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Optical forces and optical force categorizations on small magnetodielectric particles in the framework of generalized Lorenz-Mie theory.

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Abstract

A previous study has been devoted to the case of optical forces exerted on Rayleigh particles in the framework of generalized Lorenz-Mie theory (GLMT). This study is here extended to the case of small magnetodielectric particles which are characterized by their electric and magnetic Mie coefficients a_1 and b_1 respectively. A partition of optical forces between incident-scattered mixed terms and recoil terms has been established in GLMT since more than three decades. This partition is here complemented by another partition between gradient, scattering and non-standard forces. The relationship between these non-standard forces and curl forces exhibited in the framework of a dipole theory is emphasized. The case of Rayleigh particles (more generally of electric dipolar particles) previously published in the GLMT-framework is recovered as a special case of the case of small magnetodielectric particles discussed in the present paper. The case of illumination by plane waves furthermore provides complementary insights on the behavior of non-standard forces.

Keywords : generalized Lorenz-Mie theory; optical forces; scattering forces; gradient forces; non-standard forces; dipole theory.

1 Introduction.

The generalized Lorenz-Mie theory (GLMT) is a rigorous analytical theory to describe the interaction between an illuminating structured electromagnetic beam and a homogeneous spherical particle characterized by its diameter and its complex refractive index, e.g. [1] and references therein dating back to 1982 [2]. One of the issues to be considered in this framework is the study of optical forces and torques, expressed in terms of the beam shape coefficients (BSCs), usually denoted by $g_{n, TM}^m$ and $g_{n, TE}^m$ (TM standing for "Transverse Magnetic", TE for "Transverse Electric", with n ranging from 1 to infinity, and $-n \leq m \leq +n$) which encode the structure of the beam. Expressions of optical forces in the GLMT framework are originally available from [3], complemented by [4], [5], and [6] (they will serve as basic expressions in the framework of the present paper). Numerical evaluations of optical forces in the GLMT framework, using these expressions, possibly in relationship with experimental results, have afterward been provided in [7], [8], [9], [10]. Optical torques in spherical coordinates have also been discussed by Polaert et al. [11]. Concerning complementary studies in spheroidal coordinates, for both optical forces and torques, the reader may refer to [12] and to [13]. These quotations are from authors having had a direct connection with Normandie University and Coria, institutions where the GLMT has originally been derived. Many other works from various worldwide authors contributed as well to the issue and have been recently quoted in a review paper with 284 references [14].

The previous review paper has been written on the occasion of Arthur Ashkin's receipt of the 2018 Nobel prize in physics for his pioneering work in optical levitation and manipulation. After Arthur Ashkin's work, it has been traditional to think of the optical forces in terms of a partition between gradient and scattering forces, see [15] for a collection of reprints. However, strange as it may be, GLMT studies aiming to a categorization of optical forces in terms of gradient and scattering forces had to wait for a long term before a first occurrence. As far as we know, such a first occurrence would be due to Lock [16], in the course of a study devoted to the calculation of radiation trapping forces in optical tweezers using GLMT with illuminating Gaussian beams. It was then found that indeed GLMT was perfectly able to categorize optical forces in terms of gradient and scattering forces. However, the Gaussian beam in this work has been described within the weak confinement limit. As has been demonstrated later, this weak confinement limit assumption prevented the author to observe a third kind of optical forces, that we call non-standard forces and that will be discussed in the sequel, see [17] for details concerning the story of the weak confinement limit and of its implications.

A systematic study of the categorization of the optical forces in the GLMT recently started in 2020 in the case of lossless particles in the Rayleigh regime [18] and ended with an application to non-dark axisymmetric on-axis beams of the second kind and to dark axisymmetric on-axis beams [17]. The understanding of the situation required several papers which were briefly reviewed

in Section 10 of [17]. One of the results of this effort is the categorization of optical forces in terms of gradient, scattering and non-standard forces and their relationship with optical forces exhibited in the dipole theory of forces.

Relying on the understanding so gained, the present paper considers the more general situation of small magnetodielectric particles when we must retain both the first electric and magnetic Mie coefficients a_1 and b_1 respectively. This is equivalent to saying that we are working in a GLMT framework which, in another context, defines the dipolar approximation since these first electric and magnetic Mie coefficients are directly related to the induced dipole polarizabilities of the particle, e.g. [19].

Although developed in the framework of GLMT *stricto sensu* devoted to the case of homogeneous spherical particles, our approach applies as well to the cases of multilayered particles when the expressions of the BSCs are unchanged, requiring only to modify the expressions of the Mie coefficients [20], [21], and to other kinds of particles leading to expressions which are formally identical to the ones of the GLMT *stricto sensu*, namely assemblies of spheres and aggregates [22], and spheres with an eccentrically located spherical inclusion [23], [24], [25]. Other papers relevant to the issue discussed in the present paper will be quoted later when appropriate.

The paper is organized as follows. Section 2 recalls expressions of optical forces in the GLMT framework with a partition between incident-scattered terms (IS-terms) and recoil terms. Section 3 specifies the optical forces in the GLMT framework, in terms of BSCs, in the case when only the first Mie coefficients a_1 and b_1 are retained in the analysis. Section 4 provides an interpretation of the optical terms obtained in Section 2 in terms of gradient, scattering and non-standard forces. Section 5 discusses the case of electric dipolar particles as a special case of the magnetodielectric particles considered in the present paper, refines the interpretation of the non-standard forces with respect to another partition introduced in the dipole theory of forces, and discusses the special case of plane wave illumination which provides a few more insights on the behavior of non-standard forces. Section 6 deals with a conclusion together with a brief prospectus for future work.

2 Optical forces in the GLMT framework.

We consider a Cartesian coordinate system O_Pxyz with a scatterer located at the origin O_P of the coordinates. The scatterer is illuminated by a structured beam encoded by the double set of BSCs $g_{n,TM}^m$ and $g_{n,TE}^m$. The axis O_Pz is traditionally chosen to define the direction of propagation of the beam. The time dependence of the beam is assumed to read as $\exp(i\omega t)$. The optical forces may be expressed by using cross-sections denoted as $C_{pr,i}$. In the present paper, it may be convenient, by metonymy, to use the name "force" to denote the cross-sections. Although expressions for optical forces have been presented in

a paper whose most part is restricted to Gaussian beams [4], they are valid as well for arbitrary shaped beams [5], [6]. They are presented as well in textbooks devoted to the GLMT, e.g. [1]. For convenience to the reader, we shall refer to such a textbook which provides a comprehensive unified presentation of the theory (although a few misprints will have to be corrected, for which the reader may refer to [4], as will be mentioned below). The reader wanting to check the details would better begin from Section 3.11, p.66 of [1]. Furthermore, we use the normalization condition $E_0 H_0^*/2 = 1$. The validity of such a normalization is ensured by the fact that beams considered in GLMT propagate in vacuum (so that the electric E_0 and magnetic H_0 strengths are proportional with a coefficient of proportionality which is an intrinsic impedance pertaining to the set of real numbers).

In the longitudinal direction $O_p z$, the radiation pressure cross-section component $C_{pr,z}$ is then expressed by the relation (e.g. Eq.3.146 in [1]):

$$C_{pr,z} = \int_0^\pi \int_0^{2\pi} \frac{1}{2} \text{Re}(E_\varphi^i H_\theta^{s*} + E_\varphi^s H_\theta^{i*} - E_\theta^i H_\varphi^{s*} - E_\theta^s H_\varphi^{i*}) r^2 \cos \theta \sin \theta d\theta d\varphi - \int_0^\pi \int_0^{2\pi} (I_\theta^s + I_\varphi^s) r^2 \cos \theta \sin \theta d\theta d\varphi \quad (1)$$

in which the integrations are performed on the surface of a sphere S , centered at O_P , and of radius $r \gg \lambda$, \mathbf{E} and \mathbf{H} denote the electric and magnetic fields respectively, the superscripts i and s denote incident and scattered fields respectively, and (r, θ, φ) are usual spherical coordinates attached to the Cartesian coordinates (x, y, z) .

The first term in the r.h.s. represents the forward momentum removed from the beam and the second term represents minus the forward momentum given by the scatterer to the scattered wave. Because the first term results from mixed contributions between incident and scattered fields, we shall call it the incident-scattered mixed term (or IS-mixed term in short). A more traditional denomination is the one of "extinction" term in agreement with the notations $\cos \theta C_{ext}$, $\sin \theta \cos \varphi C_{ext}$ and $\sin \theta \sin \varphi C_{ext}$ borrowed below from Van de Hulst [26], but we have a preference for the more explicit IS-denomination.

