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GOOD SEQUENCES WITH UNCOUNTABLE SPECTRUM AND SINGULAR ASYMPTOTIC DISTRIBUTION

CHRISTOPHE CUNY AND FRANÇOIS PARREAU

ABSTRACT. We construct a good sequence with uncountable spectrum. As an application, we answer to a question of Lesigne, Quas Rosenblatt and Wierdl.

1. Good sequences with uncountable spectrum

Let $S = (s_n)_{n \geq 1}$ be an increasing sequence of positive integers. We say that $S$ is a good sequence if the following limit exists for every $\lambda \in \mathbb{S}^1$ ($\mathbb{S}^1 = \{ z \in \mathbb{C} : |z| = 1 \}$),

$$c(\lambda) = c_S(\lambda) := \lim_{N \to +\infty} \frac{1}{N} \sum_{n=1}^{N} \lambda^{s_n}.$$  \hspace{1cm} (1)

Equivalently, $S$ is good if, for every $\lambda \in \mathbb{S}^1$, the following limit exists

$$\lim_{N \to +\infty} \frac{1}{\pi_S(N)} \sum_{1 \leq k \leq N, k \in S} \lambda^{k},$$  \hspace{1cm} (2)

where $\pi_S(N) = \# (S \cap [1, N])$.

Good sequences have been studied by many authors. See for instance Rosenblatt and Wierdl [12] (who introduced that notion), Rosenblatt [11], Boshernitzan, Kolesnik, Quas and Wierdl [2], Lemańczyk, Lesigne, Parreau, Volný and Wierdl [8] or Cuny, Eisner and Farkas [3].

Given a good sequence $S$, we define its spectrum as the set

$$\Lambda_S := \{ \lambda \in \mathbb{S}^1 : c(\lambda) \neq 0 \}.$$  \hspace{1cm} (3)

By theorem 2.22 of [12] (due to Weyl), for any good sequence $S$, $\Lambda_S$ has Lebesgue measure 0. If moreover, $S$ has positive upper density, i.e. satisfies $\lim \sup_{N \to +\infty} (\pi_S(N)/N) > 0$, then $\Lambda_S$ is countable. See Proposition 2.12 and Corollary 2.13 of [3] for a proof based on a result of Boshernitzan published in [11]. See also [7] for more general results of that type.

On another hand, up to our knowledge, no good sequence with uncountable spectrum is known.

In [3], good sequences have been studied in connection with Wiener’s lemma. In particular they obtained the following results for good sequences, see their Proposition 2.6 and Theorem 2.10. Recall that if $\tau$ is a finite measure on $\mathbb{S}^1$, then $\hat{\tau}(n) = \int_{\mathbb{S}^1} \lambda^n d\tau(\lambda)$, for every $n \in \mathbb{Z}$.

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Proposition 1. Let $S = (s_n)_{n \geq 1}$ be a good sequence. Then, for every probability measure $\mu$ on $S^1$, we have

$$
\frac{1}{N} \sum_{n=1}^{N} |\hat{\mu}(s_n)|^2 \xrightarrow{N \to +\infty} \int_{(S^1)^2} c(\lambda_1 \lambda_2) d\mu(\lambda_1) d\mu(\lambda_2).
$$

In particular, if $S$ has countable spectrum and $\mu$ is continuous,

$$
(4) \quad \frac{1}{N} \sum_{n=1}^{N} |\hat{\mu}(s_n)|^2 \xrightarrow{N \to +\infty} 0.
$$

Remark. (4) implies that $\hat{\mu}(s_n)$ converges in density to 0, by the Koopman-von Neumann Lemma (see e.g. Lemma 2.1 of [3]).

The above considerations yield and put into perspective the following question: does there exist a good sequence with uncountable spectrum?

We answer positively to that question below. To state the result, we need some more notation.

Let $(m_j)_{j \geq 1}$ be an increasing sequence of positive integers, such that $m_{j+1}/m_j \geq 3$ for every $j \geq 1$.

