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# Synchronization in Networks of Nonlinear Systems: Contraction Metric Analysis and Deep-Learning for Feedback Estimation

Mattia Giaccagli, Samuele Zoboli, Daniele Astolfi, Vincent Andrieu and Giacomo Casadei

**Abstract**—In this work, we consider the problem of global exponential synchronization of a network of identical input-affine nonlinear time-varying systems. To this end, we tackle the problem with incremental stability tools. We propose sufficient metric-based conditions to design a distributed diffusive coupling feedback law in two frameworks. First, we consider the Euclidean scenario, where the network is assumed to be connected. Second, we tackle the Riemannian framework by assuming the presence of a leader for an undirected network. This allows considering more general systems with significant nonlinearities. In both scenarios, we propose two different control laws: a full-state feedback and a static output feedback controller. Then, we apply our design to several cases-studies. To conclude, we propose an algorithm based on deep neural networks (DNNs) to practically implement such controllers.

**Index Terms**—Synchronization, contraction, multi-agent systems, incremental stability, deep learning, deep neural network.

## I. INTRODUCTION

The problem of a group of agents trying to achieve an agreement is generally known as synchronization or consensus. Such a problem arises in a vast range of applications, e.g., power networks [1, 16], heat networks [1, 42] and robot swarms [1, 33]. As a consequence, the control community has devoted huge attention to network analysis and design of suitable control architectures. At first, systems modeled by linear ODEs (ordinary differential equations) were considered. In this context, fundamental results were obtained in [41] for homogeneous networks (i.e. networks where the agents dynamics are identical) and in [50] for heterogeneous ones. A nonlinear distributed feedback was proposed for simple single and double integrator dynamics [1]. Nowadays, the control community is focusing on the analysis and design of control for networks of systems described by nonlinear dynamics.

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In this paper, we consider *homogeneous* networks. Most of results exploit existing techniques developed for single-agent systems, specifically adapted to deal with a distributed framework. Among them, we recall passivity-based [6, 37], dissipativity-based [46] and ISS approaches [10, 11]. High-gain techniques, inherited from high-gain observers theory or high-gain domination approaches (see, e.g., [9, 14, 26, 34]), form another notable class of solutions. Such controllers share pros and cons with their observers counterparts. This led researchers to investigate alternative tools such as nonlinear integral control [37].

In this work, we approach the synchronization problem with tools derived from incremental stability theory (see [2, 5, 17, 31]). In simple terms, a system is incrementally stable if the distance between trajectories starting from different initial conditions asymptotically decreases to zero. Thanks to such a property (and many others), incremental stability tools have been used in a wide range of control problems, such as observer design [7, 40], output regulation [22, 36] and, not last, multi-agent synchronization [4, 51, 52]. In the latter case, however, existing results either focus on specific classes of systems, such as Lur’e with sector bound conditions [52], [4] and incrementally passive systems [37], or on generic classes of systems but addressing the synchronization task only locally, i.e., if the initial conditions of the agents are sufficiently close to each other, e.g., [3].

In the following, we investigate some metric-based conditions for the global exponential synchronization problem of homogeneous networks, in which each agent is described by a nonlinear time-varying input-affine multi-input multi-output (MIMO) ODE and we look for the existence of a nonlinear diffusive coupling, namely a static distributed state/output control feedback. The contributions are various. First, we approach the problems with an Euclidean-based contractive analysis. We consider a directed and connected network of input-affine nonlinear systems with constant input vector fields. We show that synchronization can be achieved under the design of an infinite gain margin control law that involves the solution of a Riccati-like inequality. This extends more classical results on synchronization of linear systems, see [25, Section 5]. In this context, the proposed design provides a unifying framework with respect to many literature results (see e.g. the high-gain observer approach in [13], the incremental passive approach in [37] and the results in [4, 52] when considering particular classes of systems).

Moreover, we show how the proposed design extends in terms of control gain the results in [3] (where the control gain was different for every agent) and domain of attraction (local synchronization), as well as the results in [37] in terms of network topology (which in our case is only assumed to be connected). Then, we show that in leader-synchronization scenarios with undirected networks the proposed control laws can be generalized further by means of Riemannian metrics (see, e.g. [2, 19, 31, 40]). This allows considering more general classes of systems with significant nonlinearities. However, in the Riemannian case, the proposed control design requires the solution of a nonlinear partial differential equation (PDE). This turns out to be a difficult task in practice. To deal constructively with such a complexity, we provide the following additional results. i) First, we show that synchronization can be achieved in a regional context under less stringent assumptions (i.e. without the so-called ‘‘Killing assumption’’). ii) Thanks to this relaxation, we provide a formulation of a practical Deep Neural Network (DNN)-based algorithm to check the solvability of such a PDE. The use of DNNs for solving PDEs is not new to the Machine Learning community, see, e.g., [8, 38]. In this context, we recall the recent results on the use of Deep Learning tools for the estimation of a Riemannian metric [47, 49]. We highlight that this work is an extension of the authors’ conference work [19], where preliminary results on the synchronization of leader-connected undirected networks of SISO time-invariant nonlinear systems were proposed. This is covered in Section V.

The rest of the paper is organized as follows. Preliminaries and framework are presented in Section II. In Section III, some aspects on synchronization of linear systems are recalled, to better frame the results of the paper. Sufficient conditions based on Euclidean contraction are discussed in Section IV while Riemannian conditions are given in Section V. Some specific classes of systems dynamics are then analyzed in Section VI, showing that existing high-gain and LMI based designs are recovered by the proposed conditions. Finally, a constructive DNNs-based algorithm to approximately solve the metric estimation problem is given in Section VII. To show the potential of this approach, we consider the problem of synchronizing a network of Lorentz oscillators. Conclusions and perspectives are provided in Section VIII.

*Notation:*  $\mathbb{N}$ , resp.  $\mathbb{R}, \mathbb{C}$ , is the set of natural, resp. real, complex, numbers,  $\mathbb{R}_{\geq 0} := [0, +\infty), \mathbb{R}_{> 0} := (0, +\infty)$ . The symbol  $\mathbf{1}$  denotes a column vector whose entries are all 1 (dimension is clear from the context). The symbol  $\otimes$  denotes the Kronecker product. Given  $N$  matrices  $A_i \in \mathbb{R}^{n_i \times n}$  for  $i = 1, \dots, N$ , we denote with  $\text{col}\{A_1, \dots, A_N\}$  the matrix  $(A_1^\top, \dots, A_N^\top)^\top \in \mathbb{R}^{(n_1 + \dots + n_N) \times n}$ . Given  $N$  square matrices  $A_i \in \mathbb{R}^{n_i \times n_i}$  for  $i = 1, \dots, N$ , we denote with  $\text{diag}\{A_1, \dots, A_N\}$  the square matrix with  $A_1, \dots, A_N$  on the main diagonal and zeros everywhere else. The symbol  $I_n$  indicates the identity matrix of dimension  $n \times n$  (subscript is omitted when it’s clear from the context). Given a square

matrix  $A$ , we denote  $\text{He}\{A\} := A + A^\top$  and  $\text{spec}\{A\}$  denotes its spectrum.  $|\cdot|$  denotes the standard Euclidean norm. Given a vector  $x \in \mathbb{R}^n$  and a set  $\mathcal{S} \subset \mathbb{R}^n$ , we denote the Euclidean distance of  $x$  with respect to  $\mathcal{S}$  as  $|x|_{\mathcal{S}} := \inf_{z \in \mathcal{S}} |x - z|$ . Given a 2-tensor  $P : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$  taking symmetric values and a vector field  $f : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$  both  $C^1$ , we denote the Lie derivative of the tensor  $P$  along  $f$  as  $L_f P(x, t)$ , defined as

$$\begin{aligned} L_f P(x, t) &:= P(x, t) \frac{\partial f}{\partial x}(x, t) + \frac{\partial f^\top}{\partial x}(x, t) P(x, t) + \mathfrak{d}_f P(x, t) \\ \mathfrak{d}_f P(x, t) &:= \lim_{h \rightarrow 0} \frac{P(X(x, t+h, t), t) - P(x, t)}{h} + \frac{\partial P}{\partial t}(x, t), \end{aligned}$$

where and  $X(x, t, t_0)$  is the solution of

$$\frac{\partial}{\partial t} X(x, t, t_0) = f(X(x, t, t_0), t), \quad X(x, t, t) = x,$$

for all  $t \geq t_0$  in time-existence of solutions. Given a 2-tensor  $P : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$  and a vector field  $g : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$  both  $C^1$  (resp., a  $C^1$  matrix function  $g : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^{n \times m}$ ), we say that  $g$  is a ‘‘Killing vector’’ field (or that it satisfies the ‘‘Killing vector property’’) with respect to  $P$ , if  $L_g P(x, t) = 0$  (resp.  $L_{g_i} P(x, t) = 0$  for all  $i = 1, \dots, m$ , with  $g_i$  denoting the  $i$ -th column of  $g$ ) for all  $(x, t) \in \mathbb{R}^n \times \mathbb{R}$ . Note that the Killing vector property is trivially satisfied in case both  $P, g$  are constant matrices.

## II. PRELIMINARIES AND PROBLEM STATEMENT

### A. Graph theory

In a general framework, a communication graph is described by a triplet  $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$  in which  $\mathcal{V} = \{v_1, v_2, \dots, v_N\}$  is a set of  $N \subset \mathbb{N}$  vertexes (or nodes),  $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$  is the set of edges  $e_{jk}$  that models the interconnection between the vertexes with the flow of information from vertex  $j$  to vertex  $k$  weighted by the  $(k, j)$ -th entry  $a_{kj} \geq 0$  of the adjacency matrix  $\mathcal{A} \in \mathbb{R}^{N \times N}$ . We denote by  $\mathcal{L} \in \mathbb{R}^{N \times N}$  the Laplacian matrix of the graph, defined as

$$\ell_{kj} = -a_{kj} \quad \text{for } k \neq j, \quad \ell_{kj} = \sum_{i=1}^N a_{ki} \quad \text{for } k = j,$$

where  $\ell_{j,k}$  is the  $(j, k)$ -th entry of  $\mathcal{L}$ . We denote with  $\mathcal{N}_i$  the set of in-neighbors of node  $i$ , i.e. the set  $\mathcal{N}_i := \{j \in \{1, \dots, N\} | e_{ji} \in \mathcal{E}\}$ . A time-invariant graph is said to be connected if and only if  $\mathcal{L}$  has only one trivial eigenvalue  $\lambda_1(\mathcal{L}) = 0$  and all other eigenvalues  $\lambda_2(\mathcal{L}), \dots, \lambda_N(\mathcal{L})$  have strictly positive parts (see [23]).

