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Rearranged Stochastic Heat Equation

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Abstract

The purpose of this work is to provide an explicit construction of a strong Feller semigroup on the space of probability measures over the real line that additionally maps bounded measurable functions into Lipschitz continuous functions, with a Lipschitz constant that blows up in an integrable manner in small time. Our construction relies on a rearranged version of the stochastic heat equation on the circle driven by a coloured noise. Formally, this stochastic equation writes as a reflected equation in infinite dimension, a topic that is known to be challenging. Under the action of the rearrangement, the solution is forced to live in a space of quantile functions that is isometric to the space of probability measures on the real line. We prove the equation to be solvable by means of an Euler scheme in which we alternate flat dynamics in the space of random variables on the circle with a rearrangement operation that projects back the random variables onto the subset of quantile functions. A first challenge is to prove that this scheme is tight. A second one is to provide a consistent theory for the limiting reflected equation and in particular to interpret in a relevant manner the reflection term. The last step in our work is to establish the aforementioned Lipschitz property of the semigroup by adapting earlier ideas from the Bismut-Elworthy-Li formula in stochastic analysis.

Keywords: Measure-valued Diffusions, Wasserstein Diffusions, Reflected SPDE, Common Noise Mean Field Models, Rearrangement Inequalities, Bismut-Elworthy-Li formula.

AMS Classification: 60H15, 60G57, 47D07, 60J35.

1 Introduction

1.1 Motivation

Mean-field models with common noise. Our work is motivated by recent developments in the theory of mean-field models, at the intersection of stochastic analysis,

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calculus of variations and control and game theories. Although mean field models have a long history, stemming from statistical mechanics (see the pioneering work [39]), the problems studied in recent years are, in comparison, of an increasing complexity. For example, the solutions of control or game problems give rise, in the mean-field regime, to partial differential equations posed on the space of probability measures, whose understanding remains an active area of research in the case of control and with even more open questions in the case of games (see [17, 18, 20, 36] and the references therein for a recent state of the art on these questions).

From a probabilistic point of view, mean-field models lead, as soon as they evolve with time, to the study of dynamics with values in the space of probability measures. Although the latter are understood as evolutions of the law of a typical random particle, representative of the mean-field continuum, these probability measures remain most often deterministic. For example, they may be governed by non-linear Fokker-Planck equations or, depending on the terminology, may obey nonlinear Markovian dynamics, see for instance the seminal work by McKean [56] and the monograph [40]. Nevertheless, many recent works have underlined the interest in considering random dynamics on the space of probability measures. For example, from a modelling point of view, the nonlinear Fokker-Planck equations become stochastic when the particles composing the mean-field continuum are all subjected to a common noise, see for instance the earlier works [26, 49, 48, 73] and also the more recent monographs [17, 19] for a systematic study within the framework of mean-field control and games. On a purely mathematical level, the presence of a common noise also raises interesting challenges. Although it is possible, in standard cases, to adapt the usual techniques of mean-field models to establish the well-posedness of stochastic Fokker-Planck equations, the understanding of the impact of common noise is in fact rather limited. In particular, there is currently no catalogue listing the varying effects of common noise on the statistical behaviour of solutions, unlike the theory of finite-dimensional diffusion processes, in which the impact of noise has been widely studied in finite or infinite time.

Models with a smoothing effect. Typically - and this is the framework of this paper - it may be relevant to ask about the possible regularisation properties of the semigroup induced, on the space of functionals of a probability measure, by a stochastic Fokker-Planck equation or by a mean-field model with a common noise. Although the expected properties are certainly limited when the common noise is of finite dimension (since the ambient space is of infinite dimension), the question takes a turn when the common noise is allowed to be infinite-dimensional. In other words, it is reasonable to imagine that a sufficiently "large" common noise could indeed provide regularisation phenomena. There is an example in the literature. The Fleming-Viot process with mutations induced by diffusions is a process with values in the space of probability measures whose semigroup is strong Feller and maps bounded functions into Lipschitz continuous functions, see [70]. The generator, which acts on functionals of probability measures, contains two parts: a first-order term that coincides with the operator coming from a deterministic linear Fokker-Planck equation and a second-order term yielded by the "sampling replacement" rule characterising the Moran and Fleming-Viot models. Notably, this second-order term

should be regarded as being induced by a form of common noise in the model. However, it must be stressed that the smoothing property is rather poor in small time, as the Lipschitz constant of the functions returned by the semigroup may blow up exponentially fast in small time. Although it may seem anecdotal, such a limitation on the rate at which the regularisation occurs renders this noise almost impossible to use in concrete situations. In finite dimension, noise is typically used to regularise transport models driven by singular velocity fields. In this case, the usual techniques rely on the fact that the (first order) derivatives of the transition kernel diverge in an integral manner in small time. This is also an essential point in the theory of nonlinear parabolic partial differential equations. These observations serve as a guideline for the rest of the paper.

1.2 Diffusions with values in the space of probability measures

Wasserstein diffusions. The search for common noise(s) that would be able to force some practicable smoothing properties on the space of probability measures is in fact connected to another question that has been addressed by a series of authors for almost fifteen years: what should be a Brownian motion on the space of probability measures? Whilst there has not yet been an answer to this question that may be called canonical, the existing candidates are usually referred to as "Wasserstein diffusions". This terminology echoes the notion of Wasserstein space, which is defined as the space $\mathcal{P}_2(\mathbb{R}^d)$ of probability measures (on $\mathbb{R}^d$, for some $d \geq 1$) having a finite second moment, and which is equipped with the 2-Wasserstein distance $\mathcal{W}_2$. Many works from calculus of variations have demonstrated the interest to endow the Wasserstein space with a kind of Riemannian structure, see for instance the earlier works [38, 61, 62] and the book [1]. In this approach, the tangent space at a probability measure $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ is the closure in $L^2(\mathbb{R}^d, \mathbb{R}^d; \mu)$ (the space of μ-square integrable functions from $\mathbb{R}^d$ into itself) of smooth compactly supported gradient vector fields on $\mathbb{R}^d$. Accordingly, the *Wasserstein derivative* or intrinsic gradient of a functional defined on $\mathcal{P}_2(\mathbb{R}^d)$ reads, at any $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, as the gradient of a real-valued function (i.e., a potential) defined on $\mathbb{R}^d$. Roughly speaking, this potential corresponds to the so-called flat or functional derivative used in the formulation of the generator of the aforementioned Fleming-Viot process, see [25, 27, 70]. Wasserstein diffusions are usually expected to be diffusion processes with values in $\mathcal{P}_2(\mathbb{R})$ that are consistent with the metric $\mathcal{W}_2$. For instance, the small time large deviations are expected to have $\mathcal{W}_2^2$ as rate functional and the local variance (or quadratic variation) in the corresponding chain rule (or Itô formula) is expected to derive from the Riemannian metric. Whilst the Fleming-Viot process is not a $\mathcal{W}_2$-Wasserstein diffusion, examples are known. The most famous one is the $1d$ Wasserstein diffusion constructed by von Renesse and Sturm in [66]. The key point in [66] is to introduce a parametrised class of *entropy* probability measures on $\mathcal{P}([0,1])$ - the space of probability measures on the unit interval - and then to consider, under each of these probability measures, the Markov process associated with the Dirichlet form generated by the Riemannian metric. The entropy probability measures are constructed in two steps: the first one is to equip the space of quantile functions on $[0,1]$ with a Poisson-Dirichlet measure and the second one is to transfer the latter measure onto $\mathcal{P}([0,1])$ by means of the natural isometry that

exists between the space of quantile functions equipped with the L^2-norm and the space $\mathcal{P}([0,1])$ equipped with $\mathcal{W}_2$. The same isometry plays a key role in our work, however we use slightly different quantile functions.

Although the work [66] has had a great impact in the field, it is fair to say that the Wasserstein diffusion that is constructed therein remains of a difficult approach. In particular, the definition *via* a Dirichlet form does not allow for generic starting points and, to the best of our knowledge, there has not been any systematic analysis of the properties of the semigroup. We refer to [5, 71] for particle approximations of this Wasserstein diffusion and to [32] for a log-Sobolev inequality. Several works have been written in the wake of [66]. For example, in [41, 42], Konarovskyi has proposed an alternative construction in one dimension, leading to another definition of the Wasserstein diffusion. From the particle system perspective, this approach aims at evolving a cloud of massive random particles, with the heavier particles having smaller fluctuations. The particles aggregate and thus become heavier as they collide. As opposed to the construction based on Dirichlet forms, the model allows one to consider arbitrary initial conditions, but the collision rules force the dynamics to instantaneously take its values in the set of finitely supported probability measures. The analysis has been pushed further in [46], but many questions remain open, starting with uniqueness: at this stage, it is not known whether the way the cloud of particles passes, instantaneously at time 0, from a continuum to a finite collection is unique. We refer to [43] and the references therein for an extension allowing for fragmentation. We also refer to [55] for a mollification of the coalescing dynamics, for which uniqueness holds true. And, last but not least, the $1d$ dynamics constructed in [41, 42] are somehow extended to the higher dimensional setting in [28] but using the theory of Dirichlet forms in the spirit of [66] (although the latter work is only in one dimension). Remarkably, the Dirichlet form that is introduced in [28] indeed induces, in the $1d$ case, the same generator (acting on functionals of a probability measure) as in [41, 42].

Connection with the Dean-Kawasaki equation. The aforementioned works are connected with stochastic Fokker-Planck equations. In [41, 42, 66], it is shown that the various forms of Wasserstein diffusions under study each induce generators (acting on functionals of a probability measure) sharing some similarities with the generator of the so-called Dean-Kawasaki equation. Formally, the latter is a stochastic version of a standard Fokker-Planck equation (of order 1 or 2 depending on the cases) that includes an additional noisy term whose local quadratic variation derives exactly from the Riemannian metric on $\mathcal{P}(\mathbb{R}^d)$ (with $d = 1$ in [41, 42, 66]). However, it has been proved in [44, 45] that the Dean-Kawasaki equation, in its strict version, cannot be solvable except in trivial cases where it reduces to a finite dimensional particle system (which requires in particular the initial distribution to be finitely supported). This negative result has an interesting consequence: to make the Dean-Kawasaki equation well-posed, some extra correction is needed in the dynamics, which is exactly what is done in [41, 42, 66]. However, the latter works demonstrate that, so far, there has not been any canonical choice for such a correction.

The very spice of the Dean-Kawasaki equation may be explained as follows. When

the solution is at some probability measure $\mu \in \mathcal{P}(\mathbb{R})$, a typical particle in the mean-field continuum, located at some point $x \in \mathbb{R}$, should be subjected to the value at this point x of a cylindrical Wiener noise on $L^2(\mathbb{R}, \mathbb{R}; \mu)$, which makes no sense in general. This seemingly naive remark suggests that Dean-Kawasaki dynamics can be approached by replacing the cylindrical noise by a coloured noise. To a certain extent, this idea is the basis of the two contributions [30] and [54]. In [54], the author succeeded to prove that the resulting semigroup has an (albeit weak) mollification effect on functions over the space of probability measures.

1.3 Our contribution

Smoothing properties of the Ornstein-Ulhenbeck process. Unlike many of the aforementioned works, our aim is not to provide another candidate for being a Wasserstein diffusion. Our primary motivation in this contribution is to construct as explicitly as possible a probability-measure valued process having sufficiently strong smoothing properties. Although not discussed further within this text, our long-term goal is indeed to propose a corresponding theory of linear or nonlinear parabolic Partial Differential Equations (PDEs) on the space of probability measures and to exhibit, in this context, second order operators allowing to smooth the singularities that may appear in the corresponding hyperbolic PDEs. Of course, such a probability-measure valued process should share some similarities with Wasserstein diffusions, but as we will see, the process that we introduce in this paper does not satisfy the pre-requisites for being a Wasserstein diffusion.

Our approach is based on the following two simple observations. First, Lions [51] showed in his lectures on mean-field games at the Collège de France, that in order to study certain mean-field models, it can prove useful to lift probability measures into random variables, i.e., to invert the map that sends a random variable onto its statistical distribution. Although the inverse is obviously multi-valued in general, it has been shown that Lions' lifting principle provides a very understandable picture of the Wasserstein derivative: in short, it can be represented as a Fréchet derivative on a Hilbert space of square-integrable random variables, see e.g. [37] for a complete overview of the connections between the different derivatives. Our second remark is a well-known fact from stochastic analysis: we know how to construct a Hilbert-valued diffusion process with strong smoothing properties. A simple example is the Ornstein-Ulhenbeck process driven by an appropriate operator, see for instance [21, 23, 24]. This suggests the following procedure: we should project back from the space of random variables, and onto the space of probability measures, an Ornstein-Ulhenbeck process taking values in a space of square-integrable random variables. Whilst this looks very appealing, this idea has an obvious drawback. In general, the projection should destroy the Markov nature of the dynamics, i.e., the transition probabilities started from two different random variables representing the same probability measure may not be the same.

The next step in our construction is thus inspired from the Lie-Trotter-Kato formula and related splitting methods. Precisely, we want to define a probability measure-valued process by alternating one step in the space of random variables, following some pre-

scribed Ornstein-Ulhenbeck dynamics, and one projection operation to return back from the space of random variables to the space of probability measures. The choice of the probability space carrying the random variables is not a difficulty and throughout the paper we work on the circle, $\mathbb{S} \cong \mathbb{R}/\mathbb{Z}$, equipped with the Lebesgue measure. However, the choice of an appropriate projection is much more difficult. It is in fact an essential aspect in the implementation of the splitting scheme and, as in the contributions [30, 41, 42, 55, 54, 66], it leads us to limit our study to the one-dimensional case, with the following two advantages. First, probability measures can be easily identified with quantile functions on the circle (or say to simplify 'non-decreasing functions' although this notion has to be clarified in the periodic setting, see Proposition 2.1 below), which in turn makes the choice of projection operation easier as it then suffices to send a function on the circle to an appropriate rearrangement. Second, the rearrangement operation is in fact an easy way to transform a random variable on the circle into a quantile function whilst preserving its statistical law (under the Lebesgue measure).

The resulting scheme in which we combine 'flat' dynamics and rearrangement is very much inspired by earlier works of Brenier on discretisation schemes for conservation laws, see for instance [14, 15], with the main difference that the works of Brenier are mostly for deterministic dynamics. In contrast, the dynamics that we consider are stochastic. As we choose the Laplacian to be the driving operator in the Ornstein-Ulhenbeck dynamics, we call the resulting equation the 'rearranged stochastic heat equation'.

Rearranged and reflected equations. In fact, the presence of the noise raises many subtleties in our construction. In particular, one issue is that the rearrangement operation and the Laplacian driving the Stochastic Heat Equation (SHE) do not marry well. Obviously, they do not commute. As a result, the smoothing effect of the Laplacian (which is now acting on functions defined on the circle) is weaker when the rearrangement is present. At least, this is what we observe in our computations. Of course, this has a rather dramatic consequence on the choice of the noise. One key feature of the SHE is that, after convolution with the heat kernel, the cylindrical white noise driving the SHE gives a true random function. When the SHE is rearranged (as we do here), this no longer seems to be the case. In order to remedy this problem, we need to colour the noise driving the SHE. As well-expected, this has an impact on the smoothing properties of the semigroup generated by the rearranged SHE, but nevertheless we succeed to show that the rate at which the derivative of the semigroup blows up in small time is integrable, as we initially intended. It remains an open question whether the same construction can be achieved for the SHE driven by a cylindrical white noise.

Another difficulty is to obtain a suitable formulation of the rearranged SHE. Although Brenier's works [14, 15] quite clearly suggest to see the rearrangement as a reflection and indeed to write the rearranged SHE as a reflected equation, again, the presence of the noise requires additional precautions. The study of reflected differential equations is in general more complicated in the stochastic case than in the deterministic case because the solutions are no longer of bounded variation. We refer to the seminal article [50] in the case of finite dimensional equations. Unfortunately, infinite dimensional equations are even more difficult to deal with. To the best of our knowledge, there is no general

theory that can cover our formulation of the rearranged SHE. We therefore propose a tailor-made interpretation in which the reflection term is constructed by hand. Schematically, the rearranged SHE is written as a stochastic differential equation on the space $L^2(\mathbb{S}) := L^2(\mathbb{S}, \mathrm{Leb}_{\mathbb{S}})$ (of functions on the circle that are square-integrable with respect to the Lebesgue measure) subject to a reflection term whose effect is to constrain the solution to remain in the cone of our chosen quantile functions (i.e., of functions with appropriate monotonicity properties). This representation is reminiscent of the $1d$ reflected stochastic differential equation studied by Nualart and Pardoux [59] (and extended next in [31]), in which the solution is constrained to remain positive under the effect of a reflection. Although the latter positivity constraint may be interpreted as a constraint on the monotonicity of the primitive, the rearranged SHE that we study here is not the primitive of the Nualart-Pardoux reflected equation. This can be seen quite simply in the calculations: roughly speaking, the derivation/integration and reflection operations do not commute. However, at the beginning of our work on this subject, this (finally false) correspondence between primitive equation and derivative equation led us to use (in the rearranged dynamics) a coloured noise, obtained by integration of the white noise. We think that this image will be useful to the reader (with the knowledge that the noise we eventually use below is actually less regular than the primitive of the white noise).

The form of the reflection in [59] was further specified in the later contributions [75, 76] due to Zambotti. These additional results provide a more refined description of the behaviour of the solution at the boundary of the domain. In our approach we are not able at this stage, to give a similar picture. Our construction of the reflection process and its associated integral is too elementary. In particular, we consider only the action of the reflection process on functions that are far more regular than the solution of the equation itself. Fortunately, this does not prevent us from obtaining a characterisation of the solutions, sufficient to carry out our program to the end.

In fact, Zambotti's results are based on a formula of integration by parts on the set of continuous trajectories with positive values under the law of the Bessel process of dimension 3. This integration by parts formula allows one to reinterpret the solutions of the Nualart-Pardoux equation by means of the theory of Dirichlet forms. The adaptation to our case remains completely open. Although we do not use them here, we think it appropriate to mention some of the more general works that have been published subsequently on stochastic differential equations in infinite dimension. The three papers [9, 10, 11] deal with the case where the reflection takes place on a sufficiently regular convex domain with a non-empty interior in a Hilbert space; as an example, the Nualart-Pardoux equation does not satisfy these conditions. The assumption made on the domain allows one to obtain an integration by parts formula similar to the one obtained in [75, 76] and then to use the theory of Dirichlet forms. This said, the results in [9, 10] are not directly stated for the reflected equation itself but for the corresponding elliptic infinite-dimensional Kolmogorov equation with Neumann boundary condition. The corresponding Markov process is addressed in [11]. The probabilistic approach has been pushed further in the parallel contribution [64].

Description of the results. The rearranged SHE is proven to be uniquely solvable in

the strong sense. The reader may find a definition of a solution in Definition 4.13. The main solvability result is Theorem 4.15

The proof holds in two main steps. The first one is to show the existence of a weak solution and the second one is to prove that uniqueness holds in the strong sense. Strong existence then follows from a standard adaptation of Yamada-Watanabe's theorem. As is often the case, the first step is more challenging. Weak solutions are obtained as weak limits of linear interpolations of an Euler scheme in which we alternate, on each time step, the Ornstein-Ulhenbeck dynamics in $L^2(\mathbb{S})$ with the rearrangement operation. To make it clear, the state (in $L^2(\mathbb{S})$) of the Ornstein-Ulhenbeck process is rearranged at the end of each time step. At the beginning of each new step, the previous terminal rearranged state is used as a new initial condition. Part of the challenge is to show that the scheme is tight (in the space of continuous functions). This is done in Section 3 by using several key properties of the rearrangement operation, which are presented in Section 2. Our choice to drive the Ornstein-Ulhenbeck by the Laplacian operator and to colour the noise is essential in the tightness analysis. In order to complete the proof of the existence of a weak solution, we need to give an appropriate sense to the reflection process, which is one of the goals of Section 4. As we already said, the corresponding integral is just defined with respect to processes that take values in the space of smooth functions on $\mathbb{S}$, but the resulting characterisation is sufficient to obtain strong uniqueness. The main point in the proof of strong uniqueness is to impose, in the definition of a solution, a weak form of orthogonality between the solution and the reflection. We prove that it is satisfied by any weak limit of the discrete scheme.

The second main statement of the article is Theorem 5.11, which says that the semi-group generated by our rearranged SHE is strongly Feller, i.e., maps bounded measurable functions into continuous functions. Moreover, we prove that the semi-group returns Lipschitz continuous functions, with a Lipschitz constant that blows up, as time tends to 0, in an integrable manner. This is a strong result and, as we have already alluded to, we expect it to have applications in the analysis of partial differential equations set on the space of probability measures. The proof of Theorem 5.11 draws heavily on previous works on the so-called Bismut-Elworthy-Li formula, which is an integration by parts formula for the transition probabilities of a diffusion process, see for instance [34, 35, 72] in the finite-dimensional framework and [22] and [21, Chapter 7] in infinite dimension. Such an integration by parts is in fact strongly related to Malliavin calculus, see for instance Exercise 2.3.5 in the book [60], together with the papers [12] and [57]. The transposition of the Bismut-Elworthy-Li formula to the reflected setting is known however to raise some technical difficulties. A major obstacle, due to the presence of the reflection term in the dynamics, is to prove the differentiability of the flow with respect to the initial condition. We refer to [29] for the first result in this direction (for a drifted Brownian motion with reflection in the orthant) and to [3, 4, 52, 53] for further results. None of these results (which are all in finite dimension) apply to our case. At this stage, we do not know whether similar results may be true for the rearranged SHE. Instead, in our analysis of the semigroup we use the sole property that the flow (generated by the rearranged SHE) is Lipschitz continuous with respect to the initial condition. Whilst this

is not enough to prove the exact analogue of the Bismut-Elworthy-Li formula, it does allow us to derive the expected smoothing effect of the semigroup. A key point in this respect is to use the fact that, when restricted to initial conditions parametrised by a finite-dimensional parameter, the flow is almost everywhere differentiable (with respect to the initial condition) as a consequence of Rademacher's theorem.

Comparison with recent literature and further prospects. A few weeks before we put this work on arXiv, another arXiv pre-publication was published ([65]) in which the authors introduce, on the space of probability measures, a Dirichlet form whose construction has some similarities with the construction of the rearranged SHE that we introduce here. Note that the results of the two papers do not overlap, but an in-depth study would be necessary to link the two constructions more properly. In short, the work [65] aims at projecting on the space of probability measures a Gaussian measure constructed on an L^2 space of random variables and then at considering, under this measure, the Dirichlet form generated by the Riemannian metric on $\mathcal{P}_2(\mathbb{R}^d)$ (with $d \geq 1$). For example, in $1d$, this Gaussian measure can be the invariant measure of the SHE driven by a cylindrical white noise. Although this example (in $1d$) does not fit our assumptions (since we need the noise to be coloured), it is worth noting that, if we had to write formally the generator of the rearranged SHE in this case, it would be different from the one computed in [65, Theorem 4.1]. This observation demonstrates that, even though it shares some similarities with our work, the construction achieved in [65] leads to a distinct process. In this respect, we also point out that the choice of the cylindrical noise in [65] raises some difficulties in the identification of the domain of the generator, which are reminiscent of our need to colour the noise (see [65, Remark 4.1]). Last but not least, it is clear that we will come back to the connection with PDEs in a future contribution.

We also point out that our construction based on the Lie-Trotter-Kato formula has a simple particle interpretation. At each time step of the Euler scheme, we can indeed consider a particle approximation of the SHE, as given for example by a finite volume discretisation. Then, at the end of each time step, the rearrangement operation, when implemented on the particles, simply consists in reordering the particles (in an increasing or decreasing manner depending on the precise definition of the rearrangement). Although we do not discuss this further in the rest of the article (for obvious reasons of length), this prospect is natural.

The reader may also wonder about higher dimensional extensions. Although this is indeed a natural equation, we think it is useful to recall that many of the aforementioned works (notably those concerning the construction of a Wasserstein diffusion) are also in one dimension. From this point of view, this limitation in our model should not come as a surprise. As for the possible ways to extend the construction to the case $d \geq 2$, the difficulty we would face in extending our results to the higher dimension is the choice of the rearrangement. Whilst the notion of rearrangement is simple in one dimension, the multi-dimensional case is far more complicated. One possibility is to use the tools of optimal transport ([16]), but this perspective is open at this stage. The reader may also worry about the fact that, in dimension $d \geq 2$, the stochastic heat equation (when

driven by the Laplace operator) requires a coloured noise, of a higher regularity than what we use here. In fact, this would be only the case if we considered the stochastic heat equation on a space of dimension d (typically the d-dimensional torus). Actually, our belief is that we could define the stochastic heat equation on the $1d$ torus, but regard it as a system of d equations. That said, another possibility could be to replace the Laplacian by another operator, the Laplacian being chosen here for convenience (many calculations or intermediate properties rely on this choice). Whilst we think that the generalisation to other operators with similar smoothing properties is feasible, it would nevertheless deserve a rigorous verification.

Organisation of the paper. We introduce some preliminary material in Section 2, including in particular a series of already known (but very useful) results on the symmetric rearrangement on $\mathbb{S}$ together with some frequently used notation. Section 3 is dedicated to the presentation and the analysis of the approximating splitting scheme that we employ to extract next a weak solution to the rearranged SHE. In particular, the reader will find all the required assumptions on the noise in the introduction of Section 3. Tightness is established in Proposition 3.7. The definition of a solution to the rearranged SHE is clarified in Section 4, see Definition 4.13. Existence and uniqueness are guaranteed by Theorem 4.15 . The smoothing properties of the semigroup is studied in Section 5, the main Lipschitz estimate being stated in Theorem 5.11.

2 Preliminary Material

2.1 The symmetric non-increasing rearrangement

As explained in the introduction, the construction of our rearranged stochastic heat equation with values in the space of quantile functions relies heavily on the notion of rearrangement for periodic functions.

Throughout, the circle $\mathbb{S}$ is chosen to be parametrised by the interval $(-1/2, 1/2]$ and 0 is regarded as a privileged fixed point on the circle, i.e., $\mathbb{S} := (\mathbb{R} + 1/2)/\mathbb{Z}$.

Proposition 2.1. *Given a measurable function $f : \mathbb{S} \to \mathbb{R}$, there exists a unique function, called symmetric non-increasing rearrangement of f and denoted $f^* : \mathbb{S} \to [-\infty, +\infty]$, that satisfies the following two properties:*

1. *f^* is symmetric (with respect to 0), is non-increasing and left-continuous on the interval $[0, 1/2)$, and is right-continuous at $1/2$ (left- and right-continuity being here understood for the topology on $[-\infty, +\infty]$),*

2. *the image of the Lebesgue measure $\mathrm{Leb}_{\mathbb{S}}$ by f^* is the same as the image of the Lebesgue measure by f, namely*

$$\forall a \in \mathbb{R}, \quad \mathrm{Leb}_{\mathbb{S}}\Big(\big\{x \in \mathbb{S} : f^*(x) \le a\big\}\Big) = \mathrm{Leb}_{\mathbb{S}}\Big(\big\{x \in \mathbb{S} : f(x) \le a\big\}\Big),$$

which is sometimes known as 'Cavalieri's principle'.

Intuitively, f^* should be regarded as a quantile function, the symmetrisation procedure here forcing an obvious form of 'continuous periodicity' (whose interpretation requires some care as f^* may have jumps). Indeed, it must be noted that the collection of functions f^* satisfying item 1 in the definition above are one-to-one with the set $\mathcal{P}_2(\mathbb{R})$ of probability measures on $\mathbb{R}$ that have a finite-second moment. In fact, for f^* as in item 1 and for a probability measure $\mu \in \mathcal{P}_2(\mathbb{R})$, the measure $\mathrm{Leb}_{\mathbb{S}} \circ (f^*)^{-1}$ is equal to μ if and only if $x \in [0,1] \mapsto f^*((1-x)/2)$ coincides with the usual quantile function, i.e. the usual generalised inverse of the (right-continuous) cumulative distribution function. The reader is referred to Baernstein [8] for further details, see in particular Definition 1.29 therein for the general definition of symmetric rearrangements in the Euclidean setting and Chapter 7 in the same book for a specific treatment of spherical symmetric rearrangements.

The following definition is used quite often in the rest of the text:

Definition 2.2. *A function $f : \mathbb{S} \to \mathbb{R}$ is said to be symmetric non-increasing if $f = f^*$. The collection of equivalence classes in $L^2(\mathbb{S})$ containing a symmetric non-increasing function is denoted by $U^2(\mathbb{S})$. It is a cone.*

Below, we often consider elements of $L^2_{\mathrm{sym}}(\mathbb{S})$. They are defined as functions in $L^2(\mathbb{S})$ that are Lebesgue almost everywhere symmetric. One of these elements is said to be *non-increasing* (we refrain from tautological use of the word symmetric and believe that given the context of the circle, this should not be a source of confusion) if it coincides almost everywhere with an element of $U^2(\mathbb{S})$. Notice that we may choose the latter representative to be uniquely defined as a symmetric non-increasing function. Indeed, two elements of $U^2(\mathbb{S})$ that coincide in $L^2(\mathbb{S})$ coincide in fact everywhere on $\mathbb{S}$ (courtesy of the left- and right-continuity properties). Also, the following proposition is of clear importance.

Proposition 2.3. *Both $L^2_{\mathrm{sym}}(\mathbb{S})$ and $U^2(\mathbb{S})$ are closed subsets of $L^2(\mathbb{S})$ equipped with $\|\cdot\|_2$.*

Proof. Closedness of $L^2_{\mathrm{sym}}(\mathbb{S})$ is obvious. Indeed, given a sequence $(f_n)_{n\geq 1}$ of elements of $L^2_{\mathrm{sym}}(\mathbb{S})$ that converges to some f in $L^2(\mathbb{S})$, it suffices to let n tend to ∞ in the identity

$$\int_{\mathbb{S}} f_n(x)\varphi(x)dx = \int_{\mathbb{S}} f_n(-x)\varphi(x)dx,$$

which holds true for any $\varphi \in L^2(\mathbb{S})$. Closedness of $U^2(\mathbb{S})$ is a consequence of Lemma 2.6 below, which says that, if $(f_n)_{n\geq 1}$ takes values in $U^2(\mathbb{S})$, then $\|f_n^\star - f^\star\|_2 \leq \|f_n - f\|_2$ for each $n \geq 1$, so that $(f_n)_{n\geq 1}$ also converges to $f^\star$, from which we get $f = f^\star$. □

2.2 Reformulating the main results

In what follows, our diffusion process with suitable smoothing properties on $\mathcal{P}_2(\mathbb{R})$ thus arrives via the construction of a diffusion process with values in $U^2(\mathbb{S})$. The equivalence relies on the fact that the mapping $f^* \in U^2(\mathbb{S}) \mapsto \mathrm{Leb}_{\mathbb{S}} \circ (f^*)^{-1} \in \mathcal{P}_2(\mathbb{R})$ is not only

one-to-one but is in fact an isometry when $\mathcal{P}_2(\mathbb{R})$ is equipped with the $\mathcal{W}_2$-Wasserstein distance, i.e., for any two f^*, g^* in $U^2(\mathbb{S})$,

$$\|f^* - g^*\|_2 = \mathcal{W}_2\big(\mathrm{Leb}_{\mathbb{S}} \circ (f^*)^{-1}, \mathrm{Leb}_{\mathbb{S}} \circ (g^*)^{-1}\big),$$

where

$$\mathcal{W}_2(\mu,\nu)^2 := \inf_{\pi \in \mathcal{P}(\mathbb{R}^2)): \pi \circ e_x^{-1} = \mu, \pi \circ e_y^{-1} = \nu} \int_{\mathbb{R}^2} |x-y|^2 \pi(dx,dy),$$

with $e_x : (x,y) \in \mathbb{R}^2 \mapsto x$ and $e_y : (x,y) \in \mathbb{R}^2 \mapsto y$ being the two evaluation mappings on $\mathbb{R}^2$. It is worth observing that this identity is a direct consequence of the non-expansive property stated in Lemma 2.6, since for any π as above, there exist two (measurable) functions f and g from $\mathbb{S}$ to $\mathbb{R}$ such that $\pi = \mathrm{Leb}_{\mathbb{S}} \circ (f,g)^{-1}$.

In this framework, our main results can be (re)formulated as follows:

1. We introduce a stochastic differential equation on $U^2(\mathbb{S})$ in the form of a reflected (or rearranged) stochastic equation on $L^2(\mathbb{S})$ whose reflection term forces solutions to stay within the cone $U^2(\mathbb{S})$, whenever they are initialised from $U^2(\mathbb{S})$, see Theorem 4.15. Solutions induce a Lipschitz continuous flow with values in $U^2(\mathbb{S})$. The construction of the rearranged equation relies on an Euler scheme, in which we alternate some flat dynamics in the space $L^2_{\mathrm{sym}}(\mathbb{S})$ with the rearrangement operation that projects back the solution onto $U^2(\mathbb{S})$.

2. The second main statement is Theorem 5.11, which says that the semigroup generated by our rearranged stochastic equation maps bounded measurable functions on $U^2(\mathbb{S})$ into Lipschitz continuous functions on $U^2(\mathbb{S})$. Recast on $\mathcal{P}_2(\mathbb{R})$ (through the isometry between $U^2(\mathbb{S})$ and $\mathcal{P}_2(\mathbb{R})$), we get in this way a semigroup that maps bounded measurable functions on $\mathcal{P}_2(\mathbb{R})$ into Lipschitz continuous functions (with respect to the 2-Wasserstein distance $\mathcal{W}_2$).

2.3 Key properties of the symmetric non-increasing rearrangement

In the subsection, we expand a list of properties that are satisfied by f^* and that we use quite often in the paper (including the aforementioned Lemma 2.6). The first one is obvious and just follows from item 2 in the statement of Proposition 2.1.

Lemma 2.4 (Preservation of L^p norms)**.** *With the same notations as in Proposition 2.1, we have, for any $p \in [1,\infty]$, $\|f^*\|_p = \|f\|_p$.*

The next result, called the Hardy-Littlewood inequality, is fundamental.

Lemma 2.5 (Hardy-Littlewood inequality)**.** *Let f and g be two measurable real-valued functions defined on $\mathbb{S}$ such that $\|f\|_p < \infty$ and $\|g\|_q < \infty$, for $p,q \in [1,\infty]$, with $1/p + 1/q = 1$. Then,*

$$\int_{\mathbb{S}} f(x)g(x)dx \le \int_{\mathbb{S}} f^*(x)g^*(x)dx.$$

We refer to [8, Corollary 2.16] for a general statement in the Euclidean setting, but stated under the conditions that f and g are non-negative, and to [8, Section 7.3] or [6, 7] for a version without non-negativity constraints that is specifically stated on the circle.

