delta \equiv_{\text{num}} e^* K_{\mathcal{F}}$. By studying the geometry of general log leaves in [AD14], Araujo and Druel obtained a version of Kobayashi-Ochiai theorem for $\mathbb{Q}$-Fano foliations ([AD14, Theorem 1.2]). We also refer to [Hör14, Corollary 1.2] for a more general statement. This motivates us to consider classification problem for (X, Δ), where X is a variety and Δ a Weil $\mathbb{Q}$-divisor.

When an algebraically integrable foliation $\mathcal{F}$ is $\mathbb{Q}$-Fano, we have the equality

$$K_{\tilde{F}} + \Delta \equiv_{\text{num}} i_{\mathcal{F}}(e^* H).$$

Hence one may very well try to establish a pair version of Theorem 1.1. In fact, Fujino and Miyamoto proved the following:

Proposition 1.2 ([FM21, Corollary 1.3])**.** *Let (X, Δ) be a projective semi-log canonical pair such that X is connected.. Assume that $(K_X + \Delta)$ is not nef and that $(K_X + \Delta) \equiv_{\text{num}} rD$ for some Cartier divisor D on X with $r > n = \dim(X)$. Then X is isomorphic to $\mathbb{P}^n$ with $\mathcal{O}_X(D) = \mathcal{O}_{\mathbb{P}^n}(-1)$ and (X, Δ) is Kawamata log terminal.*

The result of Fujino and Miyamoto assumes mild singularites on the pair (X, Δ) and a divisibility condition of the log canonical bundle $K_X + \Delta$. However, with a foliation $\mathcal{F}$, its log general leaf $(\tilde{F}, \tilde{\Delta})$ is *a priori* just normal. On the other hand, the classical results of classification theory in [BS95, Chapter 7.2] do not need divisibility assumption. However we do need $-K_X$ is very positive. Thus one may try to weaken the conditions and consider the classification problems:

(1) Classify the triple (X, Δ, L) where (X, Δ) is log canonical, L is ample and $K_X + (\dim(X) - 1)L \notin \mathrm{Pseff}(X)$;
(2) Classify the pair (X, L) where X is a projective variety with singularities wilder than normal, L is ample and $K_X + (\dim(X) - 1)L \notin \mathrm{Pseff}(X)$.

In order to achieve these goals, we study the more general class of *quasi-polarized varieties* and follow an approach of Andreatta in [And13]. For a quasi-polarized variety (X, L) where X is $\mathbb{Q}$-factorial and has canonical singularities, we may run a MMP which contracts all L-trivial extremal rays and get a polarized variety (X', L') (see Lemma 3.2). By using Andreatta's result Theorem 3.3 which describes the general fibers of extremal contractions, we can reduce the problem of classifying (X', L') with high nefvalue to the problem of classifying polarized variety with Δ-genus zero. We have the following classification:

Theorem 1.3. *Let X be a variety with canonical $\mathbb{Q}$-factorial singularities and L a nef and big line bundle on X. Suppose $K_X + (n-1)L \notin \mathrm{Pseff}(X)$. Then we have one of the following cases:*

(1) $(X, L) \sim_{\text{bir}} (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$;
(2) (X, L) *is birational equivalent to a* $(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$*-bundle over a smooth curve* C;
(3) $(X, L) \sim_{\text{bir}} (Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$, *where* $Q \subset \mathbb{Q}^{n+1}$ *is a hyperquadric;*
(4) $(X, L) \sim_{\text{bir}} (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$;

(5) $(X,L) \sim_{\text{bir}} C_n(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$, *where* $C_n(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ *is a generalized cone over* $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$

This generalizes the results of Beltrametti and Sommese [BS95, Proposition 7.2.2 and Theorem 7.2.4]. The drawback of letting L be nef and big is that after running MMP we don't have isomorphism and even have indeterminacies.

For a normal variety X, we have modifications $\mu : X' \to X$ for X such that X' has mild singularities and $K_{X'}$ is μ-ample. A good reference for these modifications is [Kol13, Chapter 1]. For a polarized variety (X,L), with X normal, we may take a canonical modifications $\mu : X' \to X$ for X and consider the quasi-polarized variety (X', μ^*L). By applying the previous result, we have the following classification.

Theorem 1.4. *Let* (X,L) *be a polarized normal variety of dimension* n. *Suppose that* K_X *is* $\mathbb{Q}$*-Cartier and* $K_X + (n-1)L \notin \mathrm{Pseff}(X)$. *Then we have one of the following cases:*

(1) $(X,L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$;
(2.i) $(X,L) \cong (\mathbb{P}(\mathcal{V}), \mathcal{O}_{\mathbb{P}(\mathcal{V})}(1))$, *where* $\mathcal{E}$ *is a rank* n *ample vector bundle over a smooth curve* C;
(2.ii) $(X,L) \cong C_n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a))$ *be a generalized cone with* $a \geq 3$;
(3) $(X,L) \cong (Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$, *where* $Q \subset \mathbb{Q}^{n+1}$ *is a hyperquadric;*
(4) $(X,L) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$;
(5) $(X,L) \cong C_n(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$, *a generalized cone over* $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$.

In Theorem 1.4, we note that even if in the proof we have taken a modification, in the resulting list we have isomorphism. The reason is that L is ample and birational equivalences between normal polarized varieties are always isomorphisms.

For a log canonical pair (X,Δ) with $(K_X+\Delta) + (\dim(X)-1)L \notin \mathrm{Pseff}(X)$, a first observation is that if Δ is $\mathbb{Q}$-Cartier, we will have $K_X + (\dim(X)-1)L \notin \mathrm{Pseff}(X)$. Hence we will have a list for (X,L) similar to Theorem 1.4. However in this list the Picard number $\rho(X) \leq 2$. Hence for Δ to be an irreducible divisor or more generally reduced divisor, we don't have to many choice. We may thus give a list for (X,Δ,L).

Proposition 1.5. *Let* (X,Δ) *be a log canonical pair, with* $\Delta \neq 0$ *a reduced divisor. Suppose that* L *is an ample line bundle on* X *and* $(K_X+\Delta)+(n-1)L \notin \mathrm{Pseff}(X)$, *where* $n = dim(X)$. *Then* (X,Δ,L) *is one of the following:*

(1) $(X,L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, $\Delta \equiv_{\text{num}} H$ *is a prime divisor where* H *is a hyperplane of* $\mathbb{P}^n$;
(2.i) *There is a* $(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$*-bundle* $(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1))$ *over a smooth curve* C, *and a birational morphism* $\mu : \mathbb{P}(E) \to X$ *such that* $\mu^*(L) \cong \mathcal{O}_{\mathbb{P}(E)}(1)$ *and* $\Delta = \sum F_i$ *is a finite sum where* $F_i \cong \mu(\mathbb{P}^{n-1})$ *are images of distinct general fibers of* π *by* μ;
(2.ii) $(X,L) = (\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1)), \mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a)\oplus\mathcal{O}_{\mathbb{P}^1}(1))}(1))$ *with* $a > 1$ *and* $\Delta = D$ *is irreducible, where* D *is the unique section of* $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \to \mathbb{P}^1$ *such that* $D \equiv_{\text{num}} \mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a)\oplus\mathcal{O}_{\mathbb{P}^1}(1))}(1)) - af$, *where* f *is a general fiber;*

(3.i) $(X, L) \cong (Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$, *where* $Q \subset \mathbb{P}^{n+1}$ *is a* $\mathrm{rk}(Q) = 3$ *hyperquadric, the boundary divisor* Δ *is a hyperplane in* Q *and* $[\Delta] = \frac{1}{2}[H \cap Q]$ *where* H *is a hyperplane in* $\mathbb{P}^{n+1}$*;*

(3.ii) $(X, L) \cong (Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$, *where* $Q \subset \mathbb{P}^{n+1}$ *is a* $\mathrm{rk}(Q) = 4$ *hyperquadirc. If we write* $Q = \mathrm{Proj}\left(\frac{\mathbb{C}[x_0, \ldots, x_{n+1}]}{(x_0x_1 - x_2x_3)}\right)$*, then* $\Delta = D$ *is prime and* D *is the cone with vertex* $\mathbb{P}^{n-3}$ *over* $\mathbb{P}^1 \times \mathrm{pt}$ *or* $\mathrm{pt} \times \mathbb{P}^1$*. In particular,* $D \cong \mathbb{P}^{n-1}$*.*

Finally we turn to non-normal varieties with semi-log canonical singularities. We have the following classification.

Theorem 1.6. *Let* X *be a non-normal slc projective variety of dimension* n *and* L *an ample line bundle over* X*. Suppose that* $K_X + (n-1)L \notin \mathrm{Pseff}(X)$*. Let* $\pi : \bar{X} \to X$ *be the normalization of* X *and* $D \subset X$*,* $\bar{D} \subset \bar{X}$ *the conductors. Then we have:*

There is a nodal curve C'*,a rank* n*-vector bundle* E'*, distinct fibers* $F_1, F_2, \ldots, F_m$ *of* $\mathbb{P}(E')$ *and a birational morphism* $\mu : \mathbb{P}(E') \to X$ *such that* $\mu^*(L) = \mathcal{O}_{\mathbb{P}(E')}(1)$ *and* $D = \sum_{1 \le i \le m} \mu(F_i)$

We see that Theorem 1.6 shortens the list in Proposition 1.5 rather than increasing it. In fact there is a degree 2 morphism $\bar{D}^\nu \to D^\nu$, where $\bar{D}^\nu$ and D^ν are the normalizations of $\bar{D}$ and D respectively. Hence we need $(L'|_{\bar{D}^\nu})^{n-1}$ to be divisible by 2 which gives more restrictions on $(\bar{X}, \bar{D})$ than the assumption in Proposition 1.5.

Remark 1.7. The classification in Theorem 1.3 is already known for even when X' is klt (*cf.* [And13, Proposition 3.5]). My personal contribution in the classification is to use modifications to get Theorem 1.4 and Theorem 1.6.

1.1. **Plan of the article.** The article is organized as following. In Section 2, we recall some basic notions and facts that we need. In Section 3, we prove Theorem 1.4 by running an MMP (Lemma 3.2) to reduce the problem to check which member in the list of classification results of Fujita, Beltrametti-Sommese satisfies our non pseudo-effective hypothesis. In Section 4, we prove Theorem 1.4 thanks to canonical modifications and use similar methods to prove Proposition 1.5. In Section 5, for a polarized slc variety (X, L), we use Proposition 1.5 on the triple $(\bar{X}, \bar{D}, L')$, where $(\bar{X}, \bar{D})$ is the normalization of X and the conductor divisor on $\bar{X}$ and L' is the pullback of L, to get Theorem 1.6.

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2. Notations and general setup

We work over $\mathbb{C}$. For general definitions we refer to [Har77].

A scheme in the article will always be projective over $\mathbb{C}$. A variety is a reduced and irreducible scheme over $\mathbb{C}$. The name *point* does not necessarily refer to closed point.

For two Cartier divisor D_1 and D_2, we denote by $D_1 \sim D_2$ the linear equivalence and by $D_1 \equiv_{\text{num}} D_2$ the numerical equivalence. We have similar notations for $\mathbb{Q}$-Cartier Weil-divisors.

A vector bundle $\mathcal{V}$ of rank r over X is a locally free sheaf of rank r. We set

$$\mathbb{P}(\mathcal{V}) := \operatorname{Proj}(\oplus_{n\geq 0}\operatorname{Sym}^n(\mathcal{V}))$$

to be its *projectivisation*.

We follow the positivity notions of divisors and vector bundles in [Laz04a][Laz04b]. We will use a generalized version of *pseudo-effectivenss* for reflexive sheaves by Höring-Peternell:

Definition 2.1 ([HP19, Definition 2.1.])**.** Let X be a normal projective variety and $\mathcal{E}$ a reflexive sheaf on X. We say that $\mathcal{E}$ is *pseudo-effective* if there exists an ample divisor H on X satisfying the following: For any $c > 0$ there exists integers $j > 0$ and $i > jc$ such that

$$H^0(X, S^{[i]}(\mathcal{E}) \otimes \mathcal{O}_X(jH)) \neq 0$$

where $S^{[i]}(\mathcal{E})$ is the double dual of $\operatorname{Sym}^i(\mathcal{E})$.

When $\mathcal{E}$ itself is a line bundle, we have that $\mathcal{E}$ is pseudo-effective in the above sense is equivalent to $\mathcal{E}$ is pseudo-effective in the usual sense of [Laz04a, Definition 2.2.25].

We recall the definition of polarized and quasi-polarized varieties.

