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# GOOD SEQUENCES WITH UNCOUNTABLE SPECTRUM AND SINGULAR ASYMPTOTIC DISTRIBUTION

CHRISTOPHE CUNY AND FRANÇOIS PARREAU

ABSTRACT. We construct a good sequence with uncountable spectrum. As an application, we answer to a question of Lesigne, Quas Rosenblatt and Wierdl.

## 1. GOOD SEQUENCES WITH UNCOUNTABLE SPECTRUM

Let  $S = (s_n)_{n \geq 1}$  be an increasing sequence of positive integers. We say that  $S$  is a *good sequence* if the following limit exists for every  $\lambda \in \mathbb{S}^1$  ( $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ ),

$$(1) \quad c(\lambda) = c_S(\lambda) := \lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{n=1}^N \lambda^{s_n}.$$

Equivalently,  $S$  is good if, for every  $\lambda \in \mathbb{S}^1$ , the following limit exists

$$(2) \quad \lim_{N \rightarrow +\infty} \frac{1}{\pi_S(N)} \sum_{1 \leq k \leq N, k \in S} \lambda^k,$$

where  $\pi_S(N) = \#(S \cap [1, N])$ .

Good sequences have been studied by many authors. See for instance Rosenblatt and Wierdl [12] (who introduced that notion), Rosenblatt [11], Boshernitzan, Kolesnik, Quas and Wierdl [2], Lemańczyk, Lesigne, Parreau, Volný and Wierdl [8] or Cuny, Eisner and Farkas [3].

Given a good sequence  $S$ , we define its spectrum as the set

$$(3) \quad \Lambda_S := \{\lambda \in \mathbb{S}^1 : c(\lambda) \neq 0\}.$$

By theorem 2.22 of [12] (due to Weyl), for any good sequence  $S$ ,  $\Lambda_S$  has Lebesgue measure 0. If moreover,  $S$  has positive upper density, i.e. satisfies  $\limsup_{N \rightarrow +\infty} (\pi_S(N)/N) > 0$ , then  $\Lambda_S$  is countable. See Proposition 2.12 and Corollary 2.13 of [3] for a proof based on a result of Boshernitzan published in [11]. See also [7] for more general results of that type.

On another hand, up to our knowledge, no good sequence with uncountable spectrum is known.

In [3], good sequences have been studied in connection with Wiener's lemma. In particular they obtained the following results for good sequences, see their Proposition 2.6 and Theorem 2.10. Recall that if  $\tau$  is a finite measure on  $\mathbb{S}^1$ , then  $\hat{\tau}(n) = \int_{\mathbb{S}^1} \lambda^n d\tau(\lambda)$ , for every  $n \in \mathbb{Z}$ .

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**Proposition 1.** *Let  $S = (s_n)_{n \geq 1}$  be a good sequence. Then, for every probability measure  $\mu$  on  $\mathbb{S}^1$ , we have*

$$\frac{1}{N} \sum_{n=1}^N |\hat{\mu}(s_n)|^2 \xrightarrow{N \rightarrow +\infty} \int_{(\mathbb{S}^1)^2} c(\lambda_1 \bar{\lambda}_2) d\mu(\lambda_1) d\mu(\lambda_2).$$

*In particular, if  $S$  has countable spectrum and  $\mu$  is continuous,*

$$(4) \quad \frac{1}{N} \sum_{n=1}^N |\hat{\mu}(s_n)|^2 \xrightarrow{N \rightarrow +\infty} 0.$$

**Remark.** (4) implies that  $\hat{\mu}(s_n)$  converges in density to 0, by the Koopman-von Neumann Lemma (see e.g. Lemma 2.1 of [3]).

The above considerations yield and put into perspective the following question: does there exist a good sequence with uncountable spectrum ?

We answer positively to that question below. To state the result, we need some more notation.

Let  $(m_j)_{j \geq 1}$  be an increasing sequence of positive integers, such that  $m_{j+1}/m_j \geq 3$  for every  $j \geq 1$ .

