



HAL
open science

Calculi for Basic Atomic Logics

Guillaume Aucher

► **To cite this version:**

Guillaume Aucher. Calculi for Basic Atomic Logics. [Research Report] Université de Rennes. 2024. hal-03800002v2

HAL Id: hal-03800002

<https://hal.science/hal-03800002v2>

Submitted on 30 Jun 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution - NonCommercial - NoDerivatives 4.0 International License

Calculi for Basic Atomic Logics

Guillaume Aucher
Univ Rennes, CNRS, IRISA, IRMAR,
263, Avenue du Général Leclerc,
35042 Rennes Cedex, France
guillaume.aucher@univ-rennes.fr

June 30, 2024

Abstract

Atomic logics are based on Dunn’s gaggle theory and generalize modal logic and the Lambek calculus. Sound and complete Hilbert, display and sequent calculi for basic atomic logics with a Kripke-style relational semantics are introduced. These calculi can be automatically computed from the definition of the connectives defining a basic atomic logic. Sufficient conditions for the cut admissibility of these calculi are given based on the shape of the connectives considered. Also, a novel Hilbert axiomatization of modal logic is found out by applying our general results to modal logic. Finally, we prove that basic atomic logics are PSPACE-complete and compact.

1 Introduction

The main reason why practitioners turn to non-classical logics and modal logics instead of classical logic is that these logics often remain decidable while providing sufficient expressive power [41]. However, from a theoretical and formal point of view, non-classical logics are still disorganized and scattered and somehow miss a common formal ground. In response to that situation, a number of frameworks and approaches have been proposed or developed further, such as the algebraic approach to logics [26, 27, 21] or the category-theoretical approach based on abstract model theory and “institutions” [7, 13], or frameworks such as the “labelled deductive systems” of Gabbay [20] or the “basic logic” of Sambin & al. [40], *etc.* Within that thread of research, which is closely related to “universal logic” [10], the framework of atomic and molecular logics provides a uniform and generic way to explore and study non-classical logics, on the basis of a generalized Kripke-style relational semantics. Atomic and molecular logics were introduced in [5] and can be (somehow) seen as normal forms for logics. In a sense, they are a generalization and an ‘incarnation’ into a logical framework of Dunn’s Gaggle theory [14, 15]. One can define and compute automatically notions of bisimulations for any atomic and molecular logic and the model theory of non-classical logics can be developed in a systematic way [6]. Likewise, they allow us to develop in a systematic and uniform way the proof theory of non-classical logics as we shall see.

There are five main contributions in this article. The first is to introduce Hilbert (Fig. 4), display (Fig. 6) and sequent calculi for basic atomic logics which are sound and complete w.r.t. a Kripke-style relational semantics. An important feature of our approach is that, like for their bisimulation notions, all our calculi can be automatically computed from the definition

of the connectives of the atomic logics. The second main contribution is to introduce a novel axiomatization of modal logic (Fig. 5), by applying our general results to it. The third is to give sufficient conditions on sets of atomic connectives (Definition 10) for obtaining a proper display calculus admitting the cut rule. The fourth main contribution is to somehow show that ‘any logic whose connectives are monotone in their arguments can be given a Kripke-style relational semantics’ (Theorem 6). The fifth is to prove the decidability, PSPACE-completeness and compactness of basic atomic logics (Theorem 1 and Corollary 1).

Structure of the article. We introduce atomic logics in Section 2. In Section 3, we redefine in a simpler and more concise way the group actions on atomic connectives introduced in [2]. In Section 4, we introduce our Hilbert calculi and in Section 5 our display calculi for basic atomic logics. In Section 6, we show how, in case the atomic connectives are so-called purely displayable, we obtain sequent calculi for our basic atomic logics without any structural connective; we illustrate this general result with the Lambek calculus. We also briefly discuss Lyndon theorem and the role of monotony with respect to the Kripke-style relational semantics. We end in Section 7 by discussing related work and conclude. All the proofs are in the appendix.

2 Atomic logics

Molecular logics are logics whose primitive connectives are compositions of connectives of atomic logics. Atomic logics are logics in which the truth conditions of connectives are defined by first-order formulas of the form $\forall x_1 \dots x_n (\pm_1 Q_1 x_1 \vee \dots \vee \pm_n Q_n x_n \vee \pm R x_1 \dots x_n x)$ or $\exists x_1 \dots x_n (\pm_1 Q_1 x_1 \wedge \dots \wedge \pm_n Q_n x_n \wedge \pm R x_1 \dots x_n x)$ where the \pm_i s and \pm are either empty or \neg . Likewise, propositional letters are defined by first-order formulas of the form $\pm R x$. They are viewed as 0-ary connectives (which is why we note them R and not Q) and the \pm in front of them stands for the fact that they can stand for literals. We will represent the structure of these formulas by means of so-called *skeletons* whose various arguments capture the different features and patterns from which they can be redefined completely. At the same time, atomic logics are generalizations of Dunn’s gaggle logics from the relevant logic tradition, [2], with dimensions associated to formulas.

Notations We first introduce some notations. \mathbb{N}^* denotes the set of natural numbers without 0 and, for all $n \in \mathbb{N}^*$, $[[1; n]]$ denotes the set of natural numbers $\{1, \dots, n\}$. \mathfrak{S}_n denotes the group of permutations over the set $\{1, \dots, n\}$. Permutations are generally denoted σ, τ , the identity permutation is denoted Id and σ^- stands for the inverse permutation of the permutation σ . For example, the permutation $\sigma = (3, 1, 2)$ is the permutation that maps 1 to 3, 2 to 1 and 3 to 2. We recall that $\mathbb{Z}/2\mathbb{Z}$ denotes the field of the integers modulo 2 (also known as the dihedral group of order 2).¹ When viewed as a multiplicative group, its elements will be denoted in the sequel $+$ and $-$ and its operation \cdot is such that $+\cdot- = -\cdot+ = -$ and $-\cdot- = +\cdot+ = +$. For brevity, when we use the notation $+, -$, we often omit the \cdot and write for example $+- = -$ for $+\cdot- = -$. See [37] for relevant details on group theory.

For example, the skeleton of the connectives \square of modal logic, represented by the first-order formula $\bullet_1(x) \triangleq \forall y (P y \vee \neg R x y)$, is $((1, 1), +, +, -, (2, 1))$, which we also write for better readability $((1, 1), \forall, +, -, (2, 1))$. The so-called ‘dimension signature’ $(1, 1)$ corresponds to the fact that this connective takes as input a formula of dimension 1, represented by the predicate P of arity 1, and

¹The group $\mathbb{Z}/2\mathbb{Z}$ was used to define atomic connectives by Espejo-Boix [19, Definition 3]. He used both the additive and multiplicative operations of the field $\mathbb{Z}/2\mathbb{Z}$ to reformulate the central group action for atomic logics [2, Definition 18] in terms of a matrix product over $\mathbb{Z}/2\mathbb{Z}^{n+1}$.

yields another formula of dimension 1, because the first-order formula has a single free variable representing the state where the resulting formula is evaluated. The ‘quantification signature’ \forall corresponds to the universal quantification \forall in front of the formula and is represented by $+$ ($-$ represents the existential quantification \exists). The ‘tonicity signature’ is $+$ because there is no negation in front of P . The $-$ in the skeleton corresponds to the negation \neg in front of $\neg Rxy$. Finally, by convention, the natural order for elements appearing in a relation is $Ryzx$ or Ryx , and more generally $Rx_1 \dots x_n x$, where the free variable x denotes the state where the formula is evaluated. So in this example, Ryx is transformed into Rxy , which explains the introduction of the permutation $(2, 1) \in \mathfrak{S}_2$ which swaps x and y .

Definition 1 (Atomic connectives and skeletons). The set SKE_0 of *propositional letter skeletons* and the sets SKE_n of *skeletons of arity* $n \in \mathbb{N}^*$ are defined as follows:

$$\begin{aligned}\text{SKE}_0 &\triangleq \mathbb{N}^* \times \mathbb{Z} / 2\mathbb{Z}^2 \\ \text{SKE}_n &\triangleq \mathbb{N}^{*n+1} \times \mathbb{Z} / 2\mathbb{Z}^{n+2} \times \mathfrak{S}_{n+1}.\end{aligned}$$

The set of *atomic skeletons* is the set $\text{SKE} \triangleq \bigcup_{n \in \mathbb{N}} \text{SKE}_n$. They can be represented by tuples $(k_1, \dots, k_{n+1}, \mathbb{A}, \pm_1, \dots, \pm_n, \pm, \sigma)$, or (k, \mathbb{A}, \pm) if it is a propositional letter skeleton, where $(k_1, \dots, k_{n+1}) \in \mathbb{N}^{*n+1}$, often denoted \bar{k} , is called the *dimension signature*, $\mathbb{A} \in \{+, -\}$ is called the *quantification signature*, $(\pm_1, \dots, \pm_n) \in \{+, -\}^n$ is called the *tonicity signature*, $\pm \in \{+, -\}$ is called the *relation signature* and σ is called the *permutation signature*. The tuple $(\bar{k}, \mathbb{A}, \pm_1, \dots, \pm_n)$ is called the *signature* or *trace* of the skeleton; the tonicity and quantification signatures are in correspondence with Dunn’s notion of trace [15] (see [2, Definition 15] for details). The quantification signature \mathbb{A} will often be denoted \forall if it is $+$ and \exists if it is $-$. The *arity* of a propositional letter skeleton is 0 and its *dimension* is k . The *input dimensions* and (*output*) *dimension* of a connective skeleton $\bullet \in \text{SKE}_n$ of arity $n \in \mathbb{N}^*$ are k_1, \dots, k_n and k_{n+1} respectively.

Let \mathbb{I} be an arbitrary but fixed set; in this article we assume that $\mathbb{N} \subseteq \mathbb{I}$. The set ATM_n of *atomic connectives of arity* $n \in \mathbb{N}$ is defined as follows:

$$\text{ATM}_n \triangleq \{(\bullet, i) \mid i \in \mathbb{I}, \bullet \in \text{SKE}_n\}$$

The set of *atomic connectives* is the set $\text{ATM} \triangleq \bigcup_{n \in \mathbb{N}} \text{ATM}_n$. Those of arity 0, ATM_0 , are also called *propositional letters*. The arity, signature, quantification signature, dimension signature, tonicity signature, relation signature, permutation signature and input and output dimensions of an atomic connective (\bullet, i) are the same as its skeleton \bullet .

If \mathbb{C} is a set of atomic connectives, its set of propositional letters is denoted \mathbb{C}_0 . Propositional letters are denoted p, p_1, p_2, \dots, p_i , etc, skeletons are denoted $\bullet, \bullet_1, \bullet_2, \dots, \bullet_i$, etc. and connectives $\odot, \odot_1, \odot_2, \dots, \odot_i$, etc. The quantification signature of a connective \odot or a skeleton \bullet is denoted $\mathbb{A}(\odot) = \mathbb{A}(\bullet)$ and the j^{th} element of the tonicity signature of \odot and \bullet is denoted $\pm_j(\odot) = \pm_j(\bullet)$. \square

Definition 2 (Residuated skeletons and connectives). Two atomic skeletons $(\bar{k}, \mathbb{A}, \pm_1, \dots, \pm_n, \pm, \sigma)$ and $(\bar{k}', \mathbb{A}', \pm'_1, \dots, \pm'_n, \pm', \sigma')$ are *residuated* when they are of equal arity $n \in \mathbb{N}^*$ and there is $\tau \in \mathfrak{S}_{n+1}$ such that $\sigma' = \tau \circ \sigma$ and $(k'_{\tau(1)}, \dots, k'_{\tau(n+1)}) = (k_1, \dots, k_{n+1})$. Two atomic connectives $(\bullet, i), (\bullet', i') \in \text{ATM}_n$ are residuated when $i = i'$ and their skeletons \bullet and \bullet' are residuated.

Example 1. The skeleton of the connectives \Box_i of modal logic, for $i \in \mathbb{I}$, represented by the first-order formula $\bullet_1(x) \triangleq \forall y(Py \vee \neg Rxy)$, is $((1, 1), \forall, +, -, (2, 1))$. Indeed, the dimension signature

(1,1) corresponds to the fact that this connective takes as input a formula of dimension 1, represented by the predicate P of arity 1, and yields another formula of dimension 1, because the first-order formula has a single free variable representing the state where the resulting formula is evaluated. The quantification signature \forall corresponds to the universal quantification \forall in front of the formula. The tonicity signature is $+$ because there is no negation in front of P . The $-$ in the skeleton corresponds to the negation \neg in front of $\neg Rxy$. Finally, by convention, the natural order for elements appearing in a relation is $Ryzx$ or Ryx , and more generally $Rx_1 \dots x_n x$, where the free variable x denotes the state where the formula is evaluated. So in this example, Ryx is transformed into Rxy , which explains the introduction of the permutation $(2, 1) \in \mathfrak{S}_2$ which swaps x and y .

For another example, the skeleton of the implication \supset of the Lambek calculus, represented by the FO formula $\bullet_3(x) \triangleq \forall yz(\neg Py \vee Qz \vee \neg Rxyz)$, is $((1, 1, 1), \forall, -, +, -, (2, 3, 1))$ and the skeleton of the fusion \otimes of the Lambek calculus, represented by the FO formula $\bullet_2(x) \triangleq \exists yz(Py \wedge Qz \wedge Ryzx)$, is $((1, 1, 1), \exists, +, +, +, \text{Id})$. Note the permutations associated with the relations: $(2, 3, 1)$ for $Rxyz$ and $\text{Id} = (1, 2, 3)$ for $Ryzx$ because the latter order (y, z, x) is the natural order. As one can also easily notice, \supset and \otimes are residuated.

The full list of permutations of \mathfrak{S}_2 and \mathfrak{S}_3 as well as all unary and binary signatures of dimension $(1, 1)$ and $(1, 1, 1)$ are given in Fig. 1.

Definition 3 (Atomic language). Let $C \subseteq \text{ATM}$ be a set of atomic connectives. The *atomic language* \mathcal{L}_C associated to C is the smallest set that contains the propositional letters of C and that is closed under the atomic connectives of C while respecting the dimensions constraint. That is,

- $C_0 \subseteq \mathcal{L}_C$;
- for all $\odot \in C$ of arity $n > 0$ and of dimension signature (k_1, \dots, k_{n+1}) and for all $\varphi_1, \dots, \varphi_n \in \mathcal{L}_C$ of dimensions k_1, \dots, k_n respectively, we have that $\odot(\varphi_1, \dots, \varphi_n) \in \mathcal{L}_C$ and $\odot(\varphi_1, \dots, \varphi_n)$ is of *dimension* k_{n+1} .

Elements of \mathcal{L}_C are called *atomic formulas* and are denoted φ, ψ, \dots . The *dimension of a formula* $\varphi \in \mathcal{L}_C$ is denoted $k(\varphi)$. A set of atomic connectives C is *plain* if for all $\odot \in C$ of dimension signature (k_1, \dots, k_{n+1}) with $n > 0$ there are propositional letters $p_1, \dots, p_n \in C_0$ of dimensions k_1, \dots, k_n respectively. *In the sequel, we assume that all sets of connectives C are plain.*

For example, the set of atomic connectives $C = \{p, \odot, \otimes\}$ where p is a propositional letter of dimension 1, \odot a connective of arity 1 and dimension $(2, 1)$ and \otimes the connective of arrow logic of dimension signature $(2, 2, 2)$ is not a plain set of connectives. Our assumption that all sets of connectives C considered are plain makes sense. Indeed, we want all connectives of C to appear in some formula of \mathcal{L}_C . If C was not plain then there would be a connective of C which would be necessarily composed with another connective of C , if we want such a connective to appear in a formula of \mathcal{L}_C . For example \otimes could only happen in a formula of the form $\odot p \otimes \odot p$. Yet, in that case, we should instead view C as a set of *molecular* connectives (see next section).

Definition 4 (Atomic C -models and C -frames). Let $C \subseteq \text{ATM}$ be a set of atomic connectives. An *(atomic) C -model* is a tuple $\mathcal{M} = (W, \mathcal{R})$ where W is a non-empty set and \mathcal{R} is a set of relations over W such that each n -ary connective $\odot \in C$ of dimension signature (k_1, \dots, k_{n+1}) is associated to a $k_1 + \dots + k_{n+1}$ -ary relation $R_\odot \in \mathcal{R}$ and such that for all connectives $\odot, \odot' \in C$ we have that $R_\odot = R_{\odot'}$ iff \odot and \odot' are residuated. An *assignment* is a tuple $(w_1, \dots, w_k) \in W^k$ for some $k \in \mathbb{N}^*$, generally denoted \bar{w} . The set of assignments of a C -model \mathcal{M} is denoted $\omega(\mathcal{M}, C)$.

A *pointed C-model* (\mathcal{M}, \bar{w}) is a C-model \mathcal{M} together with an assignment \bar{w} and, in that case, we say that (\mathcal{M}, \bar{w}) is of *dimension* k . The class of all pointed C-models is denoted \mathcal{C}_C .

A *(pointed) atomic C-frame* is a (pointed) atomic (C-ATM₀)-model. The class of all pointed C-frames is denoted \mathcal{F}_C .

