

On the ratios of urban mobility, Part 1: the HoTer model of travel demand and network flows

Abstract

Mobility systems in urbanized territories have been featured out in Travel Demand Models by state variables of land-use occupation, trip generation, trip distribution, modal split and network assignment, with emphasis on causal relationships between the variables and on spatial detail for each kind of variables. The article is aimed to provide notional averages, say ratios, for each kind of variables, and to state the causal relationships between the variables as simple analytical formulas between the ratios. This is achieved by going along the classical four steps of travel demand modeling, in a theoretical way for an idealized territory satisfying three postulates of homogeneity: namely, at block level, at link level and of indefinite spatial extension. The said formulas constitute rules of thumb linking the mobility ratios of spatial density of human occupation, trip emission rates, average trip lengths, modal shares, generalized trip cost per length unit, together with traffic variables of speed, flow rate and vehicular density at the link level. The model is stated in eight steps, namely (i) territorial composition, (ii) trip generation, (iii) trip lengths and traffic formation, (iv) quality of service, (v) trip distribution using a gravity model, (vi) modal split by multinomial logit, (vii) traffic laws, (viii) traffic equilibrium. It is followed by a Discussion of the model outreach and limitations. Areas of further research include traffic laws, impact assessment and economic analysis.

Keywords

Spatial homogeneity; State laws; Four-step travel demand model; Traffic equilibrium

Highlights

- Idealized territory with geometric regularities
- Mobility ratios as spatial averages in idealized territory
- Local balance of generated and carried traffic
- Gravity distribution yields simple formula of average axial trip length
- Monomodal traffic equilibrium as a single equation in axial speed only

1/ Introduction

Background. In the planning of urban mobility systems, a basic principle is to make transport supply consistent and congruent with travel demand: the lines of infrastructure are expected to match the “desire lines” of people’s trips between places and the flow capacity of a given link, be it road or rail, is expected to match the flow level that aggregates the trip demands of the individual trip-makers [1]. In a given territory, the mobility system develops in a progressive way and the matching principle is applied incrementally, along steps of “transport project”, “network development”, “traffic management scheme”, “demand management scheme” etc [2]. Planning studies usually rely upon the simulation of system states, including “observed states” as well as “scenarios”, by using a Travel Demand Model (TDM) of which a major outcome is the “traffic state” of supply and demand in

interaction [3]. Under the model assumptions, the notion of traffic equilibrium involves (i) the satisfaction of the travel demands of the individuals that constitute the territorial population, (ii) the adequacy of people's trip paths with the network flows of vehicles (up to the occupancy rate of vehicles by people), (iii) the consistency of local flows and local travel times (through speed-flow relationships) [4]. The equilibrium conditions apply at disaggregated levels: on a link- or path- basis on the supply side, on an individual- or origin-destination pair basis on the demand side [5], possibly with temporal detail [6]. While such disaggregation is a major strength of TDMs for practical applications, it casts some veil upon the overall correspondence between demand-related variables and supply-related ones. Street-sharing policies of multimodal transport, for instance, involve layout set-ups at the local level of roadway links, both for traffic and for parking [7]. Which set-ups would meet modal split targets set at the territorial level? More generally speaking, what are the respective measures of supply factors and demand features in territorial mobility? What are the ratios between them?

Research questions and article's objective. This raises the upstream question of which system state variables express supply factors and demand features, and of what systemic relationships link two or more of the variables. As previously stated, answers are available in the scientific body of TDMs, under refined forms and in a disaggregate way [3].

The article's aim is to deal with aggregate variables and to put the relationships in simple form. The tradeoff is between detail and refinement, on the one hand, and simple interpretation and easy insight, on the other hand. The article scope is that of a four-step model, from trip generation and spatial distribution to modal split and network assignment. Multiple modes are considered in an abstract way, with special emphasis on roadway modes and their joint use of streets. Vehicle parking is considered along with the traffic of people and vehicles. Regarding time of day, we are primarily interested in the peak period (morning or evening) associated with the commute between home and work places.

Approach and contribution. Our statement of the mobility system mirrors that in TDMs, both for state variables for the causal relationships in the four-step scheme [3]. We keep to the levels of analysis regarding individual trips, modes and network links, as well as to traffic assignment zones (TAZ) and origin-destination pairs (O-D). To circumvent spatial heterogeneity we postulate (1) that the territory is made up of identical rectangular blocks acting as TAZs, (2) that the roadway network is made of identical links according to either one of the two orthogonal directions, (3) that the set of blocks and the link network extend indefinitely in space. The idealized territory is pervaded with homogeneity – hence the name of the model, HoTer for homogenous territory. The homogeneity postulates and the associated regular shapes relate the HoTer model to the field of network geometry in transportation science, pioneered by Van Nes's study of network structures [8] and the studies of Daganzo [9], Badia et al. of transit services along roadway grids with shape regularities [10,11,12].

Under our set of postulates, we discover simple rules for network assignment, trip distribution and modal split: (i) the principle that the axial traffic generated by a block balances the traffic carried by an edge link, per trip purpose and per mode, (ii) assuming a gravity model of trip distribution with exponential decay with respect to travel impedance, the average modal trip length is straightforwardly related to the modal link cost, (iii) modal shares from an origin zone are proportional to their respective trip distribution parameters.

These relations in simple form constitute “rules of thumb” for the analysis of territorial mobility and the demand-supply adequacy.

Article structure. The rest of the article is organized in three parts of decreasing lengths: first comes the statement of the model, then a discussion and lastly a short conclusion.

Table of notation

Indices

b block

a link

z axial direction

$[r, s]$ destination block relative to origin block by r horizontal shifts and s vertical shifts

i purpose of activity

m mode of transport

h time period (and its duration)

X type of human occupation of space

Geometric features

L_x (resp. L_y) block length along (x') (resp. (y'))

L_z, L_{-z} link length along z and orthogonal axial direction denoted $-z$

S_b land area of block b

$D_{i,m,h}^{(z)}$ Average trip length along z according to purpose, mode and period

Space occupation and traffic

δ_X spatial density of occupancy type X

$O_b^{(X)}$ number of occupants of type X in block b

$\mu_{i,m,h}$ trip emission rate per person during h

$G_{i,m,h}^{(b)}$ trip number generated by b for i using m during h

$\zeta_{i,m,h}^{+/-}$ on-street parking rate of m for i during h , (-) at origin and (+) at destination

$T_{i,m,h}^{(z|b)}$ traffic generated by b along z for (i, m, h)

$\tilde{T}_{i,m,h}^{(a)}$ traffic carried by a for (i, m, h)

$N_{i,m,h}^{(a)}$ (resp. $\tilde{N}_{i,m,h}^{(a)}$) number of people (resp. vehicles) on link a for (i, m, h)

$q_{i,m,h}^{(a)}$ (resp. $\tilde{q}_{i,m,h}^{(a)}$) flow rate of people (resp. vehicles) on link a for (i, m, h)

$k_{i,m,h}^{(a)}$ (resp. $\tilde{k}_{i,m,h}^{(a)}$) spatial density of people (resp. vehicles) on link a for (i, m, h)

$\tau_{i,m,h}$ occupancy rate in persons per vehicle

$P_{m,h}^{+/-}$ on-street parking load for m during h , (+) to destination or (-) from origin

Quality of service and economic features

$v_{m,h}^{(z)}$ running speed along z for m during h

$\eta_{m,h}^{(z)}$ link time along z for m during h

$w_{m,h}$ pedestrian access time on m at h

$t_{i,m,h}^{(z)}$ trip time along z for i using m during h

$\alpha_{i,m,h}$ value-of-time for (i, m, h)

$\omega_{i,m,h}$ money cost per unit length for (i, m, h)

$c_{i,m,h}^{(z)}$ trip money cost along z for i using m during h

$g_{i,m,h}^{(z)}$ trip generalized cost along z for i using m during h

$\gamma_{i,m,h}$ decay rate of origin-destination flow with respect to trip generalized cost

$\rho_{i,m,h}^{(z)}$ geometric ratio of trip distribution along z for (i, m, h)

$\hat{\rho}_{i,m,h}^{(z)}$ total amount of destination shares of trips emitted from current block for (i, m, h) along z

$\beta_{m|i,h}$ specific utility of mode m for (i, h)

$\theta_{i,h}$ logit parameter of mode choice for (i, h)

$U_{m|i,h}^{[r,s]}$ utility of destination $[r, s]$ using mode m for (i, h)

$\pi_{m|i,h}$ modal share of mode m for people trips on (i, h)

$\tilde{U}_{m|i,h}$ multimodal utility for (i, h)

Traffic laws and traffic equilibrium

$V_0^{(z)}$ monomodal speed function with respect to vehicular density

v_0, \dot{v} Parameters of affine linear $V_0^{(z)}$

\bar{v}_m policy-driven limit speed

$k_0^{(z)}$ inverse function of $V_0^{(z)}$

$q_0^{(z)}$ flow rate function with respect to speed

$\mathbf{v}_M^{(z)}$ multimodal vector of modal speeds according to axial directions

$\mathbf{V}_M^{(z)}$ multimodal vector function of modal speeds with respect to vector of modal vehicular densities

$\tilde{\mathbf{k}}_M^{(z)}$ multimodal vector of modal vehicular densities

\mathbf{w}_M multimodal vector of access times

$\mathbf{P}_M^+, \mathbf{P}_M^-$ multimodal vectors of on-street parking loads

\mathbf{W}_M multimodal vector function of access times with respect to $\mathbf{P}_M^+, \mathbf{P}_M^-$

2/ Model statement

2.1/ Territorial constitution

A block is a rectangle shaped area with geometric dimensions L_x along direction (x') and L_y along the other direction (y'). The two directions are orthogonal and span the two dimensions of the geographical space.

The block is indexed by b . Its perimeter is $L_b \equiv 2(L_x + L_y)$ and its area $S_b \equiv L_x \cdot L_y$.

The block edges are roadway links along their specific axial direction. Denoting by $z \in \{x, y\}$ a generic direction, an edge a along z_a has length denoted $L_a \equiv L_{z_a}$. The length of an orthogonal link is then denoted L_{-a} .

At the block level, the human occupation of type X is measured by its spatial density δ_X , yielding a number $O_b^{(X)} \equiv \delta_X \cdot S_b$ of block occupants for that type. Occupant types include residents (population P) and people who have the block as workplace (E for Employed). Thus, the notion of block occupation pertains to home or workplace: it differs from real-time presence.

We shall consider a block as a TAZ. Assuming that the trips tailed at, or headed to, the block access to it via its edges in a uniform way proportional to the edge lengths, then any edge a conveys a share L_a/L_b of b 's trips. Assuming further that the two blocks on either side of a are identical, then link a can be considered as a TAZ of its own, with associated occupants in number of

$$O_a^{(X)} \equiv 2 \frac{L_a}{L_b} O_b^{(X)} \quad (1)$$

The final postulate of territorial constitution is that the geographical space is paved by identical blocks of very large number in either direction. Thus the notions of city area, population, number of jobs, are irrelevant: the basic description of space occupation are the spatial densities δ_X , i.e., ratios of the local "populations" of type X to the ground area.

2.2/ Trip generation based on emission rates

2.2.1 Trip purposes and block trip emissions

Activity purposes i make a set I including Home, Work, Study at school or university, Shopping, Leisure etc. For simplicity we postulate that each purpose i is supported by a specific occupation type X_i . On average over the individual occupants of that type, per mode of transport m and time period h along the day, the individual trip rate is denoted $\mu_{i,m,h}^-$ for emissions (and $\mu_{i,m,h}^+$ for attractions).

Thus, the number of such trips generated by block b amounts to

$$G_{i,m,h}^{(b)} \equiv \mu_{i,m,h}^- O_b^{(X_i)} \quad (2)$$

At the link level, the number of generated trips is

$$G_{i,m,h}^{(a)} \equiv \mu_{i,m,h}^- O_a^{(X_i)} \quad (3)$$

Hence the notion of trip generation rate pertains to individual occupants. Per time unit, the related emission rates are the following, for $z \in \{a, b\}$ a generic TAZ and letting h also denote the period duration:

$$\hat{G}_{i,m,h}^{(z)} \equiv \frac{1}{h} G_{i,m,h}^{(z)} \quad (4)$$

The trip generation model consists in the specification of the supporting type, purpose, mode and period, and of the trip generation rates per purpose, mode and period. We shall later on make the mode feature endogenous.

2.2.2 On the balance sheet of emissions and attractions

A block as a trip emitter sends people to other blocks as trip attractors. By the symmetry of homogenous blocks, at the block level the amount and composition of the attracted trips will match those of the emitted trips. It holds that, $\forall i, m, h$:

$$G_{i,m,h}^{(b)+} = G_{i,m,h}^{(b)-} \quad (5)$$

The balance holds for people trips per purpose and transport mode. It also holds per period, up to the time shift between trip departures and arrivals. As for vehicle trips, for private vehicles the parking mode may differ between the origin and destination. For instance, a Home to Work trip using a private car may have an off-street parking slot on the Home side but not on the Work side, then requiring on-street parking at the destination.

We shall denote by $\zeta_{i,m,h}^-$ the share of on-street parking at the trip origin and $\zeta_{i,m,h}^+$ that at the trip destination. Denote also by $\tau_{i,m,h}$ the occupancy rate of the vehicle.

At the block level, the balance sheet of on-street parking for purpose i , mode m and time period h amounts to

$$\Delta P_{i,m,h}^{(b)} = (\zeta_{i,m,h}^+ - \zeta_{i,m,h}^-) G_{i,m,h}^{(b)} / \tau_{i,m,h} \quad (6)$$

2.3/ Traffic formation: trip lengths and link flows

2.3.1 Trip lengths and generated traffic

At the block level, over the trips generated per purpose, mode and period, let us denote the average trip length by $D_{i,m,h}$ and its decomposition between the axial directions by $D_{i,m,h}^{(x)}$ and $D_{i,m,h}^{(y)}$, respectively.