In this article, two different types of graphs will be considered: directed and undirected. In particular, let us identify the Laplacian of the network as

$$\mathcal{L} = \begin{pmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{pmatrix} \quad (1)$$

where  $\mathcal{L}_{11}$  is a scalar,  $\mathcal{L}_{12}$  is a  $N - 1$  row vector,  $\mathcal{L}_{21}$  is a  $N - 1$  column vector and  $\mathcal{L}_{22}$  is a  $(N - 1) \times (N - 1)$  matrix. The following then holds.

**Lemma 1.** *Let the graph  $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$  be directed and connected. Then, there exists a strictly positive real number*

$\mu > 0$  and a symmetric positive definite matrix  $Q = Q^\top \succ 0$  such that the Laplacian  $\mathcal{L}$  satisfies

$$\text{He}\{Q(\mathcal{L}_{22} - \mathbf{1}\mathcal{L}_{12})\} \succeq \mu I. \quad (2)$$

*Proof.* Since the graph is connected, the Laplacian  $\mathcal{L}$  has one zero eigenvalue and  $N - 1$  strictly positive eigenvalues (see [25, Theorem 5.1]). Consider the matrix

$$\mathcal{R} = \mathcal{R}^{-1} = \begin{pmatrix} 1 & 0 \\ \mathbf{1} & -I \end{pmatrix}.$$

and let  $\nu^\top = (1 \ 0 \ \dots \ 0)$  and note that  $\mathcal{R}\nu = \mathbf{1}$ . Consider the change of coordinates on the Laplacian defined by

$$\tilde{\mathcal{L}} := \mathcal{R}\mathcal{L}\mathcal{R}^{-1} = \begin{pmatrix} 0 & -\mathcal{L}_{12} \\ 0 & \mathcal{L}_{22} - \mathbf{1}\mathcal{L}_{12} \end{pmatrix}.$$

Since  $\mathcal{R}$  is full rank, the new matrix  $\tilde{\mathcal{L}}$  has still one eigenvector associated with the eigenvalue zero, and all the others strictly positive definite. In particular, the zero eigenvalue is associated with the eigenvector  $\nu$ , since  $\mathcal{R}\nu = \mathbf{1}$  and  $\tilde{\mathcal{L}}\mathbf{1} = 0$ . Because of the block-diagonal structure of  $\tilde{\mathcal{L}}$ , all the eigenvalues of  $\mathcal{L}_{22} - \mathbf{1}\mathcal{L}_{12}$  are strictly positive. Hence (2) holds.  $\square$

For an undirected graph, instead, we have the following property proved in [23].

**Lemma 2.** *Consider the graph  $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$  be undirected and leader-connected (i.e. it contains at least one spanning tree with the leader as a root). Then the Laplacian  $\mathcal{L}$  can be partitioned as*

$$\mathcal{L} = \begin{pmatrix} 0 & 0 \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{pmatrix}. \quad (3)$$

Moreover, there exists a strictly positive real number  $\mu > 0$  such that

$$\mathcal{L}_{22} \succeq \mu I. \quad (4)$$

### B. Synchronization with nonlinear diffusive coupling

The dynamics of each agent of the network is described by a general time-varying nonlinear ODE

$$\dot{x}_i = f(t, x_i, u_i), \quad y_i = h(t, x_i), \quad i = 1, \dots, N, \quad (5)$$

where  $x_i \in \mathbb{R}^{n_x}$  is the state of node  $i$ ,  $u_i \in \mathbb{R}^{n_u}$  is the control action on node  $i$ , and  $y_i \in \mathbb{R}^{n_y}$  is the output which is exchanged to the neighborhoods of the network. We denote the state of the entire network as

$$\mathbf{x} := \text{col}\{x_1^\top, \dots, x_N^\top\}^\top \in \mathbb{R}^{Nn_x}. \quad (6)$$

Furthermore, we denote with  $\mathcal{X}_i(x_i^\circ, t, t_0)$  the trajectory of agent  $i$  starting from the initial condition  $x_i^\circ$  and initial time  $t_0 \in \mathbb{R}$  evaluated at time  $t \geq t_0$ , and with  $\mathcal{X}(\mathbf{x}^\circ, t, t_0)$  the trajectory of the entire network (6) evaluated at initial condition  $\mathbf{x}^\circ \in \mathbb{R}^{Nn_x}$ , initial time  $t_0 \in \mathbb{R}$  at time  $t \geq t_0$ . Our synchronization objective is to design a nonlinear diffusive

coupling, namely a distributed feedback control law of the form

$$\begin{aligned} u_i &= \sum_{j \in \mathcal{N}_i} a_{ij} [\varphi(x_i, y_j, t) - \varphi(x_i, y_i, t)] \\ &= - \sum_{j=1}^N \ell_{ij} \varphi(x_i, y_j, t) \end{aligned} \quad (7)$$

for all  $i = 1, \dots, N$ , for some  $C^1$  function  $\varphi : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \times \mathbb{R} \rightarrow \mathbb{R}^{n_u}$ , that stabilizes the dynamics (16) on the so-called *synchronization manifold*  $\mathcal{D}$  defined as

$$\mathcal{D} := \{\mathbf{x} \in \mathbb{R}^{Nn_x} \mid x_i = x_j, \text{ for all } i, j \in \{1, \dots, N\}\}, \quad (8)$$

where the states of all the agents of the network agree with each other. By construction, the  $i$ -th agent uses only the output information  $y_j$  of its neighborhoods  $j \in \mathcal{N}_i$  and its own local information  $(y_i, x_i)$ . Furthermore, the control action  $u_i$  is equal to zero on the synchronization manifold. In other words, when consensus is achieved, no correction term is needed for each individual agent. As a consequence, stabilizing all the agents on a desired equilibrium point is generally not a valid solution in such a framework. We formalize our synchronization problem as follows.

**Problem 1** (Network synchronization). *Let the function  $\varphi$  be such that the manifold  $\mathcal{D}$  defined in (8) is globally uniformly exponentially stable for the closed-loop system*

$$\dot{x}_i = f(t, x_i, - \sum_{j \in \mathcal{N}_i} \ell_{ij} \varphi(x_i, y_j, t)), \quad i = 1, \dots, N,$$

namely, there exist positive constants  $k$  and  $\lambda > 0$  such that for all  $(\mathbf{x}^\circ, t_0)$  in  $\mathbb{R}^{Nn_x} \times \mathbb{R}$  and for all  $t \geq t_0$  in the time domain of existence of solutions  $\mathcal{T} \subseteq \mathbb{R}$  we have

$$|\mathcal{X}(\mathbf{x}^\circ, t, t_0)|_{\mathcal{D}} \leq k \exp(-\lambda(t - t_0)) |\mathbf{x}^\circ|_{\mathcal{D}}. \quad (9)$$

Then, we say that the distributed feedback control law (7) solves the global exponential synchronization problem for the network (6).

### III. NETWORK SYNCHRONIZATION FOR LINEAR SYSTEMS

To better contextualize the results of our paper, it is useful to recall some important aspects on synchronization of homogeneous network of linear systems, see, e.g. [25, Chapter 5]. Consider a network where each agent is described by

$$\dot{x}_i = Ax_i + Bu_i, \quad y_i = Cx_i, \quad i = 1, \dots, N, \quad (10)$$

where  $x \in \mathbb{R}^{n_x}$  is the state,  $u \in \mathbb{R}^{n_u}$  is the control action,  $y \in \mathbb{R}^{n_y}$  is a linear combination of the state and  $A, B, C$  are matrices of appropriate dimension. The following result holds, see [25, Proposition 5.2].

**Proposition 1** (Synchronization of linear systems). *Consider a connected network  $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$  where each agent is described by (10). Assume there exists a matrix  $K$  such that the matrix  $A - \lambda_{\mathcal{L}} BKC$  is Hurwitz for all  $\lambda_{\mathcal{L}} \in \text{spec}\{\mathcal{L}\} \setminus$*

$\{0\}$ . Then the network in closed-loop with the distributed control law

$$u_i = K \sum_{j \in \mathcal{N}_i} a_{ij}(y_j - y_i) = -K \sum_{j=1}^n \ell_{ij} y_j \quad (11)$$

achieves synchronization, i.e.  $\lim_{t \rightarrow +\infty} |x_i(t) - x_j(t)| = 0$  for all  $(i, j) \in \mathcal{V} \times \mathcal{V}$ .

In other words, for a connected network of linear systems, the synchronization problem can be seen as a *robust* (or simultaneous) stabilization problem. With ‘‘robust’’ we mean that the stabilization problem must be achieved for any strictly positive eigenvalue of the Laplacian  $\lambda_{\mathcal{L}}$ , which can be seen as a gain acting on the control term. To fulfill this requirement, a solution is given by employing an *infinite gain margin* feedback (see e.g. [43, Section 3]), that is, a feedback law that achieves stability in the presence of an uncertain factor in front of the gain matrix  $B$ .

For linear systems, a general sufficient condition can be stated as follows.

**Lemma 3.** *Suppose there exists a symmetric positive definite matrix  $P = P^\top \succ 0$ , matrices  $D, E$  and positive real numbers  $\varepsilon, \rho > 0$  such that the following holds*

$$\begin{aligned} PA + A^\top P - \rho C^\top E^\top EC &\preceq -\varepsilon P, \\ PBD &= C^\top E^\top. \end{aligned} \quad (12)$$

Then, the feedback gain  $K = DE$  is an infinite-gain margin static output feedback law for the triplet  $(A, B, C)$ , namely  $A - \kappa BKC$  is Hurwitz for any  $\kappa \in [\frac{\rho}{2}, \infty)$ .