We associate with $(m_j)_{j \geq 1}$ the sequence $S = (s_n)_{n \geq 1}$ made out of the integers (an empty sum is assumed to be 0)

$$
(5) \quad \left\{ m_k + \sum_{1 \leq j \leq k-1} \omega_j m_j : k \geq 1, (\omega_1, \ldots, \omega_{k-1}) \in \{-1, 0, 1\}^{k-1} \right\}
$$

in increasing order. Notice that our assumption on $(m_j)_{j \geq 1}$ implies that all the integers in (5) are positive and distinct.

Denote by $\| \cdot \|$ the distance to the nearest integer: $\|t\| := \min\{|m - t| : m \in \mathbb{Z}\}$ for every $t \in \mathbb{R}$.

Theorem 2. Let $(m_j)_{j \geq 1}$ be an increasing sequence of positive integers, such that $m_{j+1}/m_j \geq 3$ for every $j \geq 1$ and define $S$ as above. Then, $S$ is good and

$$
\Lambda := \left\{ e^{2i\pi \theta} : \theta \in [0, 1) \setminus \mathbb{Q}, \sum_{j \geq 1} \|m_j \theta\|^2 < \infty \right\} \subset \Lambda_S.
$$

Proof. For every $k \geq 1$, consider the following set of integers

$$
M_k := \left\{ \sum_{1 \leq j \leq k-1} \omega_j m_j : (\omega_j)_{1 \leq j < k} \in \{-1, 0, 1\}^{k-1} \right\}.
$$
For every $k \geq 1$ and every $\theta \in [0,1)$, set

$$L_k(\theta) := \prod_{1 \leq j \leq k-1} \frac{1}{3} (1 + 2 \cos(2\pi m_j \theta))$$

$$= \frac{1}{3^{k-1}} \prod_{1 \leq j \leq k-1} (1 + e^{-2i\pi m_j \theta} + e^{2i\pi m_j \theta})$$

$$= \frac{1}{3^{k-1}} \sum_{x \in M_k} e^{2i \pi x}.$$

Let $\theta \in [0,1)$. As $-1/3 \leq (1 + 2 \cos(2\pi m_j))/3 \leq 1$ for all $j$, if $1 + 2 \cos(2\pi m_j)$ is infinitely often non positive, then $(L_k(\theta))_{k \geq 1}$ converges to $0$.

Assume now that $1 + 2 \cos(2\pi m_j) > 0$ for $j \geq J$, for some integer $J$. Then, the convergence of $(L_k(\theta))_{k \geq 1}$ follows from the convergence of $(\prod_{j=J}^{k-1} (1 + 2 \cos(2\pi m_j))/3)_{k \geq J}$ which is clear since we have an infinite product of positive terms less than or equal to 1. Moreover this infinite product converges, i.e. the limit is non-zero, if and only if

$$\sum_{k=J}^{\infty} \left[ 1 - \frac{1}{3} (1 + 2 \cos(2\pi m_k \theta)) \right] = \sum_{k=J}^{\infty} \frac{2}{3} (1 - \cos(2\pi m_k \theta)) < +\infty,$$

which is equivalent to $\sum_{k=J}^{\infty} \|m_k \theta\|^2 < +\infty$.

If $\theta$ is in the set $\Lambda$ defined by (6) the above condition is satisfied and moreover, as $\theta$ is then irrational, the product $\prod_{j=1}^{\infty} (1 + 2 \cos(2\pi \theta m_j))/3$ does not vanish.

Hence in any case $(L_k(\theta))_{k \geq 1}$ converges, say to $L(\theta)$, and $L$ does not vanish on $\Lambda$.

We wish to prove that for every $\theta \in [0,1)$, $(\frac{1}{N} \sum_{n=1}^{N} e^{2i\pi n \theta})_{N \geq 1}$ converges to $L(\theta)$.

Let $N \geq 1$. Since $(s_n)_{n \geq 1}$ is the increasing sequence made out of the numbers given by (5), we can write $s_{N+1} = m_{k_N} + \sum_{1 \leq j \leq k_N-1} \omega_j(N) m_j$.