We now turn to the well-known property of non-expansion:

Lemma 2.6 (Non-expansion property)**.** *Let f and g be two measurable real-valued functions such that $\|f\|_p < \infty$ and $\|g\|_p < \infty$, for $p \in [1, \infty]$. Then,*

$$\|f^* - g^*\|_p \leq \|f - g\|_p.$$

We refer to [8, Corollary 2.23] for the Euclidean setting (which requires f and g to be positive valued) and to [8, Section 7.3] for the extension to the spherical setting (which no longer requires f and g to be positive valued).

The following statement is taken from [6, 7], see also [8, Theorem 8.1].

Lemma 2.7 (Riesz rearrangement inequality)**.** *Let f, g and h be three measurable real-valued functions on $\mathbb{S}$, such that $\|f\|_p < \infty$, $\|g\|_q < \infty$ and $\|h\|_r < \infty$ for $p, q, r \in [1, \infty]$ with $1/p + 1/q + 1/r = 1$. Then,*

$$\int_{\mathbb{S}} \int_{\mathbb{S}} f(x) g(x-y) h(y) dx dy \leq \int_{\mathbb{S}} \int_{\mathbb{S}} f^*(x) g^*(x-y) h^*(y) dx dy.$$

We conclude this list of inequalities with:

Lemma 2.8 (Pólya–Szegő inequality.)**.** *Let f be a Lipschitz continuous real-valued function on $\mathbb{S}$. Then, for any real $p \in [1, \infty]$, f^* is also Lipschitz continuous and*

$$\|Df^*\|_p \leq \|Df\|_p,$$

where Df and $Df^\star$ are almost everywhere defined derivatives of f and f^* respectively.

Preservation of the Lipschitz property follows from the spherical version of [8, Theorem 3.6]. The inequality is taken from [8, Theorem 7.4].

2.4 The heat kernel and the rearrangement operator

In the rest of the work, we often compose the rearrangement operator and the heat kernel. In this preliminary subsection, we thus address several basic properties of this composition, which we invoke quite often in this text.

First, it is worth recalling that the periodic heat semigroup (with specific diffusivity parameter 1) on the circle $\mathbb{S}$, which we denote $(e^{t\Delta})_{t \geq 0}$, has the following kernel (see Dym and McKean p.63 [33]):

$$\Gamma(t, x) := \frac{1}{\sqrt{4\pi t}} \sum_{n \in \mathbb{Z}} \exp\left\{-\frac{(x-n)^2}{4t}\right\}, \quad t > 0, \ x \in \mathbb{S}. \tag{2.1}$$

The following lemma says that $\Gamma(t, \cdot)$ is a symmetric non-increasing function on the circle, namely $\Gamma(t, \cdot)$ is equal to $\Gamma(t, \cdot)^*$, the rearrangement operation $*$ here acting on the x-variable.

Lemma 2.9. *For any $t > 0$, the function $x \mapsto \Gamma(t,x)$ is non-increasing on $(0,1/2)$ and non-decreasing on $(-1/2,0)$.*

The proof of Lemma 2.9 is not so trivial, because of the series underpinning the expression of Γ. The reader will find a general discussion on spherical heat kernels in the recent paper [58], but specific (and much easier) computations that suffice for the proof of the above statement can be found in [2].

We conclude this subsection by proving that the periodic heat semigroup preserves the symmetric non-increasing property of functions on the circle:

Lemma 2.10. *Let f belonging to $L^1(\mathbb{S})$ be symmetric non-increasing on the circle (i.e., $f = f^*$). Then, for any $t > 0$, the convolution $f * \Gamma(t,\cdot)$ is also symmetric non-increasing on the circle.*

Proof. We let $h = f * \Gamma(t,\cdot)$, for a given $t > 0$. Lemmas 2.7 and 2.9 yield

$$\int_{\mathbb{S}}\int_{\mathbb{S}} f(x)\Gamma(t,x-y)h(y)dxdy \le \int_{\mathbb{S}}\int_{\mathbb{S}} f(x)\Gamma(t,x-y)h^*(y)dxdy,$$

which can be rewritten as (using the symmetry of the heat kernel)

$$\|h\|_2^2 = \int_{\mathbb{S}}\int_{\mathbb{S}} f(x)\Gamma(t,y-x)h(y)dxdy \le \int_{\mathbb{S}}\int_{\mathbb{S}} f(x)\Gamma(t,y-x)h^*(y)dxdy = \langle h, h^*\rangle_2.$$

It remains to see that, by the preservation of L^p norms,

$$\|h-h^*\|_2^2 = \|h\|_2^2 + \|h^*\|_2^2 - 2\langle h,h^*\rangle_2 = 2\|h\|_2^2 - 2\langle h,h^*\rangle_2 \le 0,$$

which yields $h = h^*$ almost everywhere. Since h is continuous (by convolution), so is h^* (see [8, Subsection 2.4]). Therefore, h and h^* coincide. □

2.5 Some notation

The paper makes an intense use of tools from functional and Fourier analysis. We thus introduce a few related notations that we use repeatedly in the text. The space of continuous functions from a metric space $\mathcal{X}$ to another metric space $\mathcal{Y}$ is denoted $\mathcal{C}(\mathcal{X},\mathcal{Y})$. When working on the Euclidean space $\mathbb{R}^k$, for an integer $k \ge 1$, we denote by $\mathcal{C}_0^\infty(\mathbb{R}^k)$ the space of infinitely differentiable real-valued functions on $\mathbb{R}^k$ with a compact support.

We recall that $\mathbb{S}$ is the circle parametrised by the interval of length 1. Also, we let

$$e_m^{\mathfrak{R}} : x \in \mathbb{S} \mapsto \sqrt{2}\cos(2m\pi x), \quad e_m^{\mathfrak{I}} : x \in \mathbb{S} \mapsto \sqrt{2}\sin(2m\pi x),$$

for any (non-negative) integer m, form the complete Fourier basis on $L^2(\mathbb{S})$, where $L^2(\mathbb{S})$ is the space of square integrable functions on $\mathbb{S}$. Quite often, we just use the even (cosine) Fourier functions, which prompts us to use the shorter notation e_m for $e_m^{\mathfrak{R}}$.

The Lebesgue measure on $\mathbb{S}$ is denoted $\mathrm{Leb}_{\mathbb{S}}$ with $d\mathrm{Leb}_{\mathbb{S}}(x)$ written as dx. For any real $p \ge 1$, we call $\|\cdot\|_p$ the L^p norm on the space of measurable functions f on $(\mathbb{S}, \mathrm{Leb}_{\mathbb{S}})$ with

$\int_{\mathbb{S}} |f(x)|^p dx < \infty$. Similarly, when $p = \infty$, the notation $\|\cdot\|_\infty$ is used for the L^∞ (supremum) norm, i.e. $\|f\|_\infty := \operatorname{essup}\{f(x) : x \in \mathbb{S}\}$. The inner product in $L^2(\mathbb{S})$ is denoted $\langle \cdot, \cdot \rangle_2$. For an element $f \in L^1(\mathbb{S})$ and for a non-negative integer m, we call $\hat{f}_m^{\mathfrak{R}} := \int_{\mathbb{S}} f(x) e_m^{\mathfrak{R}}(x) dx$ the cosine Fourier mode of f of index m and $\hat{f}_m^{\mathfrak{I}} := \int_{\mathbb{S}} f(x) e_m^{\mathfrak{I}}(x) dx$ the sine Fourier mode of f of index m. When f is Lebesgue almost everywhere (written a.e. hereafter) symmetric, i.e. $f(-x) = f(x)$ a.e., all the sine Fourier modes are 0 and we can just write $\hat{f}_m = \langle f, e_m \rangle$ in place of $\hat{f}_m^{\mathfrak{R}}$. In that case, $\hat{f}^m$ is a real number. As we already mentioned, these symmetric functions are used quite often in the text and we denote by $L^2_{\text{sym}}(\mathbb{S})$ the set of functions f in $L^2(\mathbb{S})$ that are a.e. symmetric. More generally, for a parameter $\mu \in \mathbb{R}$, we denote by $H^\mu_{\text{sym}}(\mathbb{S})$ the Sobolev space of a.e. symmetric functions f such that

$$\|f\|_{2,\mu}^2 := \sum_{m \in \mathbb{N}_0} (m \vee 1)^{2\mu} \hat{f}_m^2 < \infty,$$

($\mathbb{N}$ is the collection of natural numbers 1, 2, 3, ... The set $\mathbb{N}_0$ is defined as $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$). Of course, $H^0_{\text{sym}}(\mathbb{S})$ is nothing but $L^2_{\text{sym}}(\mathbb{S})$. The inner product on $H^\mu_{\text{sym}}(\mathbb{S})$ is denoted $\langle f, g \rangle_{2,\mu} := \sum_{m \in \mathbb{N}_0} (m \vee 1)^{2\mu} \hat{f}_m \hat{g}_m$.

For any integer $k \geq 1$, we denote by $\mathcal{C}^k(\mathbb{S})$ the space of k-times continuously differentiable functions on $\mathbb{S}$.

We now introduce some standard notations. For a real number x, we write $\lfloor x \rfloor$ for the floor of x, $\lceil x \rceil$ for the ceiling of x and $x_+ := \max(x, 0)$ (resp. $x_- = \min(-x, 0)$) for the positive (resp. negative) part of x. For two reals x and y, we let $x \vee y := \max(x, y)$ and $x \wedge y := \min(x, y)$. Moreover, for a differentiable real-valued function on $\mathbb{S}$, we write Df for the derivative of f. And, we let $\Delta := D^2$.

As for constants that are used in the various inequalities, they are usually written in the form $c_{a,b}$ or $C_{a,b}$, where the subscripts are quantities on which the current constant depends, and are implicitly allowed to vary from line to line (as long as they just depend on the same quantities a and b).

3 Approximation Scheme and its Estimates

The construction of our reflected stochastic heat equation relies on a discretisation scheme in which we alternate one random move in the Hilbert space $L^2_{\text{sym}}(\mathbb{S})$ of square-integrable symmetric random variables on $\mathbb{S}$ and a rearrangement, forcing the output of the discretisation scheme to stay within the subset of symmetric non-increasing functions $U^2(\mathbb{S})$.

Definition of the noise. The randomisation in $L^2_{\text{sym}}(\mathbb{S})$ obeys a standard Euler scheme, with Gaussian increments. This prompts us to introduce the following Wiener process, $(W_t)_{t \geq 0}$:

$$W_t := B_t^0 e_0 + \sum_{m \in \mathbb{N}} m^{-\lambda} B_t^m e_m \equiv \sum_{m \in \mathbb{N}_0} \lambda_m B_t^m e_m, \quad t \geq 0, \tag{3.1}$$

where $\lambda > 1/2$ and the sequence $(\lambda_m)_{m \in \mathbb{N}_0}$ is given by $\lambda_0 := 1$ and $\lambda_m := m^{-\lambda}$ for $m \in \mathbb{N}$. Here, the processes $\{(B_t^m)_{t \geq 0}\}_{m \in \mathbb{N}_0}$ are independent standard Brownian motions constructed on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Whilst it would be desirable to work with

the widest possible range for λ, we here restrict the analysis to $\lambda > 1/2$. In particular, this choice precludes the white noise. In contrast, our choice $\lambda > 1/2$ forces the sequence $(\lambda_m)_{m\in\mathbb{N}_0}$ to be square summable. In particular, the process $(W_t)_{t\geq 0}$ can be equivalently defined as an $L^2_{\mathrm{sym}}(\mathbb{S})$-valued Brownian motion with covariance function

$$Q : (f,g) \in \left(L^2_{\mathrm{sym}}(\mathbb{S})\right)^2 \mapsto s \wedge t \sum_{m\in\mathbb{N}_0} \lambda_m^2 \hat{f}^m \hat{g}^m = s \wedge t \, \langle f, g\rangle_{2,-\lambda}. \tag{3.2}$$

Definition of the scheme. The approximation scheme is constructed via composition of the stochastic convolution associated with W and the rearrangement operator $*$ defined in Proposition 2.1. Given a stepsize $h \in (0,1)$, we define the sequence $(X_n^h)_{n\in\mathbb{N}_0}$ by,

$$\begin{aligned} X_{n+1}^h =& \left(e^{h\Delta} X_n^h + \int_0^h e^{(h-s)\Delta} dW_s^{n+1}\right)^* \\ X_0^h =& X_0, \end{aligned} \tag{3.3}$$

where

$$W_s^{n+1} := W_{s+nh} - W_{nh}, \tag{3.4}$$

and X_0 is an $U^2(\mathbb{S})$-valued random variable that is assumed to be independent of $(W_t)_{t\geq 0}$ (see **Assumption on** X_0 for a clear formulation).

In Subsections 3.1 and 3.2, the dependence of W^{n+1} on h is suppressed in the notation, since h is kept fixed. It is only in the forthcoming Subsection 3.3 that h becomes variable as we let the stepsize of the scheme tend to 0. To precise, we use below the notation $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ to denote a filtration (satisfying the usual conditions) such that X_0 is $\mathcal{F}_0$-measurable and $(W_t)_{t\geq 0}$ is a Q-Brownian motion with respect to $(\mathcal{F}_t)_{t\geq 0}$.

Reminders about the stochastic convolution. It is worthwhile to recall several properties of the stochastic convolution that will be used within this article. In the interest of brevity, the general conditions under which the following properties hold are not recalled; they are satisfied for the Laplacian and this is enough for the present purpose. The interested reader is referred to cited articles and the proceeding literature for generalisations and to Da Prato and Zabczyk [24, Ch.5] for a comprehensive introduction.

For an $\mathcal{F}_0$-measurable initial condition X_0 with values in $L^2_{\mathrm{sym}}(\mathbb{S})$, the stochastic convolution provides a weak solution to the linear equation,

$$dX_t = \Delta X_t dt + dW_t, \quad t \geq 0,$$

written on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and a time interval $[0,T]$, for a given time horizon $T > 0$. That is to say, that for all $t \in [0,T]$ and $\varphi \in \mathcal{C}^2(\mathbb{S})$, the process $\hat{X} := (\hat{X}_t)_{t\geq 0}$ defined by,

$$\hat{X}_t := e^{t\Delta} X_0 + \int_0^t e^{(t-s)\Delta} dW_s, \quad t \geq 0,$$

satisfies, $\mathbb{P}$ almost surely,

$$\langle \hat{X}_t, \varphi \rangle = \langle X_0, \varphi \rangle + \int_0^t \langle \hat{X}_s, \Delta\varphi \rangle ds + \langle W_t, \varphi \rangle, \quad t \geq 0.$$

By a result of Kotelenez, [47, Theorem 2, p.146], the process $\hat{X}$ has a version with continuous sample paths. From Zangeneh, [77, Corollary 1, p.345], this version is adapted. Additionally, from Salavati and Zangeneh, [68, Theorem 6, p.4], the following pathwise estimate holds for the pth power of the norm, $p \geq 2$,

$$\|\hat{X}_t\|_2^p \leq \|\hat{X}_0\|_2^p + p \int_0^t \|\hat{X}_s\|_2^{p-2} \langle \hat{X}_s, dW_s \rangle + \frac{p(p-1)}{2} \int_0^t \|\hat{X}_s\|_2^{p-2} d[W]_s, \tag{3.5}$$

where

$$[W]_t = \sum_{m \in \mathbb{N}_0} \lambda_m^2 t, \quad t \geq 0, \tag{3.6}$$

is the standard bracket of $(W_t)_{t \geq 0}$.

Furthermore, noting that the stochastic convolution is generally not a martingale (or even a semimartingale), one may not apply immediately the maximal inequalities or other martingale theorems. However, in the present setting, due to a theorem of Zangeneh [78, Theorem 2, p.147], see also [68, Theorem 5, p.4], the following estimate holds, for $p \geq 2$, $T > 0$,

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} \left\| \int_0^t e^{(t-s)\Delta} dW_s \right\|_2^p \right] \leq c_p \mathbb{E}\left[[W]_T^{p/2} \right]. \tag{3.7}$$

Subsections 3.1 and 3.2 are dedicated to the proving of some key estimates on the scheme that will be useful when letting the stepsize h tend to 0. Tightness is addressed in Subsection 3.3.

Distributional derivative of the noise. Several times in the text, we use the distributional derivative of W. For any $t \geq 0$, we thus let

$$w_t := DW_t = -2\pi \sum_{m \in \mathbb{N}} m^{1-\lambda} B_t^m e_m^{\mathfrak{I}}, \quad t \geq 0. \tag{3.8}$$

Since $(W_t)_{t \geq 0}$ is a Q-Brownian motion with values in $L^2_{\text{sym}}(\mathbb{S})$, we easily get that $(w_t)_{t \geq 0}$ is a Brownian motion with values in $H^{-1}_{\text{anti-sym}}(\mathbb{S})$, the latter being defined as the dual of the space $H^1_{\text{anti-sym}}(\mathbb{S})$ of anti-symmetric periodic functions with a square-integrable generalised gradient. We do not use the form of the covariance matrix, but it could be explicitly computed, following (3.2).

Assumption on X_0. Throughout the rest of the paper, we assume that X_0 is an $\mathcal{F}_0$-measurable random variable with values in $U^2(\mathbb{S})$, satisfying

$$\forall p \geq 1, \quad \mathbb{E}\left[\|X_0\|_2^{2p} \right] < \infty. \tag{3.9}$$

3.1 L^p estimates of the solution

We start with some preliminary estimates for the L^p norm of the process $(X_n^h)_{n\in\mathbb{N}_0}$. Firstly, the following estimate for W is given:

Lemma 3.1. *For $p > 0$ (and for h being the stepsize of the scheme and λ the exponent colouring the noise),*

$$\mathbb{E}\left[\left\|\int_0^h e^{(h-s)\Delta}dW_s\right\|_2^{2p}\right] \leq c_{p,\lambda}h^p. \tag{3.10}$$

When $p = 1$,

$$c_{1,\lambda} = \sum_{m\in\mathbb{N}_0}\lambda_m^2 = \frac{d}{dt}[W]_t. \tag{3.11}$$

Proof. From Theorem 4.36 in [24], p114 (refer to p.96 therein for related notation):

$$\begin{aligned}\mathbb{E}\left[\left\|\int_0^h e^{(h-s)\Delta}dW_s\right\|_2^{2p}\right] &\leq c_p\mathbb{E}\left[\left(\int_0^h \sum_{m,n\in\mathbb{N}_0}|\langle e^{(h-s)\Delta}\lambda_m e_m, e_n\rangle|^2 ds\right)^p\right] \\ &= c_p\left(\int_0^h \sum_{m\in\mathbb{N}_0}\lambda_m^2 e^{-8\pi^2(h-s)m^2}ds\right)^p \\ &\leq c_p h^p\left(\sum_{m\in\mathbb{N}_0}\lambda_m^2\right)^p =: c_{p,\lambda}h^p,\end{aligned} \tag{3.12}$$

where we used the obvious bound $e^{-r} \leq 1$, for $r \geq 0$, to pass to the final line. The second identity in (3.11) follows from (3.6), concluding the proof. □

As a consequence, we have:

Lemma 3.2. *For $T > 0$ and $p \geq 2$ (and for h being the stepsize of the scheme and λ the exponent colouring the noise),*

$$\sup_{n\in\mathbb{N}_0:nh\leq T}\mathbb{E}\left[\|X_n^h\|_2^p\right] \leq c_{p,\lambda,T}\left(1 + \mathbb{E}\left[\|X_0\|_2^p\right]\right). \tag{3.13}$$

Proof. The first step follows from the fact that the rearrangement preserves L^p norms.

$$\mathbb{E}\left[\|X_n^h\|_2^p\right] = \mathbb{E}\left[\left\|e^{h\Delta}X_{n-1}^h + \int_0^h e^{(h-s)\Delta}dW_s^n\right\|_2^p\right]. \tag{3.14}$$

The mild solution to the stochastic heat equation started from X_{n-1}^h and driven by $(W_r^n)_{0\leq r\leq h}$ (see (3.4)) is denoted here by

$$\hat{X}_s^{h,n-1} := e^{s\Delta}X_{n-1}^h + \int_0^s e^{(s-r)\Delta}dW_r^n, \quad s \in [0,h]. \tag{3.15}$$

Then, by estimate (3.5),

$$\begin{aligned}&\mathbb{E}\left[\|X_n^h\|_2^p\right]\\&\quad\le \mathbb{E}\left[\left\|X_{n-1}^h\right\|_2^p + p\int_0^h \|\hat{X}_s^{h,n}\|_2^{p-2}\langle \hat{X}_s^{h,n}, dW_s^n\rangle + \tfrac{p(p-1)}{2}\int_0^h \|\hat{X}_s^{h,n}\|_2^{p-2} d[W^n]_s\right].\end{aligned} \tag{3.16}$$

One may remove the martingale terms that contribute zero (by induction over the index n in (3.14), the left-hand side therein is obviously finite, and then the left-hand side in (3.15) has a finite p-moment for any $p \ge 1$). It remains to control $\mathbb{E}[\|\hat{X}_s^{h,n}\|_2^q]$ for $q \ge 0$.

$$\begin{aligned}\|\hat{X}_s^{h,n}\|_2^q = \left\|e^{s\Delta}X_{n-1}^h + \int_0^s e^{(s-r)\Delta}dW_r^n\right\|_2^q &\le c_q\left(\left\|e^{s\Delta}X_{n-1}^h\right\|_2^q + \left\|\int_0^s e^{(s-r)\Delta}dW_r^n\right\|_2^q\right)\\ &\le c_q\left(\|X_{n-1}^h\|_2^q + \left\|\int_0^s e^{(s-r)\Delta}dW_r^n\right\|_2^q\right),\end{aligned} \tag{3.17}$$

where the last bound follows from the contraction property of the heat semigroup. In light of Lemma 3.1,

$$\mathbb{E}\left[\|\hat{X}_s^{h,n}\|_2^q\right] \le c_q\left(\mathbb{E}\left[\|X_{n-1}^h\|_2^q\right] + c_{q,\lambda}h^{q/2}\right). \tag{3.18}$$

Choosing $q = p-2$ and injecting the above bound in (3.16), we obtain

$$\mathbb{E}\left[\|X_n^h\|_2^p\right] \le \mathbb{E}\left[\|X_{n-1}^h\|_2^p\right] + c_{p,\lambda}h\mathbb{E}\left[\left\|X_{n-1}^h\right\|_2^{p-2} + h^{(p-2)/2}\right], \tag{3.19}$$

and then, using the assumption $h < 1$ together with the obvious bound $a^{p-2} \le 1 + a^p$, for $a \ge 0$, we get

$$\mathbb{E}\left[\|X_n^h\|_2^p\right] \le \left(1 + c_{p,\lambda}h\right)\mathbb{E}\left[\|X_{n-1}^h\|_2^p\right] + c_{p,\lambda}h.$$

The conclusion follows from the discrete version of Gronwall's lemma. □

3.2 Uniform estimate for the derivatives of the iterates

We now provide some estimates for the spatial derivative of the functions $(X_n^h)_{n\ge 0}$. Whilst this may look quite easy to do, thanks to the presence of the heat semigroup in the formula (3.3), the analysis is in fact more complicated due to the additional presence of the rearrangement. Part of the proof is thus to estimate the concomitant effect of the periodic heat semigroup and the rearrangement transformation. This is given in the following lemma:

Lemma 3.3. *Let u belong to $L^1(\mathbb{S})$ and U be uniformly distributed on $[0,1]$ on the same space $(\Omega, \mathcal{F}, \mathbb{P})$ as before. Then,*

$$\mathbb{E}\left[\|De^{hU\Delta}u^*\|_2^2\right] \le \mathbb{E}\left[\|De^{hU\Delta}u\|_2^2\right]. \tag{3.20}$$

Of course, the two expectations in the statement could be rewritten as

$$\mathbb{E}\big[\|De^{hU\Delta}u^*\|_2^2\big] = \int_0^1 \|De^{hs\Delta}u^*\|_2^2 ds, \quad \mathbb{E}\big[\|De^{hU\Delta}u\|_2^2\big] = \int_0^1 \|De^{hs\Delta}u\|_2^2 ds,$$

but the notation using the expectation symbol is convenient for the rest of the analysis.

Proof. Recalling the notation Γ in (2.1) for the heat kernel, we have the following two identities:

$$\begin{aligned}\|De^{s\Delta}u^*\|_2^2 &= \int_{\mathbb{S}}\Big(\int_{\mathbb{S}} D\Gamma(s,x-y)u^*(y)dy\Big)^2 dx \\ &= \int_{\mathbb{S}}\int_{\mathbb{S}}\int_{\mathbb{S}} D\Gamma(s,x-y)u^*(y)D\Gamma(s,x-z)u^*(z)dydzdx.\end{aligned}$$

Observe that

$$\begin{aligned}\int_{\mathbb{S}} D\Gamma(s,x-y)D\Gamma(s,x-z)dx = D_yD_z\int_{\mathbb{S}}\Gamma(s,x-y)\Gamma(s,x-z)dx &= D_yD_z\Big[\Gamma(2s,y-z)\Big] \\ &= -D_x^2\Gamma(2s,y-z).\end{aligned}$$

Therefore,

$$\begin{aligned}\|De^{s\Delta}u^*\|_2^2 &= -\int_{\mathbb{S}}\int_{\mathbb{S}} u^*(y)D_x^2\Gamma(2s,y-z)u^*(z)dydz \\ &= -\int_{\mathbb{S}}\int_{\mathbb{S}} u^*(y)D_t\Gamma(2s,y-z)u^*(z)dydz.\end{aligned} \tag{3.21}$$

By integrating the above equality in s over $[0,h]$ and by applying both Cavalieri's principle and Riesz' rearrangement inequality, we get

$$\begin{aligned}\int_0^h \|De^{s\Delta}u^*\|_2^2 ds &= \frac{1}{2}\Big[\int_{\mathbb{S}} u^*(y)^2dy - \int_{\mathbb{S}}\int_{\mathbb{S}} u^*(y)\Gamma(2h,y-z)u^*(z)dydz\Big] \\ &\le \frac{1}{2}\Big[\int_{\mathbb{S}} u(y)^2dy - \int_{\mathbb{S}}\int_{\mathbb{S}} u(y)\Gamma(2h,y-z)u(z)dydz\Big] \\ &= \int_0^h \|De^{s\Delta}u\|_2^2 ds,\end{aligned} \tag{3.22}$$

the last line being obtained by reverting back the computations in (3.21). This completes the proof. □

Next, Lemma 3.3 is applied to the scheme. We claim:

Lemma 3.4 (Uniform estimate for the schemes' derivatives)**.** *For $\delta \in [0,\lambda-\frac{1}{2})$ (with λ as in (3.1)), and for $n \in \mathbb{N}$ (with h being the stepsize of the scheme),*

$$\mathbb{E}\Big[\|DX_n^h\|_2^2\Big] \le \frac{1}{hn}\mathbb{E}\big[\|X_0\|_2^2\big] + c_{\delta,\lambda}\big[h^\delta + (hn)^{\frac{1}{2}} + (hn)^{\min(\lambda-\frac{1}{2},\frac{1}{2})}\big]. \tag{3.23}$$

Proof. Following the notation of Lemma 3.3, we introduce, as an artefact of the proof method, a uniformly distributed $[0,1]$-valued random variable U independent of the filtration $\mathbb{F}$. By Lemma 2.8 (the Pólya–Szegő inequality),

$$\mathbb{E}\Big[\|DX_n^h\|_2^2\Big] \le \mathbb{E}\Bigg[\Bigg\|D\Big(e^{h\Delta}X_{n-1}^h + \int_0^h e^{(h-s)\Delta}dW_s^n\Big)\Bigg\|_2^2\Bigg].$$

Recalling the notation (3.8) for the weak derivative w of W (and adapting accordingly the notation for increments over the time discretisation (3.4)), we get

$$\begin{aligned}\mathbb{E}\Big[\|DX_n^h\|_2^2\Big] &\le \mathbb{E}\Bigg[\Bigg\|D\Big(e^{h\Delta}X_{n-1}^h + \int_0^h e^{(h-s)\Delta}dW_s^n\Big)\Bigg\|_2^2\Bigg]\\ &= \mathbb{E}\Bigg[\|De^{h\Delta}X_{n-1}^h\|_2^2 + \Bigg\|\int_0^h e^{(h-s)\Delta}dw_s^n\Bigg\|_2^2\Bigg]\\ &\le \mathbb{E}\Bigg[\|De^{hU\Delta}X_{n-1}^h\|_2^2 + \Bigg\|\int_0^h e^{(h-s)\Delta}dw_s^n\Bigg\|_2^2\Bigg],\end{aligned}$$

where we used the independence of $De^{h\Delta}X_{n-1}^h$ and $\int_0^h e^{(h-s)\Delta}dw_s^n$ to pass from the first to the second line and the simple fact that U is $[0,1]$-valued to pass from the penultimate to the last line (together with the obvious fact that D and Δ commute). We now make use of Lemma 3.3. Using the scheme (3.3), with n replaced by $n-2$ therein, we obtain

$$\begin{aligned}\mathbb{E}\Big[\|DX_n^h\|_2^2\Big] &\le \mathbb{E}\Bigg[\Big\|De^{h(U+1)\Delta}X_{n-2}^h\Big\|_2^2 + \sum_{k=1}^{2}\Bigg\|e^{(k-1)hU\Delta}\int_0^h e^{(h-s)\Delta}dw_s^{n+1-k}\Bigg\|_2^2\Bigg]\\ &\le \mathbb{E}\Bigg[\|De^{2hU\Delta}X_{n-2}^h\|_2^2 + \sum_{k=1}^{2}\Bigg\|e^{(k-1)hU\Delta}\int_0^h e^{(h-s)\Delta}dw_s^{n+1-k}\Bigg\|_2^2\Bigg].\end{aligned}$$

By iteration,

$$\begin{aligned}&\mathbb{E}\Big[\|DX_n^h\|_2^2\Big]\\ &\le \mathbb{E}\Bigg[\|De^{nhU\Delta}X_0\|_2^2 + \sum_{k=1}^{n}\Bigg\|e^{(k-1)hU\Delta}\int_0^h e^{(h-s)\Delta}dw_s^{n+1-k}\Bigg\|_2^2\Bigg]\\ &\le \mathbb{E}\Big[\|De^{nhU\Delta}X_0\|_2^2\Big] + 4\pi^2\sum_{k=1}^{n}\mathbb{E}\Bigg[\sum_{m\ge1}e^{-8\pi^2(k-1)hUm^2}\int_0^h e^{-8\pi^2(h-s)m^2}m^{2(1-\lambda)}ds\Bigg]\\ &\le \mathbb{E}\Big[\|De^{nhU\Delta}X_0\|_2^2\Big] + \sum_{k=1}^{n}\mathbb{E}\Bigg[\sum_{m\ge1}e^{-8\pi^2(k-1)hUm^2}(1-e^{-8\pi^2hm^2})m^{-2\lambda}\Bigg].\end{aligned}$$

Next, recall that $\lambda > \frac{1}{2}$, and for $\delta \le 1$, $(1-e^{-x}) \le x^\delta$ for $x \ge 0$. Then, handing the first

summand indexed in k with $\delta \in [0, \lambda - \frac{1}{2})$, and the remaining one with $\delta = 1$,

$$\begin{aligned}
\mathbb{E}\left[\|DX_n^h\|_2^2\right] &\le \mathbb{E}\left[\|De^{nhU\Delta}X_0\|_2^2\right] + c_{\delta,\lambda}h^\delta + ch\sum_{k=2}^{n}\mathbb{E}\left[\sum_{m\ge1} e^{-8\pi^2(k-1)hUm^2}m^{2(1-\lambda)}\right] \\
&\le \mathbb{E}\left[\|De^{nhU\Delta}X_0\|_2^2\right] + c_{\delta,\lambda}h^\delta + ch\sum_{k=1}^{n-1}\mathbb{E}\left[\sum_{m\ge1} e^{-8\pi^2khUm^2}m^{2(1-\lambda)}\right] \\
&\le \mathbb{E}\left[\|De^{nhU\Delta}X_0\|_2^2\right] + c_{\delta,\lambda}h^\delta + ch\sum_{k=1}^{n-1}\mathbb{E}\left[\sum_{m\ge1} e^{-8\pi^2khUm^2}m^{2(1-\lambda)_+}\right].
\end{aligned}$$

Using the following Gaussian estimate

$$\begin{aligned}
\sum_{m\ge1} e^{-8\pi^2khUm^2}m^{2(1-\lambda)_+} &\le \int_0^\infty e^{-8\pi^2khUx^2}\left(x^{2(1-\lambda)_+}+1\right)dx \\
&\le c_\lambda\left((khU)^{-\frac{1}{2}} + (khU)^{-\frac{1}{2}-(1-\lambda)_+}\right),
\end{aligned} \tag{3.24}$$

we obtain

$$\begin{aligned}
\mathbb{E}\left[\|DX_n^h\|_2^2\right] &\le \mathbb{E}\left[\|De^{nhU\Delta}X_0\|_2^2\right] + c_{\delta,\lambda}h^\delta + c_\lambda h\sum_{k=1}^{n}\mathbb{E}\left[(khU)^{-\frac{1}{2}} + (khU)^{-\frac{1}{2}-(1-\lambda)_+}\right] \\
&\le \mathbb{E}\left[\|De^{nhU\Delta}X_0\|_2^2\right] + c_{\delta,\lambda}\left(h^\delta + (hn)^{\frac{1}{2}} + (hn)^{\frac{1}{2}-(1-\lambda)_+}\right).
\end{aligned}$$

Note that $\frac{1}{2} - (1-\lambda)_+ = \min(\lambda - \frac{1}{2}, \frac{1}{2})$.

For the final step, letting $(\widehat{X}_0^m)_{m\in\mathbb{N}_0}$ denote the (cosine) Fourier coefficients of X_0 (recalling that X_0 is assumed to be symmetric),

$$\begin{aligned}
\mathbb{E}\left[\|De^{nhU\Delta}X_0\|_2^2\right] &= \mathbb{E}\left[\sum_{m\in\mathbb{N}_0} 4\pi^2m^2e^{-8\pi^2nhm^2U}|\widehat{X}_0^m|^2\right] \\
&= 4\pi^2\mathbb{E}\sum_{m\in\mathbb{N}_0}\left(\int_0^1 m^2e^{-8\pi^2nhm^2s}ds\right)|\widehat{X}_0^m|^2 \le \frac{1}{nh}\mathbb{E}\sum_{m\in\mathbb{N}_0}|\widehat{X}_0^m|^2 = \frac{1}{nh}\mathbb{E}\left[\|X_0\|_2^2\right].
\end{aligned}$$

□

3.3 Tightness

Under the same assumption as before on X_0 (namely, X_0 takes values in $U^2(\mathbb{S})$ and $\mathbb{E}[\|X_0\|_2^{2p}] < \infty$ for any $p \ge 1$), we shall address the tightness properties of the scheme, see Proposition 3.7 for the main statement of this subsection. We warn the reader that the proof is technical. It is possible to skip it at first and come back to it later.