We associate with  $(m_j)_{j \geq 1}$  the sequence  $S = (s_n)_{n \geq 1}$  made out of the integers (an empty sum is assumed to be 0)

$$(5) \quad \left\{ m_k + \sum_{1 \leq j \leq k-1} \omega_j m_j : k \geq 1, (\omega_1, \dots, \omega_{k-1}) \in \{-1, 0, 1\}^{k-1} \right\}$$

in increasing order. Notice that our assumption on  $(m_j)_{j \geq 1}$  implies that all the integers in (5) are positive and distinct.

Denote by  $\|\cdot\|$  the distance to the nearest integer:  $\|t\| := \min\{|m - t| : m \in \mathbb{Z}\}$  for every  $t \in \mathbb{R}$ .

**Theorem 2.** *Let  $(m_j)_{j \geq 1}$  be an increasing sequence of positive integers, such that  $m_{j+1}/m_j \geq 3$  for every  $j \geq 1$  and define  $S$  as above. Then,  $S$  is good and*

$$(6) \quad \Lambda := \left\{ e^{2i\pi\theta} : \theta \in [0, 1) \setminus \mathbb{Q}, \sum_{j \geq 1} \|m_j \theta\|^2 < \infty \right\} \subset \Lambda_S.$$

**Proof.** For every  $k \geq 1$ , consider the following set of integers

$$M_k := \left\{ \sum_{1 \leq j \leq k-1} \omega_j m_j : (\omega_\ell)_{1 \leq \ell < k} \in \{-1, 0, 1\}^{k-1} \right\}.$$

For every  $k \geq 1$  and every  $\theta \in [0, 1)$ , set

$$\begin{aligned}
 (7) \quad L_k(\theta) &:= \prod_{1 \leq j \leq k-1} \frac{1}{3} (1 + 2 \cos(2\pi m_j \theta)) \\
 &= \frac{1}{3^{k-1}} \prod_{1 \leq j \leq k-1} (1 + e^{-2i\pi m_j \theta} + e^{2i\pi m_j \theta}) \\
 (8) \quad &= \frac{1}{3^{k-1}} \sum_{x \in M_k} e^{2i\pi x}.
 \end{aligned}$$

Let  $\theta \in [0, 1)$ . As  $-1/3 \leq (1 + 2 \cos(2\pi \theta m_j))/3 \leq 1$  for all  $j$ , if  $1 + 2 \cos(2\pi \theta m_j)$  is infinitely often non positive, then  $(L_k(\theta))_{k \geq 1}$  converges to 0.

Assume now that  $1 + 2 \cos(2\pi \theta m_j) > 0$  for  $j \geq J$ , for some integer  $J$ . Then, the convergence of  $(L_k(\theta))_{k \geq 1}$  follows from the convergence of  $(\prod_{j=J}^k (1 + 2 \cos(2\pi \theta m_j))/3)_{k \geq J}$  which is clear since we have an infinite product of positive terms less than or equal to 1. Moreover this infinite product converges, i.e. the limit is non-zero, if and only if

$$\sum_{k=J}^{\infty} \left[ 1 - \frac{1}{3} (1 + 2 \cos(2\pi m_k \theta)) \right] = \sum_{k=J}^{\infty} \frac{2}{3} (1 - \cos(2\pi m_k \theta)) < +\infty,$$

which is equivalent to  $\sum_{k=J}^{\infty} \|m_k \theta\|^2 < +\infty$ .

If  $\theta$  is in the set  $\Lambda$  defined by (6) the above condition is satisfied and moreover, as  $\theta$  is then irrational, the product  $\prod_{j=1}^{J-1} (1 + 2 \cos(2\pi \theta m_j))/3$  does not vanish.

Hence in any case  $(L_k(\theta))_{k \geq 1}$  converges, say to  $L(\theta)$ , and  $L$  does not vanish on  $\Lambda$ .

We wish to prove that for every  $\theta \in [0, 1)$ ,  $(\frac{1}{N} \sum_{n=1}^N e^{2i\pi s_n \theta})_{N \geq 1}$  converges to  $L(\theta)$ .

Let  $N \geq 1$ . Since  $(s_n)_{n \geq 1}$  is the increasing sequence made out of the numbers given by (5), we can write  $s_{N+1} = m_{k_N} + \sum_{1 \leq j \leq k_N-1} \omega_j(N) m_j$ .