Definition 5 (Atomic logics). Let $C \subseteq \text{ATM}$ be a set of atomic connectives and let $\mathcal{M} = (W, \mathcal{R})$ be a C-model. We define the *interpretation function of \mathcal{L}_C in \mathcal{M}* , denoted $\llbracket \cdot \rrbracket^{\mathcal{M}} : \mathcal{L}_C \rightarrow \bigcup_{k \in \mathbb{N}^*} W^k$, inductively as follows: for all propositional letters $p \in C$ of dimension k , all connectives $\odot \in C$ of skeleton $(k_1, \dots, k_{n+1}, \mathbb{A}, \pm_1, \dots, \pm_n, \pm, \sigma)$ of arity $n > 0$, for all $\varphi_1, \dots, \varphi_n \in \mathcal{L}_C$,

$$\begin{aligned} \llbracket p \rrbracket^{\mathcal{M}} &\triangleq \begin{cases} R_p & \text{if } \pm = + \\ W^k - R_p & \text{if } \pm = - \end{cases} \\ \llbracket \odot(\varphi_1, \dots, \varphi_n) \rrbracket^{\mathcal{M}} &\triangleq f_{\odot}(\llbracket \varphi_1 \rrbracket^{\mathcal{M}}, \dots, \llbracket \varphi_n \rrbracket^{\mathcal{M}}) \end{aligned}$$

where the function f_{\odot} is defined as follows. For all $W_1 \in \mathcal{P}(W^{k_1}), \dots, W_n \in \mathcal{P}(W^{k_n})$, $f_{\odot}(W_1, \dots, W_n) \triangleq \{\bar{w}_{n+1} \in W^{k_{n+1}} \mid \mathcal{C}^{\odot}(W_1, \dots, W_n, \bar{w}_{n+1})\}$ where $\mathcal{C}^{\odot}(W_1, \dots, W_n, \bar{w}_{n+1})$ is called the *truth condition* of \odot and is defined as follows:

- if $\mathbb{A} = \forall$: “ $\forall \bar{w}_1 \in W^{k_1} \dots \bar{w}_n \in W^{k_n}$
 $(\bar{w}_1 \upharpoonright_1 W_1 \vee \dots \vee \bar{w}_n \upharpoonright_n W_n \vee R_{\odot}^{\pm\sigma} \bar{w}_1 \dots \bar{w}_n \bar{w}_{n+1})$ ”;
- if $\mathbb{A} = \exists$: “ $\exists \bar{w}_1 \in W^{k_1} \dots \bar{w}_n \in W^{k_n}$
 $(\bar{w}_1 \upharpoonright_1 W_1 \wedge \dots \wedge \bar{w}_n \upharpoonright_n W_n \wedge R_{\odot}^{\pm\sigma} \bar{w}_1 \dots \bar{w}_n \bar{w}_{n+1})$ ”;

where, for all $j \in \llbracket 1; n \rrbracket$, $\bar{w}_j \upharpoonright_j W_j \triangleq \begin{cases} \bar{w}_j \in W_j & \text{if } \pm_j = + \\ \bar{w}_j \notin W_j & \text{if } \pm_j = - \end{cases}$ and

$R_{\odot}^{\pm\sigma} \bar{w}_1 \dots \bar{w}_{n+1}$ holds iff $\pm R_{\odot} \bar{w}_{\sigma(1)} \dots \bar{w}_{\sigma(n+1)}$ holds, with the notations $+R_{\odot} \triangleq R_{\odot}$ and $-R_{\odot} \triangleq W^{k_1 + \dots + k_{n+1}} - R_{\odot}$. We extend the definition of the interpretation function $\llbracket \cdot \rrbracket^{\mathcal{M}}$ to C-frames as follows: for all $\varphi \in \mathcal{L}_C$ and all C-frames \mathfrak{F} ,

$$\llbracket \varphi \rrbracket^{\mathfrak{F}} \triangleq \bigcap \left\{ \llbracket \varphi \rrbracket^{(\mathfrak{F}, V)} \mid V \text{ a set of } n\text{-ary relations over } W \text{ such that } (\mathfrak{F}, V) \text{ is a C-model} \right\}$$

Finally, if $\mathcal{E}_C \subseteq \mathcal{C}_C$ is a class of pointed C-models, the *satisfaction relation* $\Vdash \subseteq \mathcal{E}_C \times \mathcal{L}_C$ is defined as follows: for all $\varphi \in \mathcal{L}_C$ and all $(\mathcal{M}, \bar{w}) \in \mathcal{E}_C$, $((\mathcal{M}, \bar{w}), \varphi) \in \Vdash$ iff $\bar{w} \in \llbracket \varphi \rrbracket^{\mathcal{M}}$. We usually write $(\mathcal{M}, \bar{w}) \Vdash \varphi$ instead of $((\mathcal{M}, \bar{w}), \varphi) \in \Vdash$ and we say that φ is *true* in (\mathcal{M}, \bar{w}) . The logic $(\mathcal{L}_C, \mathcal{E}_C, \Vdash)$ is the *atomic logic associated to \mathcal{E}_C and C*. Logics of the form $(\mathcal{L}_C, \mathcal{C}_C, \Vdash)$ are called *basic atomic logics*.

The \pm sign in $R_{\odot}^{\pm\sigma}$ is the \pm sign in $(\bar{k}, \mathbb{A}, \pm_1, \dots, \pm_n, \pm, \sigma)$. The permutations σ in the skeletons of the connectives do not play an important role in this article, but they are necessary to completely define the truth conditions of the connectives. They play an important role in the proof theory of atomic logics, in particular in the display rules and the residuation phenomena (see [2] for more details).

Example 2. A simple example of an atomic logic is modal logic, where $C = \{p, \top, \perp, \neg, \wedge, \vee, \diamond_i, \square_i \mid i \in \mathbb{I}\}$. We spell this example out in some detail:

- p is a proposition letter of dimension 1 and \top, \perp are the proposition letters $((1, \exists, +), 0)$ and $((1, \forall, -), 0)$ respectively;
- \neg is the connective $((1, 1), \exists, -, +, \text{Id}), 0$;

Permutations of \mathfrak{S}_2	Unary signatures
$\tau_1 = (1, 2)$	$t_1 = ((1, 1), \exists, +)$
$\tau_2 = (2, 1)$	$t_2 = ((1, 1), \forall, +)$
	$t_3 = ((1, 1), \forall, -)$
	$t_4 = ((1, 1), \exists, -)$
Permutations of \mathfrak{S}_3	Binary signatures
$\sigma_1 = (1, 2, 3)$	$s_1 = ((1, 1, 1), \exists, (+, +))$
$\sigma_2 = (3, 2, 1)$	$s_2 = ((1, 1, 1), \forall, (+, -))$
$\sigma_3 = (2, 3, 1)$	$s_3 = ((1, 1, 1), \forall, (-, +))$
$\sigma_4 = (2, 1, 3)$	$s_4 = ((1, 1, 1), \forall, (+, +))$
$\sigma_5 = (3, 1, 2)$	$s_5 = ((1, 1, 1), \exists, (+, -))$
$\sigma_6 = (1, 3, 2)$	$s_6 = ((1, 1, 1), \exists, (-, +))$
	$s_7 = ((1, 1, 1), \exists, (-, -))$
	$s_8 = ((1, 1, 1), \forall, (-, -))$

Figure 1: Permutations of \mathfrak{S}_2 and \mathfrak{S}_3 and ‘orbits’ of unary and binary signatures

- $\wedge, \vee, \diamond_i, \square_i$ are the connectives $((1, 1, 1), \exists, +, +, +, \text{Id}, 0)$, $((1, 1, 1), \forall, +, +, -, \text{Id}, 0)$, $((1, 1, 1), \exists, +, +, (2, 1), i)$ and $((1, 1, 1), \forall, +, -, (2, 1), i)$ respectively;
- the C-models $\mathcal{M} = (W, \mathcal{R}) \in \mathcal{E}_C$ are such that $R_{\neg} \triangleq \{(w, w) \mid w \in W\}$, $R_{\wedge} = R_{\vee} \triangleq \{(w, w, w) \mid w \in W\}$, $R_{\diamond_i} = R_{\square_i}$ for all $i \in \mathbb{I}$ and $R_{\top} = R_{\perp} = W$.

With these conditions on the C-models of \mathcal{E}_C , for all $(\mathcal{M}, w) \in \mathcal{E}_C$, for all $i \in \mathbb{I}$,

$w \in \llbracket \diamond_i \varphi \rrbracket^{\mathcal{M}}$	iff	$\exists v (v \in \llbracket \varphi \rrbracket^{\mathcal{M}} \wedge R_{\diamond_i} wv)$
$w \in \llbracket \square_i \varphi \rrbracket^{\mathcal{M}}$	iff	$\forall v (v \in \llbracket \varphi \rrbracket^{\mathcal{M}} \vee -R_{\square_i} wv)$
$w \in \llbracket \wedge(\varphi, \psi) \rrbracket^{\mathcal{M}}$	iff	$\exists vu (v \in \llbracket \varphi \rrbracket^{\mathcal{M}} \wedge u \in \llbracket \psi \rrbracket^{\mathcal{M}} \wedge R_{\wedge} vuw)$
	iff	$w \in \llbracket \varphi \rrbracket^{\mathcal{M}} \wedge w \in \llbracket \psi \rrbracket^{\mathcal{M}}$
$w \in \llbracket \vee(\varphi, \psi) \rrbracket^{\mathcal{M}}$	iff	$\forall vu (v \in \llbracket \varphi \rrbracket^{\mathcal{M}} \vee u \in \llbracket \psi \rrbracket^{\mathcal{M}} \vee -R_{\vee} vuw)$
	iff	$w \in \llbracket \varphi \rrbracket^{\mathcal{M}} \vee w \in \llbracket \psi \rrbracket^{\mathcal{M}}$
$w \in \llbracket \neg \varphi \rrbracket^{\mathcal{M}}$	iff	$\exists v (v \notin \llbracket \varphi \rrbracket^{\mathcal{M}} \wedge R_{\neg} vw)$
	iff	$w \notin \llbracket \varphi \rrbracket^{\mathcal{M}}$
$w \in \llbracket \top \rrbracket^{\mathcal{M}}$	iff	$w \in R_{\top}$
	iff	always
$w \in \llbracket \perp \rrbracket^{\mathcal{M}}$	iff	$w \in W - R_{\perp}$
	iff	never

Other examples of atomic connectives are given in Fig. 2, 3. Many others can be found in [2, 5], but the lists in these other articles are obviously not exhaustive.

The following theorem can be easily proven by polynomially reducing the satisfiability problem of basic atomic logic to the satisfiability problem of modal logic.

Theorem 1. *Every basic atomic logic is decidable and PSPACE-complete.*

Boolean atomic and molecular logics. Atomic and molecular logics do not include Boolean connectives as primitive connectives. In fact, they can be defined in terms of specific atomic connectives.

Atomic skeleton	Truth condition	Connective in the literature
The existentially positive orbit		
$(t_1, +, \tau_1) \varphi$	$\exists y (y \in \llbracket \varphi \rrbracket \wedge Ryx)$	$\diamond^- \varphi$ [35] \diamond_\downarrow [14]
$(t_2, -, \tau_2) \varphi$	$\forall y (y \in \llbracket \varphi \rrbracket \vee -Rxy)$	$\square \varphi$ [29]
The universally positive orbit		
$(t_2, +, \tau_1) \varphi$	$\forall y (y \in \llbracket \varphi \rrbracket \vee Ryx)$	$+_\downarrow \varphi$ [14] [17, p. 401]
$(t_1, -, \tau_2) \varphi$	$\exists y (y \in \llbracket \varphi \rrbracket \wedge -Rxy)$	[14]
The existentially negative orbit		
$(t_4, +, \tau_1) \varphi$	$\exists y (y \notin \llbracket \varphi \rrbracket \wedge Ryx)$	$?\varphi$ [14][17, p. 402] $\boxminus_1 \varphi$ [14][11, Def. 10.7.7]
$(t_4, +, \tau_2) \varphi$	$\exists y (y \notin \llbracket \varphi \rrbracket \wedge Rxy)$	$?\downarrow \varphi$ [14][18] [17, p. 402] $\boxminus_2 \varphi$ [11, Def. 10.7.7]
The universally negative orbit		
$(t_3, +, \tau_1) \varphi$	$\forall y (y \notin \llbracket \varphi \rrbracket \vee Ryx)$	φ^\perp [14, 16] φ° [23] $\diamond_1^- \varphi$ [11, Def. 10.7.2]
$(t_3, +, \tau_2) \varphi$	$\forall y (y \notin \llbracket \varphi \rrbracket \vee Rxy)$	$\sim \varphi$ [22] ${}^\perp \varphi$ [14, 16] ${}^\circ \varphi$ [23] $\diamond_2^- \varphi$ [11, Def. 10.7.2]
The symmetrical existentially positive orbit		
$(t_1, -, \tau_1) \varphi$	$\exists y (y \in \llbracket \varphi \rrbracket \wedge -Ryx)$	[14]
$(t_2, +, \tau_2) \varphi$	$\forall y (y \in \llbracket \varphi \rrbracket \vee Rxy)$	$+\varphi$ [14] [17, p. 402] φ^* [11, Def. 7.1.19]
The symmetrical universally positive orbit		
$(t_2, -, \tau_1) \varphi$	$\forall y (y \in \llbracket \varphi \rrbracket \vee -Ryx)$	$\square^- \varphi$ [35] \square_\downarrow [14]
$(t_1, +, \tau_2) \varphi$	$\exists y (y \in \llbracket \varphi \rrbracket \wedge Rxy)$	$\diamond \varphi$ [29]
The symmetrical existentially negative orbit		
$(t_4, -, \tau_1) \varphi$	$\exists y (y \notin \llbracket \varphi \rrbracket \wedge -Ryx)$	$?\varphi$ [14][11, Ex. 1.4.5] φ^1 [23]
$(t_4, -, \tau_2) \varphi$	$\exists y (y \notin \llbracket \varphi \rrbracket \wedge -Rxy)$	$?\downarrow \varphi$ [14] [11, Ex. 1.4.5] ${}^1 \varphi$ [23]
The symmetrical universally negative orbit		
$(t_3, -, \tau_1) \varphi$	$\forall y (y \notin \llbracket \varphi \rrbracket \vee -Ryx)$	[14]
$(t_3, -, \tau_2) \varphi$	$\forall y (y \notin \llbracket \varphi \rrbracket \vee -Rxy)$	$\neg_l \varphi$ [30, 36] $\perp \varphi$ [18]

Figure 2: The unary atomic connectives of dimension signature $(1, 1)$

Definition 6 (Boolean connectives). The *Boolean connectives* called *conjunctions*, *disjunctions*, *negations* and *Boolean constants* (of dimension k) are the atomic connectives denoted, respectively $\text{BOOL} \triangleq \{\wedge_k, \vee_k, \rightarrow_k, \top_k, \perp_k, \neg_k \mid k \in \mathbb{N}^*\}$ where

$$\begin{aligned}
\wedge_k &\triangleq ((k, k, k, \exists, +, +, +, \text{Id}), 0) & \top_k &\triangleq ((k, \exists, +), 0) \\
\vee_k &\triangleq ((k, k, k, \forall, +, +, -, \text{Id}), 0) & \perp_k &\triangleq ((k, \forall, -), 0) \\
\neg_k &\triangleq ((k, k, \exists, -, +, \text{Id}), 0) & \rightarrow_k &\triangleq ((k, k, k, \forall, -, +, (3, 1, 2)), 0)
\end{aligned}$$

Atomic skeleton	Truth condition	Connective in the literature
The conjunction orbit $\mathcal{O}_{\alpha_3}(\otimes)$		
$\varphi (s_1, +, \sigma_1) \psi$	$\exists yz (y \in \llbracket \varphi \rrbracket \wedge z \in \llbracket \psi \rrbracket \wedge Ryzx)$	$\varphi \otimes \psi$ [31]
$\varphi (s_2, -, \sigma_2) \psi$	$\forall yz (y \in \llbracket \varphi \rrbracket \vee z \notin \llbracket \psi \rrbracket \vee -Rxyz)$	φ / ψ [31]
$\varphi (s_2, -, \sigma_3) \psi$	$\forall yz (y \in \llbracket \varphi \rrbracket \vee z \notin \llbracket \psi \rrbracket \vee -Rzxy)$	$\varphi \subset \psi$ [36]
$\varphi (s_1, +, \sigma_4) \psi$	$\exists yz (y \in \llbracket \varphi \rrbracket \wedge z \in \llbracket \psi \rrbracket \wedge Rzxy)$	
$= \psi (s_1, +, \sigma_1) \varphi$		
$\varphi (s_3, -, \sigma_5) \psi$	$\forall yz (y \notin \llbracket \varphi \rrbracket \vee z \in \llbracket \psi \rrbracket \vee -Rxyz)$	$\varphi \supset \psi$ [38]
$= \psi (s_2, -, \sigma_2) \varphi$		
$\varphi (s_3, -, \sigma_6) \psi$	$\forall yz (y \notin \llbracket \varphi \rrbracket \vee z \in \llbracket \psi \rrbracket \vee -Ryxz)$	$\varphi \setminus \psi$ [31]
$= \psi (s_2, -, \sigma_3) \varphi$		

Figure 3: Some binary atomic connectives of dimension signature $(1, 1, 1)$

In any \mathbf{C} -model $\mathcal{M} = (W, \mathcal{R})$ containing Boolean connectives, the associated relation of any \vee_k, \wedge_k or \rightarrow_k is $R_{\wedge_k} = R_{\vee_k} = R_{\rightarrow_k} \triangleq \{(\bar{w}, \bar{w}, \bar{w}) \mid \bar{w} \in W^k\}$, the associated relation of \neg_k is $R_{\neg_k} \triangleq \{(\bar{w}, \bar{w}) \mid \bar{w} \in W^k\}$ and the associated relation of any \top_k or \perp_k is $R_{\perp_k} = R_{\top_k} \triangleq W^k$. We will often omit the subscript k in $\wedge_k, \vee_k, \rightarrow_k, \top_k, \perp_k, \neg_k$ when it is clear from the context and simply write $\wedge, \vee, \rightarrow, \top, \perp, \neg$. We denote by \mathbf{BOOL}^+ the set of all atomic connectives which are residuated with some connective of \mathbf{BOOL} and \mathbf{BOOL}_k those of dimension k .

We say that a set of atomic connectives \mathbf{C} is *complete for truth constants, conjunction and disjunction (resp. negation)* when it contains all truth constants, conjunctions and disjunctions $\top_k, \perp_k, \wedge_k, \vee_k$ (resp. Boolean negation \neg_k), for k ranging over all input types and output types of the atomic connectives of \mathbf{C} . We say that a set \mathbf{C} of atomic connectives is *Boolean* when it contains all conjunctions, disjunctions, material implications, constants as well as negations $\wedge_k, \vee_k, \rightarrow_k, \top_k, \perp_k, \neg_k$, for k ranging over all input dimensions and output dimensions of the connectives of \mathbf{C} . The *Boolean completion* of a set of atomic connectives \mathbf{C} is the smallest set of connectives including \mathbf{C} which is Boolean. A *Boolean atomic logic* is an atomic logic such that its set of connectives is Boolean. \square

Proposition 1. *Let \mathbf{C} be a set of atomic connectives containing Boolean connectives. and let $\mathcal{M} = (W, \mathcal{R})$ be a \mathbf{C} -model. Then, for all $k \in \mathbb{N}^*$, all $\varphi, \psi \in \mathcal{L}_{\mathbf{C}}$, if $k(\varphi) = k(\psi) = k$, then*

$$\begin{array}{lll}
\llbracket \top_k \rrbracket^{\mathcal{M}} \triangleq W^k & \llbracket \varphi \wedge_k \psi \rrbracket^{\mathcal{M}} \triangleq \llbracket \varphi \rrbracket^{\mathcal{M}} \cap \llbracket \psi \rrbracket^{\mathcal{M}} \\
\llbracket \perp_k \rrbracket^{\mathcal{M}} \triangleq \emptyset & \llbracket \varphi \vee_k \psi \rrbracket^{\mathcal{M}} \triangleq \llbracket \varphi \rrbracket^{\mathcal{M}} \cup \llbracket \psi \rrbracket^{\mathcal{M}} \\
\llbracket \neg_k \varphi \rrbracket^{\mathcal{M}} \triangleq W^k - \llbracket \varphi \rrbracket^{\mathcal{M}} & \llbracket \varphi \rightarrow_k \psi \rrbracket^{\mathcal{M}} \triangleq (W^k - \llbracket \varphi \rrbracket^{\mathcal{M}}) \cup \llbracket \psi \rrbracket^{\mathcal{M}}
\end{array}$$

It turns out that Boolean negation can also be simulated systematically at the level of atomic connectives by applying a transformation on them. The Boolean negation of a formula then boils down to taking the Boolean negation of the outermost connective of the formula. This transformation is defined as follows.