Definition 2.2. Let (X, L) be a pair consisting of a projective variety X and a line bundle L over X. We call it

(1) a *quasi-polarized variety* if L is nef and big;
(2) a *polarized variety* if L is ample.

For a quasi-polarised variety (X, L) of dimension n, its Δ-*genus* is defined to be

$$\Delta(X, L) := n + L^n - h^0(X, L)$$

Let (X, L) be a quasi-polarized variety, we define the *nefvalue* $\tau(L)$ of L to be

$$\tau(L) := \inf\{t \in \mathbb{R} : K_X + tL \text{ is nef}\}.$$

By Kawamata's rationality theorem, we know that $\tau(L)$ is a rational number or ∞.

We now give our notions for birational equivalence and isomorphisms between quasi-polarized varieties.

Definition 2.3. Let (X_1, L_1) and (X_2, L_2) be two pairs consisting of a variety X_i and a line bundle L_i on X_i. We say that

(1) (X_1, L_1) is *isomorphic* to (X_2, L_2), if there exists an isomorphism $\phi : X_1 \to X_2$ such that $\phi^*(L_2)$ is isomorphic to L_1. We denote this by $(X_1, L_1) \cong (X_2, L_2)$.
(2) (X_1, L_1) and (X_2, L_2) are *birationally equivalent*, if there exists a variety X and two birational morphism $\phi_i : X \to X_i$ such that $\phi_1^*(L_1)$ is isomorphic to $\phi_2^*(L_2)$. We denote this by $(X_1, L_1) \sim_{\text{bir}} (X_2, L_2)$.

In the article we will repeatedly encounter *generalized cones*. We thus recall the notion of generalized cone here.

Definition 2.4 (Generalized cone)**.** We follow the construction in [BS95, 1.1.8.] Let V be a projective scheme of dimension n and L a very ample line bundle over V. Fix $N \geq n$ an integer. Set $E := \oplus^{N-n}\mathcal{O}_V$ and $p : \mathbb{P}(E \oplus L) \to V$. We denote $\mathbb{P}(E \oplus L)$ by X. Note that $E \oplus L$ is globally generated and we have for the tautological bundle $\xi := \mathcal{O}_{\mathbb{P}(E\oplus L)}(1)$ of $\mathbb{P}(E \oplus L)$ a surjective morphism $p^*(E \oplus L) \to \xi$. Hence we have a surjective morphism

$$H^0(V, E \oplus L) \otimes_{\mathbb{C}} \mathcal{O}_X \twoheadrightarrow \xi.$$

The above morphism corresponds to a unique morphism

$$\phi_{|\xi|} : X \to \mathbb{P}(H^0(V, E \oplus L)).$$

We take the Stein factorization of ϕ:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\psi_{|\xi|}"'] & \mathbb{P}(H^0(V, E \oplus L)) \\ C_N(V, L) \arrow[ur] & \end{tikzcd}$$

and call $C_N(V, L)$ the *generalized cone of dimension* N on (V, L). As ξ is big, the scheme $C_N(V, L)$ has dimension N. Set $\xi_L := \mathcal{O}_{\mathbb{P}(H^0(V,E\oplus L))}(1)|_{C_N(V,L)}$, then ξ_L is ample.

For the notions and results in birational geometry and the minimal model program, we refer to the standard [KM98] and [Kol13].

3. Canonical polarized varieties

In this section, we consider quasi-polarized varieties (X, L) with canonical singularities. First we give a lemma to show how the condition $K_X + (n-1)L \notin \mathrm{Pseff}(X)$ is related to the nefvalue of L.

Lemma 3.1. *Let (X, L) be quasi-polarized variety of dimension n with canonical singularities. Suppose that $\tau(L)$ is finite. If $K_X + (n-1)L \notin \mathrm{Pseff}(X)$, we have that $\tau(L) > n-1$.*

Proof. We know that $\mathrm{Pseff}(X') = \overline{\mathrm{Big}(X)}$ is a closed cone. Hence there exists an ample $\mathbb{Q}$-divisor A, such that $K_X + (n-1)L + A$ is not pseudo-effective. If $\tau(L) \leq n-1$, we have

$$K_X + (n-1)L + A = (K_X + \tau(L)L) + (n-1-\tau(L))L + A.$$

That is, $K_X + (n-1)L + A$ is a sum of a nef and an ample divisor, which is ample, a contradiction. □

When L is ample, its nefvalue $\tau(L)$ is finite. However, when L is just nef and big, we have some subtleties. By the cone theorem (*cf.* [Fuj11, Theorem 1.1.]), we know that

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X \geq 0} + \sum \mathbb{R}_{\geq 0}[C_j]$$

where C_j are K_X-negative rational curves and the sum is over countably many j.

(1) For every K_X-negative extremal ray $R = \mathbb{R}_{\geq 0}[C]$, we have that $L \cdot C > 0$. By the rationality theorem (*cf.* [KM98, Complement 3.6]), there exists a K_X-negative extremal C_0 such that $\tau(L) = -\dfrac{K_X \cdot C_0}{L \cdot C_0} < \infty$. Hence $\tau(L) = \infty$ only if there exists an L-trivial K_X-negative extremal ray.

(2) There exists a K_X-negative extremal ray R such that $L \cdot R = 0$. By the contraction theorem (*cf.* [Fuj11, Theorem 1.1.(4)]), we consider the contraction with respect to R, $\mathrm{cont}_R : X \to Z$. Note that there exists a line bundle L_Z on Z such that $L \cong \mathrm{cont}_R^*(L_Z)$.

Hence we may consider to run a K_X-MMP to contract every L-trivial extremal rays to get a (X', L') satisfying the case (1). We now precise how to do this.

Lemma 3.2. *Let X be a variety with canonical $\mathbb{Q}$-factorial singularities and L a big and nef line bundle on X. Suppose that $K_X+(n-1)L \notin \mathrm{Pseff}(X)$. Then (X, L) is birationally equivalent to a quasi-polarized variety (X', L'), where X' is a normal projective variety with canonical $\mathbb{Q}$-factorial singularities, $K_{X'} + (n-1)L' \notin \mathrm{Pseff}(X')$ and*

(1) Either $\tau(L')$ is finite;

(2) or there is a Mori fiber space structure $\phi : X' \to W$ and a rational number $\tau > (n-1)$ such that L' is ϕ-ample and $K_{X'} + \tau L' \sim_{\mathbb{Q},\phi} 0$.

Proof. We apply the terminal modification then a small $\mathbb{Q}$-factorialization to X (*cf.*[Kol13, Theorem 1.33, Corollary 1.37]). We get a modification $f : Y \to X$ such that Y has $\mathbb{Q}$-factorial terminal singularities. Set $L_Y = f^*L$. We have that L_Y is nef and big and $K_Y + (n-1)L_Y \notin \mathrm{Pseff}(Y)$. By [And13, Lemma 4.1.], we can find an effective $\mathbb{Q}$-divisor Δ on Y such that

$$\Delta \sim_{\mathbb{Q}} (n-1)L_Y \text{ and } (Y, \Delta) \text{ is klt.}$$

Now consider the pair (Y, Δ). We have that $K_Y + \Delta \notin \mathrm{Pseff}(Y)$. By [BCHM10, Corollary 1.3.3], we can run a $(K_Y + \Delta)$-MMP to get

$$(Y, \Delta) = (Y_0, \Delta_0) \dashrightarrow (Y_1, \Delta_1) \dashrightarrow \cdots \dashrightarrow (Y_s, \Delta_s),$$

with Y_s a Mori fiber space.

Suppose that the map $\phi_i : Y_i \dashrightarrow Y_{i+1}$ is associated with a $(K_{Y_i} + \Delta_i)$-negative extremal ray R_i. By [And13, Proposition 4.2.], for every $i = 0, 1, \ldots, s$, we have that

(1) Y_i is $\mathbb{Q}$-factorial terminal;

(2) $\Delta_i \cdot R_i = 0$;

(3) There exists nef and big line bundles L_i on Y_i and $\Delta_i \sim_{\mathbb{Q}} (n-1)L_i$.

It is then obvious $K_{Y_i} + (n-1)L_i \notin \mathrm{Pseff}(Y_i)$.

We then set $(X', L') := (Y_s, L_s)$.

(1) If (Y_s, Δ_s) has no K_{Y_s}-negative extremal ray R such that $L_s \cdot R = 0$, by Kawamata rationality theorem there exists a $K_{X'}$-negative extremal curve C_0 such that $\tau(L') = -\dfrac{K_{X'} \cdot C_0}{L' \cdot C_0}$. Hence the nefvalue of L' is finite.

(2) Otherwise, we consider the Mori fiber space $\phi_s : Y_s \to W$ obtained in the above $(K_Y + \Delta)$-MMP. Let $R_s := \mathrm{NE}(\phi_s)$ be the extremal ray of ϕ_s. We claim that $L_s \cdot R_s > 0$. Suppose by contradiction that $L_s \cdot R_s = 0$. Then by the contraction theorem, there exists L_W such that $\phi_s^*(L_W) = L_s$. As $\dim(W) < \dim(Y_s)$, we have that $L_s^n = \phi_s^*(L_W^n) = 0$ contradicting L_s to be nef and big. As R_s is a $(K_{Y_s} + \Delta_s)$-negative extremal ray, we have that $(K_{Y_s} + (n-1)L_s) \cdot R_s < 0$. Hence the $\tau > 0$ such that $K_{Y_s} + \tau L_s \sim_{\mathbb{Q},\phi} 0$ satisfies that $\tau > (n-1)$.

$\square$

The following relative Kobayashi-Ochiai criterion by Andreatta is the major tool to give further classification for the second situation in the above lemma.

Theorem 3.3 ([And95, Theorem 2.1.]). *Let X be a projective variety with klt singularities and let L be a line bundle on X. Let $\phi : X \to Z$ be a surjective morphism with connected fibers between normal varieties. Suppose that L is ϕ-ample and $K_X+\tau L \sim_{\mathbb{Q},\phi} 0$ for some $\tau \in \mathbb{Q}^+$. Let $F_1 = \phi^{-1}(z)$ be a non-trivial fiber, $F \subset F_1$ be one of its irreducible components, F' be the normalization of F and let L' be the pullback of L on F'. Let $\lfloor \tau \rfloor$ be the integral part of τ and $\tau' = \lceil \tau \rceil = -\lfloor -\tau \rfloor$.*

(I,1) $\dim(F) \geq \tau - 1$;
(I,2) If $\dim(F) < \tau$, *then* $F \cong \mathbb{P}^{\tau'-1}$ *and* $L|_F = \mathcal{O}_{\mathbb{P}^{\tau'-1}}(1)$;
(I,3) If $\dim(F) < \tau + 1$, *then* $\Delta(F', L') = 0$,

If moreover $\dim(F) > \dim(X) - \dim(Z)$, *then*

(II,1) $\dim(F) \geq \tau$;
(II,2) If $\dim(F) = \tau$, *then* $F \cong \mathbb{P}^{\tau}$ *and* $L|_F = \mathcal{O}_{\mathbb{P}^{\tau}}(1)$;
(II,3) If $\dim(F) < \tau + 1$, *then* $\Delta(F', L') = 0$,

If all components of the fiber F_1 satisfy that $\dim(F) < \tau$, *in case (I.2) or* $\dim(F) \leq \tau$ *in case (II.3), then the fiber is actually irreducible.*

A direct result of the above Theorem is the following lemma which classifies the (X', L') in the case (2) of Lemma 3.2.

Lemma 3.4. *Let (X, L) be a quasi-polarized variety of dimension n. Suppose X has canonical $\mathbb{Q}$-factorial singularities and $K_X + (n-1)L \notin \mathrm{Pseff}(X)$. Suppose that there exists a K_X-negative extremal ray $R = \mathbb{R}_{\geq 0}[C_0]$ such that $L \cdot C_0 > 0$. Then (X, L) is the one of the following*

(1) $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, *and* $\tau = n + 1$;
(2) (X, L) *is isomorphic to a* $(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$*-bundle over a smooth curve C and* $\tau = n$;
(3) $\Delta(X, L) = 0$, $K_X + \tau L \sim_{\mathbb{Q}} \mathcal{O}_X$ *and* $n - 1 < \tau \leq n$.

Proof. Let $\phi : X \to Z$ be the Mori contraction of the extremal ray R. Set $t > 0$ to be the rational number such that $(K_X + tL) \cdot C_0 = 0$. Let F be a general fiber of ϕ, then $(K_X + (n-1)L)|_F \notin \mathrm{Pseff}(F)$. As $\overline{\mathrm{NE}}(F) = \mathbb{R}_{\geq 0}[C_0]$, we have that $(K_X + (n-1)L) \cdot C_0 < 0$. Thus $t > (n-1)$.