In order to proceed, we define the following linear interpolation $(\tilde{X}_t^h)_{t\ge0}$ of the scheme:

$$\tilde{X}_t^h := (\lceil t/h\rceil - t/h)X_{\lfloor t/h\rfloor}^h + (t/h - \lfloor t/h\rfloor)X_{\lceil t/h\rceil}^h. \tag{3.25}$$

Lemma 3.2 (applied with $T+1$ instead of T) gives us the following bound:

Corollary 3.5. *For an initial condition $X_0 \in U^2(\mathbb{S})$ with finite moments of any order, for a real $T > 0$ and for any real $p \geq 1$, we have*

$$\sup_{t \leq T} \mathbb{E}\left[\|\tilde{X}_t^h\|_2^{2p}\right] \leq C_{p,\lambda,T,\mathbb{E}[\|X_0\|_2^{2p}]}, \tag{3.26}$$

where the last parameter $\mathbb{E}[\|X_0\|_2^{2p}]$ in the constant is here to say that the constant depends on X_0 through the moment $\mathbb{E}[\|X_0\|_2^{2p}]$.

The next result is an intermediary lemma that we use below in order to establish Kolmogorov-Chentsov's criterion.

Lemma 3.6 (Sufficient condition for relative compactness)**.** *Let f and g be functions such that, $f, g : (0, \infty) \to (0, \infty)$, f and $-g$ are non-increasing, $f(\varepsilon) \to \infty$ and $g(\varepsilon) \to 0$ as $\varepsilon \to 0$.*

Suppose that $A \subset C([0,T], \mathcal{X})$, with $(\mathcal{X}, \|\cdot\|)$ being a normed vector space and T being a positive real, is such that, for some $0 < \alpha < 1$ and some threshold $\varepsilon_0 > 0$ and for all $\varepsilon \in (0, \varepsilon_0)$ and $x \in A$,

$$\|x_t - x_s\| \leq f(\varepsilon)(t-s)^\alpha \quad \text{for } 0 < \varepsilon < s < t \leq T, \tag{3.27}$$

$$\|x_s - x_0\| \leq g(\varepsilon) \quad \text{for } 0 < s < \varepsilon. \tag{3.28}$$

Then, A is equicontinuous.

Proof. Fix $\delta > 0$ and choose $(s,t) \in [0,T]^2$ such that $|t-s| \leq \delta$. Then, for $\varepsilon \in (0, \varepsilon_0)$ such that $\delta \geq f(\varepsilon)^{-\frac{1+\alpha}{\alpha}}$, we have

$$\|x_t - x_s\| \leq \delta^{-\frac{\alpha}{1+\alpha}} \delta^\alpha = \delta^{\frac{\alpha^2}{1+\alpha}} \quad \text{for } 0 < \varepsilon < s < t, \tag{3.29}$$

$$\|x_s - x_0\| \leq 2g(f^{-1}(\delta^{-\frac{\alpha}{1+\alpha}})) \quad \text{for } 0 < s < \varepsilon, \tag{3.30}$$

where $f^{-1}(a) := \sup\{x \in (0,\infty) : f(x) \geq a\}$, for $a \in (0, \infty)$. Since $f(x) \to \infty$ as $x \to 0$, $f^{-1}(a)$ is well-defined (in $(0, \infty]$) for any $a \in (0, \infty)$. Moreover, it is quite standard to check that $f^{-1}(a) \to 0$ as $a \to \infty$. We deduce that, in both (3.29) and (3.30), the right-hand side tends to 0 with δ, which provides a uniform modulus of continuity for any $(x_t)_{0 \leq t \leq T}$ in A. □

Proposition 3.7. *For any finite time horizon $T > 1$, the linear interpolation schemes $\{\tilde{X}^h\}_{h \in (0,1)} := \{(\tilde{X}_t^h)_{t \geq 0}\}_{h \in (0,1)}$ induce a tight family of probability measures on the space of continuous functions $\mathcal{C}([0,T], L^2_{\text{sym}}(\mathbb{S}))$. Moreover, for any $p \geq 1$, there exists a constant $C_{p,\lambda,T,\mathbb{E}[\|X_0\|_2^{2p}]}$, independent of h, such that*

$$\mathbb{E}\Big[\sup_{n: nh \leq T+h} \|X_n^h\|_2^{2p}\Big] \leq C_{p,\lambda,T,\mathbb{E}[\|X_0\|_2^{2p}]}.$$

Proof. The proof is to verify Kolmogorov-Chentsov's criterion, by first establishing equicontinuity via Lemma 3.6, which is in fact rather demanding, and finally confirming that, for

any $t \in [0,T]$, the random variables $\{X_t^h\}_{h\in(0,1)}$ induce a tight collection of probability measures on $L^2_{\mathrm{sym}}(\mathbb{S})$. Throughout the proof, we use the notation $N_0 := \lceil T/h \rceil$.

First step. Consider the quantity $\sup_{n\in\{0,\cdots,N_0\}} \|X_n^h - X_0\|_2$. This will be controlled by splitting the problem via the triangle inequality,

$$\sup_{n\in\{0,\cdots,N_0\}} \|X_n^h - X_0\|_2 \le \sup_{n\in\{0,\cdots,N_0\}} \left\{ \|X_n^h - e^{nh\Delta}X_0\|_2 + \|e^{nh\Delta}X_0 - X_0\|_2 \right\}.$$

To handle the first summand, one begins by use of the non-expansion property of the rearrangement (together with the fact that X_0 itself is non-increasing on the circle and that the periodic heat semigroup preserves symmetric non-increasing property of functions, see Lemma 2.10):

$$\begin{aligned}
\|X_n^h - e^{nh\Delta}X_0\|_2^2 &\le \left\| e^{h\Delta}X_{n-1}^h - e^{nh\Delta}X_0 + \int_{(n-1)h}^{nh} e^{(nh-s)\Delta} dW_s \right\|_2^2 \\
&\le \left\| X_{n-1}^h - e^{(n-1)h\Delta}X_0 \right\|_2^2 + \left\| \int_{(n-1)h}^{nh} e^{(nh-s)\Delta} dW_s \right\|_2^2 \\
&\quad + 2\left\langle e^{h\Delta}X_{n-1}^h - e^{nh\Delta}X_0, \int_{(n-1)h}^{nh} e^{(nh-s)\Delta} dW_s \right\rangle.
\end{aligned}$$

The second inequality above follows from the contractive property of the heat semigroup. By iteration,

$$\begin{aligned}
\|X_n^h - e^{nh\Delta}X_0\|_2^2 &\le \sum_{k=1}^{n} \left\| \int_{(k-1)h}^{kh} e^{(kh-s)\Delta} dW_s \right\|_2^2 \\
&\quad + 2\sum_{k=1}^{n} \left\langle e^{h\Delta}X_{k-1}^h - e^{kh\Delta}X_0, \int_{(k-1)h}^{kh} e^{(kh-s)\Delta} dW_s \right\rangle.
\end{aligned} \tag{3.31}$$

The idea to estimate the above is to view the right-hand side as the value at time n of a process. To be precise, one studies the regularity of two discrete processes, T^1 and T^2 (indexing by h is omitted), defined respectively as

$$\begin{aligned}
&\left(T_n^1 := \sum_{k=1}^{n} \left\| \int_{(k-1)h}^{kh} e^{(kh-s)\Delta} dW_s \right\|_2^2 \right)_{n\ge 0}, \\
&\left(T_n^2 := 2\sum_{k=1}^{n} \left\langle e^{h\Delta}X_{k-1}^h - e^{kh\Delta}X_0, \int_{(k-1)h}^{kh} e^{(kh-s)\Delta} dW_s \right\rangle \right)_{n\ge 0}.
\end{aligned} \tag{3.32}$$

For the regularity of $(T_n^1)_{n\ge 0}$, observe that for any exponent $p \ge 1$, by Lemma 3.1,

$$\mathbb{E}\left[\left\| \int_{(k-1)h}^{kh} e^{(kh-s)\Delta} dW_s \right\|_2^{2p} \right] \le c_{p,\lambda} h^p.$$

Consequently, by the generalised means inequality, we get, for $0 \le m < n$,

$$\mathbb{E}\left[\left(\sum_{k=m+1}^{n} \left\| \int_{(k-1)h}^{kh} e^{(kh-s)\Delta} dW_s \right\|_2^2 \right)^p \right] \le c_{p,\lambda} \big(h(n-m) \big)^p,$$

which one may rewrite as

$$\mathbb{E}\Big[\big|T_n^1 - T_m^1\big|^p\Big] \le c_{p,\lambda}\big(h(n-m)\big)^p.$$

The second process in (3.32), $(T_n^2)_{n\ge 0}$, defines a martingale,

$$\big(T_n^2\big)_{n\ge 1} := \left(2\sum_{k=1}^{n}\Big\langle e^{h\Delta}X_{k-1}^h - e^{kh\Delta}X_0, \int_{(k-1)h}^{kh} e^{(kh-s)\Delta}dW_s\Big\rangle\right)_{n\ge 1},$$

with the convention that $T_0^2 = 0$. Therefore, by the Burkholder-Davis-Gundy inequality,

$$\begin{aligned}\mathbb{E}\Big[\big|T_n^2 - T_m^2\big|^p\Big] &\le \mathbb{E}\left[\left|\sum_{k=m+1}^{n}\Big\langle e^{h\Delta}X_{k-1}^h - e^{kh\Delta}X_0, \int_{(k-1)h}^{kh} e^{(kh-s)\Delta}dW_s\Big\rangle\right|^p\right]\\ &\le \mathbb{E}\left[\left(\left[\dot{\sum_{k=m+1}}\Big\langle e^{h\Delta}X_{k-1}^h - e^{kh\Delta}X_0, \int_{(k-1)h}^{kh} e^{(kh-s)\Delta}dW_s\Big\rangle\right]_n\right)^{p/2}\right],\end{aligned}$$

where the notation $[\cdot]_n$ denotes the quadratic variation up the nth instant (note that here, this is from the $(m+1)^{st}$ instant). This may be estimated by writing:

$$\begin{aligned}&\left[\dot{\sum_{k=m+1}}\Big\langle e^{h\Delta}X_{k-1}^h - e^{kh\Delta}X_0, \int_{(k-1)h}^{kh} e^{(kh-s)\Delta}dW_s\Big\rangle\right]_n\\ &= \sum_{k=m+1}^{n}\left(\Big[\widehat{e^{h\Delta}X_{k-1}^h - e^{kh\Delta}X_0}^{\,0}\Big]^2 h\right.\\ &\qquad\left. + \sum_{\ell\in\mathbb{N}}\Big[\widehat{e^{h\Delta}X_{k-1}^h - e^{kh\Delta}X_0}^{\,\ell}\Big]^2\int_{(k-1)h}^{kh}\ell^{-2\lambda}e^{-8\pi^2(kh-s)\ell^2}ds\right)\\ &\le h\sum_{k=m+1}^{n}\sum_{\ell\in\mathbb{N}_0}\Big[\widehat{e^{h\Delta}X_{k-1}^h - e^{kh\Delta}X_0}^{\,\ell}\Big]^2 = h\sum_{k=m+1}^{n}\Big\|e^{h\Delta}X_{k-1}^h - e^{kh\Delta}X_0\Big\|_2^2.\end{aligned} \tag{3.33}$$

Applying the generalised means inequality and using, from Corollary 3.5, the fact that $\sup_{k=0,\cdots,N_0}\mathbb{E}[\|X_{kh}^h\|_2^p] \le C_{p,\lambda,T,\mathbb{E}[\|X_0\|_2^p]}$, one obtains

$$\begin{aligned}\mathbb{E}\Big[\big|T_n^2 - T_m^2\big|^p\Big] &\le \mathbb{E}\left[\left|\sum_{k=m+1}^{n}\Big\langle e^{h\Delta}X_{k-1}^h - e^{kh\Delta}X_0, \int_{(k-1)h}^{kh} e^{(kh-s)\Delta}dW_s\Big\rangle\right|^p\right]\\ &\le c_{p,\lambda,T,\mathbb{E}[\|X_0\|_2^p]}\big(h(n-m)\big)^{\frac{p}{2}}.\end{aligned}$$

Returning to (3.31), via application of the Kolmogorov-Chentsov continuity theorem [see Theorem 1.2.1 in [67]] (to the linear interpolation of the two processes in (3.32)) that, for $\alpha \in (0, (\frac{p}{2}-1)/2p)$,

$$\|X_n^h - e^{nh\Delta}X_0\|_2 \le \Xi^h(nh)^\alpha, \qquad n \in \{0,\cdots,N_0\},$$

almost surely for a (non-negative) random variable Ξ^h with a finite $L^{2p}(\mathbb{P})$-moment that satisfies $\mathbb{E}[(\Xi^h)^{2p}] \le c_{p,\lambda,T,\mathbb{E}[\|X_0\|_2^{2p}]}$. Consequently,

$$\|X_n^h - X_0\|_2 \le \Xi^h(nh)^\alpha + w(nh), \qquad n \in \{0,\cdots,N_0\},$$

where $w(x)$ is a random variable that depends on X_0, that tends almost surely to 0 with x and that is dominated by $2\|X_0\|_2$. Notice that this implies in particular that

$$\mathbb{E}\Big[\sup_{n:nh\le T+h}\|X_n^h\|_2^{2p}\Big]\le c_{p,\lambda,T,\mathbb{E}[\|X_0\|_2^{2p}]}.$$

In turn, for $t\in[0,T]$ and for $n:=\lfloor t/h\rfloor$,

$$\begin{aligned}\|\tilde X_t^h-X_0\|_2&\le\Big(n+1-\frac{t}{h}\Big)\|X_n^h-X_0\|_2+\Big(\frac{t}{h}-n\Big)\|X_{n+1}^h-X_0\|_2\\&\le\Xi^h\Big[\Big(n+1-\frac{t}{h}\Big)(nh)^\alpha+\Big(\frac{t}{h}-n\Big)\big((n+1)h\big)^\alpha\Big]\\&\quad+\Big(n+1-\frac{t}{h}\Big)w(nh)+\Big(\frac{t}{h}-n\Big)w\big((n+1)h\big).\end{aligned}\tag{3.34}$$

Assuming without any loss of generality that w is non-decreasing (changing $w(x)$ into $\sup_{y\in[0,x]}w(y)$), we get

$$\|\tilde X_t^h-X_0\|_2\le\Xi^h\big((n+1)h\big)^\alpha+w\big((n+1)h\big).$$

If $n=\lfloor t/h\rfloor\ge1$, then $nh\le t<(n+1)h\le 2nh\le 2t$ and thus

$$\|\tilde X_t^h-X_0\|_2\le\Xi^h(2t)^\alpha+w(2t).\tag{3.35}$$

If $n=\lfloor t/h\rfloor=0$, then $t<h$ and (3.34) yields

$$\|\tilde X_t^h-X_0\|_2\le\frac{t}{h}\Big(\Xi^h h^\alpha+w(h)\Big).\tag{3.36}$$

We distinguish the following two cases in (3.36). If $t\le h^2$, i.e. $\sqrt{t}\le h$, then

$$\|\tilde X_t^h-X_0\|_2\le\sqrt{t}\frac{\sqrt{t}}{h}\Big(\Xi^h+w(1)\Big)\le\sqrt{t}\Big(\Xi^h+w(1)\Big).\tag{3.37}$$

If $h^2\le t\le h$, then $h\le\sqrt{t}$ and (3.36) yields

$$\|\tilde X_t^h-X_0\|_2\le\Big[\Xi^h\sqrt{t}^\alpha+w\big(\sqrt{t}\big)\Big].\tag{3.38}$$

We now collect (3.35), (3.37) and (3.38). Recalling that the random function w is dominated by $2\|X_0\|_2$, we can modify the random variable Ξ^h (in such a way that its $2p$ moment remains less than some constant $c_{p,\lambda,T,\mathbb{E}[\|X_0\|_2^{2p}]}$) so that

$$\|\tilde X_t^h-X_0\|_2\le\Xi^h t^{\alpha/2}+w\big(2(\sqrt{t}\vee t)\big),\qquad t\in[0,T].$$

Modifying again the choice of w, this may be simplified into

$$\|\tilde X_t^h-X_0\|_2\le\Xi^h t^{\alpha/2}+w(t),\qquad t\in[0,T].\tag{3.39}$$

The above bound describes the small time behaviour of the process. It remains to address the situation away from initialisation.

Second step. From here, we take $p \geq 2$. This phase of the proof establishes that for fixed $N \leq N_0$ and $N \leq m < n$,

$$\begin{aligned}\mathbb{E}\big[\|X_n^h - X_m^h\|_2^{2p}|\mathcal{F}_{Nh}\big] &\leq c_{p,\lambda}(nh-mh)^p + c_p\mathbb{E}\big[\|X_m^h - e^{(n-m)h\Delta}X_m^h\|_2^{2p}|\mathcal{F}_{Nh}\big] \\ &\leq c_{p,\lambda}(nh-mh)^p + c_p(nh-mh)^p\mathbb{E}\big[\|DX_m^h\|_2^{2p}|\mathcal{F}_{Nh}\big].\end{aligned} \tag{3.40}$$

To begin, apply the triangle and generalised means inequalities:

$$\begin{aligned}&\mathbb{E}\Big[\|X_n^h - X_m^h\|_2^{2p}\,\Big|\mathcal{F}_{Nh}\Big] \\ &\quad\leq 2^{2p-1}\mathbb{E}\Big[\Big\|X_n^h - e^{(n-m)h\Delta}X_m^h\Big\|_2^{2p} + \Big\|X_m^h - e^{(n-m)h\Delta}X_m^h\Big\|_2^{2p}\,\Big|\mathcal{F}_{Nh}\Big].\end{aligned} \tag{3.41}$$

The second summand in the above right hand side is simpler to handle in this step and follows from a readily-established heat semigroup estimate. For $u \in L^2(\mathbb{S})$ with a generalised derivative Du in $L^2(\mathbb{S})$,

$$\begin{aligned}\|e^{(t-s)\Delta}u - u\|_2^2 &= \int_0^{t-s}\Big\langle \partial_r e^{r\Delta}u, e^{(t-s)\Delta}u - u\Big\rangle dr \\ &= -\int_0^{t-s}\Big\langle De^{r\Delta}u, De^{(t-s)\Delta}u - Du\Big\rangle dr \leq 2(t-s)\|Du\|_2^2,\end{aligned} \tag{3.42}$$

which is enough for the desired estimation. For the first summand in (3.41), one proceeds via the following sequence of inequalities. Starting with the non-expansion property of the rearrangement (recall that the heat semigroup preserves the symmetric non-increasing property),

$$\Big\|X_n^h - e^{(n-m)h\Delta}X_m^h\Big\|_2^{2p} \leq \Big\|e^{h\Delta}X_{n-1}^h + \int_0^h e^{(h-s)\Delta}dW_s^n - e^{(n-m)h\Delta}X_m^h\Big\|_2^{2p}, \tag{3.43}$$

which may be estimated by means of (3.5), by considering the process

$$\hat{X}_s^{h,n-1} := e^{s\Delta}\Big[X_{n-1}^h - e^{(n-1-m)h\Delta}X_m^h\Big] + \int_0^s e^{(s-r)\Delta}dW_r^n, \quad s \in [0,h].$$

Following the same sequence of inequalities as in (3.16), (3.17), (3.18) and (3.19), we obtain[1]

$$\begin{aligned}\mathbb{E}\Big[\Big\|X_n^h - e^{(n-m)h\Delta}X_m^h\Big\|_2^{2p}\,\Big|\mathcal{F}_{Nh}\Big] &\leq \mathbb{E}\Big[\Big\|X_{n-1}^h - e^{(n-1-m)h\Delta}X_m^h\Big\|_2^{2p}\,\Big|\mathcal{F}_{Nh}\Big] \\ &\quad + c_{p,\lambda}h\Big(\mathbb{E}\Big[\Big\|X_{n-1}^h - e^{(n-1-m)h\Delta}X_m^h\Big\|_2^{2p-2}\,\Big|\mathcal{F}_{Nh}\Big] + h^{p-1}\Big),\end{aligned}$$

[1] Although the reader may find the computations reminiscent of (3.31), the objective is in fact different. In (3.31), the goal is to apply Kolmogorov-Chentsov's theorem to the process $(\|X_n^h - e^{nh\Delta}X_0\|_2^2)_{0\leq\lfloor n/h\rfloor\leq T}$. The purpose here is obviously not the same.

which gives, by iteration,

$$\mathbb{E}\Big[\Big\|X_n^h - e^{(n-m)h\Delta}X_m^h\Big\|_2^{2p}\Big|\mathcal{F}_{Nh}\Big] \le c_{p,\lambda}h\sum_{k=m+1}^{n-1}\Big(\mathbb{E}\Big[\Big\|X_k^h - e^{(k-m)h\Delta}X_k^h\Big\|_2^{2p-2}\Big|\mathcal{F}_{Nh}\Big] + h^{p-1}\Big).$$

We proceed by induction on p, assuming for a while that p is an integer (greater than or equal to 1). When $p = 1$, the above inequality yields

$$\mathbb{E}\Big[\Big\|X_n^h - e^{(n-m)h\Delta}X_m^h\Big\|_2^{2}\Big|\mathcal{F}_{Nh}\Big] \le c_{1,\lambda}h(m-n).$$

Then, by induction, we get, for any $p \in \mathbb{N}$,

$$\mathbb{E}\Big[\Big\|X_n^h - e^{(n-m)h\Delta}X_m^h\Big\|_2^{2p}\Big|\mathcal{F}_{Nh}\Big] \le c_{p,\lambda}\big(h(m-n)\big)^p. \tag{3.44}$$

When p is a real greater than or equal to 1, we may apply the above inequality to $\lceil p\rceil$ and then get the same conclusion as above by Hölder's inequality applied with exponent $\lceil p\rceil/p$. By combining (3.41), (3.42) and (3.44), we obtain (3.40).

Third step. Having established that

$$\mathbb{E}\left[\|X_n^h - X_m^h\|_2^{2p}|\mathcal{F}_{Nh}\right] \le c_{p,\lambda}(nh-mh)^p + c_p(nh-mh)^p\mathbb{E}\left[\|DX_m^h\|_2^{2p}|\mathcal{F}_{Nh}\right], \tag{3.45}$$

one needs to control the term, $\mathbb{E}\left[\|DX_m^h\|_2^{2p}|\mathcal{F}_{Nh}\right]$. As in the proof of Lemma 3.4, for ease of notation, an independent uniform random variable $\tilde{U}$ is introduced (independence being understood as independence with respect to the filtration $\mathbb{F}$). For the sake of clarity, we feel better to construct $\tilde{U}$ on a separate probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and then to elevate the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to a product with $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. In this way, one may denote by $\tilde{\mathbb{E}}$ the integral with respect to the $\tilde{\mathbb{P}}$ component of the product measure. Then, by application of the Pólya–Szegő inequality (Lemma 2.8) followed by use of the contractive property of the heat semigroup,

$$\begin{aligned}\|DX_m^h\|_2^2 &\le \Big\|D\Big(e^{h\Delta}X_{m-1}^h + \int_{(m-1)h}^{mh} e^{(mh-s)\Delta}dW_s\Big)\Big\|_2^2\\ &\le \tilde{\mathbb{E}}\Big[\|De^{h\tilde{U}\Delta}X_{m-1}^h\|_2^2\Big] + \Big\|\int_{(m-1)h}^{mh} e^{(mh-s)\Delta}dw_s\Big\|_2^2\\ &\quad + 2\Big\langle D\Big(e^{h\Delta}X_{m-1}^h\Big), \int_{(m-1)h}^{mh} e^{(mh-s)\Delta}dw_s\Big\rangle,\end{aligned}$$

where we used the same notation as in (3.8). In turn, by inequality (3.20),

$$\begin{aligned}\tilde{\mathbb{E}}\Big[\|De^{h\tilde{U}\Delta}X_{m-1}^h\|_2^2\Big] \le \tilde{\mathbb{E}}\Bigg[&\|De^{h(1+\tilde{U})\Delta}X_{m-2}^h\|_2^2 + \Big\|e^{h\tilde{U}\Delta}\int_{(m-2)h}^{(m-1)h} e^{((m-1)h-s)\Delta}dw_s\Big\|_2^2\\ &+ 2\Big\langle D\Big(e^{h(1+\tilde{U})\Delta}X_{m-2}^h\Big), \int_{(m-2)h}^{(m-1)h} e^{((m-1)h+h\tilde{U}-s)\Delta}dw_s\Big\rangle\Bigg],\end{aligned}$$

and then,

$$\begin{aligned}\|DX_m^h\|_2^2 \le \tilde{\mathbb{E}}\Big[&\|De^{h(1+\tilde{U})\Delta}X_{m-2}^h\|_2^2\\ &+\Big\|e^{h\tilde{U}\Delta}\int_{(m-2)h}^{(m-1)h}e^{((m-1)h-s)\Delta}dw_s\Big\|_2^2+\Big\|\int_{(m-1)h}^{mh}e^{(mh-s)\Delta}dw_s\Big\|_2^2\\ &+2\Big\langle D\Big(e^{h(1+\tilde{U})\Delta}X_{m-2}^h\Big),\int_{(m-2)h}^{(m-1)h}e^{((m-1)h+h\tilde{U}-s)\Delta}dw_s\Big\rangle\\ &+2\Big\langle D\Big(e^{h\Delta}X_{m-1}^h\Big),\int_{(m-1)h}^{mh}e^{(mh-s)\Delta}dw_s\Big\rangle\Big].\end{aligned}$$

Noticing that

$$\tilde{\mathbb{E}}\Big[\|De^{h(1+\tilde{U})\Delta}X_{m-2}^h\|_2^2\Big]\le\tilde{\mathbb{E}}\Big[\|De^{2h\tilde{U}\Delta}X_{m-2}^h\|_2^2\Big],$$

we can apply once again (3.20), but with h replaced by $2h$. And then, by iteration, we get for any $N<m\le N_0$,

$$\begin{aligned}\|DX_m^h\|_2^2 &\le \tilde{\mathbb{E}}\Big[\|De^{h(1+(m-N-1)\tilde{U})\Delta}X_N^h\|_2^2\Big]\\ &+\tilde{\mathbb{E}}\Big[\sum_{k=1}^{m-N}\Big\|e^{h(k-1)\tilde{U}\Delta}\int_{(m-k)h}^{(m-k+1)h}e^{((m-k+1)h-s)\Delta}dw_s\Big\|_2^2\Big]\\ &+2\sum_{k=1}^{m-N}\tilde{\mathbb{E}}\Big[\Big\langle D\Big(e^{h(1+(k-1)\tilde{U})\Delta}X_{m-k}^h\Big),\int_{(m-k)h}^{(m-k+1)h}e^{((m-k+1)h+h(k-1)\tilde{U}-s)\Delta}dw_s\Big\rangle\Big]\\ &=:\tilde{\mathbb{E}}\Big[\|De^{h(1+(m-N-1)\tilde{U})\Delta}X_N^h\|_2^2\Big]+\tilde{\mathbb{E}}\Big[\sum_{k=1}^{m-N}\Big(R_k^1+2R_k^2\Big)\Big].\end{aligned}\tag{3.46}$$

Next, calculate the pth moment of both $R^1:=\tilde{\mathbb{E}}[\sum_{k=1}^{m-N}R_k^1]$ and $R^2:=\tilde{\mathbb{E}}[\sum_{k=1}^{m-N}R_k^2]$ (conditional on $\mathcal{F}_{Nh}$). For the case of R^1, we first notice that, for $k\ge 2$,

$$\tilde{\mathbb{E}}\big[R_k^1\big]=4\pi^2\tilde{\mathbb{E}}\sum_{\ell\in\mathbb{N}}\Big(e^{-4\pi^2h(k-1)\tilde{U}\ell^2}\int_{(m-k)h}^{(m-k+1)h}\ell^{(1-\lambda)}e^{-4\pi^2((m-k+1)h-s)\ell^2}dB_s^\ell\Big)^2,$$

and then, by Jensen inequality (in the form $(\tilde{\mathbb{E}}\sum_{\ell\in\mathbb{N}}x_\ell^2y_\ell^2)^p\le(\tilde{\mathbb{E}}\sum_{\ell\in\mathbb{N}}x_\ell^2)^{p-1}\tilde{\mathbb{E}}\sum_{\ell\in\mathbb{N}}x_\ell^2y_\ell^{2p}$),

$$\begin{aligned}\Big(\tilde{\mathbb{E}}\big[R_k^1\big]\Big)^p\le c\Big(&\tilde{\mathbb{E}}\sum_{\ell\in\mathbb{N}}\ell^{2(1-\lambda)}e^{-8\pi^2(k-1)h\tilde{U}\ell^2}\Big)^{p-1}\\ &\times\tilde{\mathbb{E}}\Big[\sum_{\ell\in\mathbb{N}}\ell^{2(1-\lambda)}e^{-8\pi^2(k-1)h\tilde{U}\ell^2}\Big(\int_{(m-k)h}^{(m-k+1)h}e^{-4\pi^2((m-k+1)h-s)\ell^2}dB_s^\ell\Big)^{2p}\Big].\end{aligned}$$

Therefore,

$$
\begin{aligned}
&\mathbb{E}\Big[\Big(\tilde{\mathbb{E}}\big[R_k^1\big]\Big)^p\Big|\mathcal{F}_{Nh}\Big] \\
&\le c\Bigg(\tilde{\mathbb{E}}\sum_{\ell\in\mathbb{N}}\ell^{2(1-\lambda)}e^{-8\pi^2(k-1)h\tilde{U}\ell^2}\Bigg)^{p-1} \\
&\quad\times\tilde{\mathbb{E}}\Bigg[\sum_{\ell\in\mathbb{N}}\ell^{2(1-\lambda)}e^{-8\pi^2(k-1)h\tilde{U}\ell^2}\mathbb{E}\Bigg\{\Bigg(\int_{(m-k)h}^{(m-k+1)h}e^{-4\pi^2((m-k+1)h-s)\ell^2}dB_s^\ell\Bigg)^{2p}\Bigg|\mathcal{F}_{Nh}\Bigg\}\Bigg] \\
&\le ch^p\Bigg(\tilde{\mathbb{E}}\sum_{\ell\in\mathbb{N}}\ell^{2(1-\lambda)}e^{-8\pi^2(k-1)h\tilde{U}\ell^2}\Bigg)^p.
\end{aligned}
$$

We recall from (3.24) that, for $2\le k\le m-N$,

$$
\begin{aligned}
\Bigg(\tilde{\mathbb{E}}\sum_{\ell\in\mathbb{N}}\ell^{2(1-\lambda)}e^{-8\pi^2(k-1)h\tilde{U}\ell^2}\Bigg)^p &\le c_{p,\lambda}\Bigg(\tilde{\mathbb{E}}\Big(\big((k-1)h\tilde{U}\big)^{-\frac{1}{2}}+\big((k-1)h\tilde{U}\big)^{-\frac{1}{2}-(1-\lambda)_+}\Big)\Bigg)^p \\
&\le c_{p,\lambda}\Big\{1+\big((k-1)h\big)^{-\frac{1}{2}-(1-\lambda)_+}\Big\}^p,
\end{aligned}
$$

from which we deduce that, for $2\le k\le m-N$,

$$
\mathbb{E}\Big[\Big(\tilde{\mathbb{E}}\big[R_k^1\big]\Big)^p\Big|\mathcal{F}_{Nh}\Big]\le c_{p,\lambda}h^p\Big\{1+\big((k-1)h\big)^{-\frac{1}{2}-(1-\lambda)_+}\Big\}^p. \tag{3.47}
$$

When $k=1$ (and $m-N\ge 1$),

$$
\begin{aligned}
\mathbb{E}\Big[\Big(\tilde{\mathbb{E}}\big[R_1^1\big]\Big)^p\Big|\mathcal{F}_{Nh}\Big] &= 4\pi^2\mathbb{E}\Bigg[\Bigg\{\sum_{\ell\in\mathbb{N}}\Bigg(\int_{(m-1)h}^{mh}\ell^{1-\lambda}e^{-4\pi^2(mh-s)\ell^2}dB_s^\ell\Bigg)^2\Bigg\}^p\Bigg|\mathcal{F}_{Nh}\Bigg] \\
&= 4\pi^2\mathbb{E}\Bigg[\Bigg\{\sum_{\ell\in\mathbb{N}}\ell^{-(\lambda+\frac{1}{2})}\Bigg(\int_{(m-1)h}^{mh}\ell^{\frac{5}{4}-\frac{\lambda}{2}}e^{-4\pi^2(mh-s)\ell^2}dB_s^\ell\Bigg)^2\Bigg\}^p\Bigg|\mathcal{F}_{Nh}\Bigg].
\end{aligned}
$$

By Jensen inequality,

$$
\begin{aligned}
\mathbb{E}\Big[\Big(\tilde{\mathbb{E}}\big[R_1^1\big]\Big)^p\Big|\mathcal{F}_{Nh}\Big] &\le c_{p,\lambda}\sum_{\ell\in\mathbb{N}}\ell^{-(\lambda+\frac{1}{2})}\ell^{(\frac{5}{2}-\lambda)p}\mathbb{E}\Bigg[\Bigg(\int_{(m-1)h}^{mh}e^{-4\pi^2(mh-s)\ell^2}dB_s^\ell\Bigg)^{2p}\Bigg|\mathcal{F}_{Nh}\Bigg] \\
&\le c_{p,\lambda}\sum_{\ell\in\mathbb{N}}\ell^{-(\lambda+\frac{1}{2})}\ell^{(\frac{5}{2}-\lambda)p}\Bigg(\int_0^h e^{-8\pi^2 s\ell^2}ds\Bigg)^p \\
&\le c_{p,\lambda}\sum_{\ell\in\mathbb{N}}\ell^{-(\lambda+\frac{1}{2})}\ell^{(\frac{1}{2}-\lambda)p}\big(1-\exp(-8\pi^2h\ell^2)\big)^p \\
&\le c_{p,\lambda}h^{\min(\lambda-\frac{1}{2},1)\frac{p}{2}},
\end{aligned} \tag{3.48}
$$

where we used the inequality $1-\exp(-x)\le c_\lambda x^{\min(\lambda-\frac{1}{2},1)\frac{1}{2}}$.Therefore, by defining $b_{k,h}:=$

$1+((k-1)h)^{-\frac{1}{2}-(1-\lambda)_+}$, we deduce that, from (3.48) and (3.47),

$$\begin{aligned}
&\mathbb{E}\big[(R^1)^p\big|\mathcal{F}_{Nh}\big]\\
&\le c_p\mathbb{E}\Big[\Big(\tilde{\mathbb{E}}\big[R^1_1\big]\Big)^p\big|\mathcal{F}_{Nh}\Big]+c_p\mathbb{E}\Big[\Big(\tilde{\mathbb{E}}\Big[\sum_{k=2}^{m-N}R^1_k\Big]\Big)^p\big|\mathcal{F}_{Nh}\Big]\\
&\le c_{p,\lambda}h^{\min(\lambda-\frac{1}{2},1)\frac{p}{2}}+c_p\Big(\sum_{k=2}^{m-N}b_{k,h}\Big)^{p-1}\Big(\sum_{k=2}^{m-N}b_{k,h}b_{k,h}^{-p}\mathbb{E}\Big[\Big(\tilde{\mathbb{E}}\big[R^1_k\big]\Big)^p\big|\mathcal{F}_{Nh}\Big]\Big)\\
&\le c_{p,\lambda}h^{\min(\lambda-\frac{1}{2},1)\frac{p}{2}}+c_{p,\lambda}h^p\Big(\sum_{k=2}^{m-N}b_{k,h}\Big)^p.
\end{aligned}$$

Inserting the value of $b_{k,h}$, we obtain

$$\begin{aligned}
&\mathbb{E}\big[(R^1)^p\big|\mathcal{F}_{Nh}\big]\\
&\le c_{p,\lambda}h^{\min(\lambda-\frac{1}{2},1)\frac{p}{2}}+c_{p,\lambda}h^p\Big(m-N+\sum_{k\ge1}^{m-N}(kh)^{-\frac{1}{2}-(1-\lambda)_+}\Big)^p\\
&\le c_{p,\lambda}h^{\min(\lambda-\frac{1}{2},1)\frac{p}{2}}+c_{p,\lambda}h^p\Big(m-N+h^{-1}\int_0^{h(m-N)}x^{-\frac{1}{2}-(1-\lambda)_+}dx\Big)^p\\
&\le c_{p,\lambda}h^{\min(\lambda-\frac{1}{2},1)\frac{p}{2}}+c_{p,\lambda,T}\big(h(m-N)\big)^{p(\frac{1}{2}-(1-\lambda)_+)}.
\end{aligned}\tag{3.49}$$

Recalling that $\frac{1}{2}-(1-\lambda)_+=\min(\lambda-\frac{1}{2},\frac{1}{2})$ and using the bound $hm\le T+1$, we eventually get that

$$\mathbb{E}\big[(R^1)^p\big|\mathcal{F}_{Nh}\big]\le c_{p,\lambda,T}\big(h(m-N)\big)^{\min(\lambda-\frac{1}{2},\frac{1}{2})\frac{p}{2}}.\tag{3.50}$$

This gives a bound for R^1 in (3.46).