Concerning the second term, it only involves scattered field components, expressed in terms of scattered intensities I_θ^s and I_φ^s expressed by Eq.(3.107) in [1]. Because the momentum given by the scatterer to the scattered wave is lost by the scatterer, this term has been called the recoil term in the literature. Eq.(1) therefore expresses a first partition of optical forces in terms of IS-mixed forces and recoil forces, a partition already introduced in 1985 in [3], although we here use a different language to express it. Such a partition will later occur as well for transverse optical forces discussed below. It must be noted that such a dichotomy between IS-mixed forces and recoil forces does occur as well for

torques as discussed in the framework of the dipole theory of forces by Nieto-Vesperinas in [27] and [28], and as evidenced in the framework of GLMT by Eq.(2) of [11].

Eq.(1) may be rewritten as (Eq.3.147 in [1]):

$$C_{pr,z} = \overline{\cos\theta} C_{ext} - \overline{\cos\theta} C_{sca} \quad (2)$$

in which $\overline{\cos\theta}$ indicates integrations weighted by $\cos\theta$, C_{ext} is the extinction cross-section (Eq.3.142 in [1]), and C_{sca} is the scattering cross-section (Eq.3.137 in [1]). Eq.2 uses a symbolic notation borrowed to and generalized, from a plane wave notation to an arbitrary wave notation, from [26]. The first term still represents an IS-mixed term in agreement with the fact that it is related to both absorption (of the Incident light) and scattering (through the Scattered wave), while the recoil term is only related to scattering.

These terms may be evaluated in terms of BSCs according to (Eqs.3.158 and 3.155 in [1]):

$$\begin{aligned} \overline{\cos\theta} C_{ext} &= \frac{\lambda^2}{\pi} \sum_{n=1}^{\infty} \sum_{p=-n}^{+n} \left\{ \frac{1}{(n+1)^2} \frac{(n+1+|p|)!}{(n-|p|)!} \right. \\ &\quad \left. \text{Re}[(a_n + a_{n+1}^*) g_{n, TM}^p g_{n+1, TM}^{p*} + (b_n + b_{n+1}^*) g_{n, TE}^p g_{n+1, TE}^{p*}] \right. \\ &\quad \left. - p \frac{2n+1}{n^2} \frac{(n+|p|)!}{(n+1)^2 (n-|p|)!} \text{Re}[i(a_n + b_n^*) g_{n, TM}^p g_{n, TE}^{p*}] \right\} \end{aligned} \quad (3)$$

$$\begin{aligned} \overline{\cos\theta} C_{sca} &= -\frac{2\lambda^2}{\pi} \sum_{n=1}^{\infty} \sum_{p=-n}^{+n} p \frac{2n+1}{n^2(n+1)^2} \frac{(n+|p|)!}{(n-|p|)!} \\ &\quad \left[\text{Re}(i a_n b_n^* g_{n, TM}^p g_{n, TE}^{p*}) - \frac{1}{(n+1)^2} \frac{(n+1+|p|)!}{(n-|p|)!} \right. \\ &\quad \left. \text{Re}(a_n a_{n+1}^* g_{n, TM}^p g_{n+1, TM}^{p*} + b_n b_{n+1}^* g_{n, TE}^p g_{n+1, TE}^{p*}) \right] \end{aligned} \quad (4)$$

As a whole, from Eqs.2, 3 and 4, we obtain (Eq.3.159 in [1]):

$$\begin{aligned} C_{pr,z} &= \frac{\lambda^2}{\pi} \sum_{n=1}^{\infty} \sum_{p=-n}^{+n} \left\{ \frac{1}{(n+1)^2} \frac{(n+1+|p|)!}{(n-|p|)!} \right. \\ &\quad \left. \text{Re}[(a_n + a_{n+1}^* - 2a_n a_{n+1}^*) g_{n, TM}^p g_{n+1, TM}^{p*} + (b_n + b_{n+1}^* - 2b_n b_{n+1}^*) g_{n, TE}^p g_{n+1, TE}^{p*}] \right. \\ &\quad \left. + p \frac{2n+1}{n^2(n+1)^2} \frac{(n+|p|)!}{(n-|p|)!} \text{Re}[i(2a_n b_n^* - a_n - b_n^*) g_{n, TM}^p g_{n, TE}^{p*}] \right\} \end{aligned} \quad (5)$$

Similarly, the transverse cross-section components along x and y may be decomposed into IS-mixed and recoil terms according to (Eqs.3.162 and 3.163 in [1]):

$$C_{pr,x} = \overline{\sin \theta \cos \varphi} C_{ext} - \overline{\sin \theta \cos \varphi} C_{sca} \quad (6)$$

$$C_{pr,y} = \overline{\sin \theta \sin \varphi} C_{ext} - \overline{\sin \theta \sin \varphi} C_{sca} \quad (7)$$

The quantities of Eq.6 may be expressed in terms of BSCs as follows (respectively: Eq.3.180 in [1] with the subscript in the rightmost summation corrected from $m = p - 1$ to $m = p - 1 \neq 0$ as in Eq.158 of [4], Eq.3.174 in [1]):

$$\begin{aligned} \overline{\sin \theta \cos \varphi} C_{ext} &= \frac{\lambda^2}{2\pi} \sum_{p=1}^{\infty} \sum_{n=p}^{\infty} \sum_{m=p-1 \neq 0}^{\infty} \frac{(n+p)!}{(n-p)!} \\ &\left\{ \text{Re}(S_{mn}^{p-1} + S_{nm}^{-p}) \left[\frac{1}{m^2} \delta_{m,n+1} - \frac{1}{n^2} \delta_{n,m+1} \right] \right. \\ &\left. + \frac{2n+1}{n^2(n+1)^2} \delta_{nm} \text{Re}(T_{mn}^{p-1} - T_{nm}^{-p}) \right\} \end{aligned} \quad (8)$$

$$\begin{aligned} \overline{\sin \theta \cos \varphi} C_{sca} &= \frac{\lambda^2}{\pi} \sum_{p=1}^{\infty} \sum_{n=p}^{\infty} \sum_{m=p-1 \neq 0}^{\infty} \frac{(n+p)!}{(n-p)!} \\ &\{ [\text{Re}(U_{mn}^{p-1} + U_{nm}^{-p})] \left[\frac{1}{m^2} \delta_{m,n+1} - \frac{1}{n^2} \delta_{n,m+1} \right] \right. \\ &\left. + \frac{2n+1}{n^2(n+1)^2} \delta_{nm} [\text{Re}(V_{mn}^{p-1} - V_{nm}^{-p})] \right\} \end{aligned} \quad (9)$$

in which (respectively: Eqs.3.177 and 3.178 in [1], with the rightmost subscript of Eq.3.178 corrected from TE to TM , as in Eq.156 of [4]):

$$S_{nm}^p = (a_n + a_m^*) g_{n, TM}^p g_{m, TM}^{p+1*} + (b_n + b_m^*) g_{n, TE}^p g_{m, TE}^{p+1*} \quad (10)$$

$$T_{nm}^p = -i(a_n + b_m^*) g_{n, TM}^p g_{m, TE}^{p+1*} + i(b_n + a_m^*) g_{n, TE}^p g_{m, TM}^{p+1*} \quad (11)$$

and (e.g. Eqs.3.167 and 3.168 in [1]):

$$U_{nm}^p = a_n a_m^* g_{n, TM}^p g_{m, TM}^{p+1*} + b_n b_m^* g_{n, TE}^p g_{m, TE}^{p+1*} \quad (12)$$

$$V_{nm}^p = i b_n a_m^* g_{n, TE}^p g_{m, TM}^{p+1*} - i a_n b_m^* g_{n, TM}^p g_{m, TE}^{p+1*} \quad (13)$$

Then, from Eqs.6, 8 and 9, we obtain as a whole (see Eqs.3.181 and 3.182 in [1]):

$$C_{pr,x} = \frac{\lambda^2}{2\pi} \sum_{p=1}^{\infty} \sum_{n=p}^{\infty} \sum_{m=p-1 \neq 0}^{\infty} \frac{(n+p)!}{(n-p)!} \quad (14)$$

$$\left\{ [\text{Re}(S_{mn}^{p-1} + S_{nm}^{-p} - 2U_{mn}^{p-1} - 2U_{nm}^{-p})] \left[\frac{1}{m^2} \delta_{m,n+1} - \frac{1}{n^2} \delta_{n,m+1} \right] + \frac{2n+1}{n^2(n+1)^2} \delta_{nm} [\text{Re}(T_{mn}^{p-1} - T_{nm}^{-p} - 2V_{mn}^{p-1} + 2V_{nm}^{-p})] \right\}$$

Establishing the expression for $C_{pr,y}$ is fully similar. It is found that $\frac{\sin \theta}{\sin \varphi} C_{sca}$ and $\frac{\sin \theta}{\sin \varphi} C_{ext}$ are deduced from $\frac{\sin \theta}{\cos \varphi} C_{sca}$ and $\frac{\sin \theta}{\cos \varphi} C_{ext}$ respectively, by changing Re to Im. The final expression for $C_{pr,y}$ is therefore identical to (14) but with Re replaced by Im as well.

