



**HAL**  
open science

## Normalization errors in CHSH Bell inequalities

Gerrit Coddens

► **To cite this version:**

| Gerrit Coddens. Normalization errors in CHSH Bell inequalities. 2022. hal-03804940v2

**HAL Id: hal-03804940**

**<https://hal.science/hal-03804940v2>**

Preprint submitted on 10 Oct 2022 (v2), last revised 30 Oct 2022 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Normalization errors in CHSH Bell inequalities

Gerrit Coddens (a)

Laboratoire des Solides Irradiés,  
Institut Polytechnique de Paris, UMR 7642, CNRS-CEA-Ecole Polytechnique,  
28, Route de Saclay, F-91128-Palaiseau CEDEX, France

(a) *retired research physicist of LSI*

7th October 2022

**Abstract.** We show why a type of CHSH Bell-type inequality cannot be applied in the usual way to the experiments it has been designed for as this introduces a normalization error. The traditional normalization of the probabilities is based on a confusion between two different definitions of probability, with the effect of erroneously assuming that they are identical.

**PACS.** 03.65.Ta, 03.65.Ud, 03.67.-a

We want to point out a normalization error in the application of CHSH Bell inequalities to the experiments for which they have been designed, like e.g. the famous experiments carried out by Aspect *et al.* [1, 2]. We will not dwell here on the derivation of the inequality or on further experiments with improved set-ups that have been carried out after these first experiments to rule out experimental loopholes.

Probabilities are defined as the number of desired outcomes divided by the total number of all possible outcomes. Let us call the set of all possible events or outcomes the event space. To teach probabilities one often draws the event space in the form of a probability tree, consisting of points and branches connecting these points. The two end points of a branch  $PQ$  on such a tree correspond to probabilities  $p(P)$  and  $p(Q)$ . We assume that  $P$  is here the lower point of the branch where it sprouts from the tree. The branch itself corresponds to the conditional probability  $p(Q|P)$  and we calculate  $p(Q) = p(Q|P)p(P)$  in travelling upwards along the tree. In Fig. 1 we show the tree for the part:

$$p(a_1 \wedge b_1) - p(a_1 \wedge b_2) - p(a_2 \wedge b_1) - p(a_2 \wedge b_2), \quad (1)$$

which occurs within the CHSH Bell inequality:

$$0 \leq p(a_1 \wedge b_1) - p(a_1 \wedge b_2) - p(a_2 \wedge b_1) - p(a_2 \wedge b_2) + p(a_2) + p(b_2) \leq 1, \quad (2)$$

for the case that the polarizers deliver answers 1 for the transmission and 0 for the absorption of photons. There are other types of polarizers which deliver answers +1 and -1. Both types have been used by the teams led by Aspect. We focus here on the type that yields answers 0 or 1. We assume that the set-ups of these experiments are well-known.

The polarizer  $A$  in the left arm of the set-up can be in the settings  $A_1$  (angle  $\alpha_1$ ) or  $A_2$  (angle  $\alpha_2$ ). The polarizer  $B$  in the right arm of the set-up can be in the settings  $B_1$  (angle  $\beta_1$ ) or  $B_2$  (angle  $\beta_2$ ). In Eqs. 1-2  $a_j$  stands for the event that polarizer  $A$  in setting  $A_j$  yields the answer 1 for a photon. The same holds *mutatis mutandis* for  $b_k$ . We have in total four possible configurations for the combined settings of the two polarizers. We can select the angles  $\alpha_j, j \in \{1, 2\}$  and  $\beta_k, k \in \{1, 2\}$  with random generators in such a way that both settings  $A_j$  occur with equal probability 1/2, with the same *mutatis mutandis* for the settings  $B_k$ . And if the two random generators are statistically entirely independent the four combined settings  $(A_j, B_k)$  will each occur with probability 1/4.

The probabilities  $p(a_j)$  for transmission by filter  $A$  in settings  $A_j$  are both 1/2. The probabilities  $p(b_k)$  for absorption by filter  $B$  in settings  $B_k$  are also both 1/2. The conditional probabilities  $p(b_k|a_j)$  for transmission by filter  $B$  provided there is transmission by filter  $A$  is  $p(b_k|a_j) = \cos^2(\alpha_j - \beta_k)$ . We have further also  $p(\neg b_k|\neg a_j) = \cos^2(\alpha_j - \beta_k)$ ,  $p(\neg b_k|a_j) = \sin^2(\alpha_j - \beta_k)$ ,  $p(b_k|\neg a_j) = \sin^2(\alpha_j - \beta_k)$ . The quantities  $p(a_j \wedge b_k)$  in Eqs. 1-2 are the probabilities for joint transmission  $p(b_k|a_j)p(a_j) = \frac{1}{2} \cos^2(\alpha_j - \beta_k)$  in the configuration  $(A_j, B_k)$ . Hence, in the setting  $(A_j, B_k)$  the probability for simultaneous transmission is  $\frac{1}{2} \cos^2(\alpha_j - \beta_k)$ . The event space for such a single setting  $(A_j, B_k)$  is illustrated in Fig. 1 for the case  $(A_1, B_2)$ . It is symbolized by the rectangle in dashed lines. This rectangle corresponds