Let m be a divisible enough integer such that mK_X is a Cartier divisor and mt is an integer. The line bundle $mK_X + mtL$ is ϕ-numerically trivial. By the contraction theorem, we know that $K_X + tL \sim_{\mathbb{Q},\phi} 0$. As $\mathrm{NE}(X/Z) = R$, we have that L is ϕ-ample. Thus we are in the situation of Theorem 3.3.

We first show that ϕ is not birational. Suppose by contradiction that $\phi : X \to Z$ is birational. Let F be a component of a non trivial fiber $F_1 = \phi^{-1}(z)$. By Theorem 3.3 (II,1), we have that $\dim(F) \geq t > n - 1$. Thus $\phi(X)$ is a singleton, a contradiction.

By Theorem 3.3, we know that $\dim(F) \geq t - 1 > n - 2$. Thus we have that either $\dim(F) = n$ or $\dim(F) = n - 1$.

(1) If $\dim(F) = n$, we have that $F = X$ and $Z = \{z\}$. Then $K_X + tL \sim_{\mathbb{Q}} \mathcal{O}_X$ and $\tau = t$. If $t > n$, Theorem 3.3 (I.2) implies that $(X, L) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ and

$\tau = n+1$. If $n-1 < t \leq n$, we have that $\dim(F) = n < t+1$. By Theorem 3.3 (I.3), we know that $\Delta(X, L) = 0$.

(2) $\dim(F) = n-1$. Let $F' \subset F_1$ be another component of F_1. Then Theorem 3.3 implies $\dim(F') \geq n-1$. On the other hand we can not have $\dim(F') = n$, for this would imply that $F = F' = X$ which has dimension n, a contradiction. Hence by Theorem 3.3 again, we know that F_1 is irreducible and $F = F_1$. As ϕ is not birational, by semi-continuity of dimensions of fibers (*cf.* for example [Sta22, Tag 02FZ]), for any point z', the fiber $\phi^{-1}(z')$ has positive dimension. By Theorem 3.3 and repeating the argument for F and F_1, we know that $\phi^{-1}(z')$ is irreducible with dimension $n-1$. Then Theorem 3.3 (I,2) implies that for every fiber $\phi^{-1}(z')$, we have that $(\phi^{-1}(z'), L_{\phi^{-1}(z')}) \cong (\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$. Thus we know that (X, L) is isomorphic to a $(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$-bundle over a smooth curve C and $\tau = n$.

□

We are now left in the case (3) of Lemma 3.4. In this case, we have the following:

Lemma 3.5. *Let (X, L) be a quasi-polarized variety of dimension n with $\Delta(X, L) = 0$. Suppose that X has canonical $\mathbb{Q}$-factorial singularities, and that the nefvalue $\tau = \tau(L)$ of L satisfies $n-1 < \tau(L) \leq n$. If $K_X + \tau L \sim_{\mathbb{Q}} \mathcal{O}_X$, then there exists a birational morphism $\mu : X \to Y$ such that*

(1) Y has canonical singularities, $\mu^*(K_Y) = K_X$;
(2) there exists an ample line bundle A on Y such that $\mu^*(A) = L$;
(3) $\Delta(Y, A) = 0$ and $K_Y + \tau A \equiv_{\text{num}} \mathcal{O}_Y$.

Proof. We have that $L - K_X \sim_{\mathbb{Q}} 2\tau L$ which is nef and big. Hence we may apply the basepoint-free theorem for L ([KM98, Theorem 3.3.]), to get that for all sufficient large integer b, the linear system $|bL|$ has no basepoints. We fix a such integer b_0. Now consider the graded algebra

$$R(X, L) =: \bigoplus_{n \geq 0} H^0(X, nL).$$

We have a canonical rational map $\mu : X \to \text{Proj}(R(X, L)) =: Y$. As $\text{Bs}(|b_0 L|) = \emptyset$, we know that μ has no indeterminacy and $R(X, L)$ is finite generated (*cf.* [Deb01, Proposition 7.6.]). Hence the ring $R(X, b_0 L)$ is integral and normal. As L is big, the morphism μ is birational and $b_0 L = \mu^*(A_1)$(*cf.* [Deb01, Lemma 7.10.]) for some ample Cartier divisor A_1. With the same argument for the integer b_0+1, we get another ample Cartier divisor A_2 such that $(b_0+1)L = \mu^*(A_2)$. By setting $A := A_2 - A_1$, we get (2).

We now take a divisible enough m such that mK_X is Cartier, the number $m\tau$ is an integer and $mK_X + m\tau L \sim_{\mathbb{Z}} 0$. Denote by E the exceptional locus of μ and by $\nu : Y \setminus \mu(E) \to X \setminus E$ the inverse of μ. We have that

$$\mathcal{O}_Y(mK_Y)|_{Y \setminus \mu(E)} \sim \nu^*(\mathcal{O}_X|_{X \setminus E}) \sim \nu^*(-m\tau L|_{X \setminus E}) \sim -m\tau A|_{Y \setminus \mu(E)}$$

We have that the rank one reflexive sheaf $\mathcal{O}_Y(mK_Y)$ and the line bundle $-m\tau A$ agree outside a subset whose codimension is at least 2. Hence $\mathcal{O}_Y(mK_Y)$ is a line bundle and K_Y is $\mathbb{Q}$-Cartier. We thus have the equalities $K_Y = -\tau A$ and $\mu^*(K_Y) = K_X$. Hence

μ is crepant and Y has canonical singularities. We get (1). By projection formula, we have that

$$K_Y + \tau A = \mu_*(K_X + \tau L) = \mathcal{O}_Y.$$

Thus $\Delta(Y, A) = n + A^n - h^0(Y, A) = n + L^n - h^0(X, L) = 0$. $\square$

Hence it rest for us to classify the polarized variety (X, L) with L ample, $n - 1 < \tau(L) \leq n$, $\Delta(X, L) = 0$ and $K_X + \tau(L)L \cong \mathcal{O}_X$. We have the following

Lemma 3.6. *Let (X, L) be a polarized variety with L ample, $n-1 < \tau(L) \leq n$, $\Delta(X, L) = 0$ and $K_X + \tau(L)L \equiv_{\text{num}} \mathcal{O}_X$. Suppose that X has canonical singularities. Then one of the following occurs:*

(1) $(X, L) \cong (Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$, where $Q \subset \mathbb{Q}^{n+1}$ is a hyperquadric;
(2) (X, L) is a $\mathbb{P}^{n-1}$-bundle over $\mathbb{P}^1$ and the restriction of L to each fiber is $\mathcal{O}_{P^{n-1}}(1)$;
(3) $(X, L) \cong (\mathbb{P}^2, \mathcal{O}^2_{\mathbb{P}}(2))$;
(4) $(X, L) \cong C_n(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ is a generalized cone over $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$

Proof. If $\tau(L) = n$, we have that $K_X + nL \equiv_{\text{num}} \mathcal{O}_X$. Then Theorem 1.1 implies that $(X, L) \cong (Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$, where $Q \subset \mathbb{P}^{n+1}$ is a hyperquadric. Hence we are in case (1).

From now on we may assume that $\tau(L) < n$. As L is ample, we have that

$$K_X + nL \equiv_{\text{num}} (n - \tau(L))L$$

is ample.

By Fujita's classification theorem for polarized varieties with Δ-genus zero (*cf.* [Fuj90, Theorem 5.10 and Theorem 5.15] [BS95, Proposition 3.1.2.]), we know that besides the four cases given above in Lemma 3.6, there are two more possibilities for (X, L):

(i) Either $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$,
(ii) or (X, L) is a generalized cone over (V, L_V), where $V \subset X$ is a smooth submanifold, $L|_V = L_V$ is very ample and $\Delta(V, L_V) = 0$.

Case (i) is impossible, since $\tau(\mathcal{O}_{\mathbb{P}^n}(1)) = n + 1$. Hence we need to investigate case (ii). Set $r := n - \dim(V)$. From Definition 2.4 we have the following diagram

$$\begin{tikzcd}
 & \mathbb{P}(\mathcal{O}_V^{\oplus r}) = V \times \mathbb{P}^{r-1} \arrow[r, "\mathrm{pr}_2"] \arrow[d, hook, "i"] & \mathbb{P}^{r-1} \arrow[d, hook] \\
\mathbb{P}(L_V) \arrow[r] \arrow[dr, "\cong"'] & \mathbb{P}(\mathcal{O}_V^{\oplus r} \oplus L_V) \arrow[r, "\psi_{|\xi|}"] \arrow[d, "\pi"] & C_n(V, L_V) = X \\
 & V &
\end{tikzcd}$$

where $\xi = \mathcal{O}_{\mathbb{P}(\mathcal{O}_V^{\oplus r} \oplus L_V)}(1)$ is the tautological bundle. The identification of $V \cong \mathbb{P}(L_V)$ is given by the quotient morphism $\mathcal{O}_V^{\oplus r} \oplus L_V \twoheadrightarrow L_V$.

We claim that outside $\mathbb{P}(\mathcal{O}_V^{\oplus r})$ the morphism $\psi_{|\xi|}$ induces an isomorphism onto its image. Take $z \in C_n(V, L)$ such that $\psi_{|\xi|}^{-1}(z)$ has positive dimension. In particular, there exists a curve C_1 such that $\psi_{|\xi|}(C_1) = \{z\}$. Since $\mathcal{O}_V^{\oplus r} \oplus L_V$ is globally generated, we know that $\psi_{|\xi|}$ restricted to each fiber of π is an embedding. Hence π maps C_1

bijectively to its image C. By generic smoothness, we have an open subset $U \subset C$ such that $\pi : C_0 := \pi^{-1}(U) \to U$ is an isomorphism. We may regard C_0 as a section of π defined over U. That is

$$\begin{tikzcd}
\mathbb{P}((\mathcal{O}_V^{\oplus r} \oplus L_V)|_U) \arrow[r] \arrow[d] & \mathbb{P}(\mathcal{O}_V^{\oplus r} \oplus L_V) \arrow[d] \\
U \arrow[u, bend left, "\sigma"] \arrow[r] & V
\end{tikzcd}$$

The section σ is defined by a quotient $\rho = (\rho_1, \rho_2) : (\mathcal{O}_V^{\oplus r} \oplus L_V)|_U \twoheadrightarrow M$, with M a line bundle on U. The morphism ρ has a decomposition into $\rho_1 : \mathcal{O}_V^{\oplus r} \to M$ and $\rho_2 : L_V \to M$. As $\psi_{|\xi|} \circ \sigma(U) = \{z\}$, we know that $M \cong \sigma^*(\xi|_{\mathbb{P}((\mathcal{O}_V^{\oplus r} \oplus L_V)|_U)})$ is trivial. As $h^0(\mathrm{Hom}_{\mathcal{O}_U}(L_V|_U, \mathcal{O}_U)) = h^0(U, L_V^\vee|_U) = 0$, we have that $\rho_2 = 0$. Hence the quotient is given by $\rho_1 : \mathcal{O}_U^{\oplus n-1} \to \mathcal{O}_U$. Hence $C_0 = U \subset \mathbb{P}(\mathcal{O}_V^{\oplus r})$ and $C = \overline{C_0} \subset \mathbb{P}(\mathcal{O}_V^{\oplus r})$.