The integers $s_1, \ldots, s_N$ may be split into consecutive blocks

$$m_1 + M_1, \ldots, m_{k_N-1} + M_{k_N-1}, W_N,$$

where $W_N = \{ \ell \in m_{k_N} + M_{k_N} : \ell \leq s_N \}$.

As each block $M_k$ consists in $3^{k-1}$ integers, we have

$$\frac{3^{k-1} - 1}{2} \leq N < \frac{3^{k-1} - 1}{2}.$$  

We may furthermore split $W_N$ into translates of blocks $M_k$. Namely, if $\omega_{k_N-1}(N) \neq -1$, then $W_N$ begins with $m_{k_N} - m_{k_N-1} + M_{k_N-1}$, if $\omega_{k_N-1}(N) = 1$ another block $m_{k_N} + 0 \times m_{k_N-1} + M_{k_N-1}$ follows, and so on. More precisely, $W_N$ is the disjoint union

$$W_N = \bigcup_{1 \leq j \leq k_N-1} \bigcup_{\omega < \omega_j(N)} \left( m_{k_N} + \sum_{\ell = j+1}^{k_N-1} \omega(\omega_j(N) m_\ell + \omega m_j + M_j) \right).$$
Hence, by (8),
\begin{equation}
\sum_{n=1}^{N} e^{2i\pi s_n\theta} = \sum_{j=1}^{k_N-1} 3^{j-1} e^{2i\pi m_j\theta} L_j(\theta) + \sum_{j=1}^{k_N-1} \sum_{\omega<\omega_j(N)} 3^{j-1} e^{2i\pi u_j(\omega)\theta} L_j(\theta),
\end{equation}
where $u_j(\omega) = m_{k_N} + \sum_{\ell=j+1}^{k_N-1} \omega_\ell(N)m_\ell + \omega m_j$.

Let us first assume that $L(\theta) = 0$. Then we have
\[
\frac{1}{N} \left| \sum_{n=1}^{N} e^{2i\pi s_n\theta} \right| \leq \frac{1}{N} \sum_{j=1}^{k_N-1} 3^{j-1} |L_j(\theta)| + \frac{1}{N} \sum_{j=1}^{k_N-1} \sum_{\omega<\omega_j(N)} 3^{j-1} |L_j(\theta)| \to 0 \quad \text{as} \quad N \to +\infty,
\]
where the convergence follows from a Cesàro type argument.

Assume now that $L(\theta) \neq 0$. Then, $e^{2i\pi r_n\theta} \to 1$ as $n \to +\infty$.

Fix $\varepsilon > 0$. Let $r \in \mathbb{N}$ be such that $e^{-r} < \varepsilon$, and let $d \in \mathbb{N}$ be such that $|1 - e^{2i\pi m_j\theta}| < \varepsilon/(r+1)$ and $|L(\theta) - L_j(\theta)| < \varepsilon$ for every $j \geq d$.

For every $N$ such that $k_N \geq d + r$, we have on one hand, since $(L_n(\theta))_{n \geq 1}$ is bounded by 1,
\begin{align}
&\sum_{j=1}^{k_N-r-1} 3^{j-1} |e^{2i\pi m_j\theta} L_j(\theta) - L(\theta)| + \sum_{j=1}^{k_N-r-1} \sum_{\omega<\omega_j(N)} 3^{j-1} |e^{2i\pi u_j(\omega)\theta} L_j(\theta) - L(\theta)| \\
&\quad \leq \sum_{j=1}^{k_N-r-1} 3^{j-1} [2 + 2 \times 2] \leq 3^{k_N-r} < 3^{k_N} \varepsilon.
\end{align}

