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# Well-proportioned universes suppress the cosmic microwave background quadrupole

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## ABSTRACT

A widespread myth asserts that all small universe models suppress the cosmic microwave background (CMB) quadrupole. In actual fact, some models suppress the quadrupole while others elevate it, according to whether their low-order modes are weak or strong relative to their high-order modes. Elementary geometrical reasoning shows that a model's largest dimension determines the rough value  $\ell_{\min}$  at which the CMB power spectrum  $\ell(\ell + 1) C_\ell/2\pi$  effectively begins; for cosmologically relevant models,  $\ell_{\min} \leq 3$ . More surprisingly, elementary geometrical reasoning shows that further reduction of a model's smaller dimensions – with its largest dimension held fixed – serves to elevate modes in the neighbourhood of  $\ell_{\min}$  relative to the high- $\ell$  portion of the spectrum, rather than suppressing them as one might naively expect. Thus among the models whose largest dimension is comparable to or less than the horizon diameter, the low-order  $C_\ell$  tend to be relatively weak in well-proportioned spaces (spaces whose dimensions are approximately equal in all directions) but relatively strong in oddly proportioned spaces (spaces that are significantly longer in some directions and shorter in others). We illustrate this principle in detail for the special cases of rectangular 3-tori and spherical spaces. We conclude that well-proportioned spaces make the best candidates for a topological explanation of the low CMB quadrupole observed by *COBE* and *WMAP*.

**Key words:** cosmic microwave background – cosmology: theory.

## 1 INTRODUCTION

First-year data from *WMAP* (Bennett et al. 2003) confirm the low cosmic microwave background (CMB) quadrupole and octopole observed earlier by *COBE* (Hinshaw et al. 1996). One possible explanation for the striking lack of large-scale power is that the universe might not be big enough to support long-wavelength fluctuations (Ellis 1971, 1975). That is, the universe may have constant curvature but with a non-trivial global topology and a finite volume. Over the past decade many researchers have investigated this hypothesis, initially in the context of flat space and a vanishing cosmological constant (Stevens, Scott & Silk 1993; de Oliveira-Costa & Smoot 1995; Levin, Scannapieco & Silk 1998) but later in the more general case (for a review see Levin 2002).

A myth has arisen that non-trivial topology invariably suppresses the quadrupole and other low-order modes. In the present article we show this myth to be false. While some non-trivial topologies do

indeed suppress the quadrupole, others do not. In fact some non-trivial topologies elevate the quadrupole. As a general principle, among spaces whose dimensions are roughly comparable to the horizon radius  $R_{\text{LSS}}$ , we find that *well-proportioned* spaces suppress the quadrupole (a well-proportioned space being one whose three dimensions are of similar magnitudes) while *oddly proportioned* spaces (with one dimension much larger or smaller than the others) elevate the quadrupole. We illustrate this principle in the case of flat 3-tori (where roughly cubical tori suppress the quadrupole, while highly oblate or prolate tori elevate it) and the case of spherical manifolds (where the binary polyhedral spaces suppress the quadrupole, while typical lens spaces elevate it).

## 2 MODE SUPPRESSION: FACT AND FALLACY

First the fallacy. We start with a simply connected space  $X$  (typically  $X$  is the 3-sphere  $S^3$ , Euclidean space  $E^3$  or hyperbolic 3-space  $H^3$ ). We construct a closed manifold  $M = X/\Gamma$  by taking the quotient of  $X$  under the action of a group  $\Gamma$ . For the sake of discussion, we assume that the quotient space  $M$  is fairly small, say smaller than the last scattering surface. Because  $M$  is small, the low-order

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(long-wavelength) eigenmodes of the Laplacian are suppressed. So, according to the myth, the CMB quadrupole (and perhaps also the octopole and other low- $\ell$  CMB multipoles) are suppressed.

Now the fact. The low-order modes of the Laplacian are indeed suppressed. Where the myth goes wrong is in neglecting the fact that high-order modes are also suppressed. According to the Weyl asymptotic formula (Weyl 1911), for sufficiently large wavenumbers  $k$ , the number of modes up through  $k$  is roughly proportional to the volume of  $M$ . The smaller the manifold  $M$ , the fewer modes it supports. The question then becomes: Are the low-order modes suppressed more or less severely than the high-order modes? If the low-order modes are suppressed more severely than the high-order ones, the CMB quadrupole will be suppressed relative to the rest of the power spectrum. If the low-order modes are suppressed less severely, the CMB quadrupole will be elevated relative to the rest of the power spectrum. Sections 3 and 4 will show that well-proportioned spaces suppress the low-order modes more severely, while oddly proportioned spaces suppress them less severely.