For the treatment of R^2 (also in (3.46)), using the symmetry of the Laplace operator and Burkholder-Davis-Gundy inequality (see Theorem 4.36 in [24]), one may write

$$\begin{aligned}
&\mathbb{E}\Big[\big(R^2\big)^p\big|\mathcal{F}_{Nh}\Big]\\
&=\mathbb{E}\Bigg[\Bigg(\sum_{k=1}^{m-N}\tilde{\mathbb{E}}\Big[\Big\langle D\Big(e^{h(1+(k-1)\tilde{U})\Delta}X^h_{m-k}\Big),\int_{(m-k)h}^{(m-k+1)h}e^{((m-k+1)h+h(k-1)\tilde{U}-s)\Delta}dw_s\Big\rangle\Big]\Bigg)^p\Bigg|\mathcal{F}_{Nh}\Bigg]\\
&=\mathbb{E}\Bigg[\Bigg(\sum_{k=1}^{m-N}\Big[\Big\langle DX^h_{m-k},\tilde{\mathbb{E}}\int_{(m-k)h}^{(m-k+1)h}e^{((m-k+1)h+h(1+2(k-1)\tilde{U})-s)\Delta}dw_s\Big\rangle\Big]\Bigg)^p\Bigg|\mathcal{F}_{Nh}\Bigg]\\
&\le c_p\mathbb{E}\Bigg[\Bigg(\sum_{k=1}^{m-N}\Big[\big\|DX^h_{m-k}\big\|_2^2\Big[\int_{(m-k)h}^{\cdot}\tilde{\mathbb{E}}\Big(e^{((m-k+1)h+h(1+2(k-1)\tilde{U})-s)\Delta}\Big)dw_s\Big]_{(m-k+1)h}\Big]\Bigg)^{p/2}\Bigg|\mathcal{F}_{Nh}\Bigg]\\
&\le c_p\mathbb{E}\Bigg[\Bigg(\sum_{k=1}^{m-N}\Big\{\big\|DX^h_{m-k}\big\|_2^2\mathbb{E}\Big(\tilde{\mathbb{E}}\big[R^1_k\big]\Big)\Big\}\Bigg)^{p/2}\Bigg|\mathcal{F}_{Nh}\Bigg],
\end{aligned}\tag{3.51}$$

where we used stochastic Fubini's theorem to pass the expectation symbol $\tilde{\mathbb{E}}$ in the fourth line. We split (3.51) as follows:

$$\begin{aligned}
&\mathbb{E}\Big[\big(R^2\big)^p|\mathcal{F}_{Nh}\Big] \\
&\le c_p\mathbb{E}\Big[\big\|DX^h_{m-1}\big\|_2^p|\mathcal{F}_{Nh}\Big]\mathbb{E}\Big(\tilde{\mathbb{E}}\big[R^1_1\big]\Big)^{p/2} + c_p\mathbb{E}\Bigg[\Bigg(\sum_{k=2}^{m-N}\Big\{\big\|DX^h_{m-k}\big\|_2^2\mathbb{E}\Big(\tilde{\mathbb{E}}\big[R^1_k\big]\Big)\Big\}\Bigg)^{p/2}\Bigg|\mathcal{F}_{Nh}\Bigg].
\end{aligned}$$

Inserting (3.47) and (3.48) and recalling from (3.49) that $h\sum_{k=2}^{m-N} b_{k,h} \le c_{\lambda,T}$

$$\begin{aligned}
\mathbb{E}\Big[\big(R^2\big)^p|\mathcal{F}_{Nh}\Big] &\le c_{p,\lambda}\mathbb{E}\Big[\big\|DX^h_{m-1}\big\|_2^p|\mathcal{F}_{Nh}\Big]h^{\min(\lambda-\frac{1}{2},1)\frac{p}{4}} \\
&\quad + c_{p,\lambda}\mathbb{E}\Bigg[\Bigg(h\sum_{k=2}^{m-N}\Big\{\big\|DX^h_{m-k}\big\|_2^2\Big(1+\big((k-1)h\big)^{-\frac{1}{2}-(1-\lambda)_+}\Big)\Big\}\Bigg)^{p/2}\Bigg|\mathcal{F}_{Nh}\Bigg] \\
&\le c_{p,\lambda}\mathbb{E}\Big[\big\|DX^h_{m-1}\big\|_2^p|\mathcal{F}_{Nh}\Big]h^{\min(\lambda-\frac{1}{2},1)\frac{p}{4}} \\
&\quad + c_{p,\lambda}h\sum_{k=2}^{m-N}\Big(1+\big((k-1)h\big)^{-\frac{1}{2}-(1-\lambda)_+}\Big)\mathbb{E}\Big[\big\|DX^h_{m-k}\big\|_2^p|\mathcal{F}_{Nh}\Big].
\end{aligned}$$

Collect the above bounds for R^2 and the bound (3.50) for R^1 and plug them into (3.46). Assuming that $(m-N)h \le \varepsilon$ for some $\varepsilon \in [h,1)$ (which is possible since $h<1$), recalling that $\frac{1}{2}-(1-\lambda)_+ = \min(\frac{1}{2},\lambda-\frac{1}{2})$ and repeating (3.49), one has

$$\begin{aligned}
&\sup_{m\ge N:h(m-N)\le\varepsilon}\mathbb{E}\Big[\big\|DX^h_m\big\|_2^{2p}|\mathcal{F}_{Nh}\Big] \\
&\le c_{p,\lambda,T}\Bigg(1+\big\|DX^h_N\big\|_2^{2p}+\varepsilon^{\min(\frac{1}{2},\lambda-\frac{1}{2})\frac{1}{2}}\sup_{m\ge N:h(m-N)\le\varepsilon}\mathbb{E}\Big[\big\|DX^h_m\big\|_2^{2p}|\mathcal{F}_{Nh}\Big]\Bigg),
\end{aligned}$$

from which we deduce that, for $\varepsilon \le \varepsilon_{p,\lambda,T}$ small enough (the threshold $\varepsilon_{p,\lambda,T}$ being strictly positive and only depending on p, λ and T),

$$\sup_{m\ge N:h(m-N)\le\varepsilon}\mathbb{E}\Big[\big\|DX^h_m\big\|_2^{2p}|\mathcal{F}_{Nh}\Big] \le c_{p,\lambda,T}\Big(1+\big\|DX^h_N\big\|_2^{2p}\Big),$$

observing from a straightforward induction that $\mathbb{E}\big[\big\|DX^h_m\big\|_2^{2p}|\mathcal{F}_{Nh}\big]$ is finite for $m>N$. (Notice that, when $h \ge \varepsilon_{p,\lambda,T}$, the above is obviously true since the set of indices in the left-hand side reduces to the singleton $\{N\}$.)

For ε being fixed, we can find an integer I_ε, depending only on ε and T, together with integers $N_0 = N < N_1 < \cdots < N_{I_\varepsilon}$, such that $hN_{I_\varepsilon} \ge T+1$ and $h(N_k - N_{k-1}) \le \varepsilon$ for $k = 1,\cdots,I_\varepsilon$. For each $k = 1,\cdots,I_\varepsilon$, we get

$$\sup_{N_{k-1}\le m\le N_k}\mathbb{E}\Big[\big\|DX^h_m\big\|_2^{2p}|\mathcal{F}_{N_{k-1}h}\Big] \le c_{p,\lambda,T}\Big(1+\big\|DX^h_{N_{k-1}}\big\|_2^{2p}\Big).$$

By induction, we get

$$\sup_{m\geq N: hm\leq T+1} \mathbb{E}\Big[\big\|DX_m^h\big\|_2^{2p} \,|\, \mathcal{F}_{Nh}\Big] \leq c_{p,\lambda,T}\Big(1+\big\|DX_N^h\big\|_2^{2p}\Big).$$

Finally, back to (3.45), for $m,n\geq N$ such that hm and hn are less than T,

$$\begin{aligned}\mathbb{E}\big[\big\|X_n^h - X_m^h\big\|_2^{2p}|\mathcal{F}_{Nh}\big] &\leq c_{p,\lambda}(nh-mh)^p + c_p(nh-mh)^p\mathbb{E}\big[\big\|DX_m^h\big\|_2^{2p}|\mathcal{F}_{Nh}\big]\\ &\leq c_{p,\lambda}(nh-mh)^p + c_{p,\lambda,T}(nh-mh)^p\Big(1+\big\|DX_N^h\big\|_2^{2p}\Big).\end{aligned}$$

Fourth Step. Recall that $\tilde{X}^h$ denotes the linear interpolation of the schemes X^h. Deduce from the previous step that, for $s,t\geq Nh$ (throughout s and t are always less than a fixed T), for a fixed $N\geq 1$,

$$\mathbb{E}\Big[\big\|\tilde{X}_t^h - \tilde{X}_s^h\big\|_2^{2p}|\mathcal{F}_{Nh}\Big] \leq c_{p,\lambda,T}(t-s)^p\Big(1+\|DX_N^h\|^{2p}\Big).$$

Therefore, one may apply Kolmogorov-Chentsov's theorem under the conditional probability measure $\mathbb{P}(\cdot|\mathcal{F}_{Nh})$. It says that

$$\mathbb{E}\Bigg[\sup_{Nh\leq s<t\leq T}\left(\frac{\|\tilde{X}_t^h-\tilde{X}_s^h\|_2}{|t-s|^\alpha}\right)^{2p}\Bigg|\mathcal{F}_{Nh}\Bigg] \leq c_{p,\lambda,T}\Big(1+\|DX_N^h\|^{2p}\Big),$$

for $\alpha\in(0,(p-1)/2p)$.

By Jensen inequality, one obtains (choosing for instance $p=2$)

$$\mathbb{E}\Bigg[\sup_{Nh\leq s<t\leq T}\left(\frac{\|\tilde{X}_t^h-\tilde{X}_s^h\|_2}{|t-s|^\alpha}\right)^{2}\Bigg] \leq c_{2,\lambda,T}\mathbb{E}\Big(1+\|DX_N^h\|^{2}\Big),$$

for $\alpha=1/4$.

Fix $\varepsilon>0$ (which is distinct from ε used in the previous step). For $h<\varepsilon$, there is an integer N such that $Nh\leq\varepsilon\leq 2Nh$. Recalling Lemma 3.4, for $\delta\in[0,\lambda-\frac{1}{2})$, $\mathbb{E}[\|DX_N^h\|_2^2]\leq\frac{1}{Nh}\mathbb{E}[\|X_0\|_2^2]+c_{\delta,\lambda,T}(h^\delta+(Nh)^{\min(\lambda-\frac{1}{2},\frac{1}{2})})$, one has

$$\mathbb{E}\Bigg[\sup_{\varepsilon\leq s<t\leq T}\left(\frac{\|\tilde{X}_t^h-\tilde{X}_s^h\|_2}{|t-s|^\alpha}\right)^{2}\Bigg] \leq c_{\lambda,T,\mathbb{E}[\|X_0\|_2^2]}\varepsilon^{-1}. \tag{3.52}$$

If $\varepsilon\leq h$, then, for $s\in(\varepsilon,h)$ and $t>h$,

$$\begin{aligned}\|\tilde{X}_t^h-\tilde{X}_s^h\|_2 &\leq \|\tilde{X}_t^h-\tilde{X}_h^h\|_2+\|\tilde{X}_s^h-\tilde{X}_h^h\|_2\\ &\leq \|\tilde{X}_t^h-\tilde{X}_h^h\|_2+(h-s)\|X_h^h-X_0\|_2 \leq \frac{\tilde{\Xi}^h}{\varepsilon}|t-s|^\alpha,\end{aligned}$$

with $\mathbb{E}[(\tilde{\Xi}^h)^2]\leq C$ (with C independent of h). The same holds true if $t<h$. Therefore, (3.52) is valid for all $\varepsilon>0$.

Then, by Markov's inequality,

$$\mathbb{P}\Bigg(\bigcup_{k\geq 1}\Bigg\{\sup_{1/k\leq s<t\leq T}\frac{\|\tilde{X}_t^h-\tilde{X}_s^h\|_2}{|t-s|^\alpha}>\frac{k^2}{\varepsilon}\Bigg\}\Bigg)\leq C\varepsilon.$$

Recalling the conclusion of the first step and deducing from the above that, on an event A_ε^h of probability greater than $1-C\varepsilon$, for any $k\geq 1$,

$$\begin{aligned}&\|\tilde{X}_t^h-\tilde{X}_0\|_2\leq \Xi^h t^\alpha+w(t),\quad t\in[0,T],\\&\|\tilde{X}_t^h-\tilde{X}_s^h\|_2\leq\frac{k^2}{\varepsilon}|t-s|^\alpha;\quad s,t\in\Big[\frac{1}{k},T\Big],\end{aligned}\tag{3.53}$$

where Ξ^h has finite moments of any order, uniformly with respect to $h\in(0,1)$. Without any loss of generality, one may take k to be any real greater than 1, and not only such integers. Changing the definition of A_ε^h, we can also assume without any loss of generality that $\Xi^h\leq 1/\varepsilon$ on A_ε^h.

Here we also recall, that almost surely, $w(t)$ converges to 0 as t tends to 0. By Egorov's theorem, we deduce that, on an event B_ε of probability greater than $1-\varepsilon$, $w(1/n)$ converges uniformly to 0 as n tends to ∞. Since w was chosen to be non-decreasing in t (see the end of the first step), this says that $\lim_{t\to 0}\sup_{\omega\in B_\varepsilon}w(t)=0$ (the presence of ω comes from the fact that w is a random function).

Restrict attention to $A_\varepsilon^h\cap B_\varepsilon$. Then, we can replace $w(t)$ by $\sup_{\omega\in B_\varepsilon}w(t)$ in (3.53). By Lemma 3.6, we deduce that on $A_\varepsilon^h\cap B_\varepsilon$, the paths $\{(\tilde{X}_t^h)_{0\leq t\leq T}\}_{0<h<1}$ are contained within a common subset of equicontinuous functions from $[0,T]$ to $L^2(\mathbb{S})$. Since $\mathbb{P}(A_\varepsilon^h\cap B_\varepsilon)\geq 1-(C+1)\varepsilon$ and $\varepsilon>0$ is arbitrary, this proves that, for any $\varepsilon>0$, there exists a Borel subset $E_\varepsilon\subset\mathcal{C}([0,T],L^2(\mathbb{S}))$ of equicontinuous functions such that $\mathbb{P}(\{(\tilde{X}_t^h)_{0\leq t\leq T}\in E_\varepsilon\})\geq 1-\varepsilon$ for any $h\in(0,1)$.

Conclusion. In order to prove tightness, it thus remains to show that, for all $t\in[0,T]$, the family $\{\tilde{X}_t^h\}_{0<h<1}$ is tight on $L^2(\mathbb{S})$. We invoke Lemma 3.4. It says that we can find a constant $c_{\lambda,T}$ such that, for any $h\in(0,1)$ and any $n\in\mathbb{N}$ with $nh\leq T+1$,

$$\mathbb{E}\Big[\|DX_n^h\|_2^2\Big]\leq\frac{1}{hn}\mathbb{E}\big[\|X_0\|_2^2\big]+c_{\lambda,T}.\tag{3.54}$$

Fix $h\in(0,1)$ and then $t\in[h,T]$. Letting $n:=\lfloor t/h\rfloor\geq 1$, we have $nh\leq t<nh+h<2nh$, which implies $1/nh\leq 2/t$. By (3.25) and by convexity, (3.54) yields

$$\mathbb{E}\Big[\|D\tilde{X}_t^h\|_2^2\Big]\leq\mathbb{E}\Big[\|DX_{nh}^h\|_2^2\Big]+\mathbb{E}\Big[\Big\|DX_{(n+1)h}^h\Big\|_2^2\Big]\leq\frac{4}{t}\mathbb{E}\big[\|X_0\|_2^2\big]+2c_{\lambda,T}.$$

Recalling that the embedding from $H^1(\mathbb{S})$ into $L^2(\mathbb{S})$ is compact, we deduce that, for any $\varepsilon>0$ and any $t\in(0,T]$, there exists a compact subset $K_{\varepsilon,t}$ of $L^2(\mathbb{S})$ (with $K_{\varepsilon,t}$ independent of h) such that

$$t\geq h\Rightarrow\mathbb{P}\big(\big\{\tilde{X}_t^h\in K_{\varepsilon,t}\big\}\big)\geq 1-\varepsilon.\tag{3.55}$$

Moreover, since $L^2(\mathbb{S})$ is Polish, we deduce that the law of X_0 is tight on $L^2(\mathbb{S})$. Therefore, for any $\varepsilon > 0$, we can find a compact subset $K_{\varepsilon,0}$ of $L^2(\mathbb{S})$ (with $K_{\varepsilon,0}$ independent of h) such that

$$\mathbb{P}\big(\{X_0^h \in K_{\varepsilon,0}\}\big) \geq 1 - \varepsilon. \tag{3.56}$$

For $t \in [0,T]$, we let $\tilde{K}_{\varepsilon,t} := \{sx + (1-s)y,\ s \in [0,1],\ x \in K_{\varepsilon,0},\ y \in K_{\varepsilon,t}\}$, which is a compact subset of $L^2(\mathbb{S})$.

By (3.25) again, we have, for $t \in (0,h)$, $\tilde{X}_t^h \in \tilde{K}_{\varepsilon,t}$ on the event $\{X_0^h \in K_{\varepsilon,0}\} \cap \{\tilde{X}_h^h \in K_{\varepsilon,h}\}$. Therefore, (3.55) and (3.56) yield

$$\mathbb{P}\big(\{\tilde{X}_t^h \in \tilde{K}_{\varepsilon,t}\}\big) \geq 1 - 2\varepsilon, \quad t \in (0,h). \tag{3.57}$$

Combining (3.55), (3.56) and (3.57), we deduce that, for every $t \in [0,T]$, the family $\{\tilde{X}_t^h\}_{0<h<1}$ is tight on $L^2(\mathbb{S})$. This completes the proof. □

4 Limiting Dynamics: Characterisation and Well-posedness

This section addresses the weak limits of the schemes. As discussed in the introduction, it is expected that those weak limits, say denoted by X, should satisfy a reflected stochastic differential equation in infinite dimension, understood in the sense,

$$dX_t = \Delta X_t dt + dW_t + d\eta_t, \quad t \geq 0, \tag{4.1}$$

for an initial condition X_0 satisfying the assumption stated in the beginning of Section 3. Here, $(\eta_t)_{t\geq 0}$ should be understood as a forcing term that reflects the process X into the cone U^2, see Definition 2.2. As we said in the introduction, reflected stochastic partial differential equations were already considered in Röckner, Zhu and Zhu [64], but the results proven therein do not apply to our setting. Without access to these results, our approach follows the application of limit theorems to the tested/weak behaviour of the schemes. In order to study the forcing term in this case, we restrict the analysis to a smaller class of test functions in the weak formulation, this class being nonetheless sufficient to demonstrate uniqueness of the limit. Below, we often refer to (4.1) - with accompanying conditions on the process η - as the **rearranged stochastic heat equation (or rearranged SHE in short)**. The reader impatient for the exact solution concept at which we arrive should skip momentarily to Definition 4.13, with caution that the fourth condition contains an integral that is defined en route.

The purpose of this section is thus to identify conditions satisfied by any weak limit that are, in the end, sufficient to prove that the weak limit (of the schemes) is in fact unique. This goal is reached in a series of five subsections. In Subsection 4.1, we prove that weak limits satisfy an equation of the form (4.1). In Subsection 4.2, we define an integral with respect to the reflection process $(\eta_t)_{t\geq 0}$. The main feature of this integral is that it must be non-decreasing with respect to integrand processes valued in the cone $U^2(\mathbb{S})$. Moreover, in Subsection 4.3 we establish an orthogonality property between X and η that is key to proving uniqueness of the weak limit. The rigorous definition of a

solution to (4.1) together with the main statement of its existence and uniqueness are given in Subsection 4.4, see in particular Definition 4.13 and Theorem 4.15. We end the section with a proof of the Lipschitz regularity of the flow induced by the solution in Subsection 4.5.

4.1 Testing of the weak limits

Our analysis of the weak limits as solutions of a reflected SPDE relies on the following statement:

Proposition 4.1. *Let $(X_t, W_t)_{t\geq 0}$ be a weak limit (over $\mathcal{C}([0,\infty), L^2(\mathbb{S}))$ equipped with the topology of uniform convergence on compact subsets) of the processes $\{(\tilde{X}_t^h, W_t)_{t\geq 0}\}_{h>0}$ as h tends to 0, this weak limit being constructed on the same filtered probability space $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P})$ as the scheme itself and the second component $(W_t)_{t\geq 0}$ of the weak limit abusively denoted the same as the noise in the scheme.*

Then, $(X_t, W_t)_{t\geq 0}$ is $\mathbb{F}$-adapted, $(X_t)_{t\geq 0}$ is $U^2(\mathbb{S})$-valued (i.e., each X_t has symmetric and non-increasing values, see (2.2)) and $(W_t)_{t\geq 0}$ is an $L^2_{\mathrm{sym}}(\mathbb{S})$-valued Q-Brownian motion with respect to $\mathbb{F}$. Moreover, there exists an $\mathbb{F}$-adapted continuous process $(\eta_t)_{t\geq 0}$ with values in $H^{-2}_{\mathrm{sym}}(\mathbb{S})$ such that, with probability 1, for any $u \in H^2_{\mathrm{sym}}(\mathbb{S})$:

1. *for any $s, t \geq 0$ with $t - s > 0$,*

$$\langle X_t - X_s, u\rangle = \int_s^t \langle X_r, \Delta u\rangle dr + \langle W_t - W_s, u\rangle + \langle \eta_t - \eta_s, u\rangle, \tag{4.2}$$

2. *if u is non-increasing in the sense of Definition 2.2, then the path $(\langle \eta_t, u\rangle)_{t\geq 0}$ is non-decreasing (with t) and starts from 0 at time 0.*

The hypothesis that the weak limit can be constructed on the same space $(\Omega, \mathcal{A}, \mathbb{P})$ as in Section 3 can be made without loss of generality. In short, this just requires the probability space to be 'rich enough'[2] (e.g., it is an atomless Polish probability space), which as additional assumption, is not a hindrance for us. The claim that $(\Omega, \mathcal{A}, \mathbb{P})$ can be equipped with the filtration $\mathbb{F}$ requires a little more care: $\mathbb{F}$ cannot be any given filtration, which is a common feature with weak limits of processes. We clarify the choice of $\mathbb{F}$ in the proof below. We do this only for the convenience of using the same notation $\mathbb{F}$ for this specific choice, as we are convinced that there is no risk of confusion for the reader. Similarly, denoting the second component of the weak limit by $(W_t)_{t\geq 0}$ is also rather abusive, but is justified by the fact that the second component's law in any weak limit remains that of a Q-Brownian motion with values in $L^2_{\mathrm{sym}}(\mathbb{S})$, see (3.1) and (3.2).

To facilitate identification of the limiting dynamics, a transformation of the scheme is considered that effectively removes difficulties arising from the presence of the noise term. In order to simplify the exposition, some additional notation is introduced.

[2] By 'rich enough', we mean that, for any distribution μ on a Polish space $\mathcal{S}$, we can construct a μ-distributed random element from Ω to $\mathcal{S}$.

Let $V^h = (V_n^h)_{n\geq 0}$ denote the unique solution to the scheme (3.3) without the rearrangement operation (and with the same notation as in (3.4) for W^{n+1}):

$$V_{n+1}^h = e^{h\Delta} V_n^h + \int_0^h e^{(h-s)\Delta} dW_s^{n+1}, \quad V_0^h = X_0. \tag{4.3}$$

Equivalently, $V_n^h = V_{nh}$, where V is the solution to the stochastic heat equation with X_0 as initial condition, i.e.

$$V_t = e^{t\Delta} X_0 + \int_0^t e^{(t-s)\Delta} dW_s, \quad t \geq 0, \tag{4.4}$$

which has values in $L^2_{\mathrm{sym}}(\mathbb{S})$. The scheme (3.3) may be rewritten as

$$X_{n+1}^h = \left(V_{n+1}^h + e^{h\Delta}(X_n^h - V_n^h)\right)^*, \quad n \geq 0. \tag{4.5}$$

The so-called shifted scheme, $X^h - V^h = (X_n^h - V_n^h)_{n\geq 0}$ is denoted $Y^h = (Y_n^h)_{n\geq 0}$, so that the above may be rewritten as:

$$Y_{n+1}^h = \left(V_{n+1}^h + e^{h\Delta} Y_n^h\right)^* - V_{n+1}^h, \quad n \geq 0. \tag{4.6}$$

Following the notation (3.25), we introduce the linear interpolations

$$\begin{aligned} \tilde{Y}_t^h &:= (\lceil t/h \rceil - t/h) Y_{\lfloor t/h \rfloor}^h + (t/h - \lfloor t/h \rfloor) Y_{\lceil t/h \rceil}^h, \\ \tilde{V}_t^h &:= (\lceil t/h \rceil - t/h) V_{\lfloor t/h \rfloor}^h + (t/h - \lfloor t/h \rfloor) V_{\lceil t/h \rceil}^h, \quad t \geq 0, \end{aligned} \tag{4.7}$$

which leads to

$$\tilde{Y}_t^h = \tilde{X}_t^h - \tilde{V}_t^h, \quad t \geq 0. \tag{4.8}$$

Remark 4.2. The subsequent proof of Proposition 4.1 shows that similar notations (to the ones introduced directly above) can be used to clarify the shape of the reflection process $(\eta_t)_{t\geq 0}$ in (4.2). Indeed, denoting again by $(V_t)_{t\geq 0}$ the solution to the stochastic heat equation with X_0 as initial condition and with $(W_t)_{t\geq 0}$ as driving noise (recalling that we abusively chose to keep this notation in the limit setting), i.e., $(V_t)_{t\geq 0}$ solves (4.4), and then letting $Y_t := X_t - V_t$, for $t \geq 0$, one has, with probability 1, for any $v \in H^2_{\mathrm{sym}}(\mathbb{S})$,

$$\forall t \geq 0, \quad \langle \eta_t, v \rangle = \langle Y_t, v \rangle - \int_0^t \langle Y_r, \Delta v \rangle dr. \tag{4.9}$$

We now turn to:

Proof of Proposition 4.1. Throughout the proof, we fix $T > 0$, and we study the weak limits, as h tends to 0, of $\{\tilde{X}^h, W\}$ when the latter is restricted to the time interval $[0, T]$. This is sufficient for our purposes.

We start with the following observation that holds true, for a given $h > 0$, with probability 1 and for any $u \in H^2_{\mathrm{sym}}(\mathbb{S})$,

$$\begin{aligned} \left\langle Y_{n+1}^h - Y_n^h, u \right\rangle &= \left\langle Y_{n+1}^h - e^{h\Delta} Y_n^h, u \right\rangle + \left\langle (e^{h\Delta} - I) Y_n^h, u \right\rangle \\ &= \left\langle Y_{n+1}^h - e^{h\Delta} Y_n^h, u \right\rangle + \int_0^h \left\langle e^{s\Delta} Y_n^h, \Delta u \right\rangle ds, \end{aligned}$$

where we used the identity $\partial_s e^{s\Delta} = \Delta e^{s\Delta}$. Rearranging, and working under the additional assumption that u is non-increasing, we use the rewritten shifted scheme (4.6) to show:

$$\begin{aligned}\langle Y_{n+1}^h - Y_n^h, u\rangle - \int_0^h \left\langle e^{s\Delta} Y_n^h, \Delta u\right\rangle ds &= \langle Y_{n+1}^h - e^{h\Delta} Y_n^h, u\rangle \\ &= \left\langle \left(V_{n+1}^h + e^{h\Delta} Y_n^h\right)^* - \left(V_{n+1}^h + e^{h\Delta} Y_n^h\right), u\right\rangle \geq 0,\end{aligned} \tag{4.10}$$

the last line following from the Hardy-Littlewood inequality (Lemma 2.5).

We rewrite the second term in the left-hand side:

$$\begin{aligned}\int_0^h \langle e^{s\Delta} Y_n^h, \Delta u\rangle ds &= \int_0^h \langle (e^{s\Delta} - I) Y_n^h, \Delta u\rangle ds + h\langle Y_n^h, \Delta u\rangle \\ &= \int_0^h \langle Y_n^h, (e^{s\Delta} - I)\Delta u\rangle ds + h\langle Y_n^h, \Delta u\rangle.\end{aligned} \tag{4.11}$$

Summing over the time mesh n, we get for any $(s,t) \in [0,T]^2$, with the two notations $N_s := n^{-1}\lfloor sn\rfloor$ and $N_t := n^{-1}\lfloor tn\rfloor$,

$$\begin{aligned}&\left|\sum_{n=N_s}^{N_t} \int_0^h \left\langle e^{r\Delta} Y_n^h, \Delta u\right\rangle dr - \int_s^t \left\langle \tilde{Y}_r^h, \Delta u\right\rangle dr\right| \\ &\leq \left|\sum_{n=N_s}^{N_t} \int_0^h \left\langle Y_n^h, \left(e^{r\Delta} - I\right)\Delta u\right\rangle dr\right| + \left|\sum_{n=N_s}^{N_t} h\left\langle Y_n^h, \Delta u\right\rangle - \int_s^t \langle \tilde{Y}_r^h, \Delta u\rangle dr\right| \\ &\leq c_T \sup_{0\leq r\leq h} \left\|(e^{r\Delta} - I)\Delta u\right\|_2 \sup_{n\in\{0,\cdots,N_t\}} \|Y_n^h\|_2 + \left|\sum_{n=N_s}^{N_t} h\langle Y_n^h, \Delta u\rangle - \int_s^t \langle \tilde{Y}_r^h, \Delta u\rangle dr\right| \\ &=: T_1^h(t) + T_2^h(s,t).\end{aligned} \tag{4.12}$$

Since $\Delta u \in L^2(\mathbb{S})$, we know that

$$\lim_{h\searrow 0} \sup_{0\leq r\leq h} \left\|(e^{r\Delta} - I)\Delta u\right\|_2 = 0.$$

Together with Proposition 3.7 (recalling that $Y_n^h = (X_n^h - V_n^h)_{n\geq 0}$), we deduce that

$$\forall \varepsilon > 0, \quad \lim_{h\searrow 0} \mathbb{P}\Big(\Big\{\sup_{0\leq t\leq T} T_1^h(t) \geq \varepsilon\Big\}\Big) = 0. \tag{4.13}$$

Similarly, by tightness of $\{\tilde{X}^h\}_{h\in(0,1)}$ on $\mathcal{C}([0,\infty), L^2_{sym}(\mathbb{S}))$, we deduce that $\{\tilde{Y}^h\}_{h\in(0,1]}$ is also tight on $\mathcal{C}([0,\infty), L^2_{sym}(\mathbb{S}))$, from which we easily get that (since $\tilde{Y}^h$ is the linear interpolation of Y^h)

$$\forall \varepsilon > 0, \quad \lim_{h\searrow 0} \mathbb{P}\Big(\Big\{\sup_{0\leq s<t\leq T} T_2^h(s,t) \geq \varepsilon\Big\}\Big) = 0. \tag{4.14}$$

Therefore, returning to (4.12), the last two displays (4.13) and (4.14) yield

$$\forall \varepsilon > 0, \quad \lim_{h\searrow 0} \mathbb{P}\left(\sup_{0\leq s<t\leq T} \left|\sum_{n=N_s}^{N_t} \int_0^h \left\langle e^{r\Delta} Y_n^h, \Delta u\right\rangle dr - \int_s^t \left\langle \tilde{Y}_r^h, \Delta u\right\rangle dr\right| \geq \varepsilon\right) = 0. \tag{4.15}$$

It remains to insert (4.15) into (4.10). Together, the latter two yield

$$\forall (s,t) \in [0,T]^2 : s < t, \quad \langle \tilde{Y}_t^h - \tilde{Y}_s^h, u\rangle - \int_s^t \left\langle \tilde{Y}_r^h, \Delta u\right\rangle dr \geq T_3^h(s,t), \tag{4.16}$$

with

$$\forall \varepsilon > 0, \quad \lim_{h \searrow 0} \mathbb{P}\left(\sup_{0 \leq s < t \leq T} \left| T_3^h(s,t) \right| \geq \varepsilon \right) = 0. \tag{4.17}$$

Now we let h tend to 0. Following the statement, we slightly abuse notation and write $(X_t, W_t)_{0 \leq t \leq T}$ a weak limit of $\{(\tilde{X}_t^h, W_t)_{0 \leq t \leq T}\}_{h \in (0,1]}$. We denote by $\mathbb{F}$, the usual augmentation of the filtration generated by $(X_t, W_t)_{0 \leq t \leq T}$. There is no difficulty in proving that $(W_t)_{0 \leq t \leq T}$ is an $L^2_{\text{sym}}(\mathbb{S})$-valued Q-Brownian motion with respect to $\mathbb{F}$. Along the same subsequence, $\{(\tilde{X}_t^h, W_t, \tilde{V}_t^h, \tilde{Y}_t^h)_{0 \leq t \leq T}\}_{h \in (0,1]}$ converges to $(X_t, W_t, V_t, Y_t)_{0 \leq t \leq T}$, where $(V_t)_{0 \leq t \leq T}$ solves the heat equation (4.4) (driven by the limit process W) and $Y_t = X_t - V_t$, for $t \in [0,T]$ (in particular, $Y_0 = 0$). Obviously, $(X_t, W_t, V_t, Y_t)_{0 \leq t \leq T}$ is $\mathbb{F}$-adapted. By Proposition 2.3, $(X_t)_{0 \leq t \leq T}$ takes values in $U^2(\mathbb{S})$ (since $(\tilde{X}_t^h)_{0 \leq t \leq T}$ does, for any $h > 0$). In turn, $(Y_t)_{0 \leq t \leq T}$ is valued in $L^2_{\text{sym}}(\mathbb{S})$ since $(V_t)_{0 \leq t \leq T}$ takes values in $L^2_{\text{sym}}(\mathbb{S})$. Moreover, using (4.16) and (4.17), we obtain, when $u \in H^2_{\text{sym}}(\mathbb{S})$ is non-increasing,

$$\forall (s,t) \in [0,T]^2 : s < t, \quad \langle Y_t - Y_s, u\rangle - \int_s^t \langle Y_r, \Delta u\rangle dr \geq 0, \tag{4.18}$$

which prompts us to let, for any $v \in H^2_{\text{sym}}(\mathbb{S})$,

$$\forall t \in [0,T], \quad \langle \eta_t, v\rangle := \langle Y_t, v\rangle - \int_0^t \langle Y_r, \Delta v\rangle dr.$$

Then, (4.18) says that, for any $u \in H^2_{\text{sym}}(\mathbb{S})$, with u being non-increasing, the process $(\langle \eta_t, u\rangle)_{0 \leq t \leq T}$ is a.s. non-decreasing. By a separability argument, we deduce that, a.s., for any $u \in H^2_{\text{sym}}(\mathbb{S})$, with u being non-increasing, the process $(\langle \eta_t, u\rangle)_{0 \leq t \leq T}$ is non-decreasing. This proves item 2 in the statement.

