



**HAL**  
open science

## Bilipschitz equivalence of polynomials

Arnaud Bodin

► **To cite this version:**

Arnaud Bodin. Bilipschitz equivalence of polynomials. *Mathematical News / Mathematische Nachrichten*, 2021, 294 (7), pp.1295-1310. 10.1002/mana.201900152 . hal-03800601

**HAL Id: hal-03800601**

**<https://hal.science/hal-03800601v1>**

Submitted on 12 Jul 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# BILIPSCHITZ EQUIVALENCE OF POLYNOMIALS

ARNAUD BODIN

ABSTRACT. We study a family of two variables polynomials having moduli up to bilipschitz equivalence: two distinct polynomials of this family are not bilipschitz equivalent. However any level curve of the first polynomial is bilipschitz equivalent to a level curve of the second.

## 1. GLOBAL BILIPSCHITZ EQUIVALENCE

Let  $\mathbb{K}$  be  $\mathbb{R}$  or  $\mathbb{C}$ . For polynomial maps  $f, g : \mathbb{K}^n \rightarrow \mathbb{K}$  we introduce two notions of bilipschitz equivalence: a level equivalence (a hypersurface ( $f = c$ ) is sent to a hypersurface ( $g = c'$ )) and a global equivalence (any level ( $f = c$ ) is sent to another level ( $g = c'$ )).

- $\mathbb{K}^n$  is endowed with the Euclidean canonical metric.
- A map  $\Phi : \mathbb{K}^n \rightarrow \mathbb{K}^n$  is *Lipschitz* if there exists  $K > 0$  such that for all  $x, y \in \mathbb{K}^n$ :

$$\|\Phi(x) - \Phi(y)\| \leq K\|x - y\|.$$

- A map  $\Phi : \mathbb{K}^n \rightarrow \mathbb{K}^n$  is *bilipschitz* if it is a homeomorphism, Lipschitz and  $\Phi^{-1}$  is also Lipschitz. Equivalently,  $\Phi$  is bijective and there exists  $K > 0$  such that  $\frac{1}{K}\|x - y\| \leq \|\Phi(x) - \Phi(y)\| \leq K\|x - y\|$ .
- Two sets  $\mathcal{C}$  and  $\mathcal{C}'$  of  $\mathbb{K}^2$  are *bilipschitz equivalent* if there exists a bilipschitz map  $\Phi : \mathbb{K}^n \rightarrow \mathbb{K}^n$  such that  $\Phi(\mathcal{C}) = \mathcal{C}'$ .
- Two functions  $f, g : \mathbb{K}^n \rightarrow \mathbb{K}$  are *right-bilipschitz equivalent* if there exists a bilipschitz map  $\Phi : \mathbb{K}^n \rightarrow \mathbb{K}^n$  such that  $g \circ \Phi = f$ .
- Two functions  $f, g : \mathbb{K}^n \rightarrow \mathbb{K}$  are *left-right-bilipschitz equivalent* if there exist a bilipschitz map  $\Phi : \mathbb{K}^n \rightarrow \mathbb{K}^n$  and a bilipschitz map  $\Psi : \mathbb{K} \rightarrow \mathbb{K}$  such that  $g \circ \Phi = \Psi \circ f$ .

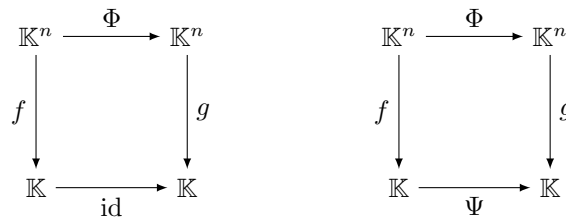


FIGURE 1. Two commutative diagrams. On the left: right-bilipschitz equivalence. On the right: left-right-bilipschitz equivalence.

Remarks:

---

*Date:* February 25, 2020.

*2010 Mathematics Subject Classification.* Primary 58K60 ; Sec. 12E05.

*Key words and phrases.* Bilipschitz geometry, Polynomials, Moduli.

*Acknowledgment.* This work was supported by the ANR project “LISA” (ANR-17-CE40-0023-01).

- A map  $\Phi : \mathbb{K}^n \rightarrow \mathbb{K}^n$  can be  $C^1$  but not Lipschitz. Hence (bi-)Lipschitz is *not* an intermediate case between smooth and continuous. This is due to the non-compactness: for instance  $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ ,  $x \mapsto x^2$  is  $C^1$  but not Lipschitz.
- For similar reasons an algebraic automorphism of  $\mathbb{K}^n$  does not necessarily provide a bilipschitz equivalence. For instance  $f(x, y) = y$  and  $g = y + x^2$  are algebraically equivalent using the map  $\Phi : (x, y) \mapsto (x, y - x^2)$ , but  $\Phi$  is not bilipschitz.

It is clear that bilipschitz equivalence implies topological equivalence (i.e. when  $\Phi$  and  $\Psi$  are only homeomorphisms). The main question is: does topological equivalence implies bilipschitz equivalence? The answer is negative.

We will actually prove more. A theorem of Fukuda asserts that in a family of polynomials there is only a finite number of different types, up to topological equivalence, see [4], [3]. However the following theorem proves that the family of polynomials  $f_s(x, y) = x(x^2y^2 - sxy - 1)$  has moduli for bilipschitz equivalence, i.e. any two polynomials in this family are not right-bilipschitz equivalent.

**Theorem 1.** *Consider the family of polynomial in  $\mathbb{K}[x, y]$ :*

$$f_s(x, y) = x(x^2y^2 - sxy - 1).$$

- $\mathbb{K} = \mathbb{R}$ . *Any two polynomials  $f_s$  and  $f_{s'}$  with  $s, s' \in \mathbb{R}$ ,  $s \neq s'$  are not right-bilipschitz equivalent. However the special levels ( $f_0 = 0$ ) and ( $f_1 = 0$ ) are bilipschitz equivalent and the generic levels ( $f_0 = 1$ ) and ( $f_1 = 1$ ) are bilipschitz equivalent.*
- $\mathbb{K} = \mathbb{C}$ . *Fix  $s \in \mathbb{C}$ , with  $s^2 + 3 \neq 0$ . For all but finitely many  $s' \in \mathbb{C}$ ,  $f_s$  and  $f_{s'}$  are not right-bilipschitz equivalent. However, if  $s^2 + 4 \neq 0$  and  $s'^2 + 4 \neq 0$ , the polynomials  $f_s$  and  $f_{s'}$  are topologically equivalent.*

This is a version at infinity of a result by Henry and Parusiński, [5]. Our polynomials  $f_s$  have only one special level ( $f_s = 0$ ) which plays the role of the singular level of the local examples of [5]. We recall that for a polynomial map  $f : \mathbb{K}^n \rightarrow \mathbb{K}$  there is a notion of *generic levels* ( $f = c$ ) and a finite number of *special levels* whose topology is not the generic one. Special levels can be due to the presence of a singular point or to singularity at infinity as this the case in our examples. We will in fact prove a non bilipschitz equivalence “at infinity”, after defining that two functions are bilipschitz equivalent at infinity if they are bilipschitz equivalent outside some compact sets.

*Acknowledgments.* I thank Vincent Grandjean, Anne Pichon and Patrick Popescu-Pampu for their encouragements and the referees for their comments.

## 2. LEVELS ARE BILIPSCHITZ EQUIVALENT

Lemmas 2 and 3 in this section will prove the bilipschitz real equivalence of theorem 1. Let

$$f_s(x, y) = x(x^2y^2 - sxy - 1)$$

which, in this section, is considered as a family of polynomials in  $\mathbb{R}[x, y]$ .

**Lemma 2.** *The levels ( $f_0 = 0$ ) and ( $f_1 = 0$ ) are bilipschitz equivalent, that is to say there exists a bilipschitz map  $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that  $\Phi((f_0 = 0)) = (f_1 = 0)$ .*

In other words, the (unique) special fibers of  $f_0$  and  $f_1$  are bilipschitz equivalent.

*Proof.*

**Definition of  $\Phi$ .**

– Let  $\sigma = \frac{\sqrt{5}+1}{2}$  be the positive root of  $z^2 - z - 1 = 0$ . Let  $\tau = \frac{\sqrt{5}-1}{2}$  be the positive root of  $z^2 + z - 1 = 0$ .

– We define a map  $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  by the following formulas:

– For  $(x, y) \in (xy = 1)$  we define:

$$\Phi(x, y) = (ax, by) \quad \text{with } ab = \sigma,$$

such that  $(a, b)$  depends on  $(x, y)$  in the following way:

$$\begin{cases} (a, b) = (\sigma, 1) & \text{if } |x| \leq \frac{1}{2} \\ (a, b) = (1, \sigma) & \text{if } |x| \geq 2 \end{cases}$$

and extended to a smooth map for  $\frac{1}{2} \leq |x| \leq 2$  so that the relation  $ab = \sigma$  is always satisfied on  $(xy = 1)$ .