The reader might still feel uneasy. After all, if passing from  $X$  to the quotient space  $M$  only removes modes – never adds them – how can the quadrupole possibly get elevated? The answer is that the *relative* strength of the modes, not their absolute strength, determines the shape of the power spectrum. As an analogy, imagine reducing all prices in the world by a factor of 10 while simultaneously reducing all wages, all savings and all debts by the same factor of 10. Obviously nothing has changed and life proceeds as before. However, if in that same scenario your own salary is merely halved, then you are five times wealthier than before in spite of your reduced pay. The same situation occurs with oddly proportioned spaces: the low-order modes get suppressed but the high-order modes get suppressed even more, so in a relative sense the low-order modes come out stronger.

### 3 TORI

Consider a 3-torus made from a rectangular box of width  $L_x$ , length  $L_y$  and height  $L_z$ . An orthonormal basis for its space of eigenmodes takes the form of planar waves

$$\Upsilon_{\mathbf{k}}(\mathbf{x}) = e^{2\pi i \mathbf{k} \cdot \mathbf{x}} \quad (1)$$

for wavevectors

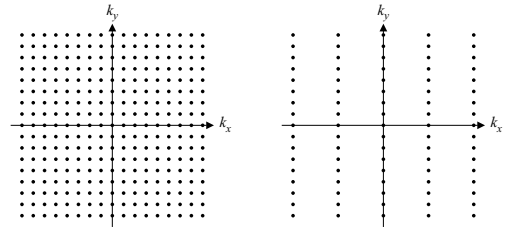
$$\mathbf{k} = \left( \frac{n_x}{L_x}, \frac{n_y}{L_y}, \frac{n_z}{L_z} \right), \quad (2)$$

where  $n_x$ ,  $n_y$  and  $n_z$  are integers (see Riazuelo et al. 2004 for full details).

#### 3.1 Cubic 3-torus

First consider a cubic 3-torus of size  $L_x = L_y = L_z = L$ . The allowable wavevectors (2) form a cubic lattice in the dual space (Fig. 1, left). The modulus  $k = |\mathbf{k}|$  of a wavevector  $\mathbf{k}$  is inversely proportional to the wavelength. For the cubic 3-torus, the shortest wavevectors, corresponding to the longest wavelength, are the six vectors  $\mathbf{k} = (\pm 1/L, 0, 0)$ ,  $(0, \pm 1/L, 0)$  and  $(0, 0, \pm 1/L)$ , each of modulus  $1/L$ . For larger moduli  $k$ , corresponding to shorter wavelengths, the number of modes of modulus  $k \leq k_{\max}$  is simply the number of lattice points within a ball of radius  $k_{\max}$ , which grows in rough proportion to the ball's volume  $\frac{4}{3}\pi k_{\max}^3$ .

This cubic 3-torus will serve as the standard to which other 3-tori will be compared in Sections 3.2–3.5. We assume that this *reference 3-torus* has a fixed size  $L$  slightly larger than the diameter of



**Figure 1.** In a cubic 3-torus the allowed wavevectors  $\mathbf{k}$  form a cubic lattice (left). Shrinking the width of the torus by a factor  $L_x/L$  stretches the lattice in the  $k_x$  direction by the inverse factor  $L/L_x$  (right). Even though the average mode density drops by a factor of  $L_x/L$ , the wavevectors in the  $k_y k_z$  plane remain fixed. For ease of illustration the  $k_z$  coordinate is not shown.

the last scattering surface. The following sections investigate how shrinking its dimensions suppresses its modes. Section 3.2 shrinks all dimensions equally. Section 3.3 shrinks one dimension while leaving the remaining two fixed. Section 3.4 shrinks two dimensions while leaving one fixed. Finally, Section 3.5 considers the general case of shrinking all dimensions, but by different amounts.

#### 3.2 Small cubic 3-torus

We start with the reference 3-torus of size  $L$  (Section 3.1) and shrink all its dimensions  $L_x$ ,  $L_y$  and  $L_z$  simultaneously from their original length  $L$  to some new length  $L' < L$ . As the 3-torus shrinks, the modulus of the lowest modes increases from  $k = 1/L$  to  $k' = 1/L'$ . Indeed the moduli of all modes increase by the same factor  $L/L'$ , so the *relative* shape of the mode spectrum remains unchanged, even though that spectrum now occurs for shorter wavelengths (larger  $k$ ).

