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GOOD SEQUENCES WITH UNCOUNTABLE SPECTRUM AND SINGULAR ASYMPTOTIC DISTRIBUTION

CHRISTOPHE CUNY AND FRANÇOIS PARREAU

Abstract. We construct a good sequence with uncountable spectrum. As an application, we answer to a question of Lesigne, Quas Rosenblatt and Wierdl.

1. Good sequences with uncountable spectrum

Let $S = (s_n)_{n \geq 1}$ be an increasing sequence of positive integers. We say that $S$ is a good sequence if the following limit exists for every $\lambda \in S^1$ ($S^1 = \{z \in \mathbb{C} : |z| = 1\}$),

$$c(\lambda) = c_S(\lambda) := \lim_{N \to +\infty} \frac{1}{N} \sum_{n=1}^{N} \lambda^{s_n}.$$  \hfill (1)

Equivalently, $S$ is good if, for every $\lambda \in S^1$, the following limit exists

$$\lim_{N \to +\infty} \frac{1}{\pi_S(N)} \sum_{1 \leq k \leq N, k \in S} \lambda^k,$$

where $\pi_S(N) = \# (S \cap [1, N])$.

Good sequences have been studied by many authors. See for instance Rosenblatt and Wierdl [12] (who introduced that notion), Rosenblatt [11], Boshernitzan, Kolesnik, Quas and Wierdl [2], Lemańczyk, Lesigne, Parreau, Volný and Wierdl [8] or Cuny, Eisner and Farkas [3].

Given a good sequence $S$, we define its spectrum as the set

$$\Lambda_S := \{\lambda \in S^1 : c(\lambda) \neq 0\}.$$  \hfill (3)

By theorem 2.22 of [12] (due to Weyl), for any good sequence $S$, $\Lambda_S$ has Lebesgue measure 0. If moreover, $S$ has positive upper density, i.e. satisfies $\limsup_{N \to +\infty} (\pi_S(N)/N) > 0$, then $\Lambda_S$ is countable. See Proposition 2.12 and Corollary 2.13 of [3] for a proof based on a result of Boshernitzan published in [11]. See also [7] for more general results of that type.

On another hand, up to our knowledge, no good sequence with uncountable spectrum is known.

In [3], good sequences have been studied in connection with Wiener’s lemma. In particular they obtained the following results for good sequences, see their Proposition 2.6 and Theorem 2.10. Recall that if $\tau$ is a finite measure on $S^1$, then $\hat{\tau}(n) = \int_{S^1} \lambda^n d\tau(\lambda)$, for every $n \in \mathbb{Z}$.

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Proposition 1. Let $S = (s_n)_{n \geq 1}$ be a good sequence. Then, for every probability measure $\mu$ on $S^1$, we have

$$\frac{1}{N} \sum_{n=1}^{N} |\hat{\mu}(s_n)|^2 \xrightarrow{N \to +\infty} \int_{S^1} c(\lambda_1 \lambda_2) d\mu(\lambda_1) d\mu(\lambda_2).$$

In particular, if $S$ has countable spectrum and $\mu$ is continuous,

$$\frac{1}{N} \sum_{n=1}^{N} |\hat{\mu}(s_n)|^2 \xrightarrow{N \to +\infty} 0.$$

Remark. (4) implies that $\hat{\mu}(s_n)$ converges in density to 0, by the Koopman-von Neumann Lemma (see e.g. Lemma 2.1 of [3]).

The above considerations yield and put into perspective the following question: does there exist a good sequence with uncountable spectrum?

We answer positively to that question below. To state the result, we need some more notation.

Let $(m_j)_{j \geq 1}$ be an increasing sequence of positive integers, such that $m_{j+1}/m_j \geq 3$ for every $j \geq 1$.

We associate with $(m_j)_{j \geq 1}$ the sequence $S = (s_n)_{n \geq 1}$ made out of the integers (an empty sum is assumed to be 0)

$$\left\{ m_k + \sum_{1 \leq j \leq k-1} \omega_j m_j : k \geq 1, (\omega_1, \ldots, \omega_{k-1}) \in \{-1, 0, 1\}^{k-1} \right\}$$

in increasing order. Notice that our assumption on $(m_j)_{j \geq 1}$ implies that all the integers in (5) are positive and distinct.

Denote by $\| \cdot \|$ the distance to the nearest integer: $\| t \| := \min \{ |m - t| : m \in \mathbb{Z} \}$ for every $t \in \mathbb{R}$.