And on the other hand, as $k_N - r \geq d$, when $k_N - r \leq j \leq k_N$ we have $|1 - e^{2i\pi m_j\theta}| < \varepsilon/(r+1)$ and
\[
|1 - e^{2i\pi u_j(\omega)\theta}| \leq \sum_{\ell=k_N-r}^{k_N-1} |1 - e^{2i\pi m_j\theta}| < \varepsilon,
\]
for every choice of $\omega$. So,
\begin{align}
&\sum_{j=k_N-r}^{k_N-1} 3^{j-1} |e^{2i\pi m_j\theta} L_j(\theta) - L(\theta)| + \sum_{j=k_N-r}^{k_N-1} \sum_{\omega<\omega_j(N)} 3^{j-1} |e^{2i\pi u_j(\omega)\theta} L_j(\theta) - L(\theta)| \\
&\quad < \sum_{j=k_N-r}^{k_N-1} 3^{j-1} [2 \varepsilon + 2 \times 2 \varepsilon] < 3^{k_N} \varepsilon.
\end{align}

Gathering (11) and (12), we get
\[
\left| \sum_{n=1}^{N} e^{2i\pi s_n\theta} - NL(\theta) \right| < 2 \cdot 3^{k_N} \varepsilon.
\]

Finally, in view of (9),
\[
\limsup_{N \to +\infty} \left| \frac{1}{N} \sum_{n=1}^{N} e^{2i\pi s_n\theta} - L(\theta) \right| \leq 12 \varepsilon,
\]
and the announced result follows since $\varepsilon$ is arbitrarily small. \qed
It follows from Theorem 2 that, in order to produce a good sequence with uncountable spectrum, it is sufficient to exhibit an increasing sequence of integers \((m_j)_{j \geq 1}\), with \(m_{j+1}/m_j \geq 3\) for every \(j \geq 1\), and such that the subgroup of \(S^1\)
\begin{equation}
H_2 = H_2((m_j)_{j \geq 1}) := \{e^{2i\pi\theta} : \theta \in [0, 1), \sum_{j \geq 1} \|m_j\theta\|^2 < \infty\},
\end{equation}
be uncountable.

It turns out that those type of subgroups have been studied in [6] (and [10]).

A similar subgroup, defined by \(H_1 := \{e^{2i\pi\theta} : \theta \in [0, 1), \sum_{j \geq 1} \|m_j\theta\| < \infty\}\), studied in [6] in connection with \(H_2\), has also been considered by Erdős and Taylor [4] and by Bergelson et al. [1] in connection with IP-rigidity.

In the above papers, sufficient conditions have been obtained for \(H_2\) or \(H_1\) to be uncountable.

To state the results concerning \(H_2\) subgroups, we shall need a strengthening of the lacunarity condition. We say that \((m_j)_{j \geq 1}\) satisfies assumption \((A)\) if one of the conditions \((A_1)\) or \((A_2)\) below is satisfied:

\begin{align*}
(A_1) & \quad \sum_{j \geq 1} \left( \frac{m_j}{m_{j+1}} \right)^2 < \infty \\
(A_2) & \quad \forall j \geq 1 \quad m_j/m_{j+1} \text{ and } m_{j+1}/m_j \xrightarrow{j \to +\infty} \infty.
\end{align*}

**Proposition 3.** Let \((m_j)_{j \geq 1}\) be a sequence of integers satisfying assumption \((A)\). Then, \(H_2((m_j)_{j \geq 1})\) is uncountable.

The proposition was proved by the second author [10] (see also section 4.2 of [6]) under \((A_1)\) (notice that the condition \(\inf_{j \in \mathbb{N}} m_{j+1}/m_j \geq 3\) used in [10] and [6] is not restrictive for the uncountability of \(H_2\)). Actually, it is proved in [10] and [6] that \(H_2\) supports a continuous (singular) probability measure given by a symmetric Riesz product. A proof of the uncountability of \(H_2\) under \((A_1)\) can also be derived from the proof of Theorem 5 in [4], which states that \(H_1\) is uncountable when \(\sum_{j \geq 1} m_{j+1}/m_j < \infty\).

Under condition \((A_2)\), the proposition follows from Theorem 3 in [4] which states that \(H_1 \subset H_2\) is uncountable. We use their argument below in the proofs of Proposition 5 and Theorem 6.

We are now able to state our main result, which follows in a straightforward way from Proposition 3 and Theorem 2.

**Theorem 4.** Let \((m_j)_{j \geq 1}\) be an increasing sequence of positive integers, such that \(m_{j+1}/m_j \geq 3\) for every \(j \geq 1\) and define \(S\) as above. If assumption \((A)\) is satisfied then \(S\) is a good sequence and it has uncountable spectrum.