If the original size  $L$  and the final size  $L'$  were both significantly greater than the diameter of the last scattering surface, then we would have no reason to expect any particular effect on the CMB quadrupole. However, in the cosmologically interesting case that  $L$  is slightly larger than the last scattering surface while  $L'$  is significantly smaller, then, even though the former lowest mode at  $k = 1/L$  would have contributed strongly to the CMB quadrupole  $C_2$ , the new lowest mode at  $k' = 1/L'$  does not, and so the quadrupole is suppressed. If we were to continue shrinking the 3-torus, we would eventually lose support for the octopole  $C_3$ , then  $C_4$  and so on.

Nevertheless, we emphasize that for the cosmologically interesting case that  $L'$  is comparable to half the diameter of the last scattering surface, the quadrupole remains significant even though it is suppressed relative to the reference 3-torus. This weak but non-trivial quadrupole will play a role in our analysis of the general case in Section 3.5.

#### 3.3 Oblate 3-torus

To construct an oblate 3-torus, shrink the width  $L_x$  of the 3-torus while leaving its length and height constant at  $L_y = L_z = L$ . Shrinking the 3-torus by a factor of  $L_x/L$  stretches the lattice of allowable wavevectors (2) in the dual space (Fig. 1, right) by the inverse ratio  $L/L_x$ . The lowest modes still have modulus  $k = 1/L$  as in the reference 3-torus (Section 3.1), but their multiplicity has dropped from 6 to 4, because now only the four wavevectors  $\mathbf{k} = (0, \pm 1/L, 0)$  and  $(0, 0, \pm 1/L)$  retain that modulus. The other two wavevectors that shared that modulus for the cubic 3-torus, namely  $(\pm 1/L, 0, 0)$ , have been stretched out to  $(\pm 1/L_x, 0, 0)$  for the oblate 3-torus.

For generic modes ( $k \gg 1/L_x$ ), the mode density (defined as the number of modes per unit volume in  $k_x k_y k_z$  space) drops by a factor

of  $L_x/L$ . In other words, the mode density for the oblate 3-torus (Fig. 1, right) is only  $L_x/L$  times that of the cubic reference 3-torus (Fig. 1, left). Visually, the mode density corresponds to the density of the dots in Fig. 1. The crucial point here is that, unlike the lowest modes, which depend strongly on the discreteness of the lattice, the overall strength of the high-order modes ( $k \gg 1/L_x$ ) depends only on the mode density.

Comparing the results of the preceding two paragraphs, we see that the multiplicity of the lowest-order mode  $k = 1/L$  has dropped by a constant factor  $\frac{4}{6}$  (independent of  $L_x!$ ), while the overall density of modes has dropped by  $L_x/L$ . Thus the *relative* strength of the lowest mode, compared to the overall spectrum, is

$$\frac{4/6}{L_x/L} = \frac{2L}{3L_x}.$$

As  $L_x$  gets small, the ratio  $2L/3L_x$  gets arbitrarily large, thus elevating the relative importance of the lowest mode. Similar reasoning applies to the other low-order modes.

### 3.4 Prolate 3-torus

To construct a prolate 3-torus, shrink the width  $L_x$  and the length  $L_y$  of the 3-torus while leaving its height constant at  $L_z = L$ . This stretches the lattice of allowable wavevectors (2) in the dual space by a factor of  $L/L_x$  in the  $x$  direction and by a factor of  $L/L_y$  in the  $y$  direction, for a total expansion of  $L^2/(L_xL_y)$ . The lowest modes still have modulus  $k = 1/L$  as in the reference 3-torus (Section 3.1) and the oblate 3-torus (Section 3.3), but their multiplicity has dropped to 2, because now only the two wavevectors  $\mathbf{k} = (0, 0, \pm 1/L)$  retain that modulus.

