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GOOD SEQUENCES WITH UNCOUNTABLE SPECTRUM AND SINGULAR ASYMPTOTIC DISTRIBUTION

CHRISTOPHE CUNY AND FRANÇOIS PARREAU

ABSTRACT. We construct a good sequence with uncountable spectrum. As an application, we answer to a question of Lesigne, Quas Rosenblatt and Wierdl.

1. GOOD SEQUENCES WITH UNCOUNTABLE SPECTRUM

Let $S = (s_n)_{n \geq 1}$ be an increasing sequence of positive integers. We say that S is a *good sequence* if the following limit exists for every $\lambda \in \mathbb{S}^1$ ($\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$),

$$(1) \quad c(\lambda) = c_S(\lambda) := \lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{n=1}^N \lambda^{s_n}.$$

Equivalently, S is good if, for every $\lambda \in \mathbb{S}^1$, the following limit exists

$$(2) \quad \lim_{N \rightarrow +\infty} \frac{1}{\pi_S(N)} \sum_{1 \leq k \leq N, k \in S} \lambda^k,$$

where $\pi_S(N) = \#(S \cap [1, N])$.

Good sequences have been studied by many authors. See for instance Rosenblatt and Wierdl [12] (who introduced that notion), Rosenblatt [11], Boshernitzan, Kolesnik, Quas and Wierdl [2], Lemańczyk, Lesigne, Parreau, Volný and Wierdl [8] or Cuny, Eisner and Farkas [3].

Given a good sequence S , we define its spectrum as the set

$$(3) \quad \Lambda_S := \{\lambda \in \mathbb{S}^1 : c(\lambda) \neq 0\}.$$

By theorem 2.22 of [12] (due to Weyl), for any good sequence S , Λ_S has Lebesgue measure 0. If moreover, S has positive upper density, i.e. satisfies $\limsup_{N \rightarrow +\infty} (\pi_S(N)/N) > 0$, then Λ_S is countable. See Proposition 2.12 and Corollary 2.13 of [3] for a proof based on a result of Boshernitzan published in [11]. See also [7] for more general results of that type.

On another hand, up to our knowledge, no good sequence with uncountable spectrum is known.

In [3], good sequences have been studied in connection with Wiener's lemma. In particular they obtained the following results for good sequences, see their Proposition 2.6 and Theorem 2.10. Recall that if τ is a finite measure on \mathbb{S}^1 , then $\hat{\tau}(n) = \int_{\mathbb{S}^1} \lambda^n d\tau(\lambda)$, for every $n \in \mathbb{Z}$.

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Proposition 1. *Let $S = (s_n)_{n \geq 1}$ be a good sequence. Then, for every probability measure μ on \mathbb{S}^1 , we have*

$$\frac{1}{N} \sum_{n=1}^N |\hat{\mu}(s_n)|^2 \xrightarrow{N \rightarrow +\infty} \int_{(\mathbb{S}^1)^2} c(\lambda_1 \bar{\lambda}_2) d\mu(\lambda_1) d\mu(\lambda_2).$$

In particular, if S has countable spectrum and μ is continuous,

$$(4) \quad \frac{1}{N} \sum_{n=1}^N |\hat{\mu}(s_n)|^2 \xrightarrow{N \rightarrow +\infty} 0.$$

Remark. (4) implies that $\hat{\mu}(s_n)$ converges in density to 0, by the Koopman-von Neumann Lemma (see e.g. Lemma 2.1 of [3]).

The above considerations yield and put into perspective the following question: does there exist a good sequence with uncountable spectrum ?

We answer positively to that question below. To state the result, we need some more notation.

Let $(m_j)_{j \geq 1}$ be an increasing sequence of positive integers, such that $m_{j+1}/m_j \geq 3$ for every $j \geq 1$.