Theorem 2. Let $(m_j)_{j \geq 1}$ be an increasing sequence of positive integers, such that $m_{j+1}/m_j \geq 3$ for every $j \geq 1$ and define $S$ as above. Then, $S$ is good and

$$\Lambda := \left\{ e^{2\pi i \theta} : \theta \in [0, 1) \backslash \mathbb{Q}, \sum_{j \geq 1} \|m_j \theta\|^2 < \infty \right\} \subset \Lambda_S.$$

Proof. For every $k \geq 1$, consider the following set of integers

$$M_k := \left\{ \sum_{1 \leq j \leq k-1} \omega_j m_j : (\omega_k)_{1 \leq k \leq} \in \{-1, 0, 1\}^{k-1} \right\}.$$
For every $k \geq 1$ and every $\theta \in [0,1)$, set
\begin{equation}
L_k(\theta) := \prod_{1 \leq j \leq k-1} \frac{1}{3} \left(1 + 2 \cos(2\pi m_j \theta)\right) = \frac{1}{3^{k-1}} \prod_{1 \leq j \leq k-1} \left(1 + e^{-2i\pi m_j \theta} + e^{2i\pi m_j \theta}\right)
\end{equation}
\begin{equation}
= \frac{1}{3^{k-1}} \sum_{x \in M_k} e^{2i\pi x}.
\end{equation}

Let $\theta \in [0,1)$. As $-1/3 \leq (1 + 2 \cos(2\pi m_j))/3 \leq 1$ for all $j$, if $1 + 2 \cos(2\pi m_j)$ is infinitely often non positive, then $(L_k(\theta))_{k \geq 1}$ converges to $0$.

Assume now that $1 + 2 \cos(2\pi m_j) > 0$ for $j \geq J$, for some integer $J$. Then, the convergence of $(L_k(\theta))_{k \geq 1}$ follows from the convergence of $(\prod_{j=J}^{k-1} (1 + 2 \cos(2\pi m_j))/3)_{k \geq J}$ which is clear since we have an infinite product of positive terms less than or equal to $1$. Moreover this infinite product converges, i.e. the limit is non-zero, if and only if
\[
\sum_{k=J}^{\infty} \left|1 - \frac{1}{3} (1 + 2 \cos(2\pi m_k \theta))\right| = \frac{2}{3} \sum_{k=J}^{\infty} (1 - \cos(2\pi m_k \theta)) < +\infty,
\]
which is equivalent to $\sum_{k=J}^{\infty} \|m_k \theta\|^2 < +\infty$.

If $\theta$ is in the set $\Lambda$ defined by (6) the above condition is satisfied and moreover, as $\theta$ is then irrational, the product $\prod_{j=1}^{J-1} (1 + 2 \cos(2\pi \theta m_j))/3$ does not vanish.

Hence in any case $(L_k(\theta))_{k \geq 1}$ converges, say to $L(\theta)$, and $L$ does not vanish on $\Lambda$.

We wish to prove that for every $\theta \in [0,1)$, $(\frac{1}{N} \sum_{n=1}^{N} e^{2i\pi s_n \theta})_{N \geq 1}$ converges to $L(\theta)$.

Let $N \geq 1$. Since $(s_n)_{n \geq 1}$ is the increasing sequence made out of the numbers given by (5), we can write $s_{N+1} = m_k + \sum_{1 \leq j \leq k-1} \omega_j(N)m_j$.

The integers $s_1, \ldots, s_N$ may be split into consecutive blocks
\[m_1 + M_1, \ldots, m_{k_N - 1} + M_{k_N - 1}, W_N,\]
where $W_N = \{\ell \in m_{k_N} + M_{k_N} : \ell \leq s_N\}$.

As each block $M_k$ consists in $3^{k-1}$ integers, we have
\begin{equation}
\frac{3^{k-1} - 1}{2} \leq N < \frac{3^{k-1} - 1}{2}.
\end{equation}