For generic modes [ $k \gg \max(1/L_x, 1/L_y)$ ], the mode density drops by a factor of  $L_xL_y/L^2$ . In other words, as we pass from the cubic reference 3-torus to the prolate 3-torus, the lattice now stretches in both the  $k_x$  and  $k_y$  directions in  $k_xk_yk_z$  space, and so the mode density (the ‘dot density’ in Fig. 1) drops by

$$\frac{L_x/L_y}{L/L} = \frac{L_xL_y}{L^2}.$$

Comparing the results of the preceding two paragraphs, we see that the multiplicity of the lowest-order mode  $k = 1/L$  has dropped by a constant factor  $\frac{2}{6}$  while the overall density of modes has dropped by  $L_xL_y/L^2$ . Thus the relative strength of the lowest mode, compared to the overall spectrum, is

$$\frac{2/6}{L_xL_y/L^2} = \frac{L^2}{3L_xL_y}.$$

As  $L_x$  and  $L_y$  get small, the ratio  $L^2/(3L_xL_y)$  gets arbitrarily large, thus elevating the relative importance of the low-order modes.

### 3.5 Generic small 3-torus

Consider the case of a general rectangular 3-torus of dimensions  $L_x, L_y$  and  $L_z$ . For cosmological interest, assume that all dimensions are comparable to or smaller than the diameter of the last scattering surface, and in particular less than the size of the reference 3-torus, that is  $L_x, L_y, L_z < L$ .

To understand this general 3-torus, imagine passing from the reference 3-torus of size  $(L, L, L)$  to the general 3-torus of size  $(L_x, L_y, L_z)$  in two steps: first shrink isotropically from size  $(L, L, L)$  to size  $(L_{\max}, L_{\max}, L_{\max})$ , where  $L_{\max} = \max\{L_x, L_y, L_z\}$ ; then shrink anisotropically from  $(L_{\max}, L_{\max}, L_{\max})$  to  $(L_x, L_y, L_z)$ .

The first (isotropic) shrinking takes the reference 3-torus  $(L, L, L)$  to a small cubic 3-torus  $(L_{\max}, L_{\max}, L_{\max})$ . Such shrinking suppresses the quadrupole, but if the shrinking is not too severe – say  $L_{\max}$  remains comparable to half the horizon radius – the quadrupole remains significant (Section 3.2). The value of  $L_{\max}$  fixes the modulus  $k_{\min} = 1/L_{\max}$  of the lowest modes of the torus, which in turn fix the value  $\ell_{\min}$  at which the CMB power spectrum effectively begins.

The second (anisotropic) shrinking takes the small cubic 3-torus  $(L_{\max}, L_{\max}, L_{\max})$  to the general 3-torus  $(L_x, L_y, L_z)$ . The largest dimension of the torus remains fixed at  $L_{\max}$ , so the modulus of the lowest mode remains at  $k_{\min}$ , as defined in the previous paragraph, while its multiplicity drops by at most a factor of 3 (Section 3.4). As the remaining dimensions of the 3-torus shrink, the high- $k$  (short-wavelength) part of the mode spectrum is suppressed in proportion to the volume of the 3-torus (Sections 3.3 and 3.4). Thus, whenever this anisotropic shrinking decreases the volume of the 3-torus by more than a factor of 3, we can expect the *relative* strength of the lowest modes to increase relative to the high modes. Note, though, that the CMB power spectrum still effectively begins at the same  $\ell_{\min}$  as before.

In summary, the first (isotropic) shrinking pushes the entire mode spectrum uniformly towards shorter wavelengths, moving the lowest modes towards higher  $k$ . The second (anisotropic) shrinking then holds the (new) lowest modes fixed (with at most a factor of 3 decrease in multiplicity) while it further suppresses all short-wavelength modes, in proportion to the volume of the 3-torus. If the first (isotropic) shrinking is mild while the second (anisotropic) shrinking is severe (volume changes by more than a factor of 3), then the latter effect trumps the former, and the lowest CMB multipoles ( $\ell \sim \ell_{\min}$ ) are elevated relative to their higher- $\ell$  neighbours in the Sachs–Wolfe plateau.

The preceding geometrical arguments constitute a complete and rigorous proof that the ordinary Sachs–Wolfe component of the CMB behaves as claimed. Nevertheless, readers may be interested to know that careful CMB simulations, taking into account the integrated Sachs–Wolfe and Doppler components as well as the ordinary Sachs–Wolfe component, confirm these conclusions (Riazuelo et al. 2004).