We also derive the following proposition which complements Proposition 1. It can be shown as an abstract consequence of the existence of a good sequence with uncountable spectrum, but we shall give explicit examples.
Proposition 5. There exist a good sequence \((s_n)_{n \geq 1}\) and a continuous measure \(\mu\) on \(\mathbb{S}^1\), such that \(\frac{1}{N} \sum_{n=1}^{N} |\hat{\mu}(s_n)|^2\) converges to some positive number.

Proof. We construct such a measure for each sequence \(S\) associated with a sequence \((m_j)_{j \geq 1}\) satisfying \((A_2)\) and \(\inf_{j \in \mathbb{N}} m_{j+1}/m_j \geq 3\).

Under this assumption, choose a subsequence \((m_{j_k})_{k \geq 1}\) such that \(j_1 > 1\) and \(m_{j_k}/m_{j_k-1} > 2^{k+2}\) for all \(j \geq j_k\). For every sequence \(\eta = (\eta_k)_{k \geq 1} \in \{0, 1\}^{\mathbb{N}}\), let

\[\theta(\eta) = \sum_{k=1}^{\infty} \frac{\eta_k}{m_{j_k}}.\]

Given \(j \geq 1\), let \(k\) be the smallest integer such that \(j_k > j\). Since \(m_{j_k}/m_{j_k}\) is an integer when \(\ell < k\), we have

\[
\|m_{j_k}\theta(\eta)\| \leq m_j \sum_{\ell \geq k} \frac{1}{m_{j_{\ell}}} \leq 2 \frac{m_j}{m_{j_k}},
\]

and in particular \(\|m_{j_k}\theta(\eta)\| \leq 1/4\), which yields that all the terms in the products \((7)\) are positive.

We also have \(\sum_{j < j_k} m_j^2 < 2 m_{j_k-1}^2\), so if we sum up the \(\|m_{j_k}\theta(\eta)\|^2\) by blocks from \(j_k-1\) to \(j_k-1\) (or from 1 to \(j_1-1\) for the first one), we get that each partial sum is less than \(8(m_{j_k-1}/m_{j_k})^2\),

\[
\sum_{j=1}^{\infty} \|m_{j_k}\theta(\eta)\|^2 < 8 \sum_{k=1}^{\infty} \frac{m_{j_k-1}^2}{m_{j_k}^2} < \sum_{k=1}^{\infty} \frac{1}{4^k} < +\infty.
\]

and \(L(\theta(\eta)) > 0\) follows.

Now, let \(\xi = (\xi_j)_{j \geq 1}\) be a sequence of i.i.d. random variables with \(\mathbb{P}(\xi_1 = 0) = \mathbb{P}(\xi_1 = 1) = \frac{1}{2}\) and let \(\mu\) be the probability distribution of \(e^{2\pi i \theta(\xi)}\). Then, as the mapping \(\eta \mapsto e^{2\pi i \theta(\eta)}\) is one-to-one, \(\mu\) is a continuous probability measure concentrated on \(\Lambda_S\) and

\[
\frac{1}{N} \sum_{n=1}^{N} \hat{\mu}(s_n) = \mathbb{E}\left(\frac{1}{N} \sum_{n=1}^{N} e^{2\pi i \theta(\xi)}\right) \to \mathbb{E}(L(\theta(\xi))) > 0 \quad \text{as} \quad N \to +\infty.
\]

Finally, Proposition 1 ensures the convergence of \(\frac{1}{N} \sum_{n=1}^{N} |\hat{\mu}(s_n)|^2\) and the positivity of the limit follows the inequality \(\frac{1}{N} \sum_{n=1}^{N} |\hat{\mu}(s_n)|^2 \geq \left(\frac{1}{N} \sum_{n=1}^{N} \hat{\mu}(s_n)\right)^2\). \(\square\)