Definition 7 (Boolean negation). Let \odot be a n -ary connective of skeleton $(\bar{k}, \mathbb{A}, \pm_1, \dots, \pm_n, \pm, \sigma)$. The *Boolean negation* of \odot is the connective $-\odot$ of skeleton $(\bar{k}, -\mathbb{A}, -\pm_1, \dots, -\pm_n, -\pm, \sigma)$. If $\varphi = \odot(\varphi_1, \dots, \varphi_n)$ is an atomic formula, the *Boolean negation* of φ is the formula $-\varphi \triangleq -\odot(\varphi_1, \dots, \varphi_n)$. \square

Proposition 2. *Let \mathcal{C} be a set of atomic connectives such that $-\odot \in \mathcal{C}$ for all $\odot \in \mathcal{C}$. Let $\varphi \in \mathcal{L}_{\mathcal{C}}$ of dimension k and let $\mathcal{M} = (W, \mathcal{R})$ be a \mathcal{C} -model. For all $\bar{w} \in W^k$, $\bar{w} \in \llbracket -\varphi \rrbracket^{\mathcal{M}}$ iff $\bar{w} \notin \llbracket \varphi \rrbracket^{\mathcal{M}}$.*

3 Group actions and residuations

In [2], a group action of the symmetric group over the set of gaggle connectives was introduced. In this section, we are going to redefine it in a simpler and more concise way. It will turn out to play a crucial role, in particular in the expression of rule (dr3).

3.1 Some notions of group theory

Groups. A *group* (G, \circ) is a non-empty set G equipped with an associative operation $\circ : G \times G \rightarrow G$ and containing an element denoted Id_G called the *neutral element* such that: $\text{Id}_G \circ a = a = a \circ \text{Id}_G$ for all $a \in G$; for every $a \in G$, there is an element $b \in G$, also denoted a^- , such that $a \circ b = \text{Id}_G = b \circ a$. If X is a subset of a group G , then the smallest subgroup of G containing X , denoted by $\langle X \rangle$, is called the *subgroup generated by X* . For example, $\mathfrak{S}_n = \langle (1\ 2), (2\ 3), \dots, (i\ i+1), \dots, (n-1\ n) \rangle = \langle (n\ 1), (n\ 2), \dots, (n\ n-1) \rangle = \langle (n-1\ n), (1\ 2 \dots n) \rangle$.² In fact, if X is non-empty, then $\langle X \rangle$ is the set of all the words on X , that is, elements of G of the form $x_1^{\pm 1} x_2^{\pm 2} \dots x_n^{\pm n}$ where $x_1, \dots, x_n \in X$ and \pm_1, \dots, \pm_n are either $-$ or empty.

Free groups and free products. If X is a subset of a group F , then F is a *free group* with *basis* X if, for every group G and every function $f : X \rightarrow G$, there exists a unique homomorphism $\varphi : F \rightarrow G$ extending f . One can prove that a free group with basis X always exists and that X generates F . We therefore use the notation $F = \langle X \rangle$ also for free groups.

If G and H are groups, the *free product* of G and H is a group P and homomorphisms j_G and j_H such that, for every group Q and all homomorphisms $f_G : G \rightarrow Q$ and $f_H : H \rightarrow Q$, there exists a unique homomorphism $\varphi : P \rightarrow Q$ with $\varphi \circ j_G = f_G$ and $\varphi \circ j_H = f_H$. Such a group always exists and it is unique modulo isomorphism, we denote it $G * H$. This definition can be generalized canonically to the case of a finite number of groups G_1, \dots, G_n , yielding the free product $G_1 * \dots * G_n$.

Group actions. If X is a set and G a group, a (*left*) *action of G on X* is a function $\alpha : G \times X \rightarrow X$ given by $(g, x) \mapsto gx$ such that: $x\text{Id} = x$ for all $x \in X$; $(g_1 g_2)x = g_1(g_2 x)$ for all $x \in X$ and all $g_1, g_2 \in G$. If $x \in X$ and α an action of a group G on X , then the *orbit* of x under α is $\mathcal{O}_{\alpha}(x) \triangleq \{\alpha(g, x) \mid g \in G\}$. The orbits form a partition of X .

Let G and H be two groups. If α and β are actions of G and H on a set X , then the *free action* $\alpha * \beta$ is the mapping $\alpha * \beta : G * H \times X \rightarrow X$ defined by $\alpha * \beta(g, x) \triangleq \alpha(g_1, \dots, \alpha(\beta(h_{n-1}, \alpha(g_n, x))))$, where $g = g_1 h_1 \dots g_n h_n$ is the factorization of g in the free group $G * H$. This definition can be generalized canonically to the case of a finite number of actions $\alpha_1, \dots, \alpha_n$, yielding the mapping $\alpha_1 * \dots * \alpha_n$.

Then, one can easily show that if $\alpha_1, \dots, \alpha_n$ are actions of G_1, \dots, G_n on a set X respectively, then the mapping $\alpha_1 * \dots * \alpha_n$ is an action of the (free) group $G_1 * \dots * G_n$ on X .

²If $x \in \{1, \dots, n\}$ and $\sigma \in \mathfrak{S}_n$, then σ *fixes* x if $\sigma(x) = x$ and σ *moves* x if $\sigma(x) \neq x$. Let j_1, \dots, j_r be distinct integers between 1 and n . If $\sigma \in \mathfrak{S}_n$ fixes the remaining $n-r$ integers and if $\sigma(j_1) = j_2, \sigma(j_2) = j_3, \dots, \sigma(j_{r-1}) = j_r, \sigma(j_r) = j_1$ then σ is an *r -cycle*; one also says that σ is a cycle of *length* r . Denote σ by $(j_1\ j_2 \dots\ j_r)$. A 2-cycle which merely interchanges a pair of elements is called a *transposition*. See [37] for more details.

3.2 Group actions over atomic connectives

Below we define the group actions $\alpha_n, \beta_n, \gamma_n$ of the symmetric group and the dihedral group over the set of atomic connectives.³

Definition 8. Let $n \in \mathbb{N}^*$, $\bullet = (k_1, \dots, k_{n+1}, \mathbb{A}, \pm_1, \dots, \pm_n, \pm, \sigma) \in \text{SKE}_n$ and $\tau \in \mathfrak{S}_{n+1}$. For all $j \in \llbracket 0; n \rrbracket$, we first define $\Delta_j \triangleq \delta_j^\tau \pm_{\tau^{-1}(n+1)}$ where

$$\delta_j^\tau \triangleq \begin{cases} + & \text{if } j = \tau(n+1) \\ \delta_{n+1, \tau(n+1)} & \text{otherwise} \end{cases} \quad \delta_{n+1, \tau(n+1)} \triangleq \begin{cases} + & \text{if } n+1 = \tau(n+1) \\ - & \text{otherwise} \end{cases}$$

and we also set $\pm_{n+1} \triangleq +$ and $\delta_{n+1}^\tau \triangleq +$.⁴ Then, we define the function $a_n : \mathfrak{S}_{n+1} \times \text{SKE}_n \rightarrow \text{SKE}_n$ as follows:

$$a_n(\tau, \bullet) \triangleq (k_{\tau^{-1}(1)}, \dots, k_{\tau^{-1}(n+1)}, \Delta_0 \mathbb{A}, \Delta_1 \pm_{\tau^{-1}(1)}, \dots, \Delta_n \pm_{\tau^{-1}(n)}, \Delta_0 \pm, \tau \circ \sigma).$$

The function a_n induces a function $\alpha_n : \text{ATM} \times \mathfrak{S}_{n+1} \rightarrow \text{ATM}$ on the set ATM of connectives of arity n defined by $\alpha_n((\bullet, i), \tau) \triangleq (a_n(\tau, \bullet), i)$. Likewise, we define the functions $\beta_n : \mathbb{Z}/2\mathbb{Z} \times \text{ATM} \rightarrow \text{ATM}$ and $\gamma_n : \mathbb{Z}/2\mathbb{Z} \times \text{ATM} \rightarrow \text{ATM}$ by

$$\beta_n(\pm, (\bullet, i)) \triangleq \begin{cases} (-\bullet, i) & \text{if } \pm = - \\ (\bullet, i) & \text{if } \pm = + \end{cases} \quad \gamma_n(\pm, (\bullet, i)) \triangleq \begin{cases} (\sim \bullet, i) & \text{if } \pm = - \\ (\bullet, i) & \text{if } \pm = + \end{cases}$$

where $-\bullet \triangleq (k_1, \dots, k_{n+1}, -\mathbb{A}, -\pm_1, \dots, -\pm_n, -\pm, \sigma)$

and $\sim \bullet \triangleq (k_1, \dots, k_{n+1}, \mathbb{A}, \pm_1, \dots, \pm_n, -\pm, \sigma)$.

$a_n(\tau, \bullet), \alpha_n(\tau, \odot)$ and $\beta_n(\pm, \odot)$ are often denoted $\tau\bullet, \tau\odot$ and $\pm\odot$ respectively.

Remark 1. Note that if $n+1 = \tau(n+1)$ then

$$\tau\bullet = (k_{\tau^{-1}(1)}, \dots, k_{\tau^{-1}(n+1)}, \mathbb{A}, \pm_{\tau^{-1}(1)}, \dots, \pm_{\tau^{-1}(n)}, \pm, \tau \circ \sigma). \quad (\text{Res})$$

Proposition 3. For all $n \in \mathbb{N}^*$, the functions $a_n, \alpha_n, \beta_n, \gamma_n$ are group actions.

- two skeletons $\bullet, \bullet' \in \text{SKE}$ are residuated iff there are $\tau_0, \dots, \tau_m \in \mathfrak{S}_{n+1}$ such that $\bullet' = \tau_0 - \dots - \tau_m \bullet$ or $\bullet' = \sim \tau_0 - \dots - \tau_m \bullet$
- two connectives $\odot, \odot' \in \text{ATM}$ are residuated iff $\mathcal{O}_{\alpha_n * \beta_n * \gamma_n}(\odot) = \mathcal{O}_{\alpha_n * \beta_n * \gamma_n}(\odot')$
- for all $\odot \in \text{ATM}$, we have that $\mathcal{O}_{\alpha_n * \beta_n * \gamma_n}(\odot) = \mathcal{O}_{\alpha_n * \beta_n}(\odot) \sqcup \mathcal{O}_{\alpha_n * \beta_n}(\gamma_n(\odot))$.

The action α'_n from [2] differs from our corresponding action α_n here by the fact that we have that $\alpha_n(\sigma, \odot) = \alpha'_n(\sigma^-, \odot)$. This difference with our previous definition is motivated by the fact that it is preferable to use the standard permutation composition operation \circ in the natural, infix order (in [2] the postfix order for permutation product was used: $\tau \circ \sigma(j)$ was unusually defined as $\sigma(\tau(j))$).

The following theorem shows how to compute the skeleton of an atomic connective \odot' from the skeleton of another atomic connective \odot when we have that $\odot' = \tau_0 - \dots - \tau_m \odot$ for some permutations τ_0, \dots, τ_m .

³The definition of a_n is inspired but slightly different from the group action introduced by Espejo-Boix [19, Definition 2].

⁴Setting δ_{n+1}^τ to $+$ does not play a role in this definition but it might appear in the expressions and play a role if we apply the group action successively to several permutations τ (as in the proof of Proposition 3).

Theorem 2. Let $n \in \mathbb{N}^*$, let $\bullet = (k_1, \dots, k_{n+1}, \mathcal{E}, \pm_1, \dots, \pm_n, \pm, \sigma)$ be a skeleton and let $\tau_0, \dots, \tau_m \in \mathfrak{S}_{n+1}$ be such that $\tau_i(n+1) \neq n+1$ for all $i \in \llbracket 0; m \rrbracket$. We set $\pm_{n+1} \triangleq +$ and we define $\bar{\tau} \triangleq \tau_0 \circ \dots \circ \tau_m$ and for all $j \in \llbracket 0; n \rrbracket$, $\Delta_j \triangleq \Delta^j \Delta^{n+1} \pm_{\bar{\tau}(n+1)}$ where

$$\Delta^j \triangleq \Delta_0^j \Delta_1^j \dots \Delta_m^j \qquad \Delta^{n+1} \triangleq \begin{cases} \Delta_0^{n+1} \Delta_1^{n+1} \dots \Delta_{m-1}^{n+1} & \text{if } m \neq 0 \\ + & \text{if } m = 0 \end{cases}$$

$$\text{for all } i \in \llbracket 0; m \rrbracket, \text{ all } k \in \llbracket 0; n+1 \rrbracket, \Delta_i^k \triangleq \begin{cases} + & \text{if } k = \tau_0 \circ \dots \circ \tau_{m-i}(n+1) \\ - & \text{otherwise} \end{cases}.$$

Then,

$$\tau_0 - \dots - \tau_m \bullet = (k_{\bar{\tau}^-(1)}, \dots, k_{\bar{\tau}^-(n+1)}, \Delta_0 \mathcal{E}, \Delta_1 \pm_{\bar{\tau}^-(1)}, \dots, \Delta_n \pm_{\bar{\tau}^-(n)}, \Delta_0 \pm, \bar{\tau} \circ \sigma).$$

Definition 9 (Common set of connectives). A set \mathcal{C} of atomic connectives is *common* when for all pairs of residuated connectives $\odot, \odot' \in \mathcal{C}$, we have that $\mathcal{O}_{\alpha_n * \beta_n}(\odot) = \mathcal{O}_{\alpha_n * \beta_n}(\odot')$, that is, $\odot' = \tau_0 - \tau_1 \dots - \tau_m \odot$ for some $\tau_0, \dots, \tau_m \in \mathfrak{S}_{n+1}$.

Common sets of atomic connectives correspond to a very large class of logics. As far as we know, all non-classical logics which are atomic logics are based on some common set of connectives.⁵

4 Hilbert calculi

In this section on Hilbert calculi, we define the notion of provability (deducibility) from a set of formulas, *i.e.* $\Gamma \vdash_{\mathcal{P}} \varphi$ like for modal logic [12, Definition 4.4].

Let $\mathbf{L} = (\mathcal{L}, \mathcal{C}, \models)$ be a Boolean atomic logic and let $\Gamma \subseteq \mathcal{L}$ and $\varphi \in \mathcal{L}$ be of dimension k . Then, we say that φ is *provable from* Γ in a proof system \mathcal{P} for \mathcal{L} , written $\Gamma \vdash_{\mathcal{P}} \varphi$, when $\vdash_{\mathcal{P}} \varphi$ or there are $n \in \mathbb{N}^*$ and $\varphi_1, \dots, \varphi_n \in \Gamma$ such that $\vdash_{\mathcal{P}} (\varphi_1 \wedge_k \dots \wedge_k \varphi_n) \rightarrow_k \varphi$ (we use the abbreviation $\varphi \rightarrow_k \psi \triangleq (\neg_k \varphi \vee_k \psi)$ and we often omit the subscript k).

If $(\mathcal{M}, \bar{w}) \in \mathcal{C}$, then we write $(\mathcal{M}, \bar{w}) \models \Gamma$ when for all $\psi \in \Gamma$, we have $(\mathcal{M}, \bar{w}) \models \psi$. We say that φ is a *logical consequence* of Γ , written $\Gamma \models_{\mathbf{L}} \varphi$, when for all $(\mathcal{M}, \bar{w}) \in \mathcal{C}$, if $(\mathcal{M}, \bar{w}) \models \Gamma$ then $(\mathcal{M}, \bar{w}) \models \varphi$; φ is *valid*, written $\models_{\mathbf{L}} \varphi$, when for all $(\mathcal{M}, \bar{w}) \in \mathcal{C}$, we have $(\mathcal{M}, \bar{w}) \models \varphi$; φ is *satisfiable* when there is a model $(\mathcal{M}, \bar{w}) \in \mathcal{C}$ such that $(\mathcal{M}, \bar{w}) \models \varphi$. We say that φ is *fully valid* (*fully unsatisfiable*) when for all $(\mathcal{M}, \bar{w}) \in \mathcal{C}$ and for all $\bar{w}' \in \omega(\mathcal{M}, \mathcal{C})$ of the same dimension as φ (resp. for no $\bar{w}' \in \omega(\mathcal{M}, \mathcal{C})$), we have that $(\mathcal{M}, \bar{w}') \models \varphi$.⁶

Strong completeness of a Hilbert calculus is defined as usual by $\Gamma \Vdash_{\mathbf{L}} \varphi$ implies $\Gamma \vdash_{\mathcal{P}} \varphi$.

Theorem 3. Let \mathcal{C} be a common Boolean set of atomic connectives. The calculus $\mathcal{P}_{\mathcal{C}}$ of Fig. 4 is sound and strongly complete for the basic atomic logic $(\mathcal{L}_{\mathcal{C}}, \mathcal{C}_{\mathcal{C}}, \Vdash)$.