– For  $(x, y) \in (xy = -1)$  we similarly define  $\Phi(x, y) = (ax, by)$  with  $ab = \tau$ , and  $(a, b) = (\tau, 1)$  for  $|x| \leq \frac{1}{2}$ ,  $(a, b) = (1, \tau)$  for  $|x| \geq 2$  and extended in a smooth map for  $\frac{1}{2} \leq |x| \leq 2$ .

–  $\Phi(0, y) = (0, y)$  for all  $y \in \mathbb{R}$ .

–  $\Phi(x, y) = (x, y)$  for  $(x, y)$  outside a neighborhood  $\mathcal{N}$  of radius 1 of  $(xy = 1) \cup (xy = -1)$ .

–  $\Phi$  is extended on  $\mathcal{N}$  to a bilipschitz homeomorphism  $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ .

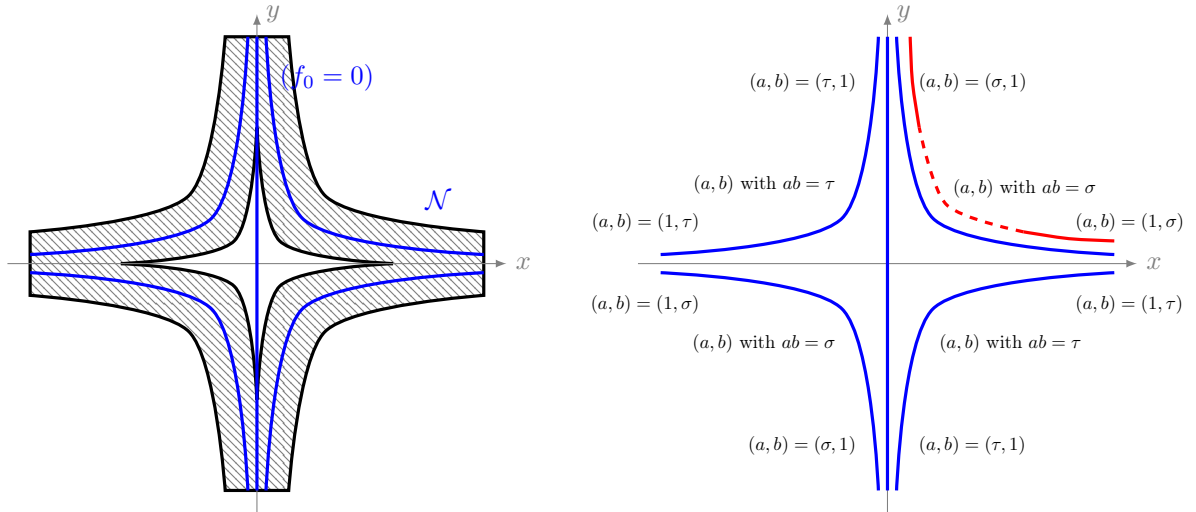


FIGURE 2. The definition of  $\Phi$ . Left: the level, a neighborhood of the level. Right: the values  $(a, b)$  for the definition of  $\Phi(x, y) = (ax, by)$  on the level.

– The only point to prove is that the formulas actually yield a bilipschitz map around the axis. For instance let  $(x_1, y_1) \in (xy = 1)$  with  $x_1 > 2$ , so that  $\Phi(x_1, y_1) = (x_1, \sigma y_1)$  and  $(x_2, y_2) \in (xy = -1)$  with  $x_2 > 2$  and  $\Phi(x_2, y_2) = (x_2, \tau y_2)$ . Then

$$\begin{aligned} \|\Phi(x_1, y_1) - \Phi(x_2, y_2)\| &= \|(x_1 - x_2, \sigma y_1 - \tau y_2)\| \\ &\leq \|(x_1 - x_2, 2\sigma(y_1 - y_2))\| \\ &\leq 2\sigma\|(x_1 - x_2, y_1 - y_2)\| \end{aligned}$$

(using that  $y_1 - y_2 = |y_1| + |y_2|$ ). A similar bound holds for  $\Phi^{-1}$  on this branch.

Then  $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is a bilipschitz homeomorphism.

**Equivalence.**

- Let  $f(x, y) = f_0(x, y) = x(x^2y^2 - 1)$  and  $g(x, y) = f_1(x, y) = x(x^2y^2 - xy - 1)$ .
- By definition of  $\Phi$ ,  $\Phi(0, y) = (0, y)$  so that the component  $(x = 0) \subset (f = 0)$  is sent by  $\Phi$  to  $(x = 0) \subset (g = 0)$ .
- Let  $(x, y) \in (xy = 1) \subset (x^2y^2 = 1) \subset (f = 0)$ . For such  $(x, y)$ ,  $\Phi(x, y) = (ax, by)$  with  $ab = \sigma$ .
- Let  $\tilde{g}(x, y) = x^2y^2 - xy - 1$ :

$$\tilde{g} \circ \Phi(x, y) = \tilde{g}(ax, by) = a^2b^2x^2y^2 - abxy - 1 = \sigma^2(xy)^2 - \sigma xy - 1.$$

As  $xy = 1$  we get:

$$\tilde{g} \circ \Phi(x, y) = \sigma^2 - \sigma - 1 = 0,$$

by definition of  $\sigma$ . Then  $\Phi(x, y) \subset (\tilde{g} = 0) \subset (g = 0)$ . A similar reasoning holds for  $(xy = -1)$ .

□

We now prove that two generic fibers are also bilipschitz equivalent.

**Lemma 3.** *The levels  $(f_0 = 1)$  and  $(f_1 = 1)$  are bilipschitz equivalent, that is to say there exists a bilipschitz map  $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that  $\Phi((f_0 = 1)) = (f_1 = 1)$ .*

*Proof.*

- **Parameterization of  $(f_0 = 1)$ .** The curve  $(f_0 = 1)$  has equation  $x^3y^2 - x - 1 = 0$  and a parameterization  $(x, y)$  is given by

$$y_+ = \sqrt{\frac{1}{x^2} + \frac{1}{x^3}} \quad \text{or} \quad y_- = -\sqrt{\frac{1}{x^2} + \frac{1}{x^3}} \quad \text{for } x \in ]-\infty, -1] \cup ]0, +\infty[.$$

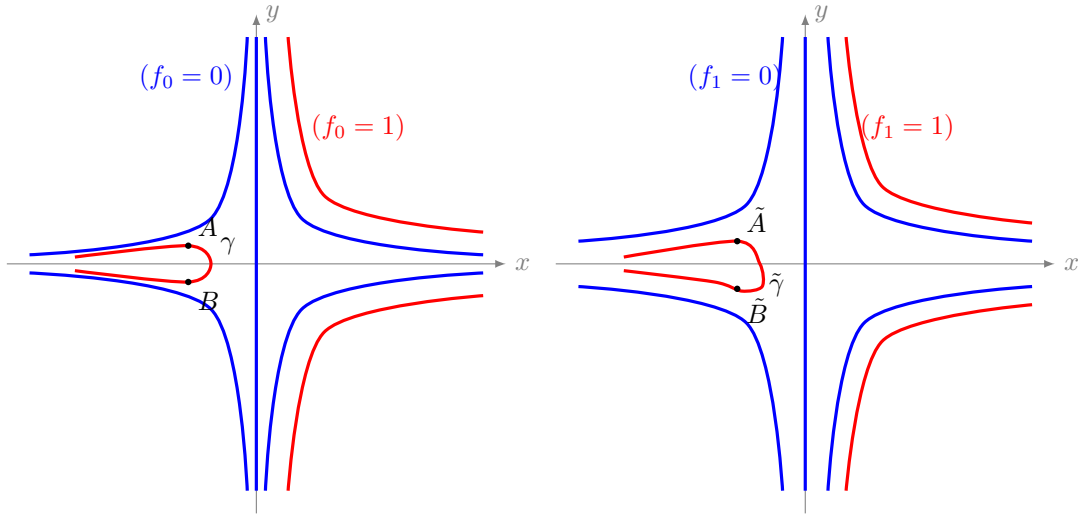


FIGURE 3. The levels  $(f_0 = 1)$  and  $(f_1 = 1)$ .