We associate with $(m_j)_{j \geq 1}$ the sequence $S = (s_n)_{n \geq 1}$ made out of the integers (an empty sum is assumed to be 0)

$$(5) \quad \left\{ m_k + \sum_{1 \leq j \leq k-1} \omega_j m_j : k \geq 1, (\omega_1, \dots, \omega_{k-1}) \in \{-1, 0, 1\}^{k-1} \right\}$$

in increasing order. Notice that our assumption on $(m_j)_{j \geq 1}$ implies that all the integers in (5) are positive and distinct.

Denote by $\|\cdot\|$ the distance to the nearest integer: $\|t\| := \min\{|m - t| : m \in \mathbb{Z}\}$ for every $t \in \mathbb{R}$.

Theorem 2. *Let $(m_j)_{j \geq 1}$ be an increasing sequence of positive integers, such that $m_{j+1}/m_j \geq 3$ for every $j \geq 1$ and define S as above. Then, S is good and*

$$(6) \quad \Lambda := \left\{ e^{2i\pi\theta} : \theta \in [0, 1) \setminus \mathbb{Q}, \sum_{j \geq 1} \|m_j \theta\|^2 < \infty \right\} \subset \Lambda_S.$$

Proof. For every $k \geq 1$, consider the following set of integers

$$M_k := \left\{ \sum_{1 \leq j \leq k-1} \omega_j m_j : (\omega_\ell)_{1 \leq \ell < k} \in \{-1, 0, 1\}^{k-1} \right\}.$$

For every $k \geq 1$ and every $\theta \in [0, 1)$, set

$$\begin{aligned}
(7) \quad L_k(\theta) &:= \prod_{1 \leq j \leq k-1} \frac{1}{3} (1 + 2 \cos(2\pi m_j \theta)) \\
&= \frac{1}{3^{k-1}} \prod_{1 \leq j \leq k-1} (1 + e^{-2i\pi m_j \theta} + e^{2i\pi m_j \theta}) \\
(8) \quad &= \frac{1}{3^{k-1}} \sum_{x \in M_k} e^{2i\pi x}.
\end{aligned}$$

Let $\theta \in [0, 1)$. As $-1/3 \leq (1 + 2 \cos(2\pi \theta m_j))/3 \leq 1$ for all j , if $1 + 2 \cos(2\pi \theta m_j)$ is infinitely often non positive, then $(L_k(\theta))_{k \geq 1}$ converges to 0.

Assume now that $1 + 2 \cos(2\pi \theta m_j) > 0$ for $j \geq J$, for some integer J . Then, the convergence of $(L_k(\theta))_{k \geq 1}$ follows from the convergence of $(\prod_{j=J}^k (1 + 2 \cos(2\pi \theta m_j))/3)_{k \geq J}$ which is clear since we have an infinite product of positive terms less than or equal to 1. Moreover this infinite product converges, i.e. the limit is non-zero, if and only if

$$\sum_{k=J}^{\infty} \left[1 - \frac{1}{3} (1 + 2 \cos(2\pi m_k \theta)) \right] = \sum_{k=J}^{\infty} \frac{2}{3} (1 - \cos(2\pi m_k \theta)) < +\infty,$$

which is equivalent to $\sum_{k=J}^{\infty} \|m_k \theta\|^2 < +\infty$.

If θ is in the set Λ defined by (6) the above condition is satisfied and moreover, as θ is then irrational, the product $\prod_{j=1}^{J-1} (1 + 2 \cos(2\pi \theta m_j))/3$ does not vanish.

Hence in any case $(L_k(\theta))_{k \geq 1}$ converges, say to $L(\theta)$, and L does not vanish on Λ .

We wish to prove that for every $\theta \in [0, 1)$, $(\frac{1}{N} \sum_{n=1}^N e^{2i\pi s_n \theta})_{N \geq 1}$ converges to $L(\theta)$.

Let $N \geq 1$. Since $(s_n)_{n \geq 1}$ is the increasing sequence made out of the numbers given by (5), we can write $s_{N+1} = m_{k_N} + \sum_{1 \leq j \leq k_N-1} \omega_j(N) m_j$.