*Proof.* Let  $P$  given by (12) and compute

$$\begin{aligned} &P(A - \kappa BKC) + (A - \kappa BKC)^\top P \\ &= PA + A^\top P - \kappa PBDEC - \kappa C^\top E^\top D^\top B^\top P \\ &= PA + A^\top P - \rho C^\top E^\top EC - (2\kappa - \rho)C^\top E^\top EC \\ &\preceq -\varepsilon P \end{aligned}$$

for all  $\kappa \geq \frac{\rho}{2}$  concluding the proof.  $\square$

We remark that condition (12) is slightly different from the one established in [29], where sufficient and necessary conditions for the existence of a static output feedback stabilizing control law are given. The conditions in [29] takes the form

$$\begin{aligned} PA + A^\top P - PBB^\top P + C^\top C + E^\top E &= 0, \\ KC - B^\top P &= -E, \end{aligned}$$

but do not necessarily have the infinite gain margin property (except for the passivity-like case  $E = 0$ ). Therefore, the proposed inequality (12) is in general more restrictive. An exhaustive discussion about necessity and feasibility of the condition (12) is out of the scope of this work. Some specific cases are discussed at the end of this section.

Coming back to our synchronization problem, we may state now the following result.

**Lemma 4** (Synchronization of linear systems). *Consider a network  $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$  with node dynamics described*

by (10) and suppose there exists a solution to (12). Then, the distributed control law (11) solves the synchronization Problem 1 with  $K = \kappa DE$  with  $\kappa \geq \frac{\rho}{2\mu}$  and  $\mu$  given by Lemma 1.

The proof of such a result is shown, for instance, in [30, Section II.B]. See also [25, Chapter 5] and references therein for more details. Finally, we may observe that the aforementioned condition (12) generalizes the following classical multi-agent synchronization results.

- For a *state-feedback* synchronization problem

$$\dot{x}_i = Ax_i + Bu_i, \quad y_i = x_i,$$

condition (12) recovers the feedback design

$$PA + A^\top P - PBB^\top P \preceq -\varepsilon P, \quad K = B^\top P, \quad (13)$$

with  $D = I$ ,  $E = B^\top P$ ,  $\rho = 1$ . Note that the ARE (13) always admits a solution if  $(A, B)$  is stabilizable.

- For the *observer-form output-feedback* synchronization problem

$$\dot{x}_i = Ax_i + u_i, \quad y_i = Cx_i,$$

the condition (12) recovers the feedback design

$$PA + A^\top P - C^\top C \preceq -\varepsilon P, \quad K = P^{-1}C^\top, \quad (14)$$

with  $D = P^{-1}C^\top$ ,  $E = I$ ,  $\rho = 1$ . The ARE (14) always admits a solution if  $(A, C)$  is detectable.

- For the square  $n_y = n_u$  MIMO output-feedback synchronization problem

$$\dot{x}_i = Ax_i + Bu_i, \quad y_i = Cx_i,$$

the condition (12) recovers the *passivity condition*

$$PA + A^\top P - C^\top C \preceq -\varepsilon P, \quad PB = C^\top, \quad (15)$$

with  $D = I_{n_u}$ ,  $E = I_{n_y}$ ,  $\rho = 1$  and  $K = I_{n_y}$ .

Therefore, the take-away message we aim to highlight in this section is that, if we aim at developing a theory for general nonlinear dynamics and generic connected networks, we need to be able to solve a robust stabilization problem as in Proposition 1. As a consequence, the key property is given by the extension of the aforementioned infinite-gain margin law in the contraction framework, see, e.g. [20]. This symmetry will be further developed in the next sections.

#### IV. NETWORK SYNCHRONIZATION WITH EUCLIDEAN CONTRACTION

In this first section, we will consider a network of  $N$  homogeneous nonlinear multi-agent systems, i.e., described by identical dynamics. The objective is to derive sufficient conditions for the synchronization of the full network. Motivated by the linear systems case, we aim to provide a design which requires no additional assumptions on the network, except for its connectivity. This is recalled in the following assumption.

**Assumption 1** (Graph network synchronization). *The graph  $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$  is directed and connected.*

Concerning the dynamics of the agents of the network, in this section we focus on time-varying nonlinear dynamics described by the following nonlinear ODE

$$\dot{x}_i = f(x_i, t) + Bu_i, \quad i = 1, \dots, N, \quad (16)$$

where  $x_i \in \mathbb{R}^{n_x}$  is the state of node  $i$ ,  $u_i \in \mathbb{R}^{n_u}$  is the control action on node  $i$ ,  $B$  is a constant matrix and  $f : \mathbb{R}^{n_x} \times \mathbb{R} \mapsto \mathbb{R}^{n_x}$  is a vector field which is  $C^1$  in the first argument and piece-wise continuous in the second. Motivated by the analysis in Section III, we consider the case in which agent (16) can exchange with their neighborhoods only an output of the form

$$y_i = Cx_i, \quad i = 1, \dots, N, \quad (17)$$

with  $y_i \in \mathbb{R}^{n_y}$  and where  $C$  is a constant matrix. In particular, our objective is to design a feedback distributed control law stabilizing the dynamics (16) on the so-called *synchronization manifold*  $\mathcal{D}$  defined in (8), where the states of all the agents of the network agree with each other. Following Section II-B, we look for a diffusive coupling law of the form (7) for all  $i = 1, \dots, N$ . Along the lines of the linear case, we focus on a control design which involves the solution of a Riccati-like inequality in order to get a control law possessing an infinite gain margin property, see, e.g. [20]. We have the following result.

**Theorem 1** (Feedback network synchronization). *Consider a network  $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$  fulfilling Assumption 1, with node dynamics (16), (17). Assume there exists a symmetric positive definite matrix  $P = P^\top \succ 0$ , two matrices  $D, E$  and two real numbers  $\varepsilon, \rho > 0$  such that*

$$\begin{aligned} L_f P(x, t) - \rho C^\top E^\top E C &\leq -\varepsilon P, \\ PBD &= C^\top E^\top. \end{aligned} \quad (18)$$

for all  $(x, t) \in \mathbb{R}^{n_x} \times \mathbb{R}$ . Then, for any  $\kappa \geq \frac{\rho}{2\mu}$ , with  $\mu$  given by Lemma 1, the distributed feedback law (7) with

$$\varphi(x, t) = \kappa Ky, \quad K = DE, \quad (19)$$

solves the synchronization Problem 1 for the network of (16).

As for the linear case, condition (18) includes many published results. For instance, in the case of output-feedback form in which  $B = I$ , we recover the high-gain observer approach proposed in [13, Proposition 1]. In the state-feedback case with  $C = I$ , we recover the results in [3, Theorem 3] which are generalized concerning the assumption on the graph and in terms of gain of the control law, which in our case is the same for each agent. When considering system dynamics expressed in Lur'e form, we recover the results of [52]. When considering the passivity condition as in (15), we recover the context of incrementally passivity with respect to a constant Euclidean metric  $P$ , see for instance [18], [36, Section 5] and we generalize the result in [37, Theorem 4] concerning the assumption on the graph.

*Proof.* The idea behind the proof is to set a *virtual leader* and show that the dynamics of the error between any other agent and such a leader exponentially goes to zero. Therefore, let

$\tilde{x} = (\tilde{x}_2^\top, \dots, \tilde{x}_N^\top)^\top$  where  $\tilde{x}_i = x_i - x_1$  represents the error between agent  $x_i$  and  $z = x_1$ . Since  $\ell_{ij} = 0$  if  $j \notin \mathcal{N}_i$ , the  $\tilde{x}_i$ -dynamics can be written as

$$\begin{aligned} \dot{\tilde{x}}_i &= f(z + \tilde{x}_i, t) - \kappa B \sum_{j=1}^N \ell_{ij} DEC(z + \tilde{x}_j) \\ &\quad - f(z, t) + \kappa B \sum_{j=1}^N \ell_{1j} DEC(z + \tilde{x}_j), \\ &\quad i = 2, \dots, N. \end{aligned}$$

Since  $\sum_{j=1}^N \ell_{ij} = 0$  for all  $i = 1, \dots, N$ , we can add the terms  $-\kappa B \left( \sum_{j=1}^N \ell_{1j} \right) DECz = 0$  and  $\kappa B \left( \sum_{j=1}^N \ell_{ij} \right) DECz = 0$ . This leads, for  $i = 2, \dots, N$ , to

$$\begin{aligned} \dot{\tilde{x}}_i &= f(z + \tilde{x}_i, t) - f(z, t) - \kappa B \sum_{j=1}^N \ell_{ij} DEC \tilde{x}_j \\ &\quad + \kappa B \sum_{j=1}^N \ell_{1j} DEC \tilde{x}_j \\ &= f(z + \tilde{x}_i, t) - f(z, t) - \kappa B \sum_{j=1}^N (\ell_{ij} - \ell_{1j}) DEC \tilde{x}_j. \end{aligned}$$

Note that in these new coordinates, the synchronization manifold defined in (8) corresponds to the origin of the  $\tilde{x}$ -dynamics. Let  $t_0$  be in  $\mathbb{R}$  and consider a solution  $\mathcal{Z}(z^\circ, t, t_0)$ ,  $\tilde{\mathcal{X}}(t, t_0) = (\tilde{\mathcal{X}}_2(t, t_0), \dots, \tilde{\mathcal{X}}_N(t, t_0))$  of the closed-loop system which is defined for all  $t$  in the time domain of definition  $\mathcal{T} \subseteq \mathbb{R}$ . Consider the following Lyapunov function  $V$

$$V(t) := \tilde{\mathcal{X}}(t, t_0)^\top (Q \otimes P) \tilde{\mathcal{X}}(t, t_0) \quad (20)$$

defined for all  $t$  in  $\mathcal{T}$ , where  $P$  solves (18) and  $Q$  is chosen as in Lemma 1. Taking the time-derivative  $\dot{V}$  and recalling that  $(Q \otimes P)^\top = (Q^\top \otimes P^\top) = (Q \otimes P)$ , it yields

$$\begin{aligned} \dot{V}(t) &= 2\tilde{\mathcal{X}}(t, t_0)^\top (Q \otimes P) \operatorname{diag}_{i=2, \dots, N} \left\{ f(\mathcal{Z} + \tilde{\mathcal{X}}_i, t) - f(\mathcal{Z}, t) \right\} \\ &\quad - 2\kappa \tilde{\mathcal{X}}(t, t_0)^\top (Q \otimes P) ((\mathcal{L}_{22} - \mathbf{1}\mathcal{L}_{12}) \otimes BDEC) \tilde{\mathcal{X}}(t, t_0). \end{aligned}$$