These expressions have been obtained by using a radiative balance of momentum. Such an approach would have been qualified as being heuristic by Bohren and Huffman ([29], p.120). From Wikipedia, "a heuristic technique is any approach to problem solving or self-discovery that employs a practical method that is not guaranteed to be optimal, perfect, or rational, but is nevertheless sufficient for reaching an immediate, short-term goal or approximation". In view of this definition, the qualification of "heuristic" by Bohren and Huffman might look pejorative. Actually, the radiative balance approach has the advantage of emphasizing in a direct way the physics supporting the calculations. Another, more abstract approach, is by using the Maxwell stress tensor. After checking, it has been established that both approaches lead exactly to the same results (possibly within irrelevant normalization prefactors), as stated for example in [30], p.66.

In view of these remarks, some statements found in the literature might be misleading. An example is a paper by Chen et al. [31] devoted to a study of pulling forces. In the last section named "Methods", the authors wrote: "We calculated the time-averaged optical force that acts on a spherical particle via a surface integral of the time-averaged Maxwell stress tensor over the surface of the sphere. The electromagnetic fields needed in the Maxwell stress tensor were computed by the rigorous and accurate generalized Lorenz-Mie theory", with no reference to GLMT associated with this sentence. The exposition provided

in the present section shows that actually expressions to calculate optical forces were already displayed in the framework of the GLMT in 1985, 25 years before. As an easy formal clue for the identification between the radiative balance approach and the Maxwellian stress tensor, the reader may compare the decomposition of the present section in terms of IS-mixed terms and recoil forces, and Eq.(16) in the Supplementary Information of [31]. An acknowledgment of the existence of two complementary approaches to the evaluation of optical forces is however provided by a paper from Zheng et al. [32] to which we shall return later.

3 Optical forces exerted on small magnetodielectric particles.

We now specify the expressions from the previous section to the case of small magnetodielectric particles when we retain only the first Mie coefficients a_1 and b_1 . From Eq.3, we then obtain, after a bit of straightforward computations in which we used $\text{Re}(z) = \text{Re}(z^*)$:

$$\overline{\cos\theta C_{ext}} = \frac{3\lambda^2}{2\pi} \text{Re}(a_1 G_E + b_1 G_H) \quad (15)$$

in which:

$$\begin{aligned} G_E &= g_{1,TM}^{-1}(g_{2,TM}^{-1*} + ig_{1,TE}^{-1*}) + g_{1,TM}^1(g_{2,TM}^{1*} - ig_{1,TE}^{1*}) + \frac{1}{3}g_{1,TM}^0 g_{2,TM}^{0*} \\ &= -i(g_{1,TM}^1 g_{1,TE}^{1*} - g_{1,TM}^{-1} g_{1,TE}^{-1*}) + g_{1,TM}^{-1} g_{2,TM}^{-1*} + g_{1,TM}^1 g_{2,TM}^{1*} + \frac{1}{3}g_{1,TM}^0 g_{2,TM}^{0*} \end{aligned} \quad (16)$$

$$G_H = g_{1,TE}^{-1}(g_{2,TE}^{-1*} - ig_{1,TM}^{-1*}) + g_{1,TE}^1(g_{2,TE}^{1*} + ig_{1,TM}^{1*}) + \frac{1}{3}g_{1,TE}^0 g_{2,TE}^{0*} \quad (17)$$

in which the subscript E stands for "Electric" because G_E is multiplicatively associated with the electric Mie coefficient a_1 , while H stands for "Magnetic" because G_H is multiplicatively associated with the magnetic Mie coefficient b_1 . For further use, we may decompose $\overline{\cos\theta C_{ext}}$ as the summation of four terms reading as:

$$\overline{[\cos\theta C_{ext}]_E}^I = \frac{-3\lambda^2}{2\pi} \text{Im}(a_1) \text{Im}(G_E) \quad (18)$$

$$[\overline{\cos \theta} C_{ext}]_E^R = \frac{3\lambda^2}{2\pi} \text{Re}(a_1) \text{Re}(G_E) \quad (19)$$

$$[\overline{\cos \theta} C_{ext}]_H^I = \frac{-3\lambda^2}{2\pi} \text{Im}(b_1) \text{Im}(G_H) \quad (20)$$

$$[\overline{\cos \theta} C_{ext}]_H^R = \frac{3\lambda^2}{2\pi} \text{Re}(b_1) \text{Re}(G_H) \quad (21)$$

in which the subscripts E and H correspond to electric and magnetic contributions respectively, associated with the electric and magnetic Mie coefficients a_1 and b_1 respectively, while the superscripts R and I are associated with real and imaginary parts of G_E and G_H . It is worth noting that G_E above identifies with the G of Eq.(24) in [18] and of Eq.(4) in [33].

Similarly, from Eq.4, we obtain:

$$\overline{\cos \theta} C_{sca} = \frac{-3\lambda^2}{\pi} \text{Re}[ia_1 b_1^* (g_{1, TM}^1 g_{1, TE}^{1*} - g_{1, TM}^{-1} g_{1, TE}^{-1*})] \quad (22)$$

whose decomposition will be considered in subsection 4.4.

For the transverse force along the x -direction, we use Eq.6 and after easy although somewhat tedious computations, we obtain:

$$C_{pr,x} = \overline{\sin \theta \cos \varphi} C_{ext} - \overline{\sin \theta \cos \varphi} C_{sca} \quad (23)$$

in which the first term in the r.h.s. is found to read as:

$$\overline{\sin \theta \cos \varphi} C_{ext} = \frac{\lambda^2}{4\pi} \text{Re}(a_1 H_1 + b_1 H_2) \quad (24)$$

in which:

$$\begin{aligned} H_1 = & g_{2, TM}^{0*} (g_{1, TM}^1 + g_{1, TM}^{-1}) \\ & - 3g_{1, TM}^0 (g_{2, TM}^{1*} + g_{2, TM}^{-1*}) \\ & - 12(g_{1, TM}^1 g_{2, TM}^{2*} + g_{1, TM}^{-1} g_{2, TM}^{-2*}) \\ & + 3i[g_{1, TM}^0 (g_{1, TE}^{-1*} - g_{1, TE}^{1*}) + g_{1, TE}^{0*} (g_{1, TM}^{-1} - g_{1, TM}^1)] \end{aligned} \quad (25)$$

$$\begin{aligned}
H_2 = & g_{2,TE}^{0*}(g_{1,TE}^1 + g_{1,TE}^{-1}) \\
& -3g_{1,TE}^0(g_{2,TE}^{1*} + g_{2,TE}^{-1*}) \\
& -12(g_{1,TE}^1 g_{2,TE}^{2*} + g_{1,TE}^{-1} g_{2,TE}^{-2*}) \\
& +3i[g_{1,TE}^0(g_{1,TE}^{1*} - g_{1,TE}^{-1*}) + g_{1,TE}^{0*}(g_{1,TE}^1 - g_{1,TE}^{-1})]
\end{aligned} \tag{26}$$

Let us note that H_1 identifies with H given in Eq.(38) of [34]. Similarly, from Eq.3.174 in [1], the second term occurring in Eq.23 is found to read as:

$$\overline{\sin \theta \cos \varphi C_{sca}} = \frac{3\lambda^2}{2\pi} \text{Re}\{ia_1 b_1^* [g_{1,TE}^{0*}(g_{1,TE}^{-1} - g_{1,TE}^1) + g_{1,TE}^0(g_{1,TE}^{-1*} - g_{1,TE}^{1*})]\} \tag{27}$$

We decompose Eq.24 as follows:

$$\overline{\sin \theta \cos \varphi C_{ext}} = [\overline{\sin \theta \cos \varphi C_{ext}}]_{Ex} + [\overline{\sin \theta \cos \varphi C_{ext}}]_{Hx} \tag{28}$$

in which:

$$[\overline{\sin \theta \cos \varphi C_{ext}}]_{Ex} = \frac{\lambda^2}{4\pi} \text{Re}(a_1 H_1) \tag{29}$$

$$[\overline{\sin \theta \cos \varphi C_{ext}}]_{Hx} = \frac{\lambda^2}{4\pi} \text{Re}(b_1 H_2) \tag{30}$$

The electric term $[\overline{\sin \theta \cos \varphi C_{ext}}]_{Ex}$ may then be decomposed according to:

$$[\overline{\sin \theta \cos \varphi C_{ext}}]_{Ex} = [\overline{\sin \theta \cos \varphi C_{ext}}]_{Ex}^R + [\overline{\sin \theta \cos \varphi C_{ext}}]_{Ex}^I \tag{31}$$

in which:

$$[\overline{\sin \theta \cos \varphi C_{ext}}]_{Ex}^I = \frac{-\lambda^2}{4\pi} \text{Im}(a_1) \text{Im}(H_1) \tag{32}$$

$$\overline{[\sin \theta \cos \varphi C_{ext}]_{Ex}^R} = \frac{\lambda^2}{4\pi} \text{Re}(a_1) \text{Re}(H_1) \quad (33)$$