Thus the block generates a traffic quantity (in person.km) of

$$T_{i,m,h}^{(b)} \equiv G_{i,m,h}^{(b)} D_{i,m,h} \quad (7a)$$

Along axial direction $z \in \{x, y\}$ the generated traffic is specifically

$$T_{i,m,h}^{(z|b)} \equiv G_{i,m,h}^{(b)} D_{i,m,h}^{(z)} \quad (7b)$$

Similarly, at the link level we have that

$$T_{i,m,h}^{(a)} \equiv G_{i,m,h}^{(a)} D_{i,m,h} \quad (8a)$$

$$T_{i,m,h}^{(z|a)} \equiv G_{i,m,h}^{(a)} D_{i,m,h}^{(z)} \quad (8b)$$

2.3.2 The balance of generated and carried axial traffic

The traffic generated by a block circulates on different links edging that block or other ones – the latter case being the most likely, all the more so for longer trips. Resorting again to the symmetry principle between identical blocks, the amount of traffic circulating on a block edge, denoted by $\tilde{T}_{i,m,h}^{(a)}$, will match the axial block traffic generated along the direction of the edge $z = z_a$:

$$\tilde{T}_{i,m,h}^{(a)} = T_{i,m,h}^{(z|b)} \quad (9)$$

The reason is that the block has two edges along either axis and each link is the edge of two blocks. To make things more obvious, there is a 1:1 correspondence between blocks and links as “western block edges” along ($x'x$) and also a 1:1 correspondence between blocks and links as “northern block edges” along ($y'y$). In either axial direction, owing to the symmetry between blocks, all blocks contribute identical amounts of generated traffic. The total amount of generated traffic is carried by the axial links and, by symmetry between them, every axial link gets an identical amount of carried traffic. Based on the 1:1 correspondence of blocks and axial links, the axial traffic generated by a block, $T_{i,m,h}^{(z|b)}$, is equal to the traffic $\tilde{T}_{i,m,h}^{(a)}$ carried by the corresponding edge link a with $z_a = z$.

However there is no ex-ante balance between the traffic generated by link a , i.e., $T_{i,m,h}^{(z|a)}$ along any axial direction z , and the link traffic $\tilde{T}_{i,m,h}^{(a)}$ carried along the axial direction z_a . The reason is that the link share L_a/L_b of the block emissions $G_{i,m,h}^{(b)}$ applies to generated traffic, but not necessarily to the carried traffic. In terms of axial direction z_a and its orthogonal direction $-z_a$, the ratio of emissions, L_z/L_{-z} , does not necessarily match that of axial trip lengths, $D_{i,m,h}^{(z)}/D_{i,m,h}^{(-z)}$.

2.3.3 Link flows of people trips

From the traffic $\tilde{T}_{i,m,h}^{(a)}$ carried by link a with axial direction z , we can derive the flow volume on this link by postulating that each trip in this flow uses link a over length L_a . Then the number of trips carried by a is simply

$$N_{i,m,h}^{(a)} = \frac{1}{L_a} \tilde{T}_{i,m,h}^{(a)} \quad (10)$$

The conversion factor L_a applies strictly to “through trips”, not to “access trips” that use their final links in a partial way only, with average length $L_a/2$ from the postulate of uniform disaggregate trip emissions. For simplicity we assume that L_a applies to all carried trips: this stands as an approximation.

From the formula (9) of $\tilde{T}_{i,m,h}^{(a)}$ and the upstream formula (8b), it holds that

$$\tilde{T}_{i,m,h}^{(a)} = \tilde{T}_{i,m,h}^{(z|b)} = \mu_{i,m,h}^- O_b^{(X_i)} D_{i,m,h}^{(z)} = \mu_{i,m,h}^- \delta_{X_i} L_z L_{-z} D_{i,m,h}^{(z)} \quad (11)$$

Combining to (10), it comes out that

$$N_{i,m,h}^{(a)} = \mu_{i,m,h}^- \delta_{X_i} L_{-z} D_{i,m,h}^{(z)} \quad (12)$$

Put in words, the formula states that any link a with axial direction z carries a number of trips equivalent to the trip emissions of a rectangular area with dimension $D_{i,m,h}^{(z)}$ along direction z and L_{-z} along the orthogonal direction $-z$, and the density δ_{X_i} specific to the purpose.

From the link flow volumes, we recover the link flow rate in persons per unit time as

$$q_{i,m,h}^{(a)} \equiv \frac{1}{h} N_{i,m,h}^{(a)} = \frac{\mu_{i,m,h}^-}{h} \delta_{X_i} L_{-z} D_{i,m,h}^{(z)} \quad (13)$$

It is a bidirectional flow rate: in either traffic direction the one way link flow rate is $q_{i,m,h}^{(a)}/2$.

2.3.4 Link flows of vehicle trips

For private modes it is straightforward to derive vehicular flows from people flows by using the occupancy rate $\tau_{i,m,h}$. Then, the vehicle flow volume and flow rate on link a are respectively:

$$\tilde{N}_{i,m,h}^{(a)} = \frac{1}{\tau_{i,m,h}} N_{i,m,h}^{(a)} \quad (14a)$$

$$\tilde{q}_{i,m,h}^{(a)} \equiv \frac{1}{h} \tilde{N}_{i,m,h}^{(a)} = \frac{1}{\tau_{i,m,h}} q_{i,m,h}^{(a)} \quad (14b)$$

Shared modes and transit modes exhibit two specificities. Firstly, their vehicles are likely to mix people trips of different purposes: then the $\tau_{i,m,h}$ are identical between the purposes and they designate the vehicle occupancy rate for all purposes, say $\tau_{m,h}$. Secondly, the transit lines may use the two axial directions of the roadway network in different ways: possibly with different line densities (one such route every n_z axial roadway line), station densities, service frequencies and vehicle sizes. Therefore the vehicle occupancy rates are specific to the axial direction, say $\tau_{m,z,h}$.

This gives rise to specific modal indicators of vehicular flow volumes and trip rates as follows:

$$\tilde{N}_{I,m,h}^{(a)} = \frac{1}{\tau_{m,z,h}} \sum_{i \in I} N_{i,m,h}^{(a)} \quad (15a)$$

$$\tilde{q}_{I,m,h}^{(a)} \equiv \frac{1}{h} \tilde{N}_{I,m,h}^{(a)} \quad (15b)$$

2.3.5 The transversal density of axial traffic

Let us select an axial direction z as the longitudinal dimension of space: then the other, orthogonal axial direction $-z$ is the transversal dimension of space. As L_{-z} measures the distance between one longitudinal link a along $z = z_a$ and the closest parallel links along z , the ratio $1/L_{-z}$ measures the 1D spatial density of longitudinal links, according to the transversal dimension. Thus, q_z/L_{-z} is the transversal spatial density of longitudinal trip flow. It is a trip flow intrinsic to the geographical space, in a deeper way than the link flow.

The longitudinal-transversal setting enables us give a geometric interpretation of the link flow formula (13), restated as follows, with $\dot{\mu}_{i,m,h} \equiv \mu_{i,m,h}^-/h$ denoting the individual trip emission rate per unit time:

$$\frac{1}{L-z} q_{i,m,h}^{(z)} = \delta_{X_i} \dot{\mu}_{i,m,h} D_{i,m,h}^{(z)} \quad (16)$$

Put in words, the transversal, 1D spatial density of longitudinal flow as demanded for purpose i on mode m during period h is equal to the 2D spatial density δ_{X_i} of human occupation X_i , times the individual trip emission time rate $\dot{\mu}_{i,m,h}$, times the longitudinal length $D_{i,m,h}^{(z)}$ required per trip.

2.3.6 Compatibility condition between mobility demand and transport supply

On the supply side, the physics of road traffic impose capacity conditions on the link flow: the well-known flow capacity is the maximum vehicle flow rate per time unit [13]. Denoting it as $\tilde{q}_{max}^{(a)}$ on link a for one mode (for all purposes on a two-way basis), it gives rise to a transversal density of maximum longitudinal vehicular flow rate of $\tilde{q}_{max}^{(a)}/L-z$. The demanded flow is compatible with the supply capacity only if, for the set I_m of trip purposes on that mode,

$$\sum_{i \in I_m} \frac{1}{L-z} q_{i,m,h}^{(z)} = \sum_{i \in I_m} \delta_{X_i} \dot{\mu}_{i,m,h} D_{i,m,h}^{(z)} \leq \frac{\tilde{q}_{max}^{(a)}}{L-z}$$

2.4/ Quality of service: link speeds and trip times

Along either axial direction z , per mode m and period h the homogenous links have identical average run times for their users, say $\eta_{m,h}^{(z)}$, hence identical average speed

$$v_{m,h}^{(z)} \equiv L_z / \eta_{m,h}^{(z)} \quad (17)$$

The modal link speeds are efficiency ratios for link usage and in turn for the individual trips.

During period h , the time spent by people on a z link for purpose i and using mode m amounts to

$$\mathcal{H}_{i,m,h}^{(z)} = N_{i,m,h}^{(z)} \eta_{m,h}^{(z)} \quad (18)$$

At any instant in the period, the number of such people on link z is simply

$$\dot{\mathcal{H}}_{i,m,h}^{(z)} \equiv \frac{\mathcal{H}_{i,m,h}^{(z)}}{h} = q_{i,m,h}^{(z)} \eta_{m,h}^{(z)} \quad (19a)$$

It is associated with a human spatial density (precisely, longitudinal density of people on move) of

$$k_{i,m,h}^{(z)} \equiv \frac{\dot{\mathcal{H}}_{i,m,h}^{(z)}}{L_z} = \frac{q_{i,m,h}^{(z)}}{v_{m,h}^{(z)}} \quad (19b)$$

Similar relations hold for vehicles: the number of mode m vehicles at any instant on a z link is

$$\dot{\mathcal{H}}_{i,m,h}^{(z)} \equiv \frac{\tilde{\mathcal{H}}_{i,m,h}^{(z)}}{h} = \tilde{q}_{i,m,h}^{(z)} \eta_{m,h}^{(z)} \quad (20a)$$

It is associated with a vehicular spatial density of

$$\tilde{k}_{m,h}^{(z)} \equiv \frac{\tilde{\mathcal{H}}_{i,m,h}^{(z)}}{L_z} = \frac{\tilde{q}_{m,h}^{(z)}}{v_{m,h}^{(z)}} \quad (20b)$$

At the individual level, the run time of a person according to purpose i and mode m decomposes with respect to the axial directions z :

$$t_{i,m,h}^{(z)} \equiv \frac{D_{i,m,h}^{(z)}}{v_{m,h}^{(z)}} \quad (21a)$$

Yielding modal run time of

$$t_{i,m,h} = t_{i,m,h}^{(x)} + t_{i,m,h}^{(y)} \quad (21b)$$

In addition to the run time, the trip time may include access time from door to vehicle and conversely from vehicle to door, and also search time for a parking slot if using the transport mode involves parking a vehicle. The full trip time is denoted by $\tilde{t}_{i,m,h}$.

2.4.1 The spatial densities of mobile units

The conventional traffic density variable $k_{i,m,h}^{(z)}$ gives rise to a geometric property of the geographical space. According to the longitudinal-transversal setting, $\tilde{k}_{m,h}^{(z)}/L_{-z}$ measures the transversal, 1D spatial density of moving people density which is a longitudinal, 1D spatial density. The joint effect of the two orthogonal spatial dimensions gives a 2D spatial density of trips. From the flow-density relation $q = k \cdot v$ and (16), it comes out that

$$\frac{1}{L_{-z}} k_{i,m,h}^{(z)} = \delta_{X_i} \dot{\mu}_{i,m,h} \frac{D_{i,m,h}^{(z)}}{v_{m,h}^{(z)}} \quad (22a)$$

Put in words, the 2D spatial density of people on move is equal to the 2D spatial density δ_{X_i} of human occupation X_i , times the individual trip emission rate $\dot{\mu}_{i,m,h}$, times the longitudinal trip time $t_{i,m,h}^{(z)} = D_{i,m,h}^{(z)}/v_{m,h}^{(z)}$.

As for vehicles, we have that

$$\frac{1}{L_{-z}} \tilde{k}_{i,m,h}^{(z)} = \delta_{X_i} \frac{\dot{\mu}_{i,m,h}}{\tau_{i,m,h}} \frac{D_{i,m,h}^{(z)}}{v_{m,h}^{(z)}} \quad (22b)$$

2.5/ Trip distribution for a given mode

2.5.1 O-D pairs and the related trip costs

Under the postulates of homogenous blocks and homogenous links per axial direction, for any mode m the parking conditions are the same for all destinations, as well as between the origins. On any origin-destination pair, the distinctive modal features consist in the run time $t_{i,m,h}$ and a variable money cost say $c_{i,m,h}$ with a directional component of $c_{i,m,h}^{(z)}$ and a per edge cost of $c_{i,m,h}^{(z,1)} = L_z \omega_{i,m,h}$.