## 4 SPHERICAL SPACES

The 3-sphere  $S^3$  supports a discrete set of modes with eigenvalues  $-k(k+2)$  and multiplicities  $(k+1)^2$  indexed by a non-negative integer  $k$ . In terms of standard  $(x, y, z, w)$  coordinates on  $R^4 \supset S^3$ , the modes are exactly the homogeneous harmonic polynomials of degree  $k$  in the variables  $x, y, z$  and  $w$ .

Taking the quotient of  $S^3$  by a finite fixed point free group  $\Gamma$  of isometries yields a spherical manifold  $M = S^3/\Gamma$  of volume  $\text{Vol}(S^3)/|\Gamma|$ . Each mode of  $M$  lifts to a  $\Gamma$ -periodic mode of  $S^3$ , and conversely each  $\Gamma$ -periodic mode of  $S^3$  defines a mode of  $M$ . In practice one works with the  $\Gamma$ -periodic modes of  $S^3$ .

The Weyl asymptotic formula asserts that the quotient manifold  $M$  has, on average,  $1/|\Gamma|$  times as many modes as  $S^3$ . The question then is this: How do the low-order modes of  $M$  compare to those of  $S^3$ ? Does  $M$  suppress the low-order modes more or less than the overall suppression factor of  $1/|\Gamma|$ ? The following subsections show that the answer depends on whether the manifold is well-proportioned or oddly proportioned.

### 4.1 Lens spaces $L(p, q)$

The lens space  $L(p, q)$  is the quotient of  $S^3$  under the action of a cyclic group  $\Gamma$ . Roughly speaking, the quotient acts in a single

direction leaving the orthogonal directions ‘un-quotiented’. In this sense a lens space is analogous to the oblate 3-torus of Section 3.3. Indeed, just as decreasing a single dimension  $L_x$  does not change the lowest mode of the oblate 3-torus, increasing the order  $p$  of the lens space (thus decreasing the volume of the manifold) does not change the lowest mode of the lens space.

The 3-sphere supports a  $k = 1$  mode of multiplicity 4, spanned by the four linear harmonic polynomials  $\{x, y, z, w\}$ . However, no linear polynomial is invariant under any fixed point free isometry of the 3-sphere (proof: a linear polynomial achieves a unique maximum on  $S^3$  which would need to be preserved, contradicting the fixed point free assumption), so no non-trivial spherical 3-manifold  $S^3/\Gamma$  admits a  $k = 1$  mode. The first potentially non-trivial mode is  $k = 2$ .

The 3-sphere supports a  $k = 2$  mode of multiplicity 9. Within that nine-dimensional space of modes, consider the mode given by the quadratic harmonic polynomial

$$P_1(x, y, z, w) = x^2 + y^2 - z^2 - w^2. \quad (3)$$

Geometrically  $P_1$  maintains a maximum along the circle  $\{x^2 + y^2 = 1, z = w = 0\}$  and a minimum along the complementary circle  $\{x = y = 0, z^2 + w^2 = 1\}$ . After a suitable change of coordinates, the generating matrix for the holonomy of an arbitrary lens space  $L(p, q)$  may be written as

$$\begin{pmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & \cos \phi & -\sin \phi \\ 0 & 0 & \sin \phi & \cos \phi \end{pmatrix} \quad (4)$$

corresponding geometrically to simultaneous rotation in the  $xy$  and  $zw$  planes. Such a rotation preserves  $P_1$ , so  $P_1$  defines a  $k = 2$  mode of every lens space  $L(p, q)$ , proving that every lens space, no matter how small its volume, has a  $k = 2$  mode of multiplicity at least 1. For non-homogeneous lens spaces [those for which  $q \not\equiv \pm 1 \pmod{p}$ ], there are no other invariant modes and the multiplicity is exactly 1.

For homogeneous lens spaces  $L(p, \pm 1)$  the generating matrix (4) takes a special form with  $\theta = \pm \phi$  and two other modes of  $S^3$  become invariant, namely  $P_2 = xz \pm yw$  and  $P_3 = xw \mp yz$ . (These modes achieve their extrema along Clifford parallels, which is why they are invariant under Clifford translations.) Thus for a homogeneous lens space the  $k = 2$  mode has multiplicity 3.

We have seen that for any lens space, homogeneous or not, the  $k = 2$  mode always has multiplicity at least 1, even though the overall mode density is suppressed in proportion to  $1/p$ . Thus as  $p$  gets large, the relative importance of the  $k = 2$  mode increases. For plausible choices of  $\Omega_{\text{total}}$  the last scattering surface has moderate size (for example when  $\Omega_{\text{total}} \approx 1.02$ , then  $R_{\text{LSS}}/R_{\text{curv}} \approx 1/2$ ) and the relatively strong  $k = 2$  mode of the lens space imprints an elevated quadrupole on the CMB power spectrum.