The integers  $s_1, \dots, s_N$  may be split into consecutive blocks

$$m_1 + M_1, \dots, m_{k_N-1} + M_{k_N-1}, W_N,$$

where  $W_N = \{\ell \in m_{k_N} + M_{k_N} : \ell \leq s_N\}$ .

As each block  $M_k$  consists in  $3^{k-1}$  integers, we have

$$(9) \quad \frac{3^{k_N-1} - 1}{2} \leq N < \frac{3^{k_N} - 1}{2}$$

We may furthermore split  $W_N$  into translates of blocks  $M_k$ . Namely, if  $\omega_{k_N-1}(N) \neq -1$ , then  $W_N$  begins with  $m_{k_N} - m_{k_N-1} + M_{k_N-1}$ , if  $\omega_{k_N-1}(N) = 1$  another block  $m_{k_N} + 0 \times m_{k_N-1} + M_{k_N-1}$  follows, and so on. More precisely,  $W_N$  is the disjoint union

$$W_N = \bigcup_{1 \leq j \leq k_N-1} \bigcup_{\omega < \omega_j(N)} \left( m_{k_N} + \sum_{\ell=j+1}^{k_N-1} \omega_\ell(N) m_\ell + \omega m_j + M_j \right).$$

Hence, by (8),

$$(10) \quad \sum_{n=1}^N e^{2i\pi s_n \theta} = \sum_{j=1}^{k_N-1} 3^{j-1} e^{2i\pi m_j \theta} L_j(\theta) + \sum_{j=1}^{k_N-1} \sum_{\omega < \omega_j(N)} 3^{j-1} e^{2i\pi u_j(\omega)\theta} L_j(\theta),$$

where  $u_j(\omega) = m_{k_N} + \sum_{\ell=j+1}^{k_N-1} \omega_\ell(N) m_\ell + \omega m_j$ .

Let us first assume that  $L(\theta) = 0$ . Then we have

$$\frac{1}{N} \left| \sum_{n=1}^N e^{2i\pi s_n \theta} \right| \leq \frac{1}{N} \sum_{j=1}^{k_N-1} 3^{j-1} |L_j(\theta)| + \frac{1}{N} \sum_{j=1}^{k_N-1} \sum_{\omega < \omega_j(N)} 3^{j-1} |L_j(\theta)| \xrightarrow{N \rightarrow +\infty} 0,$$

where the convergence follows from a Cesàro type argument.

Assume now that  $L(\theta) \neq 0$ . Then,  $e^{2i\pi m_n \theta} \xrightarrow{n \rightarrow +\infty} 1$ .

Fix  $\varepsilon > 0$ . Let  $r \in \mathbb{N}$  be such that  $e^{-r} < \varepsilon$ , and let  $d \in \mathbb{N}$  be such that  $|1 - e^{2i\pi m_j \theta}| < \varepsilon/(r+1)$  and  $|L(\theta) - L_j(\theta)| < \varepsilon$  for every  $j \geq d$ .

For every  $N$  such that  $k_N \geq d+r$ , we have on one hand, since  $(L_n(\theta))_{n \geq 1}$  is bounded by 1,

$$(11) \quad \begin{aligned} & \sum_{j=1}^{k_N-r-1} 3^{j-1} |e^{2i\pi m_j \theta} L_j(\theta) - L(\theta)| + \sum_{j=1}^{k_N-r-1} \sum_{\omega < \omega_j(N)} 3^{j-1} |e^{2i\pi u_j(\omega)\theta} L_j(\theta) - L(\theta)| \\ & \leq \sum_{j=1}^{k_N-r-1} 3^{j-1} [2 + 2 \times 2] \leq 3^{k_N-r} < 3^{k_N} \varepsilon. \end{aligned}$$