From a given block considered as trip origin, the destination blocks can be identified by the numbers of increments or decrements along each axial direction: let us put them as relative integers r and s respectively. The $[r, s]$ pair of relative integers identifies the destination block with respect to the origin block. Thus the O-D run time $t_{i,m,h}^{[r,s]}$ and money cost $c_{i,m,h}^{[r,s]}$ satisfy respectively

$$t_{i,m,h}^{[r,s]} = |r|t_{i,m,h}^{(x,1)} + |s|t_{i,m,h}^{(y,1)} \quad (23a)$$

$$c_{i,m,h}^{[r,s]} = |r|c_{i,m,h}^{(x,1)} + |s|c_{i,m,h}^{(y,1)} \quad (23b)$$

Considering a specific value of time of $\alpha_{i,m,h}$ on that mode depending on the purpose and period, the trip generalized cost is expressed as

$$g_{i,m,h}^{[r,s]} \equiv c_{i,m,h}^{[r,s]} + \alpha_{i,m,h} t_{i,m,h}^{[r,s]} \quad (24)$$

Denoting the edge generalized cost by $g_{i,m,h}^{(z,1)} \equiv c_{i,m,h}^{(z,1)} + \alpha_{i,m,h} t_{i,m,h}^{(z,1)}$, the trip generalized cost also satisfies that

$$g_{i,m,h}^{[r,s]} = |r|g_{i,m,h}^{(x,1)} + |s|g_{i,m,h}^{(y,1)} \quad (25)$$

2.5.2 Gravity model of trip distribution

A gravity model of trip distribution relates the O-D flow volumes to specific “zonal masses” at origins and destinations and also to the O-D impedance of transport. For purpose i , denoting as $E_{i,m,h}^{[b]}$ the zonal mass at the origin block and $A_{i,m,h}^{[r,s]}$ that at the destination block, postulating a decaying exponential function with parameter $\gamma_{i,m,h}$ as impedance function, the O-D volumes would state as follows: [14,15]

$$f_{i,m,h}^{[r,s]} \propto E_{i,m,h}^{[b]} A_{i,m,h}^{[r,s]} \exp(-\gamma_{i,m,h} g_{i,m,h}^{[r,s]}) \quad (26)$$

By the homogeneity postulate between blocks, both as origins and as destinations, the gravity relationship takes on the following simple form:

$$f_{i,m,h}^{[r,s]} = G_{i,m,h}^{[b]} \frac{\exp(-\gamma_{i,m,h} g_{i,m,h}^{[r,s]})}{\sum_{r,s} \exp(-\gamma_{i,m,h} g_{i,m,h}^{[r,s]})} \quad (27)$$

Denoting $e_r^{(x)} \equiv \exp(-\gamma_{i,m,h}|r|g_{i,m,h}^{(x,1)})$ and $e_s^{(y)} \equiv \exp(-\gamma_{i,m,h}|s|g_{i,m,h}^{(y,1)})$, it holds that $\exp(-\gamma_{i,m,h}g_{i,m,h}^{[r,s]}) = e_r^{(x)} \cdot e_s^{(y)}$. The summation over destination zones decomposes with respect to axial directions:

$$\sum_{r,s} e_r^{(x)} \cdot e_s^{(y)} = \sum_r \left(e_r^{(x)} \cdot \sum_s e_s^{(y)} \right) = \left(\sum_r e_r^{(x)} \right) \cdot \left(\sum_s e_s^{(y)} \right)$$

Each part is easily calculated as

$$\sum_{r \in \mathbb{Z}} \exp(-\gamma|r|g^{(z,1)}) = -1 + 2 \sum_{r>0} \left(\rho_{i,m,h}^{(z)} \right)^r = \frac{2}{1 - \rho_z} - 1 = \frac{1 + \rho_z}{1 - \rho_z}$$

Wherein $\rho_{i,m,h}^{(z)} \equiv \exp(-\gamma_{i,m,h}g_{i,m,h}^{(z,1)})$ is also denoted ρ_z . The summation is only valid if $\rho_z < 1$, i.e. if $g_{i,m,h}^{(z,1)} > 0$ which is equivalent to $\omega + \alpha/v > 0$ and in turn to $\alpha + \omega v > 0$ for $v > 0$.

Defining then $\hat{\rho}_z \equiv \frac{1 - \rho_z}{1 + \rho_z}$, we get that

$$f_{i,m,h}^{[r,s]} = G_{i,m,h}^{[b]} \hat{\rho}_x \rho_x^{|r|} \hat{\rho}_y \rho_y^{|s|} \quad (28)$$

Thus, the postulate of indefinite extension of the territory yields a simple form for gravity-based O-D flows.

2.5.3 The average trip length formula

The product form of the O-D flows between the two axial directions is a remarkable property. It enables for easy calculation of the average trip lengths:

$$\begin{aligned} D_{i,m,h}^{(x)} &= \sum_{r,s} \frac{f^{[r,s]}}{G^{[b]}} |r| L_x = \hat{\rho}_x L_x \left(\sum_r |r| \rho_x^{|r|} \right) \hat{\rho}_y \left(\sum_s \rho_y^{|s|} \right) \\ &= \hat{\rho}_x L_x 2 \sum_{r \geq 0} r \rho_x^r = L_x \hat{\rho}_x \frac{2\rho_x}{(1 - \rho_x)^2} \end{aligned}$$

Thus

$$D_{i,m,h}^{(x)} = L_x \frac{2\rho_x}{1 - \rho_x^2} = \frac{L_x}{\sinh(\gamma g_x)} \quad (29)$$

As the hyperbolic sine function \sinh is increasing with its argument and positive for positive arguments, the axial average trip length decreases with the axial edge generalized cost $g_x \equiv g_{i,m,h}^{(x,1)}$. In turn, it is an increasing function of the axial speed v_x .

On the speed domain such that $\alpha + \omega v_x \geq 0$, $v_x \mapsto \gamma g_x$ is a continuous and positive function of v_x ; in turn, so are $\sinh(\gamma g_x)$ and $D_{i,m,h}^{(x)}$. When $v_x \rightarrow 0^+$ then $\gamma g_x \approx \frac{\gamma \alpha L_x}{v_x} \rightarrow +\infty$ so that $D_{i,m,h}^{(x)} \rightarrow 0^+$, which makes it a continuous function of speed over any compact domain in \mathbb{R}^+ . The same applies to $v_x \mapsto D_{i,m,h}^{(x)}/v_x$ because when $v_x \rightarrow 0^+$, the $\sinh(\gamma g_x) \rightarrow +\infty$ in an exponential way that dominates the counteracting influence of v_x , making the trip time $D_{i,m,h}^{(x)}/v_x \rightarrow 0^+$.

When $D_{i,m,h}^{(x)}/L_x$ is fairly large, say beyond 10, then the $\sinh(\gamma g_x) = \frac{L_x}{D_{i,m,h}^{(x)}}$ satisfies that $\sinh(\gamma g_x) \leq 10\%$: it can be closely approximated by $\gamma g_x = \gamma L_x(\omega + \alpha/v)$. Thus we get the approximate formula

$$D_{i,m,h}^{(x)} \approx \frac{1}{\gamma(\omega + \alpha/v)} \quad (30)$$

2.5.4 Axial trip time depending on speed

From the definition of the axial trip time in (21a) and the distribution average length (29), we recover the average axial trip time as follows:

$$t_{i,m,h}^{(x)} = \frac{L_x}{v_x \cdot \sinh(\gamma g_x)} \quad (31a)$$

For large ratios $D_{i,m,h}^{(x)}/L_x$, it can be approximated closely by

$$t_{i,m,h}^{(x)} \approx \frac{1}{\gamma(\alpha + \omega v_x)} \quad (31b)$$

The approximation formula is a hyperbola function of axial speed v_x : as $\gamma > 0$ the direction of variations according to v_x depends on the sign of ω on the speed domain satisfying $\alpha + \omega v_x \geq 0$. If $\omega > 0$ then the approximate average time decreases with v_x . If $\omega = 0$ the average time is about constant with respect to v_x . But $\omega < 0$ makes the approximation of $t_{i,m,h}^{(x)}$ an increasing function of v_x , up to the upper bound $\alpha/|\omega|$.

More precisely, using the exact formula (31a), it is shown in Appendix A that, on the domain set $\{\alpha + \omega v_x \geq 0\}$:

- (i) if $\omega \leq 0$ then the axial time increases with v_x ,
- (ii) if $\omega > 0$ then the axial time varies with v_x in two successive ways, first increasing then decreasing, with turn point of maximum time \hat{t} at speed \hat{v} such that $\hat{\xi} \equiv \gamma \alpha L_x / \hat{v}$ is the unique solution of the following equation:

$$\tanh^{(-1)}(\hat{\xi}) - \hat{\xi} = \gamma \omega L_x$$

Under realistic parameter values the \hat{v} speed is quite small (about 1 or 2 km/h). On the range $[\hat{v}, \infty[$ the exact average axial time behaves like the approximate average.

2.5.5 Traffic variables depending on speed

Per axial direction, the link flow rate $q_{i,m,h}^{(z)}$ and the spatial density $k_{i,m,h}^{(z)}$ depend on $D_{i,m,h}^{(z)}$ in a proportional manner. As the factor $\delta_{x_i} \dot{\mu}_{i,m,h} L_{-z}$ in $q_{i,m,h}^{(z)}$ is constant, under the gravity distribution model $q_{i,m,h}^{(z)}$ varies with $v_{m,h}^{(z)}$ like $D_{i,m,h}^{(z)}$, i.e., the demanded flow rate is an increasing function of the axial speed. This property holds for all purposes hence also for the aggregate demanded flow rate.

As for spatial densities k and \tilde{k} , they depend on speed in a twofold way, proportionally to ratio D/v i.e. to the axial trip time. Between different purposes, at a given value of v some densities may be increasing with v while the other ones would be decreasing: the overall density would be a

composite function of v . Assuming positive ω_i for all purposes i and speed v above the purpose-specific turn speeds \hat{v}_i , then every purpose-specific density will decrease with respect to v and so will the aggregate density: thus the direct decreasing influence of v onto k will overcome the indirect increasing influence through demanded lengths D .

2.6/ Modal split

Here we postulate that there is a set M of modes m , of which the availability to individual trip-makers depends on the purpose. Based on this choice universe, we model the choice of travel mode at the level of the origin zone, encompassing all of the destinations, using a Multi-Nomial Logit model (MNL) [16]. As the spatial distribution per mode constitutes an MNL model of destination choice, the mode and destination choice models constitute together a nested logit model [17,18].

2.6.1 Modal utility

On modeling trip distribution for a single mode, we left aside the modal features that apply to all identical destinations and identical origins in an identical way. These features include the modal attraction $A_{i,m,h}^{[r,s]}$ of the destination block $[r,s]$ as well as parking conditions both at origin and at destination. Let us denote as $\beta_{m|i,h}$ the associated partial utility.

This leads us to the following utility for mode m , destination $[r,s]$ and origin at current block:

$$U_{m|i,h}^{[r,s]} \equiv \beta_{m|i,h} - g_{i,m,h}^{[r,s]} \quad (32)$$

The modal utility that stems from the joint availability of all destinations is estimated by the log-sum formula:

$$\tilde{U}_{m|i,h} \equiv \frac{1}{\gamma_{i,m,h}} \ln \sum_{r,s} \exp(\gamma_{i,m,h} U_{m|i,h}^{[r,s]}) = \beta_{m|i,h} - \frac{1}{\gamma_{i,m,h}} \ln(\hat{\rho}_{i,m,h}^{(x)} \hat{\rho}_{i,m,h}^{(y)}) \quad (33)$$

2.6.2 MNL model of mode choice

At the upper level, mode choice is modeled as an MNL with parameter denoted $\theta_{i,h}$, with modal options m in a subset $M_{i,h}$ of M . Thus the choice probability of every mode m , $\pi_{m|i,h}$, is proportional to $\exp(\theta_{i,h} \tilde{U}_{m|i,h})$ in the following way:

$$\pi_{m|i,h} = \frac{\exp(\theta_{i,h} \tilde{U}_{m|i,h})}{\sum_{\hat{m} \in M_{i,h}} \exp(\theta_{i,h} \tilde{U}_{\hat{m}|i,h})} \quad (34)$$

Considering the multimodal trip volume $G_{i,h}$ from the current block on purpose i and during period h , the corresponding flow on mode m is stated as:

$$G_{m|i,h} = G_{i,h} \pi_{m|i,h} = G_{i,h} \frac{\exp(\theta_{i,h} \beta_{m|i,h}) (\hat{\rho}_{i,m,h}^{(x)} \hat{\rho}_{i,m,h}^{(y)})^{-\theta_{i,h}/\gamma_{i,m,h}}}{\sum_{\hat{m} \in M_{i,h}} \exp(\theta_{i,h} \beta_{\hat{m}|i,h}) (\hat{\rho}_{i,\hat{m},h}^{(x)} \hat{\rho}_{i,\hat{m},h}^{(y)})^{-\theta_{i,h}/\gamma_{i,\hat{m},h}}} \quad (35)$$

At the individual level, the utility derived from mode choice satisfies the log-sum formula:

$$\tilde{U}_{M|i,h} = \frac{1}{\theta_{i,h}} \ln \sum_{m \in M_{i,h}} \exp(\theta_{i,h} \beta_{m|i,h}) (\hat{\rho}_{i,m,h}^{(x)} \hat{\rho}_{i,m,h}^{(y)})^{-\theta_{i,h}/\gamma_{i,m,h}} \quad (36)$$

2.6.3 Modal split at the O-D level

From the current block to destination zone identified as an $[r, s]$ pair of axial shifts, the ratio of modal flows between modes $m, \acute{m} \in M_{i,h}$ satisfies that

$$\frac{\pi_{m|i,h}^{[r,s]}}{\pi_{\acute{m}|i,h}^{[r,s]}} = \frac{f_{i,m,h}^{[r,s]}}{f_{i,\acute{m},h}^{[r,s]}} = \frac{G_{i,m,h}^{[b]} \hat{\rho}_{i,m,h}^{(x)} (\rho_{i,m,h}^{(x)})^{|r|} \hat{\rho}_{i,m,h}^{(y)} (\rho_{i,m,h}^{(y)})^{|s|}}{G_{i,\acute{m},h}^{[b]} \hat{\rho}_{i,\acute{m},h}^{(x)} (\rho_{i,\acute{m},h}^{(x)})^{|r|} \hat{\rho}_{i,\acute{m},h}^{(y)} (\rho_{i,\acute{m},h}^{(y)})^{|s|}}$$