- **Parameterization of  $(f_1 = 1)$ .** The curve  $(f_1 = 1)$  has equation  $x^3y^2 - x^2y - x - 1 = 0$ , a parameterization is given by:

$$Y_+ = \frac{1}{2x} + \frac{1}{2}\sqrt{\frac{5}{x^2} + \frac{4}{x^3}} \quad \text{or} \quad Y_- = \frac{1}{2x} - \frac{1}{2}\sqrt{\frac{5}{x^2} + \frac{4}{x^3}} \quad \text{for } x \in ]-\infty, -\frac{5}{4}] \cup ]0, +\infty[.$$

- **Definition of  $\Phi$ .**

- Case  $x > 0$ .  $\Phi$  is defined on  $(f_0 = 1)$  using the parameterization by the formula  $\Phi(x, y) = (x, Y_+)$ , for  $(x, y) \in (f_0 = 1)$  with  $x > 0$  and  $y > 0$ ;  $\Phi(x, y) = (x, Y_-)$ , for  $(x, y) \in (f_0 = 1)$  with  $x > 0$  and  $y < 0$ .
- Case  $x \leq -2$ .  $\Phi$  is defined by the same formulas  $\Phi(x, y) = (x, Y_+)$  (for  $y > 0$ ) or  $\Phi(x, y) = (x, Y_-)$  (for  $y < 0$ ).
- Case  $-2 \leq x \leq -1$ . (Note: we do not use the above formulas in the neighborhood of the point  $(-1, 0)$  because the map  $y_+ \mapsto Y_+$  is not bilipschitz near this point.) Let  $A, B$  be the two points of  $(x = -2) \cap (f_0 = 1)$ . Let  $\tilde{A}, \tilde{B}$  be their images by  $\Phi$  (i.e.  $A, B$  belong  $(x = -2) \cap (f_1 = 1)$ ). Let  $\gamma$  be the compact part of  $(f_0 = 1)$  between  $A$  and  $B$  and  $\tilde{\gamma}$  be the compact part of  $(f_1 = 1)$  between  $\tilde{A}$  and  $\tilde{B}$ . We extend  $\Phi$  in a bilipschitz way from  $\gamma$  to  $\tilde{\gamma}$ . This is possible as  $\gamma$  and  $\tilde{\gamma}$  are two compact connected components of a smooth algebraic curve.  $\Phi$  is now defined everywhere on  $(f_0 = 1)$ .
- We extend  $\Phi$  on  $\mathbb{R}^2$  to a bilipschitz map  $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ . For instance we may suppose  $\Phi$  is the identity outside a tubular neighborhood of radius 1 of  $(f_0 = 1)$ .
- **Bilipschitz on  $(f_0 = 1)$ .** It remains to justify that  $\Phi$  is actually a bilipschitz map from  $(f_0 = 1)$  to  $(f_1 = 1)$ .
  - Case  $x > 0$  and  $x \rightarrow 0$ . Hence  $y \rightarrow \pm\infty$ . Then  $y_+ \sim \frac{1}{x^{3/2}}$  and  $Y_+ \sim \frac{1}{x^{3/2}} \sim y_+$  so that the map  $\Phi(x, y_+) = (x, Y_+)$  is bilipschitz. The same applies for  $y_-$  and  $Y_-$ .
  - Case  $x \rightarrow +\infty$ . Hence  $y \rightarrow 0$ . Then  $y_+ \sim \frac{1}{x}$  and  $Y_+ \sim \frac{\sqrt{5}+1}{2} \cdot \frac{1}{x} \sim \sigma y_+$ . Then, as in the proof of proposition 2,  $\Phi(x, y_+) = (x, Y_+)$  is bilipschitz. The same applies for  $y_-$  and  $Y_- \sim \tau y_-$  with  $\tau = \frac{\sqrt{5}-1}{2}$ .
  - Case  $x \rightarrow -\infty$ . It is similar to the previous case:  $Y_+ \sim \tau y_+$ ,  $Y_- \sim \sigma y_-$ .

□

### 3. MODULI

The following theorem proves that under bilipschitz equivalence at infinity a family of polynomials can have moduli. It is a version at infinity of the example of Henry and Parusiński [5]. Two functions  $f, g : \mathbb{K}^n \rightarrow \mathbb{K}$  are *right-bilipschitz equivalent at infinity* if there exist compact sets  $C, C'$  and a bilipschitz map  $\Phi : \mathbb{K}^n \setminus C \rightarrow \mathbb{K}^n \setminus C'$  such that  $g \circ \Phi = f$ .

Using this notion, we will prove the moduli affirmation of theorem 1 with the following refinement.

**Theorem 1'.**

$$f_s(x, y) = x(x^2y^2 - sxy - 1) \in \mathbb{K}[x, y].$$

- $\mathbb{K} = \mathbb{R}$ . Any two polynomials  $f_s$  and  $f_{s'}$  with  $s, s' \in \mathbb{R}$ ,  $s \neq s'$  are not right-bilipschitz equivalent at infinity (hence not globally right-bilipschitz equivalent). Moreover they are also not left-right-equivalent if we assume  $\Phi$  analytic at infinity.
- $\mathbb{K} = \mathbb{C}$ . Fix  $s \in \mathbb{C}$ , with  $s^2 + 3 \neq 0$ . For all but finitely many (explicit)  $s' \in \mathbb{C}$ ,  $f_s$  and  $f_{s'}$  are not right-bilipschitz equivalent at infinity (hence not globally right-bilipschitz equivalent).

#### 3.1. Preliminaries.

- Let  $f_s(x, y) = x(x^2y^2 - sxy - 1) = x^3y^2 - sx^2y - x$ .

- Then  $\partial_x f_s(x, y) = 3x^2y^2 - 2sxy - 1$ .
- The equation  $3z^2 - 2sz - 1 = 0$  has discriminant  $\Delta = 4(s^2 + 3)$  and two solutions:

$$\alpha_s = \frac{s + \sqrt{s^2 + 3}}{3} \quad \text{and} \quad \beta_s = \frac{s - \sqrt{s^2 + 3}}{3}.$$

- The polar curve  $\Gamma_s : (\partial_x f_s = 0)$ , associated to the projection on the  $y$ -axis, has two components:

$$(xy = \alpha_s) \quad \text{and} \quad (xy = \beta_s),$$

parameterized by:

$$\left(\alpha_s t, \frac{1}{t}\right) \quad \text{and} \quad \left(\beta_s t, \frac{1}{t}\right) \quad t \in \mathbb{K} \setminus \{0\}.$$

- We compute the values of  $f_s$  on the polar components. Near the point at infinity  $(0 : 1 : 0)$ , that is to say for  $t \rightarrow 0$ , we compute the values of  $f_s$  on each branch of  $\Gamma_s$ :

$$f_s \left(\alpha_s t, \frac{1}{t}\right) = \alpha_s(\alpha_s^2 - s\alpha_s - 1)t,$$

and

$$f_s \left(\beta_s t, \frac{1}{t}\right) = \beta_s(\beta_s^2 - s\beta_s - 1)t.$$

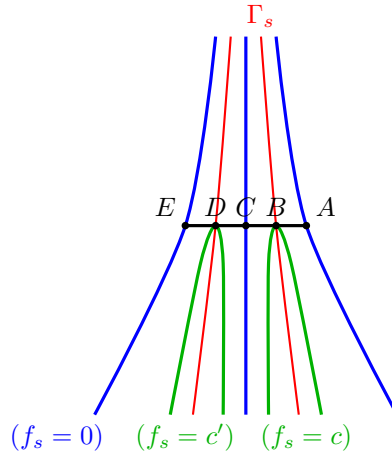
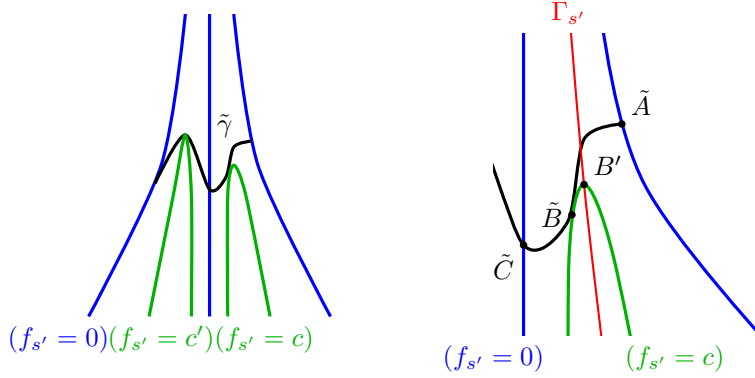
- We compare these values for two branches at a same  $y$ -value:

$$\frac{f_s \left(\alpha_s t, \frac{1}{t}\right)}{f_s \left(\beta_s t, \frac{1}{t}\right)} = \frac{\alpha_s(\alpha_s^2 - s\alpha_s - 1)}{\beta_s(\beta_s^2 - s\beta_s - 1)}.$$

- Our arguments will only focus on a neighborhood of a the point  $(0 : 1 : 0)$  at infinity. More precisely we will say that an analytic curve  $(x(t), y(t))$  tends to the point at infinity  $(0 : 1 : 0)$  if  $y(t) \rightarrow +\infty$  and  $\frac{|x(t)|}{|y(t)|} \rightarrow 0$  as  $t \rightarrow 0$ .