And on the other hand, as  $k_N - r \geq d$ , when  $k_N - r \leq j \leq k_N$  we have  $|1 - e^{2i\pi m_j \theta}| < \varepsilon/(r+1)$  and

$$|1 - e^{2i\pi u_j(\omega)\theta}| \leq \sum_{\ell=k_N-r}^{k_N} |1 - e^{2i\pi m_\ell \theta}| < \varepsilon,$$

for every choice of  $\omega$ . So,

$$(12) \quad \begin{aligned} & \sum_{j=k_N-r}^{k_N-1} 3^{j-1} |e^{2i\pi m_j \theta} L_j(\theta) - L(\theta)| + \sum_{j=k_N-r}^{k_N-1} \sum_{\omega < \omega_j(N)} 3^{j-1} |e^{2i\pi u_j(\omega)\theta} L_j(\theta) - L(\theta)| \\ & < \sum_{j=k_N-r}^{k_N-1} 3^{j-1} [2\varepsilon + 2 \times 2\varepsilon] < 3^{k_N} \varepsilon. \end{aligned}$$

Gathering (11) and (12), we get

$$\left| \sum_{n=1}^N e^{2i\pi s_n \theta} - NL(\theta) \right| < 2 \cdot 3^{k_N} \varepsilon.$$

Finally, in view of (9),

$$\limsup_{N \rightarrow +\infty} \left| \frac{1}{N} \sum_{n=1}^N e^{2i\pi s_n \theta} - L(\theta) \right| \leq 12\varepsilon,$$

and the announced result follows since  $\varepsilon$  is arbitrarily small.  $\square$

It follows from Theorem 2 that, in order to produce a good sequence with uncountable spectrum, it is sufficient to exhibit an increasing sequence of integers  $(m_j)_{j \geq 1}$ , with  $m_{j+1}/m_j \geq 3$  for every  $j \geq 1$ , and such that the subgroup of  $\mathbb{S}^1$

$$(13) \quad H_2 = H_2((m_j)_{j \geq 1}) := \{e^{2i\pi\theta} : \theta \in [0, 1), \sum_{j \geq 1} \|m_j\theta\|^2 < \infty\},$$

be uncountable.

It turns out that those type of subgroups have been studied in [6] (and [10]).

A similar subgroup, defined by  $H_1 := \{e^{2i\pi\theta} : \theta \in [0, 1), \sum_{j \geq 1} \|m_j\theta\| < \infty\}$ , studied in [6] in connection with  $H_2$ , has also been considered by Erdős and Taylor [4] and by Bergelson et al. [1] in connection with IP-rigidity.

In the above papers, sufficient conditions have been obtained for  $H_2$  or  $H_1$  to be uncountable.

To state the results concerning  $H_2$  subgroups, we shall need a strengthening of the lacunarity condition. We say that  $(m_j)_{j \geq 1}$  satisfies assumption (A) if one of the conditions  $(A_1)$  or  $(A_2)$  below is satisfied:

$$(A_1) \quad \sum_{j \geq 1} \left( \frac{m_j}{m_{j+1}} \right)^2 < \infty$$

$$(A_2) \quad \forall j \geq 1 \quad m_j | m_{j+1} \quad \text{and} \quad m_{j+1}/m_j \xrightarrow{j \rightarrow +\infty} \infty.$$

**Proposition 3.** *Let  $(m_j)_{j \geq 1}$  be a sequence of integers satisfying assumption (A). Then,  $H_2((m_j)_{j \geq 1})$  is uncountable.*

The proposition was proved by the second author [10] (see also section 4.2 of [6]) under  $(A_1)$  (notice that the condition  $\inf_{j \in \mathbb{N}} m_{j+1}/m_j \geq 3$  used in [10] and [6] is not restrictive for the uncountability of  $H_2$ ). Actually, it is proved in [10] and [6] that  $H_2$  supports a continuous (singular) probability measure given by a symmetric Riesz product. A proof of the uncountability of  $H_2$  under  $(A_1)$  can also be derived from the proof of Theorem 5 in [4], which states that  $H_1$  is uncountable when  $\sum_{j \geq 1} m_{j+1}/m_j < \infty$ .

Under condition  $(A_2)$ , the proposition follows from Theorem 3 in [4] which states that  $H_1 \subset H_2$  is uncountable. We use their argument below in the proofs of Proposition 5 and Theorem 6.

We are now able to state our main result, which follows in a straightforward way from Proposition 3 and Theorem 2.

**Theorem 4.** *Let  $(m_j)_{j \geq 1}$  be an increasing sequence of positive integers, such that  $m_{j+1}/m_j \geq 3$  for every  $j \geq 1$  and define  $S$  as above. If assumption (A) is satisfied then  $S$  is a good sequence and it has uncountable spectrum.*

We also derive the following proposition which complements Proposition 1. It can be shown as an abstract consequence of the existence of a good sequence with uncountable spectrum, but we shall give explicit examples.