Thus

$$\frac{\pi_{m|i,h}^{[r,s]}}{\pi_{\acute{m}|i,h}^{[r,s]}} = \frac{\pi_{m|i,h} \hat{\rho}_{i,m,h}^{(x)} \hat{\rho}_{i,m,h}^{(y)} \left(\frac{\rho_{i,m,h}^{(x)}}{\rho_{i,\acute{m},h}^{(x)}} \right)^{|r|} \left(\frac{\rho_{i,m,h}^{(y)}}{\rho_{i,\acute{m},h}^{(y)}} \right)^{|s|}}{\pi_{\acute{m}|i,h} \hat{\rho}_{i,\acute{m},h}^{(x)} \hat{\rho}_{i,\acute{m},h}^{(y)} \left(\frac{\rho_{i,\acute{m},h}^{(x)}}{\rho_{i,\acute{m},h}^{(x)}} \right)^{|r|} \left(\frac{\rho_{i,\acute{m},h}^{(y)}}{\rho_{i,\acute{m},h}^{(y)}} \right)^{|s|}} \quad (37a)$$

Once again, the influences of the axial shifts are under product form, i.e., they are separable. Along each direction from the origin block to a destination block, the ratio of modal flows is an exponential function of the absolute shift $|r|$ or $|s|$, with parameter

$$\ln \left(\frac{\rho_{i,m,h}^{(z)}}{\rho_{i,\acute{m},h}^{(z)}} \right) = -\gamma_{i,m,h} g_{i,m,h}^{(z,1)} + \gamma_{i,\acute{m},h} g_{i,\acute{m},h}^{(z,1)}$$

As the edge generalized costs $g_{i,m,h}^{(z,1)}$ are proportional to L_z , the inter-block axial distance $|r|L_x$ exerts an obvious influence on the argument of the exponential function:

$$|r| \ln \left(\frac{\rho_{i,m,h}^{(x)}}{\rho_{i,\acute{m},h}^{(x)}} \right) = |r|L_x \left[-\gamma_{i,m,h} \left(\omega_{i,m,h} + \frac{\alpha_{i,m,h}}{v_{m,h}^{(x)}} \right) + \gamma_{i,\acute{m},h} \left(\omega_{i,\acute{m},h} + \frac{\alpha_{i,\acute{m},h}}{v_{\acute{m},h}^{(x)}} \right) \right]$$

2.6.4 Sensitivities with respect to speeds and access times

In the modal distribution model, the edge axial parameter $\rho_{i,m,h}^{(z)}$ depends only on the speed $v_m^{(z)}$ of the mode and axial direction pair, not on other modal and axial speeds neither on the access times.

This property extends to the derived term $\hat{\rho}_z \equiv \frac{1-\rho_z}{1+\rho_z}$. From its definition, it satisfies that

$$\hat{\rho}_z \equiv \frac{1-\rho_z}{1+\rho_z} = \frac{\rho_z^{-0.5} - \rho_z^{0.5}}{\rho_z^{-0.5} + \rho_z^{0.5}} = \frac{\sinh \frac{1}{2} \gamma g_z}{\cosh \frac{1}{2} \gamma g_z} = \tanh \frac{1}{2} \gamma g_z$$

As the tanh function is increasing and $\gamma > 0$, $\hat{\rho}_z$ increases with g_z , hence it decreases with v_z . In turn, the modal joint utility of all destinations, $\tilde{U}_{m|i,h}$, is a decreasing function of $g_{i,m,h}^{(z,1)}$ and an increasing function of $v_{m,h}^{(z)}$.

The access time w_m influences $\beta_{m|i,h}$ in a decreasing manner, and its effect on $\tilde{U}_{m|i,h}$ is decreasing, too.

Coming to the mode choice probabilities, each of them can be restated equivalently as

$$\pi_{m|i,h} = \frac{1}{1 + \exp(-\theta_{i,h} \tilde{U}_{m|i,h}) \sum_{\acute{m} \in M_{i,h}, \acute{m} \neq m} \exp(\theta_{i,h} \tilde{U}_{\acute{m}|i,h})}$$

It is thus an increasing function of $\tilde{U}_{m|i,h}$; in turn, it is a decreasing function of w_m and of $g_{i,m,h}^{(z,1)}$, and an increasing function of $v_{m,h}^{(z)}$. The influences are opposite on the modal shares of the other modes.

All of the influences transfer straightforwardly to the modal flows $G_{m|i,h}$, and also to the multimodal utility $\tilde{U}_{M|i,h}$.

On an O-D pair from current block as origin to an $[r, s]$ destination block, the influence of the modal parameters on the ratio of O-D modal shares (37a) is more complex. Let us restate the ratio as

$$\frac{\pi_{m|i,h}^{[r,s]}}{\pi_{\hat{m}|i,h}^{[r,s]}} = \frac{\exp(\theta_{i,h}\beta_{m|i,h}) (\hat{\rho}_{i,m,h}^{(x)} \hat{\rho}_{i,m,h}^{(y)})^{1-\theta_{i,h}/\gamma_{i,m,h}} \left(\frac{\rho_{i,m,h}^{(x)}}{\rho_{i,m,h}^{(x)}}\right)^{|r|} \left(\frac{\rho_{i,m,h}^{(y)}}{\rho_{i,m,h}^{(y)}}\right)^{|s|}}{\exp(\theta_{i,h}\beta_{\hat{m}|i,h}) (\hat{\rho}_{i,\hat{m},h}^{(x)} \hat{\rho}_{i,\hat{m},h}^{(y)})^{1-\theta_{i,h}/\gamma_{i,\hat{m},h}} \left(\frac{\rho_{i,\hat{m},h}^{(x)}}{\rho_{i,\hat{m},h}^{(x)}}\right)^{|r|} \left(\frac{\rho_{i,\hat{m},h}^{(y)}}{\rho_{i,\hat{m},h}^{(y)}}\right)^{|s|}} \quad (37b)$$

The modal access time w_m influences $\beta_{m|i,h}$ in a decreasing manner; in turn its effect on the O-D ratio of modal shares is decreasing, too. As for modal axial speeds, an increase in $v_{m,h}^{(x)}$ increases $\rho_{i,m,h}^{(x)}$ but decreases $\hat{\rho}_{i,m,h}^{(x)}$. It is shown in Appendix B that for non-null r , the ratio varies with $g_{i,m,h}^{(x,1)}$ in two successive ways, first increasing then decreasing.

2.7/ Travel impedance functions

Impedance functions relate individual travel conditions (mostly times) to flow volumes. Run time functions pertain to traffic conditions, while access time functions pertain to parking conditions.

2.7.1 Traffic impedance functions

Given the travel mode, link run times depend on speed regulations (e.g. limit speed) and the traffic density. In its traffic direction, each vehicle requires a run length that includes its own length plus a front margin for safety purposes. The front margin increases with speed. Between homogenous vehicles in the same direction that are evenly spaced along the link, the run length is inversely proportional to the one-way vehicular density, $\frac{1}{2}\tilde{k}_{m,h}^{(z)}$. For cars, the relationship between speed and density is called the Fundamental Traffic Diagram [13]: it consists in a decreasing function $V_m^{(z)}$ linking speed to density:

$$v_{m,h}^{(z)} = V_m^{(z)}(\tilde{k}_{m,h}^{(z)}) \quad (38a)$$

Similar relationships hold for pedestrians [19,20], and also for bikes on cycle lanes [21,22]. Mixed traffic involving different vehicle types on shared lanes is more complex, e.g. bikes of lower speed impede cars of (tentatively) higher speed. To denote all kinds of traffic interactions, let us model in an abstract way the vector of modal average speeds on a link, $\mathbf{v}_{M,h}^{(z)} \equiv (v_{m,h}^{(z)} : m \in M)$, as a vector function $\mathbf{V}_M^{(z)}$ of the modal vehicular densities, $\tilde{\mathbf{k}}_{M,h}^{(z)} \equiv (\tilde{k}_{m,h}^{(z)} : m \in M)$:

$$\mathbf{v}_{M,h}^{(z)} = \mathbf{V}_M^{(z)}(\tilde{\mathbf{k}}_{M,h}^{(z)}) \quad (38b)$$

As pedestrians can avail themselves of street curbs with exclusive right-of-way, we expect their impedance function to be separable from the vehicular ones. The same isolation principle applies to bikes on cycle lanes, as well as to buses on bus lanes. Given an overall link width, a prominent issue in urban transportation planning is its allocation between the traffic of the different transport modes and also their on-street parking.

A function $V_m^{(z)}$ basically expresses the influence of vehicular densities onto the modal axial speed: in other words, it makes speed $v_m^{(z)}$ a density-driven speed. However, for motorized modes it is customary in cities to settle a limit speed, say $\bar{v}_m^{(z)}$, i.e., a policy-driven speed. The speed limitation is stated as the following condition:

$$v_m^{(z)} \leq \bar{v}_m^{(z)}$$

Mathematically, both influences can be stated in a unified function as follows:

$$\tilde{\mathbf{k}}_M^{(z)} \mapsto \tilde{V}_m^{(z)} \equiv \min \{ \bar{v}_m^{(z)}, V_m^{(z)}(\tilde{\mathbf{k}}_M^{(z)}) \} \quad (39)$$

It can be considered as an extended traffic impedance function – though less sensitive to traffic density.

2.7.2 Parking impedance functions

On an individual basis, vehicle usage requires some walk length and related pedestrian time at both trip ends to access to the vehicle from the origin endpoint and similarly from the vehicle to the destination endpoint. The walk length is short if the users can avail themselves of their own spots of private off-street parking. But, on using on-street parking, available spots may be scarce, thereby requiring longer walk time plus some search time near the destination [23,24]. Vehicles destined to a block may find their on-street parking spot on a link edging another block: however, according to the principle of block homogeneity, the demand and supply of parking spots can be estimated at the block level. We denote by $P_{m,h}^+$ the on-street parking demand of mode m at destination by the trips generated by block b during period h as follows:

$$P_{m,h}^+ = \sum_{i \in I} G_{i,m,h}^{(b)} \zeta_{i,m,h}^+ / \tau_{i,m,h} \quad (40)$$

Owing to the symmetry between the blocks, it is also the demand for on-street parking at the current block as trip destination.

Reciprocally, the trips using mode m during period h vacate the following number of on-street parking spots at the block level:

$$P_{m,h}^- = \sum_{i \in I} G_{i,m,h}^{(b)} \zeta_{i,m,h}^- / \tau_{i,m,h} \quad (41)$$

In (6) we stated the balance sheet of on-street parking for mode m during period h at the block level.

On-street parking impedances could be modeled depending on the trip end. To keep things simple, we attribute to the current trip its destination impedance plus the origin impedance of the next trip of the individual using the same vehicle. The specific parking time (including walk access and search time) is denoted by $w_m^{(h)}$. We consider a vector function $\mathbf{W}_M^{(h)}$ linking the multimodal vector of mode parking times, $\mathbf{w}_M^{(h)} \equiv (w_m^{(h)} : m \in M)$, to the multimodal vectors of on-street arrival and departure flows, $\mathbf{P}_{M,h}^+ \equiv (P_{m,h}^+ : m \in M)$ and $\mathbf{P}_{M,h}^- \equiv (P_{m,h}^- : m \in M)$:

$$\mathbf{w}_M^{(h)} = \mathbf{W}_M^{(h)}(\mathbf{P}_{M,h}^+, \mathbf{P}_{M,h}^-) \quad (42)$$

The h superscript on function $\mathbf{W}_M^{(h)}$ recalls that parking loading is a dynamic phenomenon: both the final state at the end of the period and the initial state involve the residual capacity available at the beginning of the period, which depends in turn on the overall capacity minus the on-street parking loads of the vehicles that are not moved during the period.

2.8/ Traffic equilibrium

2.8.1 The multimodal equilibrium of traffic and parking

The full HoTer model combines the sub-models introduced in the previous subsections: these are, starting from (i) block composition, (ii) the generation of trips according to activity purposes, (iii) the formation of traffic from generated trips and axial trip lengths, (iv) quality of service at link- and trip-levels, (v) trip distribution by mode, (vi) modal split, (vii) traffic and parking impedances. The sub-models link together the model variables. The graphical summary of the HoTer model (figure 1) shows that the Impedance sub-model constitutes a feedback on the quality of service conditions which, together with block constitution, determine the formation of travel demand.

The model architecture mirrors that of a typical four-step TDM, up to the consideration of aggregate variables in HoTer, especially the average trip length in place of a whole matrix of origin-destination trip flows. The cyclic chain of dependencies is typical of an equilibrium model: for the whole model to hold jointly, the state variables must satisfy all of the relations. The demand sub-models do not involve any feasibility restraint. But the Impedance sub-model involves feasibility conditions both on link flows and on parking loads: the link occupancy caused by demand flows must comply with some link traffic capacities, and similarly the change in parking loads induced by trip demand must comply with modal on-street parking capacities at the block level.