We may furthermore split $W_N$ into translates of blocks $M_k$. Namely, if $\omega_{k_N - 1}(N) \neq -1$, then $W_N$ begins with $m_{k_N} - m_{k_N - 1} + M_{k_N - 1}$, if $\omega_{k_N - 1}(N) = 1$ another block $m_{k_N} + 0 \times m_{k_N - 1} + M_{k_N - 1}$ follows, and so on. More precisely, $W_N$ is the disjoint union
\[W_N = \bigcup_{1 \leq j \leq k_N - 1} \bigcup_{\omega \leq \omega(N)} \left(m_{k_N} + \sum_{\ell = j+1}^{k_N - 1} \omega_{\ell}(N)m_{\ell} + \omega m_j + M_j\right).\]
Hence, by (8),
\begin{equation}
\sum_{n=1}^{N} e^{2\pi i n \theta} = \sum_{j=1}^{k_N-1} 3^{j-1} e^{2\pi i m_j \theta} L_j(\theta) + \sum_{j=1}^{k_N-1} \sum_{\omega < \omega_j(N)} 3^{j-1} e^{2\pi i u_j(\omega) \theta} L_j(\theta),
\end{equation}
where \( u_j(\omega) = m_{k_N} + \sum_{\ell=j+1}^{k_N-1} \omega_{\ell}(N)m_\ell + \omega m_j \).

Let us first assume that \( L(\theta) = 0 \). Then we have\[ \frac{1}{N} \left| \sum_{n=1}^{N} e^{2\pi i n \theta} \right| \leq \frac{1}{N} \sum_{j=1}^{k_N-1} 3^{j-1} |L_j(\theta)| + \frac{1}{N} \sum_{j=1}^{k_N-1} \sum_{\omega < \omega_j(N)} 3^{j-1} |L_j(\theta)| \xrightarrow{N \to +\infty} 0, \]
where the convergence follows from a Cesàro type argument.

Assume now that \( L(\theta) \neq 0 \). Then, \( e^{2\pi i m_j \theta} \xrightarrow{n \to +\infty} 1 \).

Fix \( \varepsilon > 0 \). Let \( r \in \mathbb{N} \) be such that \( e^{-r} < \varepsilon \), and let \( d \in \mathbb{N} \) be such that \( |1 - e^{2\pi i m_j \theta}| < \varepsilon/(r+1) \) and \( |L(\theta) - L_j(\theta)| < \varepsilon \) for every \( j \geq d \).

For every \( N \) such that \( k_N \geq d + r \), we have on one hand, since \( (L_n(\theta))_{n \geq 1} \) is bounded by 1,
\begin{equation}
\sum_{j=1}^{k_N-r-1} 3^{j-1} |e^{2\pi i m_j \theta} L_j(\theta) - L(\theta)| + \sum_{j=k_N-r}^{k_N-1} \sum_{\omega < \omega_j(N)} 3^{j-1} |e^{2\pi i u_j(\omega) \theta} L_j(\theta) - L(\theta)|
\end{equation}
\begin{equation}
\leq \sum_{j=k_N-r}^{k_N-1} 3^{j-1} [2 + 2 \times 2] \leq 3^{k_N-r} < 3^{k_N} \varepsilon.
\end{equation}
And on the other hand, as \( k_N - r \geq d \), when \( k_N - r \leq j \leq k_N \) we have \( |1 - e^{2\pi i m_j \theta}| < \varepsilon/(r+1) \) and\[ \left| 1 - e^{2\pi i u_j(\omega) \theta} \right| \leq \frac{k_N}{\varepsilon} |1 - e^{2\pi i m_j \theta}| < \varepsilon, \]
for every choice of \( \omega \). So,
\begin{equation}
\sum_{j=k_N-r}^{k_N-1} 3^{j-1} |e^{2\pi i m_j \theta} L_j(\theta) - L(\theta)| + \sum_{j=k_N-r}^{k_N-1} \sum_{\omega < \omega_j(N)} 3^{j-1} |e^{2\pi i u_j(\omega) \theta} L_j(\theta) - L(\theta)|
\end{equation}
\begin{equation}
< \sum_{j=k_N-r}^{k_N-1} 3^{j-1} [2\varepsilon + 2 \times 2\varepsilon] < 3^{k_N} \varepsilon.
\end{equation}
Gathering (11) and (12), we get\[ \left| \sum_{n=1}^{N} e^{2\pi i n \theta} - NL(\theta) \right| < 2 \cdot 3^{k_N} \varepsilon. \]
Finally, in view of (9),\[ \limsup_{N \to +\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i n \theta} - L(\theta) \left| \leq 12 \varepsilon, \right. \]
and the announced result follows since \( \varepsilon \) is arbitrarily small. \( \square \)
It follows from Theorem 2 that, in order to produce a good sequence with uncountable spectrum, it is sufficient to exhibit an increasing sequence of integers \((m_j)_{j \geq 1}\), with \(m_{j + 1}/m_j \geq 3\) for every \(j \geq 1\), and such that the subgroup of \(S^1\)

\[
H_2 = H_2((m_j)_{j \geq 1}) := \{ e^{2i\pi \theta} : \theta \in [0, 1), \sum_{j \geq 1} ||m_j \theta||^2 < \infty \},
\]

be uncountable.