As $V = \mathbb{P}(L_V)$ is smooth, we have the short exact sequence

$$0 \to T_{\mathbb{P}(L_V)} \to T_{\mathbb{P}(\mathcal{O}_V^{\oplus r} \oplus L_V)}|_{\mathbb{P}(L_V)} \to N_{\mathbb{P}(L_V)/\mathbb{P}(\mathcal{O}_V^{\oplus r} \oplus L_V)} \to 0.$$

We have thus

$$\omega^\vee_{\mathbb{P}(\mathcal{O}_V^{\oplus r} \oplus L_V)}|_{\mathbb{P}(L_V)} = \omega^\vee_{\mathbb{P}(L_V)} \otimes \wedge^r N_{\mathbb{P}(L_V)/\mathbb{P}(\mathcal{O}_V^{\oplus r} \oplus L_V)}. \tag{1}$$

The canonical bundle formula gives us

$$\omega_{\mathbb{P}(\mathcal{O}_V^{\oplus r} \oplus L_V)} = \pi^*(\omega_V \otimes L_V) \otimes \xi^{\otimes -(r+1)}.$$

With $\xi|_V = L_V$, we know that $\omega_{\mathbb{P}(\mathcal{O}_V^{\oplus r} \oplus L_V)}|_V = \omega_V \otimes L_V^{\otimes -r}$. Thus Equation (1) gives

$$\wedge^r N_{\mathbb{P}(L_V)/\mathbb{P}(\mathcal{O}_V^{\oplus r} \oplus L_V)} = L_V^{\otimes r}.$$

As $\mathbb{P}(L_V)$ is disjoint from the singular locus $\mathbb{P}^{r-1} \subset X$, we also have the exact sequence

$$0 \to T_{\mathbb{P}(L_V)} \to T_X|_{\mathbb{P}(L_V)} \to N_{\mathbb{P}(L_V)/X} \to 0.$$

Hence

$$\omega_X^\vee|_{\mathbb{P}(L_V)} = \omega^\vee_{\mathbb{P}(L_V)} \otimes \wedge^r N_{\mathbb{P}(L_V)/X}.$$

Note $N_{\mathbb{P}(L_V)/X} = N_{\mathbb{P}(L_V)/\mathbb{P}(\mathcal{O}_V^{\oplus r} \oplus L_V)}$. Hence $\omega_X|_V = \omega_V \otimes L^{\otimes -r}$. Then we have

$$\omega_X \otimes L^{\otimes n}|_V = \omega_V \otimes L^{\otimes (n-r)}.$$

Hence the divisor $K_V + \dim(\mathrm{V}) L_V$ is ample.

If $\dim(\mathrm{V}) \geq 2$, apply [Fuj90, Theorem 5.10] again for (V, L_V). We know that (V, L_V) is one of the following:

- $(\mathbb{P}^{\dim(V)}, \mathcal{O}_{\mathbb{P}^{\dim(V)}}(1))$; or
- $(Q, \mathcal{O}_Q(1))$, where $Q \subset \mathbb{P}^{\dim(V)+1}$ is a hyperquadric; or
- $(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$ where $\mathcal{E}$ is an ample vector bundle of rank $\dim(V)$ over $\mathbb{P}^1$; or
- $(\mathbb{P}^2, \mathcal{O}^2_{\mathbb{P}}(2))$

Suppose first that $\dim(V) = 2$. If (V, L) is $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ or $(Q, \mathcal{O}_{\mathbb{P}^3}(1))$, the divisor $K_V + 2L_V$ will not be ample. If (V, L) is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$, then $K_V + 2L_V$ is trivial on each fiber, contradicting to the fact that $K_V + 2L_V$ is ample. Hence we have $(V, L) \cong (\mathbb{P}^2, \mathcal{O}^2_{\mathbb{P}}(2))$.

If $\dim(V) = 1$, we have that $(V, L_V) \cong (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a))$ with $a \geq 3$. By the following Example 3.7 we know that for $n \geq 2$, a generalized cone $C_n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a))$ has singularities worse than canonical.

Hence when (X, L) is a generalized cone, we have that $(X, L) \cong C_n(\mathbb{P}^2, \mathcal{O}^2_{\mathbb{P}}(2))$. □

We give some characterizations of generalized cones over $\mathbb{P}^1$.

Example 3.7. Let $(X, L) = C_n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a))$ be a generalized cone with $a \geq 3$ and $n \geq 2$. We have

(1) X has klt singularities and X is not canonical;
(2) the nefvalue of L is $n - \frac{a-2}{a}$;
(3) $K_X + (n-1)L$ is not pseudo-effective.

Example 3.8. Let $(X, L) = C_n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a))$ be a generalized cone with $a \leq 2$ and $n \geq 2$. Then $K_X + (n-1)L \in \mathrm{Pseff}(X)$.

For the proof of the above two examples, see [Liu22, Lemma 3.11 and 3.12].

Proof of Theorem 1.3. The proof is by combining all the precedent results.

Proof. By Lemma 3.2, we have that $(X, L) \sim_{\mathrm{bir}} (X', L')$, where X' is a normal projective variety with canonical $\mathbb{Q}$-factorial singularity, $K_{X'} + (n-1)L' \notin \mathrm{Pseff}(X')$ and

(0-i) Either $\tau(L')$ is finite;
(0-ii) or there is a Mori fiber space structure $\phi : X' \to W$ and a rational number $\tau > (n-1)$ such that L' is ϕ-ample and $K_{X'} + \tau L' \sim_{\mathbb{Q},\phi} 0$.

In the first case, we have that $r(L') = \frac{1}{\tau(L')} > 0$, hence by Kawamata rationality theorem there exists an K'_X-negative extremal ray $R_0 = \mathbb{R}_{\geq 0}[C_0]$ such that $(r(L')K_{X'} + L') \cdot C_0 = 0$. Hence $L' \cdot C_0 > 0$. In the second case, take $R_0 = \mathbb{R}_{\geq 0}[C_0]$ be the extremal ray associated to ϕ. Then $L' \cdot C_0 > 0$.

Applying Lemma 3.4 on (X', L'), we get that (X', L') is the one of the following

(1) $(X', L') \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, and $\tau = n + 1$;
(2-i) (X', L') is isomorphic to a $(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$-bundle over a smooth curve C and $\tau = n$;
(∗) $\Delta(X', L') = 0$, $K_{X'} + \tau L' \equiv_{\mathrm{num}} \mathcal{O}_X$ and $n - 1 < \tau \leq n$.

If we are in case (∗), apply Lemma 3.5. We have a birational morphism $\mu : X' \to X''$ such that

(a) X'' has canonical singularities, $\mu^*(K_{X''}) = K_{X''}$;
(b) There exists an ample line bundle L'' on X'' such that $\mu^*(L'') = L'$;
(c) $\Delta(X'', L'') = 0$ and $K_{X''} + \tau L'' \equiv_{\mathrm{num}} \mathcal{O}_{X''}$.

In particular we have that $(X', L') \sim_{\mathrm{bir}} (X'', L'')$. Now apply Lemma 3.6 to (X'', L''). We have that (X'', L'') is isomorphic to the following pair:

(3) $(X'', L'') \cong (Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$, where $Q \subset \mathbb{Q}^{n+1}$ is a hyperquadric;
(2-ii) (X'', L'') is a $\mathbb{P}^{n-1}$-bundle over $\mathbb{P}^1$ and L restricted to each fiber is $\mathcal{O}_{P^{n-1}}(1)$;

(4) $(X'', L'') \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}}^2(2))$;
(5) $(X'', L'') \cong C_n(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}}^2(2))$ is a generalized cone over $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}}^2(2))$

Thus we get the list stated in Theorem 1.3 □

4. Normal polarized varieties

With the help of canonical modification [Kol13, Theorem 1.31], we can give a classification theorem for normal polarized varieties with $\mathbb{Q}$-Gorenstein singularities.

Proof of Theorem 1.4.

Proof. Apply [Kol13, Theorem 1.31] to the pair $(X, 0)$. We get the canonical modification $f : X' \to X$ with $K_{X'}$ being f-ample. We take a further step, taking a small $\mathbb{Q}$-factorial modification $g : Y \to X'$ of X' (*cf.* [Kol13, Corollary 1.37]). We denote the composition $g \circ f$ by μ. As g is small, we have that $K_Y = g^*(K_{X'})$ is μ-nef. Note that $\mu|_{\mu^{-1}(X_{\text{reg}})} : \mu^{-1}(X_{\text{reg}}) \to X_{\text{reg}}$ is an isomorphism.

We have that $\mu_*(\omega_Y)|_{X_{\text{reg}}} \cong \omega_X|_{X_{\text{reg}}}$ for the canonical sheaves $\omega_Y = \mathcal{O}_Y(K_Y)$ and $\omega_X = \mathcal{O}_X(K_X)$. Note that $\mu_*(\omega_Y)$ is torsion-free, so we have an injection $\mu_*(K_Y) \rightarrowtail K_X$. By the projection formula we have an injection

$$\mathcal{O}_X(\mu_*(K_Y + (n-1)\mu^*L)) \rightarrowtail \mathcal{O}_X(K_X + (n-1)L).$$

As $K_X + (n-1)L$ is not pseudo-effective, we know that neither is $K_Y + (n-1)\mu^*(L)$. We set $\mu^*(L) = M$. As M is nef and big, we know that $M \in \mathrm{Pseff}(Y)$. Note that K_Y is not pseudo-effective, hence it is not nef.

Let $R = \mathbb{R}_{\geq 0}[C]$ be a K_Y-negative extremal ray, with $C \subset Y$ a rational curve. We have that $K_Y \cdot C < 0$. As K_Y is μ-nef, we know that C is not contracted by μ. Hence $\mu(C) \subset X$ has dimension 1. The intersection number $M \cdot C = \deg(C/\mu(C))L \cdot \mu(C)$ is positive, since L is ample. Thus for any K_Y-negative extremal ray R, one has $M \cdot R > 0$. By Lemma 3.2, we obtain that $\mathrm{r}(M) > 0$ and $\tau(M) > n-1$. By Lemma 3.4 applied to (Y, M), we have one of the following cases:

(i) $(Y, M) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, and $\tau = n+1$, or
(ii) (Y, M) is isomorphic to a $(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$-bundle over a smooth curve C and $\tau = n$, or
(iii) $\Delta(Y, M) = 0$, $K_Y + \tau M \sim_{\mathbb{Q}} \mathcal{O}_Y$ and $n - 1 < \tau \leq n$.

In case (i), we have a birational morphism $\mu : \mathbb{P}^n \to X$ with $\mu^*(L) = \mathcal{O}_{\mathbb{P}^n}(1)$. We have that $\overline{\mathrm{NE}}(\mathbb{P}^n/X) = 0$ since both L and $\mathcal{O}_{\mathbb{P}^n}(1)$ are ample. By [Deb01, Proposition 1.14], the morphism μ is an isomorphism. We have case (1) in Theorem 1.4.

In case (ii), we have a birational morphism $\mu : \mathbb{P}(\mathcal{V}) \to X$, such that $K_{\mathbb{P}(\mathcal{V})}$ is μ-nef. We denote by ξ the pull-back $\mathcal{O}_{\mathbb{P}(\mathcal{V})}(1) = \mu^*(L)$. We know that ξ is nef and big.

We first note that ξ is ample if and only if μ is an isomorphism. In fact, if μ is an isomorphism, then we have ξ is ample. Conversely, if ξ is ample, we have $\overline{\mathrm{NE}}(\mathbb{P}(\mathcal{V})/X) = 0$ and hence μ is an isomorphism. In this case, we have that

$$K_{\mathbb{P}(\mathcal{V})} + (n-1)\xi = \pi^*(K_C + \det \mathcal{V}) - \xi$$

is not pseudo-effective. In fact, the general fiber f is from a covering family and we have that $K_{\mathbb{P}(\mathcal{V})} + (n-1)\xi|_f = \mathcal{O}_f(-1)$. Hence by the BDPP theorem (*cf.* [Laz04b, Theorem 11.4.19]), we know that $K_{\mathbb{P}(\mathcal{V})}+(n-1)\xi$ is not pseudo-effective. Thus we get *(2,i)*.

Now suppose that ξ is not ample. Then μ is not an ismorphism. We have the following diagram:

$$\begin{tikzcd}
(\mathbb{P}(\mathcal{V}),\xi) \arrow[r, "\mu"] \arrow[d, "\pi"] & (X,L) \\
C
\end{tikzcd} .$$

We know that $\rho(\mathbb{P}(\mathcal{V}))=2$. As $\mu\neq\pi$, we have that

$$\overline{\mathrm{NE}}(\mathbb{P}(\mathcal{V}))=\mathrm{NE}(\pi)+\mathrm{NE}(\mu).$$

We denote a general fiber of π by f. By [Ful11, Page 450], we know that $\overline{\mathrm{NE}}(\mathbb{P}(\mathcal{V}))$ has as extremal rays $\mathbb{R}_{\geq 0}\xi^{n-2}f$ and $\mathbb{R}_{\geq 0}(\xi^{n-1}+\nu^{(n-1)}\xi^{n-2}f)$ for some $\nu^{(n-1)}\in\mathbb{Q}$. Note that $\mathbb{P}^1=\xi^{n-2}f$ is contracted by π. Hence $\mathrm{NE}(\pi)=\mathbb{R}_{\geq 0}\xi^{n-2}f$. We have $K_{\mathbb{P}(\mathcal{V})}=\pi^*(K_C+\det(\mathcal{V}))-n\xi$. Hence $K_{\mathbb{P}(\mathcal{V})}\cdot\xi^{n-2}f=-n$. Thus π is the Mori contraction associated to the extremal ray $\mathbb{R}_{\geq 0}\xi^{n-2}f$. As $\overline{\mathrm{NE}}(\mu)$ is an extremal ray, we know that μ is an extremal contraction. By [KM98, Proposition 2.5.], we know that μ is either small or divisorial.