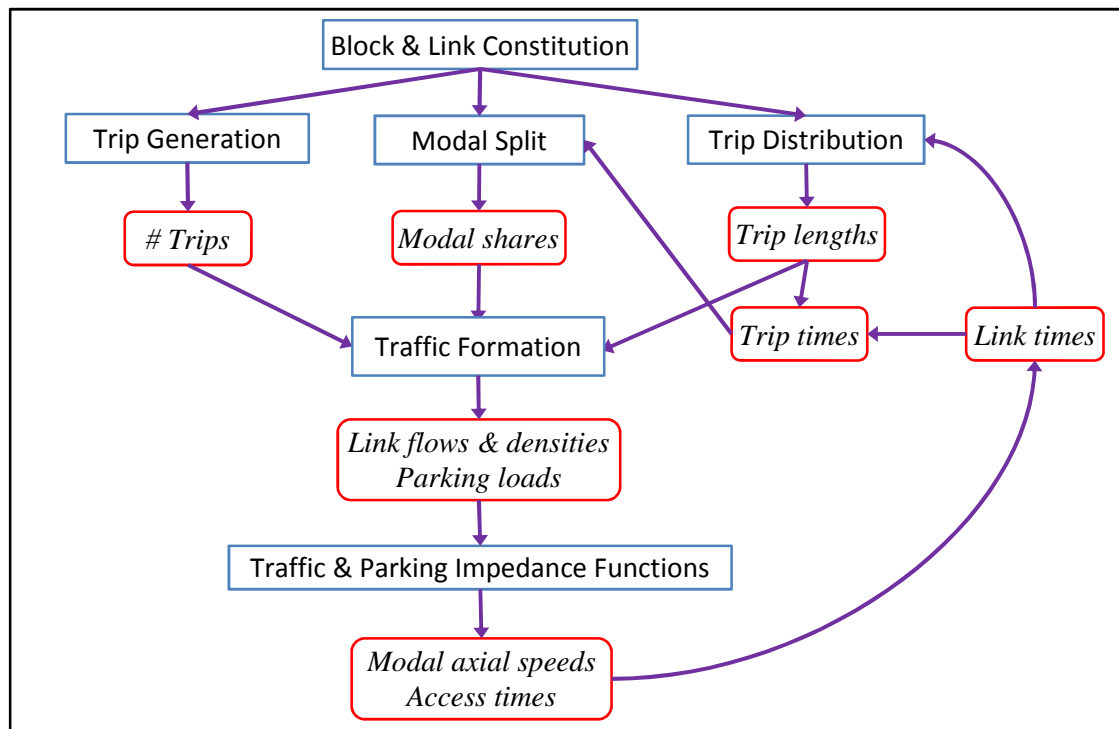


Fig.1. HoTer model architecture.

2.8.2 Mathematical formulation and analysis

To calculate the system state, i.e., the vector of state variables, it is convenient to select a subset of state variables in order to constitute a “basic” state vector. In travel demand modeling and the theory of traffic network assignment, two kinds of basic vectors are used: “primal” mathematical formulations pertain to flow variables (by purpose, mode, period, O-D pair and network link), whereas “dual” mathematical formulations pertain to time or cost variables (also by purpose, mode, period and link) [5]. To focus on the Impedance conditions, we select as basic state vector the vector of modal and axial speeds, $(\mathbf{v}_{M,h}^{(x)}, \mathbf{v}_{M,h}^{(y)})$, and that of access times, $\mathbf{w}_M^{(h)}$. Denoting by \bar{M} the number of modes, the basic state vector contains at most $3\bar{M}$ variables: $2\bar{M}$ for speeds and \bar{M} for access times (in fact $\bar{M} - 1$ dropping the walk mode). This is essentially a dual state vector since a modal axial speed $v_{m,h}^{(z)}$ is equivalent to the related modal axial link time $L_z/v_{m,h}^{(z)}$.

To state the equilibrium problem as a mathematical program, let us denote the link flow demands as functions $\mathbf{k}_D^{(M,h)}$, \mathbf{P}_D^+ , \mathbf{P}_D^- of the basic state vector $(\mathbf{v}_{M,h}^{(z)}, \mathbf{w}_M^{(h)})$:

$$\tilde{\mathbf{k}}_{M,h}^{(z)} \equiv (\tilde{k}_{m,h}^{(z)} : m \in M, z \in Z) = \mathbf{k}_D^{(M,h)}(\mathbf{v}_{M,h}^{(z)}, \mathbf{w}_M^{(h)}) \quad (43a)$$

$$\mathbf{P}_{M,h}^+ = \mathbf{P}_D^+(\mathbf{v}_{M,h}^{(z)}, \mathbf{w}_M^{(h)}) \quad (43b)$$

$$\mathbf{P}_{M,h}^- = \mathbf{P}_D^-(\mathbf{v}_{M,h}^{(z)}, \mathbf{w}_M^{(h)}) \quad (43c)$$

The influences of the basic variables onto the demanded flow variables can be traced out: for instance, a higher modal axial speed induces first a shorter modal axial run time, then a longer trip length and a higher modal share, and in turn higher modal axial trip flow¹. The increased trip flow, for a transport mode requiring vehicle parking, induces larger modal parking loads $P_{m,h}^-$ and $P_{m,h}^+$. Similarly, a lower modal access time induces a higher modal share, and in turn higher trip flows – number of generated trips, traffic quantities, link flow rates and link flow densities.

On the side of traffic supply, the traffic flowing sub-model relates the axial speeds and access times to the modal axial link densities and block parking loads, respectively, as functions $\mathbf{V}_M^{(z)}$ and $\mathbf{W}_M^{(h)}$.

We define a traffic equilibrium as a basic state vector $(\mathbf{v}_{M,h}^{(z)}, \mathbf{w}_M^{(h)})$ that satisfies the following conditions:

$$\mathbf{v}_{M,h}^{(z)} = \mathbf{V}_M^{(z)} \circ \mathbf{k}_D^{(M,z,h)}(\mathbf{v}_{M,h}^{(z)}, \mathbf{w}_M^{(h)}) \quad (44a)$$

$$\mathbf{w}_M^{(h)} = \mathbf{W}_M^{(h)} \left(\mathbf{P}_D^+(\mathbf{v}_{M,h}^{(z)}, \mathbf{w}_M^{(h)}), \mathbf{P}_D^-(\mathbf{v}_{M,h}^{(z)}, \mathbf{w}_M^{(h)}) \right) \quad (44b)$$

Thus, the mathematical program of traffic equilibrium consists in a fixed point problem.

The domain of variations is a bounded set, as the product set of speed intervals $[0, \hat{v}_m^{(z)}]$ and access time intervals $[0, \hat{w}_m]$. Regarding speeds, there are physical upper bounds $\hat{v}_m^{(z)}$ on the modal speeds

¹ and also a higher link density when $D_{m,h}^{(z)}/v_{m,h}^{(z)}$ is an increasing function of $v_{m,h}^{(z)}$

$v_m^{(z)}$ hence on the values taken by the $V_m^{(z)}$ functions: yielding the product set $\mathcal{B}_v \equiv \prod_{m,z} [0, \hat{v}_m^{(z)}]$ as domain set for speed vectors. Regarding access times, let us postulate that the $W_m^{(h)}$ functions are bounded from above by upper bounds \hat{w}_m , for instance a small number of block perimeters L_b : this yields the product set $\mathcal{B}_w \equiv \prod_m [0, \hat{w}_m]$ as domain set for access time vectors. The twofold domain set $\mathcal{B}_v \times \mathcal{B}_w$ is bounded and compact.

Concerning the continuity of the functions, we showed that the $k_D^{(m,z,h)}$ functions are continuous with respect to speeds and access times: by linearity, so are the functions \mathbf{P}_D^+ and \mathbf{P}_D^- . Postulating further that the $\mathbf{V}_M^{(z)}$ and $\mathbf{W}_M^{(h)}$ functions are continuous, then so is the mapping $(\mathbf{v}_{M,h}^{(z)}, \mathbf{w}_M^{(h)}) \mapsto [\mathbf{V}_M^{(z)} \circ \mathbf{k}_D^{(M,Z,h)}, \mathbf{W}_M^{(h)} \circ (\mathbf{P}_D^+, \mathbf{P}_D^-)]$.

Then, by Brouwer's theorem for continuous functions over compact sets [25], there exists a solution to the fixed point problem (44), i.e. a state of traffic equilibrium.

For the mathematical notion of equilibrium to have a physical meaning, we need the equilibrium speeds to be strictly positive: faced to limited flowing capacities, exceedingly high demanded densities $\mathbf{k}_D^{(M,Z,h)}$ will entail null values of the speed function $\mathbf{V}_M^{(z)}$. Such degeneracy could occur in a problem with fixed demand, i.e., fixed trip length yielding constant $\tilde{\mathbf{q}}_D^{(M,Z,h)}$ at level higher than could be accommodated by the link capacity. The resulting supplied speed would be null, contradicting the very essence of a transport mode as a service to go through space.

2.9/ Monomodal traffic equilibrium in HoTer

Here we consider a single mode of transport for trip making in the idealized territory.

2.9.1 Monomodal fundamental Traffic Equilibrium Condition (TEC) and its solution

Assuming that parking capacity is sufficient, only the speed variables are active in the determination of the system state. In the general case with multiple modes, the axial directions are only linked by the effect of modal split on trip generation and distribution: thus, in the single mode case with inelastic number of trips, each axial direction z can be addressed separately.

Dropping the indices of mode, axial direction and time period, the fundamental TEC is restated as the following fixed point problem in one unknown v :

$$v = \tilde{V}_O(\tilde{k}_D(v)) \quad (45)$$

The extended speed-density law \tilde{V}_O is decreasing with respect to vehicular density \tilde{k} . If the demanded density \tilde{k}_D is an increasing function of v then the right-hand side function $V_O \circ \tilde{k}_D$ decreases with v , whereas the left-hand side increases: then the traffic equilibrium² is a unique state. Uniqueness of solution is less trivial when function \tilde{k}_D has more complex behavior according to v : we shall see a special instance of inelastic demanded flow rate, yielding two solution states.

If the demanded density is elastic to speed then TEC (45) cannot have $v = 0$ as solution: this is because $\tilde{k}_D(0) = 0$ holds true for an elastic demanded density, combined to $\tilde{V}_O(0) > 0$.

² which exists when \tilde{k}_D is elastic owing to the properties of the multimodal model

An even more important issue pertains to whether a traffic equilibrium (v^*, k^*) is density-driven or policy-driven, i.e., limited to \bar{v} . If $v^* < \bar{v}$ then v^* is density-driven: around it \tilde{V}_0 corresponds to V_0 that is decreasing, with inverse function \tilde{k}_0 that is also decreasing: by applying it to both sides of the equilibrium condition, we get the equivalent condition $\tilde{k}_0(v) = \tilde{k}_D(v)$.

If $v^* = \bar{v}$ then, denoting $k^* \equiv \tilde{k}_D(v^*)$, we can assume that $k^* \leq \tilde{k}_0(\bar{v})$. The reason is that, if otherwise $\tilde{k}_D(\bar{v}) > \tilde{k}_0(\bar{v})$, then, as $\tilde{k}_D(0) = 0$ is below $\tilde{k}_0(0)$, the \tilde{k}_D and \tilde{k}_0 curves would have intersected previously on $]0, \bar{v}[$ at some v_2^* that would be another equilibrium state with density-driven equilibrium density $k_2^* = \tilde{k}_D(v_2^*) = \tilde{k}_0(v_2^*)$. In such a configuration we keep (v_2^*, k_2^*) as a true equilibrium and dismiss point $(\bar{v}, \tilde{k}_D(\bar{v}))$ with $\tilde{k}_D(\bar{v}) > \tilde{k}_0(\bar{v})$ as physically inconsistent.

2.9.2 Alternative TECs

The extended speed-density law \tilde{V}_0 is the minimum function of the decreasing physical density V_0 and the constant function \bar{v} . The equilibrium condition can be restated using V_0 instead of \tilde{V}_0 as the following threefold condition:

$$v \leq \bar{v} \quad (46a)$$

$$V_0 \circ \tilde{k}_D(v) \geq v \quad (46b)$$

$$(\bar{v} - v) \cdot (V_0 \circ \tilde{k}_D(v) - v) = 0 \quad (46c)$$

The last part ensures that any solution point (v^*, k^*) with $v^* < \bar{v}$ satisfies the original condition. In the middle part, the inequality corresponds to the compatibility condition at $v^* = \bar{v}$: it stems from applying the decreasing function V_0 to both sides of $\tilde{k}_D(\bar{v}) \leq \tilde{k}_0(\bar{v})$.

Using the decreasingness of the supply speed-density law V_0 , by applying \tilde{k}_0 to both sides of the middle part we get the equivalent “middle condition”

$$\tilde{k}_D(v) \leq \tilde{k}_0(v)$$

This yields an equivalent TEC on speeds v as follows:

$$v \leq \bar{v} \quad (47a)$$

$$\tilde{k}_D(v) \leq \tilde{k}_0(v) \quad (47b)$$

$$(\bar{v} - v) \cdot (\tilde{k}_0(v) - \tilde{k}_D(v)) = 0 \quad (47c)$$

This TEC on demanded and supplied densities enables one to demonstrate the existence of traffic equilibrium in a straightforward way if the demanded density is elastic: then, on the demand side, $\tilde{k}_D(0) = 0$ and $\tilde{k}_D(v) > 0$ for $v > 0$. On the supply side, $\tilde{k}_0(0) > 0$ and \tilde{k}_0 decreases with v : for large speeds, $\tilde{q}_0(v) = v \cdot \tilde{k}_0(v)$ decreases to zero whereas $\tilde{q}_D(v) = v \cdot \tilde{k}_D(v)$ keeps increasing, implying $\tilde{q}_D(v) > \tilde{q}_0(v)$ hence $\tilde{k}_D(v) > \tilde{k}_0(v)$. Thus the function $\tilde{k}_0 - \tilde{k}_D$ changes sign between zero and some finite higher value: as it is continuous, by the intermediate value theorem [45] there is an intermediate point v^* at which the function takes null value, i.e., a traffic equilibrium.