It turns out that those type of subgroups have been studied in [6] (and [10]).

A similar subgroup, defined by \(H_1 := \{ e^{2i\pi \theta} : \theta \in [0, 1), \sum_{j \geq 1} ||m_j \theta|| < \infty \}\), studied in [6] in connection with \(H_2\), has also been considered by Erdős and Taylor [4] and by Bergelson et al. [1] in connection with IP-rigidity.

In the above papers, sufficient conditions have been obtained for \(H_2\) or \(H_1\) to be uncountable.

To state the results concerning \(H_2\) subgroups, we shall need a strengthening of the lacunarity condition. We say that \((m_j)_{j \geq 1}\) satisfies assumption \((A)\) if one of the conditions \((A_1)\) or \((A_2)\) below is satisfied:

\[
(A_1) \quad \sum_{j \geq 1} \left( \frac{m_j}{m_{j+1}} \right)^2 < \infty \]

\[
(A_2) \quad \forall j \geq 1 \quad m_j |m_{j+1} \quad \text{and} \quad m_{j+1}/m_j \rightarrow \infty.
\]

**Proposition 3.** Let \((m_j)_{j \geq 1}\) be a sequence of integers satisfying assumption \((A)\). Then, \(H_2((m_j)_{j \geq 1})\) is uncountable.

The proposition was proved by the second author [10] (see also section 4.2 of [6]) under \((A_1)\) (notice that the condition \(\inf_{j \in \mathbb{N}} m_{j+1}/m_j \geq 3\) used in [10] and [6] is not restrictive for the uncountability of \(H_2\)). Actually, it is proved in [10] and [6] that \(H_2\) supports a continuous (singular) probability measure given by a symmetric Riesz product. A proof of the uncountability of \(H_2\) under \((A_1)\) can also be derived from the proof of Theorem 5 in [4], which states that \(H_1\) is uncountable when \(\sum_{j \geq 1} m_{j+1}/m_j < \infty\).

Under condition \((A_2)\), the proposition follows from Theorem 3 in [4] which states that \(H_1 \subset H_2\) is uncountable. We use their argument below in the proofs of Proposition 5 and Theorem 6.

We are now able to state our main result, which follows in a straightforward way from Proposition 3 and Theorem 2.

**Theorem 4.** Let \((m_j)_{j \geq 1}\) be an increasing sequence of positive integers, such that \(m_{j+1}/m_j \geq 3\) for every \(j \geq 1\) and define \(S\) as above. If assumption \((A)\) is satisfied then \(S\) is a good sequence and it has uncountable spectrum.

We also derive the following proposition which complements Proposition 1. It can be shown as an abstract consequence of the existence of a good sequence with uncountable spectrum, but we shall give explicit examples.
Proposition 5. There exist a good sequence \( (s_n)_{n \geq 1} \) and a continuous measure \( \mu \) on \( \mathbb{S}^1 \), such that \( \left( \frac{1}{N} \sum_{n=1}^{N} |\hat{\mu}(s_n)|^2 \right)_{N \geq 1} \) converges to some positive number.

Proof. We construct such a measure for each sequence \( S \) associated with a sequence \( (m_j)_{j \geq 1} \) satisfying (A2) and \( \inf_{j \in \mathbb{N}} m_{j+1}/m_j \geq 3 \).

Under this assumption, choose a subsequence \( (m_{j_k})_{k \geq 1} \) such that \( j_1 > 1 \) and \( m_j/m_{j-1} > 2^{k+2} \) for all \( j \geq j_k \). For every sequence \( \eta = (\eta_k)_{k \geq 1} \in \{0,1\}^{\mathbb{N}} \), let

\[
\theta(\eta) = \sum_{k=1}^{\infty} \frac{\eta_k}{m_{j_k}}.
\]

Given \( j \geq 1 \), let \( k \) be the smallest integer such that \( j_k > j \). Since \( m_j/m_{j} \) is an integer when \( \ell < k \), we have

\[
||m_j \theta(\eta)|| \leq m_j \sum_{\ell \geq k} \frac{1}{m_{j\ell}} \leq 2 \frac{m_j}{m_{j_k}},
\]

and in particular \( ||m_j \theta(\eta)|| \leq 1/4 \), which yields that all the terms in the products (7) are positive.