### 3.2. Proof in the real case.

- Fix  $t > 0$ . Let  $A, B, C, D, E$  be the following points having all  $y$ -coordinates equal to  $\frac{1}{t}$ :
  - $A \in (f_s = 0)$  with  $x_A > 0$ ,
  - $B \in \Gamma_s : (\partial_x f_s = 0)$  with  $x_B > 0$ ,
  - $C = (0, \frac{1}{t}) \in (f_s = 0)$ ,
  - $D \in \Gamma_s : (\partial_x f_s = 0)$  with  $x_D < 0$ ,
  - $E \in (f_s = 0)$  with  $x_E < 0$ .
- Let us fix  $s, s' \in \mathbb{R}$ . By contradiction let us assume that there exists a bilipschitz homeomorphism  $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that  $f_{s'} \circ \Phi = f_s$ . Let  $K$  be its bilipschitz constant. Let  $\tilde{A}, \tilde{B}, \dots$  be the image by  $\Phi$  of  $A, B, \dots$ . Let  $\gamma$  be the segment  $[AB]$  and  $\tilde{\gamma} = \Phi(\gamma)$ .
- $\Phi$  sends  $(f_s = 0)$  to  $(f_{s'} = 0)$  and, as it is a homeomorphism, it should send the component  $(x = 0)$  of  $(f_s = 0)$  to the component  $(x = 0)$  of  $(f_{s'} = 0)$ . Hence  $x_{\tilde{C}} = 0$ .
- $A, B, C, D, E$  and  $\gamma$  are all included in the disk of radius  $rt$  centered at  $C$ , where  $r$  is a constant that depends only on the fixed value  $s$ . Hence by the bilipschitz map  $\Phi$ ,  $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, \tilde{E}$  and  $\tilde{\gamma}$  are all included in a disk of radius  $Krt$  centered at  $\tilde{C}$ .
- There is an issue: the point  $B$  is on the polar curve  $\Gamma_s$  but  $\tilde{B}$  has no reason to be on  $\Gamma_{s'}$ . We will replace  $\tilde{B}$  by a point  $B'$  satisfying this condition.


 FIGURE 4. The situation for  $f_s$ .

 FIGURE 5. The situation for  $f_{s'}$ .

- Let  $c = f_s(B)$ . Let  $\tilde{X}_c$  be the part of  $(f_{s'} = c)$  in the ball of radius  $Krt$  centered at  $\tilde{C}$ . As  $f_{s'}(\tilde{B}) = f_s(B) = c$ , then  $\tilde{B} \in \tilde{X}_c$  and  $\tilde{X}_c$  is non empty. Moreover  $\tilde{X}_c$  is contained between two components of  $(f_{s'} = 0)$ :  $(x = 0)$  and one branch of  $(x^2y^2 - s'xy - 1 = 0)$ . Moreover  $\tilde{X}_c$  is strictly below  $\tilde{\gamma}$  except at  $\tilde{B}$  (because  $(f = c)$  is below  $\gamma = [AB]$  and intersects it only at  $B$ ).
- Let  $B'$  be the point of  $\tilde{X}_c$  such that  $y_{B'}$  is maximal among points of  $\tilde{X}_c$ . Then the tangent at  $B'$  is horizontal, that is to say  $\partial_x f_{s'}(B') = 0$ , hence  $B' \in \Gamma_{s'}$ . Remember also that  $B' \in \tilde{X}_c$  so that  $f_{s'}(B') = c$ .
- Partial conclusion: we constructed a point  $B' \in \Gamma_{s'} \cap (f_{s'} = c)$  such that  $\|B' - \tilde{C}\| \leq Krt$  (with  $x_{B'} > 0$ ).
- We carry on the same proof for the other side. Let  $c' = f_s(D)$ , we find a point  $D' \in \Gamma_{s'} \cap (f_{s'} = c')$  such that  $\|D' - \tilde{C}\| \leq Krt$  (with  $x_{D'} < 0$ ).
- Now both these points  $B'$  and  $D'$  are in the same disk of radius  $Krt$  centered at  $\tilde{C}$ . In particular:

$$y_{D'} - 2Krt \leq y_{B'} \leq y_{D'} + 2Krt.$$



- Let  $B' = (\alpha_{s'}t', \frac{1}{t'})$  be the coordinates of  $B'$  on the first branch of  $\Gamma_{s'}$  and  $D' = (\beta_{s'}t'', \frac{1}{t''})$  be the coordinates of  $D'$  on the second branch of  $\Gamma_{s'}$ . The former inequalities rewrite:

$$\frac{1}{t''} - 2Krt \leq \frac{1}{t'} \leq \frac{1}{t''} + 2Krt.$$

We consider  $t \rightarrow 0$ , so that  $t' \rightarrow 0$ ,  $t'' \rightarrow 0$  (a neighborhood of  $(0 : 1 : 0)$  is sent to a neighborhood of  $(0 : 1 : 0)$ ). Hence  $t'' = t' + O(tt't'') = t' + O(tt'^2)$ .

- Now

$$\begin{aligned} \frac{f_{s'}(\alpha_{s'}t', \frac{1}{t'})}{f_{s'}(\beta_{s'}t'', \frac{1}{t''})} &= \frac{\alpha_{s'}t'(\alpha_{s'}^2 - s'\alpha_{s'} - 1)}{\beta_{s'}t''(\beta_{s'}^2 - s'\beta_{s'} - 1)} \\ &= \frac{\alpha_{s'}t'(\alpha_{s'}^2 - s'\alpha_{s'} - 1)}{\beta_{s'}(t' + O(tt'^2))(\beta_{s'}^2 - s'\beta_{s'} - 1)} \rightarrow \frac{\alpha_{s'}(\alpha_{s'}^2 - s'\alpha_{s'} - 1)}{\beta_{s'}(\beta_{s'}^2 - s'\beta_{s'} - 1)} \end{aligned}$$

as  $t' \rightarrow 0$ .

- On the other hand:

$$\frac{f_{s'}(B')}{f_{s'}(D')} = \frac{c}{c'} = \frac{f_{s'}(\tilde{B})}{f_{s'}(\tilde{D})} = \frac{f_s(B)}{f_s(D)} = \frac{\alpha_s(\alpha_s^2 - s\alpha_s - 1)}{\beta_s(\beta_s^2 - s\beta_s - 1)}.$$

Finally:

$$\frac{\alpha_s(\alpha_s^2 - s\alpha_s - 1)}{\beta_s(\beta_s^2 - s\beta_s - 1)} = \frac{\alpha_{s'}(\alpha_{s'}^2 - s'\alpha_{s'} - 1)}{\beta_{s'}(\beta_{s'}^2 - s'\beta_{s'} - 1)}.$$

- The map  $s \mapsto \frac{\alpha_s(\alpha_s^2 - s\alpha_s - 1)}{\beta_s(\beta_s^2 - s\beta_s - 1)} = \frac{2(s^2+3)\alpha_s+s}{2(s^2+3)\beta_s+s}$  is strictly decreasing for  $s \in \mathbb{R}$  so that  $s = s'$ .
- Conclusion: if  $s, s' \in \mathbb{R}$ , with  $s \neq s'$ , then there exists no bilipschitz homeomorphism sending  $f_s$  to  $f_{s'}$ . Since our arguments only care about situation near  $(0 : 1 : 0)$   $f_s$  and  $f_{s'}$  are not right-bilipschitz equivalent at infinity.

**3.3. No left-right-equivalence.** We now prove that for  $s \neq s'$   $f_s$  and  $f_{s'}$  are not left-right-equivalent, if we ask the homeomorphism  $\Phi$  to be analytic near the point at infinity  $(0 : 1 : 0)$ . By contradiction we suppose that there exist bilipschitz homeomorphisms  $\Phi$  and  $\Psi$  such that  $f_{s'} \circ \Phi = \Psi \circ f_s$  and  $\Phi$  is analytic near the point at infinity  $(0 : 1 : 0)$ . We continue with the same notation as above, but we cannot conclude as before because we no longer have  $\frac{f_{s'}(B')}{f_{s'}(D')}$  equal to  $\frac{f_s(B)}{f_s(D)}$ .

- Let  $C = (0, \frac{1}{t})$  and  $\Phi(C) = \tilde{C} = (0, \frac{1}{\tilde{t}})$  ( $t > 0$ ). The map  $\frac{1}{t} \mapsto \frac{1}{\tilde{t}}$  is a bilipschitz homeomorphism. We will assume  $\Phi(0, 0) = (0, 0)$  so that  $\frac{1}{K}\frac{1}{t} \leq \frac{1}{\tilde{t}} \leq K\frac{1}{t}$  hence  $\frac{1}{K}t \leq \tilde{t} \leq Kt$ . Define  $\chi(t) = \tilde{t}$ , for  $t > 0$ , and set  $\chi(0) = 0$ . In the following we will actually only need the relation  $\frac{1}{K}t \leq \chi(t) \leq Kt$ , but in fact the map  $t \mapsto \chi(t)$  is a bilipschitz homeomorphism (with the constant  $K^3$ ).
- We assumed that the map  $\Phi$  is analytic at infinity around  $(0 : 1 : 0)$ . It implies that the map  $t \mapsto \chi(t)$  is analytic for  $t > 0$ :  $\chi(t) = a_0t^{r_0} + a_1t^{r_1} + \dots$ . The map  $\chi$  being bilipschitz it implies  $r_0 = 1$  so that  $\chi(t) = a_0t + a_1t^{r_1} + \dots$  with  $r_1 > 1$ .
- Notice that the relation  $f_{s'} \circ \Phi = \Psi \circ f_s$  implies that the map  $\Psi$  is also an analytic map.
- Recall that  $B = (\alpha_s t, \frac{1}{t})$  and  $f_s(B) = c = \alpha_s(\alpha_s^2 - s\alpha_s - 1)t$ ,  $D = (\beta_s t, \frac{1}{t})$  and  $f_s(D) = c' = \beta_s(\beta_s^2 - s\beta_s - 1)t$ .  $\Phi(B) = \tilde{B}$  and  $f_{s'}(\tilde{B}) = \tilde{c} = \Psi(c)$ ,  $\Phi(D) = \tilde{D}$  and  $f_{s'}(\tilde{D}) = \tilde{c}' = \Psi(c')$ . We found  $B' = (\alpha_{s'}t', \frac{1}{t'})$  close to  $\tilde{B}$  such that  $f_{s'}(B') = f_{s'}(\tilde{B}) = \tilde{c}$ . Hence  $\tilde{c} = \alpha_{s'}(\alpha_{s'}^2 - s'\alpha_{s'} - 1)t'$ . Similarly  $D' = (\beta_{s'}t'', \frac{1}{t''})$  is close to  $\tilde{D}$  and  $f_{s'}(D') = f_{s'}(\tilde{D}) = \tilde{c}'$ . Hence  $\tilde{c}' = \beta_{s'}(\beta_{s'}^2 - s'\beta_{s'} - 1)t''$ .