The integers s_1, \dots, s_N may be split into consecutive blocks

$$m_1 + M_1, \dots, m_{k_N-1} + M_{k_N-1}, W_N,$$

where $W_N = \{\ell \in m_{k_N} + M_{k_N} : \ell \leq s_N\}$.

As each block M_k consists in 3^{k-1} integers, we have

$$(9) \quad \frac{3^{k_N-1} - 1}{2} \leq N < \frac{3^{k_N} - 1}{2}$$

We may furthermore split W_N into translates of blocks M_k . Namely, if $\omega_{k_N-1}(N) \neq -1$, then W_N begins with $m_{k_N} - m_{k_N-1} + M_{k_N-1}$, if $\omega_{k_N-1}(N) = 1$ another block $m_{k_N} + 0 \times m_{k_N-1} + M_{k_N-1}$ follows, and so on. More precisely, W_N is the disjoint union

$$W_N = \bigcup_{1 \leq j \leq k_N-1} \bigcup_{\omega < \omega_j(N)} \left(m_{k_N} + \sum_{\ell=j+1}^{k_N-1} \omega_\ell(N) m_\ell + \omega m_j + M_j \right).$$

Hence, by (8),

$$(10) \quad \sum_{n=1}^N e^{2i\pi s_n \theta} = \sum_{j=1}^{k_N-1} 3^{j-1} e^{2i\pi m_j \theta} L_j(\theta) + \sum_{j=1}^{k_N-1} \sum_{\omega < \omega_j(N)} 3^{j-1} e^{2i\pi u_j(\omega) \theta} L_j(\theta),$$

where $u_j(\omega) = m_{k_N} + \sum_{\ell=j+1}^{k_N-1} \omega_\ell(N) m_\ell + \omega m_j$.

Let us first assume that $L(\theta) = 0$. Then we have

$$\frac{1}{N} \left| \sum_{n=1}^N e^{2i\pi s_n \theta} \right| \leq \frac{1}{N} \sum_{j=1}^{k_N-1} 3^{j-1} |L_j(\theta)| + \frac{1}{N} \sum_{j=1}^{k_N-1} \sum_{\omega < \omega_j(N)} 3^{j-1} |L_j(\theta)| \xrightarrow{N \rightarrow +\infty} 0,$$

where the convergence follows from a Cesàro type argument.

Assume now that $L(\theta) \neq 0$. Then, $e^{2i\pi m_n \theta} \xrightarrow{n \rightarrow +\infty} 1$.

Fix $\varepsilon > 0$. Let $r \in \mathbb{N}$ be such that $e^{-r} < \varepsilon$, and let $d \in \mathbb{N}$ be such that $|1 - e^{2i\pi m_j \theta}| < \varepsilon/(r+1)$ and $|L(\theta) - L_j(\theta)| < \varepsilon$ for every $j \geq d$.

For every N such that $k_N \geq d+r$, we have on one hand, since $(L_n(\theta))_{n \geq 1}$ is bounded by 1,

$$(11) \quad \begin{aligned} & \sum_{j=1}^{k_N-r-1} 3^{j-1} |e^{2i\pi m_j \theta} L_j(\theta) - L(\theta)| + \sum_{j=1}^{k_N-r-1} \sum_{\omega < \omega_j(N)} 3^{j-1} |e^{2i\pi u_j(\omega) \theta} L_j(\theta) - L(\theta)| \\ & \leq \sum_{j=1}^{k_N-r-1} 3^{j-1} [2 + 2 \times 2] \leq 3^{k_N-r} < 3^{k_N} \varepsilon. \end{aligned}$$