Example 3. If we take $\mathcal{C} = \{p, \neg, \wedge, \square, \diamond\}$ to be the set of atomic connectives of modal logic with $\diamond = ((t_1, +, \tau_2), 1)$ and $\square = ((t_2, -, \tau_2), 1)$ then \diamond and \square are residuated and therefore the \mathcal{C} -models $\mathcal{M} = (W, \mathcal{R}) \in \mathcal{E}_{\mathcal{C}}$ are such that $R_{\diamond} = R_{\square}$. The Hilbert calculus $\mathcal{P}_{\mathcal{C}}$ that we obtain and which is sound and complete for (this) modal logic is spelled out in Fig. 5. Note that this proof system is novel but it can be simplified and is in fact equivalent to the classical

⁵An exception might be Levesque's logic of only knowing [32], because we might need to refer to the complement of the epistemic accessibility relation so as to be able to capture it as an atomic logic.

⁶The distinction between validity/unsatisfiability and *full* validity/unsatisfiability plays a role in some logics which distinguish two kinds of states, such as relevance logic [39] where validity is defined w.r.t. *normal* worlds.

<ul style="list-style-type: none"> • <i>Axiom schemas:</i> 	
Any sound and complete axiomatization of propositional logic	\mathcal{P}_{PL}
For all $\odot \in \mathbf{C}$ such that $\mathbb{A}(\odot) = \exists$,	
if $\pm_j(\odot) = +$ then	
$\odot(\varphi_1, \dots, \varphi_j \vee \varphi'_j, \dots, \varphi_n) \rightarrow \odot(\varphi_1, \dots, \varphi_j, \dots, \varphi_n) \vee \odot(\varphi_1, \dots, \varphi'_j, \dots, \varphi_n)$	A1
if $\pm_j(\odot) = -$ then	
$\odot(\varphi_1, \dots, \varphi_j \wedge \varphi'_j, \dots, \varphi_n) \rightarrow \odot(\varphi_1, \dots, \varphi_j, \dots, \varphi_n) \vee \odot(\varphi_1, \dots, \varphi'_j, \dots, \varphi_n)$	A2
For all $\odot \in \mathbf{C}$ such that $\mathbb{A}(\odot) = \forall$,	
if $\pm_j(\odot) = +$ then	
$\odot(\varphi_1, \dots, \varphi_j, \dots, \varphi_n) \wedge \odot(\varphi_1, \dots, \varphi'_j, \dots, \varphi_n) \rightarrow \odot(\varphi_1, \dots, \varphi_j \wedge \varphi'_j, \dots, \varphi_n)$	A3
if $\pm_j(\odot) = -$ then	
$\odot(\varphi_1, \dots, \varphi_j, \dots, \varphi_n) \wedge \odot(\varphi_1, \dots, \varphi'_j, \dots, \varphi_n) \rightarrow \odot(\varphi_1, \dots, \varphi_j \vee \varphi'_j, \dots, \varphi_n)$	A4
For all $\odot, \odot' \in \mathbf{C}$ such that $\odot' = \tau_0 - \tau_1 \dots - \tau_m \odot$ for some $\tau_0, \dots, \tau_m \in \mathfrak{S}_{n+1}$: denoting $\bar{\tau} = \tau_0 \circ \tau_1 \circ \dots \circ \tau_m$,	
$S\left(\odot, \pm^{\bar{\tau}(1)}\varphi_{\bar{\tau}(1)}, \dots, \pm^{n+1}\odot'(\varphi_1, \dots, \varphi_n), \dots, \pm^{\bar{\tau}(n+1)}\varphi_{\bar{\tau}(n+1)}\right)$	A5
where $S(\odot, \psi_1, \dots, \psi_n, \psi) \triangleq \begin{cases} \odot(\psi_1, \dots, \psi_n) \rightarrow \psi & \text{if } \mathbb{A}(\odot) = \exists \\ \psi \rightarrow \odot(\psi_1, \dots, \psi_n) & \text{if } \mathbb{A}(\odot) = \forall \end{cases}$	
$\pm\varphi = \begin{cases} \neg\varphi & \text{if } \pm = - \\ \varphi & \text{if } \pm = + \end{cases}$ and for all $j \in \llbracket 1; n+1 \rrbracket$, $\pm^j = \begin{cases} \pm_1^j \pm_2^j \dots \pm_m^j & \text{if } m \neq 0 \\ + & \text{if } m = 0 \end{cases}$	
where for all $i \in \llbracket 1; m \rrbracket$, $\pm_i^j = \begin{cases} - & \text{if } \tau_i \circ \tau_{i+1} \circ \dots \circ \tau_m(j) = n+1 \\ + & \text{otherwise} \end{cases}$	
<ul style="list-style-type: none"> • <i>Inference rules:</i> 	
From φ and $(\varphi \rightarrow \psi)$, infer ψ	MP
For all $\odot \in \mathbf{C}$ such that $\mathbb{A}(\odot) = \exists$,	
if $\pm_j(\odot) = +$ then from $\neg\varphi_j$, infer $\neg\odot(\varphi_1, \dots, \varphi_j, \dots, \varphi_n)$	R1
if $\pm_j(\odot) = -$ then from φ_j , infer $\neg\odot(\varphi_1, \dots, \varphi_j, \dots, \varphi_n)$	R2
For all $\odot \in \mathbf{C}$ such that $\mathbb{A}(\odot) = \forall$,	
if $\pm_j(\odot) = +$ then from φ_j , infer $\odot(\varphi_1, \dots, \varphi_j, \dots, \varphi_n)$	R3
if $\pm_j(\odot) = -$ then from $\neg\varphi_j$, infer $\odot(\varphi_1, \dots, \varphi_j, \dots, \varphi_n)$	R4
For all $\odot \in \mathbf{C}$,	
if $\pm_j(\odot) = +$ then	
From $\varphi_j \rightarrow \psi_j$, infer $\odot(\varphi_1, \dots, \varphi_j, \dots, \varphi_n) \rightarrow \odot(\varphi_1, \dots, \psi_j, \dots, \varphi_n)$	R5
if $\pm_j(\odot) = -$ then	
From $\varphi_j \rightarrow \psi_j$, infer $\odot(\varphi_1, \dots, \psi_j, \dots, \varphi_n) \rightarrow \odot(\varphi_1, \dots, \varphi_j, \dots, \varphi_n)$	R6

Figure 4: Hilbert calculus $\mathcal{P}_{\mathbf{C}}$

Any sound and complete axiomatization of propositional logic	\mathcal{P}_{PL}
$\diamond(\varphi \vee \psi) \rightarrow \diamond\varphi \vee \diamond\psi$	(A1)
$\Box\varphi \wedge \Box\psi \rightarrow \Box(\varphi \wedge \psi)$	(A3)
$\diamond\neg\varphi \rightarrow \neg\Box\varphi$	(A5)
$\neg\diamond\varphi \rightarrow \Box\neg\varphi$	(A5)
From $\neg\varphi$, infer $\neg\diamond\varphi$	(R1)
From φ , infer $\Box\varphi$	(R3)
From $\varphi \rightarrow \psi$, infer $\Box\varphi \rightarrow \Box\psi$	(R5)
From $\varphi \rightarrow \psi$, infer $\diamond\varphi \rightarrow \diamond\psi$	(R5)
From φ and $\varphi \rightarrow \psi$, infer ψ	(MP)

Figure 5: Hilbert calculus $\mathcal{P}_{\mathcal{C}}$ instantiated with the atomic connectives \mathcal{C} of modal logic.

proof system of modal logic [12, Definition 1.39]. We prove in Appendix F that the axiom K, $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$, is derivable in this calculus.

We now give the details of the computations leading to the two instances of Axiom A5, $\diamond\neg\varphi \rightarrow \neg\Box\varphi$ and $\neg\diamond\varphi \rightarrow \Box\neg\varphi$. The former stems from $\Box = (1\ 2) - (1\ 2) - \text{Id}\diamond$ and the latter stems from $\diamond = (1\ 2) - (1\ 2) - \text{Id}\Box$. As for the former, we have that $\bar{\tau} = \text{Id}$, $n = 1$, $m = 2$, $\tau_0 = (1\ 2)$, $\tau_1 = (1\ 2)$ and $\tau_2 = \text{Id}$. Then, $\pm_1^1 = -$ because $\tau_1 \circ \tau_2(1) = 2 = n + 1$, $\pm_2^1 = +$ because $\tau_2(1) = 1 \neq n + 1$, so $\pm^1 = -$. Likewise, $\pm_1^2 = +$ because $\tau_1 \circ \tau_2(2) = 1 \neq n + 1$, $\pm_2^2 = -$ because $\tau_2(2) = 2 = n + 1$, so $\pm^2 = -$. So, we have that $S(\diamond, \pm^{\bar{\tau}(1)}\varphi, \pm^{\bar{\tau}(2)}\Box\varphi) = S(\diamond, \neg\varphi, \neg\Box\varphi)$, i.e. $\diamond\neg\varphi \rightarrow \neg\Box\varphi$. As for the latter, as in the former case, we have that $\pm^1 = -$ and $\pm^2 = -$. Thus, $S(\Box, \pm^1\varphi, \pm^2\diamond\varphi) = S(\Box, \neg\varphi, \neg\diamond\varphi)$, i.e. $\neg\diamond\varphi \rightarrow \Box\neg\varphi$. \square

However, we need to prove that $\mathcal{P}_{\mathcal{C}}$ is in general well-defined and, to be more precise, that the axiom A5 does not depend on the particular residuation equation $\odot' = \tau_0 - \tau_1 \dots - \tau_m \odot$ that we pick. That is, we need to prove the following.

Proposition 4. *Let \mathcal{C} be a set of atomic connectives. Then, for all $\odot, \odot' \in \mathcal{C}$ of arity $n \in \mathbb{N}^*$ and all $\tau_0^1, \dots, \tau_{m_1}^1, \tau_0^2, \dots, \tau_{m_2}^2 \in \mathfrak{S}_{n+1}$ such that $\odot' = \tau_0^1 - \tau_1^1 \dots - \tau_{m_1}^1 \odot$ and $\odot' = \tau_0^2 - \tau_1^2 \dots - \tau_{m_2}^2 \odot$, we have for all $j \in \llbracket 1; n + 1 \rrbracket$ that it holds that $\pm^{1,j} = \pm^{2,j}$, where $\pm^{1,j}$ is the \pm^j associated to $\tau_0^1, \dots, \tau_{m_1}^1$ and $\pm^{2,j}$ is the \pm^j associated to $\tau_0^2, \dots, \tau_{m_2}^2$ defined in Fig. 4.*

Corollary 1. *Every basic atomic logic is compact.*

5 Display calculi

Often, it is not very clear how and why one should choose structural connectives in a calculus. In this section, we give some proposals on how one should choose them so as to enforce cut admissibility. Basically, they should be ‘displayable enough’ and chosen so as to ensure the display property of the sequent calculus.

Definition 10 (Displayable enough set of connectives). A set of atomic connectives \mathcal{C} is *displayable enough* when for all $\odot \in \mathcal{C}$ of arity $n \in \mathbb{N}^*$ and all $i \in \llbracket 1; n \rrbracket$, there is $\odot' \in \mathcal{C}$ such that $\odot' = \tau_0 - \tau_1 \dots - \tau_m \odot$ for some $\tau_0, \dots, \tau_m \in \mathfrak{S}_{n+1}$ such that $\tau_0 \circ \dots \circ \tau_m(i) = n + 1$. It is *purely displayable* if moreover, for all $i \in \llbracket 1; n + 1 \rrbracket$, we have that $\pm^i = +$.⁷

⁷We recall that \pm^i is defined in Fig. 4.

- *Structural rules:*

$$\frac{(X [\wedge] Y) \vdash U}{(Y [\wedge] X) \vdash U} \text{ (CI} \vdash) \quad \frac{X \vdash U}{(X [\wedge] Y) \vdash U} \text{ (K} \vdash) \quad \frac{(X [\wedge] X) \vdash U}{X \vdash U} \text{ (WI} \vdash)$$

$$\frac{(X [\wedge] [\top]) \vdash U}{X \vdash U} \text{ (I} \vdash) \quad \frac{U \vdash \varphi \quad \varphi \vdash V}{U \vdash V} \text{ cut}$$

If X is empty then $(X [\wedge] Y)$ and $(Y [\wedge] X)$ are Y .

- *Axioms and introduction rules :* $p \vdash p$ for all $p \in \mathbf{C}_0$

$$\frac{S(X_1, \varphi_1) \quad \dots \quad S(X_n, \varphi_n)}{S([\odot], X_1, \dots, X_n, \odot(\varphi_1, \dots, \varphi_n))} \text{ (} \vdash \odot)$$

$$\frac{S([\odot], \varphi_1, \dots, \varphi_n, U)}{S(\odot, \varphi_1, \dots, \varphi_n, U)} \text{ (} \odot \vdash)$$

for all $\odot \in \mathbf{C}$ of skeleton $(\bar{k}, \mathbb{A}, \pm_1, \dots, \pm_n, \pm, \sigma)$ such that:

– for all $j \in \llbracket 1; n \rrbracket$, we define $S(X_j, \varphi_j) \triangleq \begin{cases} X_j \vdash \varphi_j & \text{if } \pm_j \mathbb{A} = - \\ \varphi_j \vdash X_j & \text{if } \pm_j \mathbb{A} = + \end{cases}$

such that for all $j \in \llbracket 1; n \rrbracket$, X_j is not empty and with the convention that if φ_j is empty for some j then $\odot(\varphi_1, \dots, \varphi_n)$ is also empty.

– for all $\star \in \{\odot, [\odot]\}$, $S(\star, X_1, \dots, X_n, X) \triangleq \begin{cases} \star(X_1, \dots, X_n) \vdash X & \text{if } \mathbb{A} = \exists \\ X \vdash \star(X_1, \dots, X_n) & \text{if } \mathbb{A} = \forall \end{cases}$.

- *Display rule:* for all $\odot, \odot' \in \mathbf{C}^+$ such that $\odot' = \tau_0 - \tau_1 \dots - \tau_m \odot$ for some $\tau_0, \dots, \tau_m \in \mathfrak{S}_{n+1}$, denoting $\bar{\tau} \triangleq \tau_0 \circ \tau_1 \circ \dots \circ \tau_m$,

$$\frac{S([\odot'], X_1, \dots, X_n, X_{n+1})}{S([\odot], \pm^{\bar{\tau}(1)} X_{\bar{\tau}(1)}, \dots, \pm^{\bar{\tau}(n)} X_{\bar{\tau}(n)}, \pm^{\bar{\tau}(n+1)} X_{\bar{\tau}(n+1)})} \text{ (dr3)}$$

where for all $j \in \llbracket 1; n+1 \rrbracket$, $\pm^j \triangleq \begin{cases} \pm_1^j \pm_2^j \dots \pm_m^j & \text{if } m \neq 0 \\ + & \text{if } m = 0 \end{cases}$ and for all $i \in \llbracket 1; m \rrbracket$,

$$\pm_i^j \triangleq \begin{cases} - & \text{if } \tau_i \circ \tau_{i+1} \circ \dots \circ \tau_m(j) = n+1 \\ + & \text{otherwise} \end{cases} \quad \pm X \triangleq \begin{cases} *X & \text{if } \pm = - \\ X & \text{if } \pm = + \end{cases}$$

Figure 6: Display calculus $\mathcal{P}_{\mathbf{C}, \mathbf{C}^+}^{\text{BOOL}}$

So, informally, a set of connectives is displayable enough when each argument of each of its connectives can be displayed as the sole antecedent or the sole consequent of a sequent. Note that for every common set of atomic connectives \mathbf{C} there is always a common displayable enough set of atomic connectives and even a common purely displayable set of atomic connectives \mathbf{C}^+ such that $\mathbf{C} \subseteq \mathbf{C}^+$. This concept was first introduced by Espejo-Boix [19, Theorem 5].

Definition 11 (Structures, consecutions). *Atomic structural connectives* are copies of the atomic connectives: for all sets of atomic connectives \mathbf{C} , its associated set of structural connectives is denoted $[\mathbf{C}] \triangleq \{[\odot] \mid \odot \in \mathbf{C}\}$. For all atomic connectives \odot , the *arity*, *signature*, *type signature*, *tonicity signature*, *quantification signature* of $[\odot]$ are the same as \odot . Structural connectives are denoted $[p], [p_1], [p_2], \dots$ and $[\odot], [\odot_1], [\odot_2], \dots$. For each $k \in \mathbb{N}^*$, we also introduce the (*Boolean*) *structural connective* associated to the Boolean connectives \wedge_k, \vee_k , denoted $[\wedge_k], [\vee_k]$ and often simply $[\wedge], [\vee]$ by abuse. We also denote $*_k$, and often simply $*$ by abuse, the structural connective $[\neg_k]$.

Let $(\mathbf{C}, \mathbf{C}^+)$ be a pair of sets of atomic connectives such that $\mathbf{C} \subseteq \mathbf{C}^+$. The *structural atomic language* $[\mathcal{L}_{\mathbf{C}^+}]$ associated to the pair $(\mathbf{C}, \mathbf{C}^+)$ is the smallest set that contains the atomic language $\mathcal{L}_{\mathbf{C}}$ as well as $[\mathbf{C}_0]$ and that is closed under the structural connectives of $[\mathbf{C}^+] \cup \{*\}$ while respecting the dimension constraints. Its elements are called *structures* and their *dimensions* are defined like for formulas of $\mathcal{L}_{\mathbf{C}}$. A $\mathcal{L}_{\mathbf{C}}$ -*consecution* (resp. $[\mathcal{L}_{\mathbf{C}^+}]$ -*consecution*) is an expression of the form $\varphi \vdash \psi$, $\vdash \varphi$ or $\varphi \vdash$ (resp. $X \vdash Y$, $\vdash X$ or $Y \vdash$), where $\varphi, \psi \in \mathcal{L}_{\mathbf{C}}$ (resp. $X, Y \in [\mathcal{L}_{\mathbf{C}^+}]$) are of the same dimension. The set of all $\mathcal{L}_{\mathbf{C}}$ -consecutions (resp. $[\mathcal{L}_{\mathbf{C}^+}]$ -consecutions) is denoted $\mathcal{S}_{\mathbf{C}}$ (resp. $[\mathcal{S}_{\mathbf{C}^+}]$). Elements of $\mathcal{L}_{\mathbf{C}}$ (resp. $[\mathcal{L}_{\mathbf{C}^+}]$ and $[\mathcal{S}_{\mathbf{C}^+}]$) are called *formulas* (resp. *structures* and *consecutions*); they are denoted $\varphi, \psi, \alpha, \dots$ (resp. X, Y, U, V, \dots and $X \vdash Y, U \vdash V, \dots$). Structures and consecutions are interpreted canonically over atomic \mathbf{C} -models exactly like the formulas to which they correspond. In particular, we have for all \mathbf{C} -models \mathcal{M} and all $\bar{w} \in \omega(\mathcal{M}, \mathbf{C})$ that $\bar{w} \in \llbracket *X \rrbracket^{\mathcal{M}}$ iff $\bar{w} \notin \llbracket X \rrbracket^{\mathcal{M}}$.