- $B'$  is close to  $\tilde{B}$  actually means  $\left| \frac{1}{t'} - \frac{1}{t} \right| \leq Krt$ , that implies  $|t' - \tilde{t}| \leq Krtt'\tilde{t}$ . That implies  $t' = \chi(t) + O(t^3)$ . Similarly  $t'' = \chi(t) + O(t^3)$ .
- The map  $\Psi$  is defined, for negative values, by  $c \mapsto \tilde{c}$  that is to say  $\alpha_s(\alpha_s^2 - s\alpha_s - 1)t \mapsto \alpha_{s'}(\alpha_{s'}^2 - s'\alpha_{s'} - 1)t'$ . It implies that, for  $u < 0$ , the map  $\Psi$  is defined by

$$\Psi : u \mapsto \alpha_{s'}(\alpha_{s'}^2 - s'\alpha_{s'} - 1)\chi\left(\frac{u}{\alpha_s(\alpha_s^2 - s\alpha_s - 1)}\right) + O(u^3).$$

Hence, as  $\chi(t) = a_0t + o(t)$ :

$$\Psi : u \mapsto \frac{\alpha_{s'}(\alpha_{s'}^2 - s'\alpha_{s'} - 1)}{\alpha_s(\alpha_s^2 - s\alpha_s - 1)}u + o(u).$$

Similarly  $\Psi(d) = \tilde{d}$  so that for  $u > 0$ :

$$\Psi : u \mapsto \frac{\beta_{s'}(\beta_{s'}^2 - s'\beta_{s'} - 1)}{\beta_s(\beta_s^2 - s\beta_s - 1)}u + o(u).$$

- By analyticity of  $\Psi$ , it implies that the coefficients of  $u$  are equal, whence

$$\frac{\alpha_s(\alpha_s^2 - s\alpha_s - 1)}{\beta_s(\beta_s^2 - s\beta_s - 1)} = \frac{\alpha_{s'}(\alpha_{s'}^2 - s'\alpha_{s'} - 1)}{\beta_{s'}(\beta_{s'}^2 - s'\beta_{s'} - 1)},$$

which is impossible for  $s \neq s'$  as we have seen before in section 3.2.

**3.4. No left-right-equivalence (again).** It is not clear whether  $f_s$  and  $f_{s'}$  ( $s \neq s'$ ) are or not left-right bilipschitz equivalent when no restriction is made on  $\Phi$ . However we can complicate our example in order to exclude left-right equivalence.

**Lemma 4.** *Let*

$$f_s(x, y) = x(x^4y^4 - 3sx^2y^2 + 1)$$

*be a family of polynomials in  $\mathbb{R}[x, y]$ . Then for  $s, s' > 1$ , with  $s \neq s'$ , the polynomials  $f_s$  and  $f_{s'}$  are not left-right bilipschitz equivalent.*

*Proof.* The proof is similar to the proof of section 3.3.

- The equation  $5z^4 - 9sz^2 + 1 = 0$  has 4 real solutions  $-\alpha_s < -\beta_s < \beta_s < \alpha_s$  corresponding to 4 branches of the polar curve ( $\partial_x f_s = 0$ ).
- We use the same method as before in section 3.3 with  $B = (-\alpha_s t, \frac{1}{t})$ ,  $f_s(B) = -\alpha_s(\alpha_s^4 - 3s\alpha_s^2 + 1)t = c_s t > 0$  and  $D = (\beta_s t, \frac{1}{t})$ ,  $f_s(D) = \beta_s(\beta_s^4 - 3s\beta_s^2 + 1)t = d_s t > 0$  (with  $t > 0$ ).
- This times for  $u > 0$  we have two formulas for  $\Psi$  :

$$\Psi(u) = c_{s'}\chi\left(\frac{u}{c_s}\right) + O(u^3),$$

and

$$\Psi(u) = d_{s'}\chi\left(\frac{u}{d_s}\right) + O(u^3).$$

- It implies that the bilipschitz map  $\chi$  verifies

$$\chi\left(\frac{c_s}{d_s}v\right) = \frac{c_{s'}}{d_{s'}}\chi(v) + O(v^3)$$

for all  $v > 0$  near 0.

- Then by lemma 5 below, it implies  $p = \frac{c_s}{d_s} > 1$  is equal to  $q = \frac{c_{s'}}{d_{s'}} > 1$  which is impossible if  $s \neq s'$ .

□

**Lemma 5.** *Let  $\chi : \mathbb{R} \rightarrow \mathbb{R}$  be a bilipschitz map such that*

$$\chi(pv) = q\chi(v) + O(v^3)$$

*for some constant  $p, q > 1$  and all  $v$  near 0. Then  $p = q$ .*

*Proof.* We have  $\chi(v) = q\chi(v/p) + v^3\eta(v)$ , where  $\eta(v)$  is a bounded function for  $v$  near 0. By induction it yields  $\chi(v) = q^n\chi(v/p^n) + v^3\sum_{k=0}^{n-1}\eta(v/p^k)(q/p^3)^k$ . Hence, except for the special case  $p^3 = q$  that would be treated in a similar way, we have:

$$(1) \quad \left| \chi(v) - q^n\chi\left(\frac{v}{p^n}\right) \right| \leq C v^3 \frac{1 - (q/p^3)^n}{1 - q/p^3}.$$

Let  $K > 0$  be a bilipschitz constant for  $\chi$ . As  $\chi(0) = 0$  we have  $K^{-1} < \frac{|\chi(v)|}{|v|} < K$  for all  $v \neq 0$ . In particular  $K^{-1} < p^n \frac{|\chi(v/p^n)|}{|v|} < K$ .

*Case  $p > q$ .* Then we have  $q^n\chi(v/p^n) \rightarrow 0$  as  $n \rightarrow +\infty$ . At the limit, when  $n \rightarrow +\infty$ , inequality (1) gives  $|\chi(v)| \leq C'v^3$ , which contradicts that  $\chi$  is bilipschitz.

*Case  $p < q$ .* Inequality (1) gives

$$\left| \frac{p^n}{q^n}\chi(v) - p^n\chi\left(\frac{v}{p^n}\right) \right| \leq C'v^3 \left( \frac{p^n}{q^n} - \frac{1}{p^{2n}} \right)$$

Fix  $v \neq 0$ . As  $n \rightarrow +\infty$ , the term  $p^n\chi(\frac{v}{p^n})$  does not tend towards 0, it contradicts that all the other terms  $\frac{p^n}{q^n}\chi(v)$ ,  $\frac{p^n}{q^n}$  and  $\frac{1}{p^{2n}}$  tends towards 0.

Conclusion:  $p = q$ . □

We completed the proof of theorem 1 in the real setting.

#### 4. PROOF IN THE COMPLEX CASE

The proof in the complex case at infinity is an adaptation of the local proof of Henry and Parusiński [5].

##### 4.1. Notations.

- Let  $g : \mathbb{C}^2 \rightarrow \mathbb{C}$  be a polynomial map and  $p = (x, y)$  be a point near the point at infinity  $(0 : 1 : 0)$ , that is to say  $|y| \gg 1$  and  $|x| \ll |y|$ .
- Fix  $p_0$ , let  $c = g(p_0)$ . Denote  $B(p_0, \rho)$  the open ball centered at  $p_0$  of radius  $\rho$  and  $X(p_0, \rho) = (g = c) \cap B(p_0, \rho)$ .
- Fix  $K > 0$  and denote  $\text{dist}_{p_0, \rho, K}(p, q)$  the inner distance of  $p$  and  $q$  supposed to be in the same connected component of  $X(p_0, K\rho)$ .
- Let

$$\phi(p_0, K, \rho) = \sup \frac{\text{dist}_{p_0, \rho, K}(p, q)}{\|p - q\|}$$

be the ratio between the inner and outer distances.