By replacing $(Y_t)_{0 \leq t \leq T}$ by $(X_t - V_t)_{0 \leq t \leq T}$ in the definition of $(\eta_t)_{0 \leq t \leq T}$ and by recalling that, for any $v \in H^2(\mathbb{S})$,

$$\langle V_t, v\rangle - \int_0^t \langle V_r, \Delta v\rangle dr = \langle e^{t\Delta} X_0, v\rangle + \int_0^t e^{(t-r)\Delta} dW_r,$$

we easily verify item 1 in the statement, completing the proof. □

From the bound obtained in Proposition 3.7 and the notation defined in Remark 4.2, we deduce the following result:

Proposition 4.3. *For any $p \geq 1$ and any $T > 0$, there exists constant $C_{p,\lambda,T,\mathbb{E}[\|X_0\|^{2p}]}$, such that for any weak limit as in Proposition 4.1,*

$$\mathbb{E}\Big[\sup_{t \in [0,T]} \|X_t\|_2^{2p} \Big] \leq C_{p,\lambda,T,\mathbb{E}[\|X_0\|^{2p}]}.$$

Additionally, with the notation of Remark 4.2,

$$\mathbb{E}\Big[\sup_{t \in [0,T]} \|Y_t\|_2^{2p} \Big] \leq C_{p,\lambda,T,\mathbb{E}[\|X_0\|^{2p}]}.$$

4.2 Integral with respect to the reflection process

Our next objective is to construct an integral with respect to the reflection process $(\eta_t)_{t\geq 0}$ identified in the statement of Proposition 4.1. Not only is this question interesting in its own right, but we go on to make use of the resulting integral in order to establish uniqueness of the weak limits obtained in Proposition 4.1.

The ingredients that we want to use in the construction of the integral are the following two properties, which are satisfied with probability 1:

- the path $(\eta_t)_{t\geq 0}$ has continuous values in $H^{-2}_{\mathrm{sym}}(\mathbb{S})$;
- for any non-increasing $u \in H^2_{\mathrm{sym}}(\mathbb{S})$, the path $(\langle \eta_t, u\rangle)_{t\geq 0}$ is non-decreasing.

In order to distinguish clearly between the random process $(\eta_t)_{t\geq 0}$ and one of its realisation, we denote below by $(n_t)_{t\geq 0}$ a **deterministic** trajectory that satisfies the above two features, namely

(**E1**) $t \mapsto n_t$ is a continuous function from $[0,\infty)$ to $H^{-2}_{\mathrm{sym}}(\mathbb{S})$.

(**E2**) For any non-increasing $u \in H^2_{\mathrm{sym}}(\mathbb{S})$, the function $t \in [0,\infty) \mapsto \langle \eta_t, u\rangle$ is non-decreasing.

It is important to stress again that the integral we construct below holds for a path $(n_t)_{t\geq 0}$ satisfying only the two assumptions (**E1**) and (**E2**). In particular, the integral with respect to $(\eta_t)_{t\geq 0}$ is then obtained by choosing $(n_t)_{t\geq 0}$ as the current realisation of $(\eta_t)_{t\geq 0}$. In this regard, the definition of the latter integral is pathwise.

Now, let $u \in H^2_{\mathrm{sym}}(\mathbb{S})$ be non-increasing. If we consider in addition another **deterministic** trajectory $(z_t)_{t\geq 0}$ with continuous sample paths valued in $L^2_{\mathrm{sym}}(\mathbb{S})$, (**E2**) allows us to define

$$\left(\int_0^t \langle z_r, u\rangle d\langle n_r, u\rangle\right)_{t\geq 0} \tag{4.19}$$

as a Riemann-Stieltjes integral. From this, it would be desirable to give a meaning to the as yet informally written integrals

$$\left(\int_0^t z_r \cdot dn_r\right)_{t\geq 0}, \tag{4.20}$$

where the dot $\cdot$ in the notation is intended to denote a form of duality presence between in the integrand and the integrator.

Our definition of the integral is done by analogy with Parseval's identity, setting u in (4.19) to be (cosine) elements in the Fourier basis. The next step is then to expand (4.20) along the (cosine) Fourier basis $(e_m)_{m\in\mathbb{N}_0}$, noticing that one may indeed decompose each e_m as the difference of two symmetric non-increasing functions that we denote as e_m^+ and e_m^-, defined by:

$$\begin{aligned} e_m^+(x) &:= e_m(0) + \int_0^x \left[-\mathbb{1}_{(-1/2,0]}(y)\big(De_m(y)\big)_- + \mathbb{1}_{[0,1/2)}(y)\big(De_m(y)\big)_+\right](y)dy, \\ e_m^-(x) &:= \int_0^x \left[-\mathbb{1}_{(-1/2,0]}(y)\big(De_m(y)\big)_+ + \mathbb{1}_{[0,1/2)}(y)\big(De_m(y)\big)_-\right](y)dy. \end{aligned} \tag{4.21}$$

By construction, the functions e_m^+ and e_m^- are symmetric non-increasing (courtesy of the symmetry properties of e_m) and $e_m = e_m^+ - e_m^-$. Therefore, one may set:

$$\int_s^t \langle z_r, e_m\rangle \cdot d\langle n_r, e_m\rangle := \int_s^t \langle z_r, e_m\rangle \cdot d\langle n_r, e_m^+\rangle - \int_s^t \langle z_r, e_m\rangle \cdot d\langle n_r, e_m^-\rangle. \tag{4.22}$$

A key remark is that, for any $\varepsilon > 0$, $(z_r)_{r\geq 0}$ can be replaced by $(e^{\varepsilon\Delta} z_r)_{r\geq 0}$ in the above formulation, which is a direct consequence of Lemma 2.10. The following statement is a crucial step in the construction of our integral (4.2).

Lemma 4.4. *For any integer $k \geq 1$ and any real $\varepsilon > 0$, there exists a constant $c_{k,\varepsilon}$ such that, for any two (deterministic) curves $(n_t)_{t\geq 0}$ and $(z_t)_{t\geq 0}$, with $(n_t)_{t\geq 0}$ satisfying* (**E1**) *and* (**E2**) *and with $(z_t)_{t\geq 0}$ a continuous path in $L^2_{\mathrm{sym}}(\mathbb{S})$,*

$$\forall T \geq 0,\ \forall m \in \mathbb{N}_0, \quad \sup_{t\in[0,T]} \left| \int_0^t \langle e^{\varepsilon\Delta} z_r, e_m\rangle d\langle n_r, e_m^\pm\rangle \right| \leq \frac{c_{k,\varepsilon}}{m^k \vee 1} \|n_T\|_{2,-2} \sup_{t\in[0,T]} \|z_t\|_2. \tag{4.23}$$

When $(n_t)_{t\geq 0}$ is understood as a realisation of $(\eta_t)_{t\geq 0}$, the term $\|n_T\|_{2,-2}$ becomes $\|\eta_T\|_{2,-2}$ and can be upper bounded by

$$\|\eta_T\|_{2,-2} \leq c_T \sup_{t\in[0,T]} \|Y_t\|_2, \tag{4.24}$$

with $(Y_t)_{t\geq 0}$ as in (4.9).

In (4.23), we use the notation $e_m^\pm$ to indicate that the result holds true with both e_m^+ and e_m^-. Also, note that the L^2 contributions of e_m^+ and e_m^- diverge with m. This is precisely the reason why we consider integrands of the form $(e^{\varepsilon\Delta} z_t)_{t\geq 0}$, since convolution with the heat kernel forces the higher modes of the resulting function to decay exponentially fast. In brief, for any k and ε as in the statement, we can find two constants c_k and $c_{k,\varepsilon}$ such that, for $m \in \mathbb{N}$,

$$\forall r \geq 0, \quad |\langle e^{\varepsilon\Delta} z_r, e_m\rangle| \leq c_k \left|\left\langle D^{2k} e^{\varepsilon\Delta} z_r, \frac{1}{m^{2k}} e_m\right\rangle\right| \leq c_{k,\varepsilon} \frac{1}{m^{2k}} \|z_r\|_2. \tag{4.25}$$

Obviously, the proof of Lemma 4.4 relies on the bound (4.25), whence appears the constant $c_{k,\varepsilon}$ in the statement.

Proof of Lemma 4.4. We begin with the following simple observation. Obviously, each $e_m^\pm$ belongs to $H^2_{\mathrm{sym}}(\mathbb{S})$ and it is a simple exercise to show that $\|e_m^\pm\|_{2,2} \leq c(m^2 \vee 1)$.

We invoke (**E2**), which allows one to use the following standard property of the Riemann-Stieltjes integral:

$$\begin{aligned} \sup_{t\in[0,T]} \left| \int_0^t \langle e^{\varepsilon\Delta} z_r, e_m\rangle d\langle n_r, e_m^\pm\rangle \right| &\leq \sup_{t\in[0,T]} \left|\langle e^{\varepsilon\Delta} z_t, e_m\rangle\right| \times \langle n_T, e_m^\pm\rangle \\ &\leq c_{k,\varepsilon} \frac{m^2 \vee 1}{m^{2k} \vee 1} \|n_T\|_{2,-2} \sup_{t\in[0,T]} \|z_t\|_2, \end{aligned} \tag{4.26}$$

with the last line following from (4.25) together with the bound $\|e_m^{\pm}\|_{2,2} \le c(m^2 \vee 1)$. This shows (4.23). As for the proof of (4.24), we just make use of (4.9). This completes the proof. □

Lemma 4.4 allows us to make the following definition:

Definition 4.5. *For any two (deterministic) curves $(n_t)_{t\ge 0}$ and $(z_t)_{t\ge 0}$, with $(n_t)_{t\ge 0}$ satisfying* (**E1**) *and* (**E2**) *and with $(z_t)_{t\ge 0}$ being a continuous function from $[0,\infty)$ to $L^2_{\mathrm{sym}}(\mathbb{S})$, we can define, almost surely, for any $\varepsilon > 0$ the integral process*

$$\left(\int_0^t e^{\varepsilon\Delta} z_r \cdot dn_r\right)_{t\ge 0}$$

as the limit, for the uniform topology on compact subsets:

$$\int_0^t e^{\varepsilon\Delta} z_r \cdot dn_r := \lim_{M\to\infty} \sum_{m=0}^{M} \left(\int_0^t \langle e^{\varepsilon\Delta} z_r, e_m \rangle d\langle n_r, e_m^+ \rangle - \int_0^t \langle e^{\varepsilon\Delta} z_r, e_m \rangle d\langle n_r, e_m^- \rangle \right).$$

It satisfies

$$\forall T \ge 0, \quad \sup_{t\in[0,T]} \left| \int_0^t e^{\varepsilon\Delta} z_r \cdot dn_r \right| \le c_\varepsilon \|n_T\|_{2,-2} \times \sup_{t\in[0,T]} \|z_t\|_2. \tag{4.27}$$

When $(n_t)_{t\ge 0}$ is understood as a realisation of $(\eta_t)_{t\ge 0}$, the term $\|n_T\|_{2,-2}$ becomes $\|\eta_T\|_{2,-2}$ and can be upper bounded as in (4.24).

Remark 4.6. The following three remarks are in order:

1. In Definition 4.5, not only is the convergence uniform in time t in a fixed segment $[0,T]$, for some $T > 0$, but it is also uniform with respect to the integrand $(z_t)_{0\le t\le T}$ when the latter is required to satisfy $\sup_{t\in[0,T]} \|z_t\|_2 \le A$ for some given $A > 0$. This is a direct consequence of the form of the rate of convergence given by (4.23).

2. Moreover, it is straightforward to observe that both Lemma 4.4 and Definition 4.5 extend to the case when $(z_t)_{t\ge 0}$ is a piecewise constant path with values in $L^2_{\mathrm{sym}}(\mathbb{S})$ (i.e., there exists an increasing locally-finite sequence of time indices $(t_k)_{k\ge 0}$, with $t_0 = 0$, such that $t \in [t_k, t_{k+1}) \mapsto z_t$ is constant for each $k \ge 0$).

3. Definition 4.5 could be in fact seen as a particular case of a more general definition of a Riemann-Stieltjes integral for functions with values in a Banach space. Implicitly, in our definition, the integrand is regarded as a path taking values in a space of 'very regular functions' and the integrator as a path with values in a space of distributions acting on those very regular functions. By specifying the underlying functional spaces, we could recast our construction within the larger framework studied for instance in [69]. Actually, we prefer to provide a tailor-made analysis of the integral and not to make use of such general results, for the following two reasons. First, it makes the reading much easier. Second, the Banach spaces we deal with here are standard Hilbert spaces in functional analysis, and this allows us to provide an elementary approach to the integral.

The following lemma explains the interest of the second remark right above.

Lemma 4.7. *Within the same framework as in Definition 4.5 but with $(z_t)_{t\geq 0}$ therein being piecewise constant (with the same jumping times $(t_k)_{k\geq 0}$ as in Remark 4.6), the integral process $(\int_0^t e^{\varepsilon\Delta} z_r \cdot dn_r)_{t\geq 0}$ coincides with the process defined by standard Riemann sums, i.e.,*

$$\int_0^t e^{\varepsilon\Delta} z_r \cdot dn_r = \sum_{k\geq 0: t_k \leq t} \langle e^{\varepsilon\Delta} z_{t_k}, \eta_{t_{k+1}\wedge t} - \eta_{t_k} \rangle, \quad t \geq 0.$$

In particular, if z_{t_k}, for each $k \geq 0$, is symmetric non-increasing, then

$$\forall t \geq 0, \quad \int_0^t e^{\varepsilon\Delta} z_r \cdot d\eta_r \geq 0.$$

Before we prove Lemma 4.7, we state the following important corollary.

Corollary 4.8. *Within the same framework as in Definition 4.5, with $(z_t)_{t\geq 0}$ therein being continuous, we let, for any $k \in \mathbb{N}$, $z^k = (z_t^k)_{t\geq 0}$ be the piecewise constant approximation of $z = (z_t)_{t\geq 0}$ of stepsize $1/k$, namely*

$$z_t^k := z_{\lfloor kt \rfloor / k}, \quad t \geq 0.$$

Then, the following convergence holds true, uniformly on compact subsets,

$$\int_0^t e^{\varepsilon\Delta} z_r \cdot dn_r = \lim_{k\to\infty} \int_0^t e^{\varepsilon\Delta} z_r^k \cdot dn_r, \quad t \geq 0.$$

In particular, if z_t is non-increasing for each $t \geq 0$, then

$$\forall t \geq 0, \quad \int_0^t e^{\varepsilon\Delta} z_r \cdot dn_r \geq 0.$$

We now prove Lemma 4.7.

Proof of Lemma 4.7. Throughout the proof, we use the same notation as in the statement of the lemma. Back to Definition 4.5 - but for a path of the type discussed in the second item of Remark 4.6 - we then observe that, for any integer $M \geq 1$,

$$\begin{aligned}
&\sum_{m=0}^{M} \left(\int_0^t \langle e^{\varepsilon\Delta} z_r, e_m \rangle d\langle n_r, e_m^+ \rangle - \int_0^t \langle e^{\varepsilon\Delta} z_r, e_m \rangle d\langle n_r, e_m^- \rangle \right) \\
&= \sum_{m=0}^{M} \sum_{k\geq 0: t_k \leq t} \left(\langle e^{\varepsilon\Delta} z_{t_k}, e_m \rangle \Big[\Big(\langle n_{t_{k+1}\wedge t}, e_m^+ \rangle - \langle n_{t_k}, e_m^+ \rangle \Big) - \Big(\langle n_{t_{k+1}\wedge t}, e_m^- \rangle - \langle \eta_{t_k}, e_m^- \rangle \Big) \Big] \right) \\
&= \sum_{m=0}^{M} \sum_{k\geq 0: t_k \leq t} \left(\langle e^{\varepsilon\Delta} z_{t_k}, e_m \rangle \Big(\langle n_{t_{k+1}\wedge t}, e_m \rangle - \langle n_{t_k}, e_m \rangle \Big) \right).
\end{aligned}$$

Exchanging the two sums, we get

$$\begin{aligned}
&\sum_{m=0}^{M} \left(\int_0^t \langle e^{\varepsilon\Delta} z_r, e_m \rangle d\langle n_r, e_m^+ \rangle - \int_0^t \langle e^{\varepsilon\Delta} z_r, e_m \rangle d\langle n_r, e_m^- \rangle \right) \\
&= \sum_{k\geq 0: t_k \leq t} \sum_{m=0}^{M} \left(\langle e^{\varepsilon\Delta} z_{t_k}, e_m \rangle \Big(\langle n_{t_{k+1}\wedge t}, e_m \rangle - \langle n_{t_k}, e_m \rangle \Big) \right).
\end{aligned}$$

Since, for each $k \in \mathbb{N}$, n_{t_k} belongs to $H^{-2}_{\mathrm{sym}}(\mathbb{S})$ and $e^{\varepsilon\Delta} z_{t_k}$ to $H^{2}_{\mathrm{sym}}(\mathbb{S})$, we have

$$\forall k \in \mathbb{N}, \quad \lim_{M\to\infty} \sum_{m=0}^{M} \Big(\big\langle e^{\varepsilon\Delta} z_{t_k}, e_m \big\rangle \big(\langle n_{t_{k+1}\wedge t}, e_m \rangle - \langle \eta_{t_k}, e_m \rangle \big) \Big) = \big\langle e^{\varepsilon\Delta} z_{t_k}, n_{t_{k+1}\wedge t} - n_{t_k} \big\rangle,$$

from which we easily deduce that

$$\begin{aligned} \int_0^t e^{\varepsilon\Delta} z_r \cdot dn_r &:= \lim_{M\to\infty} \sum_{m=0}^{M} \Big(\int_0^t \big\langle e^{\varepsilon\Delta} z_r, e_m \big\rangle d\langle n_r, e_m^+ \rangle - \int_0^t \big\langle e^{\varepsilon\Delta} z_r, e_m \big\rangle d\langle n_r, e_m^- \rangle \Big) \\ &= \sum_{k: t_k \le t} \big\langle e^{\varepsilon\Delta} z_{t_k}, n_{t_{k+1}\wedge t} - n_{t_k} \big\rangle, \end{aligned}$$

the convergence being uniform with respect to t in compact subsets.

In order to complete the proof, note that whenever z_{t_k} is symmetric non-increasing, so is $e^{\varepsilon\Delta} z_{t_k}$, see Lemma 2.10. By the second item in Proposition 4.1, we then obtain that $\langle e^{\varepsilon\Delta} z_{t_k}, n_{t_{k+1}\wedge t} - n_{t_k} \rangle \ge 0$. □

It remains to check Corollary 4.8.

Proof of Corollary 4.8. The first claim in the statement of Corollary 4.8 is a consequence of (4.27), using the linearity of the integral, which says that

$$\int_0^t e^{\varepsilon\Delta} z_r \cdot dn_r - \int_0^t e^{\varepsilon\Delta} z_r^k \cdot dn_r = \int_0^t e^{\varepsilon\Delta} \big(z_r - z_r^k \big) \cdot dn_r, \quad t \in [0, T],$$

together with the fact that

$$\lim_{k\to\infty} \sup_{0\le t\le T} \| z_t - z_t^k \|_2 = 0.$$

As for the second claim, it follows from Lemma 4.7, which gives

$$\forall t \ge 0, \quad \int_0^t z_r^k \cdot dn_r \ge 0,$$

for any integer $n \ge 0$, whenever z_t, for each $t \ge 0$, is symmetric and non-increasing. □

Remark 4.9. A useful observation is that (notice the presence of the factor 2 in the exponential below)

$$\int_0^t e^{2\varepsilon\Delta} z_r \cdot dn_r$$

can be written as

$$\begin{aligned} \int_0^t e^{2\varepsilon\Delta} z_r \cdot dn_r &= \int_0^t e^{\varepsilon\Delta} z_r \cdot d\big(e^{\varepsilon\Delta} n_r \big) \\ &= \lim_{M\to\infty} \sum_{m=0}^{M} \int_0^t \big\langle e^{\varepsilon\Delta} z_r, e_m \big\rangle d\langle e^{\varepsilon\Delta} \eta_r, e_m \rangle \\ &= \lim_{M\to\infty} \sum_{m=0}^{M} \int_0^t \big\langle e^{\varepsilon\Delta} z_r, e_m \big\rangle d\langle \eta_r, e^{\varepsilon\Delta} e_m \rangle, \end{aligned}$$

where it must be stressed that $(e^{\varepsilon\Delta}n_t)_{t\geq 0}$ on the first line satisfies (**E1**) and (**E2**). While (**E1**) follows from the contractive properties of the heat semigroup, (**E2**) follows from Lemma 2.10 (in words, $e^{\varepsilon\Delta}u$ is symmetric non-increasing if $u \in L^2(\mathbb{S})$ is symmetric non-increasing).

A sketch proof of these identities is as follows. By Corollary 4.8 (and with the same notation as therein), we can write

$$\int_0^t e^{2\varepsilon\Delta} z_r \cdot dn_r = \lim_{k\to\infty} \int_0^t e^{2\varepsilon\Delta} z_r^k \cdot dn_r.$$

Then, Lemma 4.7 allows one to write the right-hand side as a Riemann sum. The proof is then completed by expanding the terms in the Riemann sum in Fourier coefficients, exactly as in the proof of Lemma 4.7.

Remark 4.10. Definition 4.5 supplies us with the integral

$$\left(\int_0^t e^{\varepsilon\Delta} z_r \cdot dn_r\right)_{t\geq 0},$$

when $(z_t)_{t\geq 0}$ is a deterministic continuous path with values in $L^2_{\mathrm{sym}}(\mathbb{S})$ and $(n_t)_{t\geq 0}$ satisfies (**E1**) and (**E2**).

Importantly, one can easily replace $(z_t)_{t\geq 0}$ by the realisation of a (stochastic) continuous process $(Z_t)_{t\geq 0}$ with values in $L^2_{\mathrm{sym}}(\mathbb{S})$ and $(n_t)_{t\geq 0}$ by the same stochastic process as in Proposition 4.1. The resulting integral process is denoted

$$\left(\int_0^t e^{\varepsilon\Delta} Z_r \cdot d\eta_r\right)_{t\geq 0}.$$

It is continuous in time.

When $(Z_t)_{t\geq 0}$ is adapted to the filtration $\mathbb{F}$ used in the statement of Proposition 4.1, the integral is also adapted to $\mathbb{F}$: this is a consequence of Lemma 4.7 and Corollary 4.8.

4.3 Orthogonality of the reflection

We now come to the last property in the description of the weak limits:

Proposition 4.11. *Let $(X_t, W_t)_{t\geq 0}$ be a weak limit of the processes $\{(\tilde{X}_t^h, W_t)_{t\geq 0}\}_{h>0}$ as h tends to 0, as given by Proposition 4.1. Then, for any $t \geq s \geq 0$,*

$$\lim_{\varepsilon\searrow 0} \mathbb{E}\left[\int_s^t e^{\varepsilon\Delta} X_r \cdot d\eta_r\right] = 0, \tag{4.28}$$

To appreciate the scope of the above statement, the reader should recall that $(X_t)_{t\geq 0}$ takes symmetric non-increasing values. Therefore, Corollary 4.8 yields, almost surely,

$$\int_s^t e^{\varepsilon\Delta} X_r \cdot d\eta_r \geq 0,$$

for any $t \geq s \geq 0$. In particular, Fatou's lemma (together with the time continuity of the integral) implies that, with probability 1, for any $t \geq s \geq 0$,

$$\liminf_{\varepsilon \searrow 0} \int_s^t e^{\varepsilon\Delta} X_r \cdot d\eta_r = 0.$$

We regard this property as a (weak) form of orthogonality between X_r and $d\eta_r$, recalling that the orthogonality property is standard in reflected equations (see for instance the seminal work [50]).

Moreover, we obtain the following corollary as an important by-product of the proof.

Corollary 4.12. *Let $(X_t, W_t)_{t\geq 0}$ be a weak limit of the processes $\{(\tilde{X}_t^h, W_t)_{t\geq 0}\}_{h>0}$ as h tends to 0, as given by Proposition 4.1. Then, for any $T > 0$, with probability 1, the process $(X_t)_{0\leq t\leq T}$ takes values in $L^2([0,T], H^1_{\mathrm{sym}}(\mathbb{S}))$ and*

$$\mathbb{E} \int_0^T \|DX_t\|_2^2 dt < \infty.$$

Proof of Proposition 4.11 and Corollary 4.12.

First step. We first prove Corollary 4.12 (which is not a direct corollary of Proposition 4.11, but which comes as a consequence of the global architecture of the proof). In order to do so, we return to the scheme (3.3). For a given $h \in (0,1]$ and for any integer $n \geq 0$,

$$\begin{aligned} \mathbb{E}\big[\|X_{n+1}^h\|_2^2\big] &\leq \mathbb{E}\left[\left\| e^{h\Delta} X_n^h + \int_0^h e^{(h-s)\Delta} dW_s^{n+1} \right\|_2^2\right] \\ &= \mathbb{E}\big[\|e^{h\Delta} X_n^h\|_2^2\big] + \mathbb{E}\left[\left\| \int_0^h e^{(h-s)\Delta} dW_s^{n+1} \right\|_2^2\right]. \end{aligned} \tag{4.29}$$

Then, using the fact that $(e^{t\Delta} X_n^h)_{0\leq t\leq h}$ solves the heat equation, one has the equality

$$\mathbb{E}\left[\|e^{h\Delta} X_n^h\|_2^2 + 2\int_0^h \|De^{s\Delta} X_n^h\|_2^2 ds\right] = \mathbb{E}\big[\|X_n^h\|_2^2\big]. \tag{4.30}$$

Considering $h \leq \varepsilon$ for some $\varepsilon \in (0,1)$, we have $\mathbb{E}[\|e^{\varepsilon\Delta} DX_n^h\|_2^2] \leq \mathbb{E}[\|De^{s\Delta} X_n^h\|_2^2]$ for $s \in (0,h]$. Combining this with equations (4.29), (4.30), and Lemma 3.1 (for the definition of $c_{1,\lambda}$), we get

$$\mathbb{E}\big[\|X_{n+1}^h\|_2^2\big] - \mathbb{E}\big[\|X_n^h\|_2^2\big] + 2h\mathbb{E}\left[\|De^{\varepsilon\Delta} X_n^h\|_2^2\right] \leq c_{1,\lambda} h.$$

And then, for $t \geq s \geq 0$ and $N_t := h\lfloor t/h \rfloor$ and $N_s := h\lfloor s/h \rfloor$, we have

$$\mathbb{E}\big[\|X_{N_t}^h\|_2^2\big] - \mathbb{E}\big[\|X_{N_s}^h\|_2^2\big] + 2h \sum_{n=N_s}^{N_t-1} \mathbb{E}\left[\|De^{\varepsilon\Delta} X_n^h\|_2^2\right] \leq c_{1,\lambda} h(N_t - N_s).$$

Recalling the notation (3.25) and combining tightness of the family $\{(\tilde{X}_t^h)_{t\geq 0}\}_{h\in(0,1]}$ with Corollary 3.5 (which supplies us with uniform integrability properties), we obtain

$$\mathbb{E}\big[\|\tilde{X}_t^h\|_2^2\big] - \mathbb{E}\big[\|\tilde{X}_s^h\|_2^2\big] + 2\int_s^t \mathbb{E}\left[\|De^{\varepsilon\Delta} \tilde{X}_r^h\|_2^2\right] dr \leq c_{1,\lambda}(t-s) + \mathcal{O}_h(1), \tag{4.31}$$

with $\lim_{h\searrow 0}\mathcal{O}_h(1) = 0$ (the rate possibly depending on ε). Noticing that the function $z \mapsto D[e^{\varepsilon\Delta}z]$ is continuous from $L^2(\mathbb{S})$ into itself, we can easily take some weak limit as in the statement (as h tends to 0). We get

$$\mathbb{E}\big[\|X_t\|_2^2\big] - \mathbb{E}\big[\|X_s\|_2^2\big] + 2\int_s^t \mathbb{E}\big[\|De^{\varepsilon\Delta}X_r\|_2^2\big]\,dr \le c_{1,\lambda}(t-s). \tag{4.32}$$

Since the constant c is independent of ε, this establishes Corollary 4.12 (recalling that closed balls of $H^1(\mathbb{S})$ are closed for the L^2 norm $\|\cdot\|_2$).

Second step. For the next part, return to equation (4.2), with $u \in H^2_{\mathrm{sym}}(\mathbb{S})$ replaced by $e^{\varepsilon\Delta}u$,

$$\langle X_t - X_s, e^{\varepsilon\Delta}u\rangle - \int_s^t \langle X_s, \Delta e^{\varepsilon\Delta}u\rangle ds = \langle W_t - W_s, e^{\varepsilon\Delta}u\rangle + \langle \eta_t - \eta_s, e^{\varepsilon\Delta}u\rangle. \tag{4.33}$$

The next step is to choose $u = e_m$ and then to apply Itô's formula in order to expand $(\langle X_t, e^{\varepsilon\Delta}e_m\rangle^2)_{t\ge 0}$. To do so, it is worth recalling from (4.22) that since e_m may be written as the difference of two symmetric decreasing functions e_m^+ and e_m^-, the process $(\langle \eta_t, e^{\varepsilon\Delta}e_m\rangle)_{t\ge 0}$ may be written as the difference of two non-decreasing processes and consequently has finite variation. Therefore, due to Itô's formula, one has

$$\begin{aligned} d\langle X_t, e^{\varepsilon\Delta}e_m\rangle^2 =& 2\langle e^{\varepsilon\Delta}X_t, e_m\rangle\langle \Delta e^{\varepsilon\Delta}X_t, e_m\rangle dt + 2\langle X_t, e^{\varepsilon\Delta}e_m\rangle d\langle \eta_t, e^{\varepsilon\Delta}e_m\rangle \\ &+ 2\langle X_t, e^{\varepsilon\Delta}e_m\rangle d\langle W_t, e^{\varepsilon\Delta}e_m\rangle + d\big[\langle W_\cdot, e^{\varepsilon\Delta}e_m\rangle\big]_t, \end{aligned} \tag{4.34}$$

where, as before, the symbol $[\cdot]_t$ is used to denote the bracket.

Integrating between 0 and t, applying expectation, and then summing over $m \in \mathbb{N}_0$,

$$\begin{aligned} \mathbb{E}\big[\|e^{\varepsilon\Delta}X_t\|_2^2\big] &= \mathbb{E}\big[\|e^{\varepsilon\Delta}X_0\|_2^2\big] - 2\int_0^t \mathbb{E}\big[\|De^{\varepsilon\Delta}X_r\|_2^2\big]dr \\ &\quad + 2\mathbb{E}\Bigg[\sum_{m\in\mathbb{N}_0}\int_0^t \langle e^{\varepsilon\Delta}X_r, e_m\rangle d\langle \eta_r, e^{\varepsilon\Delta}e_m\rangle\Bigg] + \sum_{m\in\mathbb{N}_0}\mathbb{E}\big[\langle W_t, e^{\varepsilon\Delta}e_m\rangle^2\big] \\ &= \mathbb{E}\big[\|e^{\varepsilon\Delta}X_0\|_2^2\big] - 2\int_0^t \mathbb{E}\big[\|De^{\varepsilon\Delta}X_r\|_2^2\big]dr \\ &\quad + 2\mathbb{E}\Bigg[\sum_{m\in\mathbb{N}_0}\int_0^t \langle e^{\varepsilon\Delta}X_r, e_m\rangle d\langle \eta_r, e^{\varepsilon\Delta}e_m\rangle\Bigg] + \mathbb{E}\big[\|e^{\varepsilon\Delta}W_t\|_2^2\big]. \end{aligned} \tag{4.35}$$

Recalling Definition 4.5 and invoking Remark 4.9, one arrives at the equality:

$$\begin{aligned} \mathbb{E}\big[\|e^{\varepsilon\Delta}X_t\|_2^2\big] + 2\int_0^t \mathbb{E}\big[\|De^{\varepsilon\Delta}X_r\|_2^2\big]dr =& \mathbb{E}\big[\|e^{\varepsilon\Delta}X_0\|_2^2\big] + \mathbb{E}\big[\|e^{\varepsilon\Delta}W_t\|_2^2\big] \\ &+ 2\mathbb{E}\Bigg[\int_0^t e^{2\varepsilon\Delta}X_r\cdot d\eta_r\Bigg]. \end{aligned} \tag{4.36}$$

Combining with the inequality (4.32), recalling (3.11) and passing to the limit as $\varepsilon \to 0$, this implies that

$$\begin{aligned} \lim_{\varepsilon\searrow 0}\mathbb{E}\Bigg[\int_0^t e^{2\varepsilon\Delta}X_r\cdot d\eta_r\Bigg] &= \mathbb{E}\big[\|X_t\|_2^2\big] + 2\int_0^t \mathbb{E}\big[\|DX_r\|_2^2\big]dr - \mathbb{E}\big[\|X_0\|_2^2\big] - \mathbb{E}\big[\|W_t\|_2^2\big] \\ &\le c_{1,\lambda}t - \mathbb{E}\big[\|W_t\|_2^2\big] = 0, \end{aligned}$$

which completes the proof. □

4.4 Definition and uniqueness of solutions to the rearranged SHE

Based on the analysis performed so far, we have all the required ingredients to define a solution to the rearranged SHE studied in this paper.