Definition 12 (Antecedant and consequent part of a consecution). If Z is a substructure of X , then $tn(Z, X)$ is defined inductively as follows:

- if $X = Z$ then $tn(Z, X) \triangleq +$;
- if $X = *Y$ and Z appears in Y then $tn(Z, X) \triangleq -tn(Z, Y)$;
- if $X = [\odot](X_1, \dots, X_n)$ and Z appears in X_j then $tn(Z, X) \triangleq \pm_j(\odot)tn(Z, X_j)$.

If Z is a substructure of X or Y (but not both), Z is an *antecedant part* (resp. *consequent part*) of $X \vdash Y$ when $tn(Z, X) = +$ or $tn(Z, Y) = -$ (resp. $tn(Z, X) = -$ or $tn(Z, Y) = +$).

Proposition 5 (Display property). *Let $(\mathbf{C}, \mathbf{C}^+)$ be a pair of sets of atomic connectives such that $\mathbf{C} \subseteq \mathbf{C}^+$ and \mathbf{C}^+ is displayable enough. For all $[\mathcal{L}_{\mathbf{C}^+}]$ -consecutions $X \vdash Y$ provable in $\mathcal{P}_{\mathbf{C}, \mathbf{C}^+}^{BOOL}$ and for all substructures Z of $X \vdash Y$,*

- *if Z is antecedant part of $X \vdash Y$ then there exists a structure $W \in [\mathcal{L}_{\mathbf{C}^+}]$ such that $Z \vdash W$ is provably equivalent to $X \vdash Y$ in $\mathcal{P}_{\mathbf{C}, \mathbf{C}^+}^{BOOL}$;*
- *if Z is consequent part of $X \vdash Y$ then there exists a structure $W \in [\mathcal{L}_{\mathbf{C}^+}]$ such that $W \vdash Z$ is provably equivalent to $X \vdash Y$ in $\mathcal{P}_{\mathbf{C}, \mathbf{C}^+}^{BOOL}$.*

Definition 13 (Proper display calculus). A calculus is a *typed properly displayable calculus* if it satisfies the conditions C1 to C5 and C8 of [8] and two new conditions: C6' and C7'. Conditions C6' and C7' are the same as Belnap's C6 and C7 except that the substituted structure X and the part M where it is substituted should have equal dimension. By abuse and for simplicity, we will often call a 'typed proper display calculus' a proper display calculus.

Like Belnap's proper display calculi, typed proper display calculi enjoy cut-elimination (the proof is the same as the one in [8]). Then, we have the following.

$\frac{}{[\top] \vdash \top}$ ($\vdash \top$)	$\frac{X \vdash \varphi \quad Y \vdash \psi}{(X [\wedge] Y) \vdash (\varphi \wedge \psi)}$ ($\vdash \wedge$)	$\frac{U \vdash (\varphi [\vee] \psi)}{U \vdash (\varphi \vee \psi)}$ ($\vdash \vee$)
$\frac{[\top] \vdash U}{\top \vdash U}$ ($\top \vdash$)	$\frac{(\varphi [\wedge] \psi) \vdash U}{(\varphi \wedge \psi) \vdash U}$ ($\wedge \vdash$)	$\frac{\varphi \vdash X \quad \psi \vdash Y}{(\varphi \vee \psi) \vdash (X [\vee] Y)}$ ($\vee \vdash$)
$\frac{}{\perp \vdash [\perp]}$ ($\vdash \perp$)	$\frac{\varphi \vdash X}{*X \vdash \neg \varphi}$ ($\vdash \neg$)	$\frac{U \vdash (\varphi [\rightarrow] \psi)}{U \vdash (\varphi \rightarrow \psi)}$ ($\vdash \rightarrow$)
$\frac{U \vdash [\perp]}{U \vdash \perp}$ ($\perp \vdash$)	$\frac{* \varphi \vdash U}{\neg \varphi \vdash U}$ ($\neg \vdash$)	$\frac{X \vdash \varphi \quad \psi \vdash Y}{(\varphi \rightarrow \psi) \vdash (X [\rightarrow] Y)}$ ($\rightarrow \vdash$)

Figure 7: Classical introduction rules as instances of the rules ($\vdash \odot$) and ($\odot \vdash$)

Theorem 4 (Soundness and strong completeness). *Let (C, C^+) be a pair of common Boolean sets of atomic connectives such that $C \subseteq C^+$ and such that C^+ is displayable enough. The calculus $\mathcal{P}_{C, C^+}^{BOOL}$ of Fig. 6 is sound and strongly complete for the basic atomic logic based on C . Moreover, it is a proper display calculus and enjoys cut elimination.*

As shown in Fig. 7, the classical introduction rules are all instances of the rules ($\vdash \odot$), ($\odot \vdash$). The Axiom $p \vdash p$ could be replaced by axioms and inference rules for atoms p which are special instances of the rules ($\vdash \odot$) and ($\odot \vdash$) of Fig. 6. With $\odot = p$, we would have that $n = 0$ and, replacing \odot with p in ($\vdash \odot$) and ($\odot \vdash$), we would obtain the inference rules below. Note that ($\vdash p$) is in fact an axiom.

$$\frac{}{S([p], p)} (\vdash p) \qquad \frac{S([p], X)}{S(p, X)} (p \vdash)$$

$$\text{where, if } \otimes \text{ is } p \text{ or } [p], \text{ then } S(\otimes, X) \triangleq \begin{cases} \otimes \vdash X & \text{if } \mathcal{A} = \exists \\ X \vdash \otimes & \text{if } \mathcal{A} = \forall \end{cases}$$

Hence, for all $p = (\text{Id}, \pm, \mathcal{A}, k)$, if $\mathcal{A} = \exists$ then ($\vdash p$) and ($p \vdash$) would rewrite as follows:

$$\frac{}{[p] \vdash p} (\vdash p) \qquad \frac{[p] \vdash X}{p \vdash X} (p \vdash) \qquad (\text{Axiom}')$$

and if $\mathcal{A} = \forall$ then ($\vdash p$) and ($p \vdash$) would rewrite as follows:

$$\frac{}{p \vdash [p]} (\vdash p) \qquad \frac{X \vdash [p]}{X \vdash p} (p \vdash) \qquad (\text{Axiom}'')$$

Note that in both cases, the standard axiom $p \vdash p$ is derivable by applying ($p \vdash$) once again to $[p] \vdash p$ or $p \vdash [p]$. Our rules ($\vdash \top$), ($\top \vdash$), ($\perp \vdash$) and ($\vdash \perp$) are in fact instances of these rules and are the same as those of Kracht [28] and Belnap [8]. Like in the calculus **DLM** of Kracht [28], we impose some conditions on these atoms by means of the structural inference rule (**I** \vdash) so that these special atoms \top and \perp do behave as truth constants, as intended. Alternatively, one can easily prove that adding the following axioms to our calculus is enough to capture the standard truth constants \top and \perp :

$$\frac{}{\perp \vdash} (\perp \vdash) \qquad \frac{}{\vdash \top} (\vdash \top)$$

Atomic logics have four different propositional letter skeletons of dimension 1: $(1, \forall, +)$, $(1, \exists, +)$, $(1, \forall, -)$, $(1, \exists, -)$. Their eight introduction rules are the same as the eight rules $(\mathbf{1}), (\perp), (\top), (\mathbf{0})$ of Belnap's display calculus for linear logic [9, p. 19]. Hence, with appropriate structural rules, our four propositional letter skeletons of type 1 can stand for the four propositional constants of linear logic.

Rule (dr3) is meant to ensure from a semantic point of view that residuated connectives are associated to the same relation. For the case of Boolean connectives, it implies the following standard display rules. The first two blocks are just instances of (dr3) and the third block is proven from the first two using the structural rules.

Proposition 6. *Let C^+ be a Boolean set of atomic connectives. Then, the following classical inference rules (from [42, p. 29]) are all derivable in $\mathcal{P}_{C, C^+}^{BOOL}$ from (dr3):*

$$\frac{\frac{X, Y \vdash Z}{X \vdash Z, *Y}}{Y \vdash *X, Z} \qquad \frac{\frac{X \vdash Y, Z}{X, *Z \vdash Y}}{*Y, X \vdash Z} \qquad \frac{\frac{X \vdash Y}{*Y \vdash *X}}{X \vdash **Y} \qquad (\text{dr2})$$

where $X, Y \triangleq \begin{cases} (X \wedge Y) & \text{if } (X, Y) \text{ is antecedent part in the consecution} \\ (X \vee Y) & \text{if } (X, Y) \text{ is consequent part in the consecution.} \end{cases}$

Conversely, every instance of (dr3) involving the structural connectives $[\wedge]$ and $[\vee]$ is derivable in $\mathcal{P}_{C, C^+}^{BOOL}$ from (dr2).

Below is a consequence of cut elimination.

Corollary 2. *Let C_1, C_2 be common sets of atomic connectives such that $C_1 \subseteq C_2$. Then, the logics $(\mathcal{S}_{C_2}, \mathcal{C}_{C_2}, \Vdash)$ and $(\mathcal{L}_{C_2}, \mathcal{C}_{C_2}, \Vdash)$ are conservative extensions of $(\mathcal{S}_{C_1}, \mathcal{C}_{C_1}, \Vdash)$ and $(\mathcal{L}_{C_1}, \mathcal{C}_{C_1}, \Vdash)$ respectively.*

As another corollary of this result, the common set C of atomic connectives in Theorem 3 is in fact not required to be Boolean, even if the Hilbert calculus \mathcal{P}_C includes axioms and inference rules with Boolean connectives. This is because for any basic atomic logic, its extension with Boolean connectives is a conservative extension, by Corollary 2. So, even if the proof of a formula may contain extra Boolean connectives, if the conclusion of the proof does not contain any, the proven formula is nevertheless valid in the initial logic.

Corollary 3. *Let C be a common set of atomic connectives, not necessarily Boolean. The calculus \mathcal{P}_C of Fig. 4 is sound and strongly complete for the basic atomic logic $(\mathcal{L}_C, \mathcal{C}_C, \Vdash)$.*

Our display calculi are sound and complete for logics including the Boolean connectives until now. However, in the same spirit, we would like to obtain calculi for plain atomic logics, without Boolean connectives. Indeed, we consider the latter to be more primitive than Boolean atomic logics because even the Boolean connectives can be seen as particular atomic connectives, interpreted over special relations (identity relations, see Example 2). These special relations are obtained at the proof-theoretical level by imposing the validity of Gentzen's structural rules. So, below, we define sound and complete calculi for basic atomic logics, without Boolean connectives.

Definition 14 (Calculus \mathcal{P}_{C, C^+}). Let (C, C^+) be a pair of common sets of atomic connectives such that C is *without Boolean connectives*, $C \subseteq C^+$ and C^+ is displayable enough. The calculus

For all $\odot, \odot' \in \mathbf{C}$ such that $\odot' = \tau_0 - \tau_1 \dots - \tau_m \odot$ for some $\tau_0, \dots, \tau_m \in \mathfrak{S}_{n+1}$:

$$\frac{S(\odot', \psi_1, \dots, \psi_n, \psi_{n+1})}{S(\odot, \psi_{\bar{\tau}(1)}, \dots, \psi_{\bar{\tau}(n)}, \psi_{\bar{\tau}(n+1)})} \text{ (dr3')} \quad \frac{S(\psi_1, \varphi_1) \dots S(\psi_n, \varphi_n)}{S(\odot, \psi_1, \dots, \psi_n, \odot(\varphi_1, \dots, \varphi_n))} (\vdash \odot)$$

For all $p \in \mathbf{C}_0$: $p \vdash p$

where $\bar{\tau} \triangleq \tau_0 \circ \dots \circ \tau_m$ and for all $\odot \in \mathbf{C}$ of skeleton $(\bar{k}, \mathbb{A}, \pm_1, \dots, \pm_n, \pm, \sigma)$:

- for all $j \in \llbracket 1; n \rrbracket$, we set $S(\psi_j, \varphi_j) \triangleq \begin{cases} \psi_j \vdash \varphi_j & \text{if } \pm_j \mathbb{A} = - \\ \varphi_j \vdash \psi_j & \text{if } \pm_j \mathbb{A} = + \end{cases}$
such that for all $j \in \llbracket 1; n \rrbracket$, ψ_j is not empty and with the convention that if φ_j is empty for some j then $\odot(\varphi_1, \dots, \varphi_n)$ is also empty
- and $S(\odot, \psi_1, \dots, \psi_n, \psi) \triangleq \begin{cases} \odot(\psi_1, \dots, \psi_n) \vdash \psi & \text{if } \mathbb{A} = \exists \\ \psi \vdash \odot(\psi_1, \dots, \psi_n) & \text{if } \mathbb{A} = \forall \end{cases}$

Figure 8: Sequent calculus $\mathcal{P}_{\mathbf{C}, \mathbf{C}}$

$\mathcal{P}_{\mathbf{C}, \mathbf{C}^+}$ is the proof system consisting of $\mathcal{P}_{\mathbf{C}, \mathbf{C}^+}^{\text{BOOL}}$ of Fig. 6 without the structural rules, which are replaced by the following rules below.

$$\frac{X \vdash Y}{*Y \vdash *X} \quad \frac{X \vdash Y}{X \vdash **Y} \quad \frac{X \vdash Y}{**X \vdash Y} \quad (\text{dr2}')$$

Theorem 5. *Let $(\mathbf{C}, \mathbf{C}^+)$ be a pair of common sets of atomic connectives such that \mathbf{C} is without Boolean connectives, $\mathbf{C} \subseteq \mathbf{C}^+$ and \mathbf{C}^+ is displayable enough. The calculus $\mathcal{P}_{\mathbf{C}, \mathbf{C}^+}$ is sound and strongly complete for the basic atomic logic based on \mathbf{C} .*

So, even if Boolean connectives do not appear in a given atomic logic, it is often necessary to resort to negation, at least as a structural connective. Here we obtain calculi for basic atomic logics with a De Morgan structural negation. The De Morgan negation is weaker than the Boolean negation (see [18] for details on the different kinds of negation). This might seem problematic because $*$ has the semantics of a Boolean negation. In fact, this only indicates that the language without Boolean connectives is too poor to impose on $*$ to be a Boolean negation, we would need extensional connectives like Boolean conjunction or disjunction and rules connecting the negation with these connectives to ensure that. Without them, it leaves some freedom of interpretation to $*$ which is only assumed to be (at least) a De Morgan negation. In the next section we will see that negation can be completely eliminated from the calculus, even at the structural level, when the set of connectives \mathbf{C} is purely displayable.

6 Monotonicity and Kripke-style relational semantics

If \mathbf{C} is without Boolean connective and purely displayable then one does not need any structural connective (not even $*$). In that case, we have the following:

Theorem 6. *Let \mathbf{C} be a purely displayable set of atomic connectives without Boolean connectives. The sequent calculus $\mathcal{P}_{\mathbf{C}, \mathbf{C}}$ of Fig. 8 is sound and strongly complete for the basic atomic logic based on \mathbf{C} .*

Example 4 (Lambek calculus). The set of connectives of the Lambek calculus, $\mathcal{C} \triangleq \{p, \otimes, /, \backslash\}$ defined in Fig. 3, is purely displayable. Its sequent calculus $\mathcal{P}_{\mathcal{C}, \mathcal{C}}$ is simply the original Lambek calculus [31].

$$\begin{array}{c}
p \vdash p \\
\frac{\varphi \otimes \psi \vdash \chi}{\psi \vdash \varphi \backslash \chi} \text{ (dr3)} \qquad \frac{\psi \vdash \varphi \backslash \chi}{\varphi \vdash \chi / \psi} \text{ (dr3)} \\
\frac{\varphi' \vdash \varphi \quad \psi' \vdash \psi}{\varphi' \otimes \psi' \vdash \varphi \otimes \psi} (\vdash \odot) \quad \frac{\varphi \vdash \varphi' \quad \psi' \vdash \psi}{\varphi \backslash \psi' \vdash \varphi' \backslash \psi} (\vdash \odot) \quad \frac{\varphi' \vdash \varphi \quad \psi \vdash \psi'}{\varphi / \psi \vdash \varphi' / \psi'} (\vdash \odot)
\end{array}$$

Assuming that we have a single atomic connective in \mathcal{C} , then $\mathcal{P}_{\mathcal{C}, \mathcal{C}}$ consists of the single rule $(\vdash \odot)$. Informally, this tells us that ‘any logic whose connectives are monotone (they satisfy the rule $(\vdash \odot)$) can be given a Kripke-style relational semantics’. However, in general, this is not exactly true because we often do not have a single connective and because in that case the set of atomic connectives should be purely displayable to apply the theorem.