Careful CMB simulations, taking into account the integrated Sachs–Wolfe and Doppler components as well as the ordinary Sachs–Wolfe component, confirm this expectation (Uzan et al. 2004). The simulations suggest that for homogeneous lens spaces  $L(p, 1)$  the low- $\ell$  modes are elevated in a uniform way, while for non-homogeneous lens spaces  $L(p, q)$ ,  $q \not\equiv \pm 1 \pmod{p}$ , the effect on the low- $\ell$  multipoles is more erratic and depends on the position of the observer.

#### 4.2 Binary polyhedral spaces

When  $\Gamma$  is the binary tetrahedral group  $T^*$ , the quotient manifold  $M = S^3/\Gamma$  has fundamental domain a regular octahedron, 24 of which tile the 3-sphere.

When  $\Gamma$  is the binary octahedral group  $O^*$ , the quotient manifold  $M = S^3/\Gamma$  has fundamental domain a truncated cube, 48 of which tile the 3-sphere.

When  $\Gamma$  is the binary icosahedral group  $I^*$ , the quotient manifold  $M = S^3/\Gamma$  has fundamental domain a regular dodecahedron, 120 of which tile the 3-sphere.

In all three cases the fundamental domain is well proportioned and we expect, in analogy with the small cubic 3-torus of Section 3.2, that the low-order modes will be suppressed. This expectation is borne out: the lowest non-trivial modes of  $S^3/T^*$ ,  $S^3/O^*$  and  $S^3/I^*$  occur at  $k = 6, 8$  and  $12$ , respectively (Ikeda 1995). In other words, as the volume of the fundamental domain gets smaller, the lowest modes gradually disappear. One expects increasing suppression of the CMB quadrupole. A study of  $S^3/I^*$  (Luminet et al. 2003) along with work in progress on  $S^3/T^*$  and  $S^3/O^*$  confirm this expectation.

#### 4.3 Spherical summary

Lens spaces are short in only one direction (analogous to the oblate 3-torus) and retain a lowest mode at  $k = 2$  no matter how large the order  $p$ . This effect is most pronounced for the homogeneous lens spaces  $L(p, 1)$ , where every observer sees his or her own translates aligned along a single geodesic, related by Clifford translations. For a homogeneous lens space the lowest mode at  $k = 2$  always has multiplicity 3 [except for  $L(2, 1)$  where the multiplicity is 9 due to extra symmetry], and the expected CMB quadrupole rises steadily with increasing  $p$ . For a non-homogeneous lens space  $L(p, q)$ ,  $q \not\equiv \pm 1 \pmod{p}$ , the multiplicity of the  $k = 2$  mode is only 1; again the CMB quadrupole is elevated but the expected effect on other low- $\ell$  modes is more erratic and depends on the observer’s position in the space.

The binary polyhedral spaces  $S^3/T^*$ ,  $S^3/O^*$  and  $S^3/I^*$  are homogeneous and every observer may picture him- or herself as sitting at the centre of a well-proportioned fundamental domain (an octahedron, truncated cube or dodecahedron, respectively). Because the fundamental domain is well proportioned, the lowest modes ( $k < 6, 8$  or  $12$ , respectively) disappear entirely. For realistic values of the cosmological parameters ( $\Omega_{\text{total}} \approx 1.02$ ), the Poincaré dodecahedral space  $S^3/I^*$  suppresses the expected quadrupole most effectively. To achieve comparable suppression in  $S^3/T^*$  or  $S^3/O^*$  requires a slightly higher  $\Omega_{\text{total}}$ .

#### 5 CONCLUSION

The CMB quadrupole in a finite universe gets suppressed or elevated according to whether the low-order modes of the universe are weak or strong relative to its generic high-order modes. In the cosmologically interesting case of a fundamental domain whose largest dimension is comparable to the diameter of the last scattering surface, we found that in a well-proportioned universe the low-order modes will be relatively weak, while in an oddly proportioned universe they will be relatively strong. Therefore, well-proportioned spaces make the best candidates in the on-going search for a topological explanation of the low CMB quadrupole and octopole.

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