Definition 4.13. *On a given (filtered) probability space $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P})$ equipped with a Q-Brownian motion $(W_t)_{t\geq 0}$ with values in $L^2_{\mathrm{sym}}(\mathbb{S})$ (with respect to the filtration $\mathbb{F}$) and with an $\mathcal{F}_0$-measurable initial condition X_0 with values in $U^2(\mathbb{S})$ (see (2.2)) and with finite moments of any order (see (3.9)), we say that a pair of processes $(X_t, \eta_t)_{t\geq 0}$ solves the rearranged SHE (4.1) driven by $(W_t)_{t\geq 0}$ and X_0 if*

1. *$(X_t)_{t\geq 0}$ is a continuous $\mathbb{F}$-adapted process with values in $U^2(\mathbb{S})$;*
2. *$(\eta_t)_{t\geq 0}$ is a continuous $\mathbb{F}$-adapted process with values in $H^{-2}_{\mathrm{sym}}(\mathbb{S})$, starting from 0 at 0, such that, with probability 1, for any $u \in H^2_{\mathrm{sym}}(\mathbb{S})$ that is non-increasing, the path $(\langle \eta_t, u\rangle)_{t\geq 0}$ is non-decreasing;*
3. *with probability 1, for any $u \in H^2_{\mathrm{sym}}(\mathbb{S})$,*
$$\forall t \geq 0, \quad \langle X_t, u\rangle = \int_0^t \langle X_r, \Delta u\rangle dr + \langle W_t, u\rangle + \langle \eta_t, u\rangle. \tag{4.37}$$
4. *for any $t \geq 0$,*
$$\lim_{\varepsilon \searrow 0} \mathbb{E}\left[\int_0^t e^{\varepsilon\Delta} X_r \cdot d\eta_r\right] = 0. \tag{4.38}$$

Of course, the definition of the integral in (4.38) is understood as in Definition 4.5.

The next result says that, given the filtered probability space, the rearranged SHE has at most one solution.

Proposition 4.14. *Given $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P})$ equipped with a Q-Brownian motion $(W_t)_{t\geq 0}$ with values in $L^2_{\mathrm{sym}}(\mathbb{S})$ (with respect to the filtration $\mathbb{F}$) and with an $\mathcal{F}_0$-measurable initial condition X_0 with values in $U^2(\mathbb{S})$ and with finite moments of any order, there exists at most one solution $(X_t, \eta_t)_{t\geq 0}$ to the rearranged SHE (4.1) that satisfies Definition 4.13.*

Proof of Proposition 4.14. Consider two candidate solutions $(X^1_t, \eta^1_t)_{t\geq 0}$ and $(X^2_t, \eta^2_t)_{t\geq 0}$. Then, by application of Itô's formula, for any $\varepsilon > 0$, $\mathbb{P}$-a.s., for any $m \in \mathbb{N}_0$,

$$\begin{aligned} &d\langle e^{\varepsilon\Delta}(X^1_t - X^2_t), e_m\rangle^2 \\ &= 2\langle e^{\varepsilon\Delta}(X^1_t - X^2_t), e_m\rangle \left[\langle \Delta e^{\varepsilon\Delta}(X^1_t - X^2_t), e_m\rangle dt + d\langle \eta^1_t - \eta^2_t, e^{\varepsilon\Delta} e_m\rangle\right], \quad t \geq 0. \end{aligned} \tag{4.39}$$

Writing the above in integral form and summing over $m \in \mathbb{N}_0$, one obtains ($\mathbb{P}$-a.s)

$$
\begin{aligned}
&\|e^{\varepsilon\Delta}(X_t^1 - X_t^2)\|_2^2 \\
&\quad = 2\int_0^t \langle e^{\varepsilon\Delta}(X_r^1 - X_r^2), \Delta e^{\varepsilon\Delta}(X_r^1 - X_r^2)\rangle dr + 2\int_0^t e^{2\varepsilon\Delta}(X_r^1 - X_r^2)\cdot d(\eta_r^1 - \eta_r^2) \\
&\quad = -2\int_0^t \|De^{\varepsilon\Delta}(X_r^1 - X_r^2)\|_2^2 dr + 2\int_0^t e^{2\varepsilon\Delta}(X_r^1 - X_r^2)\cdot d\eta_r^1 \\
&\qquad - 2\int_0^t e^{2\varepsilon\Delta}(X_r^1 - X_r^2)\cdot d\eta_r^2 \\
&\quad \leq 2\left(\int_0^t e^{2\varepsilon\Delta}X_r^1 \cdot d\eta_r^1 + \int_0^t e^{2\varepsilon\Delta}X_r^2 \cdot d\eta_r^2\right),
\end{aligned}
\tag{4.40}
$$

where we used Corollary 4.8 to establish the last inequality.

Applying expectation and setting $\varepsilon \to 0$, we obtain from (4.38) that $X_t^1 = X_t^2$, $\mathbb{P}-a.s.$ for any $t \geq 0$. As X^1 and X^2 are modifications of each other and have continuous sample paths, they are consequently indistinguishable. Since η^1 and η^2 are defined via the processes X^1 and X^2 respectively, there exists a unique solution to (4.1). □

Proposition 4.14 could be reformulated as follows: strong uniqueness holds true for the rearranged SHE. By combining Propositions 4.1 and 4.14, we deduce from an obvious adaptation of Yamada-Watanabe argument, the following theorem, which is one the main two results of the paper:

Theorem 4.15. *Given $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P})$ equipped with a Q-Brownian motion $(W_t)_{t\geq 0}$ with values in $L^2_{\mathrm{sym}}(\mathbb{S})$ (with respect to the filtration $\mathbb{F}$) and with an $\mathcal{F}_0$-measurable initial condition X_0 with values in $L^2_{\mathrm{sym}}(\mathbb{S})$, there exists a unique solution $(X_t, \eta_t)_{t\geq 0}$ to the rearranged SHE (4.1) that satisfies Definition 4.13.*

Moreover, the processes $\{(\tilde{X}_t^h, W_t)_{t\geq 0}\}_{h>0}$, as defined in (3.3), are convergent in law (over $\mathcal{C}([0,\infty), L^2(\mathbb{S}))$ equipped with the topology of uniform convergence on compact subsets) and the limit is the law of $(X_t, W_t)_{t\geq 0}$.

Notice that, as in Proposition 4.14, we use the same noise for the scheme and for the limiting equation. However, in contrast to Proposition 4.14, there is no abuse in doing so: the first part of the statement allows us to construct the solution to the rearranged equation on the same filtered probability space (equipped with the same noise) as the scheme.

It shall prove useful to note that item 4 in Definition 4.13 may be strengthened into

Proposition 4.16. *Let $(X_t, \eta_t)_{t\geq 0}$ satisfy Definition 4.13 except item 4 therein. Then, item 4 holds true if and only if one of the following two properties below is satisfied:*

4' For any $t > 0$, there exists a sequence of positive reals $(\varepsilon_q)_{q\geq 1}$, with 0 as limit, such that, in $\mathbb{P}$-probability,

$$\lim_{q\to\infty} \int_0^t e^{\varepsilon_q\Delta} X_r \cdot d\eta_r = 0.$$

4" It holds that, for any $p \geq 1$, for any $t > 0$,

$$\lim_{\varepsilon \searrow 0} \mathbb{E}\left[\left(\int_0^t e^{\varepsilon\Delta} X_r \cdot d\eta_r\right)^p\right] = 0.$$

Proof. We proceed as follows. We consider a process $(X_t, \eta_t)_{t\geq 0}$ satisfying Definition 4.13 except item *4* therein. Obvious *4"* implies *4*, which implies in turn *4'*. The only difficulty is to prove that *4'* implies *4"*.

Assuming *4'*, we recall that, by construction, the argument inside the power function in *4"* is non-negative. We then show that, for any $t \geq 0$,

$$\forall p \geq 1, \quad \sup_{\varepsilon > 0} \mathbb{E}\left[\left(\int_0^t e^{\varepsilon\Delta} X_r \cdot d\eta_r\right)^p\right] < \infty. \tag{4.41}$$

We restart from (4.34) and we follow the derivation of (4.35) and (4.36), but without taking the expectation therein. For a given $\varepsilon > 0$, we get, with probability 1, for any $t \geq 0$,

$$\begin{aligned} \|e^{\varepsilon\Delta} X_t\|_2^2 + 2\int_0^t \|De^{\varepsilon\Delta} X_r\|_2^2 dr =& \|e^{\varepsilon\Delta} X_0\|_2^2 + \|e^{\varepsilon\Delta} W_t\|_2^2 + 2\int_0^t e^{2\varepsilon\Delta} X_r \cdot d\eta_r \\ &+ 2\sum_{m\in\mathbb{N}_0} \int_0^t \langle X_s, e^{\varepsilon\Delta} e_m\rangle d\langle W_s, e^{\varepsilon\Delta} e_m\rangle. \end{aligned} \tag{4.42}$$

Recall from the contractivity of the heat semigroup that the function

$$\varepsilon \in (0, \infty) \mapsto \int_0^t \|De^{\varepsilon\Delta} X_r\|_2^2 dr$$

is non-increasing (for any given realisation). Therefore, choosing $\varepsilon = \varepsilon_q/2$ for some $q \in \mathbb{N}_0$ in (4.42) and then taking the limit (in probability) as q tends to ∞, we deduce from item *4'* that

$$\|X_t\|_2^2 + 2\sup_{\varepsilon>0} \int_0^t \|e^{\varepsilon\Delta} DX_r\|_2^2 dr = \|X_0\|_2^2 + \|W_t\|_2^2 + 2\int_0^t X_s \cdot dW_s.$$

By taking power p and then recalling (3.11) and Proposition 4.3 together with the fact that, as a corollary to Proposition 3.7, $\|X_t\|_2$ has finite moments of any order for any t, we easily deduce from that

$$\mathbb{E}\left[\left(\sup_{\varepsilon>0} \int_0^t \|De^{\varepsilon\Delta} X_r\|_2^2\right)^p\right] < \infty. \tag{4.43}$$

Back to (4.42), we can express $\int_0^t e^{2\varepsilon\Delta} X_r \cdot d\eta_r$ in terms of all the other terms. Using the Burkolder-Davis-Gundy inequality in order to handle the stochastic integral therein, the claim (4.41) easily follows.

Now that we have (4.41), it suffices to prove item 4" with $p = 1$. The result for $p > 1$ then follows from a standard uniform integrability argument. In fact, by combining (4.41) and item 4', we already know that, for any $t > 0$,

$$\lim_{q\to\infty} \mathbb{E}\Big[\int_0^t e^{\varepsilon_q \Delta} X_r \cdot d\eta_r\Big] = 0. \tag{4.44}$$

It remains to observe that the limit $\lim_{\varepsilon\searrow 0} \mathbb{E}\int_0^t e^{\varepsilon\Delta} X_r \cdot d\eta_r$ exists necessarily. Indeed, taking expectation in (4.42), we observe that

$$\lim_{\varepsilon\searrow 0} \mathbb{E}\Big[\int_0^t e^{\varepsilon\Delta} X_r \cdot d\eta_r\Big] = \mathbb{E}\big[\|X_t\|_2^2\big] + 2\sup_{\varepsilon>0} \mathbb{E}\Big[\int_0^t \|De^{\varepsilon\Delta} X_r\|_2^2 dr\Big] - \mathbb{E}\big[\|X_0\|_2^2\big] - \mathbb{E}\big[\|W_t\|_2^2\big],$$

where we used again the contractivity of the heat semigroup to write

$$\lim_{\varepsilon\to 0} \mathbb{E}\Big[\int_0^t \|De^{\varepsilon\Delta} X_r\|_2^2 dr\Big] = \sup_{\varepsilon>0} \mathbb{E}\Big[\int_0^t \|De^{\varepsilon\Delta} X_r\|_2^2 dr\Big].$$

The proof is complete. □

Remark 4.17. The reader may wonder whether, in the proof of Proposition 4.16, we can write

$$\sup_{\varepsilon>0} \mathbb{E}\int_0^t \|De^{\varepsilon\Delta} X_r\|_2^2 dr = \mathbb{E}\int_0^t \|DX_r\|_2^2 dr.$$

Whilst it could be proved directly by weak compactness arguments in $L^2([0,t]\times\mathbb{S})$ that the right-hand side is finite and thus coincides with the left, it is worth observing that we recover the result from our earlier construction of a solution in Corollary 4.12.

4.5 Lipschitz regularity of the flow

We conclude this section with the following result, which is core to the rest of the paper.

Proposition 4.18. *Given $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P})$ equipped with a Q-Brownian motion $(W_t)_{t\geq 0}$ with values in $L^2_{\mathrm{sym}}(\mathbb{S})$ (with respect to the filtration $\mathbb{F}$), consider (X^x, η^x) and (X^y, η^y) the solutions to the rearranged SHE (4.1) with $X_0^x = x \in U^2(\mathbb{S})$ and $X_0^y = y \in U^2(\mathbb{S})$ as respective initial conditions. Then, $\mathbb{P}$-almost surely,*

$$\|X_t^x - X_t^y\|_2^2 + 2\int_0^t \|De^{\varepsilon\Delta}(X_r^x - X_r^y)\|_2^2 dr \leq \|x-y\|_2^2, \quad t \geq 0. \tag{4.45}$$

Proof. The proof is quite similar to that of uniqueness and so we only give a sketch. For

$\varepsilon > 0$, with probability 1, for any $t \geq 0$

$$
\begin{aligned}
&\sum_{m\in\mathbb{N}_0} \langle e^{\varepsilon\Delta}(X_t^x - X_t^y), e_m\rangle^2 - \langle e^{\varepsilon\Delta}(x-y), e_m\rangle^2 \\
&= 2\sum_{m\in\mathbb{N}_0} \langle e^{\varepsilon\Delta}(X_t^x - X_t^y), e_m\rangle \left[\langle \Delta e^{\varepsilon\Delta}(X_t^x - X_t^y), e_m\rangle dt + d\langle \eta_t^x - \eta_t^y, e^{\varepsilon\Delta} e_m\rangle\right] \\
&= -2\int_0^t \|De^{\varepsilon\Delta}(X_r^x - X_r^y)\|_2^2 dr + 2\int_0^t e^{2\varepsilon\Delta}(X_r^x - X_r^y)\cdot d\eta_r^x \\
&\quad - 2\int_0^t e^{2\varepsilon\Delta}(X_r^x - X_r^y)\cdot d\eta_r^y \\
&\leq 2\left(\int_0^t e^{2\varepsilon\Delta} X_r^x \cdot d\eta_r^x + \int_0^t e^{2\varepsilon\Delta} X_r^y \cdot d\eta_r^y\right).
\end{aligned}
\tag{4.46}
$$

Following the discussion immediately after Proposition 4.11, we know that

$$
\liminf_{\varepsilon\searrow 0} \int_0^t e^{2\varepsilon\Delta} X_r^x \cdot d\eta_r^x = 0.
$$

In fact, the same argument (based on Fatou's lemma) shows that

$$
\liminf_{\varepsilon\searrow 0}\left(\int_0^t e^{2\varepsilon\Delta} X_r^x \cdot d\eta_r^x + \int_0^t e^{2\varepsilon\Delta} X_r^y \cdot d\eta_r^y\right) = 0.
$$

Taking the infimum limit in (4.46) and recalling Corollary 4.12, we easily get the result. □

As an obvious (but very useful) consequence of Proposition 4.18, we have, for all $T > 0$,

$$
\sup_{0\leq t\leq T} \left\|\eta_t^x - \eta_t^y\right\|_{2,-2} \leq c_T \|x-y\|_2, \tag{4.47}
$$

for a constant c_T only depending on T. The proof follows from the identity (4.37).

Remark 4.19. Inequalities (4.45) and (4.47) say that, for each $x \in L^2_{\mathrm{sym}}(\mathbb{S})$, we can find versions of $(X_t^x)_{t\geq 0}$ and $(\eta_t^x)_t \geq 0$ such that, for any $T > 0$, the mappings $(\omega, t, x) \in \Omega \times [0,T] \times L^2_{\mathrm{sym}}(\mathbb{S}) \mapsto X_t^x(\omega) \in L^2_{\mathrm{sym}}(\mathbb{S})$ and $(\omega, t, x) \in \Omega \times [0,T] \times L^2_{\mathrm{sym}}(\mathbb{S}) \mapsto \eta_t^x(\omega) \in H^{-2}_{\mathrm{sym}}(\mathbb{S})$ are measurable, continuous in t and Lipschitz in x (with 1 and c_T as respective Lipschitz constants).

5 Smoothing Effect

Due to the forcing term η, one does not expect to easily obtain the differentiability with respect to initial condition of the solution; see however, the references [3, 4, 29, 52, 53] cited in Subsection 1.3 for positive results in this direction but in finite dimension.

However, as will be demonstrated below, it is possible to show that the semigroup flow $\{P_t\}_{t\geq 0}$ - defined by

$$
P_t f(x) := \mathbb{E}[f(X_t^x)], \quad t \geq 0,\ x \in U^2(\mathbb{S}),
$$

for f within the set of bounded measurable functions on $U^2(\mathbb{S})$ - maps bounded measurable functions into Lipschitz continuous functions on $U^2(\mathbb{S})$, at least when the parameter λ in (3.1) belongs to $(1/2,1)$. Importantly, we prove that the rate at which the Lipschitz constant of $P_t f$ blows up as t decreases to 0 is integrable, see Theorem 5.11 below together with Remark 5.12 for possible applications to infinite dimensional PDEs.

We first consider a finite dimensional reduction of the problem. For a given truncation level $M \in \mathbb{N}_0$ and for any v in $L^2_{\mathrm{sym}}(\mathbb{S})$, we let

$$v^M := \sum_{m=0}^{M} \langle v, e_m \rangle e_m(\cdot), \quad v^{*,M} := \left(\sum_{m=0}^{M} \langle v, e_m \rangle e_m(\cdot) \right)^*.$$

Clearly, $v^{*,M}$ is an element of $U^2(\mathbb{S})$ parametrised by a finite number of parameters, namely by the Fourier modes $(\langle v, e_m \rangle)_{m=0,\cdots,N}$. We let $E^M := \{v^M,\ v \in L^2_{\mathrm{sym}}(\mathbb{S})\}$ and $E^{*,M} := \{v^{*,M},\ v \in L^2_{\mathrm{sym}}(\mathbb{S})\}$. Obviously, $E^M \cong \mathbb{R}^{M+1}$. The point is then to prove that, for any $t > 0$, the mapping

$$x \in E^M \mapsto P_t f\big(x^{*,M}\big),$$

is Lipschitz continuous, with a Lipschitz constant that is independent of M. We then get the expected result by taking the limit as M tends to ∞. Reducing the dimensionality allows us to use many tools from finite dimensional analysis, notably Rademacher's theorem which we invoke quite often in this section. Together with Proposition 4.18, the latter says that, for a given $t > 0$, the flow

$$x \in E^M \mapsto X_t^{x^*}$$

is almost everywhere (for the Lebesgue measure on E^M) differentiable. This is a way to circumvent the issues of trying to establish the (everywhere) differentiability of the flow, a property that is not clear to us at this stage.

The second step is then to consider, for $x, v \in L^2_{\mathrm{sym}}(\mathbb{S})$, $\delta \in \mathbb{R}$ and $T > 0$, the difference

$$P_T f\big((x+\delta v)^{*,M}\big) - P_T f\big(x^{*,M}\big) = \mathbb{E}\Big[f\big(X_T^{(x+\delta v)^{*,M}}\big)\Big] - \mathbb{E}\Big[f\big(X_T^{x^{*,M}}\big)\Big], \tag{5.1}$$

and then to represent it *via* use of a Girsanov transformation. This adapts earlier arguments from Malliavin calculus, see for instance Bismut [12] and Norris [57], and from the proof of the so-called Bismut-Elworthy-Li formula, see Elworthy and Li [35] and Thalmaier [72].

The key idea is to consider the *shifted process*

$$\Big(X_t^{(x+\delta \frac{T-t}{T} v)^{*,M}}\Big)_{0 \le t \le T},$$

which satisfies $X_0^{(x+\delta \frac{T-t}{T} v)^{*,M}} = (x+\delta v)^{*,M}$ and $X_T^{(x+\delta \frac{T-t}{T} v)^{*,M}} = X_T^{x^{*,M}}$. It is then shown that, under a particular change of measure, the shifted process is the unique solution to the rearranged SHE started from $(x+\delta v)^{*,M}$. To make sure that the change of measure satisfies the standard conditions of Girsanov transformation, we need however to localise

the dynamics in a suitable manner and in particular to enact the *shifting* up to a stopping time, the form of which is clarified in definition (5.26).

Throughout the section, we use the convenient notation

$$\begin{aligned} y_t(v,\delta) &:= y + \delta\frac{T-t}{T}v, \\ y_t^*(v,\delta) &:= \Big(y + \delta\frac{T-t}{T}v\Big)^*, \quad t\in[0,T], \quad y,v\in E^M, \quad \delta\in\mathbb{R}. \end{aligned} \tag{5.2}$$

Pay attention that the time horizon T is implicitly understood (and thus omitted) in the two left-hand sides and that the initial condition y is manifested by the notations y_t and y_t^*. For another initial condition, say $x\in E^M$, we write $x_t(v,\delta)$ and $x_t^*(v,\delta)$. Moreover, many times below, we take derivatives with respect to the finite-dimensional variable $y\in E^M$. The gradient is denoted ∂_y. Notice also that for $y\in E^M$, both y and y^* can be regarded as elements of $L^2_{\mathrm{sym}}(\mathbb{S})$. By Lemma 2.4, $\|y^*\|_2 = \|y\|_2$ and, by Parseval identity, $\|y\|_2 = |y|$ where $|\cdot|$ stands for the standard Euclidean norm on E^M.

5.1 Shifted state process and tilted reflection process

In this subsection, we address the dynamics of the shifted state process $(X_t^{y_t^*(v,\delta)})_{0\le t\le T}$ and the corresponding forcing term $(\eta_t^{y_t^*(v,\delta)})_{0\le t\le T}$, for y and v in E^M and $\delta\in\mathbb{R}$. Until further notice, the number M of low frequency modes to which the initial condition is truncated is considered as fixed and consequently, the indexing with respect to M is omitted.

To identify the dynamics, one begins by studying the time evolution of the Fourier modes of the shifted process integrated against a test function $\varphi\in\mathcal{C}_0^\infty(E^M)$. To this end, we expand

$$\Big(\int_{E^M}\big\langle e_m, X_t^{y_t^*(v,\delta)}\big\rangle\varphi(y)dy\Big)_{0\le t\le T},$$

with $\langle e_m, X_t^{y_t^*(v,\delta)}\rangle$ standing for the mth Fourier mode of $X_t^{y_t^*(v,\delta)}$. Changing variables and recalling the dynamics (4.2), we have

$$\begin{aligned} &\int_{E^M}\big\langle X_t^{y_t^*(v,\delta)}, e_m\big\rangle\varphi(y)dy \\ &= \int_{E^M}\big\langle X_t^{y^*}, e_m\big\rangle\varphi\big(y_t(v,-\delta)\big)dy \\ &= \int_{E^M}\big\langle X_0^{y^*}, e_m\big\rangle\varphi\big(y-\delta v\big)dy + \int_{E^M}\int_0^t\big\langle X_s^{y^*}, \Delta e_m\big\rangle\varphi\big(y_s(v,-\delta)\big)ds\,dy \\ &\quad + \int_{E^M}\Big(\int_0^t\varphi\big(y_s(v,-\delta)\big)d\big\langle\eta_s^{y^*}, e_m\big\rangle\Big)dy + \int_{E^M}\Big(\int_0^t\varphi\big(y_s(v,-\delta)\big)d\big\langle W_s, e_m\big\rangle\Big)dy \\ &\quad + \tfrac{\delta}{T}\int_{E^M}\int_0^t\big\langle X_s^{y^*}, e_m\big\rangle\partial_y\varphi\big(y_s(v,-\delta)\big)\cdot v\,ds\,dy, \end{aligned} \tag{5.3}$$

where $\partial_y\varphi(y_s(v,-\delta))\cdot v$ represents the gradient of φ in the direction of v at point $y_s(v,-\delta)$. Notice that the well-posedness of the second integral in the penultimate line is guaranteed by the stochastic version of Fubini's theorem stated in [24, Theorem 4.33].

Ideally, we would like to revert back the variables in the various integrals appearing in the expansion (5.3) and hence to compute the test function φ at the generic point y instead of $y_s(v,-\delta)$. The main difficulty in this regard is to perform the change of variable in the integral

$$\int_{E^M}\left(\int_0^t \varphi\big(y_s(v,-\delta)\big)d\langle\eta_s^{y^*},e_m\rangle\right)dy \tag{5.4}$$

driven by the reflection term $(\langle\eta_t^{y^*},e_m\rangle)_{0\le t\le T}$. Notice that the existence of the time integral in the right-hand side follows from the two bounds (4.27) (applied say with $y=0$) and (4.47), the latter ensuring in particular the measurability of the mapping $(\omega,t,y)\mapsto\langle\eta_t^{y^*}(\omega),e_m\rangle$. The combination of both guarantees that, with probability 1,

$$\forall R>0,\quad \sup_{0\le t\le T}\sup_{|y|\le R}\|\eta_t^{y^*}\|_{2,-2}\le\sup_{0\le t\le T}\|\eta_t^0\|_{2,-2}+c_TR<\infty, \tag{5.5}$$

for a constant c_T only depending on T.

Before formulating a convenient change of variables for (5.4) in the next subsection, we introduce in this subsection some ingredients that are essential in our analysis. This includes defining a so-called *tilted* version $(\tilde\eta_t^{y^*})_{0\le t\le T}$ of the reflection process $(\eta_t^{y^*})_{0\le t\le T}$.

In order to carry out the analysis, we proceed below as in Subsection 4.2 and formally replace (at least for the first result) the realisation of the field $((\eta_t^y)_{0\le t\le T})_{y\in E^{*,M}}$ (note that, in this notation, y is directly assumed to be in $E^{*,M}$) by a **deterministic** flow $((n_t^y)_{0\le t\le T})_{y\in E^{*,M}}$ satisfying the following two properties:

(**F1**) For any $y\in E^{*,M}$, the trajectory $t\in[0,T]\mapsto n_t^y$ satisfies (**E1**) and (**E2**) in Subsection 4.2, with the latter two conditions being restricted in an obvious manner to the interval $[0,T]$;

(**F2**) The flow satisfies the Lipschitz condition $\sup_{0\le t\le T}\|n_t^x-n_t^y\|_{2,-2}\le c_T|x-y|$, for any $x,y\in E^{*,M}$ and for a finite constant $c\ge 0$, which is equivalent to saying that $\sup_{0\le t\le T}\|n_t^{x^*}-n_t^{y^*}\|_{2,-2}\le c_T|x-y|$, for any $x,y\in E^M$.

The reader may reformulate (5.5) accordingly.

We start with the following lemma.

Lemma 5.1. *Let $((n_t^y)_{0\le t\le T})_{y\in E^{*,M}}$ satisfy (**F1**)–(**F2**) and $((z_t^y)_{0\le t\le T})_{y\in E^{*,M}}$ be a deterministic jointly continuous flow with values in $L^2_{\mathrm{sym}}(\mathbb{S})$, i.e., the map $(t,y)\in[0,T]\times E^{*,M}\mapsto z_t^y\in L^2_{\mathrm{sym}}(\mathbb{S})$ is continuous. Then, for any $\varepsilon>0$, for any family $\{r_i^N\}_{i=0,\cdots,N}$ of subdivision points of $[0,T]$ with stepsize converging to 0 as N tends to ∞, i.e.*

$$\lim_{N\to\infty}\sup_{i=1,\cdots,N}\left|r_i^N-r_{i-1}^N\right|=0,$$

the following identity holds true (for convenience we merely write r_i for r_i^N):

$$\begin{aligned}&\sum_{m\in\mathbb{N}_0}\int_{E^M}\left\{\int_0^T\left\langle e^{\varepsilon\Delta}z_t^{y^*},e_m\right\rangle\varphi\big(y_t(v,-\delta)\big)d\left\langle n_t^{y^*},e_m\right\rangle\right\}dy\\&=\lim_{N\to\infty}\int_{E^M}\varphi(y)\sum_{i=0}^{N-1}\left[\left\langle n_{r_{i+1}}^{y^*_{r_i}(v,\delta)},e^{\varepsilon\Delta}z_{r_i}^{y^*_{r_i}(v,\delta)}\right\rangle-\left\langle n_{r_i}^{y^*_{r_i}(v,\delta)},e^{\varepsilon\Delta}z_{r_i}^{y^*_{r_i}(v,\delta)}\right\rangle\right]dy.\end{aligned}\tag{5.6}$$

Proof. By Corollary 4.8, one has for a fixed value of $m\in\mathbb{N}_0$:

$$\begin{aligned}&\int_{E^M}\left\{\int_0^T\left\langle e^{\varepsilon\Delta}z_t^{y^*},e_m\right\rangle\varphi\big(y_t(v,-\delta)\big)d\left\langle n_t^{y^*},e_m\right\rangle\right\}dy\\&=\int_{E^M}\lim_{N\to\infty}\sum_{i=0}^{N-1}\left\langle e^{\varepsilon\Delta}z_{r_i}^{y^*},e_m\right\rangle\varphi\big(y_{r_i}(v,-\delta)\big)\left[\left\langle n_{r_{i+1}}^{y^*},e_m\right\rangle-\left\langle n_{r_i}^{y^*},e_m\right\rangle\right]dy\\&=\lim_{N\to\infty}\int_{E^M}\sum_{i=0}^{N-1}\left\langle e^{\varepsilon\Delta}z_{r_i}^{y^*},e_m\right\rangle\varphi\big(y_{r_i}(v,-\delta)\big)\left[\left\langle n_{r_{i+1}}^{y^*},e_m\right\rangle-\left\langle n_{r_i}^{y^*},e_m\right\rangle\right]dy,\end{aligned}$$

the argument for exchanging the limit and the sum directly following from Lebesgue's dominated convergence theorem. From Lemma 4.4, it is indeed clear that the sum over i on the second line is uniformly bounded in N. Therefore, performing for each $i\in\{0,\cdots,N-1\}$ an obvious change of variable for the integral in y, we get

$$\begin{aligned}&\int_{E^M}\left\{\int_0^T\left\langle e^{\varepsilon\Delta}z_t^{y^*},e_m\right\rangle\varphi\big(y_t(v,-\delta)\big)d\left\langle n_t^{y^*},e_m\right\rangle\right\}dy\\&=\lim_{N\to\infty}\int_{E^M}\varphi(y)\sum_{i=0}^{N-1}\left\langle e^{\varepsilon\Delta}z_{r_i}^{y^*_{r_i}(v,\delta)},e_m\right\rangle\left[\left\langle \eta_{r_{i+1}}^{y^*_{r_i}(v,\delta)},e_m\right\rangle-\left\langle \eta_{r_i}^{y^*_{r_i}(v,\delta)},e_m\right\rangle\right]dy.\end{aligned}$$

In fact, Lemma 4.4 says more: the argument inside the limit directly above decays polynomially fast with m, uniformly in N. In particular, summing over $m\in\mathbb{N}_0$, one can exchange the sum over m and the limit over N. We obtain:

$$\begin{aligned}&\sum_{m\in\mathbb{N}_0}\left\{\int_{E^M}\int_0^T\left\langle e^{\varepsilon\Delta}z_t^{y^*},e_m\right\rangle\varphi\big(y_t(v,-\delta)\big)d\left\langle \eta_t^{y^*},e_m\right\rangle\right\}dy\\&=\lim_{N\to\infty}\int_{E^M}\varphi(y)\sum_{i=0}^{N-1}\sum_{m\in\mathbb{N}_0}\left\langle e^{\varepsilon\Delta}z_{r_i}^{y^*_{r_i}(v,\delta)},e_m\right\rangle\left[\left\langle \eta_{r_{i+1}}^{y^*_{r_i}(v,\delta)},e_m\right\rangle-\left\langle \eta_{r_i}^{y^*_{r_i}(v,\delta)},e_m\right\rangle\right]dy\\&=\lim_{N\to\infty}\int_{E^M}\varphi(y)\sum_{i=0}^{N-1}\left[\left\langle e^{\varepsilon\Delta}z_{r_i}^{y^*_{r_i}(v,\delta)},\eta_{r_{i+1}}^{y^*_{r_i}(v,\delta)}\right\rangle-\left\langle e^{\varepsilon\Delta}z_{r_i}^{y^*_{r_i}(v,\delta)},\eta_{r_i}^{y^*_{r_i}(v,\delta)}\right\rangle\right]dy,\end{aligned}$$

which is the desired result. □

The next step in the analysis of the reflection term in (5.3) aims at correcting the integrator in (5.4) in order to make it additive. The following definition clarifies the form of the corrected (or *tilted*) reflection term:

Definition 5.2. *Let $v \in E^M$ and $\delta \in \mathbb{R}$ be given and M be the same level of truncation as in (5.1). Moreover, let $((n_t^y)_{0\le t\le T})_{y\in E^{*,M}}$ satisfy (**F1**)–(**F2**), and $((z_t^y)_{0\le t\le T})_{y\in E^{*,M}}$ be a deterministic jointly continuous flow with values $L^2_{\mathrm{sym}}(\mathbb{S})$, i.e., the map $(t,y) \in [0,T]\times E^M \mapsto z_t^y \in L^2_{\mathrm{sym}}(\mathbb{S})$ is continuous. Then, for any $m \in \mathbb{N}_0$, $y \in E^M$ and $t \in [0,T]$, define*

$$\partial_y^* n_t^{m,y} := \begin{cases} \partial_y \langle n_t^{y^*}, e_m\rangle & \text{whenever the } \mathbb{R}^M\text{-valued gradient at} \\ & \text{point } y \in E^M \text{ in the right-hand side exists,} \\ 0 & \text{otherwise,} \end{cases} \tag{5.7}$$

and let

$$\langle \tilde{n}_t^{y,(v,\delta)}, e_m\rangle := \langle n_t^{y_t^*(v,\delta)}, e_m\rangle + \frac{\delta}{T}\int_0^t \Big[\partial_y^* n_s^{m,w}\cdot v\Big]_{|w=y_s(v,\delta)} ds. \tag{5.8}$$

Measurability of $(t,y) \mapsto \partial_y^* n_t^{m,y}$ is obvious and, in fact, the Lipschitz property (**F2**) says that the derivative in the definition (5.2) exists for any $t \in [0,T]$, for almost every $y \in E^M$, and thus for almost every $(t,y) \in [0,T]\times E^M$. When $((n_t^y)_{0\le t\le T})_{y\in E^M}$ is replaced by $((\eta_t^y)_{0\le t\le T})_{y\in E^{*,M}}$, it is standard to prove that the mapping

$$(t,\omega,y) \in [0,T]\times E^M\times\Omega \mapsto \partial_y^*\eta_t^{m,y}$$

is measurable with respect to $\mathcal{P}\times\mathcal{B}(E^M)$, where $\mathcal{P}$ is the progressive σ-algebra on $[0,T]\times\Omega$ (Ω being equipped with the same filtration $\mathbb{F}$ as before) and $\mathcal{B}(E^M)$ is the Borel σ-algebra on E^M. Of course, in the random setting, we will use the notation

$$\langle \tilde{\eta}_t^{y,(v,\delta)}, e_m\rangle := \langle \eta_t^{y_t^*(v,\delta)}, e_m\rangle + \frac{\delta}{T}\int_0^t \Big[\partial_y^* \eta_s^{m,w}\cdot v\Big]_{|w=y_s(v,\delta)} ds.$$

It is easy to check that $\partial_y^* n_t^y := \sum_{m\in\mathbb{N}_0}\partial^* n_t^{m,y} e_m$ as defined through the Fourier modes (5.7) is an element of $H^{-2}_{\mathrm{sym}}(\mathbb{S})$, since the series $\sum_{m\in\mathbb{N}_0} m^{-4}|\partial_y^* n_t^{m,y}|^2$ is bounded. In particular, we have

$$\|\partial_y^* n_t^y\|_{2,-2} \le C_c, \tag{5.9}$$

for a constant C_c depending on c in (**F2**).