And on the other hand, as $k_N - r \geq d$, when $k_N - r \leq j \leq k_N$ we have $|1 - e^{2i\pi m_j \theta}| < \varepsilon/(r+1)$ and

$$|1 - e^{2i\pi u_j(\omega) \theta}| \leq \sum_{\ell=k_N-r}^{k_N} |1 - e^{2i\pi m_\ell \theta}| < \varepsilon,$$

for every choice of ω . So,

$$(12) \quad \begin{aligned} & \sum_{j=k_N-r}^{k_N-1} 3^{j-1} |e^{2i\pi m_j \theta} L_j(\theta) - L(\theta)| + \sum_{j=k_N-r}^{k_N-1} \sum_{\omega < \omega_j(N)} 3^{j-1} |e^{2i\pi u_j(\omega) \theta} L_j(\theta) - L(\theta)| \\ & < \sum_{j=k_N-r}^{k_N-1} 3^{j-1} [2\varepsilon + 2 \times 2\varepsilon] < 3^{k_N} \varepsilon. \end{aligned}$$

Gathering (11) and (12), we get

$$\left| \sum_{n=1}^N e^{2i\pi s_n \theta} - NL(\theta) \right| < 2 \cdot 3^{k_N} \varepsilon.$$

Finally, in view of (9),

$$\limsup_{N \rightarrow +\infty} \left| \frac{1}{N} \sum_{n=1}^N e^{2i\pi s_n \theta} - L(\theta) \right| \leq 12\varepsilon,$$

and the announced result follows since ε is arbitrarily small. \square

It follows from Theorem 2 that, in order to produce a good sequence with uncountable spectrum, it is sufficient to exhibit an increasing sequence of integers $(m_j)_{j \geq 1}$, with $m_{j+1}/m_j \geq 3$ for every $j \geq 1$, and such that the subgroup of \mathbb{S}^1

$$(13) \quad H_2 = H_2((m_j)_{j \geq 1}) := \{e^{2i\pi\theta} : \theta \in [0, 1), \sum_{j \geq 1} \|m_j\theta\|^2 < \infty\},$$

be uncountable.

It turns out that those type of subgroups have been studied in [6] (and [10]).

A similar subgroup, defined by $H_1 := \{e^{2i\pi\theta} : \theta \in [0, 1), \sum_{j \geq 1} \|m_j\theta\| < \infty\}$, studied in [6] in connection with H_2 , has also been considered by Erdős and Taylor [4] and by Bergelson et al. [1] in connection with IP-rigidity.

In the above papers, sufficient conditions have been obtained for H_2 or H_1 to be uncountable.

To state the results concerning H_2 subgroups, we shall need a strengthening of the lacunarity condition. We say that $(m_j)_{j \geq 1}$ satisfies assumption (A) if one of the conditions (A_1) or (A_2) below is satisfied:

$$(A_1) \quad \sum_{j \geq 1} \left(\frac{m_j}{m_{j+1}}\right)^2 < \infty$$

$$(A_2) \quad \forall j \geq 1 \quad m_j | m_{j+1} \quad \text{and} \quad m_{j+1}/m_j \xrightarrow{j \rightarrow +\infty} \infty.$$

Proposition 3. *Let $(m_j)_{j \geq 1}$ be a sequence of integers satisfying assumption (A). Then, $H_2((m_j)_{j \geq 1})$ is uncountable.*

The proposition was proved by the second author [10] (see also section 4.2 of [6]) under (A_1) (notice that the condition $\inf_{j \in \mathbb{N}} m_{j+1}/m_j \geq 3$ used in [10] and [6] is not restrictive for the uncountability of H_2). Actually, it is proved in [10] and [6] that H_2 supports a continuous (singular) probability measure given by a symmetric Riesz product. A proof of the uncountability of H_2 under (A_1) can also be derived from the proof of Theorem 5 in [4], which states that H_1 is uncountable when $\sum_{j \geq 1} m_{j+1}/m_j < \infty$.

Under condition (A_2) , the proposition follows from Theorem 3 in [4] which states that $H_1 \subset H_2$ is uncountable. We use their argument below in the proofs of Proposition 5 and Theorem 6.

We are now able to state our main result, which follows in a straightforward way from Proposition 3 and Theorem 2.