We also have \( \sum_{j \leq j_k} m_j^2 < 2 m_{j_k-1}^2 \), so if we sum up the \( ||m_j \theta(\eta)||^2 \) by blocks from \( j_k - 1 \) to \( j_k - 1 \) (or from \( 1 \) to \( j_1 - 1 \) for the first one), we get that each partial sum is less than \( 8 (m_{j_k - 1}/m_{j_k})^2 \),

\[
\sum_{j=1}^{\infty} ||m_j \theta(\eta)||^2 < 8 \sum_{k=1}^{\infty} \frac{m_{j_k-1}^2}{m_{j_k}^2} < \sum_{k=1}^{\infty} \frac{1}{4^k} < +\infty.
\]

and \( L(\theta(\eta)) > 0 \) follows.

Now, let \( \xi = (\xi_j)_{j \geq 1} \) be a sequence of i.i.d. random variables with \( \mathbb{P}(\xi_1 = 0) = \mathbb{P}(\xi_1 = 1) = \frac{1}{2} \) and let \( \mu \) be the probability distribution of \( e^{2\pi i \theta(\xi)} \). Then, as the mapping \( \eta \to e^{2\pi i \theta(\eta)} \) is one-to-one, \( \mu \) is a continuous probability measure concentrated on \( \Lambda_S \) and

\[
\frac{1}{N} \sum_{n=1}^{N} \hat{\mu}(s_n) = \mathbb{E} \left( \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i \theta(\xi)} \right) \to \mathbb{E}(L(\theta(\xi))) > 0 \quad \text{as} \quad N \to +\infty.
\]

Finally, Proposition 1 ensures the convergence of \( \frac{1}{N} \sum_{n=1}^{N} |\hat{\mu}(s_n)|^2 \) and the positivity of the limit follows the inequality \( \frac{1}{N} \sum_{n=1}^{N} |\hat{\mu}(s_n)|^2 \geq \left| \frac{1}{N} \sum_{n=1}^{N} \hat{\mu}(s_n) \right|^2 \). \( \square \)

Remark. Under assumption (A1) and \( \inf_{j \in \mathbb{N}} m_{j+1}/m_j \geq 3 \), the result holds for the measure \( \mu \) constructed in [10] or [6]. Indeed then \( \mu \) is a generalized Riesz product, weak*-limit of products of trigonometric polynomials \( \hat{P}_j \) with coefficients in blocks \( \{km_j; -k_j \leq k \leq k_j \} \) and \( \hat{P}_j(m_j) = \hat{P}_j(-m_j) = \cos(\pi/(m_j + 2)) \). Then for every \( s = \sum_{1 \leq j \leq n} \omega_j m_j \) where \( |\omega_j| \leq k_j \) for all \( j \), we have \( \hat{\mu}(s) = \Pi_{1 \leq j \leq n} \hat{P}_j(\omega_j m_j) \) (see [6]). From there, the convergence of \( \frac{1}{N} \sum_{n=1}^{N} |\hat{\mu}(s_n)|^2 \) and the positivity of the limit can be proven as in Theorem 2 (we skip the details).
2. SINGULAR ASYMPTOTIC DISTRIBUTION

We now turn to a matter addressed by Lesigne, Quas, Rosenblatt and Wierdl in the preprint [9].

Let $S = (s_n)_{n \in \mathbb{N}}$ be a good sequence. Let $\lambda \in S^1$. Since $S$ is good, the sequence $(\frac{1}{N} \sum_{n=1}^{N} \delta_{\lambda^n}(m))_{N \in \mathbb{N}} = (\frac{1}{N} \sum_{n=1}^{N} \lambda^{m s_n})_{N \in \mathbb{N}}$ converges towards $c(\lambda^m)$ for any integer $m$, that is for any character on $S^1$, so that $(\frac{1}{N} \sum_{n=1}^{N} \delta_{\lambda^n})_{N \in \mathbb{N}}$ converges weakly to some probability $\nu_{S,\lambda}$.

Given a probability measure $\nu$ on $S^1$, if there exists a good sequence $S$ and $\lambda \in S^1$ such that $\nu_{S,\lambda} = \nu$, one says that $S$ represents the measure $\nu$ at the point $\lambda$.

Lesigne et al. proved several interesting results concerning the measures that can be represented by a good sequence. If so, can one take $\nu_{S,\lambda}$ is a non Rajchman probability measure?