Anyway, this has to be related to Lyndon’s theorem for first-order logic [25, Theorem 10.3.3] which states that a first-order formula $\varphi(Q_1, \dots, Q_n)$ including the predicate symbols Q_1, \dots, Q_n is isotonic (resp. antitonic) in the interpretation of the predicate symbol Q_i if, and only if, it is equivalent to a formula where Q_i occurs only positive (resp. negative) in this formula.⁸ Let $\varphi(Q_1, \dots, Q_n)(\bar{x})$ be a first-order formula with a tuple of free variables \bar{x} . If we assume that $\varphi(Q_1, \dots, Q_n)(\bar{x})$ is not only isotonic or antitonic in each predicate symbol Q_i but also *residuated*, meaning that, for all $i \in \llbracket 1; n \rrbracket$ there is a first-order formula $\varphi_i(Q_1, \dots, Q, \dots, Q_n)(\bar{y})$ with a tuple of free variables \bar{y} such that $\varphi(Q_1, \dots, Q_n)(\bar{x}) \models Q\bar{x}$ holds iff $\varphi_i(Q_1, \dots, Q, \dots, Q_n)(\bar{y}) \models Q_i\bar{y}$ holds or $Q_i\bar{y} \models \varphi_i(Q_1, \dots, Q, \dots, Q_n)(\bar{y})$ holds, then Theorem 6 implies that the protologic induced by the first-order formulas $\{\varphi(Q_1, \dots, Q_n)(\bar{x}), \varphi_i(Q_1, \dots, Q, \dots, Q_n)(\bar{y}) \mid i \in \llbracket 1; n \rrbracket\}$ has the same valid consecutions as an atomic logic, and thus is somehow equally expressive as an atomic logic.⁹

7 Related work and concluding remarks

The DLE-logics introduced by Greco et al. [24] are similar to our basic atomic logics. Their families \mathcal{F} and \mathcal{G} correspond in our framework to connectives of “quantification signatures” \exists and \forall respectively. Likewise, their order types correspond in our framework to “tonicity signatures”. In fact, several of their notions correspond to notions introduced by Dunn [14, 15]. However, there is a number of differences between their and our work. Firstly, we provide and prove the completeness of our calculi w.r.t. a Kripke-style relational semantics. Secondly, we introduce a generalized and novel form of residuation based on a group action. Thirdly, we use dimensions and we consider compositions of atomic connectives as primitive connectives in molecular logics. These extensions and generalizations are motivated at length in [5]. Besides, some logics/protologics cannot be represented without the use of dimensions, such as temporal logic [5, Example 8], arrow logic, many-dimensional logics [33] and first-order logic. This use of dimensions is crucial to represent these logics and it is also instrumental in showing that any protologic is as expressive as a molecular logic, which constitutes the main result of [5]. Lastly, one of the main differences is the fact that we are able to define *automatically* from the connectives of a given basic atomic logic display and Hilbert calculi in a generic fashion together

⁸A first-order formula is *isotonic* (resp. *antitonic*) in Q_i if it defines a class of structures which is closed under adding (resp. removing) tuples to the relation interpreting Q_i and it is *positive* (resp. *negative*) if Q_i is in the scope of an even (resp. odd) number of negations in the first-order formula.

⁹Protologics are defined in [5]. A protologic is a logic such that the truth conditions of its connectives can be defined by arbitrary first-order formulas. Thus, protologics include a very large fragment of non-classical logics.

with their *Kripke-style relational semantics* for which they are sound and complete. In particular, they do not provide a Kripke-style relational semantics to their DLE-logics, only an algebraic one which more or less mimics the axioms and inference rules of their DLE-logics.

Atomic logics are logics of residuation to which dimensions are added. Residuated logics have been extensively studied in the algebraic approach to logic [21]. It remains to propose and adapt these algebraic approaches and semantics to our atomic and molecular logics in the spirit of modal logic [12, Section 5].

We were able to axiomatize our basic atomic logics only for so-called “common” sets of atomic connectives. Even if they already cover most, if not all, existing non-classical logics, extending our results to any set of atomic connectives is an open problem. Finally, this article dealt with the proof theory of *basic* atomic logics, that is atomic logics where we do not impose any conditions on the accessibility relations of their semantics. To obtain similar results for other atomic logics, defined with specific accessibility relations, one needs to develop a correspondence theory. This is the topic of [4, 3].

Acknowledgements. I thank Raúl Espejo-Boix and Rajeev Goré for comments and discussions on the topic of this article. Raúl’s master’s thesis [19] inspired me and led to an improvement of several definitions (see the various references to his thesis in the text). Also, Raúl pointed out to me that Belnap’s conditions C1-C8 needed to be adapted to deal adequately with dimensions.

References

- [1] Guillaume Aucher. Displaying Updates in Logic. *Journal of Logic and Computation*, 26(6):1865–1912, March 2016. URL: <https://hal.inria.fr/hal-01476234>, doi:10.1093/logcom/exw001.
- [2] Guillaume Aucher. *Selected Topics from Contemporary Logics*, chapter Towards Universal Logic: Gaggles Logics, pages 5–73. Landscapes in Logic. College Publications, October 2021.
- [3] Guillaume Aucher. A Characterization of Properly Displayable Atomic Logics. Research report, Université de Rennes 1, October 2022. URL: <https://hal.inria.fr/hal-03800070>.
- [4] Guillaume Aucher. Correspondence Theory for Atomic Logics. Research report, Université de Rennes 1, October 2022. URL: <https://hal.inria.fr/hal-03800044>.
- [5] Guillaume Aucher. On the universality of atomic and molecular logics via protologics. *Logica Universalis*, 16(1):285–322, 2022. doi:10.1007/s11787-022-00298-5.
- [6] Guillaume Aucher. A van Benthem theorem for atomic and molecular logics. In Andrzej Indrzejczak and Michał Zawidzki, editors, Proceedings of the 10th International Conference on *Non-Classical Logics. Theory and Applications*, Łódź, Poland, 14-18 March 2022, volume 358 of *Electronic Proceedings in Theoretical Computer Science*, pages 84–101. Open Publishing Association, 2022. doi:10.4204/EPTCS.358.7.
- [7] Jon Barwise. Axioms for abstract model theory. *Annals of Mathematical Logic*, 7(2):221 – 265, 1974. URL: <http://www.sciencedirect.com/science/article/pii/0003484374900163>, doi:10.1016/0003-4843(74)90016-3.
- [8] Jr. Belnap, Nuel D. Display logic. *Journal of Philosophical Logic*, 11(4):375–417, 1982. doi:10.1007/BF00284976.

- [9] Nuel Belnap. Linear logic displayed. *Notre Dame J. Formal Log.*, 31(1):14–25, 1990. doi: 10.1305/ndjfl/1093635329.
- [10] Jean-Yves Béziau. *Logica Universalis*, chapter From Consequence Operator to Universal Logic: A Survey of General Abstract Logic. Birkhäuser Basel, 2007.
- [11] Katalin Bimbó and J. Michael Dunn. *Generalized Galois Logics: Relational Semantics of Nonclassical Logical Calculi*. Number 188. Center for the Study of Language and Information, 2008.
- [12] Patrick Blackburn, Maarten de Rijke, and Yde Venema. *Modal Logic*, volume 53 of *Cambridge Tracts in Computer Science*. Cambridge University Press, 2001.
- [13] Razvan Diaconescu. *Institution-independent model theory*. Springer Science & Business Media, 2008.
- [14] J Michael Dunn. Gaggle theory: an abstraction of galois connections and residuation, with applications to negation, implication, and various logical operators. In *European Workshop on Logics in Artificial Intelligence*, pages 31–51. Springer Berlin Heidelberg, 1990.
- [15] J Michael Dunn. Partial-gaggles applied to logics with restricted structural rules. In Peter Schroeder-Heister and Kosta Dosen, editors, *Substructural Logics*, pages 63–108. Clarendon Press: Oxford, 1993.
- [16] J. Michael Dunn. *Philosophy of Language and Logic*, volume 7 of *Philosophical Perspectives*, chapter Perp and star: Two treatments of negation, pages 331–357. Ridgeview Publishing Company, Atascadero, California, USA, 1993.
- [17] J. Michael Dunn and Gary M. Hardegree. *Algebraic Methods in Philosophical Logic*. Number 41 in Oxford Logic Guides. Clarendon Press: Oxford, 2001.
- [18] J. Michael Dunn and Chunlai Zhou. Negation in the context of gaggle theory. *Studia Logica*, 80(2-3):235–264, 2005. doi:10.1007/s11225-005-8470-y.
- [19] Raül Espejo-Boix. A study on actions for atomic logics. Master’s thesis, University of Rennes, <https://doi.org/10.48550/arXiv.2403.07948>, 2024.
- [20] Dov Gabbay. *Labelled Deductive Systems*, volume 1. Oxford University Press, 1996.
- [21] Nikolaos Galatos, Peter Jipsen, Tomasz Kowalski, and Hiroakira Ono. *Residuated lattices: an algebraic glimpse at substructural logics*. Studies in Logic and the Foundations of Mathematics. Elsevier, 2007.
- [22] Robert Goldblatt. Semantic analysis of orthologic. *Journal of Philosophical Logic*, 3:19–35, 1974.
- [23] Rajeev Goré. Substructural logics on display. *Logic Journal of IGPL*, 6(3):451–504, 1998.
- [24] Giuseppe Greco, Minghui Ma, Alessandra Palmigiano, Apostolos Tzimoulis, and Zhiguang Zhao. Unified correspondence as a proof-theoretic tool. *J. Log. Comput.*, 28(7):1367–1442, 2018. doi:10.1093/logcom/exw022.
- [25] Wilfrid Hodges. *Model theory*, volume 42 of *Encyclopedia of mathematics and its applications*. Cambridge University Press, 1993.

- [26] Bjarni Jónsson and Alfred Tarski. Boolean algebras with operators. part i. *American journal of mathematics*, 73(4):891–939, 1951.
- [27] Bjarni Jónsson and Alfred Tarski. Boolean algebras with operators. part ii. *American Journal of Mathematics*, 74:127–162, 1952.
- [28] Marcus Kracht. Power and weakness of the modal display calculus. In *Proof theory of modal logic*, pages 93–121. Springer, 1996.
- [29] Saul A. Kripke. Semantical analysis of modal logic, i: Normal propositional calculi. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 8:113–116, 1963.
- [30] Saul A. Kripke. *Formal Systems and Recursive Functions*, chapter Semantical Analysis of Intuitionistic Logic, I, pages 92–130. North Holland, Amsterdam, 1965.
- [31] Joachim Lambek. The mathematics of sentence structure. *American mathematical monthly*, 65:154–170, 1958.
- [32] Hector J. Levesque. All i know: A study in autoepistemic logic. *Artificial Intelligence*, 42(2):263–309, 1990. URL: <https://www.sciencedirect.com/science/article/pii/0004370290900566>, doi:10.1016/0004-3702(90)90056-6.
- [33] Maarten Marx and Yde Venema. *Multi-dimensional modal logic*, volume 4 of *Applied logic series*. Kluwer, 1997.
- [34] Elliott Mendelson. *Introduction to mathematical logic*. CRC press, 5th edition, 2010.
- [35] Arthur Prior. *Past, Present and Future*. Oxford: Clarendon Press, 1967.
- [36] Greg Restall. *An Introduction to Substructural Logics*. Routledge, 2000.
- [37] Joseph J. Rotman. *An Introduction to the Theory of Groups*, volume 148 of *Graduate texts in mathematics*. Springer, 1995.
- [38] Richard Routley and Robert K. Meyer. The semantics of entailment - III. *J. Philos. Log.*, 1(2):192–208, 1972. doi:10.1007/BF00650498.
- [39] Richard Routley and Robert K Meyer. The semantics of entailment. *Studies in Logic and the Foundations of Mathematics*, 68:199–243, 1973.
- [40] Giovanni Sambin, Giulia Battilotti, and Claudia Faggian. Basic logic: reflection, symmetry, visibility. *The Journal of Symbolic Logic*, 65(03):979–1013, 2000.
- [41] M. Vardi. From philosophical to industrial logics. *Logic and Its Applications*, pages 89–115, 2009.
- [42] Heinrich Wansing. *Displaying Modal Logic*. Kluwer Academic Publishers, Norwell, MA, USA, 1998.

A Proofs of Propositions 1, 2 and Theorem 1

The proofs of Propositions 1 and 2 are without particular difficulty, it suffices to check the definitions.

Theorem 1. *Every basic atomic logic is decidable and PSPACE-complete.*

Proof. It follows from the fact that the satisfiability problem of any basic atomic logic is polynomially reducible to the satisfiability problem of some basic modal logic, a problem which is known to be decidable. The translation is made in two steps.

1. Let φ be a formula of a basic atomic logic based on a set of atomic connectives C . Let $t_1(\varphi)$ be the formula φ where all atomic connectives have been uniformly replaced by atomic connectives of the same skeleton except that the dimension signatures for each of them is now $(1, \dots, 1, 1)$, and likewise for $t_1(C)$. Then, φ is satisfiable in the basic atomic logic based on C iff $t_1(\varphi)$ is satisfiable in the basic atomic logic based on $t_1(C)$. Indeed, from the left to right direction, it suffices to name the tuples of states in the model satisfying φ appearing as arguments of the relations as new (single) state and, from the right to left direction, it suffices to make copies of appropriate sizes of the states appearing in the relation of the model satisfying $t_1(\varphi)$ to obtain the right dimensions for the tuples of the corresponding relation in the model satisfying φ .
2. Now, for every $\odot_1 \in t_1(C)$ there is a modal connective (or a propositional letter) $t_2(\odot_1)$ which is an atomic connective such that $t_2(\odot_1) \in \mathcal{O}_{\alpha*\beta}(\odot_1)$. In particular the tonicity signature of each $t_2(\odot_1)$ is $(+, \dots, +)$. So, consider now the Boolean completion of $t_2(C) \triangleq \{t_2(\odot_1) \mid \odot_1 \in C\}$. This is a set of n -ary modal connectives in the usual sense [12]. Every formula of the atomic language based on $t_1(C)$ can be canonically translated into a formula of the atomic language based on $t_2(C)$ using appropriately the Boolean negation. Moreover, this polynomial translation preserves the satisfiability of the formula. In that case, the $t_1(C)$ -models are the same as the $t_2(C)$ -models.

Since the satisfiability problem for n -ary modal logics is known to be decidable, the satisfiability problem for basic atomic logic is also decidable. This polynomial reduction into modal logic shows as well that it is in PSPACE. The polynomial reduction of modal logic into the atomic logic of Example 2 shows that it is PSPACE-hard. Hence, we obtain the result. \square

B Proofs of Proposition 3 and Theorem 2

Proposition 7. *For all $n \in \mathbb{N}^*$, the functions $a_n, \alpha_n, \beta_n, \gamma_n$ are group actions.*

- two skeletons $\bullet, \bullet' \in SKE$ are residuated iff there are $\tau_0, \dots, \tau_m \in \mathfrak{S}_{n+1}$ such that $\bullet' = \tau_0 - \dots - \tau_m \bullet$ or $\bullet' \simeq \tau_0 - \dots - \tau_m \bullet$
- two connectives $\odot, \odot' \in ATM$ are residuated iff $\mathcal{O}_{\alpha_n*\beta_n*\gamma_n}(\odot) = \mathcal{O}_{\alpha_n*\beta_n*\gamma_n}(\odot')$
- for all $\odot \in ATM$, we have that $\mathcal{O}_{\alpha_n*\beta_n*\gamma_n}(\odot) = \mathcal{O}_{\alpha_n*\beta_n}(\odot) \sqcup \mathcal{O}_{\alpha_n*\beta_n}(\gamma_n(\odot))$.

Proof. We prove that α_n is a group action. Let $n \in \mathbb{N}^*$, let $\tau, \rho \in \mathfrak{S}_{n+1}$ and let $\bullet \in SKE_n$. Clearly, $\text{Id} \bullet = \bullet$ (see Expression (Res) in Remark 1). To prove that $a_n(\rho \circ \tau, \bullet) = a_n(\rho, a_n(\tau, \bullet))$, that is $\rho \circ \tau \bullet = \rho(\tau \bullet)$, we distinguish different cases depending on whether $n+1 = \tau(n+1)$, $n+1 = \rho(n+1)$ or not.

1. $\tau(n+1) = n+1$ and $\rho(n+1) = n+1$. The result follows straightforwardly from Expression (Res) of Remark 1.
2. $\tau(n+1) = n+1$ and $\rho(n+1) \neq n+1$. Then, by Expression (Res) of Remark 1,

$$\tau^\bullet = (k_{\tau^-(1)}, \dots, k_{\tau^-(n+1)}, \mathbb{A}, \pm_{\tau^-(1)}, \dots, \pm_{\tau^-(n)}, \pm, \tau \circ \sigma).$$

So,

$$\begin{aligned} \rho(\tau^\bullet) &= \left(k_{\tau^-(\rho^-(1))}, \dots, k_{\tau^-(\rho^-(n+1))}, \delta_0^\tau \pm_{\tau^-(\rho^-(n+1))} \mathbb{A}, \right. \\ &\quad \left. \delta_1^{\tau^-} \pm_{\tau^-(\rho^-(n+1))} \pm_{\tau^-(\rho^-(1))}, \dots, \delta_n^\tau \pm_{\tau^-(\rho^-(n+1))} \pm_{\tau^-(\rho^-(n))}, \right. \\ &\quad \left. \delta_0^\tau \pm_{\tau(\rho(n+1))} \pm, \rho \circ \tau \circ \sigma \right) \\ &= \left(k_{(\rho \circ \tau)^-(1)}, \dots, k_{(\rho \circ \tau)^-(n+1)}, \delta_0^\tau \pm_{(\rho \circ \tau)^-(n+1)} \mathbb{A}, \right. \\ &\quad \left. \delta_1^{\tau^-} \pm_{(\rho \circ \tau)^-(n+1)} \pm_{(\rho \circ \tau)^-(1)}, \dots, \delta_n^\tau \pm_{(\rho \circ \tau)^-(n+1)} \pm_{(\rho \circ \tau)^-(n)}, \right. \\ &\quad \left. \delta_0^\tau \pm_{(\rho \circ \tau)^-(n+1)} \pm, \rho \circ \tau \circ \sigma \right) \end{aligned}$$

where for all $j \in \llbracket 1; n \rrbracket$, because $\tau(n+1) = n+1$,

$$\delta_j^\tau \triangleq \begin{cases} + & \text{if } j = \tau(n+1) \\ - & \text{otherwise} \end{cases} = \begin{cases} + & \text{if } j = \rho \circ \tau(n+1) \\ - & \text{otherwise} \end{cases}$$

So, $\rho(\tau^\bullet) = \rho \circ \tau^\bullet$.