If μ is small, we have that $K_{\mathbb{P}(\mathcal{V})}=\mu^*(K_X)$. As $\rho(X)=1$, we have that $K_X\equiv_{\mathrm{num}}mL$ for some $m\in\mathbb{Q}$. Hence $K_{\mathbb{P}(\mathcal{V})}\equiv_{\mathrm{num}}m\xi$. We have that

$$m=m\xi\cdot\xi^{n-2}f=K_{\mathbb{P}(\mathcal{V})}\cdot\xi^{n-2}f=-n.$$

Thus we get that $K_X+nL\equiv_{\mathrm{num}}\mathcal{O}_X$. By Theorem 1.1, we have that $(X,L)\cong(Q,\mathcal{O}_Q(1))$ where $Q\subset\mathbb{P}^{n+1}$ is a hyperquadric. Hence we are in case (3) of Theorem 1.4.

If μ is divisorial, we denote the exceptional divisor by $E=\mathrm{exc}(\mu)$. Note that $\mathcal{V}$ is nef, since $\mu^*(L)=\xi=\mathcal{O}_{\mathcal{V}}(1)$ is nef. We have a unique exact sequence of locally free sheaves:

$$0\to\mathcal{A}\to\mathcal{V}\to\mathcal{Q}\to 0.$$

with $\mathcal{A}$ being an ample vector bundle and $\mathcal{Q}$ being numerically flat. If $l\subset\mathbb{P}(\mathcal{V})$ is a curve such that $\xi\cdot l=0$, we have that $l\subset\mathbb{P}(\mathcal{Q})$. Thus we have that $E\subset\mathbb{P}(\mathcal{Q})$. In particular, $\mathrm{rk}(\mathcal{Q})=n-1$ and $E=\mathbb{P}(\mathcal{Q})$. We denote the bundle morphism by $\pi':\mathbb{P}(\mathcal{Q})\to C$. Now we compute $E|_E$:

$$\begin{aligned}
E|_E &= (K_{\mathbb{P}(\mathcal{Q})}-K_{\mathbb{P}(\mathcal{V})})|_E\\
&= \pi'^*(K_C+\det\mathcal{Q})-(n-1)\xi|_E-(\pi^*(K_C+\det\mathcal{V})-n\xi)|_E\\
&= \pi'^*(\det\mathcal{Q}-\det\mathcal{V})+\xi|_E\\
&= \pi'^*(-\mathcal{A})+\xi|_E.
\end{aligned}$$

Take a rational curve l that is in the fiber of π'. We have that $E\cdot l=E|_E\cdot l=1$. Now write $K_{\mathbb{P}(\mathcal{V})}=\mu^*(K_X)+\lambda E$. As $\rho(X)=1$, we have that $K_X\equiv_{\mathrm{num}}mL$ for some $m\in\mathbb{Q}$. As $K_X+(n-1)L\equiv_{\mathrm{num}}(m+n-1)L\notin\mathrm{Pseff}(X)$, we have that $m+n<1$. Intersecting with l, we get that

$$-n=K_{\mathbb{P}(\mathcal{V})}\cdot l=(\mu^*(K_X)+\lambda E)\cdot l=(m\xi+\lambda E)\cdot l=m+\lambda.$$

Hence $\lambda = -m - n > -1$ and X has klt singularities. A π'-fiber is isomorphic to $\mathbb{P}^{n-2}$ and is mapped isomorphically onto its image by μ. Hence each non-trivial μ-fiber has dimension 1. As X has klt singularities, *a fortiori* $(X, 0)$ is dlt. Applying [HM07, Corollary 1.5-(1)] to the birational morphism μ, each μ-fiber is rationally chain connected. Hence a non trivial fiber has $\mathbb{P}^1$ as its normalization. We have thus a finite map $\pi'|_{\mathbb{P}^1} : \mathbb{P}^1 \to C$. Thus $C = \mathbb{P}^1$ and $\mathcal{V} = \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus(n-1)}$.

Consider the morphism $\psi : \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus(n-1)}) \to C_n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a))$. We know that ψ does not contract the extremal ray $\overline{\mathrm{NE}}(\pi)$. Hence $\overline{\mathrm{NE}}(\psi) = \overline{\mathrm{NE}}(\mu)$ and by [Deb01, Proposition 1.14] $X = C_n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a))$. As L and the restriction of $\mathcal{O}_{\mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a)))}$ to $C_n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a))$ agree outside a subscheme of codimension at least 2, we have that $(X, L) = C_n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a))$. As $K_X + (n-1)L \notin \mathrm{Pseff}(X)$, Example 3.8 implies $a \geq 3$. Now Example 3.7 shows that for all $a \geq 3$, the divisor $K_X + (n-1)L$ is not pseudo-effective and X is klt. Thus we get *(2,ii)*.

If we are in case (iii), apply Lemma 3.5 to (Y, M). We have a crepant resolution $\nu : Y \to Y_{\mathrm{can}}$ with an ample divisor A on Y_{can} such that $\nu^*(A) = M$, the Δ-genus satisfies $\Delta(Y_{\mathrm{can}}, A) = 0$ and $K_{Y_{\mathrm{can}}} + \tau A \equiv_{\mathrm{num}} \mathcal{O}_{Y_{can}}$. By Lemma 3.6, (Y_{can}, A) is isomorphic to one of the following:

(a) $(Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$, where $Q \subset \mathbb{Q}^{n+1}$ is a hyperquadric;
(b) a $\mathbb{P}^{n-1}$-bundle over $\mathbb{P}^1$ and L restricted to each fiber is $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$;
(c) $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$;
(d) a generalized cone $C_n(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ over $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$.

Case (b) is a special case of (ii) treated above. In case (a),(c) and (d), we have the following diagram

$$\begin{tikzcd}
(Y, M) \arrow[r, "\mu"] \arrow[d, "\nu"] & (X, L) \ , \\
(Y_{\mathrm{can}}, A) \arrow[ur, dashed, "h"'] &
\end{tikzcd}$$

where h is a birational map *a priori* not necessarily defined on all Y_{can}. We now show h is indeed an isomorphism and $h^*(L) = A$. Let $C \subset Y$ be a curve. We have

$$\nu^*(A) \cdot C = M \cdot C = \mu^*(L) \cdot C.$$

As A and L are both ample, we have that $\mathrm{NE}(\mu) = \mathrm{NE}(\nu)$. [Deb01, Proposition 1.14] implies that h is an isomorphism. As $h^*(L)$ agrees with A outside a subscheme of codimension at least 2, we have that $h^*(L) = A$. Hence we get case (3), (4), (5) in Theorem 1.4. □

Using similar methods, we can classify log pairs (X, Δ) with Δ a reduced Weil divisor.

Proof of Proposition 1.5.

Proof. We take a canonical modification of X then take a small $\mathbb{Q}$-factorialization. We get a birational morphism $\mu : Y \to X$ such that Y has $\mathbb{Q}$-factorial canonical singularities, K_Y is μ-nef and μ is isomorphic over regular points of X. Set $\Delta' := \mu_*^{-1}(\Delta)$. Then Δ'

is a reduced divisor. Let $\omega_Y = \mathcal{O}_Y(K_Y)$ and $\omega_X = \mathcal{O}_X(K_X)$ be the canonical sheaves. We know that

$$\mu_*(\omega_Y \otimes \mathcal{O}_Y(\Delta'))|_{X_{\mathrm{reg}}} \cong (\omega_X \otimes \mathcal{O}_X(\Delta))|_{X_{\mathrm{reg}}}.$$

The sheaf $\mu_*(\omega_Y \otimes \mathcal{O}_Y(\Delta'))$ is torsion-free, so we have an injection

$$\mu_*(\omega_Y \otimes \mathcal{O}_Y(\Delta')) \rightarrowtail \omega_X \otimes \mathcal{O}_X(\Delta).$$

Tensoring with $\mu^*(L^{\otimes n-1})$, we have an injection

$$\mu_*(\omega_Y \otimes \mathcal{O}_Y(\Delta') \otimes \mu^*(L^{\otimes n-1})) \rightarrowtail \omega_X \otimes \mathcal{O}_X(\Delta) \otimes L^{\otimes n-1}.$$

As $(K_X+\Delta)+(n-1)L$ is not pseudo-effective, neither is $(K_Y+\Delta')+(n-1)\mu^*(L)$. We set $\mu^*(L) =: M$. As Δ' is effective, the divisor $K_Y + (n-1)M$ is not pseudo-effective.

As K_Y is μ-nef and $M = \mu^*(L)$, for any K_Y-negative extremal ray R, we have that $M \cdot R > 0$. Hence we can apply Lemma 3.4 to (Y, M) and get:

(a) $(Y, M) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, and $\tau = n+1$;
(b) (Y, M) is isomorphic to a $(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$-bundle over a smooth curve C and $\tau = n$;
(c) $\Delta(Y, M) = 0$, $K_Y + \tau M \sim_{\mathbb{Q}} \mathcal{O}_Y$ and $n-1 < \tau(M) \leq n$.

If we are in case (a), the morphism μ is an isomorphism. The divisor Δ is given by $\mathcal{O}_{\mathbb{P}^n}(a)$ for some $a \geq 1$. We have that $K_X + (n-1)L + D = \mathcal{O}_{\mathbb{P}^n}(a-2)$. Hence the only possible choice is $a = 1$ and $\Delta = D$ is a hyperplane. We are thus in case (1) of Proposition 1.5.

If we are in case (b), we have a diagram

$$\begin{tikzcd} (\mathbb{P}(\mathcal{V}), \xi) \arrow[r, "\mu"] \arrow[d, "\pi"'] & (X, L) \\ C & \end{tikzcd},$$

where $\xi = \mathcal{O}_{\mathbb{P}(\mathcal{V})}(1)$ and $K_{\mathbb{P}(\mathcal{V})}$ is μ-nef.

First we assume that μ is an isomorphism. In this case the vector bundle $\mathcal{V}$ is ample and $X = \mathbb{P}(\mathcal{V})$ is $\mathbb{Q}$-factorial. Let $F = \mathbb{P}^{n-1}$ be a general fiber of π . Suppose that $\Delta|_F = \mathcal{O}_F(d)$ for some natural number $d \geq 0$. We have the following equality:

$$(K_{\mathbb{P}(\mathcal{V})} + \Delta + (n-1)\xi)|_F = (\pi^*(K_C + \det(\mathcal{V})) + \Delta - \xi)|_F = \mathcal{O}_F(d-1).$$

If $d = 0$, let D be a component of Δ, then $D|_F = \mathcal{O}_F(0)$. We claim that D is one of the general fiber. In fact, suppose by contradiction that there exists a general fiber F such that $D \cap F \neq \emptyset$ and $D \nsubseteq F$. Then there will be a curve $l \subset F \setminus D$ such that $l \cap D \neq \emptyset$. Then we have that $D \cdot l > 0$, a contradiction. Thus we have that $\Delta = \sum F_i$ is a finite sum of distinct general fibers. Let l be a rational curve in F. We have that

$$(K_{\mathbb{P}(\mathcal{V})} + \Delta + (n-1)\xi) \cdot l = -1.$$

Since F is a member of a covering family, BDPP theorem (*cf.* [Laz04b, Theorem 11.4.19]) implies that $K_{\mathbb{P}(\mathcal{V})} + (n-1)\xi + \Delta$ is not pseudo-effective. We are thus in case $(2.i)$ of Proposition 1.5.

If $d > 0$, let D be a component of Δ such that $D|_F = \mathcal{O}_F(d')$ for some $d' > 0$. By Lemma 4.1 after the proof, we have that $n = \dim(\mathbb{P}(\mathcal{V})) = 2$. We first show that

$C = \mathbb{P}^1$. The non pseudo-effective divisor in question $K_X + D + (n-1)L$ thus becomes $K_{\mathbb{P}(\mathcal{V})} + D + \xi$. We have that $(K_{\mathbb{P}(\mathcal{V})} + D + \xi)|_F = \mathcal{O}_F(d'-1)$ which is nef. As $K_{\mathbb{P}(\mathcal{V})} + D + \xi$ is not nef, we note that there will be an extremal ray R' which is not generated by the fiber of π, such that $(K_{\mathbb{P}(\mathcal{V})} + D + \xi) \cdot R' < 0$. In particular, we have that R' is $(K_{\mathbb{P}(\mathcal{V})} + D)$-negative. By the cone theorem, we know that $R' = \mathbb{R}_{\geq 0}[l]$ for a rational curve l. Note that l maps finitely onto C. Hence we have $C \cong \mathbb{P}^1$.