Theorem 6. Let $(m_j)_{j \in \mathbb{N}}$ be an increasing sequence of integers satisfying $(A_2)$ and $\inf_{j \in \mathbb{N}} m_{j+1}/m_j \geq 3$, and let $S$ be the sequence associated with it. There are uncountably many $\lambda \in \Lambda_S$ such that the weak*-limit $\nu_{S,\lambda}$ of $(\frac{1}{N} \sum_{n=1}^{N} \delta_{\lambda^n})_{N \in \mathbb{N}}$ satisfies $\limsup_{j \to +\infty} |\nu_{S,\lambda}(m_j)| = 1$.

Proof. We proceed as in the proof of Proposition 5, except that we require a stronger condition on the subsequence $(m_{j_k})_{k \geq 1}$, namely $m_j/m_{j-1} > 2^{k+2}m_{j-1}$ for all $j \geq j_k$ if $k > 1$.

For $\eta \in \{0,1\}^{\mathbb{N}_+}$, we still define $\theta(\eta) = \sum_{k \geq 1} \eta_k/m_{j_k}$. By the proof of Proposition 5, this yields an uncountable family of $\lambda = e^{2\pi i \theta(\eta)}$ in $\Lambda_S$.

For each such $\theta = \theta(\eta)$ we have $\nu_{S,\lambda}(m) = c(e^{2\pi i m \theta(\eta)}) = L(m\theta)$ for all $m \in \mathbb{Z}$. So, it will be sufficient to show that $L(m_j\theta) \to 1$ as $n \to +\infty$. Clearly, from the expression of $L(\theta)$ as an infinite product, it is equivalent to prove that $\sum_{j \geq 1} \|m_j m_j\theta\|^2$ converges to 0 as $n \to +\infty$.

Fix $n > 1$. We may apply the inequality (14) either to $\|m_j\theta\|$ or to $\|m_j m_j\theta\|$. For $j < j_n$ we get $\|m_j m_j\theta\| \leq m_j \|m_j\theta\| \leq 2 m_j m_j m_{j+1}$, and in the opposite case $\|m_j m_j\theta\| \leq m_j \|m_j\theta\| \leq 2 m_j m_j m_j m_j m_j m_{j-1}$, where $k$ is the smallest integer such that $j_k > j$. So,

$$\sum_{j=1}^{j_{n-1}} \|m_j m_j\theta\|^2 \leq \frac{4}{m_j m_{j+1}} \sum_{j=1}^{j_{n-1}} m_j \leq \frac{8}{m_j m_{j+1}} m_j m_{j-1} < \frac{1}{4^n}.$$
For $j \geq j_n$, we sum again by blocks from $j_{k-1}$ to $j_k - 1$, for $k > n$,
\[
\sum_{j_{k-1}}^{j_k-1} \|m_{j_k}m_j \theta\|^2 \leq 4 \frac{m_{j_k}^2}{m_{j_k}} \sum_{j_{k-1}}^{j_k-1} m_j^2 \leq 8 \frac{m_{j_k}^2}{m_{j_k}} m_{j_k-1}^2 < \frac{1}{4^k} \frac{m_{j_k}^2}{m_{j_k-1}^2} \leq \frac{1}{4^k},
\]
and finally
\[
\sum_{j=1}^{\infty} \|m_{j_n}m_j \theta\|^2 < \sum_{k=n}^{\infty} \frac{1}{4^k} \rightarrow 0 \quad \text{as } n \rightarrow +\infty.
\]

Let $S$ and $\lambda \in S^1$ be as in Theorem 6 and write $\nu = \nu_{S,\lambda}$.

The property $\limsup_{j \to +\infty} |\hat{\nu}(m_j)| = 1$ means precisely that $\nu$ is a Dirichlet measure, see [6] and [5] for properties of Dirichlet measures.

In particular there is then a subsequence $(n_j)_{j \geq 1}$ such that $\lambda^{n_j}$ converges towards a constant of modulus 1 in the $L^1(\nu)$ topology, and it follows that any measure absolutely continuous with respect to $\nu$ is itself a Dirichlet measure.

On the other hand, any probability measure absolutely continuous with respect to some Rajchman measure is itself a Rajchman measure.

Hence, we infer that $\nu$ is singular with respect to any Rajchman probability measure on $S^1$.

**Question.** In view of Theorem 6, one may wonder if it is possible to find a good sequence $S$ and $\lambda \in S^1$ such that
\[
0 < \limsup_{n \to +\infty} |\hat{\nu}_{S,\lambda}(n)| < 1.
\]

Another question is whether one can have $\nu_{S,\lambda}$ Rajchman and singular with respect to the Lebesgue measure.

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