The magnetic term $\overline{[\sin \theta \cos \varphi C_{ext}]_{Hx}}$ of Eq.30 may as well be decomposed according to:

$$\overline{[\sin \theta \cos \varphi C_{ext}]_{Hx}} = \overline{[\sin \theta \cos \varphi C_{ext}]_{Hx}^R} + \overline{[\sin \theta \cos \varphi C_{ext}]_{Hx}^I} \quad (34)$$

in which:

$$\overline{[\sin \theta \cos \varphi C_{ext}]_{Hx}^I} = \frac{-\lambda^2}{4\pi} \text{Im}(b_1) \text{Im}(H_2) \quad (35)$$

$$\overline{[\sin \theta \cos \varphi C_{ext}]_{Hx}^R} = \frac{\lambda^2}{4\pi} \text{Re}(b_1) \text{Re}(H_2) \quad (36)$$

Similarly, we have:

$$C_{pr,y} = \overline{\sin \theta \sin \varphi C_{ext}} - \overline{\sin \theta \sin \varphi C_{sca}} \quad (37)$$

For $\overline{\sin \theta \sin \varphi C_{ext}}$ and $\overline{\sin \theta \sin \varphi C_{sca}}$, we start from $\overline{\sin \theta \cos \varphi C_{ext}}$ and $\overline{\sin \theta \cos \varphi C_{sca}}$ respectively, and replace Re by Im [4], [1]. We then proceed similarly as above and obtain:

$$\overline{\sin \theta \sin \varphi C_{ext}} = \frac{\lambda^2}{4\pi} \text{Im}(a_1 H'_1 + b_1 H'_2) \quad (38)$$

in which:

$$\begin{aligned} H'_1 = & g_{2,TM}^{0*} (g_{1,TM}^{-1} - g_{1,TM}^1) \\ & - 3g_{1,TM}^0 (g_{2,TM}^{1*} - g_{2,TM}^{-1*}) \\ & - 12(g_{1,TM}^1 g_{2,TM}^{2*} - g_{1,TM}^{-1} g_{2,TM}^{-2*}) \\ & + 3i[g_{1,TE}^{0*} (g_{1,TM}^1 + g_{1,TM}^{-1}) - g_{1,TM}^0 (g_{1,TE}^{1*} + g_{1,TE}^{-1*})] \end{aligned} \quad (39)$$

$$\begin{aligned}
H'_2 &= g_{2,TE}^{0*}(g_{1,TE}^{-1} - g_{1,TE}^1) \\
&\quad - 3g_{1,TE}^0(g_{2,TE}^{1*} - g_{2,TE}^{-1*}) \\
&\quad - 12(g_{1,TE}^1 g_{2,TE}^{2*} - g_{1,TE}^{-1} g_{2,TE}^{-2*}) \\
&\quad + 3i[g_{1,TE}^0(g_{1,TE}^{1*} + g_{1,TE}^{-1*}) - g_{1,TE}^{0*}(g_{1,TE}^1 + g_{1,TE}^{-1})]
\end{aligned} \tag{40}$$

Let us note that the H'_1 of Eq.39 exactly identifies with the H' found in [35]. Furthermore, it is found that:

$$\overline{\sin \theta \sin \varphi C_{sca}} = \frac{3\lambda^2}{2\pi} \text{Im}\{ia_1 b_1^* [g_{1,TE}^{0*}(g_{1,TE}^{-1} + g_{1,TE}^1) - g_{1,TE}^0(g_{1,TE}^{-1*} + g_{1,TE}^{1*})]\} \tag{41}$$

This is completed by the following similar decomposition:

$$\overline{\sin \theta \sin \varphi C_{ext}} = \overline{[\sin \theta \sin \varphi C_{ext}]_{Ey}} + \overline{[\sin \theta \sin \varphi C_{ext}]_{Hy}} \tag{42}$$

in which:

$$\overline{[\sin \theta \sin \varphi C_{ext}]_{Ey}} = \frac{\lambda^2}{4\pi} \text{Im}(a_1 H'_1) \tag{43}$$

$$\overline{[\sin \theta \sin \varphi C_{ext}]_{Hy}} = \frac{\lambda^2}{4\pi} \text{Im}(b_1 H'_2) \tag{44}$$

(in which H'_1 is H' , for instance in [36]). The electric term is furthermore decomposed in two terms according to:

$$\overline{[\sin \theta \sin \varphi C_{ext}]_{Ey}} = \overline{[\sin \theta \sin \varphi C_{ext}]_{Ey}^R} + \overline{[\sin \theta \sin \varphi C_{ext}]_{Ey}^I} \tag{45}$$

in which:

$$\overline{[\sin \theta \sin \varphi C_{ext}]_{Ey}^R} = \frac{\lambda^2}{4\pi} \text{Im}(a_1) \text{Re}(H'_1) \tag{46}$$

$$\overline{[\sin \theta \sin \varphi C_{ext}]_{Ey}^I} = \frac{\lambda^2}{4\pi} \text{Re}(a_1) \text{Im}(H'_1) \tag{47}$$

Similarly, for the magnetic term $\overline{[\sin \theta \sin \varphi C_{ext}]_{Hy}}$, we have:

$$\overline{[\sin \theta \sin \varphi C_{ext}]_{Hy}} = \overline{[\sin \theta \sin \varphi C_{ext}]_{Hy}^R} + \overline{[\sin \theta \sin \varphi C_{ext}]_{Hy}^I} \quad (48)$$

in which:

$$\overline{[\sin \theta \sin \varphi C_{ext}]_{Hy}^R} = \frac{\lambda^2}{4\pi} \text{Im}(b_1) \text{Re}(H_2') \quad (49)$$

$$\overline{[\sin \theta \sin \varphi C_{ext}]_{Hy}^I} = \frac{\lambda^2}{4\pi} \text{Re}(b_1) \text{Im}(H_2') \quad (50)$$

4 Interpretation of optical forces in terms of gradient, scattering and non-standard forces.

4.1 The \mathcal{T} -transformation.

From the expressions for the electric \mathbf{E} and magnetic \mathbf{H} fields, e.g. Eqs.(85)-(90) in [4] or (3.39)-(3.50) in [1], it is found that:

$$\mathbf{H} = \mathcal{T}(\mathbf{E}) \quad (51)$$

in which the transformation \mathcal{T} changes E_0 to H_0 , $g_{n,TM}^m$ to $+g_{n,TE}^m$ and $g_{n,TE}^m$ to $-g_{n,TM}^m$, in which however E_0 and H_0 are forced to satisfy the normalization condition $E_0 H_0^* / 2 = 1$. We shall soon return to this issue when appropriate. We also have:

$$G_H = \mathcal{T}(G_E) \quad (52)$$

$$H_2 = \mathcal{T}(H_1) \quad (53)$$

$$\text{Re}(H_2') = \mathcal{T} \text{Re}(H_1') \quad (54)$$

This \mathcal{T} -transformation is actually a well-known dual transformation between \mathbf{E} and \mathbf{H} , e.g. [37], pp. 367-376, which is however here expressed in the framework of GLMT. According to this duality transformation, changing $g_{n,TM}^m$ to $+g_{n,TE}^m$ and $g_{n,TE}^m$ to $-g_{n,TM}^m$, and forgetting the electric E_0 and magnetic H_0 strengths, we obtain $\mathbf{H} \rightarrow -\mathbf{E}$.

4.2 Preliminaries.

The interpretation of the optical forces exhibited in the previous section requires the evaluation of gradients and Poynting vector components at the location of the particle (denoted by the subscript P). We begin with the evaluation of gradients. It is demonstrated that, e.g. Eq.(61) in [35]:

$$[\partial_z |\mathbf{E}|^2]_P = -2k |E_0|^2 \text{Im}(G_E) \quad (55)$$

which does satisfy the normalization convention $E_0 H_0^*/2 = 1$ of the present paper, e.g. Eq.(3.106) of [1]. Specifically, for a nonabsorbing medium in which μ and ε are real numbers, we have $E_0 = H_0 \sqrt{\mu/\varepsilon}$ in which μ and ε are the permeability and the permittivity of the vacuum (more generally of the surrounding nonabsorbing medium) respectively. This leads to $|E_0|^2 = 2\sqrt{\mu/\varepsilon}$, so that Eq.55 becomes:

$$[\partial_z |\mathbf{E}|^2]_P = -4k\eta \text{Im}(G_E) \quad (56)$$

in which $\eta = \sqrt{\mu/\varepsilon}$ is the impedance of the surrounding medium. From Eqs.51 and 52, we then have:

$$[\partial_z |\mathbf{H}|^2]_P = -2k |H_0|^2 \text{Im}(G_H) \quad (57)$$

while the normalization condition leading to $|E_0|^2 = 2\eta$ as above leads as well to $|H_0|^2 = 2/\eta$, implying:

$$[\partial_z |\mathbf{H}|^2]_P = -4k \frac{1}{\eta} \text{Im}(G_H) \quad (58)$$

in which the change $\eta \rightarrow 1/\eta$ from Eq.56 to Eq.58 ensures a coherence of units, in accordance with the relation $|H_0|^2 = |E_0|^2/\eta^2$ which implies to change η in Eq.56 to $1/\eta$ in Eq.58.