Theorem 4. *Let $(m_j)_{j \geq 1}$ be an increasing sequence of positive integers, such that $m_{j+1}/m_j \geq 3$ for every $j \geq 1$ and define S as above. If assumption (A) is satisfied then S is a good sequence and it has uncountable spectrum.*

We also derive the following proposition which complements Proposition 1. It can be shown as an abstract consequence of the existence of a good sequence with uncountable spectrum, but we shall give explicit examples.

Proposition 5. *There exist a good sequence $(s_n)_{n \geq 1}$ and a continuous measure μ on \mathbb{S}^1 , such that $(\frac{1}{N} \sum_{n=1}^N |\hat{\mu}(s_n)|^2)_{N \geq 1}$ converges to some positive number.*

Proof. We construct such a measure for each sequence S associated with a sequence $(m_j)_{j \geq 1}$ satisfying (A_2) and $\inf_{j \in \mathbb{N}} m_{j+1}/m_j \geq 3$.

Under this assumption, choose a subsequence $(m_{j_k})_{k \geq 1}$ such that $j_1 > 1$ and $m_j/m_{j-1} > 2^{k+2}$ for all $j \geq j_k$. For every sequence $\eta = (\eta_k)_{k \geq 1} \in \{0, 1\}^{\mathbb{N}^*}$, let

$$\theta(\eta) = \sum_{k=1}^{\infty} \frac{\eta_k}{m_{j_k}}.$$

Given $j \geq 1$, let k be the smallest integer such that $j_k > j$. Since m_j/m_{j_ℓ} is an integer when $\ell < k$, we have

$$(14) \quad \|m_j \theta(\eta)\| \leq m_j \sum_{\ell \geq k} \frac{1}{m_{j_\ell}} \leq 2 \frac{m_j}{m_{j_k}},$$

and in particular $\|m_j \theta(\eta)\| \leq 1/4$, which yields that all the terms in the products (7) are positive.

We also have $\sum_{j < j_k} m_j^2 < 2m_{j_k-1}^2$, so if we sum up the $\|m_j \theta(\eta)\|^2$ by blocks from j_{k-1} to $j_k - 1$ (or from 1 to $j_1 - 1$ for the first one), we get that each partial sum is less than $8(m_{j_k-1}/m_{j_k})^2$,

$$\sum_{j=1}^{\infty} \|m_j \theta(\eta)\|^2 < 8 \sum_{k=1}^{\infty} \frac{m_{j_k-1}^2}{m_{j_k}^2} < \sum_{k=1}^{\infty} \frac{1}{4^k} < +\infty.$$

and $L(\theta(\eta)) > 0$ follows.

Now, let $\xi = (\xi_j)_{j \geq 1}$ be a sequence of i.i.d. random variables with $\mathbb{P}(\xi_1 = 0) = \mathbb{P}(\xi_1 = 1) = \frac{1}{2}$ and let μ be the probability distribution of $e^{2i\pi\theta(\xi)}$. Then, as the mapping $\eta \mapsto e^{2i\pi\theta(\eta)}$ is one-to-one, μ is a continuous probability measure concentrated on Λ_S and

$$\frac{1}{N} \sum_{n=1}^N \hat{\mu}(s_n) = \mathbb{E} \left(\frac{1}{N} \sum_{n=1}^N e^{2i\pi\theta(\xi)} \right) \rightarrow \mathbb{E}(L(\theta(\xi))) > 0 \quad \text{as } N \rightarrow +\infty.$$

Finally, Proposition 1 ensures the convergence of $\frac{1}{N} \sum_{n=1}^N |\hat{\mu}(s_n)|^2$ and the positivity of the limit follows the inequality $\frac{1}{N} \sum_{n=1}^N |\hat{\mu}(s_n)|^2 \geq \left| \frac{1}{N} \sum_{n=1}^N \hat{\mu}(s_n) \right|^2$. \square