**Proposition 5.** *There exist a good sequence  $(s_n)_{n \geq 1}$  and a continuous measure  $\mu$  on  $\mathbb{S}^1$ , such that  $(\frac{1}{N} \sum_{n=1}^N |\hat{\mu}(s_n)|^2)_{N \geq 1}$  converges to some positive number.*

**Proof.** We construct such a measure for each sequence  $S$  associated with a sequence  $(m_j)_{j \geq 1}$  satisfying  $(A_2)$  and  $\inf_{j \in \mathbb{N}} m_{j+1}/m_j \geq 3$ .

Under this assumption, choose a subsequence  $(m_{j_k})_{k \geq 1}$  such that  $j_1 > 1$  and  $m_j/m_{j-1} > 2^{k+2}$  for all  $j \geq j_k$ . For every sequence  $\eta = (\eta_k)_{k \geq 1} \in \{0, 1\}^{\mathbb{N}^*}$ , let

$$\theta(\eta) = \sum_{k=1}^{\infty} \frac{\eta_k}{m_{j_k}}.$$

Given  $j \geq 1$ , let  $k$  be the smallest integer such that  $j_k > j$ . Since  $m_j/m_{j_\ell}$  is an integer when  $\ell < k$ , we have

$$(14) \quad \|m_j \theta(\eta)\| \leq m_j \sum_{\ell \geq k} \frac{1}{m_{j_\ell}} \leq 2 \frac{m_j}{m_{j_k}},$$

and in particular  $\|m_j \theta(\eta)\| \leq 1/4$ , which yields that all the terms in the products (7) are positive.

We also have  $\sum_{j < j_k} m_j^2 < 2m_{j_k-1}^2$ , so if we sum up the  $\|m_j \theta(\eta)\|^2$  by blocks from  $j_{k-1}$  to  $j_k - 1$  (or from 1 to  $j_1 - 1$  for the first one), we get that each partial sum is less than  $8(m_{j_k-1}/m_{j_k})^2$ ,

$$\sum_{j=1}^{\infty} \|m_j \theta(\eta)\|^2 < 8 \sum_{k=1}^{\infty} \frac{m_{j_k-1}^2}{m_{j_k}^2} < \sum_{k=1}^{\infty} \frac{1}{4^k} < +\infty.$$

and  $L(\theta(\eta)) > 0$  follows.

Now, let  $\xi = (\xi_j)_{j \geq 1}$  be a sequence of i.i.d. random variables with  $\mathbb{P}(\xi_1 = 0) = \mathbb{P}(\xi_1 = 1) = \frac{1}{2}$  and let  $\mu$  be the probability distribution of  $e^{2i\pi\theta(\xi)}$ . Then, as the mapping  $\eta \mapsto e^{2i\pi\theta(\eta)}$  is one-to-one,  $\mu$  is a continuous probability measure concentrated on  $\Lambda_S$  and

$$\frac{1}{N} \sum_{n=1}^N \hat{\mu}(s_n) = \mathbb{E} \left( \frac{1}{N} \sum_{n=1}^N e^{2i\pi\theta(\xi)} \right) \rightarrow \mathbb{E}(L(\theta(\xi))) > 0 \quad \text{as } N \rightarrow +\infty.$$

Finally, Proposition 1 ensures the convergence of  $\frac{1}{N} \sum_{n=1}^N |\hat{\mu}(s_n)|^2$  and the positivity of the limit follows the inequality  $\frac{1}{N} \sum_{n=1}^N |\hat{\mu}(s_n)|^2 \geq \left| \frac{1}{N} \sum_{n=1}^N \hat{\mu}(s_n) \right|^2$ .  $\square$