Since  $Q \otimes P = (Q \otimes I)(I \otimes P)$ , by Lemma 1 and by using the Mean Value Theorem, it follows that

$$\begin{aligned} \dot{V}(t) &= \\ &= 2\tilde{\mathcal{X}}(t, t_0)^\top \left[ (Q \otimes P) \operatorname{diag}_{i=2, \dots, N} \left\{ \int_0^1 \frac{\partial f}{\partial x}(\mathcal{Z} - s\tilde{\mathcal{X}}_i, t) ds \right\} \right. \\ &\quad \left. - 2\kappa (Q(\mathcal{L}_{22} - \mathbf{1}\mathcal{L}_{12}) \otimes PBDEC) \right] \tilde{\mathcal{X}}(t, t_0) \\ &\leq 2\tilde{\mathcal{X}}(t, t_0)^\top \left[ (Q \otimes I) \operatorname{diag}_{i=2, \dots, N} \left\{ \int_0^1 L_f P(\mathcal{Z} - s\tilde{\mathcal{X}}_i, t) ds \right\} \right. \\ &\quad \left. - 2\kappa (\mu I \otimes C^\top E^\top EC) \right] \tilde{\mathcal{X}}(t, t_0). \end{aligned}$$

By using (18) and by choosing any  $\kappa \geq \frac{\rho}{2\mu}$  with  $\mu$  given by Lemma 1, we get

$$\dot{V}(t) \leq -\varepsilon \tilde{\mathcal{X}}(t, t_0)^\top (Q \otimes P) \tilde{\mathcal{X}}(t, t_0) = -\varepsilon V(t). \quad (21)$$

From Gronwall's Lemma (see [24, Corollary 6.6]) this implies  $V(t) \leq \exp(-\varepsilon(t-t_0))V(t_0)$  for all  $t$  in  $\mathcal{T}$ . Using the definition of  $V$  in (20), the latter inequality gives

$$|\tilde{\mathcal{X}}(t, t_0)|^2 \leq \frac{\lambda_M(Q \otimes P)}{\lambda_m(Q \otimes P)} \exp(-\varepsilon(t-t_0)) |\tilde{\mathcal{X}}(t_0, t_0)|^2, \quad (22)$$

for all  $t$  in  $\mathcal{T}$ , where  $\lambda_m(Q \otimes P), \lambda_M(Q \otimes P)$  are respectively the minimum and the maximum (strictly positive) eigenvalues of  $Q \otimes P$ . Remembering the definition  $\tilde{x}_i = x_i - z$ , by equivalence of norms in finite dimensional spaces it follows that there exist two strictly positive real numbers  $\underline{c}, \bar{c} > 0$  such that

$$\underline{c} |\mathcal{X}(x^\circ, t, t_0)|_{\mathcal{D}} \leq |\tilde{\mathcal{X}}(t, t_0)| \leq \bar{c} |\mathcal{X}(x^\circ, t, t_0)|_{\mathcal{D}}, \quad (23)$$

for all  $t$  in  $\mathcal{T}$ , which implies (9) and concludes the proof.  $\square$

We highlight again the symmetry between the results proposed in Theorem 1 and the linear case in Section III. Both results involve the solution of a Riccati-like inequality in order to guarantee an infinite-gain margin property for the control action. This aspect allows relying on the sole assumption of the network being simply connected.

As a last remark of this section, we stress that the Riccati-like condition (18) are in general only sufficient and not necessary. Indeed, in the context of contraction analysis and incremental stability, the existence of a Riemannian metric is proved to be equivalent (under some conditions of the vector field) to the desired contraction properties [2]. However, when considering an Euclidean metric, this equivalence is lost. As a consequence, a set of Euclidean metric-based conditions is only sufficient. See also [40] for further details.

## V. LEADER SYNCHRONIZATION WITH RIEMANNIAN CONTRACTION

In Section IV the synchronization problem has been solved with an analysis based on incremental stability tools with respect to an Euclidean metric. In order to extend such a result for more general classes of systems, in this section we aim to pursue a similar analysis with a study based on Riemannian metrics, see, e.g. [2, 31, 40]. Consider a network  $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$  of  $N$  identical agents described by

$$\dot{x}_i = f(x_i, t) + g(x_i, t)u_i, \quad i = 1, \dots, N, \quad (24)$$

where  $x \in \mathbb{R}^{n_x}$  is the state,  $u_i \in \mathbb{R}^{n_u}$  is the control action and  $f : \mathbb{R}^{n_x} \times \mathbb{R} \mapsto \mathbb{R}^{n_x}, g : \mathbb{R}^{n_x} \times \mathbb{R} \mapsto \mathbb{R}^{n_x \times n_u}$  are  $C^2$  functions in the first argument and piecewise continuous in the second.

In what follows, we focus on the leader-synchronization problem. In other words, we suppose the presence of a *leader*, that is, an agent of the network (24) for which the control action is zero, as specified in the following assumption. Without loss of generality, we label the leader as the node 1.

**Assumption 2.** *The graph  $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$  is undirected and leader-connected. In other words, it contains at least one spanning tree with the leader as a root.*

Without loss of generalities, we assume  $a_{1j} = 0$ , for all  $j = 1, \dots, N$ . As a consequence, the dynamical system (25) can be rewritten as

$$\begin{aligned} \dot{x}_1 &= f(x_1, t) \\ \dot{x}_i &= f(x_i, t) + g(x_i, t)u_i, \quad i = 2, \dots, N. \end{aligned} \quad (25)$$

The control objective is to design a state-feedback distributed control law of the form

$$u_i = \sum_{j \in \mathcal{N}_i} a_{ij} \left[ \varphi(x_j, t) - \varphi(x_i, t) \right] = - \sum_{j=1}^N \ell_{ij} \varphi(x_j, t) \quad (26)$$

for all  $i = 2, \dots, N$ , for some  $C^1$  function  $\varphi : \mathbb{R}^{n_x} \times \mathbb{R} \rightarrow \mathbb{R}^{n_u}$ , stabilizing the dynamics of (25) to the synchronization manifold  $\mathcal{D}$  defined in (8). The  $i$ -th agent uses only the state information of its neighborhoods and of itself, and the control action  $u_i$  is equal to zero on the synchronization manifold  $\mathcal{D}$  in (8). To this end, we look again for a feedback design which is derived by a Riccati-like inequality. In this case, however, we follow the contraction analysis based on Riemannian metrics, see, e.g. [2]. We have the following result.

**Theorem 2** (State-feedback leader-synchronization). *Consider a network  $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$  of agents (25) and let Assumption 2 hold. Assume moreover that there exist a  $C^1$  matrix function  $P : \mathbb{R}^{n_x} \times \mathbb{R} \rightarrow \mathbb{R}^{n_x \times n_x}$  taking symmetric positive definite values, a  $C^2$  function  $\alpha : \mathbb{R}^{n_x} \times \mathbb{R} \mapsto \mathbb{R}^{n_u}$  and positive real numbers  $\underline{p}, \bar{p}, \varepsilon, \rho > 0$  such that the following conditions hold:*

- *the state control matrix function (SCMF) conditions hold for all  $(x, t) \in \mathbb{R}^{n_x} \times \mathbb{R}$*

$$\begin{aligned} L_f P(x, t) - \rho P(x, t)g(x, t)g(x, t)^\top P(x, t) &\preceq -\varepsilon P(x, t), \\ \underline{p}I &\preceq P(x, t) \preceq \bar{p}I; \end{aligned} \quad (27)$$

- *the matrix function  $g$  has the Killing vector field (see Notation) property with respect to  $P$ , i.e.,*

$$L_g P(x, t) = 0, \quad \forall (x, t) \in \mathbb{R}^{n_x} \times \mathbb{R}; \quad (28)$$

- *the following integrability condition holds*

$$\frac{\partial \alpha^\top}{\partial x}(x, t) = P(x, t)g(x, t), \quad \forall (x, t) \in \mathbb{R}^{n_x} \times \mathbb{R}. \quad (29)$$

*Then, for any  $\kappa \geq \frac{\rho}{2\mu}$ , with  $\mu$  given by Lemma 2, the distributed state-feedback control law (26) with*

$$\varphi(x, t) = \kappa \alpha(x, t), \quad (30)$$

*solves the synchronization Problem 1 for the network of (25).*

*Proof.* Similarly to the proof of Theorem 1, the main goal is to show that the dynamics of the error between any agent and the leader exponentially decreases to zero. Therefore, let us denote  $\tilde{x} = (\tilde{x}_2^\top \dots \tilde{x}_N^\top)^\top$  where  $\tilde{x}_i = x_i - x_1$  represents the error between agent  $i$  and agent 1 and  $z = x_1$ . Since  $\ell_{ij} = 0$  for all  $j \notin \mathcal{N}_i$ , the dynamics of the error  $\tilde{x}_i$  for all  $i = 2, \dots, N$  with the control law (30) can be rewritten as

$$\dot{\tilde{x}}_i = f(z + \tilde{x}_i, t) - f(z, t) - \kappa g(z + \tilde{x}_i, t) \sum_{j=1}^N \ell_{ij} \alpha(z + \tilde{x}_j, t).$$

Note that there's no term on  $g(z, t)$  since no control action is acting on the leader. Since  $\sum_{j=1}^N \ell_{ij} = 0$  for all  $i = 1, \dots, N$ , we can add the term  $\kappa g(z + \tilde{x}_i, t) \left( \sum_{j=1}^N \ell_{ij} \right) \alpha(z, t) = 0$  and get

$$\begin{aligned} \dot{\tilde{x}}_i &= f(z + \tilde{x}_i, t) - f(z, t) \\ &\quad - \kappa g(z + \tilde{x}_i, t) \sum_{j=1}^N \ell_{ij} [\alpha(z + \tilde{x}_j, t) - \alpha(z, t)]. \end{aligned} \quad (31)$$