Replacing now the speed variable by the density variable related to it through V_0 , we obtain the following equivalent TEC with respect to link density k :

$$k \geq \bar{k} \equiv \tilde{k}_O(\bar{v}) \quad (48a)$$

$$\tilde{k}_D \circ V_O(k) \leq k \quad (48b)$$

$$(\bar{k} - k) \cdot (k - \tilde{k}_D \circ V_O(k)) = 0 \quad (48c)$$

Alternatively, we can use the increasingness of the demanded link flow \tilde{q}_D with respect to v and the inverse function $q \mapsto \tilde{q}_D^{(-1)}$ to state a TEC involving flow rate functions: starting from (46),

$$q \leq \bar{q} \equiv \tilde{q}_D(\bar{v}) \quad (49a)$$

$$\tilde{q}_D \circ V_O(q/\tilde{q}_D^{(-1)}(q)) \geq q \text{ and } \tilde{q}_D^{(-1)}(q) > 0 \quad (49b)$$

$$(\bar{q} - q) \cdot \left(\tilde{q}_D \circ V_O\left(\frac{q}{\tilde{q}_D^{(-1)}(q)}\right) - q \right) = 0 \quad (49c)$$

The middle part is equivalent to $q > \tilde{q}_D(0)$ combined to either one of the following:

$$V_O(q/\tilde{q}_D^{(-1)}(q)) \geq \tilde{q}_D^{(-1)}(q)$$

$$q/\tilde{q}_D^{(-1)}(q) \leq \tilde{k}_O \circ \tilde{q}_D^{(-1)}(q)$$

$$q \leq \tilde{q}_D^{(-1)}(q) \cdot \tilde{k}_O \circ \tilde{q}_D^{(-1)}(q) = \tilde{q}_O \circ \tilde{q}_D^{(-1)}(q)$$

We can then devise an alternative TEC with respect to speeds:

$$v \leq \bar{v} \text{ and } v > 0 \quad (50a)$$

$$\tilde{q}_O(v) \geq \tilde{q}_D(v) \quad (50b)$$

$$(\bar{v} - v) \cdot (\tilde{q}_O(v) - \tilde{q}_D(v)) = 0 \quad (50c)$$

In this formulation emphasizing flow rate functions, the condition $v > 0$ is required to dismiss the trivial equality $\tilde{q}_D(0) = \tilde{q}_O(0) = 0$ that checks the other partial conditions.

2.9.3 Sensitivity analysis

The traffic equilibrium condition (47) involving offered and demanded flow spatial densities enables us to analyze the sensitivity of a (v^*, k^*) equilibrium pair of variables to the model parameters. Knowing that function \tilde{k}_O is decreasing, one has to specify the configuration of \tilde{k}_D at point v^* relatively to \tilde{k}_O .

If \tilde{k}_D is increasing at v^* , or more generally if its curve traverses that of \tilde{k}_O from below to above when v increases, then the following properties hold. On the demand side, any increase in parameters O and μ moves \tilde{k}_D upwards, therefore reducing v^* . But any increase in parameters τ , L , γ , α , ω will reduce \tilde{k}_D and increase v^* . On the supply side, any parameter change that shifts the \tilde{k}_O curve up (resp. down) will move the intersection point with \tilde{k}_D both up and to the right (resp. both down and to the left); therefore v^* is increased (resp. decreased) and k^* is decreased (resp. increased).

But if the \tilde{k}_D curve traverses that of \tilde{k}_O from above to below when v increases, then the following properties hold. On the demand side, any increase in parameters O and $\dot{\mu}$ moves \tilde{k}_D upwards, therefore increasing v^* . Any increase in parameters τ , L , γ , α , ω will reduce \tilde{k}_D and decrease v^* . On the supply side, any parameter change that shifts the \tilde{k}_O curve up (resp. down) will move the intersection point with \tilde{k}_D both up and to the left (resp. both down and to the right); therefore v^* is decreased (resp. increased) and k^* is increased (resp. decreased).

2.9.4 Geometric interpretation

The TEC on density functions of speed involves the territorial parameters in a straightforward way. Multiplying both sides of condition $\tilde{k}_D(v) \leq \tilde{k}_O(v)$ (47b) by $1/L_{-z}$, we obtain that

$$\sum_{i \in I_m} \delta_{X_i} \frac{\dot{\mu}_i D_i}{\tau_i v} \leq \frac{1}{L_{-z}} \tilde{k}_O(v)$$

Under the gravity model for every purpose, then $D_i = L_z / \sinh(\gamma_i g_{z,1|i})$, so that

$$\sum_{i \in I_m} \delta_{X_i} \frac{\dot{\mu}_i}{\tau_i v} \frac{L_z}{\sinh(\gamma_i g_{z,1|i})} \leq \frac{1}{L_{-z}} \tilde{k}_O(v)$$

By approximating $\sinh \xi \approx \xi$, it comes out that

$$\sum_{i \in I_m} \delta_{X_i} \frac{\dot{\mu}_i}{\tau_i \gamma_i (\alpha_i + \omega_i v)} \leq \frac{1}{L_{-z}} \tilde{k}_O(v)$$

The equilibrium speed is either \bar{v} or the density-driven v^* that balances the two sides of the inequality. In the right-hand side we recognize a transversal density of longitudinal flow density.

2.9.5 Special cases

If the money costs per unit length ω_i are null then, from the approximation, the demanded density is about constant:

$$\tilde{k}_D(v) \approx L_{-z} \sum_{i \in I_m} \delta_{X_i} \frac{\dot{\mu}_i}{\tau_i \gamma_i \alpha_i}$$

Then demand determines the equilibrium density k^* , while the supply side determines the equilibrium speed v^* as

$$v^* = \min\{\bar{v}, V_O(k^*)\}$$

If the values of time α_i are null, then the demanded flow \tilde{q}_D does not depend on speed:

$$\tilde{q}_D(v) = L_{-z} \sum_{i \in I_m} \delta_{X_i} \frac{\dot{\mu}_i}{\tau_i} \frac{L_z}{\sinh(\gamma_i L_z \omega_i)}$$

The equilibrium state follows using the TEC on flow rates (50):

$$\sum_{i \in I_m} \delta_{X_i} \frac{\dot{\mu}_i}{\tau_i} \frac{L_z}{\sinh(\gamma_i L_z \omega_i)} \leq \frac{1}{L_{-z}} \tilde{q}_O(v)$$

Recalling the link flow capacity $\tilde{q}_{max}^{(a)}$ introduced in §2.3.6, the existence of a physically meaningful traffic equilibrium requires that

$$\sum_{i \in I_m} \delta_{X_i} \frac{\dot{\mu}_i}{\tau_i} \frac{L_z}{\sinh(\gamma_i L_z \omega_i)} \leq \frac{1}{L_{-z}} \tilde{q}_{max}^{(a)}$$

Under that proviso, from the general behavior of traffic fundamental diagrams, function \tilde{q}_0 has a bell-shaped curve so that there are two values of speed that satisfy the partial equilibrium condition: either congested or free-flow. In the urban setting, the truncation to the policy-driven limit speed will exclude the free-flow solution.

2.9.6 Simple instance

Assuming a single trip purpose and an affine linear speed-density relationship $V_0 = v_0 - \dot{v} \cdot \tilde{k}$, then $\tilde{k}_0 = (v_0 - v)/\dot{v}$ and the axial traffic equilibrium condition takes the following simple form:

$$\frac{O\dot{\mu}}{\tau} \frac{1}{\sinh(\gamma L_z(\omega + \alpha/v))} \frac{1}{v} = \frac{v_0 - v}{\dot{v}} \quad \text{subject to } v \in [0, v_0] \quad (51)$$

Under the $\sinh \xi \approx \xi$ approximation, the axial TEC can be approximated as follows:

$$\frac{\delta_X L_{-z} \dot{\mu}}{\tau \gamma} \frac{1}{\alpha + \omega v} = \frac{v_0 - v}{\dot{v}} \quad (52)$$

If $\omega \neq 0$ then the approximate TEC is a second-degree equation in v . In Appendix C it is put in standard form, with simpler indexation and defining $B \equiv \frac{\delta_X \dot{\mu} L_{-z} \dot{v}}{\gamma \tau}$, yielding:

$$\left(v - \frac{1}{2} \left(v_0 - \frac{\alpha}{\omega}\right)\right)^2 - \frac{1}{4} \left(v_0 + \frac{\alpha}{\omega}\right)^2 + \frac{B}{\omega} = 0 \quad (53)$$

Letting $v_2 \equiv \frac{1}{2} \left(v_0 + \frac{\alpha}{\omega}\right)$, the equation has discriminant of $\Delta = (v_2)^2 - \frac{B}{\omega}$. It must be non-negative for the equation to admit a real-valued solution [26], the non-negativity condition states as

$$B \leq \omega (v_2)^2, \text{ or equivalently}$$

$$\frac{\delta_X \dot{\mu}}{\gamma \omega} \leq \tau \frac{v_2}{L_{-z} \dot{v}} v_2 \quad (54)$$

The left-hand side is a 2D spatial density of longitudinal trip length demand (as $1/\gamma\omega$ is an axial length) per unit time. The right-hand side involves the techno-economic speed v_2 , which must be positive for the gravity model to hold at all speeds up to supply maximum speed v_0 . The ratio $v_2/(L_{-z} v_1)$ is a transversal density of vehicular density $k_2 \equiv v_2/\dot{v}$: multiplying it by v_2 yields a transversal density of vehicular flow rate $q_2 \equiv k_2 \cdot v_2$. Overall, the non-negative discriminant condition states that the 2D spatial density of longitudinal demanded length per unit time must be less than the transversal density of supply longitudinal flow time rate τq_2 .

The non-negative discriminant condition mirrors the compatibility condition of demand and supply established in §2.3.6 for inelastic speed and trip length: it extends it to the endogenous case where both speed and trip length are elastic.

Under the compatibility condition, the solutions are given by the classical formula as follows, with $\varepsilon \in \{-1, +1\}$:

$$v_\varepsilon = \frac{1}{2} \left(v_0 - \frac{\alpha}{\omega}\right) + \varepsilon \sqrt{\frac{1}{4} \left(v_0 + \frac{\alpha}{\omega}\right)^2 - \frac{L_{-z} \dot{v} \delta_X \dot{\mu}}{\tau \gamma \omega}} \quad (55)$$

It is shown in Appendix C that:

- If $\omega < 0$ then $v_{(+1)}$ is not feasible and $v_{(-1)}$ is feasible only if $B \leq \alpha v_0$.
- If $\omega \in]0, \alpha/v_0[$, hence $v_2 > v_0$, then $v_{(+1)}$ is feasible for $B/\alpha \in \mathbb{R}^+ \setminus]v_0, v_2[$ and $v_{(-1)}$ is feasible only if $B \geq \alpha v_2$.
- If $\omega \geq \alpha/v_0$, hence $v_2 \leq v_0$, then $v_{(+1)}$ is feasible for every nonnegative B and $v_{(-1)}$ is feasible only if $B \geq \alpha v_0$.

From an equilibrium speed v_ε stems the equilibrium density $k_\varepsilon \equiv \tilde{k}_0(v_\varepsilon)$:

$$k_\varepsilon = \frac{v_0 - v_\varepsilon}{\dot{v}} = \frac{1}{\dot{v}} \left[\frac{1}{2} \left(v_0 + \frac{\alpha}{\omega} \right) - \varepsilon \sqrt{\frac{1}{4} \left(v_0 + \frac{\alpha}{\omega} \right)^2 - \frac{L_{-z} \dot{v} \delta_X \dot{\mu}}{\tau \gamma \omega}} \right] \quad (56)$$

If $v_\varepsilon > \bar{v}$, then the density-driven equilibrium gives place to the policy-driven equilibrium state (\bar{v}, \bar{k}) with $\bar{k} \equiv \frac{\delta_X L_{-z} \dot{\mu}}{\tau \gamma} \frac{1}{\alpha + \omega \bar{v}}$.

Figure 2 depicts the supply and demand functions in the (\tilde{k}, v) quadrant (left side) and in the (\tilde{q}, v) one (right side). It shows that the approximate flow rate and density are very close to their exact counterpart, except for very low speeds.

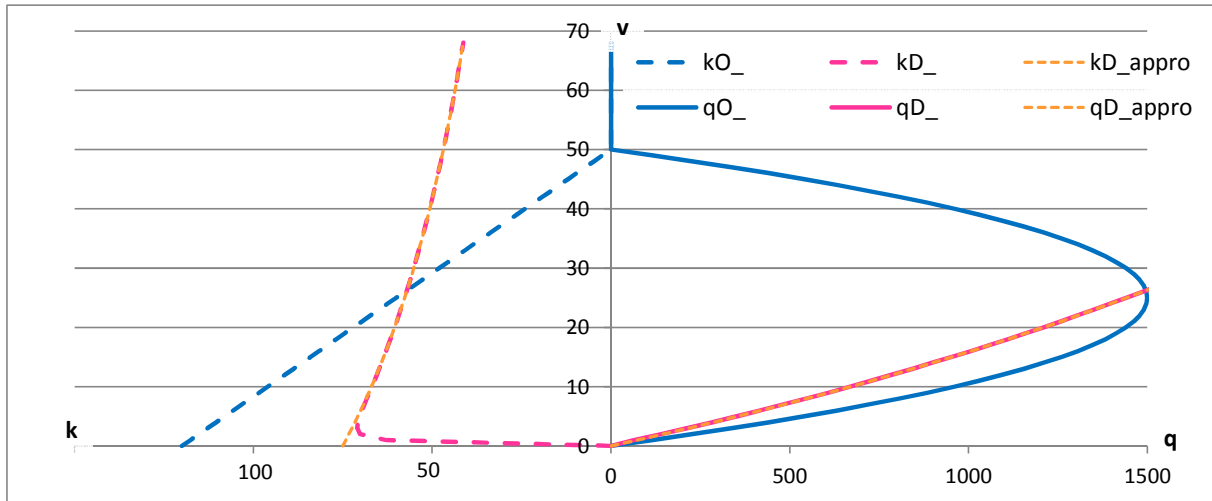


Fig. 2: Traffic equilibrium of demand and offer. Speed v on the ordinate axis determines flow rate \tilde{q} (right side) and vehicular density \tilde{k} (left side) on the abscissa axis.