Similarly, from Eq.(56) in [36] and Eq.53 above, we obtain:

$$[\partial_x |\mathbf{E}|^2]_P = \frac{-2}{3} k\eta \text{Im}(H_1) \quad (59)$$

$$[\partial_x |\mathbf{H}|^2]_P = \frac{-2}{3} \frac{k}{\eta} \text{Im}(H_2) \quad (60)$$

Concerning the derivatives with respect to y , we use Eq.(57) in [36] (with the misprint Im in the rightmost term changed to Re) and Eq.54 of the present paper to obtain:

$$[\partial_y |\mathbf{E}|^2]_P = \frac{2}{3}k\eta \text{Re}(H'_1) \quad (61)$$

$$[\partial_y |\mathbf{H}|^2]_P = \frac{2}{3}\frac{k}{\eta} \text{Re}(H'_2) \quad (62)$$

We now consider the components of the Poynting vector at P . We start from the fairly complicated expressions of the Poynting vector demonstrated in [38], [39] and the values of the components at P are thereafter evaluated. For the z -component, we obtain (see Eq.(66) in [18] and (37), (93) in [33]):

$$\begin{aligned} [S_z]_P &= 2 \text{Re}[i(g_{1,TM}^{-1}g_{1,TE}^{-1*} - g_{1,TM}^1g_{1,TE}^{1*})] \\ &= 2 \text{Re}[i(g_{1,TE}^1g_{1,TM}^{1*} - g_{1,TE}^{-1}g_{1,TM}^{-1*})] \end{aligned} \quad (63)$$

The components along x and y are given by Eq.(61) and (68) in [34] according to:

$$\begin{aligned} [S_x]_P &= \text{Re}\{i[g_{1,TM}^0(g_{1,TE}^{-1*} - g_{1,TE}^{1*}) + g_{1,TE}^0(g_{1,TM}^{1*} - g_{1,TM}^{-1*})]\} \\ &= \text{Re}\{i[g_{1,TE}^0(g_{1,TM}^{1*} - g_{1,TM}^{-1*}) + g_{1,TM}^0(g_{1,TE}^{1*} - g_{1,TE}^{-1*})]\} \\ &= \text{Re}\{i[g_{1,TE}^{0*}(g_{1,TM}^{-1} - g_{1,TM}^1) + g_{1,TM}^0(g_{1,TE}^{-1*} - g_{1,TE}^{1*})]\} \end{aligned} \quad (64)$$

$$\begin{aligned} [S_y]_P &= \text{Re}[g_{1,TE}^{0*}(g_{1,TM}^1 + g_{1,TM}^{-1}) - g_{1,TM}^0(g_{1,TE}^{1*} + g_{1,TE}^{-1*})] \\ &= \text{Re}[g_{1,TE}^0(g_{1,TM}^{-1*} + g_{1,TM}^{1*}) - g_{1,TM}^0(g_{1,TE}^{-1*} + g_{1,TE}^{1*})] \\ &= \text{Re}[g_{1,TE}^0(g_{1,TM}^{-1*} + g_{1,TM}^{1*}) - g_{1,TM}^{0*}(g_{1,TE}^{-1} + g_{1,TE}^1)] \end{aligned} \quad (65)$$

4.3 Interpretations of electric and magnetic optical forces.

Using Eq.56, we see that $[\overline{\cos\theta}C_{ext}]_E^I$ of Eq.18 is a gradient electric force which may be denoted $C_{pr,z}^{EG}$, in which the superscript G stands for "Gradient". Conversely, $[\overline{\cos\theta}C_{ext}]_E^R$ of Eq.19 is expressed in terms of G_E which is expressed by Eq.16 to be compared with Eq.63 from which we may conclude that it is the summation of a scattering term, defined as being here proportional to $[S_z]_P$, and of a non-standard force according to:

$$C_{pr,z}^{ES} = \frac{3\lambda^2}{2\pi} \text{Re}(a_1) \text{Re}[i(g_{1,TM}^{-1}g_{1,TE}^{-1*} - g_{1,TM}^1g_{1,TE}^{1*})] \quad (66)$$

$$C_{pr,z}^{ENS} = \frac{3\lambda^2}{2\pi} \text{Re}(a_1) \text{Re}[g_{1,TM}^{-1}g_{2,TM}^{-1*} + g_{1,TM}^1g_{2,TM}^{1*} + \frac{1}{3}g_{1,TM}^0g_{2,TM}^{0*}] \quad (67)$$

in which the superscript S stands for "Scattering" and the superscript NS for "Non-Standard". In the present paper, scattering forces gather all forces which are proportional to components of the Poynting vector, while non-standard forces are forces which are neither gradient nor scattering forces. This defines a *point of view* which will be opposed to a different complementary *point of view* when we later discuss the interpretation of non-standard forces. For the NS -term, see as well the summation of the terms G^{12} and G^0 of Eqs.(45)-(46) in [18].

Similarly, using Eq.58, we see that $[\overline{\cos\theta}C_{ext}]_H^I$ of Eq.20 is a gradient magnetic force which may be denoted $C_{pr,z}^{HG}$, while $[\overline{\cos\theta}C_{ext}]_H^R$ of Eq.21 is expressed in terms of G_H which is expressed by Eq.17 to be compared again with Eq.63, so that it may be again decomposed as a summation of scattering and non-standard forces according to:

$$C_{pr,z}^{HS} = \frac{3\lambda^2}{2\pi} \text{Re}(b_1) \text{Re}[i(g_{1,TE}^1g_{1,TM}^{1*} - g_{1,TE}^{-1}g_{1,TM}^{-1*})] \quad (68)$$

$$C_{pr,z}^{HNS} = \frac{3\lambda^2}{2\pi} \text{Re}(b_1) \text{Re}[g_{1,TE}^{-1}g_{2,TE}^{-1*} + g_{1,TE}^1g_{2,TE}^{1*} + \frac{1}{3}g_{1,TE}^0g_{2,TE}^{0*}] \quad (69)$$

From Eq.59, we see that $[\overline{\sin\theta\cos\varphi}C_{ext}]_{Ex}^I$ of Eq.32 is a gradient term, while, using Eqs.25 and 64, it is found that $[\overline{\sin\theta\cos\varphi}C_{ext}]_{Ex}^R$ of Eq.33 is a summation of scattering and non-standard forces according to:

$$C_{pr,x}^{ES} = \frac{3\lambda^2}{4\pi} \text{Re}(a_1) \text{Re}\{i[g_{1,TM}^0(g_{1,TE}^{-1*} - g_{1,TE}^{1*}) + g_{1,TE}^{0*}(g_{1,TM}^{-1} - g_{1,TM}^{-1})]\} \quad (70)$$

$$\begin{aligned} C_{pr,x}^{ENS} &= \frac{\lambda^2}{4\pi} \text{Re}(a_1) \text{Re}[g_{2,TM}^{0*}(g_{1,TM}^1 + g_{1,TM}^{-1}) \\ &\quad - 3g_{1,TM}^0(g_{2,TM}^{1*} + g_{2,TM}^{-1*}) \\ &\quad - 12(g_{1,TM}^1 g_{2,TM}^{2*} + g_{1,TM}^{-1} g_{2,TM}^{-2*})] \end{aligned} \quad (71)$$

Next, from Eq.60, we see that $[\overline{\sin \theta \cos \varphi C_{ext}}]_{H_x}^I$ of Eq.35 is a gradient term although, as a magnetic term, it is better expressed in terms of $[\partial_x |\mathbf{H}|^2]_P$ rather than in terms of $[\partial_x |\mathbf{E}|^2]_P$. Conversely, $[\overline{\sin \theta \cos \varphi C_{ext}}]_{H_x}^R$ of Eq.36 may be decomposed again in two terms (scattering and non-standard) according to:

$$C_{pr,x}^{HS} = \frac{3\lambda^2}{4\pi} \text{Re}(b_1) \text{Re}\{i[g_{1,TE}^0(g_{1,TM}^{1*} - g_{1,TM}^{-1*}) + g_{1,TM}^{0*}(g_{1,TE}^1 - g_{1,TE}^{-1})]\} \quad (72)$$

$$\begin{aligned} C_{pr,x}^{HNS} &= \frac{\lambda^2}{4\pi} \text{Re}(b_1) \text{Re}[g_{2,TE}^{0*}(g_{1,TE}^1 + g_{1,TE}^{-1}) \\ &\quad - 3g_{1,TE}^0(g_{2,TE}^{1*} + g_{2,TE}^{-1*}) \\ &\quad - 12(g_{1,TE}^1 g_{2,TE}^{2*} + g_{1,TE}^{-1} g_{2,TE}^{-2*})] \end{aligned} \quad (73)$$