Furthermore, consider that in this new coordinates, the synchronization manifold defined in (8) corresponds to the origin of the  $\tilde{x}$ -dynamics. Let  $t_0$  be in  $\mathbb{R}$  and consider a solution  $\mathcal{Z}(z^\circ, t, t_0)$  and  $\tilde{\mathcal{X}}(t, t_0) = (\tilde{\mathcal{X}}_2(t, t_0), \dots, \tilde{\mathcal{X}}_N(t, t_0))$  of the closed-loop system which is defined for all  $t$  in the time domain of definition  $\mathcal{T}_1 \subset \mathbb{R}$ . Consider the function  $\Gamma : [0, 1] \times \mathcal{T}_2 \times \mathbb{R} \mapsto \mathbb{R}^{Nn_x}$  which satisfies

$$\Gamma(s, t_0, t_0) = s \tilde{\mathcal{X}}(t_0, t_0)$$

and where its  $i$ -th component  $\Gamma_i$  is the solution of the following ordinary differential equation

$$\begin{aligned} \frac{\partial \Gamma_i}{\partial t}(s, t, t_0) &= f(\zeta_i, t) - f(\mathcal{Z}(z^\circ, t, t_0), t) \\ &\quad - \kappa g(\zeta_i, t) \sum_{j=1}^N \ell_{ij} (\alpha(\zeta_j, t) - \alpha(\mathcal{Z}(z^\circ, t, t_0), t)) \end{aligned}$$

where for the sake of shortness we denoted  $\zeta_i = \mathcal{Z}(z^\circ, t, t_0) + \Gamma_i(s, t, t_0)$  and where  $\mathcal{Z}(z^\circ, t, t_0)$  is the solution of the virtual leader  $z$  initialized at  $(z^\circ, t_0) \in \mathbb{R}^{n_x} \times \mathbb{R}$  and evaluated at time  $t \geq t_0$ . Finally  $\mathcal{T}_2 \subset \mathcal{T}_1$  is the time domain of definition of  $\Gamma$ . We will show in the following that  $\mathcal{T}_2 = \mathcal{T}_1$ . Consider the function  $V_i$  for  $i = 2, \dots, N$  defined for  $t$  in  $\mathcal{T}_2$  by

$$V_i(t) = \int_0^1 \frac{\partial \Gamma_i}{\partial s}(s, t, t_0)^\top P(\zeta_i, t) \frac{\partial \Gamma_i}{\partial s}(s, t, t_0) ds. \quad (32)$$

Note that we have for all  $(k, l)$  in  $\{1, \dots, n_x\}^2$

$$\begin{aligned} \frac{d}{dt} [P(\zeta_i, t)_{kl}] &= \\ \frac{\partial P_{kl}}{\partial x}(\zeta_i, t) [f(\mathcal{Z}(z^\circ, t, t_0) + \frac{\partial \Gamma_i}{\partial t}(s, t, t_0))] &+ \frac{\partial P_{kl}}{\partial t}(\zeta_i, t). \end{aligned}$$

This implies that for all vector  $\nu$  in  $\mathbb{R}^{n_x}$ ,

$$\begin{aligned} \frac{d}{dt} [\nu^\top P(\zeta_i, t) \nu] &= \nu^\top \mathfrak{d}_f P(\zeta_i, t) \nu \\ &\quad - 2\kappa \sum_{j=1}^N \left[ \ell_{ij} \sum_{i=1}^{n_u} \nu^\top \mathfrak{d}_{g_i} P(\zeta_i, t) \nu \right. \\ &\quad \left. \times (\alpha_i(\zeta_j, t) - \alpha_i(\mathcal{Z}(z^\circ, t, t_0), t)) \right]. \end{aligned}$$

By using the Killing vector assumption (28) and the integrability one on the function  $\alpha$  in (29), the time derivative of  $V_i$

becomes

$$\begin{aligned} \dot{V}_i(t) &= \int_0^1 \left[ \frac{\partial \Gamma_i^\top}{\partial s}(s, t, t_0) L_f P(\zeta_i, t) \frac{\partial \Gamma_i}{\partial s}(s, t, t_0) \right. \\ &\quad \left. - 2\kappa \frac{\partial \Gamma_i^\top}{\partial s}(s, t, t_0) \sum_{j=1}^N \ell_{ij} P(\zeta_i, t) g(\zeta_i, t) \right. \\ &\quad \left. \times g(\zeta_j, t)^\top P(\zeta_j, t) \frac{\partial \Gamma_j}{\partial s}(s, t, t_0) \right] ds. \end{aligned}$$

Let  $D_f P(\zeta, t) := \text{diag}\{L_f P(\zeta_2, t), \dots, L_f P(\zeta_N, t)\}$  and  $\Psi(\zeta, t) := \text{col}\{P(\zeta_1, t)g(\zeta_1, t), \dots, P(\zeta_N, t)g(\zeta_N, t)\}$ . By considering all the  $V_i$  we have that

$$\begin{aligned} \sum_{i=2}^N \dot{V}_i(t) &= \int_0^1 \left[ \frac{\partial \Gamma^\top}{\partial s}(s, t, t_0) D_f P(\zeta, t) \frac{\partial \Gamma^\top}{\partial s}(s, t, t_0) \right. \\ &\quad \left. - 2\kappa \frac{\partial \Gamma^\top}{\partial s}(s, t, t_0) \Psi(\zeta, t) \mathcal{L}_{22} \Psi^\top(\zeta, t) \frac{\partial \Gamma}{\partial s}(s, t, t_0) \right] ds. \end{aligned} \quad (33)$$

Hence by Lemma 2,

$$\begin{aligned} \sum_{i=2}^N \dot{V}_i(t) &\leq \int_0^1 \left[ \frac{\partial \Gamma^\top}{\partial s}(s, t, t_0) D_f P(\zeta, t) \frac{\partial \Gamma^\top}{\partial s}(s, t, t_0) \right. \\ &\quad \left. - 2\kappa \mu \frac{\partial \Gamma^\top}{\partial s}(s, t, t_0) \Psi(\zeta, t) \Psi^\top(\zeta, t) \frac{\partial \Gamma}{\partial s}(s, t, t_0) \right] ds. \end{aligned} \quad (34)$$

Therefore by selecting  $\kappa \geq \frac{\rho}{2\mu}$  with  $\rho$  satisfying the SCMF defined in (27) and  $\mu > 0$  given by Lemma 2 we get

$$\dot{V}(t) = \sum_{i=2}^N \dot{V}_i(t) \leq -\varepsilon \sum_{i=2}^N V_i(t) = -\varepsilon V(t). \quad (35)$$

From Gronwall's Lemma (see [24, Corollary 6.6]) this implies for all  $t$  in  $\mathcal{T}_2$

$$V(t) \leq \exp(-\varepsilon(t - t_0)) V(t_0). \quad (36)$$

From this inequality, we first deduce that  $\mathcal{T}_2 = \mathcal{T}_1$  since the path  $\Gamma$  has finite (Riemanniann) energy and has boundary defined in  $\mathcal{T}_1$ . Moreover,

$$\begin{aligned} V_i(t) &\geq \underline{p} \int_0^1 \frac{\partial \Gamma_i}{\partial s}(s, t, t_0)^\top \frac{\partial \Gamma_i}{\partial s}(s, t, t_0) ds \\ &\geq \underline{p} \tilde{\mathcal{X}}_i(t, t_0)^\top \tilde{\mathcal{X}}_i(t, t_0). \end{aligned}$$

Since moreover,

$$V(t_0) \leq \bar{p} \tilde{\mathcal{X}}(t_0, t_0)^\top \tilde{\mathcal{X}}(t_0, t_0)$$

it yields for all  $t$  in  $\mathcal{T}_1$

$$|\tilde{\mathcal{X}}(t, t_0)|^2 \leq \exp(-\varepsilon(t - t_0)) \frac{\bar{p}}{\underline{p}} |\tilde{\mathcal{X}}(t_0, t_0)|^2.$$

With (23), inequality (9) follows.  $\square$



As in Section IV, the state-synchronization case considered in Theorem 2 can be generalized to the output feedback context. First, we consider a network of agents described as

$$\begin{aligned} \dot{x}_1 &= f(x_1, t), & y_1 &= h(x_1, t), \\ \dot{x}_i &= f(x_i, t) + u_i, & y_i &= h(x_i, t), \end{aligned} \quad (37)$$

for all  $i = 2, \dots, N$ . The following result establishes a set of sufficient conditions for the existence of a distributed output feedback nonlinear diffuse coupling of the form (7).

**Corollary 1** (Output-feedback leader-synchronization). *Consider a network  $G = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$  of agents (37) and let satisfying Assumption 2 hold. Suppose that there exist a  $C^1$  matrix function  $P : \mathbb{R}^{n_x} \times \mathbb{R} \rightarrow \mathbb{R}^{n_x \times n_x}$  taking symmetric positive definite values, and positive real numbers  $\underline{p}, \bar{p}, \varepsilon, \rho > 0$  such that the following conditions hold.*

1) *The Output Control Matrix Function (OCMF) condition holds:*

$$L_f P(x, t) - \rho \frac{\partial h^\top}{\partial x}(x, t) \frac{\partial h}{\partial x}(x, t) \preceq -\varepsilon P(x, t), \quad (38)$$

$$\underline{p}I \preceq P(x, t) \preceq \bar{p}I,$$

for all  $(x, t) \in \mathbb{R}^{n_x} \times \mathbb{R}$  for some

2) *The vector field  $\beta : \mathbb{R}^{n_x} \times \mathbb{R} \rightarrow \mathbb{R}^{n_x}$  defined as*

$$\beta(x, t) = P^{-1}(x, t) \frac{\partial h}{\partial x}(x, t)^\top \quad (39)$$

is a Killing vector for  $P$ , i.e.

$$L_\beta P(x, t) = 0 \quad \forall (x, t) \in \mathbb{R}^{n_x} \times \mathbb{R}. \quad (40)$$

Then, for any  $\kappa \geq \frac{\rho}{2\mu}$ , with  $\mu$  given by Lemma 2, the distributed state-feedback control law (7) with

$$\varphi(x, y, t) = \kappa \beta(x, t) y \quad (41)$$

solves the synchronization Problem 1 for the network (37).