As a corollary of the bound (5.9), we get the following statement, which allows us to regard $(\tilde{n}_t^y)_{0\le t\le T}$ as a path with values in $H^{-2}_{\mathrm{sym}}(\mathbb{S})$:

Corollary 5.3. *Within the framework of Definition 5.2, for any $R > 0$,*

$$\forall R>0,\quad \sup_{0\le t\le T}\sup_{|y|\le R}\Big\|\tilde{n}_t^{y,(v,\delta)}\Big\|_{2,-2} \le \sup_{0\le t\le T}\|n_t^0\|_{2,-2} + c\big(R+\delta|v|\big) < \infty, \tag{5.10}$$

*for the same constant c as in (**F2**).*

5.2 Integrating in space the tilted reflection process

Based on the tilted process $(\tilde{\eta}_t^{y,(v,\delta)})_{0\le t\le T}$ introduced in the previous section, we provide here, a convenient change of variables for (5.4). The main step in the analysis is the following proposition.

Proposition 5.4. *Let $((n_t^y)_{0\le t\le T})_{y\in E^{*,M}}$ satisfy* (**F1**)–(**F2**), *and $((z_t^y)_{0\le t\le T})_{y\in E^{*,M}}$ be a deterministic jointly continuous flow with values in $L^2_{\mathrm{sym}}(\mathbb{S})$, i.e., the map $(t,y)\in[0,T]\times E^M\mapsto z_t^y\in L^2_{\mathrm{sym}}(\mathbb{S})$ is continuous. Then, for $\varepsilon>0$ and for a function $\varphi\in\mathcal{C}_0^\infty(E^M)$,*

$$\sum_{m\in\mathbb{N}_0}\int_{E^M}\left\{\int_0^T\left\langle e^{\varepsilon\Delta}z_t^{y^*},e_m\right\rangle\varphi\big(y_t(v,-\delta)\big)d\left\langle n_t^{y^*},e_m\right\rangle\right\}dy \tag{5.11}$$
$$=\lim_{N\to\infty}\int_{E^M}\varphi(y)\sum_{i=0}^{N-1}\left[\left\langle\tilde{n}_{r_{i+1}}^{y,(v,\delta)},e^{\varepsilon\Delta}z_{r_i}^{y^*_{r_i}(v,\delta)}\right\rangle-\left\langle\tilde{n}_{r_i}^{y,(v,\delta)},e^{\varepsilon\Delta}z_{r_i}^{y^*_{r_i}(v,\delta)}\right\rangle\right]dy.$$

Proof. We observe that it suffices to prove (5.11) for a flow $((z_t^{y^*})_{0\le t\le T})_{y\in E^M}$ that is differentiable with respect to the variable y, with a derivative that is jointly continuous in (t,y). Indeed, using a standard mollification argument in the variable y, we can approximate any flow $((z_t^{y^*})_{0\le t\le T})_{y\in E^M}$ that is only continuous in (t,y) by a flow that is regular in y (with derivatives that are jointly continuous in (t,y)) and use (4.27) in order to pass to the limit in (5.11). Below, we thus assume that $((z_t^{y^*})_{0\le t\le T})_{y\in E^M}$ is differentiable with respect to the variable y, with a derivative that is jointly continuous in (t,y).

Another key observation is that $(t,y)\in[0,T]\times E^M\mapsto n_t^{y^*}\in H^{-2}_{\mathrm{sym}}(\mathbb{S})$ is jointly continuous in (t,y) and thus uniformly continuous on $[0,T]\times\mathrm{Supp}(\varphi)$, with $\mathrm{Supp}(\varphi)$ denoting the support of φ. This follows from (**F2**) together with the fact that $t\in[0,T]\mapsto n_t^{y^*}\in\mathbb{H}^{-2}_{\mathrm{sym}}(\mathbb{S})$ is continuous for each $y\in E^M$.

Using the same notation as in Lemma 5.1, one has

$$\sum_{m\in\mathbb{N}_0}\int_{E^M}\left\{\int_0^T\left\langle e^{\varepsilon\Delta}z_t^{y^*},e_m\right\rangle\varphi\big(y_t(v,-\delta)\big)d\left\langle n_t^{y^*},e_m\right\rangle\right\}dy$$
$$=\lim_{N\to\infty}\int_{E^M}\varphi(y)\Bigg\{\sum_{i=0}^{N-1}\Bigg[\left\langle n_{r_{i+1}}^{y^*_{r_i}(v,\delta)},e^{\varepsilon\Delta}z_{r_i}^{y^*_{r_i}(v,\delta)}\right\rangle-\left\langle n_{r_{i+1}}^{y^*_{r_{i+1}}(v,\delta)},e^{\varepsilon\Delta}z_{r_i}^{y^*_{r_{i+1}}(v,\delta)}\right\rangle$$
$$+\left\langle n_{r_{i+1}}^{y^*_{r_{i+1}}(v,\delta)},e^{\varepsilon\Delta}z_{r_i}^{y^*_{r_{i+1}}(v,\delta)}\right\rangle-\left\langle n_{r_{i+1}}^{y^*_{r_{i+1}}(v,\delta)},e^{\varepsilon\Delta}z_{r_i}^{y^*_{r_i}(v,\delta)}\right\rangle$$
$$+\left\langle n_{r_{i+1}}^{y^*_{r_{i+1}}(v,\delta)},e^{\varepsilon\Delta}z_{r_i}^{y^*_{r_i}(v,\delta)}\right\rangle-\left\langle n_{r_i}^{y^*_{r_i}(v,\delta)},e^{\varepsilon\Delta}z_{r_i}^{y^*_{r_i}(v,\delta)}\right\rangle\Bigg]\Bigg\}dy.$$

Exchanging the first and third lines in the summand, we get

$$
\begin{aligned}
&\sum_{m\in\mathbb{N}_0}\int_{E^M}\Big\{\int_0^T\big\langle e^{\varepsilon\Delta}z_t^{y^*},e_m\big\rangle\varphi\big(y_t(v,-\delta)\big)d\big\langle n_t^{y^*},e_m\big\rangle\Big\}\,dy\\
&=\lim_{N\to\infty}\Bigg\{\int_{E^M}\varphi(y)\sum_{i=0}^{N-1}\Big[\Big\langle n_{r_{i+1}}^{y^*_{r_{i+1}}(v,\delta)},e^{\varepsilon\Delta}z_{r_i}^{y^*_{r_i}(v,\delta)}\Big\rangle-\Big\langle n_{r_i}^{y^*_{r_i}(v,\delta)},e^{\varepsilon\Delta}z_{r_i}^{y^*_{r_i}(v,\delta)}\Big\rangle\Big]dy\\
&\qquad+\int_{E^M}\varphi(y)\sum_{i=0}^{N-1}\Big\langle n_{r_{i+1}}^{y^*_{r_{i+1}}(v,\delta)},e^{\varepsilon\Delta}\Big(z_{r_i}^{y^*_{r_{i+1}}(v,\delta)}-z_{r_i}^{y^*_{r_i}(v,\delta)}\Big)\Big\rangle dy\\
&\qquad+\sum_{i=0}^{N-1}\int_{E^M}\Big[\varphi(y)-\varphi\Big(y+\delta\tfrac{r_{i+1}-r_i}{T}v\Big)\Big]\Big\langle n_{r_{i+1}}^{y^*_{r_i}(v,\delta)},e^{\varepsilon\Delta}z_{r_i}^{y^*_{r_i}(v,\delta)}\Big\rangle dy\Bigg\}\\
&=:\lim_{N\to\infty}\Big\{T_1^N+T_2^N+T_3^N\Big\}.
\end{aligned}
\tag{5.12}
$$

Analysis of T_1^N. By applying Definition 5.2 (at point $y_t(v,\delta)$ instead of y) and by using the fact that $(\tilde{n}_t^y)_{0\le t\le T}$ takes values in $H^{-2}_{\mathrm{sym}}(\mathbb{S})$, we get

$$
\begin{aligned}
&\lim_{N\to\infty}T_1^N\\
&=\lim_{N\to\infty}\int_{E^M}\varphi(y)\sum_{i=0}^{N-1}\Big[\Big\langle\tilde{n}_{r_{i+1}}^{y,(v,\delta)},e^{\varepsilon\Delta}z_{r_i}^{y^*_{r_i}(v,\delta)}\Big\rangle-\Big\langle\tilde{n}_{r_i}^{y,(v,\delta)},e^{\varepsilon\Delta}z_{r_i}^{y^*_{r_i}(v,\delta)}\Big\rangle\Big]dy\\
&\quad-\tfrac{\delta}{T}\lim_{N\to\infty}\int_{E^M}\varphi(y)\sum_{i=0}^{N-1}\sum_{m\in\mathbb{N}_0}\Big[\int_{r_i}^{r_{i+1}}\Big(\partial_w\Big[\Big\langle n_s^{w^*},e_m\Big\rangle\Big]_{|w=y_s(v,\delta)}\cdot v\Big)\Big\langle z_{r_i}^{y^*_{r_i}(v,\delta)},e^{\varepsilon\Delta}e_m\Big\rangle ds\Big]dy\\
&=\lim_{N\to\infty}\int_{E^M}\varphi(y)\sum_{i=0}^{N-1}\Big[\Big\langle\tilde{n}_{r_{i+1}}^{y,(v,\delta)},e^{\varepsilon\Delta}z_{r_i}^{y^*_{r_i}(v,\delta)}\Big\rangle-\Big\langle\tilde{n}_{r_i}^{y,(v,\delta)},e^{\varepsilon\Delta}z_{r_i}^{y^*_{r_i}(v,\delta)}\Big\rangle\Big]dy\\
&\quad-\tfrac{\delta}{T}\lim_{N\to\infty}\int_{E^M}\varphi(y)\sum_{i=0}^{N-1}\sum_{m\in\mathbb{N}_0}\Big[\int_{r_i}^{r_{i+1}}\Big(\partial_y\Big[\Big\langle n_s^{y^*_s(v,\delta)},e_m\Big\rangle\Big]\cdot v\Big)\Big\langle z_{r_i}^{y^*_{r_i}(v,\delta)},e^{\varepsilon\Delta}e_m\Big\rangle ds\Big]dy.
\end{aligned}
$$

Exchanging the integral in y and the sum over m (which is possible thanks to Lemma 4.4) and then performing an integration by parts in the last line, yields

$$
\begin{aligned}
&\lim_{N\to\infty}T_1^N\\
&=\lim_{N\to\infty}\int_{E^M}\varphi(y)\sum_{i=0}^{N-1}\Big[\Big\langle\tilde{n}_{r_{i+1}}^{y,(v,\delta)},e^{\varepsilon\Delta}z_{r_i}^{y^*_{r_i}(v,\delta)}\Big\rangle-\Big\langle\tilde{n}_{r_i}^{y,(v,\delta)},e^{\varepsilon\Delta}z_{r_i}^{y^*_{r_i}(v,\delta)}\Big\rangle\Big]dy\\
&\quad+\lim_{N\to\infty}\tfrac{\delta}{T}\sum_{i=0}^{N-1}\int_{r_i}^{r_{i+1}}\int_{E^M}\Big(\partial_y\varphi(y)\cdot v\Big)\Big\langle n_s^{y^*_s(v,\delta)},e^{\varepsilon\Delta}z_{r_i}^{y^*_{r_i}(v,\delta)}\Big\rangle dyds\\
&\quad+\lim_{N\to\infty}\tfrac{\delta}{T}\int_{E^M}\varphi(y)\sum_{i=0}^{N-1}\sum_{m\in\mathbb{N}_0}\Big[\int_{r_i}^{r_{i+1}}\Big\langle n_s^{y^*_s(v,\delta)},e_m\Big\rangle\Big(\partial_y\Big[\Big\langle z_{r_i}^{y^*_{r_i}(v,\delta)},e^{\varepsilon\Delta}e_m\Big\rangle\Big]\cdot v\Big)ds\Big]dy\\
&=:\lim_{N\to\infty}\Big(T_{1,1}^N+T_{1,2}^N+T_{1,3}^N\Big).
\end{aligned}
\tag{5.13}
$$

Analysis of $T_{1,3}^N + T_2^N$. Using the regularity of the flow $((z_t^y)_{0\le t\le T})_{y\in E^{*,M}}$, we write

$$\begin{aligned}
\lim_{N\to\infty} T_2^N &= \lim_{N\to\infty} \int_{E^M} \varphi(y) \sum_{i=0}^{N-1} \Big\langle n_{r_{i+1}}^{y^*_{r_{i+1}}(v,\delta)}, e^{\varepsilon\Delta}\Big(z_{r_i}^{y^*_{r_{i+1}}(v,\delta)} - z_{r_i}^{y^*_{r_i}(v,\delta)}\Big)\Big\rangle dy \\
&= -\lim_{N\to\infty} \tfrac{\delta}{T} \int_{E^M} \varphi(y) \sum_{i=0}^{N-1} (r_{i+1}-r_i) \sum_{m\in\mathbb{N}_0} \Big\langle n_{r_{i+1}}^{y^*_{r_{i+1}}(v,\delta)}, e_m\Big\rangle \Big(\partial_y\Big[\Big\langle z_{r_i}^{y^*_{r_i}(v,\delta)}, e^{\varepsilon\Delta} e_m\Big\rangle\Big]\cdot v\Big) dy \\
&\quad + \lim_{N\to\infty} \sum_{i=0}^{N-1} \mathcal{O}(r_{i+1}-r_i),
\end{aligned}$$

where $\mathcal{O}$ is the little Landau symbol (and is here implicitly understood to be uniform in N and i). And, then using the joint regularity of $((n_t^y)_{0\le t\le T})_{y\in E^{*,M}}$, we get

$$\begin{aligned}
\lim_{N\to\infty} \big(T_2^N + T_{1,3}^N\big) &= \lim_{N\to\infty} \int_{E^M} \varphi(y) \sum_{i=0}^{N-1} \sum_{m\in\mathbb{N}_0} \Bigg[\int_{r_i}^{r_{i+1}} \Big\langle n_s^{y^*_s(v,\delta)} - n_{r_{i+1}}^{y^*_{r_{i+1}}(v,\delta)}, e_m\Big\rangle \\
&\qquad\qquad \times \Big(\partial_y\Big[\Big\langle z_{r_i}^{y^*_{r_i}(v,\delta)}, e^{\varepsilon\Delta} e_m\Big\rangle\Big]\cdot v\Big)\, ds\Bigg] dy \\
&= 0.
\end{aligned}$$

Analysis of $T_{1,2}^N + T_3^N$. Adding and subtracting the quantity

$$\lim_{N\to\infty} \tfrac{\delta}{T} \sum_{i=0}^{N-1} \int_{E^M} \big(\partial_y\varphi(y)\cdot v\big)(r_{i+1}-r_i) \Big\langle n_{r_{i+1}}^{y^*_{r_i}(v,\delta)}, e^{\varepsilon\Delta} z_{r_i}^{y^*_{r_i}(v,\delta)}\Big\rangle dy,$$

we have

$$\begin{aligned}
&\lim_{N\to\infty} \Big(T_{1,2}^N + T_3^N\Big) \\
&= \lim_{N\to\infty} \sum_{i=0}^{N-1} \tfrac{\delta}{T} \int_{r_i}^{r_{i+1}} \int_{E^M} \Big(\partial_y\varphi(y)\cdot v\Big)\Bigg[\Big\langle n_s^{y^*_s(v,\delta)}, e^{\varepsilon\Delta} z_{r_i}^{y^*_{r_i}(v,\delta)}\Big\rangle - \Big\langle n_{r_{i+1}}^{y^*_{r_i}(\delta,v)}, e^{\varepsilon\Delta} z_{r_i}^{y^*_{r_i}(v,\delta)}\Big\rangle\Bigg] dy ds \\
&\quad + \lim_{N\to\infty} \sum_{i=0}^{N-1} \int_{E^M} \Bigg[\varphi(y) + \Big(\tfrac{\delta}{T}(r_{i+1}-r_i)\Big)\partial_y\varphi(y)\cdot v \\
&\qquad\qquad - \varphi\Big(y + \tfrac{\delta}{T}(r_{i+1}-r_i)v\Big)\Bigg]\Big\langle n_{r_{i+1}}^{y^*_{r_i}(v,\delta)}, e^{\varepsilon\Delta} z_{r_i}^{y^*_{r_i}(v,\delta)}\Big\rangle dy \\
&=: \lim_{N\to\infty} \Big(T_{(1,2,3),1}^N + T_{(1,2,3),2}^N\Big).
\end{aligned} \tag{5.14}$$

Analysis of $T_{(1,2,3),1}^N + T_{(1,2,3),2}^N$. We claim that the limits of the two terms in the above argument are 0 as we can write both of them in the form $\sum_{i=0}^{N-1} \mathcal{O}(r_{i+1}-r_i)$.

The limit of $(T_{(1,2,3),2}^N)_{N\ge1}$ is easily handled by using the fact that φ is smooth and by invoking the duality between $H^2_{\mathrm{sym}}(\mathbb{S})$ and $H^{-2}_{\mathrm{sym}}(\mathbb{S})$ (to handle terms of the form $\langle n_{r_{i+1}}^{y^*_{r_i}(v,\delta)}, e^{\varepsilon\Delta} z_{r_i}^{y^*_{r_i}(v,\delta)}\rangle$).

The limit of $(T^N_{(1,2,3),1})_{N\geq 1}$ is shown to be 0 by invoking the fact that the mapping $(t,y) \in [0,T] \times E^M \mapsto \eta^y_t \in \mathbb{H}^{-2}_{\mathrm{sym}}(\mathbb{S})$ is jointly continuous in (t,y) and thus uniformly continuous on $[0,T] \times \mathrm{Supp}(\varphi)$.

Conclusion. Back to (5.14), we deduce from the above analysis that

$$\lim_{N\to\infty} \left\{ T^N_1 + T^N_2 + T^N_3 \right\} = \lim_{N\to\infty} \left\{ T^N_{1,1} + T^N_{1,2} + T^N_{1,3} + T^N_2 + T^N_3 \right\} = \lim_{N\to\infty} T^N_{1,1}.$$

This completes the proof. □

Obviously, Proposition 5.4 is one key step for obtaining a change of variable in the integral (5.4). It is however not enough. In particular, it is worth observing that, at this stage, the right-hand side of (5.11) cannot be directly interpreted as an integral, even in the simplest case where $(z_t)_{0\leq t\leq T}$ is just one trajectory independent of y.

Assuming that the flow $((z^y_t)_{0\leq t\leq T})_{y\in E^{\star,M}}$ in Proposition 5.4 reduces to one single trajectory $(z_t)_{0\leq t\leq T}$, we want to rearrange the two terms in (5.11) in order to interpret each of them as an integral. As for the left-hand side in (5.11), this is quite easy. Indeed, for $(z_t)_{0\leq t\leq T}$ a continuous path with values in $L^2_{\mathrm{sym}}(\mathbb{S})$, for $\varphi \in \mathcal{C}^\infty_0(E^M)$, for $\delta \in \mathbb{R}$ and for $y, v \in E^M$, we can let

$$\tilde{z}^{y,\varphi,(v,\delta)}_t := \varphi(y_t(v,-\delta))z_t, \quad t \in [0,T]. \tag{5.15}$$

The collection $((\tilde{z}^{y,\varphi,(v,\delta)}_t)_{0\leq t\leq T})_{y\in E^M}$ (for v fixed) is a jointly continuous flow (in (t,y)) with values in $L^2_{\mathrm{sym}}(\mathbb{S})$. Therefore, Definition 4.5 (together with the bound (4.27) in order to justify that the integral in $y \in E^M$ below is well-defined) applies and says that (notice that the left-hand side below is the same as the left-hand side in (5.11))

$$\begin{aligned} &\sum_{m\in\mathbb{N}_0} \int_{E^M} \left\{ \int_0^T \left\langle e^{\varepsilon\Delta} z_t, e_m \right\rangle \varphi\big(y_t(v,-\delta)\big) d\left\langle n^{y^\star}_t, e_m \right\rangle \right\} dy \\ &= \int_{E^M} \left(\int_0^T \left\langle e^{\varepsilon\Delta} \tilde{z}^{y,\varphi,(v,\delta)}_t, dn^{y^\star}_t \right\rangle \right) dy. \end{aligned} \tag{5.16}$$

In order to handle the right-hand side of (5.11), we let

$$\tilde{n}^{\varphi,(v,\delta)}_t := \int_{E^M} \varphi(y) \tilde{n}^{y,(v,\delta)}_t dy, \tag{5.17}$$

which is regarded as an element of $H^{-2}_{\mathrm{sym}}(\mathbb{S})$, namely

$$\left\langle \tilde{n}^{\varphi,(v,\delta)}_t, z \right\rangle = \int_{E^M} \varphi(y) \left\langle \tilde{n}^{y,(v,\delta)}_t, z \right\rangle dy,$$

for z a generic element of $H^2_{\mathrm{sym}}(\mathbb{S})$. Notice that the integral in the right-hand side is well-defined by means of Corollary 5.3. Then, we observe that the argument in the limit

appearing in the right-hand side of (5.11) can be rewritten

$$\begin{aligned}&\int_{E^M}\varphi(y)\sum_{i=0}^{N-1}\left[\left\langle\tilde{n}^{y,(v,\delta)}_{r_{i+1}},e^{\varepsilon\Delta}z_{r_i}\right\rangle-\left\langle\tilde{n}^{y,(v,\delta)}_{r_i},e^{\varepsilon\Delta}z_{r_i}\right\rangle\right]dy\\&=\sum_{i=0}^{N-1}\left[\left\langle\tilde{n}^{\varphi,(v,\delta)}_{r_{i+1}},e^{\varepsilon\Delta}z_{r_i}\right\rangle-\left\langle\tilde{n}^{\varphi,(v,\delta)}_{r_i},e^{\varepsilon\Delta}z_{r_i}\right\rangle\right].\end{aligned}\tag{5.18}$$

We get the following corollary:

Corollary 5.5. *For a given $v\in E^M$, let $(\tilde{n}^{\varphi,(v,\delta)}_t)_{0\le t\le T}$ be as in (5.17). Assume that the function φ is positive-valued, then, for $\varepsilon>0$ and any $z\in L^2_{\mathrm{sym}}(\mathbb{S})$ that is non-increasing, the process $(\langle\tilde{n}^{\varphi,(v,\delta)}_t,e^{\varepsilon\Delta}z\rangle)_{0\le t\le T}$ is non-decreasing (in time).*

Proof. With z as in the statement, we apply Proposition 5.4 with $(z_t:=\mathbf{1}_{[r,s]}(t)z)_{0\le t\le T}$ for a given pair $(r,s)\in[0,T]^2$ satisfying $r<s$. By (5.15), (5.16) and (5.18), we obtain

$$\int_{E^M}\left(\int_0^T\left\langle e^{\varepsilon\Delta}\tilde{z}^{y,\varphi,(v,\delta)}_t,dn^{y^*}_t\right\rangle\right)dy=\lim_{N\to\infty}\sum_{i=0}^{N-1}\left[\left\langle\tilde{n}^{\varphi,(v,\delta)}_{r_{i+1}},e^{\varepsilon\Delta}z_{r_i}\right\rangle-\left\langle\tilde{n}^{\varphi,(v,\delta)}_{r_i},e^{\varepsilon\Delta}z_{r_i}\right\rangle\right].\tag{5.19}$$

Regardless the choice of the subdivision $\{r^N_i\}_{i=0,\cdots,N}$ (recall that we omit the superscript N in the various equations), we have

$$\begin{aligned}&\sum_{i=0}^{N-1}\left[\left\langle\tilde{n}^{\varphi,(v,\delta)}_{r_{i+1}},e^{\varepsilon\Delta}z_{r_i}\right\rangle-\left\langle\tilde{n}^{\varphi,(v,\delta)}_{r_i},e^{\varepsilon\Delta}z_{r_i}\right\rangle\right]\\&=\sum_{i=0}^{N-1}\left\{\left[\left\langle\tilde{n}^{\varphi,(v,\delta)}_{r_{i+1}},e^{\varepsilon\Delta}z\right\rangle-\left\langle\tilde{n}^{\varphi,(v,\delta)}_{r_i},e^{\varepsilon\Delta}z\right\rangle\right]\mathbf{1}_{[r,s]}(r_i)\right\}.\end{aligned}$$

If we assume that r and s belong to the collection $\{r^N_i\}_{i=0,\cdots,N}$, which can be done without any loss of generality, we get

$$\sum_{i=0}^{N-1}\left[\left\langle\tilde{n}^{\varphi,(v,\delta)}_{r_{i+1}},e^{\varepsilon\Delta}z_{r_i}\right\rangle-\left\langle\tilde{n}^{\varphi,(v,\delta)}_{r_i},e^{\varepsilon\Delta}z_{r_i}\right\rangle\right]=\left\langle\tilde{n}^{\varphi,(v,\delta)}_s,e^{\varepsilon\Delta}z\right\rangle-\left\langle\tilde{n}^{\varphi,(v,\delta)}_r,e^{\varepsilon\Delta}z\right\rangle.$$

Then, (5.19) yields

$$\left\langle\tilde{n}^{\varphi,(v,\delta)}_s,e^{\varepsilon\Delta}z\right\rangle-\left\langle\tilde{n}^{\varphi,(v,\delta)}_r,e^{\varepsilon\Delta}z\right\rangle=\int_{E^M}\left(\int_0^T\left\langle e^{\varepsilon\Delta}\tilde{z}^{y,\varphi,(v,\delta)}_t,dn^{y^*}_t\right\rangle\right)dy.$$

Recalling the definition (5.15) and using the fact that $\varphi\ge 0$, we observe that $(\tilde{z}^{y,\varphi,(v,\delta)}_t)_{0\le t\le T}$ takes values in $U^2(\mathbb{S})$. By Corollary 4.8, the right-hand side is non-negative. Assuming that $z\in H^2_{\mathrm{sym}}(\mathbb{S})$ and letting ε tend to 0, we complete the proof. □

The above corollary shows that $(\tilde{n}^{\varphi,(v,\delta)}_t)_{0\le t\le T}$ satisfies Assumption (**E2**) in Subsection 4.2. Since (**E1**) follows quite obviously from the joint continuity of the flow $((n^y_t)_{0\le t\le T})_{y\in E^{*,M}}$, we can invoke Definition 4.5 to give meaning to the integral

$$\left(\int_0^t\left\langle e^{\varepsilon\Delta}z_s,d\tilde{n}^{\varphi,(v,\delta)}_s\right\rangle\right)_{0\le t\le T},$$

for $\varepsilon > 0$ and for a continuous path $(z_t)_{0\le t\le T}$ with values in $L^2_{\mathrm{sym}}(\mathbb{S})$, at least when φ takes non-negative values. When φ takes values in $\mathbb{R}$, we may decompose it into $\varphi := \varphi^+ - \varphi^-$ (with φ^+ and φ^- being the positive and negative parts of φ) and then let

$$\int_0^t \langle e^{\varepsilon\Delta} z_s, d\tilde{n}_s^{\varphi,(v,\delta)} \rangle := \int_0^t \langle e^{\varepsilon\Delta} z_s, d\tilde{n}_s^{\varphi^+,(v,\delta)} \rangle - \int_0^t \langle e^{\varepsilon\Delta} z_s, d\tilde{n}_s^{\varphi^-,(v,\delta)} \rangle, \quad t \in [0,T],$$

but we will not make use of this latter extension.

In the end, the conclusion of Proposition 5.4 may be rewritten as follows. Under the same notation as therein and under the assumption that φ takes non-negative values and that $(z_t)_{0\le t\le T}$ is one single continuous path with values in $L^2_{\mathrm{sym}}(\mathbb{S})$,

$$\int_E \left(\int_0^T \left\langle e^{\varepsilon\Delta} \tilde{z}_t^{y,\varphi,(v,\delta)}, dn_t^{y^*} \right\rangle \right) dy = \int_0^T \langle e^{\varepsilon\Delta} z_t, d\tilde{n}_t^{\varphi,(v,\delta)} \rangle, \quad t \in [0,T]. \tag{5.20}$$

Remark 5.6. Following Remark 4.10, all the results proven in this subsection can be applied when $((n_t^y)_{0\le t\le T})_{y\in E^{*,M}}$ is chosen as the realisation of $((\eta_t^y)_{0\le t\le T})_{y\in E^{*,M}}$ (which satisfies (**F1**) and (**F2**) thanks to Remark 4.19) and $((z_t^y)_{0\le t\le T})_{y\in E^{*,M}}$ is chosen as the realisation of a random flow whose trajectories are $L^2_{\mathrm{sym}}(\mathbb{S})$-valued, continuous in time and Lipschitz continuous in space. We feel that it is not instructive to formalise this here by means of rigorous statements and we refer to the next subsection for a concrete application of the results obtained to the study of the rearranged SHE introduced in Definition 4.13.

5.3 Integrating in time the shifted tilted reflection process

We now kill two birds with one stone. Not only do we make explicit the dynamics of the shifted process $(X_t^{y_t^*(v,\delta)})_{0\le t\le T}$ (for a given $y \in E^M$), but we also clarify the notion of an integral with respect to the *shifted* version of the *tilted* reflection process, namely $(\tilde{\eta}_t^{y,(v,\delta)})_{0\le t\le T}$. In brief, we are able to define a pathwise integral (in time) with respect to $(\tilde{\eta}_t^{y,(v,\delta)})_{0\le t\le T}$ by using the procedure described in Subsection 4.2. Of course, the reader may object that we have already constructed an integral with respect to the process $(\tilde{\eta}_t^{\varphi,(v,\delta)})_{0\le t\le T}$ defined in (5.17) (replacing therein $(\tilde{n}_t^{y,(v,\delta)})_{0\le t\le T}$ by $(\tilde{\eta}_t^{y,(v,\delta)})_{0\le t\le T}$), but this was only for φ smooth. Somehow, we want here to choose φ as a Dirac mass.

The first result of this subsection is:

Proposition 5.7. *Fix $\delta \in \mathbb{R}$ and $v \in E^M$. Then, with probability 1, for any almost every $y \in E^M$, the path $(\tilde{\eta}_t^{y,(v,\delta)})_{0\le t\le T}$ satisfies assumptions* (**E1**) *and* (**E2**) *in Subsection 4.2. In particular, we can invoke Definition 4.5 in order to define an integral with respect to $(\tilde{\eta}_t^{y_t^*(v,\delta),v})_{0\le t\le T}$.*

The second one is

Proposition 5.8. *Fix $\delta \in \mathbb{R}$ and $v \in E^M$. Then, with probability 1, for almost every $y \in E^M$, one has, for any $u \in H^2_{\mathrm{sym}}(\mathbb{S})$,*

$$\begin{aligned}\big\langle X_t^{y_t^*(v,\delta),v}, u\big\rangle =& \big\langle X_0^{(y+\delta v)^*}, u\big\rangle + \int_0^t \big\langle X_s^{y_s^*(v,\delta)}, \Delta u\big\rangle ds + \int_0^t d\big\langle \tilde{\eta}_s^{y,(v,\delta)}, u\big\rangle \\ &+ \int_0^t d\langle W_s, u\rangle - \tfrac{\delta}{T}\int_0^t \partial_y\big\langle X_s^{y_s^*(v,\delta)}, u\big\rangle \cdot v\, ds,\end{aligned} \tag{5.21}$$

where it is implicitly understood that, with probability 1, for almost every $y \in E^M$, for almost every $t \in [0,T]$ and for any $u \in H^2_{\mathrm{sym}}(\mathbb{S})$, the derivative $\partial_y\langle X_s^{y_s^*(v,\delta)}, u\rangle$ exists and induces a jointly measurable mapping on $\Omega \times [0,T] \times E^M$.

Proof of Propositions 5.7 and 5.8. We prove the two statements at once.

As a preliminary remark, we observe that the existence of the derivatives, as stated in the last line of the statement, follows from Rademacher's theorem. Indeed, by [13, Theorem 4], with probability 1, for any $t \in [0,T]$, the map $y \in E^M \mapsto X_t^{y^*} \in L^2_{\mathrm{sym}}(\mathbb{S})$ is a.e. differentiable. By Fubini's theorem, it is quite easy to see that the map $(\omega, t, y) \mapsto \partial_y X_t^{y^*}(\omega)$ is hence defined up to a negligible subset of $\Omega \times [0,T] \times E^M$ and induces a jointly measurable mapping on $\Omega \times [0,T] \times E^M$. Since, for any $t \in [0,T]$, the map $y \mapsto y_t(v,\delta) = y + \delta(T-t)/T\, v$ preserves the Lebesgue measure, we deduce that $(\omega, t, y) \mapsto \partial_y X_t^{y_t^*(v,\delta)}$ is also defined up to a negligible subset of $\Omega \times [0,T] \times E^M$ and also induces a jointly measurable mapping on $\Omega \times [0,T] \times E^M$.