Next, we consider $[\overline{\sin \theta \sin \varphi C_{ext}}]_{E_y}^R$ of Eq.46 which, by virtue of Eq.61, is found to be a gradient term. Conversely, $[\overline{\sin \theta \sin \varphi C_{ext}}]_{E_y}^I$ of Eq.47, by virtue of Eq.65, may be again expressed as the summation of a scattering and of a non-standard term according to:

$$C_{pr,y}^{ES} = \frac{3\lambda^2}{4\pi} \text{Re}(a_1) \text{Re}[g_{1,TE}^{0*}(g_{1,TM}^1 + g_{1,TM}^{-1}) - g_{1,TM}^0(g_{1,TE}^{1*} + g_{1,TE}^{-1*})] \quad (74)$$

$$\begin{aligned} C_{pr,y}^{ENS} &= \frac{\lambda^2}{4\pi} \text{Re}(a_1) \text{Im}[g_{2,TM}^{0*}(g_{1,TM}^{-1} - g_{1,TM}^1) \\ &\quad - 3g_{1,TM}^0(g_{2,TM}^{1*} - g_{2,TM}^{-1*}) \\ &\quad - 12(g_{1,TM}^1 g_{2,TM}^{2*} - g_{1,TM}^{-1} g_{2,TM}^{-2*})] \end{aligned} \quad (75)$$

Similarly, $\overline{[\sin \theta \sin \varphi C_{ext}]_{Hy}^R}$ of Eq.49 is a gradient term as may be seen from Eq.62, while $\overline{[\sin \theta \sin \varphi C_{ext}]_{Hy}^I}$ of Eq.50, using Eq.65, is once more the summation of a scattering and of a non-standard term according to :

$$C_{pr,y}^{HS} = \frac{3\lambda^2}{4\pi} \text{Re}(b_1) \text{Re}[g_{1,TE}^0 (g_{1,TM}^{1*} + g_{1,TM}^{-1*}) - g_{1,TM}^{0*} (g_{1,TE}^1 + g_{1,TE}^{-1})] \quad (76)$$

$$\begin{aligned} C_{pr,y}^{HNS} &= \frac{\lambda^2}{4\pi} \text{Re}(b_1) \text{Im}[g_{2,TE}^{0*} (g_{1,TE}^{-1} - g_{1,TE}^1) \\ &\quad - 3g_{1,TE}^0 (g_{2,TE}^{1*} - g_{2,TE}^{-1*}) \\ &\quad - 12(g_{1,TE}^1 g_{2,TE}^{2*} - g_{1,TE}^{-1} g_{2,TE}^{-2*})] \end{aligned} \quad (77)$$

4.4 Interpretation of magneto-electric optical forces.

Decompositions of magneto-electric optical forces (i.e. forces related to a multiplicative contribution of both magnetic and electric Mie coefficients) are found to be better discussed here in the same section as the interpretations for a better (more aesthetic) balance of the display of the material. We then begin with $\overline{\cos \theta C_{sca}}$ of Eq.22 which may be decomposed as:

$$\overline{\cos \theta C_{sca}} = [\overline{\cos \theta C_{sca}}]_z^{EHR} + [\overline{\cos \theta C_{sca}}]_z^{EHI} \quad (78)$$

in which:

$$[\overline{\cos \theta C_{sca}}]_z^{EHR} = \frac{-3\lambda^2}{\pi} \text{Re}(a_1 b_1^*) \text{Re}[i(g_{1,TM}^1 g_{1,TE}^{1*} - g_{1,TM}^{-1} g_{1,TE}^{-1*})] \quad (79)$$

$$[\overline{\cos \theta C_{sca}}]_z^{EHI} = \frac{+3\lambda^2}{\pi} \text{Im}(a_1 b_1^*) \text{Im}[i(g_{1,TM}^1 g_{1,TE}^{1*} - g_{1,TM}^{-1} g_{1,TE}^{-1*})] \quad (80)$$

in which the superscripts EH recall us that we deal with a Magneto-Electric coupling term, while the superscripts R and I recall that we deal with the real and imaginary parts of the BSC-terms.

From Eq.63, we conclude that $[\overline{\cos \theta C_{sca}}]_z^{EHR}$ is a scattering term. Since $[\overline{\cos \theta C_{sca}}]_z^{EHR}$ is a scattering term, the comparison between Eqs.79 and 80 demonstrates that $[\overline{\cos \theta C_{sca}}]_z^{EHI}$ is not a scattering term. Furthermore, since $G_E, G_H, H_1, H_2, H'_1, H'_2$, which define gradient terms through Eqs.56, 58, 59, 60, 61, 62, do contain BSCs with $m = 0$ which do not occur in Eq.80, it follows that $[\overline{\cos \theta C_{sca}}]_z^{EHI}$ is not a gradient term neither. This term therefore displays a new kind of non-standard forces. Up to now, non-standard forces where associated with IS-mixed forces and encompassed couplings between $n = 1$ and $n = 2$ partial waves. From now on, they may be denoted as IS non-standard forces. Conversely, the non-standard forces of $[\overline{\cos \theta C_{sca}}]_z^{EHI}$ only contain BSCs associated with $n = 1$, without any coupling. They may be denoted as recoil non-standard forces.

Next, we consider $\overline{\sin \theta \cos \varphi C_{sca}}$ of Eq.27 which may be decomposed according to:

$$\overline{\sin \theta \cos \varphi C_{sca}} = [\overline{\sin \theta \cos \varphi C_{sca}}]_x^{EHR} + [\overline{\sin \theta \cos \varphi C_{sca}}]_x^{EHI} \quad (81)$$

in which:

$$[\overline{\sin \theta \cos \varphi C_{sca}}]_x^{EHR} = \frac{3\lambda^2}{2\pi} \text{Re}(a_1 b_1^*) \text{Re}\{i[g_{1,TE}^{0*}(g_{1,TM}^{-1} - g_{1,TM}^1) + g_{1,TM}^0(g_{1,TE}^{-1*} - g_{1,TE}^{1*})]\} \quad (82)$$

$$[\overline{\sin \theta \cos \varphi C_{sca}}]_x^{EHI} = \frac{-3\lambda^2}{2\pi} \text{Im}(a_1 b_1^*) \text{Im}\{i[g_{1,TE}^{0*}(g_{1,TM}^{-1} - g_{1,TM}^1) + g_{1,TM}^0(g_{1,TE}^{-1*} - g_{1,TE}^{1*})]\} \quad (83)$$

But, recalling Eq.64, we see that $[\overline{\sin \theta \cos \varphi C_{sca}}]_x^{EHR}$ is a scattering term, while, similarly as above, $[\overline{\sin \theta \cos \varphi C_{sca}}]_x^{EHI}$ denotes recoil non-standard forces. Finally, we deal with $\overline{\sin \theta \sin \varphi C_{sca}}$ from Eq.41 and, proceeding as above, we have:

$$\overline{\sin \theta \sin \varphi C_{sca}} = [\overline{\sin \theta \sin \varphi C_{sca}}]_y^{EHR} + [\overline{\sin \theta \sin \varphi C_{sca}}]_y^{EHI} \quad (84)$$

in which:

$$[\overline{\sin \theta \sin \varphi C_{sca}}]_y^{EHR} = \frac{3\lambda^2}{2\pi} \text{Im}(i a_1 b_1^*) \text{Re}[g_{1,TE}^{0*}(g_{1,TM}^{-1} + g_{1,TM}^1) - g_{1,TM}^0(g_{1,TE}^{-1*} + g_{1,TE}^{1*})] \quad (85)$$

$$\overline{[\sin \theta \sin \varphi C_{sca}]_y}^{EHI} = \frac{3\lambda^2}{2\pi} \text{Re}(ia_1 b_1^*) \text{Im}[g_{1,TE}^{0*}(g_{1,TM}^{-1} + g_{1,TM}^1) - g_{1,TM}^0(g_{1,TE}^{-1*} + g_{1,TE}^{1*})] \quad (86)$$

in which, by virtue of Eq.65, $\overline{[\sin \theta \sin \varphi C_{sca}]_y}^{EHR}$ denotes scattering forces, while $\overline{[\sin \theta \sin \varphi C_{sca}]_y}^{EHI}$ of Eq.86 denotes recoil non-standard forces.