In view of the structure (41) and the leader-connected Assumption 2, the control law of Corollary 1 takes the form  $u_1 = 0$  and

$$u_i = -\kappa \sum_{j=1}^N \ell_{ij} \beta(x_i, t) y_j, \quad i = 2, \dots, N.$$

We remark that, differently from the previous cases, in the output feedback context the  $i$ -th agent needs the knowledge of its own state  $x_i$  in order to implement the proposed control law, i.e. the local information doesn't coincide with the distributed one. Such an example extends in a non-trivial manner the case of observer-like form (14) to the Riemannian context. The proof is omitted for space reasons and it can be easily derived by adapting the steps used in the proof of Theorem 2. A preliminary version (in the time invariant context) can be also found in [19, Section 3].

The previous result can be also easily adapted to the context of a Riemannian incremental passivity condition for a network of systems of the form

$$\begin{aligned} \dot{x}_1 &= f(x_1, t), & y_1 &= h(x_1, t), \\ \dot{x}_i &= f(x_i, t) + g(x_i, t) u_i, & y_i &= h(x_i, t), \end{aligned} \quad (42)$$

for all  $i = 2, \dots, N$  when  $n_u = n_y$ . In this case, such a result is obtained under the solution of the OCMF condition in (38), while the conditions (39), (40) are replaced by

$$P(x, t) g(x, t) = \frac{\partial h^\top}{\partial x}(x, t), \quad L_g P(x, t) = 0, \quad (43)$$

for all  $(x, t) \in \mathbb{R}^{n_x} \times \mathbb{R}$ , while the function  $\varphi$  is given by  $\varphi(x, y, t) = \kappa y$ . This case extends, in the context of leader connected graphs, the Euclidean passivity condition given in [37]. An example of systems satisfying such a form is given below in Section VI-A.

We conclude this section by noting that, differently from the Euclidean case developed in Section IV, we don't have a complete extension to the Riemannian case of the general ARE condition (12): we only extended the three cases (13), (14), (15). Furthermore, only undirected leader-connected networks have been considered. We believe that a complete generalization is possible but the mathematical challenges are not trivial. With respect to the proof of Theorem 2, one of the main difficulties to address is that, the sum of  $V_i$  is not (in general) a good candidate to be a Lyapunov function for the non-leader directed case. A more complex Lyapunov function has to be designed.

## VI. CASE STUDIES

In this section, we specialize the previous conditions to specific classes of nonlinear systems. In particular, we will consider minimum-phase systems and nonlinear systems that are described in the Lur'e form with an incremental monotonic nonlinearity.

### A. Minimum-phase systems

A significant amount of results in synchronization considers systems either in normal form or for which there exists a globally defined diffeomorphism that allows to rewrite the dynamics in normal form, see, e.g. [14, 26, 34]. For such a class of systems, the zero-dynamics is generally assumed to possess a unique steady-state trajectory which is attractive with a given domain of attraction (minimum-phase). We show below that the conditions provided in Section IV recover these results. In particular, consider a network where each agent is modeled as a SISO system whose dynamics can be described by

$$\begin{aligned} \dot{z}_i &= F(z_i, y_i) \\ \dot{y}_i &= q(z_i, y_i) + u_i \end{aligned} \quad (44)$$

where  $x_i = (z_i, y_i) \in \mathbb{R}^{n_x}$  is the state, with  $i = 1, \dots, N$ . In particular, we consider systems with unitary relative degree, that is,  $z_i \in \mathbb{R}^{n_x-1}$  and  $y \in \mathbb{R}$ . The functions  $F, q$  are  $C^2$  in their arguments. The high-frequency gain (i.e. the coefficient in front of  $u$ ) is selected as 1 without loss of generality. With respect to the representation (42), we have  $f(x) = (F(z, y)^\top, q(z, y))^\top$ ,  $g(x) = (0, 1)^\top$ ,  $h(x) = (0, 1)$ .

We cast the synchronization problem to the incremental framework with the following (incremental) minimum-phase assumption.

**Assumption 3** (Incremental minimum-phase). *There exists a  $C^1$  symmetric and positive definite matrix function  $S : \mathbb{R}^{n_x-1} \mapsto \mathbb{R}^{(n_x-1) \times (n_x-1)}$  and positive real numbers  $\underline{s}, \bar{s}, \varepsilon, \ell > 0$  such that the following inequalities<sup>1</sup> hold*

$$\underline{s}I \preceq S(z) \preceq \bar{s}I, \quad L_f S(z, y) \preceq -\varepsilon S(z),$$

$$\left| \frac{\partial F}{\partial y}(z, y) \right| \leq \ell, \quad \left| \frac{\partial q}{\partial z}(z, y) \right| \leq \ell, \quad \left| \frac{\partial q}{\partial y}(z, y) \right| \leq \ell,$$

for all  $(z, y) \in \mathbb{R}^{n_x}$ .

Under the previous assumption, the problem of synchronization is solved, as established in the following result. It recovers the results of [14].

**Proposition 2** (Synchronization of minimum-phase systems). *Consider a network  $G = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$  of agents (44) and let Assumptions 2 and 3 hold. Then, there exists  $\kappa^* > 0$  such that, for any  $\kappa \geq \kappa^*$ , the distributed state-feedback control law (7) with  $\varphi(x, y, t) = \kappa y$  solves the synchronization Problem 1 for (44). Furthermore, if the metric  $S$  is Euclidean (i.e. constant), then Assumption 2 can be relaxed with Assumption 1.*

*Proof.* The proof follows by noticing that condition (43) is satisfied with the metric  $P(x) = \text{diag}(S(z), 1)$ . Then, it is completed with standard high-gain arguments in order to find a sufficiently large  $\rho$  satisfying the condition (38). As a consequence, the value of  $\kappa^*$  depends on the parameter  $\mu$  of Lemma 1 and the Lipschitz constant  $\ell$  of Assumption 3. Details are omitted for space reasons. A complete proof can be found, e.g., in the first part of [22, Proposition 6] in the context of constant output regulation. An alternative version using similar arguments can be found also in [37, Theorem 2].  $\square$

### B. Systems with monotonic nonlinearity

We consider now a connected network a network  $G = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$  of nonlinear systems having the form

$$\begin{aligned} \dot{x}_i &= Ax_i + G\psi(\eta_i, t) + Bu_i + \omega(t), & \eta_i &= Hx_i, \\ y_i &= Cx_i, \end{aligned} \quad (45)$$

$i = 1 \dots, N$ , where  $A, G, B, H, C$  are constant matrices of suitable dimension,  $\eta_i \in \mathbb{R}^{n_\eta}$  is a linear combination of the state,  $\omega : \mathbb{R} \mapsto \Omega$  is any time-varying piecewise continuous signal taking values in a compact set  $\Omega \subset \mathbb{R}^{n_x}$  and  $\psi : \mathbb{R}^{n_\eta} \times \mathbb{R} \mapsto \mathbb{R}^{n_\psi}$  is a  $C^1$  nonlinearity satisfying the following *incremental monotonic* condition. With respect to the representation (16), we have  $f(x) = Ax + G\psi(\eta, t) + \omega(t)$ . The signal  $\omega$  must be the same for all the agents of the networks.

**Assumption 4** (Monotonic). *There exists a symmetric positive matrix  $\Upsilon = \Upsilon^\top \succ 0$  such that*

$$0 \preceq \frac{\partial \psi}{\partial \eta}(\eta, t) + \frac{\partial \psi}{\partial \eta}(\eta, t)^\top \preceq \Upsilon \quad (46)$$

for all  $(\eta, t) \in \mathbb{R}^{n_\eta} \times \mathbb{R}$ .

<sup>1</sup>The notation  $L_f S(z, y)$  has to be understood as the Lie derivative of  $S$  along the vector field  $z \mapsto f(z, y)$  where  $y$  is fixed.

Incremental stability for this class of systems has been studied, for instance, in [4, 20, 48]. It is also known as quadratic stability [15]. In the following, we want to show that it is possible to rewrite the design proposed in Theorem 1 in the form of a Linear Matrix Inequality (LMI).

**Proposition 3** (Synchronization of monotonic systems). *Consider a network  $G = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$  of systems (45) and let Assumption 4 hold. Suppose there exist a symmetric positive definite matrix  $W = W^\top \succ 0$ , and two strictly positive real number  $\epsilon, \rho > 0$  such that the following LMI*

$$\begin{pmatrix} AW + W^\top A^\top - \rho BB^\top & G + W^\top H^\top & W \\ G^\top + HW & -4\Upsilon^{-1} & 0 \\ W & 0 & -\epsilon I \end{pmatrix} \preceq 0 \quad (47)$$

is satisfied. Then, the assumptions of Theorem 1 are satisfied with  $D = I$ ,  $C = I$ ,  $E = B^\top W^{-1}$  and, for any  $\kappa \geq \frac{\rho}{2\mu}$ , with  $\mu$  given by Lemma 1, the distributed feedback law (26) with

$$\varphi(x, t) = \kappa B^\top P x, \quad P = W^{-1},$$

solves the synchronization Problem 1 for the network (45). Moreover, if  $B = WC^\top$ , then the Assumptions of Theorem 1 are satisfied with  $D = I$  and  $E = I$  and synchronization is achieved with  $\varphi(x, t) = \kappa y$ .

*Proof.* The proof follows similar lines as in [20, Proposition 1] (see also [21, Lemma 2]) and it is omitted for space reasons.  $\square$

Note that a similar result has been given in [52] for nonlinearities satisfying an incremental sector bound condition.

## VII. DEEP LEARNING FOR METRIC ESTIMATION

The main limitation of the approach presented in Section V is the complexity of finding a metric  $P$  solving (27) and, at the same time, satisfying the Killing vector field property in (28). As shown in Section VI, for some classes of systems these conditions can be easily verified, in particular for constant metric. However, this is not an easy task in general, especially when considering high-dimensional systems with significant nonlinearities. In this section, we aim to provide a practical solution to such limitations. First, we relax the Killing vector assumption. Hence, we show that synchronization can still be achieved (in compact sets) when the  $L_g P(x, t)$ -term is non-zero, yet sufficiently small. Second, we circumvent the need of computing a suitable metric by relying on Deep Learning tools. We provide a general optimization-based algorithm allowing the approximation of both the metric and the integrability condition in (29) via Deep Neural Networks (DNNs). Finally, the proposed design is applied to a synchronization problem of a network of Lorentz oscillators. We present results for the state-feedback control design. Nevertheless, the same tools can be used to derive conditions for the output-feedback design. Furthermore, we consider the single-input case for simplicity. Yet, similar results can be applied for the multi-input one.