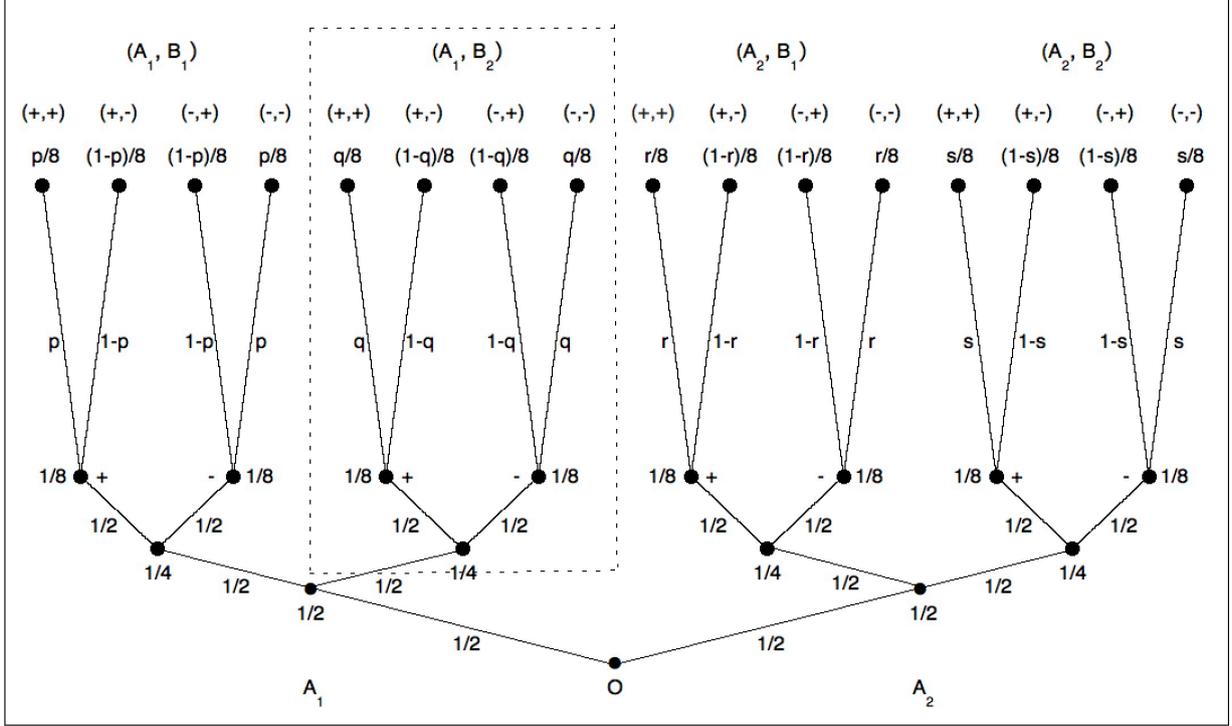


Fig. 1: Probability tree in the event space for the quantity in Eq. 1 which occurs in the CHSH Bell inequality in Eq. 2. The point  $O$  is the root of the tree. Two branches are starting from  $O$ . They represent the possibilities  $A_1$  and  $A_2$  for the setting of the filter  $A$  (in the left arm of the set-up). The end points of these branches are both the starting point for branches corresponding to the possibilities  $B_1$  and  $B_2$  for the setting of the filter  $B$  (in the right arm of the set-up). The four combined settings  $(A_j, B_k)$  occur each with the frequency  $1/4$ . The higher parts of the tree represent the possible outcomes for the photon correlation experiments in the four settings. In order not to burden the figure we have introduced the notations  $p = \cos^2(\alpha_1 - \beta_1)$ ,  $q = \cos^2(\alpha_1 - \beta_2)$ ,  $r = \cos^2(\alpha_2 - \beta_1)$ ,  $s = \cos^2(\alpha_2 - \beta_2)$ . The rectangle drawn in dashed lines represents the event space for the setting  $(A_1, B_2)$ . In this setting the other points  $(A_j, B_k)$  and their related events are not elements of physical reality, because a polarizer cannot assume two orientations at the same time. The four points only exist simultaneously in our mind and in the event space for the Bell inequality. The complete figure describes exactly the event space for the Bell inequality as explained in the main text. It shows that the experimental probabilities are normalized with respect to a smaller event space than the probabilities that occur in the Bell inequality, such that they must be normalized differently. The experimental probability is  $q/2$ , while the probability to be inserted into the Bell inequality is  $q/8$  (see main text).

to the event space for the real experiment in the setting  $(A_1, B_2)$ . The total event space for the Bell inequality must contain all four set-ups  $(A_j, B_k)$  and corresponds therefore to the whole figure (see below). In fact, this is compulsory because the inequality contains all four probabilities  $\frac{1}{2} \cos^2(\alpha_j - \beta_k)$  simultaneously and therefore the event space for the Bell inequality must allow for defining all these probabilities simultaneously. In real life the four settings  $(A_j, B_k)$  cannot exist simultaneously.