Remark. Under assumption \((A_1)\) and \(\inf_{j \in \mathbb{N}} m_{j+1}/m_j \geq 3\), the result holds for the measure \(\mu\) constructed in \([10]\) or \([6]\). Indeed then \(\mu\) is a generalized Riesz product, weak*-limit of products of trigonometric polynomials \(\hat{P}_j\) with coefficients in blocks \(\{km_j; -k_j \leq k \leq k_j\}\) and \(\hat{P}_j(m_j) = \hat{P}_j(-m_j) = \cos(\pi/(m_j + 2))\). Then for every \(s = \sum_{1 \leq j \leq n} \omega_j m_j\) where \(|\omega_j| \leq k_j\) for all \(j\), we have \(\hat{\mu}(s) = \Pi_{1 \leq j \leq n} \hat{P}_j(\omega_j m_j)\) (see \([6]\)). From there, the convergence of \(\frac{1}{N} \sum_{n=1}^{N} |\hat{\mu}(s_n)|^2\) and the positivity of the limit can be proven as in Theorem 2 (we skip the details).
2. SINGULAR ASYMPTOTIC DISTRIBUTION

We now turn to a matter addressed by Lesigne, Quas, Rosenblatt and Wierdl in the preprint [9].

Let $S = (s_n)_{n \in \mathbb{N}}$ be a good sequence. Let $\lambda \in \mathbb{S}^1$. Since $S$ is good, the sequence $(\frac{1}{N} \sum_{n=1}^{N} \delta_{\lambda^n}(m))_{N \in \mathbb{N}} = (\frac{1}{N} \sum_{n=1}^{N} \lambda^{m \cdot n})_{N \in \mathbb{N}}$ converges towards $c(\lambda^m)$ for any integer $m$, that is for any character on $\mathbb{S}^1$, so that $(\frac{1}{N} \sum_{n=1}^{N} \delta_{\lambda^n})_{N \in \mathbb{N}}$ converges weakly to some probability $\nu_{S,\lambda}$.

Given a probability measure $\nu$ on $\mathbb{S}^1$, if there exists a good sequence $S$ and $\lambda \in \mathbb{S}^1$ such that $\nu_{S,\lambda} = \nu$, one says that $S$ represents the measure $\nu$ at the point $\lambda$.

Lesigne et al. proved several interesting results concerning the measures that can be represented by a good sequence at some point $\lambda \in \mathbb{S}^1$. For instance, they proved that if $\lambda$ is not a root of unity then $\nu_{S,\lambda}$ is continuous (see their Theorem 8.1). They also proved that if a given probability measure $\nu$ on $\mathbb{S}^1$ is not Rajchman (i.e. its Fourier coefficients do not vanish at infinity) then, for almost every $\lambda$ with respect to the Haar measure, there does not exist any good sequence representing $\nu$ at $\lambda$ (see their Theorem 8.2). On the opposite, if $\nu$ is absolutely continuous with respect to the Haar measure, then for every $\lambda \in \mathbb{S}^1$ which is not a root of unity there exists a good sequence $S$ representing $\nu$ at $\lambda$ (see their Theorem 9.1).

The above results raise the following questions. Does there exist a continuous but singular probability measure $\nu$ on $\mathbb{S}^1$ that can be represented by a good sequence? If so, can one take $\nu$ to be non Rajchman?

It turns out that Theorem 2 allows to exhibit a good sequence $S$ and a point $\lambda$ such that $\nu_{S,\lambda}$ is a non Rajchman probability measure.

**Theorem 6.** Let $(m_j)_{j \in \mathbb{N}}$ be an increasing sequence of integers satisfying $(A_2)$ and $\inf_{j \in \mathbb{N}} m_{j+1}/m_j \geq 3$, and let $S$ be the sequence associated with it. There are uncountably many $\lambda \in \Lambda_S$ such that the weak* limit $\nu_{S,\lambda}$ of $(\frac{1}{N} \sum_{n=1}^{N} \delta_{\lambda^n})_{N \in \mathbb{N}}$ satisfies $\limsup_{j \to +\infty} |\nu_{S,\lambda}(m_j)| = 1$.