### A. Killing-less leader-synchronization

We now present the theoretical result relaxing the Killing vector field property used in the development of Section V. In particular, we focus on condition (28).

**Proposition 4** (Killing-less synchronization). *Consider a network  $G = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$  of systems (25). Let Assumption 2 hold and assume there exists a  $C^1$  matrix function  $P : \mathbb{R}^{n_x} \times \mathbb{R} \mapsto \mathbb{R}^{n_x \times n_x}$  taking symmetric positive definite values, positive real numbers  $p, \bar{p}, \varepsilon, \rho > 0$  and a  $C^2$  function  $\alpha : \mathbb{R}^{n_x} \times \mathbb{R} \mapsto \mathbb{R}^{n_u}$  such that (27), (29) hold. Let  $X \subset \mathbb{R}^{n_x}$  be a compact set and fix  $\kappa \geq \frac{\rho}{2\mu}$ , with  $\mu$  given by Lemma 2. Then, there exists a positive real number  $\epsilon_X > 0$  such that, if  $|L_g P(x, t)| \leq \epsilon_X$  for all  $(x, t) \in X \times \mathbb{R}$ , the trajectory of network in closed-loop with the state-feedback distributed control law (26) with (30) exponentially converges towards the synchronization manifold (8) for all time existence of solutions in  $X$ , i.e. for all  $t_f \geq t_0$  such that  $\mathcal{X}_i(x_i^o, t, t_0) \in X$  for all  $t \in [t_0, t_f]$  and all  $i = 1, \dots, N$ .*

The previous proposition relax the Killing condition by focusing on solutions evolving in a compact set  $X$ . As a consequence, if such a set  $X$  is invariant for solutions of each agent of the network, then synchronization can be achieved.

*Proof.* The proof is identical to the proof of Theorem 2 up to equation (32). Therefore, the first part is omitted. Then, by using the integrability assumption on the function  $\alpha$  in (29), the time derivative of  $V_i$  in (32) becomes

$$\begin{aligned} \dot{V}_i(t) = & \int_0^1 \left[ \frac{\partial \Gamma_i^\top}{\partial s}(s, t, t_0) L_f P(\zeta_i, t) \frac{\partial \Gamma_i}{\partial s}(s, t, t_0) \right. \\ & - 2\kappa \frac{\partial \Gamma_i^\top}{\partial s}(s, t, t_0) \sum_{j=1}^N \ell_{ij} P(\zeta_j, t) g(\zeta_j, t) \\ & \quad \times g(\zeta_j, t)^\top P(\zeta_j, t) \frac{\partial \Gamma_j}{\partial s}(s, t, t_0) \\ & \left. - 2\kappa \frac{\partial \Gamma_i^\top}{\partial s}(s, t, t_0) L_g P(\zeta_i, t) \frac{\partial \Gamma_i}{\partial s}(s, t, t_0) \sum_{j=1}^N \ell_{ij} \alpha(\zeta_j, t) \right] ds. \end{aligned}$$

Consequently, by selecting  $\kappa \geq \frac{\rho}{2\mu}$  with  $\rho$  satisfying the SCMF condition in (27),  $\mu$  given by Lemma 2, and by following similar computations as in the proof of Theorem 2, we get

$$\begin{aligned} \sum_{i=2}^N \dot{V}_i(t) \leq & -\varepsilon \int_0^1 \left[ \frac{\partial \Gamma^\top}{\partial s}(s, t, t_0) P(\zeta, t) \frac{\partial \Gamma}{\partial s}(s, t, t_0) \right. \\ & \left. - 2\kappa \frac{\partial \Gamma^\top}{\partial s}(s, t, t_0) D_g P(\zeta, t) \frac{\partial \Gamma}{\partial s}(s, t, t_0) L_{22} \Lambda(\zeta, t) \right] ds \end{aligned} \quad (48)$$

where we indicated with

$$\begin{aligned} D_g P(\zeta, t) &:= \text{diag}\{L_g P(\zeta_2, t), \dots, L_g P(\zeta_N, t)\} \\ \Lambda(\zeta, t) &:= [\alpha(\zeta_2, t)^\top, \dots, \alpha(\zeta_N, t)^\top]^\top. \end{aligned}$$

Now, for any  $X \subset \mathbb{R}^{n_x}$ , let

$$\epsilon_X := \frac{\varepsilon p}{4\kappa N^{\frac{3}{2}} \bar{\mu} \bar{\alpha}}$$

with  $\bar{\mu}$  being the largest eigenvalue of  $\mathcal{L}_{22}$  and

$$\bar{\alpha} := \sup_{x \in X, t \geq t_0} \{\alpha(x, t)\}.$$

Then if  $|L_g P(x, t)| \leq \epsilon_X$ , it follows that

$$\begin{aligned} \kappa \int_0^1 \frac{\partial \Gamma^\top}{\partial s}(s, t, t_0) D_g P(\zeta, t) \frac{\partial \Gamma}{\partial s}(s, t, t_0) L_{22} \Lambda(\zeta, t) ds \\ \leq \frac{\varepsilon p}{2} \int_0^1 \frac{\partial \Gamma^\top}{\partial s}(s, t, t_0) \frac{\partial \Gamma}{\partial s}(s, t, t_0) ds \end{aligned}$$

which by (48) it implies

$$\dot{V}(t) = \sum_{i=2}^N \dot{V}_i(t) \leq -\frac{\varepsilon}{2} \sum_{i=2}^N V_i(t) = -\frac{\varepsilon}{2} V(t).$$

The proof concludes by following the same lines of Theorem 2.  $\square$

**Remark 1.** Note that a bound on  $\epsilon_X$  can be given also by linearizing the error dynamics around the synchronization manifold  $\mathcal{D}$  defined in (8). This recovers known results on local exponential synchronization, see [3].

### B. A DNN-based algorithm for metric estimation

As mentioned in the previous section, a drawback of the proposed approach lies in the fact that metrics may not be easy to find in the Riemannian scenario. Moreover, even when a metric has been found, it may not be straightforward to design a control law satisfying the integrability condition (29). A way of overcoming such difficulties is to combine the proposed control design with Deep Learning tools. In the last years, Deep Neural Networks (DNNs) turned out to be effective tools for solving complex differential equations, see, e.g., [8, 38]. As a matter of fact, recent works began mixing learning tools and control. Such a combined framework tackles the complexity of computing control theoretic exact solutions by exploiting deep approximators, e.g., [27, 39, 53]. Hence, the idea is to set up an optimization problem for approximating the solution. In this section, we circumvent the need of analytically computing a suitable metric by learning it with a DNN. Once a suitable metric has been found, we train a second DNN to satisfy the integrability condition.

Similar approaches already appeared in the literature (see e.g., [47, 49]). In [47], the authors propose a convex optimization problem to compute a suitable metric. Yet, they successively suggest to approximate the solution via a DNN. Hence, the convex optimization is solved on a finite number of samples and the DNN provides a continuous interpolation through those points. This overcomes the need of solving such an optimization in each point of the state space. Our approach, however, is inspired by [49]. The authors propose a Siamese DNN structure [44]. The goal is to minimize

a loss function defined by the matrix conditions required for contraction. Once such a function reaches 0, the DNN provides the entries of a suitable metric for each point in the training/test datasets. Even if similar, our solution differs in three main points. First, we rely on the continuous time framework. Hence, we avoid the need of a Siamese network by computing the DNN's Jacobian. Note that, usually, such a Jacobian can be easily obtained thanks to the automatic differentiation tools provided by common libraries such as Pytorch [35]. Second, we add a separate estimator which looks for the best parameters in the cost function. It works jointly with the DNN during the optimization process. Finally, we rely on (29) instead of computing the control law via approximate integration over the geodesic. This greatly simplifies the algorithm, since geodesics are not easy to find in general.

We now describe the proposed algorithm. Let us consider the problem of finding a suitable approximation of the metric first. The neural metric is constructed as

$$P(x, \theta') = \begin{pmatrix} p_1(x, \theta') & p_2(x, \theta') & \cdots & p_n(x, \theta') \\ p_2(x, \theta') & p_{n+1}(x, \theta') & \cdots & p_{2n}(x, \theta') \\ \vdots & \vdots & \ddots & \vdots \\ p_n(x, \theta') & p_{2n}(x, \theta') & \cdots & p_M(x, \theta') \end{pmatrix},$$

where  $M = \frac{n(n+1)}{2}$ ,  $n = n_x$ , is the total number of entries to be learned, the vector  $\varrho = (p_0(x, \theta'), \dots, p_M(x, \theta'))^\top$  is the output of the neural network  $\text{DNN}_P : \mathbb{R}^{n_x} \times \mathbb{R}^{n_{\theta'}} \mapsto \mathbb{R}^M$  and  $\theta' \in \mathbb{R}^{n_{\theta'}}$  is the vector of  $\text{DNN}_P$  parameters. To train the  $\text{DNN}_P$  parameters, we relax the Killing-vector assumption and rely on Proposition 4. We set up an optimization problem asking for the minimization of the following cost function

$$J_P(x, \theta') = \sum_{i=1}^4 w_i J_i(x, \theta'), \quad (49)$$

being  $w = (w_1, \dots, w_4)$  a vector of (positive) scalar weights and

$$J_i(x, \theta') = \ln \left( \max \left( \Re \left\{ \lambda_M(M_i) \right\}, 0 \right) + 1 \right),$$

with  $i = 1, \dots, 4$ ,  $\lambda_M$  being the maximum eigenvalue and  $M_i$  defined as

$$\begin{aligned} M_1 &= L_f P(x, \theta') - \rho P(x, \theta') g(x) g^\top(x) P(x, \theta') + \varepsilon I \\ M_2 &= L_g P(x, \theta') - \epsilon I \\ M_3 &= -L_g P(x, \theta') - \epsilon I \\ M_4 &= -P(x, \theta') + pI \end{aligned}$$

where  $\rho, \epsilon, p > 0$  are positive scalars with  $\varepsilon > \epsilon$  and where  $\Re\{\lambda\}$  is the real part of the complex number  $\lambda \in \mathbb{C}$ . Note that each cost  $J_i$  serves the purpose of satisfying a particular condition for the neural metric. While  $J_1$  provides a positive cost if the contraction condition (27) is not satisfied,  $J_2$  and  $J_3$  encourage the boundedness of  $L_g P$ , thus relaxing the Killing vector condition (28), and  $J_4$  steers the solution towards positive definite matrices, see (27). Note that the upperbound is always satisfied as we optimize our algorithm