5 Electric dipolar particles, dipole theory and interpretation of non-standard terms.

5.1 Electric dipolar particles.

The interest of considering electric dipolar particles (in the usual sense, i.e. retaining only the electric Mie coefficient a_1) is twofold. First, it will allow one to recover results already published in the literature and therefore will provide a confirmation (although only partial) of our results concerning magnetodielectric particles. Second, it will allow one, referring to the dipole theory of forces, to obtain some clue about the interpretation of non-standard forces and therefore will prepare us to a more elaborate understanding of them. Let us furthermore note that the Rayleigh limit of GLMT has been proven to be equivalent to the Rayleigh limit of the dipole theory of forces [33], [36]. In the same way, the GLMT limit of magnetodielectric particles studied in the present paper must be equivalent to the corresponding dipole theory approach published by Nieto-Vesperinas et al. [40]. Both theories however use different languages and establishing a complete dictionary between these languages would require an extra-work which is outside of the scope of the present paper. There is however at least one dramatic difference between both approaches, namely the GLMT expresses the optical forces in terms of BSCs pertaining to small order partial waves (namely for $n = 1$ and 2) while the dipole theory of forces expresses the forces in terms of illuminating total fields (encompassing all partial wave orders).

For the first issue, namely retrieving expressions for the electric dipolar case from expressions for the magnetodielectric case, it is sufficient to set $b_1 = 0$ in the expressions for the magnetodielectric case. We then observe that the recoil forces which are all magneto-dielectric forces are equal to 0 as well as all the magnetic terms of the IS-mixed forces. This implies in particular that all recoil non-standard forces are zero. We are then left with:

$$C_{pr,z} = \overline{\cos \theta} C_{ext} = \frac{3\lambda^2}{2\pi} \text{Re}(a_1 G_E) = \overline{[\cos \theta C_{ext}]_E}^I + \overline{[\cos \theta C_{ext}]_E}^R \quad (87)$$

$$C_{pr;x} = \overline{\sin \theta \cos \varphi} C_{ext} = \frac{\lambda^2}{4\pi} \text{Re}(a_1 H_1) = [\overline{\sin \theta \cos \varphi} C_{ext}]_{Ex}^I + [\overline{\sin \theta \cos \varphi} C_{ext}]_{Ex}^R \quad (88)$$

$$C_{pr;y} = \overline{\sin \theta \sin \varphi} C_{ext} = \frac{\lambda^2}{4\pi} \text{Im}(a_1 H_1) = [\overline{\sin \theta \sin \varphi} C_{ext}]_{Ey}^R + [\overline{\sin \theta \sin \varphi} C_{ext}]_{Ey}^I \quad (89)$$

in which $[\overline{\cos \theta} C_{ext}]_E^I$, $[\overline{\sin \theta \cos \varphi} C_{ext}]_{Ex}^I$ and $[\overline{\sin \theta \sin \varphi} C_{ext}]_{Ey}^R$ (Eqs.18, 32 and 46 respectively) have been found to be gradient terms, while $[\overline{\cos \theta} C_{ext}]_E^R$, $[\overline{\sin \theta \cos \varphi} C_{ext}]_{Ex}^R$ and $[\overline{\sin \theta \sin \varphi} C_{ext}]_{Ey}^I$ (Eqs.19, 33, 47 respectively) have been found to be a summation of scattering and IS non-standard terms. The interpretation of these non-standard terms require us to make a detour through the dipole theory of forces.

5.2 Dipole theory of forces and interpretations.

In this framework, the optical forces exerted on electric dipolar particles (in particular Rayleigh particles) read as [41]:

$$F_k = \frac{2\pi}{\eta} \text{Re}[\alpha E_i \partial_k E_i^*] \quad (90)$$

in which the F_k 's are expressed in m^2 so that they can be more easily compared with the $C_{pr,k}$ of the GLMT when dealing with numerical comparisons, e.g. [42], [35]. The conversion from Gaussian units used in [41] to S.I. units used in Eq.90 is originally explained in [42], [35], see as well [33], [36] to complete the discussion of the dipole theory which is reproduced below. For Rayleigh particles (when the electric Mie coefficient a_1 is expanded in a power series in terms of the size parameter and we retain only the terms with the smallest orders), the quantity α is related to the electric Mie coefficient a_1 by the relation:

$$\alpha = \frac{-3i}{2k^3} a_1 \quad (91)$$

in which [16], and [26], pp.143-144:

$$a_1 = \frac{2i}{3} \frac{n_p^2 - 1}{n_p^2 + 2} s^3 + O(is^5) + \frac{4}{9} \left(\frac{n_p^2 - 1}{n_p^2 + 2} \right)^2 s^6 \quad (92)$$

in which s is the size parameter $\pi d/\lambda$, so that we retain:

$$\text{Im}(a_1) = \frac{2}{3} \frac{n_p^2 - 1}{n_p^2 + 2} s^3 \quad (93)$$

$$\text{Re}(a_1) = \frac{4}{9} \left(\frac{n_p^2 - 1}{n_p^2 + 2} \right)^2 s^6$$

Furthermore, α is usually written as:

$$\alpha = \alpha_R - i\alpha_I \quad (94)$$

in which α_R is the real part of α and α_I is minus its imaginary part. From Eq.90, we may then demonstrate, through somewhat standard manipulations, that F_k may be rewritten as:

$$F_k = \frac{2\pi}{\eta} \text{Re}[\alpha(\partial_k |\mathbf{E}|^2 - E_i^* \partial_i E_k + i\omega\mu\varepsilon_{ijk} E_i^* H_j)] \quad (95)$$

in which ε_{ijk} is the Cartesian Levi-Civita tensor. Also, using Eq.94, we may then show that F_k may be decomposed as:

$$F_k = F_k^g + F_k^{ng} \quad (96)$$

in which F_k^g denotes gradient forces (see Eq.100 below) and F_k^{ng} denotes non-gradient forces, according to:

$$F_k^g = \frac{2\pi}{\eta} \alpha_R \partial_k |\mathbf{E}|^2 - \frac{2\pi}{\eta} \alpha_R \text{Re}[(\mathbf{E}^* \cdot \nabla) E_k] + \frac{4\pi}{\eta} \omega\mu\alpha_R \text{Im} S_k^c \quad (97)$$

in which S_k^c is the k th-component of the complex Poynting vector $\mathbf{S}^c = (\mathbf{E} \times \mathbf{H}^*)/2$ reading as:

$$S_k^c = \frac{1}{2} \varepsilon_{ijk} E_i^* H_j \quad (98)$$

and:

$$F_k^{ng} = -\frac{2\pi}{\eta} \alpha_I \text{Im}(\mathbf{E}^* \cdot \nabla) E_k + \frac{4\pi}{\eta} \omega \mu \alpha_I \text{Re} S_k^c \quad (99)$$

It may afterward be established, with further manipulations, that F_k^g may actually be reduced to:

$$F_k^g = \frac{\pi}{\eta} \alpha_R \partial_k |\mathbf{E}|^2 \quad (100)$$

showing that it indeed denotes gradient forces which have been shown to be equal to the gradient forces of the Rayleigh limit of GLMT.

Concerning F_k^{ng} of Eq.99, it had sometimes been called a scattering term, the definition of scattering term being that it is proportional (in the Rayleigh regime) to the sixth power s^6 of the size parameter. In the terminology used in the present paper, only the second term of Eq.99 is a scattering term, the definition of scattering term being now that it is proportional to the (time averaged) Poynting vector. It is however important to remark that the first term contains as well a scattering term. It is actually the summation of a scattering term and of the non-standard contribution. In other words, the scattering term in the Rayleigh limit of the GLMT, using the terminology used in the present paper, is the summation of the second term of Eq.99 and of a (hidden) contribution pertaining to the first term.

This hidden contribution may be revealed by using the results presented in [33], [36]. For instance, let us consider Eq.(81) of [33]:

$$\begin{aligned} (\mathbf{E}^* \cdot \nabla) E_z &= \frac{-ik}{3} |E_0|^2 g_{2, TM}^0 g_{1, TM}^{0*} \quad (101) \\ &- ik |E_0|^2 [(g_{2, TM}^1 g_{1, TM}^{1*} + g_{2, TM}^{-1} g_{1, TM}^{-1*}) \\ &- i(g_{1, TE}^1 g_{1, TM}^{1*} - g_{1, TE}^{-1} g_{1, TM}^{-1*})] \end{aligned}$$

Taking the imaginary parts of both sides of Eq.101 and using $\text{Im}(z) = -\text{Re}(iz)$, we see that the last term of it is proportional to $[S_z]_P$ as given by Eq.63. This shows that $\text{Im}[(\mathbf{E}^* \cdot \nabla) E_z]$ does indeed contain a scattering term. Similar considerations hold for the transverse components along x and y when we compare Eqs.64 and 65 with the last terms of Eqs.(89) and (90) of [36]. Anticipating on the proof of the aforementioned conjecture concerning the identification between GLMT and dipole theory for magnetodielectric particles, we may more or less safely guess that magnetic non-standard forces may be generated by $\text{Re}[\alpha H_i \partial_k H_i^*]$ and magneto-electric non-standard forces by $\text{Re}[\alpha E_i \partial_k H_i^*]$ and/or $\text{Re}[\alpha H_i \partial_k E_i^*]$.