- Denote

$$\psi(p_0, K, \rho) = \sup_{\rho' \leq \rho} \phi(p_0, K, \rho').$$

- Finally let

$$Y(\rho, K, A) = \{p \mid \psi(p, K, \rho) \geq A\}.$$

be the set of points  $p$  where the curvature of the curve  $(g = c)$  is large.

- Let  $\Phi : \mathbb{C}^2 \rightarrow \mathbb{C}^2$  be a bilipschitz homeomorphism at infinity such that  $\tilde{g} \circ \Phi = g$ . Let  $L$  be a bilipschitz constant of  $\Phi$ .

- Once  $\Phi$  is fixed, we add a tilde to denote an object in then target space, for instance

$$\tilde{Y}(\rho, K, A) = \{\tilde{p} \mid \psi(\tilde{p}, K, \rho) \geq A\}$$

is the set of points  $\tilde{p}$  in the target space where the curvature of the curve ( $\tilde{g} = \tilde{c}$ ) is large.

We have the following lemma saying that points with large curvature are sent to points of large curvature by a bilipschitz map:

**Lemma 6** ([5], Lemma 2.1). *For  $K \geq L^2$ :*

$$\tilde{Y}(L^{-1}\rho, K, AL^2) \subset \Phi(Y(\rho, K, A)) \subset \tilde{Y}(L\rho, K, AL^{-2}).$$

And a variant:

**Lemma 7** ([5], Lemma 2.2). *Let  $\delta > 0$  and*

$$Y(\delta, K, M, A) = \{p \mid \psi(p, M\|p\|^{-1+\delta}, K) \geq A\}.$$

*If  $K \geq L^2$  then:*

$$\tilde{Y}(\delta, K, ML^{-\delta}, AL^2) \subset \Phi(Y(\delta, K, M, A)) \subset \tilde{Y}(\delta, K, ML^{+\delta}, AL^{-2}).$$

Remarks:

- There are two distinct uses of the norm:
  - $\|p - q\|$ : distance between two “near” points: a “small” number.
  - $\|p\|$ : distance to the origin: a “large” number. We will use it for  $\frac{1}{\|p\|}$  in order to get a “small” number.
- If we denote  $\tilde{p} = \Phi(p)$ , then the bilipschitz property implies:  $L^{-1}\|p\| \leq \|\tilde{p}\| \leq L\|p\|$  for some bilipschitz constant  $L$ , hence also:

$$L^{-1}\|p\|^{-1} \leq \|\tilde{p}\|^{-1} \leq L\|p\|^{-1}.$$

- Notice that in our definition of  $Y(\delta, K, M, A)$  of lemma 7 there is a term in  $\|p\|^{-1+\delta}$  while in [5] the term is  $\|p\|^{1+\delta}$ . After this modification, the proof is the same as in [5].
- We will restrict ourselves to a neighborhood of the point at infinity  $(0 : 1 : 0)$ , in particular we may suppose  $|y| \gg |x|$  so that morally  $\|p\| = \|(x, y)\| \simeq |y|$  (this is an equality in the case  $\|\cdot\| = \|\cdot\|_\infty$ ).

Fix  $s \in \mathbb{C}$  and denote  $f_s(x, y) = x(x^2y^2 - sxy - 1)$ . Let us denote  $U = \{(x, y) \mid |\partial_x f_s| < |\partial_y f_s|\}$ .

**Lemma 8** (compare to [5], Lemma 3.2). *Let  $(x(t), y(t)) \in U$  with  $y(t) = \frac{1}{t}$ . Then for  $s^2 + 3 \neq 0$ :*

$$x(t) = \gamma t + O(t^3) \quad \text{and} \quad f_s(x(t), y(t)) = \gamma(\gamma^2 - s\gamma - 1)t + O(t^3),$$

*with*

$$\gamma = \alpha_s \text{ or } \gamma = \beta_s \text{ a solution of } 3z^2 - 2sz - 1 = 0.$$

In this section we now suppose  $s^2 + 3 \neq 0$ .

*Proof.* Let  $u = xy$ . On  $U$  the inequality  $|\partial_x f_s| < |\partial_y f_s|$  yields  $|3u^2 - 2su - 1| < |x|^2|2u - s|$ . In a neighborhood of the point at infinity  $(0 : 1 : 0)$  we first prove that  $|x(t)|$  is bounded as  $t \rightarrow +\infty$ . If this is not the case, then write  $x(t) = a_0 t^{r_0} + a_1 t^{r_1} + \dots$  with  $r_i \in \mathbb{Q}$ ,  $r_i < r_{i+1}$  and here  $r_0 < 0$ . As  $y(t) = 1/t$ , then  $u(t) \sim a_0 t^{r_0-1} \rightarrow +\infty$ . Then  $|3u^2 - 2su - 1| < |x|^2|2u - s|$  implies  $r_0 \leq -1$  in contradiction with  $\frac{x(t)}{y(t)} \rightarrow 0$ . Now,

as  $|x(t)|$  is bounded, inequality  $|3u^2 - 2su - 1| < |x|^2|2u - s|$  implies that  $|u(t)|$  is also bounded. Write again  $x(t) = a_0t^{r_0} + a_1t^{r_1} + \dots$  and using that  $u(t)$  is bounded gives  $r_0 \geq 1$ :  $x(t) = a_0t + a_1t^{r_1} + \dots$  and  $u(t) = a_0 + a_1t^{r_1-1} + \dots$  ( $a_0 \in \mathbb{C}$ ). We plug  $u(t)$  in the inequality  $|3u^2 - 2su - 1| < |x|^2|2u - s|$ :

$$|3(a_0 + a_1t^{r_1-1} + \dots)^2 - 2s(a_0 + a_1t^{r_1-1} + \dots) - 1| = O(t^2).$$

It implies:

$$3a_0^2 - 2sa_0 - 1 = 0$$

and

$$6a_0a_1t^{r_1-1} - 2sa_1t^{r_1-1} = O(t^2).$$

We may suppose  $a_1 \neq 0$  and we now prove  $r_1 \geq 3$ . Otherwise  $6a_0 = 2s$ , that is to say  $s = 3a_0$ , but  $a_0$  is a solution of  $3z^2 - 2sz - 1 = 0$ . This is only possible if  $s^2 + 3 = 0$ . So that  $x(t) = \gamma t + O(t^3)$  as required, where  $\gamma$  is a solution of  $3z^2 - 2sz - 1 = 0$ . Then  $f_s(x(t), y(t)) = \gamma(\gamma^2 - s\gamma - 1)t + O(t^3)$ .  $\square$

**Lemma 9** (compare to [5], Lemma 3.3). *Let  $0 < \delta < 1$  and  $C > 0$ . On the set:*

$$\{p = (x, y) \mid \exists p_0 = (x_0, y_0) \in U, f_s(p) = f_s(p_0), |y - y_0| \leq C|y_0|^{-1+\delta}\},$$

if we denote  $y(t) = \frac{1}{t}$ , then

$$(2) \quad x(t) = O(t)$$

and

$$(3) \quad f_s(x(t), y(t)) = \gamma(\gamma^2 - s\gamma - 1)t + O(t^{2-\delta}).$$

*Proof.* We denote  $y(t) = \frac{1}{t}$  and  $y(t_0) = \frac{1}{t_0}$ . As  $|y - y_0| \leq C|y_0|^{-1+\delta}$ , we have  $|\frac{1}{t} - \frac{1}{t_0}| \leq C|t_0|^{1-\delta}$  hence  $|t_0/t - 1| \leq C|t_0|^{2-\delta}$  hence  $t_0/t \rightarrow 1$ , i.e.  $t \sim t_0$ . Then  $|t_0/t - 1| \leq C'|t|^{2-\delta}$  so that  $t_0 = t + O(t^{2-\delta})$ .

Now by hypothesis and by lemma 8,

$$f_s(x(t), y(t)) = f_s(x(t_0), y(t_0)) = \gamma(\gamma^2 - s\gamma - 1)t_0 + O(t_0^3) = \gamma(\gamma^2 - s\gamma - 1)t + O(t^{2-\delta}).$$

So that

$$f_s(x(t), y(t)) = x(t)(x(t)^2y(t)^2 - sx(t)y(t) - 1) = \gamma(\gamma^2 - s\gamma - 1)t + O(t^{2-\delta}).$$

We start over the computations of lemma 8. Set  $x(t) = a_0t^{r_0} + a_1t^{r_1} + \dots$  and  $y(t) = 1/t$ . Then

$$(4) \quad \frac{x(t)}{t} \left( \frac{x(t)^2}{t^2} - s \frac{x(t)}{t} - 1 \right) = \gamma(\gamma^2 - s\gamma - 1) + O(t^{1-\delta})$$

We cannot have  $r_0 > 1$  since we would have  $\frac{x(t)}{t} \rightarrow 0$  (as  $t \rightarrow 0$ ) and the left-hand side of equation (4) would also tends to 0. We cannot either have  $r_0 < 1$ , since we would have  $|\frac{x(t)}{t}| \rightarrow +\infty$  and the left-hand side of equation (4) would also tends to infinity. Then  $r_0 = 1$  and  $a_0(a_0^2 - sa_0 - 1) = \gamma(\gamma^2 - s\gamma - 1)$ , so that  $x(t) = O(t)$ .  $\square$