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**Algorithm 1** DNN-based controller learning

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- 1: Input: Dataset of  $(x, f(x), g(x), \frac{\partial f}{\partial x}(x), \frac{\partial g}{\partial x}(x))$ ,  $\text{DNN}_P, \text{DNN}_\alpha$ ;
- 2: **while**  $J_P(x, \theta') \neq 0$  **do**
- 3:   Train  $\text{DNN}_P$  and the estimator with (49);
- 4: **end while**
- 5: Train the  $\text{DNN}_\alpha$  with (50);
- 6: Set the distributed law

$$u_i = -\kappa \sum_{j=1}^N \ell_{ij} \text{DNN}_\alpha(x_j, \theta'');$$


---

in a compact set  $X$ . The natural logarithm is used as a regularization term between costs  $J_i$ . It allows the rescaling of widely different costs to similar values and a more precise selection of their importance through the weight vector  $w$ . In parallel to the  $\text{DNN}_P$ , we train a parameter estimator outputting the values of  $\rho, \varepsilon, \epsilon, p$ . The estimator and  $\text{DNN}_P$  work together, trying to minimize (49). Note that if the cost reaches 0, all the contraction conditions are satisfied for the dataset and the learned estimator outputs, hence learning can be stopped.

The second step is to find a suitable law satisfying the integrability condition (29). We train the parameters  $\theta'' \in \mathbb{R}^{n_{\theta''}}$  of the second network  $\text{DNN}_\alpha : \mathbb{R}^{n_x} \times \mathbb{R}^{n_{\theta''}} \mapsto \mathbb{R}^{n_u}$  such that

$$J_\alpha(x, \theta'') = \left| \frac{\partial \text{DNN}_\alpha}{\partial x}(x, \theta'') - g(x)^\top P(x, \theta') \right|^2 \quad (50)$$

is minimized.

Finally, the controller is synthesized as in (30) with the approximation  $\alpha(x) \approx \text{DNN}_\alpha(x, \theta'')$ . We rely on the robustness properties of contractive systems (see [45]) to compensate for the DNNs' approximation errors. The full learning procedure is summarized by Algorithm 1.

### C. Leader-synchronization of a network of Lorenz oscillators with DNNs

In the following, we apply the proposed algorithm to a leader-synchronization problem<sup>2</sup>. We consider a network of  $N = 6$  identical Lorenz attractors. Such systems are particularly interesting, since they can present a chaotic behavior. Each agent  $i = 1, \dots, N$  is described by the following dynamics

$$\begin{cases} \dot{x}_{i,1} = a(x_{i,2} - x_{i,1}) + u_i \\ \dot{x}_{i,2} = x_{i,1}(b - x_{i,3}) - x_{i,2} + (2 + \sin(x_{i,1}))u_i \\ \dot{x}_{i,3} = x_{i,1}x_{i,2} - cx_{i,3} \end{cases}$$

with  $x_i = (x_{i,1}, x_{i,2}, x_{i,3}) \in \mathbb{R}^3$  and where  $a, b, c$  are positive scalars. Similarly to [12, Section 5], we pick  $a = 10$ ,

<sup>2</sup>The code for reproducing the experiments proposed in this section can be found at <https://github.com/SamueleZoboli/Control-learning-multiagent-lorenz.git>

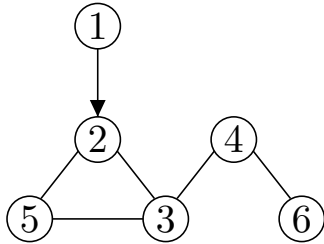


Fig. 1: Considered network of Lorentz’s oscillators

$b = 28$ ,  $c = \frac{8}{3}$ , guaranteeing the chaotic behavior. We consider the control matrix  $g(x) = (1 \ 2 + \sin(x_{i,1}) \ 0)^\top$  to exclude the possibility of feedback linearizing solutions. The agents communicate with each other following the leader-connected graph represented in Figure 1.

We code and train our fully-connected DNNs and estimator using PyTorch [35]. For the metric network, we select an architecture composed of 4 hidden layers, with size 30, 20, 20, 10 respectively and tanh activation functions. The output layer passes through a saturation function as a final activation, limiting the single elements of the metric. The second network presents 3 hidden layers, with size 30, 20, 10 respectively and tanh activation functions. We select the identity function as output layer activation function.

We select a weight vector  $w = (1, 10, 10, 20)$ , directing the learning towards positive matrices first and successively satisfying the Killing-less assumptions and the contraction condition. We train both the networks and the estimator using Adam optimizer [28]. The learning rate for the metric network and the estimator is set as  $3 \times 10^{-3}$ , while  $DNN_\alpha$  uses a learning rate of  $5 \times 10^{-3}$ . The DNNs learning rate are scheduled according to a cosine annealing policy [32], while the estimator one remains constant. We train the neural metric and the estimator over 100 epochs (yet stopped after 15 epochs due to the cost reaching 0) and the second DNN over 200 epochs. For both of the learning phases (the metric learning and the integrability learning), the dataset is composed by  $2 \times 10^5$  samples coming from a Gaussian distribution  $\mathcal{N}(0, 10)$ . We use 80% of the dataset as the training set, with a batch size of 512. The remaining 20% is used as test set.

We select a  $\kappa = 5$  and we apply the controller in a noisy-measurements scenario, i.e.,  $u_i = \varphi(x_i + \nu_i)$  where  $\nu_i \sim \mathcal{N}(0, 0.2)$  represents some Gaussian measurement noise. This allows testing the robustness properties of the proposed neural control law. Each agents’ initial condition is randomly sampled from a Gaussian distribution  $\mathcal{N}(0, 20)$ . Figure 2 and Figure 3 show the controller performances once the DNNs have been trained. Figure 2 presents the mean and standard deviation between agents of the norm of the error with respect to the leader trajectory. Figure 3 directly shows the state trajectories of each agent. As synchronization is achieved, we can see that the DNN optimized with (49) provides a suitable approximation of the metric, while the one trained with (50) effectively learns a primitive of  $g(x)^\top P(x, \theta')$ . The parameter estimator provided a decay rate  $\varepsilon \approx 4.7$  and  $\rho \approx 36.3$ . From Figure 3 it’s possible to see that the agents

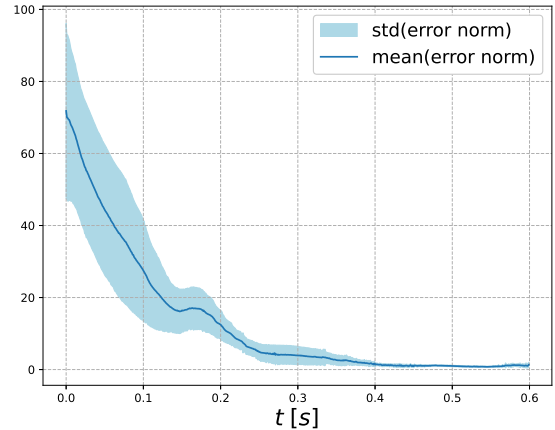


Fig. 2: Evolution of the mean error norm between agents with respect to the leader

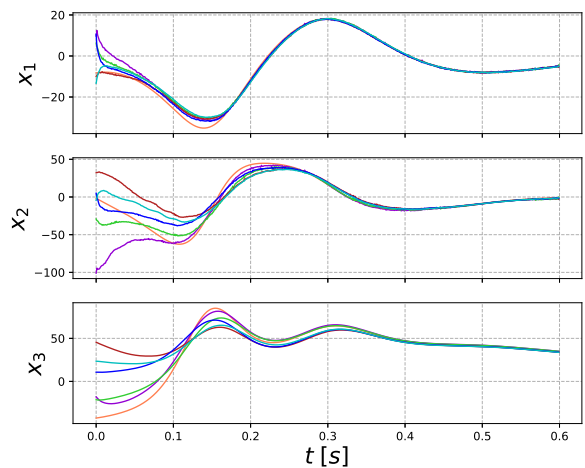


Fig. 3: States’ evolution of agents  $a_i$ ,  $i = 1, \dots, N$ , orange is leader

quickly synchronize, despite having significantly different initial conditions. It has to be noticed that such an approach depends on the generalization capabilities of the DNN. Since it is trained on a finite number of data, we cannot guarantee perfect behavior for all the state space. Some considerations about the neighborhoods of the sampled points could be done, similarly to [49, Theorem 1]. Nevertheless, due to robustness properties of contracting systems, training a neural metric on a sufficiently big dataset is a valuable tool to tackle the complexity of the proposed solution. This is proved by the experimental results.

## VIII. CONCLUSIONS

In this work, we considered the problem of state-synchronization for a homogeneous connected network. To this end, we derived sufficient conditions for the design of a distributed nonlinear diffusive coupling control law. First, by means of Euclidean metric-based incremental stability tools, we achieved network-consensus between the agents,

globally in the domain of attraction. Then, by means of a Riemannian metric-based analysis, we achieved synchronization of undirected graphs in the presence of a leader. Static state and output feedback laws have been investigated. Then we showed that our design applies for classical case studies, such as minimum phase systems and linear systems coupled with a monotonic nonlinearity. To conclude, we showed that the proposed conditions for the Riemannian case can be further relaxed by asking for synchronization in compact sets. We provided a constructive algorithm based on Deep Learning tools to estimate the metric. We applied such a design for the synchronization of a network of Lorentz oscillators. Future studies will involve the generalization of the Riemannian case without the presence of a leader and in case of directed networks and the design of (possibly robust) dynamic output-feedback control actions.

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