Parameter values are set up as follows: $\delta_X = 4000$ p/km², $\dot{\mu} = 0.15$ trip/p.h, $L_{-z} = 0.3$ km, $L_z = 0.5$ km, $\gamma = 0.2/\text{€}$, $\omega = 0.15\text{€}/\text{km}$, $\alpha = 10\text{€}/\text{h}$, $\tau = 1.2$ p/veh, $v_0 = 50\text{km}/\text{h}$, $\dot{v} = 0.42$ km²/h.veh.

3/ Discussion

3.1/ Summary (and contribution)

The HoTer model is an abstract, theoretical model of mobility physics in an idealized territory made of identical urban blocks, edged by identical links according to either axial direction. In this setting, we modeled the formation of modal traffic flows at the link level and of parking loads at the block level by following the classical four steps of travel demand modeling. Each step of analysis was cast into a mathematical formula involving physical ratios of urban mobility:

0/ the block as a territorial unit involves human occupation of a ground area (the spatial density of human occupation is the ratio between a number of occupants of a given kind and the ground area).

1/ trip generation involves trip emission rates on a daily basis; these are ratios between a number of emitted trips and a number of people over a given stretch of time.

2/ per trip purpose, transport mode and time period, the spatial distribution of trips between blocks as origins and destinations was summarized by the average trip length according to either axial direction: a such length is a ratio between a quantity of traffic and a number of trips.

3/ the modal split model yields modal shares, i.e., ratios between the number of trips using any mode and the total number of trips.

4/ per mode and period, the assignment of traffic to network links yields link flow volumes as ratios between the traffic carried by the link and the link length, both of which according to the axial direction. Traffic assignment also involves link speeds that are ratios of traveled length and spent time, at the disaggregate level of individual trip-makers, as well as ratios between the flow rate and the spatial density of vehicles, at the aggregate level of link traffic.

Overall, the formation of traffic was modeled as a combination of physical ratios. Furthermore, by standard economic modeling of trip distribution (gravity model) and modal choice (multinomial logit discrete choice), we endogenized the average trip lengths per purpose, mode and axial direction, the modal shares and, ultimately, the modal link flow rates and speeds. We stated the issue of traffic equilibrium between travel demand and the related supply capacities at the link level. Traffic equilibrium is feasible only if the urban mobility ratios are mutually compatible and jointly consistent: the compatibility condition involves the period duration, the rates of vehicle occupancy by individual trip-makers and the occupancy rate (at the link level) of infrastructure capacity by the vehicles of the transport modes.

All in all, the HoTer model features out a threefold relation between the idealized territory and its mobility: the land-use occupation and the related generation of trips, at the local level, the territorial extension and the spatial distribution of trips, at the global level, and, back to the local level, the interaction of link flows and supply set-ups in terms of flow capacity, speed regulation etc.

3.2/ Outreach and comparison to existing methodology

The HoTer model lies at an intermediate position between three scientific fields: the four step methodology of travel demand modeling, the monocentric modeling in urban economics and the more recent field of network geometry in transportation science.

First and foremost, HoTer is deeply rooted in the four-step methodology of travel demand modeling: through the considered variables of physical or economic nature, and also the relationships between the variables, the causalities involved in the four steps, be they of physical constitution or of microeconomic behaviors of individual trip-makers. The original contribution in HoTer consists in the homogeneity postulates and the subsequent derivation of state laws linking the mobility variables. The model variables constitute statistical summaries, more precisely statistical averages over an idealized space. Related attempts in homogenous space are [27,28].

The state laws between idealized spatial averages are, in essence, specific averaging relationships. They may be called “rules of thumb” linking the mobility variables.

For instance, the trip distribution model involves the spatial complexity of origin-destination pairs in standard travel demand modeling: in HoTer the related statistical summary consists in the average trip length for either axial direction in 2D space. The related endogenization according to a gravity axiom is summarized in a state law linking average trip length to the generalized trip cost per unit length.

Thus the HoTer model constitutes an idealization of four-step TDMs: it captures the causalities and puts them as rules of thumb. While the rules ignore spatial variations, they provide summaries useful to gain insight in a quick preliminary analysis. Such insight should stand as a complement, not as a substitute, to applied studies implementing TDM sensitive to the detailed conditions of mobility demand and transportation supply, hence to the spatial variations of the mobility variables [3].

As a theoretical model of urban mobility, more precisely as an idealized representation of it, the HoTer model can be compared to the monocentric model in urban economics [29,30].

A monocentric model typically features out the residential locations of urban households around a central business district as the common workplace: home to work commuting makes the individual trip-makers spend time and cost, therefore influencing the household budget and its bidding ability on the housing market, and in turn the market prices (land rents) and the households' locations, lot sizes and housing expenses [30]. These features of land-use economics are absent from HoTer, as well as the differentiation of space along a radial axis emanating from the central workplace. The figuration of home-to-work transport and travel in the monocentric model has enabled for many economic analyses of urban mobility [30,31,32]. The HoTer model constitutes an alternative framework for such analyzes. Its specific set of assumptions enable for spatially distributed workplaces and more generally activity places, for the explicit consideration of different transport modes and their joint occupation of the roadway. Here again the respective situations of monocentric modeling and HoTer modeling would be as complements, rather than substitutes.

While the field of TDM dates back from the 1950s (Alan Voorhees [33]) and that of monocentric modeling from the 1960s (Alonzo [29], Muth [34] and Mills [35]), network geometry is a more recent field as it was pioneered in the 2000s by Van Nes [8] and Daganzo [9] and enjoyed steady expansion in the 2010s and onwards. Van Nes [8] modeled modal and multimodal transport networks with specific shape regularities by small subsets of structural parameters: he devised optimal design rules for roadway networks and also for transit infrastructures and services. Daganzo [9] modeled a street network with rectangular shape using some structural parameters: using such infrastructural grid he designed an optimal network of bus lines according to either axial direction in terms of line spatial density, station density per line and line fleet size. Subsequent works [10,11,12] addressed more elaborated spatial patterns, with special emphasis on the urban gradients of demand density and line density from center to periphery.

Thus the economics of urban transit networks has been a central topic in network geometry. In most works, travel demand is given by mode and by origin-destination pair, often with some generic relation between the O-D flow and its geometric features (cf. the $[r,s]$ shifts in the HoTer distribution model). HoTer's postulates of spatial coverage by blocks and network composition out of identical links clearly affiliate the model to the field of network geometry. Its focus on travel demand starting from block composition is original in this stream. More common is the modeling of modal split.

Another common feature is the simple modeling of the speed-density law of roadway traffic: the aggregate representation of a roadway sub-network using a single speed-density relationship is

known as a Macroscopic Fundamental Traffic Diagram (MFD, [36,37]): in HoTer the fundamental traffic diagram at the link level also applies at the global level, owing to the homogeneity postulates that transfer from micro- to macro- level.

3.3/ Limitations and research directions

Our geometric postulates of identical blocks and indefinite spatial extension, while inducing theoretical properties, obviously put strong limitations on the descriptive power of the HoTer model. However, the postulates may be relevant for sufficiently homogenous sub-territories, in which the urban fabric of population density and job density would be fairly homogenous, across a stretch of space that would be sufficiently large, i.e., with axial dimensions significantly longer than the average trip length. In such conditions we would expect the trip generation, trip distribution and modal split models to achieve some practical relevance, especially for private transport modes on roadways.

The remaining geometric postulate of identical links according to either axial dimension is even more heroic. The distinction of two axes is useful to explicitly model the two dimensions of geographical space and, potentially, to differentiate two kinds of traffic scenes at the link level. Yet, in real-world urban roadway networks there is a hierarchy of links according to traffic functions: from local access, to flow arteries and up to long-range through traffic (on highways). The distinction of at least two levels of roads is certainly a valuable direction for model development.

In the same vein of channeling person trips in large traffic flows, another area of further research consists in modeling the transit network more explicitly, along the lines of network geometry contributions [9,10,11,38].

Coming to traffic laws, here we have only considered those of running trips along road links. While this is typical of macroscopic fundamental traffic diagram modeling, our homogeneity postulates lead to identical traffic junctions, therefore enabling for explicit modeling. As a starting point, it is straightforward to derive directional flows turning from branch to branch at junctions.

A related development would be to go beyond the static representation of traffic. A bottleneck model is an obvious solution to model the dynamics of trip-making and of local flows, and also to consider departure time choice as an additional travel decision for trip-makers.

There are several other directions of economic research using HoTer, including:

- the segmentation of demand in multiple classes [39,40]: notably to differentiate people depending on their mobility levels, separating workers from non-workers, motorized from non-motorized people, among motorized people those with electric cars vs. those with thermic cars.
- impact assessment: from the economics of transport mode operations and development, to the socioeconomic impacts of demand surplus [38], accessibility and social inclusion benefits [27], to the environmental impacts of transport mode emissions of GHG, of air pollutants and noise, and up to the resulting effects on residents' health [41].
- the design of public policy packages concerning mobility and its impacts: including the provision of transport infrastructure and transit services [38], infrastructural layout to allocate road space between modes at running and at on-street parking, traffic management schemes with speed regulation and junction control, the pricing of transport modes, schemes of tradable mobility permits [42,43]...

As traffic conditions are modeled more physically in HoTer than in monocentric models, theoretical analyses of impact assessment and policy design would likely bear more resemblance to the real-world conditions of mobility in territories – except for the analysis of radial patterns from center to periphery.

The block homogeneity postulate in HoTer and the resulting property that the carried traffic balances the generated traffic at the block level are especially suitable to theoretical studies of sustainable mobility plans at the neighborhood level.

4/ Conclusion

In this article, we modeled as statistical averages over an idealized space several significant features of urban mobility, from land-use, to trip generation and trip distribution over space, to modal split and network assignment, and up to link flows and parking loads. The idealization consists in three postulates of homogeneity, namely at block level, at link level and for space coverage using blocks. We stated the relations between the model variables as analytical formulas that are rules of thumb suitable for dimensional analysis. The HoTer model, acronym for Homogenous Territory, is a theoretical construct that mirrors classical four step travel demand modeling. It is also rooted in network geometry and macroscopic fundamental traffic diagrams. It constitutes a counterpart to monocentric models of land-use economics for theoretical studies of urban mobility about multimodal traffic composition, traffic equilibrium, impact assessment and policy design.

Directions of further research include (i) the hierarchization of network links and transport modes, (ii) the economics of mobility services, (iii) demand segmentation, (iv) impact assessment, and (v) the design of policy packages.

5/ Appendices

5.1/ Appendix A: Axial run time depending on speed (complement to §2.5)

Turning to the axial run time per trip, $t_z = D_z/v_z$, it is a function of axial speed v_z as follows:

$$t_z = \frac{L_z}{v_z \sinh(\gamma g_z)} \quad (\text{A.1})$$

Wherein $g_z = L_z(\omega + \alpha/v_z)$.

Its partial derivative with respect to speed is

$$\begin{aligned} \frac{\partial t_z}{\partial v_z} &= -\frac{L_z}{v_z^2 \sinh(\gamma g_z)} - \frac{L_z}{v_z} \cdot \frac{\cosh(\gamma g_z)}{(\sinh(\gamma g_z))^2} \cdot \left(-\frac{\gamma \alpha L_z}{v_z^2}\right) \\ &= \frac{L_z}{v_z^2} \cdot \frac{\cosh(\gamma g_z)}{(\sinh(\gamma g_z))^2} \cdot \left(\frac{\gamma \alpha L_z}{v_z} - \tanh(\gamma g_z)\right) \end{aligned}$$

Its sign depends on whether $\xi \equiv \gamma \alpha L_z/v_z$ is greater or smaller than $\tanh(Y + \xi)$, with $Y \equiv \gamma \omega L_z$. As the hyperbolic tangent function keeps to values below 1, if $v_z \leq \gamma \alpha L_z$ then $\gamma \alpha L_z/v_z \geq 1$ and the partial derivative is positive and the axial time increases with v_z .

Furthermore, if $\omega \leq 0$ then for $Y + \xi \geq 0$ we have that

$$\xi - \tanh(Y + \xi) = (Y + \xi) - \tanh(Y + \xi) - Y$$

$$\begin{aligned} &\geq (Y + \xi) - \tanh(Y + \xi) \\ &\geq 0 \end{aligned}$$

Since $y \geq \tanh(y)$ when $y \geq 0$.

When $v_z > \gamma \alpha L_z$, i.e. $\xi < 1$, the condition that $\xi = \tanh(Y + \xi)$ is equivalent to

$$\tanh^{(-1)}(\xi) - \xi = Y$$

The function $\varphi: \xi \mapsto \tanh^{(-1)}(\xi) - \xi$ is nonnegative for $\xi \in [0,1[$ and it increases from 0 to $+\infty$: this ensures that there exists a single solution $\xi_Y \in [0,1[$ to the equation $\tanh^{(-1)}(\xi) - \xi = Y$ for any positive ω (figure 3): for $\xi \leq \xi_Y$ we have that $\varphi(\xi) \leq \varphi(\xi_Y) = Y$ thereby making the partial derivative positive, whereas $\xi > \xi_Y$ implies $\varphi(\xi) > \varphi(\xi_Y) = Y$ thereby making the partial derivative negative. It follows that the axial run time function increases up to $\hat{v}_z = \gamma \alpha L_z / \xi_Y$ and then decreases, making \hat{v}_z the argument of the maximum run time.

At the turning point, we have that $\tanh(\gamma g_z) = \tanh(\xi_Y + Y) = \xi_Y = \frac{\gamma \alpha L_z}{\hat{v}_z}$. Using the fact that $\sinh(y) = \tanh(y) \cosh(y)$, we get that

$$\hat{t}_z = \frac{L_z}{\hat{v}_z \tanh(\gamma g_z) \cosh(\gamma g_z)} = \frac{1}{\gamma \alpha \cosh(\gamma g_z)}$$

i.e. a formula for the maximum run time.