Thus (X, L) is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ and Lemma 4.2 implies that (X, Δ, L) is either in cases $(2.i)$, $(2.ii)$ of Proposition 1.5 or (X, L) is a hyperquadric of rank 4, which will be dealt in the following case $(c1)$.

Assume, from now on, that μ is not an isomorphism. We know that

$$\overline{\mathrm{NE}}(\mathbb{P}(\mathcal{V})) = \overline{\mathrm{NE}}(\mu) + \overline{\mathrm{NE}}(\pi).$$

and π contracts the extremal ray $\mathbb{R}_{\geq 0}\xi^{n-2}f$. The birational morphism μ is either small or divisorial.

If μ is small, by construction, we have that $K_{\mathbb{P}(\mathcal{V})} + \Delta' = \mu^*(K_X + \Delta)$. Let F be a general fiber of π. We have that $\Delta'|_F = \mathcal{O}_F(d)$ for some integer $d \geq 0$. As $K_X + \Delta$ is $\mathbb{Q}$-Cartier and $\rho(X) = 1$, we have that $K_X + \Delta \equiv_{\mathrm{num}} mL$ for some $m \in \mathbb{Q}$. Hence $K_{\mathbb{P}(\mathcal{V})} + \Delta' \equiv_{\mathrm{num}} \mu^* mL$. Intersect with $\xi^{n-2}f$. We get that $-n + d = m$. Hence $K_X + \Delta + (n-1)L \equiv_{\mathrm{num}} (d-1)L$. Thus $d = 0$.Hence $d = 0$. If we write $\Delta' = \sum D_i'$ with D_i' distinct prime divisors. We have that $D_i'|_F = \mathcal{O}_F(0)$. Thus the D_i''s are distinct general fibers. As $D_i' = \mu_*^{-1}(D_i)$ by definition, we get that $D_i' \to D_i = \mu(D_i')$ has degree 1. Thus we are in case $(2.i)$ of Proposition 1.5.

If μ is divisorial, we denote the exceptional divisor by $E = \mathrm{exc}(\mu)$. [KM98, Proposition 3.36.] implies that X is $\mathbb{Q}$-factorial. In particular K_X is $\mathbb{Q}$-Cartier. We have a unique exact sequence of locally free sheaves:

$$0 \to \mathcal{A} \to \mathcal{V} \to \mathcal{Q} \to 0$$

with $\mathcal{A}$ is an ample vector bundle and $\mathcal{Q}$ is numerically flat. And we know that $E = \mathbb{P}(\mathcal{Q})$ and $E \cdot \xi^{n-2}f = 1$. Let F be a general fiber of π. There exists a $d \geq 0$ such that $\Delta'|_F = \mathcal{O}_F(d)$. As $K_X + \Delta$ is $\mathbb{Q}$-Cartier and $\rho(X) = 1$, there exists an $m \in \mathbb{Q}$ such that $K_X + \Delta \equiv_{\mathrm{num}} mL$. Then $K_X + \Delta + (n-1)L \equiv_{\mathrm{num}} (m+n-1)L \notin \mathrm{Pseff}(X)$. Hence $m + n < 1$. We now have

$$K_{\mathbb{P}(\mathcal{V})} + \Delta' = \mu^*(K_X + \Delta) + \lambda E.$$

Intersect both sides with $\xi^{n-2}f$. We get that $-n + d = m + \lambda$. Since $-(m+n) > -1$, we have that $\lambda \geq -1 + d$.

Now we claim that $d = 0$. Suppose by contradiction that $d \geq 1$. Then we have that $(K_{\mathbb{P}(\mathcal{V})} + \Delta' + (n-1)\xi)|_F = \mathcal{O}_F(d-1)$. As $K_{\mathbb{P}(\mathcal{V})} + \Delta' + (n-1)\xi$ is not nef, we know that $\mathrm{NE}(\mu)$ is an $(K_{\mathbb{P}(\mathcal{V})} + \Delta')$-negative extremal ray. Note that $(\mathbb{P}(\mathcal{V}), \Delta')$ is log canonical. By the cone theorem, there is a rational curve l whose class $[l]$ is in $\overline{\mathrm{NE}}(\mu)$. As l maps finitely onto C, we know that $C \cong \mathbb{P}^1$. Hence $(X, L) = C_n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a))$. As $K_X + (n-1)L$ is not pseudo-effective, Example 3.8 implies that $a \geq 3$. Example 3.7 implies that $K_X \equiv_{\mathrm{num}} (-n + \dfrac{a-2}{a})L$. Suppose that $\Delta \equiv_{\mathrm{num}} m_2 L$ for some $m_2 \in \mathbb{Q}^+$. For $\mathbb{P}^1 \subset F$ mapped isomorphic to its image, we have that $m_2 = m_2\xi \cdot \mathbb{P}^1 = \mu^*(\Delta) \cdot \mathbb{P}^1 = \Delta \cdot \mu(\mathbb{P}^1) \in \mathbb{N}$.

Hence $m_2 \geq 1$ and $K_X + \Delta + (n-1)L = \dfrac{a-2}{a}L \in \mathrm{Pseff}(X)$, a contradiction. This proves the claim.

Write $\Delta = \sum D_i$ and $\Delta' = \sum D'_i$. Then each D'_i is a general fiber. As $D'_i = \mu_*^{-1}(D_i)$ by definition, we get that $D'_i \to D_i$ has degree 1. Thus we have that $D_i \cong \mu(\mathbb{P}^{n-1})$ are the images of distinct general fibers of π and we are in case $(2.i)$ of Proposition 1.5.

If we are in case (c), apply Lemma 3.5 to (Y, M). We have a crepant resolution $\nu : Y \to Y_{\mathrm{can}}$ with an ample divisor A on Y_{can} such that $\nu^*(A) = M$, the Δ-genus $\Delta(Y_{\mathrm{can}}, A) = 0$ and $K_{Y_{\mathrm{can}}} + \tau A \equiv_{\mathrm{num}} \mathcal{O}_{Y_{can}}$. By Lemma 3.6, we have one of the following cases:

- (c1) $(Y_{\mathrm{can}}, A) \cong (Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$, where $Q \subset \mathbb{Q}^{n+1}$ is a hyperquadric;
- (c2) (Y_{can}, A) is a $\mathbb{P}^{n-1}$-bundle over $\mathbb{P}^1$ and the restriction of L to each fiber is $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$;
- (c3) $(Y_{\mathrm{can}}, A) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$;
- (c4) $(Y_{\mathrm{can}}, A) \cong C_n(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ is a generalized cone over $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$.

We have the following diagram

$$\begin{tikzcd} (Y, M) \arrow[r, "\mu"] \arrow[d, "\nu"] & (X, L) \\ (Y_{\mathrm{can}}, A) \arrow[ur, "h"'] & \end{tikzcd},$$

such that h is an isomorphism and $\mu^*(L) = M = \nu^*(A)$ with (Y_{can}, A) being one of the above four pairs.

In case $(c1)$, after an automorphism of $\mathbb{P}^{n+1} = \mathrm{Proj}(\mathbb{C}[x_0, ... x_{n+1}])$, the hyperquadric Q is given by the homogeneous ideal $I_r = (\sum_{0 \leq i \leq r} x_i^2) \subset \mathbb{C}[x_0, ... x_{n+1}]$ for some $r \geq 2$. By [Har77, Exercise II.6.5], the class group $\mathrm{Cl}(Q)$ of Q is the following:

- When $r = 2$, $\frac{1}{2}[\mathcal{O}_Q(1)]$ is an integral divisor and $\mathrm{Cl}(Q) = \mathbb{Z} \cdot \frac{1}{2}[\mathcal{O}_Q(1)]$. Suppose that $\Delta = k \cdot \frac{1}{2}[\mathcal{O}_Q(1)]$. Write $\Delta = \sum D_i$. Then each . Then
$$K_X + (n-1)L + \Delta = (\tfrac{k}{2} - 1)\mathcal{O}_Q(1).$$
It is not pseudo-effective if and only if $k = 1$. Thus $\Delta = D$ is irreducible and is numerically equivalent to a hyperplane $\mathbb{P}^{n-1}$ in Q. We are thus in case $(3.i)$ of Proposition 1.5.
- When $r = 3$, $\mathrm{Cl}(Q) \cong \mathbb{Z} \oplus \mathbb{Z}$. Note that here we can write
$$Q = \mathrm{Proj}\left(\frac{\mathbb{C}[x_0, \ldots, x_{n+1}]}{(x_0x_1 - x_2x_3)}\right),$$
which is a cone of vertex $\mathbb{P}^{n-3} = \{x_1 = x_2 = x_3 = 0\} \subset \mathbb{P}^{n+1}$ with base $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3 = \{x_4 = \cdots = x_{n+1} = 0\} \subset \mathbb{P}^{n+1}$ (*cf.* [Har77, Exercise I.5.12.(d)]). If we consider the inclusions $\mathbb{P}^3 \subset \mathbb{P}^4 \subset \cdots \subset \mathbb{P}^n \subset \mathbb{P}^{n+1}$, then Q is also obtained by taking projective cone in the sense of [Har77, Exercise I.2.10] of $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ successively. By [Har77, Exercise II.6.3.(a)], we know $\mathrm{Cl}(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathrm{Cl}(Q)$. For a hyperplane $H \subset \mathbb{P}^{n+1}$, $H \cap Q$ has type $(1,1)$. The cone over $\mathbb{P}^1 \times \mathrm{pt}$ has type $(1,0)$ and the cone over $\mathrm{pt} \times \mathbb{P}^1$ has type $(0,1)$. Thus Δ has type $(1,0)$ or type $(0,1)$ and is irreducible. We are thus in case $(3.ii)$ of Proposition 1.5.

- When $r \geq 4$, $\mathrm{Cl}(Q) = \mathbb{Z} \cdot [\mathcal{O}_Q(1)]$. Hence $\Delta = d[\mathcal{O}_Q(1)]$, and
$$K_X + (n-1)L + \Delta \equiv_{\mathrm{num}} \mathcal{O}_Q(d-1)$$
is pseudo-effective. Thus this situation is excluded.

The case $(c2)$ is treated in case (b). The case $(c3)$ does not happen.
In case $(c4)$, we consider the following diagram

$$\begin{tikzcd}
E = \mathbb{P}^2 \times \mathbb{P}^{n-3} \arrow[r, "\mathrm{pr}_2"] \arrow[d, hook, "i"] & \mathbb{P}^{n-3} \arrow[d, hook] \\
T = \mathbb{P}(\mathcal{O}^{\oplus n-2} \oplus \mathcal{O}(2)) \arrow[r, "\psi_{|\xi|}"] \arrow[d, "\pi"] & C_n(\mathbb{P}^2, \mathcal{O}(2)) = X \\
\mathbb{P}^2 &
\end{tikzcd}.$$