**Lemma 10** (compare to [5], Corollary 3.4). *Let  $Y = Y(\delta, K, M, A) = \{p \mid \psi(p, M\|p\|^{-1+\delta}, K) \geq A\}$  where  $0 < \delta < 1$ ,  $M > 0$  and  $A, K$  are sufficiently large constants. Then the formulas (2) and (3) holds for  $(x(t), y(t)) \in Y$  with  $y(t) = \frac{1}{t}$ .*

*Proof.* The proof is the same as in [5]: for  $p_0 = (x_0, y_0) \in Y$  there exists  $p = (x, y) \in U$  such that

$$\|p - p_0\| \leq KM \|p_0\|^{-1+\delta}.$$

As  $\frac{1}{2}|y_0| \leq \|p_0\| \leq 2|y_0|$  (since  $x_0 \leq y_0$ ), it implies  $|y - y_0| \leq C|y_0|^{-1+\delta}$  and lemma 9 applies.  $\square$

**Lemma 11** (compare to [5], Proposition 3.5). *Let  $Y = Y(\delta, K, M, A)$ , where  $0 < \delta < 1$ ,  $M > 0$  and  $A, K$  are sufficiently large constants. Suppose that  $p_1$  and  $p_2$  are in  $Y$  and there exists a  $0 < \delta_1 < 1$  such that  $\|p_1 - p_2\| \leq \|p_1\|^{-1+\delta_1}$ . Then for  $\max\{\delta, \delta_1\} < \delta_2 < 1$  and in a sufficiently small neighborhood of the point at infinity  $(0 : 1 : 0)$ :*

$$\left| \frac{f_s(p_1)}{f_s(p_2)} - a \right| \leq \|p_1\|^{-1+\delta_2},$$

with

$$a \in \left\{ 1, \frac{\alpha_s(\alpha_s^2 - s\alpha_s - 1)}{\beta_s(\beta_s^2 - s\beta_s - 1)}, \frac{\beta_s(\beta_s^2 - s\beta_s - 1)}{\alpha_s(\alpha_s^2 - s\alpha_s - 1)} \right\}.$$

*Proof.* Let  $p_1 = (x_1(t), y_1(t))$  and  $p_2 = (x_2(t'), y_2(t'))$  be two points in  $Y$ . Then by lemma 10

$$\begin{aligned} f_s(x_1(t), y_1(t)) &= \gamma(\gamma^2 - s\gamma - 1)t + O(t^{2-\delta}), \\ f_s(x_2(t'), y_2(t')) &= \gamma'(\gamma'^2 - s\gamma' - 1)t' + O(t'^{2-\delta}), \end{aligned}$$

where  $\gamma$  and  $\gamma'$  are in  $\{\alpha_s, \beta_s\}$ .

Now as  $\|p_1 - p_2\| \leq \|p_1\|^{-1+\delta_1}$  it implies  $|y_1 - y_2| \leq 2|y_1|^{-1+\delta_1}$ , as in the proof of lemma 9 we get  $t' = t + O(t^{2-\delta_1})$ . Whence

$$f_s(x_2(t'), y_2(t')) = \gamma'(\gamma'^2 - s\gamma' - 1)t + O(t^{2-\delta_1}) + O(t^{2-\delta}).$$

Then

$$\frac{f_s(p_1)}{f_s(p_2)} = \frac{\gamma(\gamma^2 - s\gamma - 1)}{\gamma'(\gamma'^2 - s\gamma' - 1)} + O(t^{1-\delta_1}) + O(t^{1-\delta}).$$

Then for  $\delta_2 > \max\{\delta, \delta_1\}$  with  $\delta_2 < 1$  and in neighborhood of the point at infinity  $(0 : 1 : 0)$  we get:

$$\left| \frac{f_s(p_1)}{f_s(p_2)} - a \right| \leq \frac{1}{2}|t|^{1-\delta_2} \leq \|p_1\|^{-1+\delta_2},$$

where  $a = \gamma/\gamma'$ .  $\square$

**Lemma 12** (compare to [5], Lemma 3.6). *Let  $K$  and  $A$  sufficiently large and  $0 < \delta < 1$ . Fix  $s$  with  $s^2 + 3 \neq 0$ . Then  $Y = Y(\delta, K, M, A)$  is nonempty and contains the polar curve  $\Gamma_s$ . Moreover all the limits of  $f_s(p_1)/f_s(p_2)$  given in lemma 11 can be obtained by taking  $p_1$  and  $p_2$  along the branches of  $\Gamma_s$  associated to the point at infinity  $(0 : 1 : 0)$ .*

*Proof.* Fix  $\delta$  and  $K$ . Let  $\pi_c : (f_s = c) \rightarrow \mathbb{C}$  be the projection  $(x, y) \mapsto y$ . It is a triple covering branched at the points  $\Gamma_s \cap (f_s = c)$ . These points are of coordinates

$$\left(\alpha_s t, \frac{1}{t}\right) \quad \text{and} \quad \left(\beta_s t', \frac{1}{t'}\right) \quad \text{with} \quad f_s\left(\alpha_s t, \frac{1}{t}\right) = f_s\left(\beta_s t', \frac{1}{t'}\right) = c.$$

As  $f_s\left(\alpha_s t, \frac{1}{t}\right) = \alpha_s t(\alpha_s^2 - s\alpha_s - 1)$  it implies

$$t = \frac{c}{\alpha_s(\alpha_s^2 - s\alpha_s - 1)} \quad \text{and similarly} \quad t' = \frac{c}{\beta_s(\beta_s^2 - s\beta_s - 1)}.$$

For  $s^2 + 3 \neq 0$ ,  $\alpha_s \neq \beta_s$  and it also implies  $t \neq t'$  hence  $|y(t) - y(t')|$  is of order  $y(t)$ , that is to say two points of ramifications are far enough. Let  $p_0 = (x_0, y_0)$  be a point of ramification

of  $\pi_c$ . Let  $\mathcal{V} = \{y \mid |y - y_0| \leq \epsilon|y_0|\}$ , with  $\epsilon$  sufficiently small such that no other ramification point projects in  $\mathcal{V}$ . For a sufficiently large  $p_0$  (i.e. small  $c$ ),  $X(p_0, KM\|p_0\|^{-1+\delta}) \subset \pi_c^{-1}(\mathcal{V})$ . Now let  $p = (x, y)$  such that:

$$|y - y_0| \leq |y_0|^{-1+\delta},$$

then by lemma 9,  $x = O(\frac{1}{y})$  whence

$$\|p - p_0\| \leq 2\|p_0\|^{-1+\delta}.$$

Let  $\mathcal{V}_\delta = \{y \mid |y - y_0| \leq \epsilon|y_0|^{-1+\delta}\}$ , by the above inequality we get  $\pi_c^{-1}(\mathcal{V}_\delta) \subset X(p_0, KM\|p_0\|^{-1+\delta})$ . We restrict the triple branched covering  $\pi_c$  to a map  $\tilde{\pi}_c$  from  $\pi_c^{-1}(\mathcal{V}_\delta)$  composed by only two components of the triple cover. Let  $y \in \mathcal{V}_\delta$  such that  $|y - y_0| = \frac{1}{2}|y_0|^{-1+\delta}$ . Let  $p_1 = (x_1, y)$ ,  $p_2 = (x_2, y)$  be the two points of  $\tilde{\pi}_c^{-1}(y)$ . These two points are in  $\tilde{\pi}_c^{-1}(\mathcal{V}_\delta)$  which is a connected set. Any curve  $\gamma$  in  $\tilde{\pi}_c^{-1}(\mathcal{V}_\delta)$  from  $p_1$  to  $p_2$  passes through  $p_0$ , hence the projection of  $\gamma$  by  $\tilde{\pi}_c$  passes through  $y_0$ . Hence the inner distance (in  $(f_s = c)$ ) of  $p_1$  and  $p_2$  is greater or equal than  $2|y - y_0|$ , it yields:

$$\text{dist}_{p_0, M\|p_0\|^{-1+\delta}, K}(p_1, p_2) \geq 2|y - y_0| = |y_0|^{-1+\delta} = |t|^{1-\delta},$$

where we denote  $y_0 = \frac{1}{t}$ . By lemma 9 we have  $x_1 = O(t)$  and  $x_2 = O(t)$ , so that

$$\|p_1 - p_2\| \leq C|t|.$$

Then

$$\frac{\text{dist}_{p_0, M\|p_0\|^{-1+\delta}, K}(p_1, p_2)}{\|p_1 - p_2\|} \geq \frac{1}{C|t|^\delta} \xrightarrow[t \rightarrow 0]{} +\infty.$$

Then  $\psi(p_0, M\|p_0\|^{-1+\delta}, K) \rightarrow +\infty$ , as  $p_0$  tends to the point at infinity  $(0 : 1 : 0)$ . It means that the branch of  $\Gamma_s$  near this point at infinity is included in  $Y(\delta, K, M, A)$ .