First step. We begin by expanding

$$\int_{E^M} \big\langle X_t^{y_t^*(v,\delta)}, u\big\rangle \varphi(y) dy,$$

for a function $\varphi \in \mathcal{C}_0^\infty(E^M)$ with non-negative values and for a given element $u \in H^2_{\mathrm{sym}}(\mathbb{S})$. Notice that, once (5.21) has been proven to hold true, for a given u, with probability 1 and for almost every y, it is easy to get the result with probability 1, for almost every y and for any $u \in H^2_{\mathrm{sym}}(\mathbb{S})$. It suffices to use the separability $H^2_{\mathrm{sym}}(\mathbb{S})$. By a change of variables, we get, for any $u \in H^2_{\mathrm{sym}}(\mathbb{S})$, with probability 1,

$$\begin{aligned}\int_{E^M} \big\langle X_t^{y_t^*(\delta,v)}, u\big\rangle \varphi(y) dy &= \int_{E^M} \big\langle X_t^{y^*}, u\big\rangle \varphi\big(y_t(v,-\delta)\big) dy \\ &= \int_{E^M} \big\langle X_0^{y^*}, u\big\rangle \varphi(y - \delta v) dy + \int_{E^M}\int_0^t \big\langle X_s^{y^*}, \Delta u\big\rangle \varphi\big(y_s(v,-\delta)\big) ds dy \\ &\quad + \int_{E^M}\Big(\int_0^t \varphi\big(y_s(v,-\delta)\big) d\big\langle \eta_s^{y^*}, u\big\rangle\Big) dy + \int_{E^M}\Big(\int_0^t \varphi\big(y_s(v,-\delta)\big) d\big\langle W_s, u\big\rangle\Big) dy \\ &\quad + \tfrac{\delta}{T}\int_{E^M}\int_0^t \big\langle X_s^{y^*}, u\big\rangle\, \partial_y \varphi\big(y_s(v,-\delta)\big) \cdot v\, ds dy.\end{aligned} \tag{5.22}$$

Here is the key point. By (5.15) and (5.20), we can perform a change of variable in the penultimate line (with $z_t = u$). As for the last line, we can make an integration by parts,

recalling the (pathwise) Lipschitz property of the flow, see Proposition 4.18. We get

$$\begin{aligned}
&\int_{E^M} \langle X_t^{y_t^*(v,\delta)}, u\rangle \varphi(y)dy \\
&= \int_{E^M} \langle X_0^{(y+\delta v)^*}, u\rangle \varphi(y)dy + \int_{E^M}\int_0^t \langle X_s^{y_s^*(v,\delta)}, \Delta u\rangle \varphi(y)dsdy \\
&\quad + \langle \tilde{\eta}_t^{\varphi,(v,\delta)}, u\rangle + \langle W_t, u\rangle \int_{E^M} \varphi(y)dy - \tfrac{\delta}{T}\int_{E^M}\int_0^t \varphi(y)\partial_y \langle X_s^{y_s^*(v,\delta)}, u\rangle \cdot v dsdy.
\end{aligned} \tag{5.23}$$

Second step. Assume $\varphi \geq 0$. By invoking Lemma 5.9 below and then by taking the supremum over $u \in H^2_{\mathrm{sym}}(\mathbb{S})$ in (5.23) right above, we claim

$$\|\tilde{\eta}_t^{\varphi,(v,\delta)} - \tilde{\eta}_s^{\varphi,(v,\delta)}\|_{2,-2} \leq \zeta_R(|t-s|)\int_{E^M} \varphi(y)dy, \quad (s,t) \in [0,T],$$

for a random field ζ_R, with values in $(0,\infty)$ and with $\zeta_R(\rho) \to 0$, almost surely, as $\rho \to 0$.

Back to the definition of $\tilde{\eta}^{\varphi,v}$ in (5.17), we obtain that, for any $u \in H^2_{\mathrm{sym}}(\mathbb{S})$, with $\|u\|_{2,2} \leq 1$,

$$\left|\int_{E^M} \varphi(y)\langle \tilde{\eta}_t^{y,(v,\delta)} - \tilde{\eta}_s^{y,(v,\delta)}, u\rangle dy\right| \leq \zeta_R(|t-s|)\int_{E^M} \varphi(y)dy, \quad (s,t) \in [0,T]^2,$$

and then, we deduce that, with probability 1, for almost every $y \in E^M$, for s,t in a dense countable subset of $[0,T]$, for u in a dense countable subset of the unit ball of $H^2_{\mathrm{sym}}(\mathbb{S})$,

$$\left|\langle \tilde{\eta}_t^{y,(v,\delta)}, u\rangle - \langle \tilde{\eta}_s^{y,(v,\delta)}, u\rangle\right| \leq \zeta_R(|t-s|).$$

Therefore, with probability 1, for almost every $y \in E^M$, for s,t in a dense countable subset of $[0,T]$,

$$\left\|\tilde{\eta}_t^{y,(v,\delta)} - \tilde{\eta}_s^{y,(v,\delta)}\right\|_{2,-2} \leq \zeta_R(|t-s|),$$

and then we have a continuous extension to the whole $[0,T]$.

Third step. We observe in a similar manner from Corollary 5.5 that, with probability 1, for almost every y, for s,t in dense countable subset of $[0,T]$, for u in a countable subset of $U^2(\mathbb{S})$,

$$\left\langle \tilde{\eta}_t^{y,(v,\delta)} - \tilde{\eta}_s^{y,(v,\delta)}, u\right\rangle \geq 0.$$

And then, by density, we get that the continuous extension satisfies

$$\left\langle \tilde{\eta}_t^{y,(v,\delta)} - \tilde{\eta}_s^{y,(v,\delta)}, u\right\rangle \geq 0,$$

for all $s,t \in [0,T]$ and $u \in U^2(\mathbb{S})$. This completes the proof of Proposition 5.7. In particular, we can construct, almost surely, for almost every $y \in E^M$, an integral with respect to the process $(\tilde{\eta}_t^{y,(v,\delta)})_{0\leq t\leq T}$ as the latter satisfies (**E1**) and (**E2**) in Subsection 4.2. By combining Lemma 4.7 and Corollary 5.3, we deduce that the identity (5.17) is preserved, despite the additional extension by continuity. Proposition 5.8 then follows by inserting (5.17) into (5.23).

□

In the proof of Proposition 5.7, we made use of the following statement:

Lemma 5.9. *Let $\delta \in \mathbb{R}$ and $v \in E^M$. Then, for any $R > 0$, there exists a random field ζ_R, with values in $(0,\infty)$ and with $\zeta_R(\rho) \to 0$ almost surely as $\rho \to 0$, such that, for $|y| \le R$,*

$$\left\| X_t^{y_t^*(v,\delta)} - X_s^{y_s^*(v,\delta)} \right\|_2 \le \zeta_R(|t-s|), \quad (s,t) \in [0,T].$$

Proof. The proof follows from Remark 4.19, which says that the flow $((X_t^y)_{0\le t\le T})_{y\in E^{*,M}}$ is jointly continuous in (t,y). □

5.4 Dynamics under new probability measure

For $\delta \in \mathbb{R}$ and $v \in E^M$ as before (with $M \in \mathbb{N}$ fixed), Proposition 5.8 prompts us to implement Girsanov transformation in such a way that, for almost every $y \in E^M$, under a new probability measure $\mathbb{Q}^{y,(v,\delta)}$ depending on y, the process $(\widetilde{W}_t^{y,(v,\delta)})_{t\ge 0}$ defined via Fourier modes by

$$\left\langle \widetilde{W}_t^{y,(v,\delta)}, e_m \right\rangle := \langle W_t, e_m \rangle - \tfrac{\delta}{T} \int_0^t \lambda_m^{-1} \partial_y \left\langle X_t^{y_t^*(v,\delta)}, e_m \right\rangle \cdot v ds, \quad t \ge 0, \quad m \in \mathbb{N}_0,$$

becomes a Q-Wiener process.

Of course, it must be stressed that, in the definition of $((\langle \widetilde{W}_t^{y,(v,\delta)}, e_m \rangle)_{0\le t\le T})_{y\in E^M}$, the integral process only exists for almost every $\omega \in \Omega$ and for almost every $y \in E^M$, see the last line in Proposition 5.8. In order to remedy this issue, we use below the notation

$$\chi_t^{m,y,(v,\delta)} := \begin{cases} \partial_y \left\langle e_m, X_t^{y_t^*(v,\delta)} \right\rangle \cdot v & \text{if the derivative exists,} \\ 0 & \text{otherwise,} \end{cases} \qquad m \in \mathbb{N}_0, \quad y, v \in E^M,$$

which allows one to extend the derivative when it does not exist.

Intuitively, the guess is that, for almost every $y \in E^M$, the process $(X_t^{y_t^*(v,\delta)})_{0\le t\le T}$ solves under the new probability measure the various items of Definition 4.13. As we will see next, the main difficulty is in fact to verify (4.38). In order to define the new probability measure, it is convenient to define here the processes $L^{y,(v,\delta)} = (L_t^{y,(v,\delta)})_{0\le t\le T}$ as

$$L_t^{y,(v,\delta)} := \sum_{m\in\mathbb{N}_0} \int_0^t \lambda_m^{-1} \chi_s^{m,y,(v,\delta)} dB_s^m,$$

for $t \in [0,T]$ an $y, v \in E^M$. While we wish to let

$$\frac{d\mathbb{Q}^{y,(v,\delta)}}{d\mathbb{P}} := \mathcal{E}_T \left\{ \tfrac{\delta}{T} \sum_{m\in\mathbb{N}_0} \int_0^{\cdot} \lambda_m^{-1} \chi_s^{m,y,(v,\delta)} dB_s^m \right\} = \mathcal{E}_T \left\{ \tfrac{\delta}{T} L_{\cdot}^{y,(v,\delta)} \right\}, \tag{5.24}$$

where $\mathcal{E}_T$ is a shorter notation for the Doléans-Dade exponential at time T, we are not able to prove, at this stage of the proof, that this defines indeed a new probability measure. For this reason, we employ a localisation argument by introducing the stopping time

$$\tau_{y,(v,\delta)} := \inf \left\{ t \ge 0 : \int_0^t \sum_{m\in\mathbb{N}_0} \left| \lambda_m^{-1} \chi_s^{m,y,(v,\delta)} \right|^2 ds \ge \frac{2T^2}{\delta^2} \right\} \wedge T. \tag{5.25}$$

Then, we obviously have that

$$\mathbb{E}\Big[\exp\Big\{\tfrac{1}{2}(\tfrac{\delta}{T})^2\big[L^{y,(v,\delta)}_{\cdot\wedge\tau_{y,(v,\delta)}}\big]_T\Big\}\Big] = \mathbb{E}\Big[\exp\Big\{\frac{\delta^2}{2T^2}\int_0^{T\wedge\tau_{y,(v,\delta)}}\sum_{m\in\mathbb{N}_0}|\lambda_m^{-1}\chi_s^{m,y,(v,\delta)}|^2 ds\Big\}\Big] \le e.$$

Consequently, denoting (note that we omit the subscript $(y,(v,\delta))$ in the index τ in the left-hand side)

$$\frac{d\mathbb{Q}^{y,(v,\delta),\tau}}{d\mathbb{P}} := \mathcal{E}_T\Big\{\frac{\delta}{T}\int_0^{\cdot\wedge\tau_{y,(v,\delta)}}\sum_{m\in\mathbb{N}_0}\lambda_m^{-1}\chi_s^{m,y,(v,\delta)}dB_s^m\Big\} =: \mathcal{E}_T\big\{\tfrac{\delta}{T}L^{y,(v,\delta)}_{\cdot\wedge\tau_{y,(v,\delta)}}\big\}, \tag{5.26}$$

one has a well-defined change of measure and the process $(\widetilde{W}_t^{y,(v,\delta),\tau})_{t\ge0}$ defined in Fourier modes by

$$\big\langle\widetilde{W}_t^{y,(v,\delta),\tau}, e_m\big\rangle := \langle W_t, e_m\rangle - \tfrac{\delta}{T}\int_0^{t\wedge\tau_{y,(v,\delta)}}\lambda_m^{-1}\chi_s^{m,y,(v,\delta)}ds, \quad t\ge0 \quad m\in\mathbb{N}_0, \tag{5.27}$$

is a Q-Wiener process under $\mathbb{Q}^{y,(v,\delta),\tau}$.

Here is the main statement of this subsection:

Proposition 5.10. *Let $y,v\in E^M$ and $\delta\in\mathbb{R}$. For $\tau_{y,(v,\delta)}$ as in (5.25), let (omitting the parameters (v,δ) in the notation $y^*(v,\delta)$)*

$$\widetilde{X}_t^{y,(v,\delta),\tau} := \begin{cases} X_t^{y_t^*}, & t\in[0,\tau_{y,(v,\delta)}] \\ X_t^{y^*_{\tau_{y,(v,\delta)}}}, & t\in[\tau_{y,(v,\delta)},T] \end{cases},$$

$$\tilde{\eta}_t^{y,(v,\delta),\tau} := \begin{cases} \tilde{\eta}_t^{y,(v,\delta)}, & t\in[0,\tau_{y,(v,\delta)}] \\ \eta_t^{y^*_{\tau_{y,(v,\delta)}}} - \eta_{\tau_{y,(v,\delta)}}^{y^*_{\tau_{y,(v,\delta)}}} + \tilde{\eta}_{\tau_{y,(v,\delta)}}^{y,(v,\delta),\tau}, & t\in[\tau_{y,(v,\delta)},T] \end{cases}.$$

Then, for almost every $y\in E^M$, the process $(\widetilde{X}_t^{y,(v,\delta),\tau},\tilde{\eta}_t^{y,(v,\delta),\tau})_{0\le t\le T}$ satisfies Definition 4.13 under the probability measure $\mathbb{Q}^{y,(v,\delta),\tau}$.

Wile the notation looks complicated, the interpretation of $(\widetilde{X}_t^{y,(v,\delta),\tau},\tilde{\eta}_t^{y,(v,\delta),\tau})_{0\le t\le T}$ is in fact quite simple: the shifting $y_t(v,\delta)$ is enacted up until time $t=\tau_{y,(v,\delta)}$.

Proof. Item 1 in Definition 4.13 is easily checked by means of Proposition 4.18 and Remark 4.19. Item 2 follows from Proposition 5.7 and item 3 from Proposition 5.8. Notice that, in both cases, the properties are proved on $[\tau_{y,(v,\delta)},T]$ by applying Definition 4.13 itself for the solution restarted from the random initial condition $X_{\tau_{y,(v,\delta)}}^{y^*_{\tau_{y,(v,\delta)}}}$ (omitting the notation (v,δ) in $y(v,\delta)$) and driven by the shifted version $(W_{t+\tau_{y,(v,\delta)}}-W_{\tau_{y,(v,\delta)}})_{t\ge0}$ of the noise.

The main difficulty is to check item 4 in Definition 4.13. As above, it is in fact easily verified on $[\tau_{y,(v,\delta)},T]$ by applying Definition 4.13 for the solution restarted from the random initial condition $X_{\tau_{y,(v,\delta)}}^{y^*_{\tau_{y,(v,\delta)}}}$ (for instance, we may invoke item 4 for the restarted

solution on the interval $[\tau_{y,(v,\delta)}, T+\tau_{y,(v,\delta)}]$ and then use the non-decreasing property of the integral to get the result on $[\tau_{y,(v,\delta)}, T]$). The key point is thus to prove that, for almost every $y \in E^M$.

$$\lim_{\varepsilon \searrow 0} \mathbb{E}_{\mathbb{Q}^{y,(v,\delta),\tau}} \left[\int_0^{\tau_{y,(v,\delta)}} e^{2\varepsilon\Delta} X_s^{y_s^*(v,\delta)} \cdot d\tilde{\eta}_s^{y,(v,\delta)} \right] = 0.$$

By Proposition 4.16, it suffices to prove that the above convergence holds, for almost every $y \in E^M$, in $\mathbb{Q}^{y,(v,\delta),\tau}$ probability, along a subsequence. Moreover, by the localisation procedure (5.25), we have a bound on the moments of the density $d\mathbb{Q}^{y,(v,\delta),\tau}/d\mathbb{P}$ and it suffices to establish the convergence in $\mathbb{P}$ probability only. Actually, since the integral is non-decreasing in time, it is sufficient to address the convergence for the integral on the entire $[0,T]$. The argument is as follows. For $\varphi \in \mathcal{C}_0^\infty(E^M)$ with non-negative values, Proposition 5.4 yields, with probability 1,

$$\begin{aligned}
&\sum_{m\in\mathbb{N}_0} \int_{E^M} \int_0^T \left\langle e^{\varepsilon\Delta} X_t^{y^*}, e_m \right\rangle \varphi\big(y_t(v,-\delta)\big) d\left\langle \eta_t^{y^*}, e_m \right\rangle dy \\
&= \lim_{N\to\infty} \int_{E^M} \varphi(y) \sum_{i=0}^{N-1} \left[\left\langle \tilde{\eta}_{r_{i+1}}^{y,(v,\delta)}, e^{\varepsilon\Delta} X_{r_i}^{y_{r_i}^*(v,\delta)} \right\rangle - \left\langle \tilde{\eta}_{r_i}^{y,(v,\delta)}, e^{\varepsilon\Delta} X_{r_i}^{y_{r_i}^*(v,\delta)} \right\rangle \right] dy.
\end{aligned}$$

In the above identity, the first line is handled as (5.16). As for the second line, we can invoke Proposition 5.7 and Corollary 4.8, and then regard the sum therein as a Riemann sum associated with the integral

$$\int_0^T \big(e^{\varepsilon\Delta} X_t^{y_t^*(v,\delta)} \big) \cdot d\tilde{\eta}_t^{y,(v,\delta)}.$$

Recalling the inequality (4.27) together with Corollary 5.3 and the fact that, with probability 1, the flow $((X_t^y)_{0\le t\le T})_{y\in E^{*,M}}$ is jointly continuous (with values in $L^2_{\mathrm{sym}}(\mathbb{S})$), we can easily exchange the limit (over N) and the integral (in y) in the second line. Therefore, we have, with probability 1,

$$\begin{aligned}
&\int_{E^M} \left(\int_0^T \left[\varphi\big(y_t^*(v,-\delta)\big) e^{\varepsilon\Delta} X_t^{y^*} \right] \cdot d\eta_t^{y^*} \right) dy \\
&= \int_{E^M} \varphi(y) \left(\int_0^T \big(e^{\varepsilon\Delta} X_t^{y_t^*(v,\delta)} \big) \cdot d\tilde{\eta}_t^{y,(v,\delta)} \right) dy.
\end{aligned} \tag{5.28}$$

By combining Proposition 4.3, (4.24), (4.27), Corollary 5.3 and Proposition 4.18, we can easily have a bound for (the expectation below is taken under $\mathbb{P}$)

$$\mathbb{E}\left[\int_0^T \left[\varphi\big(y_t(v,-\delta)\big) e^{\varepsilon\Delta} X_t^{y^*} \right] \cdot d\eta_t^{y^*} \right]$$

that is uniform with respect to $\varepsilon \in (0,1)$ and to y in compact subsets of E^M. Therefore, by item 4 in Definition (4.13), we deduce that the expectation of the left-hand side in (5.28) tends to 0 (with ε). Then, the expectation of the right-hand side also tends to 0

(with ε). Recalling that φ is non-negative valued and assuming that φ matches 1 on a given compact subset of E^M, we deduce from Fatou's lemma that, for any $R > 0$,

$$\int_{E^M} \mathbf{1}_{\{|y|\le R\}} \liminf_{\varepsilon \searrow 0} \mathbb{E}\Big[\int_0^T \big(e^{\varepsilon\Delta} X_t^{y_t^*(v,\delta)}\big) \cdot d\tilde{\eta}_t^{y,(v,\delta)}\Big] dy = 0.$$

Therefore, for almost every $y \in E^M$, there exists a subsequence $(\varepsilon_q)_{q\ge 1}$ (possibly depending on y), with 0 as limit, such that

$$\lim_{q\to\infty} \mathbb{E}\Big[\int_0^T \big(e^{\varepsilon_q\Delta} X_t^{y_t^*(v,\delta)}\big) \cdot d\tilde{\eta}_t^{y,(v,\delta)}\Big] = 0,$$

which implies convergence in probability, as we claimed. □

5.5 Regularity

Now we arrive at the main statement of this section, which asserts that the semigroup generated by the solution of the rearranged SHE maps bounded functions into Lipschitz functions:

Theorem 5.11. *Assume that λ in (3.1) belongs to $(1/2, 1)$. Let $((X_t^{x^*})_{t\ge 0})_{x\in L^2_{\mathrm{sym}}(\mathbb{S})}$ be the flow generated by the rearranged SHE, as defined in Remark 4.19. Then, there exists a constant c_λ, only depending on the exponent λ such that, for any $t > 0$, for any bounded (measurable) function $f : L^2_{\mathrm{sym}}(\mathbb{S}) \to \mathbb{R}$, the function*

$$x \in L^2_{\mathrm{sym}}(\mathbb{S}) \mapsto \mathbb{E}\Big[f\big(X_t^{x^*}\big)\Big]$$

is Lipschitz continuous with $c_\lambda t^{-(1+\lambda)/2}$ as Lipschitz constant.

Remark 5.12. It is important to notice that, for $\lambda \in (1/2, 1)$, the exponent $(1+\lambda)/2$ is strictly less than 1. This guarantees that the rate at which the Lipschitz constant blows up when time becomes small is integrable. This observation is expected to have important applications for the analysis of related partial differential equations on $\mathcal{P}(\mathbb{R})$ when driven by the generator of the process $((X_t^x)_{t\ge 0})_{x\in L^2_{\mathrm{sym}}(\mathbb{S})}$.

Proof.
First step. We start with a bounded measurable function $f : L^2_{\mathrm{sym}}(\mathbb{S}) \to \mathbb{R}$. We are also given a threshold M as in (5.1), a time horizon $T > 0$ and an element $v \in E^M$. By Proposition 5.10, we know that for almost every $y \in E^M$, under $\mathbb{Q}^{y,(v,\delta),\tau}$, the process $(\widetilde{X}_t^{y,(v,\delta),\tau}, \tilde{\eta}_t^{y,(v,\delta),\tau})_{0\le t\le T}$ is the unique solution to the rearranged SHE started from $y+\delta v$ and driven by the tilted noise (5.27). By an obvious adaptation of the Yamada-Watanabe theorem (to which we already alluded before the proof of Theorem 4.15), we deduce that not only uniqueness holds in the strong sense (as guaranteed by Theorem 4.15) but it also holds in the weak sense. Therefore,

$$\mathbb{E}\Big[f\big(X_T^{(y+\delta v)^*}\big)\Big] = \mathbb{E}_{\mathbb{Q}^{y,(v,\delta),\tau}}\Big[f\big(\widetilde{X}_T^{y,(v,\delta),\tau}\big)\Big], \tag{5.29}$$

where we recall the notations (5.25) and (5.26).

Second step. Following the notation (5.25), (pay attention that the initial condition below is fixed and does not evolve with time) we let

$$\tilde{\tau}_{y,(v,\delta)} := \inf\left\{t \geq 0 : \int_0^t \sum_{m\in\mathbb{N}_0} \left|\lambda_m^{-1}\partial_y\langle e_m, X_s^{(y+\delta v)^*}\rangle \cdot v\right|^2 ds \geq \frac{2T^2}{\delta^2}\right\} \wedge T, \tag{5.30}$$

recalling that the derivative in the integral is well defined, with probability 1, for almost every (s,y). Since the law of $(\widetilde{X}_t^{y,(v,\delta),\tau})_{0\leq t\leq T}$ under $\mathbb{Q}^{y,(v,\delta),\tau}$ is the same as the law of $(X_t^{y+\delta v})_{0\leq t\leq T}$ under $\mathbb{P}$, one has

$$\begin{aligned}
\mathbb{Q}^{y,(v,\delta),\tau}\big(\{\tau_{y,(v,\delta)} \leq T\}\big) &= \mathbb{P}\big(\{\tilde{\tau}_{y,(v,\delta)} \leq T\}\big) \\
&\leq \mathbb{P}\left(\left\{\int_0^T \sum_{m\in\mathbb{N}_0} \left|\lambda_m^{-1}\partial_y\langle e_m, X_s^{(y+\delta v)^*}\rangle \cdot v\right|^2 ds \geq \frac{2T^2}{\delta^2}\right\}\right) \\
&\leq \frac{\delta^2}{2T^2}\mathbb{E}\left[\int_0^T \sum_{m\in\mathbb{N}_0} \left|\lambda_m^{-1}\partial_y\langle e_m, X_s^{(y+\delta v)^*}\rangle \cdot v\right|^2 ds\right] \\
&= \frac{\delta^2}{2T^2}\mathbb{E}\left[\int_0^T \sum_{m\in\mathbb{N}_0} \left|(1\vee m)^\lambda\partial_y\langle e_m, X_s^{(y+\delta v)^*}\rangle \cdot v\right|^2 ds\right].
\end{aligned} \tag{5.31}$$

To estimate the above, return to Proposition 4.18. Taking expectation in (4.45), we obtain for $\mu \in \mathbb{R}$ and $z \in E^M$,

$$\mathbb{E}\left[\int_0^T \|De^{\varepsilon\Delta}(X_r^{(y+\mu z)^*} - X_r^{y^*})\|_2^2 dr\right] \leq \mu^2\|z\|_2^2.$$

Letting $\varepsilon \searrow 0$ (by Fatou's lemma or by Corollary 4.12),

$$\mathbb{E}\left[\int_0^T \sum_{m\in\mathbb{N}_0} m^2\left|\langle (X_t^{(x+\mu z)^*} - X_t^{z^*}), e_m\rangle\right|^2 dt\right] \leq \mu^2\|z\|_2^2. \tag{5.32}$$

Dividing through by μ^2 and sending μ to zero, we deduce from Fatou's lemma that the last term in (5.31) is $\mathcal{O}(\delta^2\|v\|_2^2)$ (for almost every $y \in E^M$), where the (big) Landau symbol $\mathcal{O}(\cdot)$ is uniform with respect to y, δ and v.

Then, returning to equation (5.29), one has that, for almost every $y \in E^M$,

$$\mathbb{E}\Big[f\big(X_T^{(y+\delta v)^*}\big)\Big] \tag{5.33}$$

$$\begin{aligned}
&= \mathbb{E}_{\mathbb{Q}^{y,(v,\delta),\tau}}\Big[f\big(\widetilde{X}_T^{y,(v,\delta),\tau}\big)\mathbf{1}_{\{T=\tau_{y,(v,\delta)}\}}\Big] + \mathbb{E}_{\mathbb{Q}^{y,(v,\delta),\tau}}\Big[f\big(\widetilde{X}_T^{y,(v,\delta),\tau}\big)\mathbf{1}_{\{\tau_{y,(v,\delta)}<T\}}\Big] \\
&= \mathbb{E}_{\mathbb{Q}^{y,(v,\delta),\tau}}\Big[f\big(X_T^{y^*}\big)\mathbf{1}_{\{T=\tau_{y,(v,\delta)}\}}\Big] + \mathcal{O}\big(\delta^2\|v\|_2^2\big) \\
&= \mathbb{E}_{\mathbb{Q}^{y,(v,\delta),\tau}}\Big[f\big(X_T^{y^*}\big)\Big] + \mathcal{O}\big(\delta^2\|v\|_2^2\big).
\end{aligned} \tag{5.34}$$

Third step. By the second step, for $\delta \in \mathbb{R}$ and $v \in E^M$, and for almost every $y \in E^M$,

$$\begin{aligned}\Big|\mathbb{E}\Big[f\big(X_T^{(y+\delta v)^*}\big)\Big] - \mathbb{E}\big[f\big(X_T^{y^*}\big)\big]\Big| &\le \Big|\mathbb{E}_{\mathbb{Q}^{y,(v,\delta),\tau}}\Big[f\big(X_T^{y^*}\big)\Big] - \mathbb{E}\big[f\big(X_T^{y^*}\big)\big]\Big| + \mathcal{O}\big(\delta^2\|v\|_2^2\big)\\ &\le \|f\|_\infty \mathrm{d}_{TV}\big(\mathbb{Q}^{y,(v,\delta),\tau},\mathbb{P}\big) + \mathcal{O}\big(\delta^2\|v\|_2^2\big).\end{aligned}$$

where d_{TV} is the distance in total variation (see [74, p. 22]) By Pinsker inequality (see [74, Eq. (22.25)]), we get

$$\Big|\mathbb{E}\Big[f\big(X_T^{(y+\delta v)^*}\big)\Big] - \mathbb{E}\big[f\big(X_T^{y^*}\big)\big]\Big| \le \|f\|_\infty \sqrt{2\,\mathbb{E}_{\mathbb{Q}^{y,(v,\delta),\tau}}\Big(\ln\Big[\frac{d\mathbb{Q}^{y,(v,\delta),\tau}}{d\mathbb{P}}\Big]\Big)} + \mathcal{O}\big(\delta^2\|v\|_2^2\big).$$

By (5.26) and (5.32),

$$\begin{aligned}&\Big|\mathbb{E}\Big[f\big(X_T^{(y+\delta v)^*}\big)\Big] - \mathbb{E}\big[f\big(X_T^{y^*}\big)\big]\Big|\\ &\le \|f\|_\infty \sqrt{2\mathbb{E}_{\mathbb{Q}^{y,(v,\delta),\tau}}\Big(\ln\Big[\mathcal{E}_T\big\{\tfrac{\delta}{T}L^{y,(v,\delta)}_{\cdot\wedge\tau_{y,(v,\delta)}}\big\}\Big]\Big)} + \mathcal{O}\big(\delta^2\|v\|_2^2\big)\\ &= \|f\|_\infty \frac{\delta}{T}\Bigg(\mathbb{E}_{\mathbb{Q}^{y,(v,\delta),\tau}}\Bigg[\int_0^{\tau_{y,(v,\delta)}} \sum_{m\in\mathbb{N}_0} \Big|\lambda_m^{-1}\partial_y\big\langle e_m, X_t^{y_t^*(v,\delta)}\big\rangle\cdot v\Big|^2 dt\Bigg]\Bigg)^{1/2} + \mathcal{O}\big(\delta^2\|v\|_2^2\big).\end{aligned}$$

Since the law of $(X_t^{y_t^*(v,\delta)} = \tilde{X}_t^{y,(v,\delta),\tau})_{0\le t\le \tau_{y,(v,\delta)}}$ under $\mathbb{Q}^{y,(v,\delta),\tau}$ is the same as the law of $(X_t^{y+\delta v})_{0\le t\le T}$ under $\mathbb{P}$, we get

$$\begin{aligned}&\Big|\mathbb{E}\Big[f\big(X_T^{(y+\delta v)^*}\big)\Big] - \mathbb{E}\big[f\big(X_T^{y^*}\big)\big]\Big|\\ &\le \|f\|_\infty \frac{\delta}{T}\Bigg(\mathbb{E}\Bigg[\int_0^{\tau_{y,(v,\delta)}} \sum_{m\in\mathbb{N}_0} \Big|\lambda_m^{-1}\partial_y\big\langle e_m, X_t^{(y+\delta v)^*}\big\rangle\cdot v\Big|^2 dt\Bigg]\Bigg)^{1/2} + \mathcal{O}\big(\delta^2\|v\|_2^2\big)\\ &\le \|f\|_\infty \frac{\delta}{T}\Bigg(\mathbb{E}\Bigg[\int_0^{T} \sum_{m\in\mathbb{N}_0} \Big|\lambda_m^{-1}\partial_y\big\langle e_m, X_t^{(y+\delta v)^*}\big\rangle\cdot v\Big|^2 dt\Bigg]\Bigg)^{1/2} + \mathcal{O}\big(\delta^2\|v\|_2^2\big).\end{aligned} \tag{5.35}$$

By Cauchy-Schwarz inequality and by (4.45)–(5.32) and Proposition 4.18,

$$\begin{aligned}\mathbb{E}\Bigg[\int_0^T \sum_{m\in\mathbb{N}_0} \Big|\lambda_m^{-1}\partial_y\big\langle e_m, X_t^{y+\delta v}\big\rangle\cdot v\Big|^2 dt\Bigg] &= \mathbb{E}\Bigg[\int_0^T \sum_{m\in\mathbb{N}_0} \Big|(1\vee m)^\lambda\partial_y\big\langle e_m, X_t^{(y+\delta v)^*}\big\rangle\cdot v\Big|^2 dt\Bigg]\\ &\le \Bigg(\mathbb{E}\Bigg[\int_0^T \sum_{m\in\mathbb{N}_0} (1\vee m)^2\Big|\partial_y\big\langle e_m, X_t^{(y+\delta v)^*}\big\rangle\cdot v\Big|^2 dt\Bigg]\Bigg)^{\lambda}\\ &\quad\times\Bigg(\mathbb{E}\Bigg[\int_0^T \sum_{m\in\mathbb{N}_0} \Big|\partial_y\big\langle e_m, X_t^{(y+\delta v)^*}\big\rangle\cdot v\Big|^2 dt\Bigg]\Bigg)^{1-\lambda}\\ &\le c^{1-\lambda}\|v\|_2^2 T^{1-\lambda},\end{aligned} \tag{5.36}$$

for a constant c independent of y, v and λ.

By combining (5.35) and (5.36), we get for every $\delta \in \mathbb{R}$ and $v \in E^M$ and for almost every $y \in E^M$,

$$\Big|\mathbb{E}\Big[f\big(X_T^{(y+\delta v)^*}\big)\Big] - \mathbb{E}\big[f\big(X_T^{y^*}\big)\big]\Big| \le c_\lambda \|f\|_\infty T^{-(1+\lambda)/2}\delta\|v\|_2 + \mathcal{O}\big(\delta^2\|v\|_2^2\big). \tag{5.37}$$

Fourth step. We now assume that f itself is Lipschitz continuous. By Lipschitz continuity of the flow $(X_T^{y^*})_{y\in E^M}$, the mapping $y \in E^M \mapsto \mathbb{E}[f(X_T^{y^*})]$ is Lipschitz continuous (with respect to the Euclidean norm on E^M, which coincides in fact with the $L^2(\mathbb{S})$-norm by Parseval's identity). Therefore, (5.37) holds true for any $\delta \in \mathbb{R}$ and $y, v \in E^M$. Dividing (5.37) by δ and then letting δ tend to 0, we get that the almost everywhere gradient of $y \in E^M \mapsto \mathbb{E}[f(X_T^{y^*})]$ is less than $c_\lambda\|f\|_\infty T^{-(1+\lambda)/2}$. Therefore, the Lipschitz constant of $y \in E^M \mapsto \mathbb{E}[f(X_T^{y^*})]$ is less than $c_\lambda\|f\|_\infty T^{-(1+\lambda)/2}$, with E^M being equipped with the $L^2_{\mathrm{sym}}(\mathbb{S})$-norm. Approximating any $y \in L^2_{\mathrm{sym}}(\mathbb{S})$ by a sequence $(y^M \in E^M)_{M\ge1}$, we deduce that the result remains true when the function $y \mapsto \mathbb{E}[f(X_T^{y^*})]$ is considered on the entire $L^2_{\mathrm{sym}}(\mathbb{S})$.

It remains to pass from a Lipschitz function f to a merely bounded (measurable) function, but this may be regarded as a consequence of standard results in measure theory, see for instance [63, Lem 2.2 p.160]. □

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