Since T is a projective bundle, by [Ful98, Theorem 3.3.(b)] we have that
$$\mathrm{Cl}(T) = \mathbb{Z}[\pi^*(\mathcal{O}_{\mathbb{P}^2}(1))] \oplus \mathbb{Z}[\xi].$$
On the other hand, since $E = \mathrm{exc}(\psi_{|\xi|})$ is contracted, we know that the homomorphism $(\psi_{|\xi|})_* : \mathrm{Cl}(T) \to \mathrm{Cl}(X)$ is surjective and $\mathrm{rk}(\mathrm{Cl}(X)) = 1$. We have that $\psi_{|\xi|}^*(L) = \xi$. Thus $(\psi_{|\xi|})_*([\xi]) = [L] \neq 0$. To determine $\mathrm{Cl}(X)$, one just need to know the image $(\psi_{|\xi|})_*\pi^*([\mathcal{O}_{\mathbb{P}^2}(1)])$. Let H be a Weil divisor on T such that $\mathcal{O}_T(H) = \pi^*(\mathcal{O}_{\mathbb{P}^2}(1))$. For example, we can take H to be $\pi^{-1}(l)$ where $l \subset \mathbb{P}^2$ is a linear subspace. Then it's easy to see that $H \neq E$. Set $G := (\psi_{|\xi|})_*H$. As L is ample, the class $[L]$ is non-zero in $\mathrm{Cl}(X) \otimes \mathbb{Q}$. Take $m \in \mathbb{Q}$ such that $[G] = m[L]$ in $\mathrm{Cl(X)} \otimes \mathbb{Q}$. We have that
$$\psi_{|\xi|}^*(G) \sim_{\mathbb{Q}} (\psi_{|\xi|})_*^{-1}(G) + aE, \tag{2}$$
with $(\psi_{|\xi|})_*^{-1}(G) = H$. By the canonical bundle formula, we have that
$$K_T = \pi^*(\mathcal{O}_{\mathbb{P}^2}(-1)) - (n-1)\xi \text{ and}$$
$$K_E = \mathrm{pr}_1^*(\mathcal{O}_{\mathbb{P}^2}(-2)) - (n-2)\xi|_E.$$
Hence we have that
$$\mathcal{O}_E(E) = \mathrm{pr}_1^*(\mathcal{O}_{\mathbb{P}^2}(-2)) \otimes \mathrm{pr}_2^*(\mathcal{O}_{\mathbb{P}^{n-3}}(1)).$$
Let $C_1 = \mathbb{P}^1 \times \{\mathrm{pt}\} \subset E$. Then $E \cdot C_1 = -2$. We intersect both sides of Equation (2) with C_1. As $(\psi_{|\xi|})_*(C_1) = 0$, by the projection we get that $(\psi_{|\xi|})^*(G) \cdot C_1 = 0$. By applying the projection formula to the morphism $\pi|_H : H \to \mathbb{P}^2$, we get that $H \cdot C_1 = 1$. Hence $a = \frac{1}{2}$. Thus we have that
$$m[\xi] = \pi^*[\mathcal{O}_{\mathbb{P}^2}(1)] + \frac{1}{2}E. \tag{3}$$
Let $F = \mathbb{P}^{n-2}$ be a fiber of π such that $F \cap E \neq \emptyset$. Then $E \cap F = \mathbb{P}^{n-3} \subset \mathbb{P}^{n-2} = F$. Take $C_2 = \mathbb{P}^1 \subset F$ and intersect both side of Equation (3) with C_2. We have that $\xi \cdot C_2 = 1$ and $\pi^*[\mathcal{O}_{\mathbb{P}^2}(1)] \cdot C_2 = 0$ and $E \cdot C_2 = 1$. Thus we get that $m = \frac{1}{2}$. Hence we know that $\mathrm{Cl}(X) \otimes \mathbb{Q} = \mathbb{Q} \cdot \frac{1}{2}[L]$. Let D be a component of Δ. Suppose that
$$(\psi_{|\xi|})_*^{-1}[D] = m_1\pi^*[(\mathcal{O}_{\mathbb{P}^2}(1))] + m_2[\xi]$$

for some natural numbers m_1, m_2. We have that $D = (\psi_{|\xi|})_*(\psi_{|\xi|})_*^{-1}D \sim_{\mathbb{Q}} (\frac{m_1}{2} + m_2)L$. Hence $\frac{m_1}{2} + m_2 \geq \frac{1}{2}$. Being a generalized cone, X is $\mathbb{Q}$-factorial. For the $\mathbb{Q}$-Cartier divisor K_X we have

$$K_X = (\psi_{|\xi|})_*(K_T) = (\psi_{|\xi|})_*(\pi^*(\mathcal{O}(-1)) - (n-1)\xi) = -(n - \frac{1}{2})[L].$$

Now $K_X + D + (n-1)L \equiv_{\text{num}} (\frac{m_1 - 1}{2} + m_2)L$ is pseudo-effective. We thus exclude case $(c4)$.

□

Lemma 4.1. *Let $(X, D) = (\mathbb{P}(\mathcal{V}), D)$ be a log canonical pair, where $\pi : \mathbb{P}(\mathcal{V}) \to C$ is a projective bundle over a smooth curve C and $\mathcal{V}$ is an ample vector bundle of rank n. If for a general fiber F, we have that $D|_F = \mathcal{O}_F(d)$ for some $d > 0$. Then* $\dim(X) = \dim(\mathbb{P}(\mathcal{V})) = 2$.

Proof. We have that

$$(K_{\mathbb{P}(\mathcal{V})} + D + (n-1)\xi)|_F = (\pi^*(K_C + \det(\mathcal{V})) + D - \xi)|_F = \mathcal{O}_F(d-1).$$

We take a *thrifty dlt modification* for $(\mathbb{P}(\mathcal{V}), D)$ as in [Kol13, Corollary 1.36.], *i.e.*, a proper birational morphism $f : \mathbb{P}(\mathcal{V})^{\text{dlt}} \to \mathbb{P}(\mathcal{V})$ with a boundary divisor Δ^{dlt} such that:

(1) $(\mathbb{P}(\mathcal{V})^{\text{dlt}}, \Delta^{\text{dlt}})$ has dlt singularities;
(2) $f^*(K_{\mathbb{P}(\mathcal{V})} + D) \sim_{\mathbb{Q}} K_{\mathbb{P}(\mathcal{V})^{\text{dlt}}} + \Delta^{\text{dlt}}$;
(3) $K_{\mathbb{P}(\mathcal{V})^{\text{dlt}}} + \Delta^{\text{dlt}}$ is f-nef;
(4) $\mathbb{P}(\mathcal{V})^{\text{dlt}}$ is $\mathbb{Q}$-factorial.

Thus we have that

$$\mathbb{P}(\mathcal{V})^{\text{dlt}} \xrightarrow{f} \mathbb{P}(\mathcal{V}) \xrightarrow{\pi} C .$$

We set that $g = \pi \circ f$ and $\xi' = f^*\xi$. Then we have that

$$f^*(K_{\mathbb{P}(\mathcal{V})} + D + (n-1)\xi) \equiv_{\text{num}} K_{\mathbb{P}(\mathcal{V})^{\text{dlt}}} + \Delta^{\text{dlt}} + (n-1)\xi'$$

As f is surjective, we have that f_* preserves numerical equivalence. By the projection formula we have that

$$f_*(K_{\mathbb{P}(\mathcal{V})^{\text{dlt}}} + \Delta^{\text{dlt}} + (n-1)\xi') \equiv_{\text{num}} f_*f^*(K_{\mathbb{P}(\mathcal{V})} + D + (n-1)\xi) = K_{\mathbb{P}(\mathcal{V})} + D + (n-1)\xi.$$

Hence $K_{\mathbb{P}(\mathcal{V})^{\text{dlt}}} + \Delta^{\text{dlt}} + (n-1)\xi'$ cannot be pseudo-effective. So there exists an extremal ray R of $\overline{\text{NE}}(\mathbb{P}(\mathcal{V})^{\text{dlt}})$ such that

$$(K_{\mathbb{P}(\mathcal{V})^{\text{dlt}}} + \Delta^{\text{dlt}} + (n-1)\xi') \cdot R < 0.$$

For $0 < \epsilon \ll 1$, we have that

$$(K_{\mathbb{P}(\mathcal{V})^{\text{dlt}}} + (1-\epsilon)\Delta^{\text{dlt}} + (n-1)\xi') \cdot R < 0. \tag{4}$$

We note that $\xi' \cdot R = f^*(L) \cdot R > 0$, for otherwise any curve l such that $[l] \in R$ is contracted by f, which means $(K_{\mathbb{P}(\mathcal{V})^{\text{dlt}}} + \Delta^{\text{dlt}}) \cdot R \geq 0$, a contradiction. Hence R is in fact a $(K_{\mathbb{P}(\mathcal{V})^{\text{dlt}}} + \Delta^{\text{dlt}})$-negative extremal ray.

By the contraction theorem (*cf.* [Fuj11, Theorem 1.1.(4)], we get the contraction morphism $\mathrm{cont}_R : \mathbb{P}(\mathcal{V})^{\mathrm{dlt}} \to Y$ which contracts the ray R. Let $S \subset \mathbb{P}(\mathcal{V})^{\mathrm{dlt}}$ be a fiber of cont_R. If $\dim(S) \geq 2$, there exists a curve $l \subset S$ that is contracted to a point by g. As $K_{\mathbb{P}(\mathcal{V})^{\mathrm{dlt}}} + \Delta^{\mathrm{dlt}}$ is f-nef, the $K_{\mathbb{P}(\mathcal{V})^{\mathrm{dlt}}} + \Delta^{\mathrm{dlt}}$-negative curve l can not be contracted to a point by f. Hence l maps finitely onto a curve $l' \subset F$. Now we have that

$$\begin{aligned}(K_{\mathbb{P}(\mathcal{V})^{\mathrm{dlt}}} + \Delta^{\mathrm{dlt}} + (n-1)\xi') \cdot l &= (f^*(K_{\mathbb{P}(\mathcal{V})} + D + (n-1)\xi)) \cdot l \\ &= \deg(l/l')(K_{\mathbb{P}(\mathcal{V})} + D + (n-1)\xi) \cdot l \\ &= \deg(l/l')\mathcal{O}_F(d-1) \cdot l' \\ &\geq 0,\end{aligned}$$

a contradiction. Hence any fiber of cont_R has dimension at most 1.

Let $E \subset \mathrm{exc}(\mathrm{cont}_R)$ be an irreducible component of the exceptional locus of cont_R. We thus have that

$$\dim(E) - \dim(\mathrm{cont}_R(E)) \leq 1.$$

For $0 < \epsilon \ll 1$, the pair $(\mathbb{P}(\mathcal{V})^{\mathrm{dlt}}, (1-\epsilon)\Delta^{\mathrm{dlt}})$ has klt singularities (*cf.* [KM98, Proposition 2.41.]). For small ϵ, the divisor $-(K_{\mathbb{P}(\mathcal{V})^{\mathrm{dlt}}} + (1-\epsilon)\Delta^{\mathrm{dlt}})$ is still cont_R-ample. The estimate of the length of extremal ray by Kawamata (*cf.* [Deb01, Theorem 7.46.]) for klt pairs shows that the rational curves $l \in R$ cover E and there exists a rational curve $l_\epsilon \in R$ such that

$$0 < -(K_{\mathbb{P}(\mathcal{V})^{\mathrm{dlt}}} + (1-\epsilon)\Delta^{\mathrm{dlt}}) \cdot l_\epsilon \leq 2.$$

For any curve l whose class $[l]$ is in R, we have that $\xi' \cdot l \geq 1$. Combining these two inequalities with Equation (4), we have that

$$0 > (K_{\mathbb{P}(\mathcal{V})^{\mathrm{dlt}}} + (1-\epsilon)\Delta^{\mathrm{dlt}} + (n-1)\xi') \cdot l_\epsilon \geq -2 + (n-1).$$

Hence $n = 2$. □

Lemma 4.2. *Set $(X, L) := (\mathbb{P}(\mathcal{V}), \mathcal{O}_{\mathbb{P}(\mathcal{V})}(1))$, where $\mathcal{V}$ is a rank 2 ample vector bundle over $\mathbb{P}^1$. Suppose that Δ is a reduced divisor on X and $K_X+\Delta+L$ is not pseudo-effective. Then we have one of the following:*

1. *Either $\Delta = \sum D_i$ where $D_i \cong \mathbb{P}^1$ are distinct fibers of the structure map $\pi : \mathbb{P}(\mathcal{V}) \to \mathbb{P}^1$; or*
2. *$(X, L) = (\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1)), \mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a)\oplus\mathcal{O}_{\mathbb{P}^1}(1))}(1))$ with $a > 1$ and D is the unique section of $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \to \mathbb{P}^1$ such that*
$$D \equiv_{\mathrm{num}} \mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a)\oplus\mathcal{O}_{\mathbb{P}^1}(1))}(1)) - af,$$
where f is a general fiber; or
3. *$\mathcal{V} = \mathcal{O}_{\mathbb{P}}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$.*

Proof. As $\mathcal{V}$ is ample, we know that $\mathcal{V} \cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$ with $a, b > 0$. We may suppose that $a \geq b > 0$. If $a = b = 1$, then $(X, L) \cong (\mathbb{P}^1 \times \mathbb{P}^1, (1,1))$ which is a rank 4 hyperquadric in $\mathbb{P}^3$. Thus we are in case (3) of Lemma 4.2.