The two-term splitting of Eq.99 was first presented in 2003 by Arias-Gonzalez and Nieto-Vesperinas [43] when studying gradient, scattering and absorption force contributions for light fields expressed by their paraxial form. It is important however to mention again that scattering forces in [43] are defined as being proportional to the sixth power s^6 of the size parameter, which is different from the more precise point of view taken in this paper where scattering forces are defined as being proportional to the (time-averaged) Poynting vector. Furthermore, the authors of [43] did not attempt to provide a physical explanation for the first term of Eq.99. In 2009, Albaladejo et al. [44] showed, by using a simple vector identity, that $\text{Im}[(\mathbf{E}^* \cdot \nabla)\mathbf{E}]$ is proportional to $\nabla \times (\mathbf{E} \times \mathbf{E}^*)$ or, equivalently, to the time average spin density, and concluded that the *total scattering force* contained a contribution "associated to the nonuniform distribution of the spin density of the light field" (see also [45] for a comment on the aforementioned reference, with a reply in [46]). Because of its physical origins, the first term of Eq.99 has then been named spin-curl forces or, alternatively, "polarization gradient forces" by Marago et al. in a review paper in 2013 [47]. Curl forces, i.e. "curl forces due to spin", are discussed as well in Gao et al. [48] in the context of magnetodielectric particles, particularly in their Eqs.(7)-(9), while Berry discussed the decomposition of the Poynting "current" in terms of orbital and spin parts [49]. The categorization using the terminology "spin-curl" forces is however not compatible with the terminology used in the present paper since it is actually, *in the terminology used in the present paper*, a summation between a term which is a scattering term and a non-standard term.

From this subsection, it should be clear, to insist on the issue, that the decomposition in terms of gradient, scattering and spin-curl forces does not identify with our decomposition in terms of gradient, scattering and non-standard forces for two reasons (i) the meaning of the expression "scattering forces" in the two decompositions is different (if only due to the scattering hidden term in the spin-curl forces) and (ii) spin-curl forces do not identify with non-standard forces. While spin-curl forces are known since one decade, non-standard forces have been introduced (under the name of axicon forces) in 2020 in a paper devoted to the study of optical forces exerted on small particles by Bessel beams [18]. The fact that spin-curl forces do not identify with non-standard forces has been pointed out in [36], bottom of the first column of page 10, and has been more generally discussed in section 5 of the same paper. It must furthermore be noted that, as a by-product of the present work, spin-curl forces may be expressed in terms of BSCs in the GLMT framework, revealing that they depend only on $n = 1$ and $n = 2$ partial waves. They depend on $n = 1$ partial waves via their hidden scattering contribution (which therefore is a non-coupling contribution), and they depend as well on $n = 1$ and $n = 2$ partial waves due to the fact that they contain non-standard contributions (which therefore are coupling contributions).

5.3 Plane waves.

A complementary insight concerning non-standard forces is obtained if we consider the special case of illuminating plane waves. This insight is going to be related with an interesting property of non-standard forces, namely that they are non-gradient forces involving a coupling between ($n = 1$)- and ($n = 2$)-partial waves, see Eqs.67, 69, 71, 73, 75, 77. For plane waves, however, such couplings are annihilated for a reason to be soon specified, with the question to know what would be the consequences for non-standard forces.

Indeed, plane waves are axisymmetric beams of the first kind whose BSCs satisfy the following relations [38], [39], [17], [50]:

$$g_{n,TM}^m = g_{n,TE}^m = 0 \text{ for } m \neq \pm 1 \quad (102)$$

$$g_{n,TM}^1 = \frac{g_{n,TM}^{-1}}{K} = -i\varepsilon g_{n,TE}^1 = \frac{i\varepsilon g_{n,TE}^{-1}}{K} = \frac{g_n}{2} \quad (103)$$

which defines a set of uni-index BSCs denoted g_n , with K denoting the polarization state of the beam and $\varepsilon = \pm 1$ defining the propagation direction. More specifically, for plane waves propagating in the positive z -direction, we may take $K = 1$, $\varepsilon = -1$ while the uni-index BSCs g_n reduce to a phase factor $e(i\phi_0)$ which does not depend on the subscript n [1]. Then, inserting these results in the various optical forces investigated above, we find that:

$$C_{pr,x} = C_{pr,y} = 0 \quad (104)$$

so that transverse forces are zero, as we should have expected. The longitudinal force is however different from 0 and reads as:

$$C_{pr,z} = \frac{3\lambda^2}{2\pi} \text{Re}\left(\frac{a_1 + b_1}{4} - a_1 b_1^*\right) \quad (105)$$

resulting from an IS-mixed force reading as:

$$\overline{\cos\theta} C_{ext} = \frac{3\lambda^2}{8\pi} \text{Re}(a_1 + b_1) \quad (106)$$

and from a recoil force reading as:

$$\overline{\cos\theta}C_{sca} = \frac{3\lambda^2}{2\pi} \text{Re}(a_1 b_1^*) \quad (107)$$

Returning to Eqs.19-20 and to their interpretations, it is found that $\overline{\cos\theta}C_{ext}$ may be decomposed into electric and magnetic gradient forces $[\overline{\cos\theta}C_{ext}]_E^I$ and $[\overline{\cos\theta}C_{ext}]_H^I$ respectively, which are zero as we should have expected, and non-gradient forces which may be decomposed into scattering and IS non-standard forces which, using Eqs.66, 67, 68, 69, are found to read as:

$$C_{pr,z}^{ES} = C_{pr,z}^{ENS} = \frac{3\lambda^2}{4\pi} \text{Re}(a_1) \quad (108)$$

$$C_{pr,z}^{HS} = C_{pr,z}^{HNS} = \frac{3\lambda^2}{4\pi} \text{Re}(b_1) \quad (109)$$

Therefore, the IS non-standard forces are not equal to 0 but, due to the special expressions of the uni-index BSCs g_n in the case of plane waves, they became exactly equal to the scattering forces so that, if we prefer, we may state that, in the present case, the IS-mixed forces are only constituted from scattering forces. A similar collapse of non-standard forces (named axicon forces at this time) to scattering forces has also been observed in the case of Bessel beams when the axicon angle is equal to 0, see Eq.(68) and associated comments in [18]. This makes sense because, under such circumstances, the angular spectrum decomposition of the beams is made of plane waves propagating perpendicularly to the direction of propagation.

Concerning the recoil force, it has been decomposed into $[\overline{\cos\theta}C_{sca}]_z^{EHR}$ and $[\overline{\cos\theta}C_{sca}]_z^{EHI}$, see Eq.78, reading as, from Eqs.79 and 80:

$$[\overline{\cos\theta}C_{sca}]_z^{EHR} = \frac{3\lambda^2}{2\pi} \text{Re}(a_1 b_1^*) \quad (110)$$

$$[\overline{\cos\theta}C_{sca}]_z^{EHI} = 0 \quad (111)$$

in which $[\overline{\cos\theta}C_{sca}]_z^{EHR}$ has been identified as a scattering force. Concerning $[\overline{\cos\theta}C_{sca}]_z^{EHI}$, it is found to be 0 so that there is no recoil non-standard force. As a whole, non-standard force vanish in the case of plane wave or collapse to scattering forces.

6 Conclusion.

In the framework of GLMT, we have discussed the optical forces exerted by arbitrary shaped beams on magnetodielectric particles (in which only the electric a_1 and magnetic b_1 Mie coefficients are different from 0). Two categorizations of the optical forces have been introduced. The first one is a decomposition of the optical forces in terms of IS-mixed forces (more traditionally named extinction forces) and of recoil forces which have been introduced in the GLMT-framework in an approximate way in 1985 [3], and in a completely rigorous way in 1988 [4]. The second categorization is new. It has been introduced in a study of optical forces exerted on small particles (retaining only the electric Mie coefficient a_1), e.g. [50] and references therein, and is confirmed in the present paper. It distinguishes between gradient (conservative) forces, and non-gradient forces (non-conservative forces) which are constituted of scattering forces and of non-standard forces. It is remarkable to this categorization in terms of gradient, scattering, and non-standard forces does occur both for IS and for recoil forces. In the case of electric dipolar particles (more specifically Rayleigh particles), non-standard forces are related (but not identical) to spin-curl forces, a denomination originally attributed to Albaladejo et al [44]. Furthermore, in the GLMT framework, all kinds of forces (gradient, scattering, spin-curl and non-standard forces) may be expressed in terms of low order partial waves of order $n = 1$ and $n = 2$, a fact which is not explicitly revealed in the dipole theory of forces. Since both GLMT (for dipolar particles) and dipole theory of forces are rigorous theories, they must be equivalent. However, because these two theories use different languages, the proof of equivalence has still to be produced, in order to reveal the connection between both languages. The present work is viewed as a step toward the categorization of optical forces in a GLMT framework, for arbitrary sized particles. Such a study for arbitrary sized particles has been recently been published by Zheng et al. [32] but using a less refined categorization which does not explicitly evidence the existence of non-standard forces. Our current effort is then expected to produce a complementary point of view on the issue.

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