We see thus that the event space needed to derive the inequality is larger than the individual event spaces in the four individual real-life experiments. This implies that the total number of outcomes in the event space we use to normalize the probabilities in real-life single set-ups is different from the total number of outcomes in the theoretical event space on which the Bell inequality is being proved.

The probabilities in the Bell inequality must therefore be normalized differently in the event space for the inequality than the probabilities in the event spaces for isolated experiments. Consequently, according to Fig. 1, the probabilities

to be fed into the Bell inequality must be attributed the values  $\frac{1}{8} \cos^2(\alpha_j - \beta_k)$  rather than  $\frac{1}{2} \cos^2(\alpha_j - \beta_k) = \frac{1}{8} \cos^2(\alpha_j - \beta_k)/(1/4)$ , where  $1/4$  is the probability for selecting the configuration  $(A_j, B_k)$ . In the Bell inequality, the real-world probability becomes a conditional probability, whereby the condition is the selection of the configuration  $(A_j, B_k)$ . The violation of the Bell inequality in Eq. 2 appears this way as due to a normalization error in the definitions of the quantities which occur in Eq. 1. With the correct normalization the inequality is not violated. We could rephrase this by stating that there has been a confusion between two different definitions of probability, by tacitly assuming that they were identical.

The error becomes most obvious by noticing that the sum of all probabilities in the top line in Fig. 1 is 1 as it should be, while following the values used by Aspect *et al.* it would be 4. One may object that our drawing constitutes a leading argument and that the probabilities might be conceived as corresponding to overlapping realities, such that they should not be juxtaposed and summed as suggested by Fig. 1. However, this is not true. Let  $S_{jk}$  be the set of hidden variables needed to describe the configuration  $(A_j, B_k)$ . The set  $S_{jk}$  contains then couples of numbers  $(\lambda_j, \mu_k)$  where  $\lambda_j$  are hidden variables for filter  $A$  in its setting  $A_j$  and  $\mu_k$  are hidden variables for filter  $B$  in its setting  $B_k$ . This shows that the sets  $S_{jk}$  are disjoint because their elements  $(\lambda_j, \mu_k)$  just have different meanings (even if there might be numerical coincidences), such that the representation in Fig. 1 is correct [3]. Furthermore Fig. 1 also represents the correct measurement protocol. The photons and their parameters intervening in the four top branches of the tree can be considered to be different in each branch as in the real measurement protocol or identical in all four branches as assumed in the derivation of the Bell inequality [3]. The difference between the two choices will not change the structure of the tree. It will also not change the accurately measured probabilities, as explained by Larsson [4]. Finally, each argument stipulating that one should reason differently than we have done in the construction of the tree, could be rebutted by invoking the violation of the inequality that might result from it as a proof that the alternative reasoning is wrong. Our approach differs from the traditional approach by considering also hidden variables in the set-up, which is unusual but still open to classical understanding. How this can work out without running into contradictions using a concept of macroscopic “fields” is discussed in Remark 1 of [3], which is a much more detailed version of the present paper.

Everybody knows that probability calculus is full of pitfalls, as illustrated by the ferocious polemics that have historically surrounded the correct solution of the Monty Hall problem by Marilyn Vos Savant. Also Bertrand’s paradox illustrates this. It is therefore not recommended practice to question special relativity on the basis of some results obtained by probability calculus. That might be considered as a rather heedless attack on special relativity. The weak element in such a confrontation is *a priori* not the theory of relativity but the probability calculus. Furthermore, if there were really a flaw in special relativity then it must be replaced by something else and the existence of entanglement could imply such a flaw. It could e.g. signal a different connectivity for space-time than assumed or faster-than-light transfer of information. Not many persons seem to bother about resolving such issues of all-out importance. It looks as though it is considered that blatant contradictions in the foundations of physics are not very much of a problem. For the rebuttal of other experiments which claim that the existence of entanglement has been proved, other approaches are needed [3].

## References

1. Aspect, A.; Dalibard, J.; and Roger, G.; Experimental Test of Bell’s Inequalities Using Time-Varying Analyzers. *Phys. Rev. Lett.* **49**, 1804 (1982).
2. Aspect, A.; Grangier, P.; Roger, G.; Experimental Realization of Einstein-Podolsky-Rosen-Bohm Gedankenexperiment: A New Violation of Bell’s Inequalities, *Phys. Rev. Lett.* **49**, 91 (1982).
3. Coddens, G.; Inadequacy of the CHSH Bell inequality for an application to its associated photon correlation experiments, <https://hal.archives-ouvertes.fr/hal-03482574v6>.
4. Larsson, J.-A.; Loopholes in Bell inequality test of local realism. *J. Phys. A* **47**, 424003 (2014).