Remark. Under assumption (A_1) and $\inf_{j \in \mathbb{N}} m_{j+1}/m_j \geq 3$, the result holds for the measure μ constructed in [10] or [6]. Indeed then μ is a generalized Riesz product, weak*-limit of products of trigonometric polynomials P_j with coefficients in blocks $\{km_j; -k_j \leq k \leq k_j\}$ and $\hat{P}_j(m_j) = \hat{P}_j(-m_j) = \cos(\pi/(m_j + 2))$. Then for every $s = \sum_{1 \leq j \leq n} \omega_j m_j$ where $|\omega_j| \leq k_j$ for all j , we have $\hat{\mu}(s) = \prod_{1 \leq j \leq n} \hat{P}_j(\omega_j m_j)$ (see [6]). From there, the convergence of $\frac{1}{N} \sum_{n=1}^N |\hat{\mu}(s_n)|^2$ and the positivity of the limit can be proven as in Theorem 2 (we skip the details).

2. SINGULAR ASYMPTOTIC DISTRIBUTION

We now turn to a matter addressed by Lesigne, Quas, Rosenblatt and Wierdl in the preprint [9].

Let $S = (s_n)_{n \in \mathbb{N}}$ be a good sequence. Let $\lambda \in \mathbb{S}^1$. Since S is good, the sequence $(\frac{1}{N} \sum_{n=1}^N \hat{\delta}_{\lambda^{s_n}}(m))_{N \in \mathbb{N}} = (\frac{1}{N} \sum_{n=1}^N \lambda^{ms_n})_{N \in \mathbb{N}}$ converges towards $c(\lambda^m)$ for any integer m , that is for any character on \mathbb{S}^1 , so that $(\frac{1}{N} \sum_{n=1}^N \delta_{\lambda^{s_n}})_{N \in \mathbb{N}}$ converges weakly to some probability $\nu_{S,\lambda}$.

Given a probability measure ν on \mathbb{S}^1 , if there exists a good sequence S and $\lambda \in \mathbb{S}^1$ such that $\nu_{S,\lambda} = \nu$, one says that S represents the measure ν at the point λ .

Lesigne et al. proved several interesting results concerning the measures that can be represented by a good sequence at some point $\lambda \in \mathbb{S}^1$. For instance, they proved that if λ is not a root of unity then $\nu_{S,\lambda}$ is continuous (see their Theorem 8.1). They also proved that if a given probability measure ν on \mathbb{S}^1 is not Rajchman (i.e. its Fourier coefficients do not vanish at infinity) then, for almost every λ with respect to the Haar measure, there does not exist any good sequence representing ν at λ (see their Theorem 8.2). On the opposite, if ν is absolutely continuous with respect to the Haar measure, then for every $\lambda \in \mathbb{S}^1$ which is not a root of unity there exists a good sequence S representing ν at λ (see their Theorem 9.1).

The above results raise the following questions. Does there exist a continuous but singular probability measure ν on \mathbb{S}^1 that can be represented by a good sequence? If so, can one take ν to be non Rajchman?

It turns out that Theorem 2 allows to exhibit a good sequence S and a point λ such that $\nu_{S,\lambda}$ is a non Rajchman probability measure.

Theorem 6. *Let $(m_j)_{j \in \mathbb{N}}$ be an increasing sequence of integers satisfying (A_2) and $\inf_{j \in \mathbb{N}} m_{j+1}/m_j \geq 3$, and let S be the sequence associated with it. There are uncountably many $\lambda \in \Lambda_S$ such that the weak*-limit $\nu_{S,\lambda}$ of $(\frac{1}{N} \sum_{n=1}^N \delta_{\lambda^{s_n}})_{N \in \mathbb{N}}$ satisfies $\limsup_{j \rightarrow +\infty} |\hat{\nu}_{S,\lambda}(m_j)| = 1$.*

Proof. We proceed as in the proof of Proposition 5, except that we require a stronger condition on the subsequence $(m_{j_k})_{k \geq 1}$, namely $m_j/m_{j-1} > 2^{k+2}m_{j_{k-1}}$ for all $j \geq j_k$ if $k > 1$.