**Remark.** Under assumption  $(A_1)$  and  $\inf_{j \in \mathbb{N}} m_{j+1}/m_j \geq 3$ , the result holds for the measure  $\mu$  constructed in [10] or [6]. Indeed then  $\mu$  is a generalized Riesz product, weak\*-limit of products of trigonometric polynomials  $P_j$  with coefficients in blocks  $\{km_j; -k_j \leq k \leq k_j\}$  and  $\hat{P}_j(m_j) = \hat{P}_j(-m_j) = \cos(\pi/(m_j + 2))$ . Then for every  $s = \sum_{1 \leq j \leq n} \omega_j m_j$  where  $|\omega_j| \leq k_j$  for all  $j$ , we have  $\hat{\mu}(s) = \prod_{1 \leq j \leq n} \hat{P}_j(\omega_j m_j)$  (see [6]). From there, the convergence of  $\frac{1}{N} \sum_{n=1}^N |\hat{\mu}(s_n)|^2$  and the positivity of the limit can be proven as in Theorem 2 (we skip the details).

## 2. SINGULAR ASYMPTOTIC DISTRIBUTION

We now turn to a matter addressed by Lesigne, Quas, Rosenblatt and Wierdl in the preprint [9].

Let  $S = (s_n)_{n \in \mathbb{N}}$  be a good sequence. Let  $\lambda \in \mathbb{S}^1$ . Since  $S$  is good, the sequence  $(\frac{1}{N} \sum_{n=1}^N \hat{\delta}_{\lambda^{s_n}}(m))_{N \in \mathbb{N}} = (\frac{1}{N} \sum_{n=1}^N \lambda^{ms_n})_{N \in \mathbb{N}}$  converges towards  $c(\lambda^m)$  for any integer  $m$ , that is for any character on  $\mathbb{S}^1$ , so that  $(\frac{1}{N} \sum_{n=1}^N \delta_{\lambda^{s_n}})_{N \in \mathbb{N}}$  converges weakly to some probability  $\nu_{S,\lambda}$ .

Given a probability measure  $\nu$  on  $\mathbb{S}^1$ , if there exists a good sequence  $S$  and  $\lambda \in \mathbb{S}^1$  such that  $\nu_{S,\lambda} = \nu$ , one says that  $S$  represents the measure  $\nu$  at the point  $\lambda$ .

Lesigne et al. proved several interesting results concerning the measures that can be represented by a good sequence at some point  $\lambda \in \mathbb{S}^1$ . For instance, they proved that if  $\lambda$  is not a root of unity then  $\nu_{S,\lambda}$  is continuous (see their Theorem 8.1). They also proved that if a given probability measure  $\nu$  on  $\mathbb{S}^1$  is not Rajchman (i.e. its Fourier coefficients do not vanish at infinity) then, for almost every  $\lambda$  with respect to the Haar measure, there does not exist any good sequence representing  $\nu$  at  $\lambda$  (see their Theorem 8.2). On the opposite, if  $\nu$  is absolutely continuous with respect to the Haar measure, then for every  $\lambda \in \mathbb{S}^1$  which is not a root of unity there exists a good sequence  $S$  representing  $\nu$  at  $\lambda$  (see their Theorem 9.1).

The above results raise the following questions. Does there exist a continuous but singular probability measure  $\nu$  on  $\mathbb{S}^1$  that can be represented by a good sequence? If so, can one take  $\nu$  to be non Rajchman?

It turns out that Theorem 2 allows to exhibit a good sequence  $S$  and a point  $\lambda$  such that  $\nu_{S,\lambda}$  is a non Rajchman probability measure.

**Theorem 6.** *Let  $(m_j)_{j \in \mathbb{N}}$  be an increasing sequence of integers satisfying  $(A_2)$  and  $\inf_{j \in \mathbb{N}} m_{j+1}/m_j \geq 3$ , and let  $S$  be the sequence associated with it. There are uncountably many  $\lambda \in \Lambda_S$  such that the weak\*-limit  $\nu_{S,\lambda}$  of  $(\frac{1}{N} \sum_{n=1}^N \delta_{\lambda^{s_n}})_{N \in \mathbb{N}}$  satisfies  $\limsup_{j \rightarrow +\infty} |\hat{\nu}_{S,\lambda}(m_j)| = 1$ .*

**Proof.** We proceed as in the proof of Proposition 5, except that we require a stronger condition on the subsequence  $(m_{j_k})_{k \geq 1}$ , namely  $m_j/m_{j-1} > 2^{k+2}m_{j_{k-1}}$  for all  $j \geq j_k$  if  $k > 1$ .