Finally we have already proved in subsection 3.1 that the list of values  $f_s(p_1)/f_s(p_2)$  on  $\Gamma_s$  is the required one.  $\square$

We conclude by the proof of the theorem in the complex case.

*Proof of theorem 1'.* Fix  $s$ . By lemma 7 the set  $Y$  for  $f_s$  is sent into a set  $\tilde{Y}$  for  $f_{s'}$ . The polar curve  $\Gamma_s$  is included in  $Y$  (lemma 12) and on this polar curve  $f_s(p_1)/f_s(p_2)$  tends to a  $\frac{\alpha_s(\alpha_s^2 - s\alpha_s - 1)}{\beta_s(\beta_s^2 - s\beta_s - 1)}$  for instance (lemma 11). On the one hand  $f_{s'}(\tilde{p}_1)/f_{s'}(\tilde{p}_2)$  tends to the same value, because the bilipschitz homeomorphism  $\Phi$  sends the levels of  $f_s$  to the levels of  $f_{s'}$ . On the other hand  $\tilde{p}_1, \tilde{p}_2$  are in  $\tilde{Y}$  so that  $f_{s'}(\tilde{p}_1)/f_{s'}(\tilde{p}_2)$  is in

$$\left\{ 1, \frac{\alpha_{s'}(\alpha_{s'}^2 - s'\alpha_{s'} - 1)}{\beta_{s'}(\beta_{s'}^2 - s'\beta_{s'} - 1)}, \frac{\beta_{s'}(\beta_{s'}^2 - s'\beta_{s'} - 1)}{\alpha_{s'}(\alpha_{s'}^2 - s'\alpha_{s'} - 1)} \right\}.$$

This is only possible for a finite set of values  $s'$ .  $\square$

## 5. TOPOLOGICAL EQUIVALENCE

To complete the complex part of theorem 1 we prove the topological equivalence of any two polynomials.

**Lemma 13.** *Consider the following family of polynomials in  $\mathbb{C}[x, y]$ :*

$$f_s(x, y) = x(x^2y^2 - sxy - 1)$$

with  $s^2 + 4 \neq 0$ . For any  $s$  and  $s'$  the polynomials  $f_s$  and  $f_{s'}$  are topologically equivalent.

This family is similar to examples in [1] of polynomials that are topologically equivalent but not algebraically equivalent. Recall that two polynomials  $f, g : \mathbb{K}^n \rightarrow \mathbb{K}$  are *topologically equivalent* if there exist a homeomorphism  $\Phi : \mathbb{K}^n \rightarrow \mathbb{K}^n$  and a homeomorphism  $\Psi : \mathbb{K} \rightarrow \mathbb{K}$  such that  $g \circ \Phi = \Psi \circ f$ . We will use the following result that is global version of Lê-Ramanujam  $\mu$ -constant theorem. See [2] for the two variables case and [3] for any number of variables.

**Theorem 14.** *Let  $\{f_s\}_{s \in [0,1]}$  be a continuous family of complex polynomials with isolated singularities (in the affine space and at infinity), with  $n \neq 3$  variables. Suppose that the following integers are constant w.r.t. the value of  $s$ :*

- $\deg f_s$ , the degree,
- $\#\mathcal{B}_s$ , the number of irregular values,
- $\chi(f_s = c_{gen})$ , the Euler characteristic of a generic fiber.

Then  $f_0$  and  $f_1$  are topologically equivalent.

*Proof of lemma 13.*

- **Degree.** It is clear that the degree of the  $f_s$  is independent of  $s$ .
- **Affine singularities.** We search for points  $(x, y)$  where both derivatives vanish.  $\partial_x f_s(x, y) = 3x^2y^2 - 2sxy - 1$  and  $\partial_y f_s(x, y) = x^2(2xy - s)$ . If  $x = 0$  then  $\partial_x f_s(x, y) \neq 0$ . So that  $\partial_y f_s(x, y) = 0$  implies  $2xy - s = 0$ . We plug  $xy = s/2$  in  $\partial_x f_s(x, y) = 0$  and get  $s^2 + 4 = 0$ . Notice that  $s^2 + 4 = 0$  gives also the values where  $f_s$  is not a reduced polynomial. Conclusion: for  $s^2 + 4 \neq 0$ , the polynomials  $f_s$  has no affine singularities (nor affine critical values), so that its global affine Milnor number is  $\mu_s = 0$ .
- **Singularities at infinity.** The two points at infinity for this family are  $P_1 = (0 : 1 : 0)$  and  $P_2 = (1 : 0 : 0)$ . Let  $F_s(x, y, z) = x(x^2y^2 - sxyz^2 - z^4) - cz^5$  be the homogenization of  $f_s(x, y) - c$ .
  - **Milnor number at  $P_1$ .** We localize  $F_s$  at  $P_1 = (0 : 1 : 0)$  to get  $g_s(x, z) = F_s(x, 1, z) = x(x^2 - sxz^2 - z^4) - cz^5$ . We compute the local Milnor of  $g_s$  at  $(0, 0)$ . For instance we may use the Newton polygon of  $g_s$  and Kouchnirenko formula. We get, for any  $s$  (with  $s^2 + 4 \neq 0$ ) and depending on  $c$ :

$$\mu(g_s) = 8 \text{ if } c \neq 0 \quad \text{and} \quad \mu(g_s) = 10 \text{ if } c = 0.$$

Hence the value 0 is an irregular value at infinity and the jump of Milnor number is  $\lambda_{P_1} = 10 - 8 = 2$ .

- **Milnor number at  $P_2$ .** At  $P_2 = (1 : 0 : 0)$  we get  $h_s(y, z) = F_s(1, y, z) = y^2 - syz^2 - z^4 - cz^5$ . The local Milnor number of  $h_s$  at  $(0, 0)$  is independent of  $s$  and  $c$ :

$$\mu(h_s) = 3.$$

So that there is no irregular values at infinity for this point and  $\lambda_{P_2} = 0$ .

- Then the Milnor number at infinity is  $\lambda_s = \lambda_{P_1} + \lambda_{P_2} = 2$  and the only irregular value at infinity is 0.
- **Conclusion.** For all  $s$  the only irregular value is 0:  $\mathcal{B}_s = \{0\}$ , the Euler characteristic of a generic fiber given by  $\chi_s = 1 - \mu_s - \lambda_s = -1$  is also constant. Then by theorem 14 any  $f_s$  and  $f_{s'}$  are topologically equivalent.

□



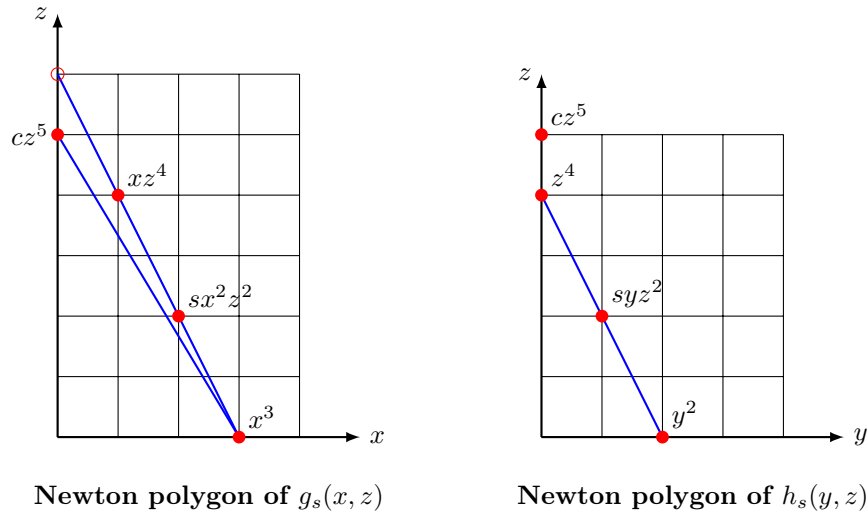


FIGURE 6. Computation of Milnor number at infinity.

## REFERENCES

- [1] Arnaud Bodin. Classification of polynomials from  $\mathbb{C}^2$  to  $\mathbb{C}$  with one critical value. *Math. Z.*, 242(2):303–322, 2002.
- [2] Arnaud Bodin. Invariance of Milnor numbers and topology of complex polynomials. *Comment. Math. Helv.*, 78(1):134–152, 2003.
- [3] Arnaud Bodin and Mihai Tibăr. Topological equivalence of complex polynomials. *Adv. Math.*, 199(1):136–150, 2006.
- [4] Takuo Fukuda. Types topologiques des polynômes. *Inst. Hautes Études Sci. Publ. Math.*, (46):87–106, 1976.
- [5] Jean-Pierre Henry and Adam Parusiński. Existence of moduli for bi-Lipschitz equivalence of analytic functions. *Compositio Math.*, 136(2):217–235, 2003.

*E-mail address:* [arnaud.bodin@univ-lille.fr](mailto:arnaud.bodin@univ-lille.fr)

UNIVERSITÉ DE LILLE, CNRS, LABORATOIRE PAUL PAINLEVÉ, 59000 LILLE, FRANCE