For $\eta \in \{0, 1\}^{\mathbb{N}^*}$, we still define $\theta(\eta) = \sum_{k \geq 1} \eta_k/m_{j_k}$. By the proof of Proposition 5, this yields an uncountable family of $\lambda = e^{2i\pi\theta(\eta)}$ in Λ_S .

For each such $\theta = \theta(\eta)$ we have $\hat{\nu}_{S,\lambda}(m) = c(e^{2i\pi m\theta(\eta)}) = L(m\theta)$ for all $m \in \mathbb{Z}$. So, it will be sufficient to show that $L(m_{j_n}\theta) \rightarrow 1$ as $n \rightarrow +\infty$. Clearly, from the expression of $L(\theta)$ as an infinite product, it is equivalent to prove that $\sum_{j \geq 1} \|m_{j_n} m_j \theta\|^2$ converges to 0 as $n \rightarrow +\infty$.

Fix $n > 1$. We may apply the inequality (14) either to $\|m_j \theta\|$ or to $\|m_{j_n} \theta\|$. For $j < j_n$ we get $\|m_{j_n} m_j \theta\| \leq m_j \|m_{j_n} \theta\| \leq 2m_{j_n} m_j / m_{j_{n+1}}$, and in the opposite case $\|m_{j_n} m_j \theta\| \leq m_{j_n} \|m_j \theta\| \leq 2m_{j_n} m_j / m_{j_k}$ where k is the smallest integer such that $j_k > j$. So,

$$\sum_{j=1}^{j_n-1} \|m_{j_n} m_j \theta\|^2 \leq 4 \frac{m_{j_n}^2}{m_{j_{n+1}}^2} \sum_{j=1}^{j_n-1} m_j^2 \leq 8 \frac{m_{j_n}^2}{m_{j_{n+1}}^2} m_{j_{n-1}}^2 < \frac{1}{4^n}.$$

For $j \geq j_n$, we sum again by blocks from j_{k-1} to $j_k - 1$, for $k > n$,

$$\sum_{j_{k-1}}^{j_k-1} \|m_{j_n} m_j \theta\|^2 \leq 4 \frac{m_{j_n}^2}{m_{j_k}^2} \sum_{j_{k-1}}^{j_k-1} m_j^2 \leq 8 \frac{m_{j_n}^2}{m_{j_k}^2} m_{j_{k-1}}^2 < \frac{1}{4^k} \frac{m_{j_n}^2}{m_{j_{k-1}}^2} \leq \frac{1}{4^k},$$

and finally

$$\sum_{j=1}^{\infty} \|m_{j_n} m_j \theta\|^2 < \sum_{k=n}^{\infty} \frac{1}{4^k} \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

□

Let S and $\lambda \in \mathbb{S}^1$ be as in Theorem 6 and write $\nu = \nu_{S,\lambda}$.

The property $\limsup_{j \rightarrow +\infty} |\hat{\nu}(m_j)| = 1$ means precisely that ν is a *Dirichlet measure*, see [6] and [5] for properties of Dirichlet measures.

In particular there is then a subsequence $(n_j)_{j \geq 1}$ such that λ^{n_j} converges towards a constant of modulus 1 in the $L^1(\nu)$ topology, and it follows that any measure absolutely continuous with respect to ν is itself a Dirichlet measure.

On the other hand, any probability measure absolutely continuous with respect to some Rajchman measure is itself a Rajchman measure.

Hence, we infer that ν is singular with respect to any Rajchman probability measure on \mathbb{S}^1 .

Question. In view of Theorem 6, one may wonder if it is possible to find a good sequence S and $\lambda \in \mathbb{S}^1$ such that

$$0 < \limsup_{n \rightarrow +\infty} |\hat{\nu}_{S,\lambda}(n)| < 1.$$

Another question is whether one can have $\nu_{S,\lambda}$ Rajchman and singular with respect to the Lebesgue measure.

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