3. $\tau(n+1) \neq n+1$ and $\rho(n+1) = n+1$. The proof of this case is similar to the previous one.
4. $\tau(n+1) \neq n+1$ and $\rho(n+1) \neq n+1$. Then,

$$\begin{aligned} \tau^\bullet &= \left(k_{\tau^-(1)}, \dots, k_{\tau^-(n+1)}, \delta_0^\tau \pm_{\tau^-(n+1)} \mathbb{A}, \right. \\ &\quad \left. \delta_1^{\tau^-} \pm_{\tau^-(n+1)} \pm_{\tau^-(1)}, \dots, \delta_n^\tau \pm_{\tau^-(n+1)} \pm_{\tau^-(n)}, \right. \\ &\quad \left. \delta_0^\tau \pm_{\tau^-(n+1)} \pm, \tau \circ \sigma \right) \\ \rho(\tau^\bullet) &= \left(k_{\tau^-(\rho^-(1))}, \dots, k_{\tau^-(\rho^-(n+1))}, \delta_0^\rho \left(\delta_{\rho^-(n+1)}^\tau \pm_{\tau^-(n+1)} \pm_{\tau^-(\rho^-(n+1))} \right) \left(\delta_0^\tau \pm_{\tau^-(n+1)} \mathbb{A} \right), \right. \\ &\quad \left. \delta_1^\rho \left(\delta_{\rho^-(n+1)}^\tau \pm_{\tau^-(n+1)} \pm_{\tau^-(\rho^-(n+1))} \right) \left(\delta_{\rho^-(1)}^\tau \pm_{\tau^-(n+1)} \pm_{\tau^-(\rho^-(1))} \right), \dots, \right. \\ &\quad \left. \delta_n^\rho \left(\delta_{\rho^-(n+1)}^\tau \pm_{\tau^-(n+1)} \pm_{\tau^-(\rho^-(n+1))} \right) \left(\delta_{\rho^-(n)}^\tau \pm_{\tau^-(n+1)} \pm_{\tau^-(\rho^-(n))} \right), \right. \\ &\quad \left. \delta_0^\rho \left(\delta_{\rho^-(n+1)}^\tau \pm_{\tau^-(n+1)} \pm_{\tau^-(\rho^-(n+1))} \right) \left(\delta_0^\tau \pm_{\tau^-(n+1)} \pm, \rho \circ \tau \circ \sigma \right) \right) \\ &= \left(k_{(\rho \circ \tau)^-(1)}, \dots, k_{(\rho \circ \tau)^-(n+1)}, \delta_{\rho^-(n+1)}^\tau \pm_{(\rho \circ \tau)^-(n+1)} \mathbb{A}, \right. \\ &\quad \left. \delta_1^\rho \delta_{\rho^-(1)}^\tau \delta_{\rho^-(n+1)}^\tau \pm_{(\rho \circ \tau)^-(n+1)} \pm_{(\rho \circ \tau)^-(1)}, \dots, \right. \\ &\quad \left. \delta_n^\rho \delta_{\rho^-(n)}^\tau \delta_{\rho^-(n+1)}^\tau \pm_{(\rho \circ \tau)^-(n+1)} \pm_{(\rho \circ \tau)^-(n)}, \right. \\ &\quad \left. \delta_{\rho^-(n+1)}^\tau \pm_{(\rho \circ \tau)^-(n+1)} \pm, \rho \circ \tau \circ \sigma \right) \\ &= \rho \circ \tau^\bullet \end{aligned}$$

where for all $j \in \llbracket 0; n \rrbracket$, $\delta_j^\rho \triangleq \begin{cases} + & \text{if } j = \rho(n+1) \\ - & \text{otherwise} \end{cases}$.

(a) Assume that $\rho^-(n+1) = \tau(n+1)$ (*). Then, $\rho \circ \tau(n+1) = n+1$.

Then, $\pm_{(\rho \circ \tau)^-(n+1)} = \pm_{n+1} = +$ and $\delta_{\rho^-(n+1)}^\tau = +$ by definition because $\rho^-(n+1) = \tau(n+1)$. Moreover, for all $j \in \llbracket 1; n \rrbracket - \{\rho(n+1)\}$, $\rho^-(j) \neq \tau(n+1)$ because of (*). So, $\delta_{\rho^-(j)}^\tau = -$ and $\delta_j^\rho = -$. Thus, for all $j \in \llbracket 1; n \rrbracket - \{\rho(n+1)\}$, we have that $\delta_j^\rho \delta_{\rho^-(j)}^\tau \delta_{\rho^-(n+1)}^\tau = +$. And for $j_0 \triangleq \rho(n+1)$, we have that $\delta_{\rho^-(j_0)}^\tau = \delta_{n+1}^\tau = +$ by definition and therefore $\delta_{j_0}^\rho \delta_{\rho^-(j_0)}^\tau \delta_{\rho^-(n+1)}^\tau = + + + = +$. So, eventually,

$$\begin{aligned} \rho(\tau \bullet) &= (k_{(\rho \circ \tau)^-(1)}, \dots, k_{(\rho \circ \tau)^-(n+1)}, \mathbb{E}, \pm_{(\rho \circ \tau)^-(1)}, \dots, \pm_{(\rho \circ \tau)^-(n)}, \pm, \rho \circ \tau \circ \sigma) \\ &= \rho \circ \tau \bullet \end{aligned}$$

(b) Assume that $\rho^-(n+1) \neq \tau(n+1)$. Then, $\rho \circ \tau(n+1) \neq n+1$.

Then, $\delta_{\rho^-(n+1)}^\tau = -$ by definition, because $\rho^-(n+1) \neq \tau(n+1)$. Moreover, for all $j \in \llbracket 1; n \rrbracket - \{\rho \circ \tau(n+1)\}$, we have that $j \neq \rho(\tau(n+1))$, so $\rho^-(j) \neq \tau(n+1)$.

- If $j = \rho(n+1)$, then $\rho^-(j) = n+1$. So, $\delta_{\rho^-(j)}^\tau = \delta_{n+1}^\tau = +$ and $\delta_j^\rho = +$ by definition. Thus, $\delta_j^\rho \delta_{\rho^-(j)}^\tau = +$.
- If $j \neq \rho(n+1)$, then $\delta_j^\rho = -$. Moreover, $\rho^-(j) \neq n+1$ and $\rho^-(j) \neq \tau(n+1)$, so $\delta_{\rho^-(j)}^\tau = -$. Thus, $\delta_j^\rho \delta_{\rho^-(j)}^\tau = +$.

So, for all $j \in \llbracket 1; n \rrbracket - \{\rho \circ \tau(n+1)\}$, we have that $\delta_j^\rho \delta_{\rho^-(j)}^\tau \delta_{\rho^-(n+1)}^\tau = +- = -$. For $j_0 \triangleq \rho \circ \tau(n+1)$, we have that $j_0 = \rho(\tau(n+1)) \neq \rho(n+1)$ because $\tau(n+1) \neq n+1$. So, $\delta_{j_0}^\rho = -$. Moreover, $\rho^-(j_0) = \tau(n+1)$. Hence, $\delta_{\rho^-(j_0)}^\tau = +$. Therefore, $\delta_{j_0}^\rho \delta_{\rho^-(j_0)}^\tau \delta_{\rho^-(n+1)}^\tau = - + - = +$. So, finally, for all $j \in \llbracket 1; n \rrbracket$,

$$\begin{aligned} \delta_j^\rho \delta_{\rho^-(j)}^\tau \delta_{\rho^-(n+1)}^\tau &= \begin{cases} + & \text{if } j = \rho \circ \tau(n+1) \\ - & \text{otherwise} \end{cases} \\ &= \begin{cases} + & \text{if } j = \rho \circ \tau(n+1) \\ \delta_{n+1, (\rho \circ \tau)^-(n+1)} & \text{otherwise} \end{cases} \\ &= \delta_j^{\rho \circ \tau}. \end{aligned}$$

So, eventually,

$$\begin{aligned} \rho(\tau \bullet) &\triangleq (k_{(\rho \circ \tau)^-(1)}, \dots, k_{(\rho \circ \tau)^-(n+1)}, \Delta_0 \mathbb{E}, \Delta_1 \pm_{(\rho \circ \tau)^-(1)}, \dots, \Delta_n \pm_{(\rho \circ \tau)^-(n)}, \\ &\quad \Delta_0 \pm, \rho \circ \tau \circ \sigma) \end{aligned}$$

where for all $j \in \llbracket 0; n \rrbracket$, $\Delta_j \triangleq \delta_j^{\rho \circ \tau} \pm_{(\rho \circ \tau)^-(n+1)}$. That is, $\rho(\tau \bullet) = \rho \circ \tau \bullet$.

As for the proof of the three items, they follow from [2, Proposition 32]. Only the first item really requires explanations. We know from [2, Proposition 32] that $\alpha_n * \beta_n * \gamma_n$ is transitive over ATM. The first item is then proven from the fact that \sim permutes with any other actions of some τ or $-$ and the fact that \sim is idempotent. \square

Theorem 2. Let $n \in \mathbb{N}^*$, let $\bullet = (k_1, \dots, k_{n+1}, \mathbb{E}, \pm_1, \dots, \pm_n, \pm, \sigma)$ be a skeleton and let $\tau_0, \dots, \tau_m \in \mathfrak{S}_{n+1}$ be such that $\tau_i(n+1) \neq n+1$ for all $i \in \llbracket 0; m \rrbracket$. We set $\pm_{n+1} \triangleq +$ and we define $\bar{\tau} \triangleq \tau_m \circ \dots \circ \tau_0$ and for all $j \in \llbracket 0; n \rrbracket$, $\Delta_j \triangleq \Delta^j \Delta^{n+1} \pm_{\bar{\tau}(n+1)}$ where

$$\Delta^j \triangleq \Delta_0^j \Delta_1^j \dots \Delta_m^j \quad \Delta^{n+1} \triangleq \begin{cases} \Delta_0^{n+1} \Delta_1^{n+1} \dots \Delta_{m-1}^{n+1} & \text{if } m \neq 0 \\ + & \text{if } m = 0 \end{cases}$$

$$\text{for all } i \in \llbracket 0; m \rrbracket, \text{ all } k \in \llbracket 0; n+1 \rrbracket, \Delta_i^k \triangleq \begin{cases} + & \text{if } k = \tau_m \circ \dots \circ \tau_{i+1} \circ \tau_i(n+1) \\ - & \text{otherwise} \end{cases}.$$

Then,

$$\tau_m - \dots - \tau_0 \bullet = \left(k_{\bar{\tau}^{-}(1)}, \dots, k_{\bar{\tau}^{-}(n+1)}, \Delta_0 \mathbb{E}, \Delta_1 \pm_{\bar{\tau}^{-}(1)}, \dots, \Delta_n \pm_{\bar{\tau}^{-}(n)}, \Delta_0 \pm, \bar{\tau} \circ \sigma \right).$$

Proof. It is by induction on m . The base case $m = 0$ holds by Definition 8. Now we prove the induction step for $m + 1$.

$$\begin{aligned} -\tau_m - \dots - \tau_0 \bullet &= \left(k_{\bar{\tau}^{-}(1)}, \dots, k_{\bar{\tau}^{-}(n+1)}, -\Delta_0 \mathbb{E}, -\Delta_1 \pm_{\bar{\tau}^{-}(1)}, \dots, -\Delta_n \pm_{\bar{\tau}^{-}(n)}, -\Delta_0 \pm, \bar{\tau} \circ \sigma \right) \\ &\quad \left(k_{\bar{\tau}^{-}(1)}, \dots, k_{\bar{\tau}^{-}(n+1)}, -\Delta^0 \Delta^{n+1} \pm_{\bar{\tau}^{-}(n+1)} \mathbb{E}, \right. \\ &\quad \quad -\Delta^1 \Delta^{n+1} \pm_{\bar{\tau}^{-}(n+1)} \pm_{\bar{\tau}^{-}(1)}, \dots, -\Delta^n \Delta^{n+1} \pm_{\bar{\tau}^{-}(n+1)} \pm_{\bar{\tau}^{-}(n)}, \\ &\quad \quad \left. -\Delta^0 \Delta^{n+1} \pm_{\bar{\tau}^{-}(n+1)} \pm, \bar{\tau} \circ \sigma \right) \end{aligned}$$

- If $j_0 \triangleq \tau_{m+1}^-(n+1) \neq n+1$ then

$$\begin{aligned} &\tau_{m+1} - \tau_m - \dots - \tau_0 \bullet \\ &= \left(k_{\bar{\tau}^{-}(\tau_{m+1}^-(1))}, \dots, k_{\bar{\tau}^{-}(\tau_{m+1}^-(n+1))}, \right. \\ &\quad \delta_0^\tau \left(-\Delta^{j_0} \Delta^{n+1} \pm_{\bar{\tau}^{-}(n+1)} \pm_{\bar{\tau}^{-}(j_0)} \right) - \Delta^0 \Delta^{n+1} \pm_{\bar{\tau}^{-}(n+1)} \mathbb{E}, \\ &\quad \delta_1^\tau \left(-\Delta^{j_0} \Delta^{n+1} \pm_{\bar{\tau}^{-}(n+1)} \pm_{\bar{\tau}^{-}(j_0)} \right) - \Delta^{\tau_{m+1}^-(1)} \Delta^{n+1} \pm_{\bar{\tau}^{-}(n+1)} \pm_{\bar{\tau}^{-}(\tau_{m+1}^-(1))}, \\ &\quad \dots, \\ &\quad \delta_n^\tau \left(-\Delta^{j_0} \Delta^{n+1} \pm_{\bar{\tau}^{-}(n+1)} \pm_{\bar{\tau}^{-}(j_0)} \right) - \Delta^{\tau_{m+1}^-(n)} \Delta^{n+1} \pm_{\bar{\tau}^{-}(n+1)} \pm_{\bar{\tau}^{-}(\tau_{m+1}^-(n))}, \\ &\quad \delta_0^\tau \left(-\Delta^{j_0} \Delta^{n+1} \pm_{\bar{\tau}^{-}(n+1)} \pm_{\bar{\tau}^{-}(j_0)} \right) - \Delta^0 \Delta^{n+1} \pm_{\bar{\tau}^{-}(n+1)} \pm, \\ &\quad \left. \tau_{m+1} \circ \bar{\tau} \circ \sigma \right) \\ &= \left(k_{(\tau_{m+1} \circ \bar{\tau})^{-}(1)}, \dots, k_{(\tau_{m+1} \circ \bar{\tau})^{-}(n+1)}, \right. \\ &\quad \delta_0^\tau \Delta^{j_0} \pm_{\bar{\tau}^{-}(j_0)} \Delta^0 \mathbb{E}, \\ &\quad \delta_1^\tau \Delta^{j_0} \pm_{\bar{\tau}^{-}(j_0)} \Delta^{\tau_{m+1}^-(1)} \pm_{\bar{\tau}^{-}(\tau_{m+1}^-(1))}, \\ &\quad \dots, \\ &\quad \left. \delta_n^\tau \Delta^{j_0} \pm_{\bar{\tau}^{-}(j_0)} \Delta^{\tau_{m+1}^-(n)} \pm_{\bar{\tau}^{-}(\tau_{m+1}^-(n))}, \delta_0^\tau \Delta^{j_0} \pm_{\bar{\tau}^{-}(j_0)} \Delta^0 \pm, \tau_{m+1} \circ \bar{\tau} \circ \sigma \right). \end{aligned}$$

Now, $\tau_{m+1} \circ \bar{\tau}$ corresponds to $\bar{\tau}$ for the step $m+1$. Observing that Δ^{j_0} corresponds to Δ^{n+1} for the step $m+1$ (because $j_0 = \tau_m \circ \dots \circ \tau_i(n+1)$ iff $n+1 = \tau_{m+1} \circ \tau_m \circ \dots \circ \tau_i(n+1)$) and that for all $j \in \llbracket 1; n \rrbracket$, δ_j^τ and $\Delta^{\tau_{m+1}(j)}$ correspond to Δ_{m+1}^j and $\Delta_0^j \dots \Delta_m^j$ respectively, we have that $\delta_j^\tau \Delta^{\tau_{m+1}(j)}$ corresponds to δ_j^τ for the step $m+1$ and, likewise, $\delta_0^\tau \Delta^0$ corresponds to Δ^0 for the step $m+1$. So, we obtain the result.

- if $\tau_{m+1}(n+1) = n+1$ then

$$\begin{aligned} & \tau_{m+1} - \tau_m - \dots - \tau_0^\bullet = \\ & \left(k_{\bar{\tau}^-(\tau_{m+1}^-(1))}, \dots, k_{\bar{\tau}^-(\tau_{m+1}^-(n+1))}, -\Delta^0 \Delta^{n+1} \pm_{\bar{\tau}^-(n+1)} \mathbb{E}, \right. \\ & - \Delta^{\tau_{m+1}^-(1)} \Delta^{n+1} \pm_{\bar{\tau}^-(n+1)} \pm_{\bar{\tau}^-(\tau_{m+1}^-(1))}, \dots, -\Delta^{\tau_{m+1}^-(n)} \Delta^{n+1} \pm_{\bar{\tau}^-(n+1)} \pm_{\bar{\tau}^-(\tau_{m+1}^-(n))}, \\ & \left. - \Delta^0 \Delta^{n+1} \pm_{\bar{\tau}^-(n+1)} \pm, \sigma \circ \bar{\tau}^- \right) \end{aligned}$$

Now, Δ^{n+1} for the step m is equal to Δ^{n+1} for the step $m+1$ because $\tau_{m+1}(n+1) = n+1$. Likewise, $\bar{\tau}(n+1)$ for the step m is equal to $\bar{\tau}(n+1)$ for the step $m+1$ (where, for the step $m+1$, $\bar{\tau}$ is in fact $\tau_{m+1} \circ \bar{\tau}$). Moreover, for all $j \in \llbracket 1; n \rrbracket$, $-\Delta^{\tau_{m+1}^-(j)}$ for the step m is Δ^j for the step $m+1$ (because $\tau_{m+1}^-(j) = \tau_m \circ \dots \circ \tau_i(n+1)$ iff $j = \tau_{m+1} \circ \tau_m \circ \dots \circ \tau_i(n+1)$) and $\Delta_{m+1}^j = -$ since $j \neq n+1 = \tau_{m+1}(n+1)$. Likewise, $-\Delta^0$ for the step is Δ^0 for the step $m+1$, for the same reasons. So, we obtain the result in that case too. \square

C Proofs of Theorem 3, Proposition 4 and Corollary 1

The proof of Theorem 3 follows the same steps as in [2]. The proof needs to be changed and is different from the proof in [2] because we need to take the dimensions into account as well as a different notion of provability/deducibility and because we do not have at our disposal all the connectives of the orbits associated to the set of connectives \mathbf{C} since we assume that \mathbf{C} is common. In this proof, we will often identify connectives with their skeletons to highlight the main ideas.

Theorem 3. *Let \mathbf{C} be a common Boolean set of atomic connectives. The calculus $\mathcal{P}_{\mathbf{C}}$ of Fig. 4 is sound and strongly complete for the basic atomic logic $(\mathcal{L}_{\mathbf{C}}, \mathcal{C}_{\mathbf{C}}, \Vdash)$.*

In this section, the set of atomic connectives \mathbf{C} considered is such that it contains a connective of input or output dimension 1 and a propositional letter of dimension 1.

We provide the soundness and completeness proofs of Theorem 3. We adapt the proof methods introduced in [1], based on a Henkin construction, to our more abstract and general setting. We start by the soundness proof.