**Proof.** We proceed as in the proof of Proposition 5, except that we require a stronger condition on the subsequence $(m_{j_k})_{k \geq 1}$, namely $m_j/m_{j-1} > 2^{k+2} m_{j-1}$ for all $j \geq j_k$ if $k > 1$.

For $\eta \in \{0, 1\}^{\mathbb{N}}$, we still define $\theta(\eta) = \sum_{k \geq 1} \eta_k/m_{j_k}$. By the proof of Proposition 5, this yields an uncountable family of $\lambda = e^{2\pi i \theta(\eta)}$ in $\Lambda_S$.

For each such $\theta = \theta(\eta)$ we have $\hat{v}_{S,\lambda}(m) = c(e^{2\pi i m \theta(\eta)}) = L(m \theta)$ for all $m \in \mathbb{Z}$. So, it will be sufficient to show that $L(m_j \theta) \to 1$ as $n \to +\infty$. Clearly, from the expression of $L(\theta)$ as an infinite product, it is equivalent to prove that $\sum_{j \geq 1} \|m_j \cdot \theta\|^2$ converges to 0 as $n \to +\infty$.

Fix $n > 1$. We may apply the inequality (14) either to $\|m_j \cdot \theta\|$ or to $\|m_j \cdot \theta\|$. For $j < j_n$ we get $\|m_{j_n} m_j \cdot \theta\| \leq m_j \|m_j \cdot \theta\| \leq 2 m_j m_{j + 1} / m_{j + 2}$, and in the opposite case $\|m_{j_n} m_j \cdot \theta\| \leq m_{j_n} \|m_j \cdot \theta\| \leq 2 m_{j_n} m_j / m_{j + 1}$ where $k$ is the smallest integer such that $j_k > j$. So,

$$\sum_{j=1}^{j_n-1} \|m_{j_n} m_j \cdot \theta\|^2 \leq \frac{4}{m_{j + 1}^2} \sum_{j=1}^{j_n-1} m_j^2 \leq 8 \frac{m_{j_n}^2}{m_{j + 1}^2} m_{j - 1}^2 < \frac{1}{4^n}. $$
For $j \geq j_n$, we sum again by blocks from $j_{k-1}$ to $j_k - 1$, for $k > n$,

$$\sum_{j_{k-1}}^{j_k-1} \|m_{j_k}m_j\theta\|^2 \leq 4 \frac{m_{j_n}^2}{m_{j_k}^2} \sum_{j=n}^{j_k-1} m_j^2 \leq 8 \frac{m_{j_n}^2}{m_{j_k}^2} m_{j_k-1}^2 < \frac{1}{4^k} \frac{m_{j_n}^2}{m_{j_k}^2} \leq \frac{1}{4^k},$$

and finally

$$\sum_{j=1}^{\infty} \|m_{j_n}m_j\theta\|^2 < \sum_{k=n}^{\infty} \frac{1}{4^k} \to 0 \quad \text{as } n \to +\infty.$$

Let $S$ and $\lambda \in S^1$ be as in Theorem 6 and write $\nu = \nu_{S,\lambda}$.

The property $\limsup_{j \to +\infty} |\hat{\nu}(m_j)| = 1$ means precisely that $\nu$ is a Dirichlet measure, see [6] and [5] for properties of Dirichlet measures.

In particular there is then a subsequence $(n_j)_{j \geq 1}$ such that $\lambda^{n_j}$ converges towards a constant of modulus 1 in the $L^1(\nu)$ topology, and it follows that any measure absolutely continuous with respect to $\nu$ is itself a Dirichlet measure.

On the other hand, any probability measure absolutely continuous with respect to some Rajchman measure is itself a Rajchman measure.

Hence, we infer that $\nu$ is singular with respect to any Rajchman probability measure on $S^1$.

**Question.** In view of Theorem 6, one may wonder if it is possible to find a good sequence $S$ and $\lambda \in S^1$ such that

$$0 < \limsup_{n \to +\infty} |\hat{\nu}(m_j)| < 1.$$

Another question is whether one can have $\nu_{S,\lambda}$ Rajchman and singular with respect to the Lebesgue measure.

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**References**


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