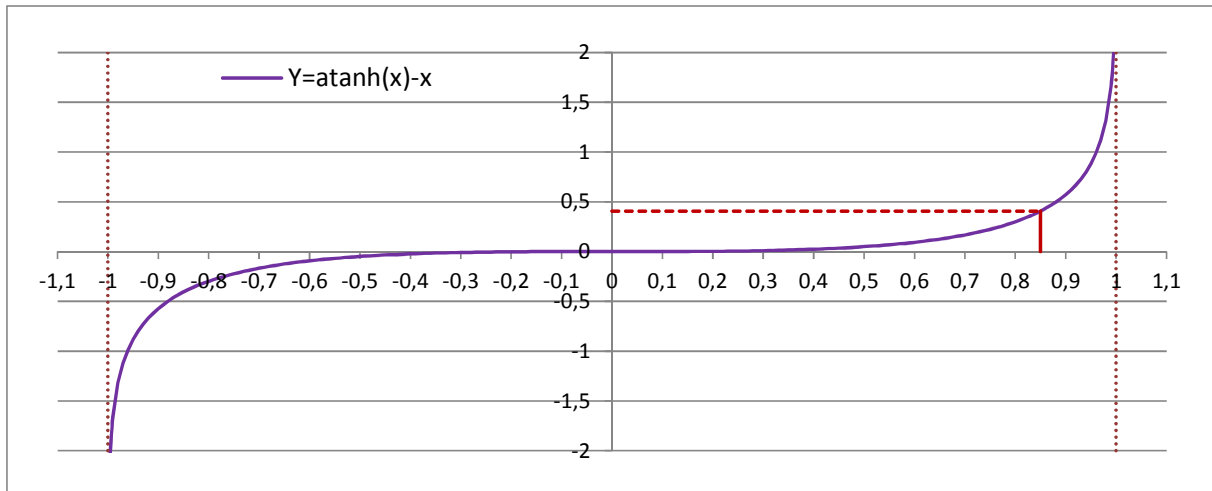


Fig. 3: Relation between the ξ and Y variables.

5.2/ Appendix B: complement to §2.6.4

Following §2.6.4, we look into the influence of modal axial speed $v_m^{(x)}$ onto the ratio of modal shares, $\pi_{m|i,h}^{[r,s]} / \pi_{\hat{m}|i,h}^{[r,s]}$ between mode m and another mode \hat{m} , on an O-D pair from current block to destination $[r, s]$. Focusing on axial direction $z = (x'x)$ without loss of generality, $v_m^{(x)}$ influences only the numerator part of the ratio in (37b), through $g_{x,1|m}$ denoted g_x . The influence of g_x is twofold in the following product form:

$$g_x \mapsto (\hat{\rho}_x)^{1-\frac{\theta}{\gamma}} (\rho_x)^{|r|} = \left(\tanh\left(\frac{1}{2}\gamma g_x\right) \right)^{1-\frac{\theta}{\gamma}} \exp(-|r|\gamma g_x) \quad (\text{B.1})$$

From the nesting order with mode choice at the upper level and destination choice at the lower one, for economic consistency it must hold that $\theta < \gamma$ [44], so that $1 - \frac{\theta}{\gamma} > 0$ and the influence of g_x on $(\hat{\rho}_x)^{1-\frac{\theta}{\gamma}}$ is increasing, whereas that on $(\rho_x)^{|r|}$ is decreasing except if $r = 0$.

Assume from now that $r \neq 0$ and denote r instead of $|r|$ for convenience. The influence of g_x on the product in (B.1) is equivalent to that of $\xi \equiv \gamma g_x$ onto the log function,

$$\xi \mapsto f(\xi) \equiv \left(1 - \frac{\theta}{\gamma}\right) \ln \tanh\left(\frac{1}{2}\xi\right) - r\xi$$

The derived function with respect to ξ is

$$\frac{df}{d\xi} = \left(1 - \frac{\theta}{\gamma}\right) \frac{1 - (\tanh\frac{1}{2}\xi)^2}{\tanh\frac{1}{2}\xi} - r\xi$$

The condition $\frac{df}{d\xi} \geq 0$ is equivalent to the condition on $y \equiv \tanh\frac{1}{2}\xi$ that

$$\frac{1}{y} - y \geq \frac{2r}{1 - \frac{\theta}{\gamma}}$$

On $[0,1]$ the function $y \mapsto \frac{1}{y} - y$ decreases from $+\infty$ to 0, hence there is a unique y_r such that

$$\frac{1}{y_r} - y_r = \frac{2r}{1 - \frac{\theta}{\gamma}}$$

For $y \in]0, y_r]$, i.e. $\xi \in]0, 2 \tanh^{(-1)} y_r]$, $\frac{df}{d\xi} \geq 0$ whereas for $\xi > 2 \tanh^{(-1)} y_r$ we have $\frac{df}{d\xi} < 0$.

To sum up, the influence of g_x onto the modal share of its mode against another mode is increasing up to $g_x = \frac{2}{\gamma} \tanh^{(-1)} y_r$ but decreasing beyond it. Accordingly, the influence of v_x onto the modal share is first decreasing then increasing.

5.3/ Appendix C: solution of approximate traffic equilibrium instance (complement to §2.9.6)

The approximate TEC stated in (52), with simpler indexation and defining $B \equiv \frac{\delta_X \dot{\mu}_{L-z} \dot{v}}{\gamma\tau}$, is put in standard form along the following steps:

$$\begin{aligned} (\alpha + \omega v)(v_0 - v) &= B \\ \omega v^2 - v(\omega v_0 - \alpha) - \alpha v_0 + B &= 0 \\ v^2 - v\left(v_0 - \frac{\alpha}{\omega}\right) - \frac{\alpha}{\omega} v_0 + \frac{B}{\omega} &= 0 \\ \left(v - \frac{1}{2}\left(v_0 - \frac{\alpha}{\omega}\right)\right)^2 - \frac{1}{4}\left(v_0 - \frac{\alpha}{\omega}\right)^2 - \frac{\alpha}{\omega} v_0 + \frac{B}{\omega} &= 0 \\ \left(v - \frac{1}{2}\left(v_0 - \frac{\alpha}{\omega}\right)\right)^2 - \frac{1}{4}\left(v_0 + \frac{\alpha}{\omega}\right)^2 + \frac{B}{\omega} &= 0 \end{aligned} \tag{C.1}$$

Generally speaking, solving a 2nd degree equation involves a twofold solution set: parameterizing it by $\varepsilon \in \{-1, +1\}$, we obtain the roots of the approximate TEC as

$$v_\varepsilon = \frac{1}{2} \left(v_0 - \frac{\alpha}{\omega} \right) + \varepsilon \sqrt{(v_2)^2 - \frac{B}{\omega}} \quad (\text{C.2})$$

But the equation was obtained not only by using the sinh approximation but also by extending the affine linear density-speed relation from segment $[0, v_0]$ to the full real line. Therefore we need to select the solutions that belong to the segment. The condition states as follows:

$$\frac{1}{2} \left(v_0 - \frac{\alpha}{\omega} \right) + \varepsilon \sqrt{\Delta} \in [0, v_0] \quad (\text{C.3})$$

$$\begin{aligned} \varepsilon \sqrt{\Delta} - \frac{\alpha}{2\omega} &\in \left[-\frac{v_0}{2}, \frac{v_0}{2} \right] \\ \left| \varepsilon \sqrt{\Delta} - \frac{\alpha}{2\omega} \right| &\leq \frac{v_0}{2} \end{aligned} \quad (\text{C.4})$$

$$\Delta + \left(\frac{\alpha}{2\omega} \right)^2 - \frac{\alpha}{\omega} \varepsilon \sqrt{\Delta} \leq \left(\frac{v_0}{2} \right)^2$$

Recalling that $\Delta = (v_2)^2 - B/\omega$ and $(v_2)^2 = \frac{1}{4}v_0^2 + \frac{1}{2}v_0 \frac{\alpha}{\omega} + \frac{1}{4} \left(\frac{\alpha}{\omega} \right)^2$, the condition is equivalent to

$$\begin{aligned} \frac{1}{2}v_0 \frac{\alpha}{\omega} + \frac{1}{2} \left(\frac{\alpha}{\omega} \right)^2 - \frac{B}{\omega} &\leq \frac{\alpha}{\omega} \varepsilon \sqrt{\Delta} \\ \frac{\alpha}{\omega} v_2 - \frac{B}{\omega} &\leq \frac{\alpha}{\omega} \varepsilon \sqrt{\Delta} \end{aligned} \quad (\text{C.5})$$

To go further, we need to specify whether $\omega > 0$ or $\omega < 0$.

When $\omega > 0$, dividing both sides of (C.5) by α/ω , we get that

$$v_2 - \frac{B}{\alpha} \leq \varepsilon \sqrt{\Delta} \quad (\text{C.6})$$

Consider for now $\varepsilon = +1$. Then (C.6) holds true if $B > \alpha v_2$ that yields a negative LHS which is less than a nonnegative RHS. Otherwise, if $B \leq \alpha v_2$, by squaring both sides and dividing by B/α , we get

$$\begin{aligned} B \leq \alpha v_2 \text{ and } (v_2)^2 - 2v_2 \frac{B}{\alpha} + \left(\frac{B}{\alpha} \right)^2 &\leq (v_2)^2 - \frac{B}{\omega} \\ B \leq \alpha v_2 \text{ and } \frac{B}{\alpha} \leq 2v_2 - \frac{\alpha}{\omega} = v_0 \\ B &\leq \alpha \min\{v_2, v_0\} \end{aligned} \quad (\text{C.7})$$

If $\alpha \leq \omega v_0$ then $v_2 \leq v_0$ and (C.3) holds true both for $B \geq \alpha v_2$ and for $B \leq \alpha v_2$, i.e., everywhere. If $\alpha > \omega v_0$ then $v_2 > v_0$ and the admissible set of B/α is $\mathbb{R}^+ \setminus]v_0, v_2[$.

Coming to $\varepsilon = -1$, (C.6) requires that

$$v_2 + \sqrt{\Delta} \leq \frac{B}{\alpha} \quad (\text{C.8})$$

$$\frac{B}{\alpha} - v_2 \geq \sqrt{\Delta}$$

$$\begin{aligned}
(v_2)^2 - 2v_2 \frac{B}{\alpha} + \left(\frac{B}{\alpha}\right)^2 &\geq (v_2)^2 - \frac{B}{\omega} \\
\frac{B}{\alpha} &\geq 2v_2 - \frac{\alpha}{\omega} = v_0
\end{aligned} \tag{C.9}$$

It is thus required that $B \geq \alpha \max\{v_2, v_0\}$.

To sum up the case where $\omega > 0$:

- If $B \leq \alpha \min\{v_2, v_0\}$ then $v_{(+1)}$ is feasible,
- If $B \geq \alpha \max\{v_2, v_0\}$ then both $v_{(+1)}$ and $v_{(-1)}$ are feasible,
- If $v_2 \leq v_0$ then $v_{(+1)}$ is feasible also for $B/\alpha \in]v_2, v_0[$, making $v_{(+1)}$ feasible for all real nonnegative values of B .

For completeness, let us address also the case where $\omega < 0$, as this may depict professional purposes with some private gain for the trip-maker to travel on greater lengths. Then, (C.5) gives rise to

$$v_2 - \frac{B}{\alpha} \geq \varepsilon \sqrt{\Delta} \tag{C.10}$$

If $\varepsilon = +1$, then (C.10) requires that

$$\begin{aligned}
B \leq \alpha v_2 \quad \text{and} \quad (v_2)^2 - 2v_2 \frac{B}{\alpha} + \left(\frac{B}{\alpha}\right)^2 &\geq (v_2)^2 - \frac{B}{\omega} \\
B \leq \alpha v_2 \quad \text{and} \quad \frac{B}{\alpha} &\geq 2v_2 - \frac{\alpha}{\omega} = v_0
\end{aligned}$$

The two partial conditions contradict one another since $v_2 < v_0$ as $\omega < 0$. Thus $v_{(+1)}$ is not feasible under $\omega < 0$.

If $\varepsilon = -1$, then (C.8) is equivalent to

$$v_2 + \sqrt{\Delta} \geq \frac{B}{\alpha}$$

Which sets up an upper bound on B and gives rise to

$$\sqrt{\Delta} \geq \frac{B}{\alpha} - v_2$$

Thus, (C.8) holds true if $B \leq \alpha v_2$. If $B \geq \alpha v_2$, then we can square both sides of the inequality to obtain

$$\begin{aligned}
(v_2)^2 - \frac{B}{\omega} &\geq (v_2)^2 - 2v_2 \frac{B}{\alpha} + \left(\frac{B}{\alpha}\right)^2 \\
\frac{B}{\alpha} &\leq 2v_2 - \frac{\omega}{\alpha} = v_0
\end{aligned}$$

It means that the values of B/α between v_2 and v_0 yield feasible $v_{(-1)}$, too. Thus the solution $v_{(-1)}$ is feasible under $\omega < 0$ as long as $B \leq \alpha v_0$.

To conclude the discussion of the solutions to (C.3) for all nonzero values of ω , let us bring together the results of the two cases $\omega < 0$ and $\omega > 0$:

- If $\omega < 0$ then $v_{(+1)}$ is not feasible and $v_{(-1)}$ is feasible only if $B \leq \alpha v_0$.
- If $\omega \in]0, \alpha/v_0[$, hence $v_2 > v_0$, then $v_{(+1)}$ is feasible for $B/\alpha \in \mathbb{R}^+ \setminus]v_0, v_2[$ and $v_{(-1)}$ is feasible only if $B \geq \alpha v_2$.
- If $\omega \geq \alpha/v_0$, hence $v_2 \leq v_0$, then $v_{(+1)}$ is feasible for every nonnegative B and $v_{(-1)}$ is feasible only if $B \geq \alpha v_0$.

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