Hence in the rest we only consider $a > b$. We follow the convention in [Har77, Notation V.2.8.1] in this proof. Set $e := a - b \geq 0$. Set $\mathcal{W} := \mathcal{V} \otimes \mathcal{O}_{\mathbb{P}^1}(-a)$. We have that $X_e := \mathbb{P}(\mathcal{W}) \cong \mathbb{P}(\mathcal{V})$. We denote by $p : \mathbb{P}(\mathcal{W}) \to \mathbb{P}^1$ the projection. By [Har77, Lemma II.7.9], we know that $(X,L)\cong(X_e,\mathcal{O}_{X_e}(1)\otimes p^*(\mathcal{O}_{\mathbb{P}^1}(a)))$. We denote the general fiber of p by f'. Note that $\mathcal{W}$ satisfies the assumption in [Har77, Proposition 2.8.]. Hence there exists a section C_0 of p such that $\mathcal{O}_{X_e}(1)\cong\mathcal{O}_{X_e}(C_0)$. [Har77, Proposition 2.9.] implies $C_0^2=-e$. Hence if $a\neq b$, we have that C_0 is unique. We know that $L\equiv_{\mathrm{num}} C_0+af'$ and [Har77, Lemma 2.10.] implies that $K_{X_e}\sim-2C_0+(-2-e)f'$. Let D be a component of Δ. Then K_X+D+L is not pseudo-effective. Assume that $D\sim xC_0+d'f'$, with x,d' being integers. We have that

$$K_X+L+D\equiv_{\mathrm{num}}(x-1)C_0+(d'+b-2)f'.$$

As D is a prime divisor, [Har77, Corollary V.2.18-(b)] implies one of the following:

(i) $x=0,d'=1$, and $K_X+L+D\equiv_{\mathrm{num}}-C_0+(b-1)f'\notin\mathrm{Pseff}(X)$;
(ii) $x=1,d'=0$ and $K_X+L+D\equiv_{\mathrm{num}}(b-2)f'$, which is not pseudo-effective if and only if $b=1$;
(iii) $x>0,d'>xe$. Note that $d'+b-2\geq 0$. So we have that K_X+L+D, being a positive combination of effective divisors, is pseudo-effective;
(iv) $e>0,x>0$, and $d'=xe$. Again we have that $d'+b-2\geq 0$. So K_X+L+D, being a positive combination of effective divisors, is pseudo-effective.

In case (i), the divisor D is a fiber of p, which maps isomorphically to a fiber of π under the canonical isomorphism $\mathbb{P}(\mathcal{W})\cong\mathbb{P}(\mathcal{V})$.

In case (ii), as D is irreducible, [Har77, Proposition V.2.20-(a)] implies $D=C_0$. Hence D is the unique section of $\pi:\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a)\oplus\mathcal{O}_{\mathbb{P}^1}(1))\to\mathbb{P}^1$ such that $D\equiv_{\mathrm{num}}\xi-\pi^*(\mathcal{O}_{\mathbb{P}^1}(a))$. If we have another component D' of Δ, we know that $D'\equiv_{\mathrm{num}} f$. Then

$$K_X+L+D+D'\equiv_{\mathrm{num}}0$$

is pseudo-effective, a contradiction. Thus $\Delta=D$ is irreducible and we are in case (2) of Lemma 4.2.

Write $\Delta=\sum D_i$. If we don't have any component D of Δ such that $D\sim C_0$, each D_i will be a fiber. Then

$$K_X+L+\Delta\equiv_{\mathrm{num}}-C_0+(b-2+k)f'$$

is not pseudo-effective, where k is the number of components of Δ. Hence we are in case (1) of Lemma 4.2. ☐

5. Semi-log canonical polarized varieties

The hypothesis for the pair (X,Δ) in Proposition 1.5 alludes to the normalization of a slc variety together with its conductor divisors. In this section, we will show how to use Proposition 1.5 to classifying polarized slc varieties.

For the basic definition and statements of slc varieties we refer to [Kol13, Chapter 5]. See also [Liu22, Chapter 2.5] for an account in our setup.

We recall the definition of *conductor*.

Definition 5.1 (conductor)**.** Let X be a reduced scheme and $\pi:\bar{X}\to X$ its normalization. The *conductor ideal*

$$\mathfrak{cond}_X:=\mathcal{H}om_{\mathcal{O}_X}(\pi_*\mathcal{O}_{\bar{X}},\mathcal{O}_X)$$

is the largest ideal sheaf of $\mathcal{O}_X$ such that it is also an ideal sheaf of $\pi_*\mathcal{O}_{\bar{X}}$. As π is finite, we have a unique ideal sheaf $\mathfrak{cond}_{\bar{X}}$ of $\bar{X}$ that corresponds to $\mathfrak{cond}_X$.

We define the conductor schemes to be

$$D := \mathrm{Spec}(\mathcal{O}_X / \mathfrak{cond}_X) \text{ and } \bar{D} := \mathrm{Spec}(\mathcal{O}_{\bar{X}} / \mathfrak{cond}_{\bar{X}})$$

They fit into the Cartesian square

$$\begin{tikzcd} \bar{D} \arrow[r] \arrow[d] & \bar{X} \arrow[d] \\ D \arrow[r] & X \end{tikzcd} .$$

Note that when X is demi-normal, the conductors D and $\bar{D}$ are reduced divisors. We now give our definition of slc singularities.

Definition 5.2 ([Kol13, Definition-Lemma 5.10])**.** Let (X, Δ) be a pair with X demi-normal. Let $\pi : \bar{X} \to X$ be its normalization, the conductors $\bar{D}$ and D as in Definition 5.1. The pair (X, Δ) is called *semi-log canonical* or *slc* if $(\bar{X}, \bar{D} + \bar{\Delta})$ is log canonical.

Proof of Theorem 1.6.

Proof. We know by definition that $(\bar{X}, \bar{D})$ is log canonical. Note that the absolute normalization $\pi : \bar{X} \to X$ is finite (*cf.* [Sta22, Tag 0BXR]). Hence $\pi^*(L)$ is ample. We have that

$$\pi^*(K_X + (n-1)L) = K_{\bar{X}} + \bar{D} + (n-1)\pi^*(L).$$

Let $C \subset X$ be a movable curve in X such that $(K_X + (n-1)L) \cdot C < 0$. Let $C' \subset \bar{X}$ be a movable curve that dominates C. Then by the projection formula we have that

$$K_{\bar{X}} + \bar{D} + (n-1)\pi^*(L)) \cdot C' = \deg(C'/C)(K_X + (n-1)) \cdot C < 0.$$

Hence by the BDPP theorem, the divisor $K_{\bar{X}} + \bar{D} + (n-1)\pi^*(L)$ is not pseudo-effective.

Note that D and $\bar{D}$ are reduced. We denote by $\bar{D}^\nu$, D^ν respectively their normalizations. Then π induces a degree 2 map $\nu : \bar{D}^\nu \to D^\nu$ and there is a Galois involution $\tau : \bar{D}^\nu \to \bar{D}^\nu$ which is generically fixed point free (*cf.* [Kol13, 5.2]). Thus we have the following diagram

$$\begin{tikzcd} \bar{D}^\nu \arrow[loop left, "\tau"] \arrow[r] \arrow[d, "\nu"] & \bar{D} \arrow[r] \arrow[d] & \bar{X} \arrow[d, "\pi"] \\ D^\nu \arrow[r] & D \arrow[r] & X \end{tikzcd} .$$

Since $\nu : \bar{D}^\nu \to D^\nu$ has degree 2, we have by the projection formula that

$$\pi^*(L)|_{\bar{D}^\nu}^{n-1} = \deg(\nu) \cdot (L|_{D^\nu}^{n-1}) \in 2\mathbb{Z}.$$

We now apply Proposition 1.5 to $(\bar{X}, \bar{D}, \pi^*(L))$. We have one of the following:

(1) $(\bar{X}, \pi^*L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$,. The conductor $\bar{D} = H$ is a prime divisor where H is a hyperplane of $\mathbb{P}^n$;

(2.i) There is a $(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$-bundle $(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1))$ over a smooth curve C, and a birational morphism $\mu : \mathbb{P}(E) \to \bar{X}$ such that $\mu^*(\pi^*L) \cong \mathcal{O}_{\mathbb{P}(E)}(1)$ and $\bar{D} = \sum F_i$ is a finite sum where $F_i \cong \mu(\mathbb{P}^{n-1})$ are images of distinct general fibers by μ and $\deg(\mathbb{P}^{n-1}/F_i) = 1$;

(2.ii) $(\bar{X}, \pi^*L) = (\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1)), \mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a)\oplus\mathcal{O}_{\mathbb{P}^1}(1))}(1))$ with $a > 1$ and $\bar{D} = C$, where C is the unique section of $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \to \mathbb{P}^1$ such that

$$C \equiv_{\text{num}} \mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a)\oplus\mathcal{O}_{\mathbb{P}^1}(1))}(1)) - af,$$

where f is a general fiber;

(3.i) $(\bar{X}, \pi^*L) \cong (Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$, where $Q \subset \mathbb{P}^{n+1}$ is a $\text{rk}(Q) = 3$ hyperquadric, the divisor $\bar{D}$ is a hyperplane in Q and $[\bar{D}] = \dfrac{1}{2}[H \cap Q]$ where H is a hyperplane in $\mathbb{P}^{n+1}$;

(3.ii) $(\bar{X}, \pi^*L) \cong (Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$, where $Q \subset \mathbb{P}^{n+1}$ is a $\text{rk}(Q) = 4$ hyperquadirc. If we write $Q = \text{Proj}\left(\dfrac{\mathbb{C}[x_0, \ldots, x_{n+1}]}{(x_0x_1 - x_2x_3)}\right)$, then $\bar{D}$ is prime and $\bar{D}$ is the cone with vertex $\mathbb{P}^{n-3}$ over $\mathbb{P}^1 \times \text{pt}$ or $\text{pt} \times \mathbb{P}^1$. In particular, $\bar{D} \cong \mathbb{P}^{n-1}$.

In case (1), we have that $\bar{D}^\nu = \bar{D} \cong \mathbb{P}^{n-1}$ is smooth and $\pi^*(L)|_{\bar{D}^\nu} = \mathcal{O}_{\mathbb{P}^{n-1}}(1)$. As $\pi^*(L)|_{\bar{D}^\nu}^{n-1} = 1$ is odd. We exclude case (1).

In case $(2.i)$, we have that $\bar{D} = \sum_{1\leq i\leq k} F_i$ for a natural number k and the morphism $\mu : \coprod_{1\leq i\leq k} \mathbb{P}^{n-1} \to \bar{D}$ factors through $\bar{D}^\nu \to \bar{D}$ (*cf.* [Sta22, Tag 035Q]-(4)). Hence $\pi^*(L)|_{\bar{D}^\nu}^{n-1} = k$ and k is even. As $\deg(\mathbb{P}^{n-1}/F_i) = 1$, we have that

$$\pi^*(L)|_{F_i}^{n-1} = \deg(\mathbb{P}^{n-1}/F_i)(\mathcal{O}_{\mathbb{P}(E)}(1)|_{\mathbb{P}^{n-1}})^{n-1} = 1.$$

Thus each irreducible component of D has pre-image consisting of two of the F_i's. We have thus the diagram

$$\begin{tikzcd} \mathbb{P}(E) \arrow[r, "\mu"] \arrow[d, "p"'] & \bar{X} \arrow[r, "\pi"] & X \\ C & & \end{tikzcd}.$$

Set $k = 2m$. We write $D = \sum_{1\leq i\leq m} D_i$. We denote the two components of $\bar{D}$ that are mapped onto D_i by $F_{i,1}$ and F_{i_2}. Let $x_{i,1}$ (*resp.* $x_{i,2}$) be the point of C such that $\mu(p^{-1}(x_{i,1})) = F_{i,1}$ (*resp.* $\mu(p^{-1}(x_{i,2})) = F_{i,2}$). As C is smooth, we may glue $x_{i,1}$ and $x_{i,2}$. We thus get a nodal curve C' together with a quotient morphism $q : C \to C'$ such that there exists a rank n vector bundle E' on C' satisfying $q^*(E') = E$. The morphism $\pi \circ \mu$ thus factors through $\mathbb{P}(E)$, *i.e.* we have the following commutative diagram:

$$\begin{tikzcd} \mathbb{P}(E) \arrow[r, "\mu"] \arrow[d] & \bar{X} \arrow[d, "\pi"] \\ \mathbb{P}(E') \arrow[r, "\mu'"] & X \end{tikzcd}.$$

The morphism μ' is birational. If we denote $x_i = p(x_{i,1})$ and F_i the fiber of x_i in $\mathbb{P}(E')$, we have that $D_i = \mu'(F_i)$. Thus we have the result of Theorem 1.6.

In case $(2.ii)$, we have that $\pi^*(L) \cdot C = a - e = 1$. Hence we exclude this case.

In case $(3.i)$, the conductor $\bar{D}$ is irreducible and $\pi^*(L)|_{\bar{D}^n}^{n-1} = 1$. Hence we also exclude this case.

In case $(3.ii)$, the conductor $\bar{D} \cong \mathbb{P}^{n-1}$ and $\pi^*(L)|_{\bar{D}} = \mathcal{O}_{\mathbb{P}^{n-1}}(1)$ which is not divisible by 2. Hence we exclude this case, too. $\square$

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