For  $\eta \in \{0, 1\}^{\mathbb{N}^*}$ , we still define  $\theta(\eta) = \sum_{k \geq 1} \eta_k/m_{j_k}$ . By the proof of Proposition 5, this yields an uncountable family of  $\lambda = e^{2i\pi\theta(\eta)}$  in  $\Lambda_S$ .

For each such  $\theta = \theta(\eta)$  we have  $\hat{\nu}_{S,\lambda}(m) = c(e^{2i\pi m\theta(\eta)}) = L(m\theta)$  for all  $m \in \mathbb{Z}$ . So, it will be sufficient to show that  $L(m_{j_n}\theta) \rightarrow 1$  as  $n \rightarrow +\infty$ . Clearly, from the expression of  $L(\theta)$  as an infinite product, it is equivalent to prove that  $\sum_{j \geq 1} \|m_{j_n} m_j \theta\|^2$  converges to 0 as  $n \rightarrow +\infty$ .

Fix  $n > 1$ . We may apply the inequality (14) either to  $\|m_j \theta\|$  or to  $\|m_{j_n} \theta\|$ . For  $j < j_n$  we get  $\|m_{j_n} m_j \theta\| \leq m_j \|m_{j_n} \theta\| \leq 2m_{j_n} m_j / m_{j_{n+1}}$ , and in the opposite case  $\|m_{j_n} m_j \theta\| \leq m_{j_n} \|m_j \theta\| \leq 2m_{j_n} m_j / m_{j_k}$  where  $k$  is the smallest integer such that  $j_k > j$ . So,

$$\sum_{j=1}^{j_n-1} \|m_{j_n} m_j \theta\|^2 \leq 4 \frac{m_{j_n}^2}{m_{j_{n+1}}^2} \sum_{j=1}^{j_n-1} m_j^2 \leq 8 \frac{m_{j_n}^2}{m_{j_{n+1}}^2} m_{j_{n-1}}^2 < \frac{1}{4^n}.$$



For  $j \geq j_n$ , we sum again by blocks from  $j_{k-1}$  to  $j_k - 1$ , for  $k > n$ ,

$$\sum_{j_{k-1}}^{j_k-1} \|m_{j_n} m_j \theta\|^2 \leq 4 \frac{m_{j_n}^2}{m_{j_k}^2} \sum_{j_{k-1}}^{j_k-1} m_j^2 \leq 8 \frac{m_{j_n}^2}{m_{j_k}^2} m_{j_{k-1}}^2 < \frac{1}{4^k} \frac{m_{j_n}^2}{m_{j_{k-1}}^2} \leq \frac{1}{4^k},$$

and finally

$$\sum_{j=1}^{\infty} \|m_{j_n} m_j \theta\|^2 < \sum_{k=n}^{\infty} \frac{1}{4^k} \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

□

Let  $S$  and  $\lambda \in \mathbb{S}^1$  be as in Theorem 6 and write  $\nu = \nu_{S,\lambda}$ .

The property  $\limsup_{j \rightarrow +\infty} |\hat{\nu}(m_j)| = 1$  means precisely that  $\nu$  is a *Dirichlet measure*, see [6] and [5] for properties of Dirichlet measures.

In particular there is then a subsequence  $(n_j)_{j \geq 1}$  such that  $\lambda^{n_j}$  converges towards a constant of modulus 1 in the  $L^1(\nu)$  topology, and it follows that any measure absolutely continuous with respect to  $\nu$  is itself a Dirichlet measure.

On the other hand, any probability measure absolutely continuous with respect to some Rajchman measure is itself a Rajchman measure.

Hence, we infer that  $\nu$  is singular with respect to any Rajchman probability measure on  $\mathbb{S}^1$ .

**Question.** In view of Theorem 6, one may wonder if it is possible to find a good sequence  $S$  and  $\lambda \in \mathbb{S}^1$  such that

$$0 < \limsup_{n \rightarrow +\infty} |\hat{\nu}_{S,\lambda}(n)| < 1.$$

Another question is whether one can have  $\nu_{S,\lambda}$  Rajchman and singular with respect to the Lebesgue measure.

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