Lemma 1. *The calculus $\mathcal{P}_{\mathbf{C}}$ is sound for the Boolean basic atomic logic $(\mathcal{L}_{\mathbf{C}}, \mathcal{C}_{\mathbf{C}}, \Vdash)$.*

Proof. It follows the same line as in [2] and relies on the results of Dunn's gaggle theory. Rule (dr3) is a combination of the rules (dr1) and (dr2) of [2] and its soundness follows from the soundness of these two other rules. \square

The completeness proof uses a canonical model built up from maximal $\mathcal{P}_{\mathbf{C}}$ -consistent sets. First, we define the notions of $\mathcal{P}_{\mathbf{C}}$ -consistent set and maximal $\mathcal{P}_{\mathbf{C}}$ -consistent set.

Definition 15 ((Maximal) \mathcal{P}_C -consistent set). Let $k \in \mathbb{N}^*$. We denote by \mathcal{L}_C^k the sublanguage of \mathcal{L}_C where all formulas are of dimension k .

- A k - \mathcal{P}_C -consistent set is a subset Γ of \mathcal{L}_C^k such that there are no $\varphi_1, \dots, \varphi_n \in \Gamma$ such that $\vdash_{\mathcal{H}} \neg(\varphi_1 \wedge \dots \wedge \varphi_n)$. If $\varphi \in \mathcal{L}_C^k$, we also say that φ is k - \mathcal{P}_C -consistent when the set $\{\varphi\}$ is k - \mathcal{P}_C -consistent.
- A maximal k - \mathcal{P}_C -consistent set is a k - \mathcal{P}_C -consistent set Γ of \mathcal{L}_C^k such that there is no $\varphi \in \mathcal{L}_C^k$ satisfying both $\varphi \notin \Gamma$ and $\Gamma \cup \{\varphi\}$ is k - \mathcal{P}_C -consistent. \square

Lemma 2 (Cut lemma). Let Γ be a maximal k - \mathcal{P}_C -consistent set. For all $\varphi \in \Gamma$ and all $\psi \in \mathcal{L}_C^k$, if $\vdash_{\mathcal{H}} (\varphi \rightarrow \psi)$ then $\psi \in \Gamma$.

Proof. First, we show that $\Gamma \cup \{\psi\}$ is k - \mathcal{P}_C -consistent. Assume towards a contradiction that it is not the case. Then, there are $\psi_1, \dots, \psi_m \in \Gamma$ such that $\vdash_{\mathcal{H}} \neg(\psi \wedge \psi_1 \wedge \dots \wedge \psi_m)$ (*). Then, by the propositional axioms \mathcal{P}_{PL} , we have that $\vdash_{\mathcal{H}} ((\varphi \rightarrow \psi) \rightarrow (\neg(\psi \wedge \psi_1 \wedge \dots \wedge \psi_m) \rightarrow \neg(\psi_1 \wedge \dots \wedge \psi_m \wedge \varphi)))$. By assumption, $\vdash_{\mathcal{H}} (\varphi \rightarrow \psi)$. Therefore, by Modus Ponens, $\vdash_{\mathcal{H}} (\neg(\psi \wedge \psi_1 \wedge \dots \wedge \psi_m) \rightarrow \neg(\psi_1 \wedge \dots \wedge \psi_m \wedge \varphi))$. Now, applying again Modus Ponens with (*), we have that $\vdash_{\mathcal{H}} \neg(\psi_1 \wedge \dots \wedge \psi_m \wedge \varphi)$. However, since $\varphi, \psi_1, \dots, \psi_m \in \Gamma$, we have that Γ is not k - \mathcal{P}_C -consistent. This is impossible. Thus, $\Gamma \cup \{\psi\}$ is k - \mathcal{P}_C -consistent. Now, since Γ is a maximal k - \mathcal{P}_C -consistent set, this implies that $\psi \in \Gamma$. \square

Lemma 3 (Lindenbaum lemma). Any k - \mathcal{P}_C -consistent set can be extended into a maximal k - \mathcal{P}_C -consistent set.

Proof. Let $\varphi_1, \varphi_2, \dots, \varphi_n, \dots$ be an enumeration of \mathcal{L}_C^k (it exists because C is countable). We define the sets Γ_n inductively as follows:

$$\begin{aligned} \Gamma_0 &\triangleq \Gamma \\ \Gamma_{n+1} &\triangleq \begin{cases} \Gamma_n \cup \{\varphi_n\} & \text{if } \Gamma_n \cup \{\varphi_n\} \text{ is } k\text{-}\mathcal{P}_C\text{-consistent} \\ \Gamma_n & \text{otherwise.} \end{cases} \end{aligned}$$

Then, we define the subset Γ^+ of \mathcal{L} as follows: $\Gamma^+ = \bigcup_{n \in \mathbb{N}} \Gamma_n$.

We show that Γ^+ is a maximal k - \mathcal{P}_C -consistent set. Clearly, for all $n \in \mathbb{N}$, Γ_n is k - \mathcal{P}_C -consistent by definition of Γ_n . So, if Γ^+ was not k - \mathcal{P}_C -consistent, there would be a $n_0 \in \mathbb{N}$ such that Γ_{n_0} is not k - \mathcal{P}_C -consistent, which is impossible. Now, assume towards a contradiction that Γ^+ is not a maximal k - \mathcal{P}_C -consistent set. Then, there is $\varphi \in \mathcal{L}_C^k$ such that $\varphi \notin \Gamma^+$ and $\Gamma \cup \{\varphi\}$ is k - \mathcal{P}_C -consistent. But there is $n_0 \in \mathbb{N}$ such that $\varphi = \varphi_{n_0}$. Because $\varphi \notin \Gamma^+$, we also have that $\varphi_{n_0} \notin \Gamma_{n_0+1}$. So, $\Gamma_{n_0} \cup \{\varphi_{n_0}\}$ is not k - \mathcal{P}_C -consistent by definition of Γ^+ . Therefore, $\Gamma^+ \cup \{\varphi\}$ is not k - \mathcal{P}_C -consistent either, which is impossible. \square

Lemma 4. The following formulas are provable in \mathcal{P}_C : for all $\varphi, \varphi' \in \mathcal{L}_C$,

$$(\varphi \rightarrow \varphi) \tag{1}$$

$$(\neg\neg\varphi \rightarrow \varphi) \tag{2}$$

$$(\varphi \rightarrow (\varphi' \rightarrow (\varphi \wedge \varphi'))) \tag{3}$$

$$((\varphi \wedge \varphi') \rightarrow \varphi') \tag{4}$$

$$(((\varphi \vee \varphi') \wedge (\varphi \vee \neg\varphi')) \rightarrow \varphi) \tag{5}$$

$$(\varphi \rightarrow ((\varphi \wedge \neg\varphi') \vee (\varphi \wedge \varphi'))) \tag{6}$$

Proof. Since Expressions (1)–(6) are all validities of propositional logic, they are also provable in \mathcal{P}_C by \mathcal{P}_{PL} (and MP). \square

Lemma 5. Let $\odot(\varphi_1, \dots, \varphi_n) \in \mathcal{L}_C^k$. For all $\pm \in \mathbb{Z}/2\mathbb{Z}$, we define

$$\pm\varphi_j \triangleq \begin{cases} \varphi_j & \text{if } \pm = + \\ \neg\varphi_j & \text{if } \pm = - \end{cases}.$$

- If $\bullet = (k_1, \dots, k_{n+1}, \exists, \pm_1, \dots, \pm_n, \pm, \sigma)$ and $\odot(\varphi_1, \dots, \varphi_n)$ is $k\text{-}\mathcal{P}_C$ -consistent then for all $j \in \llbracket 1; n \rrbracket$, $\pm_j\varphi_j$ is $k_j\text{-}\mathcal{P}_C$ -consistent;
- if $\bullet = (k_1, \dots, k_{n+1}, \forall, \pm_1, \dots, \pm_n, \pm, \sigma)$ and $\neg\odot(\varphi_1, \dots, \varphi_n)$ is $k\text{-}\mathcal{P}_C$ -consistent then for all $j \in \llbracket 1; n \rrbracket$, $-\pm_j\varphi_j$ is $k_j\text{-}\mathcal{P}_C$ -consistent.

Proof. We prove the first item by contraposition. Assume that $\pm_j\varphi_j$ is $k_j\text{-}\mathcal{P}_C$ -inconsistent. If $\pm_j = +$ then $\vdash_{\mathcal{H}} \neg\varphi_j$. If $\pm_j = -$ then $\vdash_{\mathcal{H}} \neg\neg\varphi_j$ and therefore $\vdash_{\mathcal{H}} \varphi_j$ because $\vdash_{\mathcal{H}} \neg\neg\varphi_j \rightarrow \varphi_j$. So, in both cases, applying Rules R1 and R2 respectively, we obtain that $\vdash_{\mathcal{H}} \neg\odot(\varphi_1, \dots, \varphi_j, \dots, \varphi_n)$ and thus $\odot(\varphi_1, \dots, \varphi_n)$ is $k\text{-}\mathcal{P}_C$ -inconsistent. This proves the first item. The proof of the second item is dual, using rules R3 and R4 instead. \square

Definition 16 (Canonical model). Let $C \subseteq \text{ATM}$. The *canonical model associated to C* is the tuple $\mathcal{M}^c \triangleq (W^c, \mathcal{R}^c)$ where W^c is the set of all maximal $k\text{-}\mathcal{P}_C$ -consistent sets of \mathcal{L}_C^k , for k ranging over the output dimensions of the connectives of C , and \mathcal{R}^c is a set of relations R_\odot over W^c , associated to the connectives $\odot \in C$ (of skeleton \bullet) and defined by:

- if $\bullet = p = (k, \mathcal{A}, \pm)$ then for all maximal $k\text{-}\mathcal{P}_C$ -consistent set Γ , $\Gamma \in R_p^\pm$ iff $p \in \Gamma$;
- if $\bullet = (k_1, \dots, k_{n+1}, \exists, \pm_1, \dots, \pm_n, \pm, \sigma)$ then for all $j \in \llbracket 1; n+1 \rrbracket$ and all maximal $k_j\text{-}\mathcal{P}_C$ -consistent sets Γ_j ,
 $(\Gamma_1, \dots, \Gamma_{n+1}) \in R_\odot^{\pm\sigma}$ iff for all $\varphi_1, \dots, \varphi_n \in \mathcal{L}_C$, if $\varphi_1 \uparrow_1 \Gamma_1$ and ... and $\varphi_n \uparrow_n \Gamma_n$ then $\odot(\varphi_1, \dots, \varphi_n) \in \Gamma_{n+1}$;
- if $\bullet = (k_1, \dots, k_{n+1}, \forall, \pm_1, \dots, \pm_n, \pm, \sigma)$ then for all $j \in \llbracket 1; n+1 \rrbracket$ and all maximal $k_j\text{-}\mathcal{P}_C$ -consistent sets Γ_j ,
 $(\Gamma_1, \dots, \Gamma_{n+1}) \notin R_\odot^{\pm\sigma}$ iff for all $\varphi_1, \dots, \varphi_n \in \mathcal{L}_C$, if $\odot(\varphi_1, \dots, \varphi_n) \in \Gamma_{n+1}$ then $\varphi_1 \uparrow_1 \Gamma_1$ or ... or $\varphi_n \uparrow_n \Gamma_n$;

where for all $j \in \llbracket 1; n \rrbracket$, $\varphi_j \uparrow_j \Gamma_j \triangleq \begin{cases} \varphi_j \in \Gamma_j & \text{if } \pm_j = + \\ \varphi_j \notin \Gamma_j & \text{if } \pm_j = - \end{cases}$. \square

Lemma 6 (Truth lemma). For all $\varphi \in \mathcal{L}_C^k$, for all maximal $k\text{-}\mathcal{P}_C$ -consistent sets Γ , we have that $(\mathcal{M}^c, \Gamma) \Vdash \varphi$ iff $\varphi \in \Gamma$.

Proof. By induction on φ . The base case $\varphi = p \in \text{ATM}_0$ holds trivially by definition of \mathcal{M}^c .

- Case $\neg\varphi$.

Assume that $\neg\varphi \in \Gamma$ and assume towards a contradiction that it is not the case that $\mathcal{M}^c, \Gamma \Vdash \neg\varphi$. Then, $\mathcal{M}^c, \Gamma \Vdash \varphi$. So, by Induction Hypothesis, $\varphi \in \Gamma$. Now, $\vdash_{\mathcal{H}} \neg(\varphi \wedge \neg\varphi)$ (that is Expression (1), $\vdash_{\mathcal{H}}(\varphi \rightarrow \varphi)$) and $\neg\varphi \in \Gamma$ by assumption. Thus, Γ is not $k\text{-}\mathcal{P}_C$ -consistent, which is impossible. Therefore, $\mathcal{M}^c, \Gamma \Vdash \neg\varphi$.

Conversely, assume that $\mathcal{M}^c, \Gamma \Vdash \neg\varphi$. Then, it is not the case that $\mathcal{M}^c, \Gamma \Vdash \varphi$, so, by Induction Hypothesis, $\varphi \notin \Gamma$. Since Γ is a maximal $k\text{-}\mathcal{P}_C$ -consistent set, this implies that $\Gamma \cup \{\varphi\}$ is not $k\text{-}\mathcal{P}_C$ -consistent. So, there are $\varphi_1, \dots, \varphi_n \in \Gamma$ such that $\vdash_{\mathcal{H}} \neg(\varphi_1 \wedge \dots \wedge \varphi_n \wedge \varphi)$. Now, because of Expression (2), we have that $\vdash_{\mathcal{H}} (\neg\neg\varphi \rightarrow \varphi)$. So, by MP and axiom A3, we have that $\vdash_{\mathcal{H}} \neg(\varphi_1 \wedge \dots \wedge \varphi_n \wedge \neg\neg\varphi)$. That is, $\vdash_{\mathcal{H}} ((\varphi_1 \wedge \dots \wedge \varphi_n) \rightarrow \neg\varphi)$ (*). Then, by Expression (3) and an iterative application of the cut lemma, we have that $\neg\varphi \in \Gamma$.

- Cases $(\varphi \wedge \psi)$ and $(\varphi \vee \psi)$.

We prove the following fact. This will prove this induction step because $\mathcal{M}^c, \Gamma \Vdash \varphi \wedge \psi$ iff $\mathcal{M}^c, \Gamma \Vdash \varphi$ and $\mathcal{M}^c, \Gamma \Vdash \psi$, iff $\varphi \in \Gamma$ and $\psi \in \Gamma$ by induction hypothesis.

Fact 1. *For all maximal $k\text{-}\mathcal{P}_C$ -consistent sets Γ , $(\varphi \wedge \psi) \in \Gamma$ iff $\varphi \in \Gamma$ and $\psi \in \Gamma$, and $(\varphi \vee \psi) \in \Gamma$ iff $\varphi \in \Gamma$ or $\psi \in \Gamma$.*

Proof. Assume that $\varphi \in \Gamma$ and $\psi \in \Gamma$. Then, since $\vdash_{\mathcal{H}} (\varphi \rightarrow (\psi \rightarrow (\varphi \wedge \psi)))$, we have by a double application of the cut lemma that $(\varphi \wedge \psi) \in \Gamma$. Conversely, assume that $(\varphi \wedge \psi) \in \Gamma$. Then, since $\vdash_{\mathcal{H}} ((\varphi \wedge \psi) \rightarrow \varphi)$, we have that $\varphi \in \Gamma$ by the cut lemma. Likewise, since $\vdash_{\mathcal{H}} ((\varphi \wedge \psi) \rightarrow \psi)$ by Expression (4), we have that $\psi \in \Gamma$. The second part of the proof is proven dually using the fact proven in the previous induction step for \neg that for all maximal $k\text{-}\mathcal{P}_C$ -consistent sets Γ , it holds that $\varphi \notin \Gamma$ iff $\neg\varphi \in \Gamma$. \square

- Case $\odot(\varphi_1, \dots, \varphi_n)$ with $\bullet = (k_1, \dots, k_{n+1}, \mathbb{A}, \pm_1, \dots, \pm_n, \pm, \sigma)$.

1. We deal with the subcase $\mathbb{A} = \exists$.

Assume that $\odot(\varphi_1, \dots, \varphi_n) \in \Gamma$. We have to show that $\mathcal{M}^c, \Gamma \Vdash \odot(\varphi_1, \dots, \varphi_n)$, i.e., there are $\Gamma_1, \dots, \Gamma_n \in \mathcal{M}^c$ such that $R_{\odot}^{\pm\sigma} \Gamma_1 \dots \Gamma_n \Gamma$ and $\Gamma_1 \Vdash \llbracket \varphi_1 \rrbracket$ and \dots and $\Gamma_n \Vdash \llbracket \varphi_n \rrbracket$. We build these maximal $k\text{-}\mathcal{P}_C$ -consistent sets $\Gamma_1, \dots, \Gamma_n$ thanks to (pseudo) Algorithm 1 (because it does not terminate). This algorithm is such that if $\odot(\boxtimes_1 \Gamma_1, \dots, \boxtimes_n \Gamma_n) \in \Gamma$ then for all $\varphi_1^m, \dots, \varphi_n^m \in \mathcal{L}_C^k$, there are $(\pm'_1, \dots, \pm'_n) \in \{+, -\}^n$ such that $\odot((\boxtimes_1 \Gamma_1^m) \times_1 (\pm'_1 \varphi_1^m), \dots, (\boxtimes_n \Gamma_n^m) \times_n (\pm'_n \varphi_n^m)) \in \Gamma$. This is due to Expressions (5), (6) of Lemma 4 and Axioms A1 and A2. What happens is that each $\boxtimes_j \Gamma_j$ is decomposed into disjunctions $((\boxtimes_j \Gamma_j) \wedge \varphi_n) \vee ((\boxtimes_j \Gamma_n) \wedge \neg\varphi_n)$ and conjunctions $((\boxtimes_j \Gamma_j) \vee \varphi_n) \wedge ((\boxtimes_j \Gamma_j) \vee \neg\varphi_n)$ depending on whether $\pm_j = +$ or $\pm_j = -$. Then, each decomposition of $\boxtimes_j \Gamma_n$ is replaced in Expression $\odot(\boxtimes_1 \Gamma_1, \dots, \boxtimes_n \Gamma_n)$. This is possible thanks to rules R5 and R6 and this yields a new expression (*). This new expression (*) belongs to Γ because Γ is a maximal $k\text{-}\mathcal{P}_C$ -consistent set, by the cut lemma. Then, we decompose again (*) iteratively by applying Axioms A1 and A2. For each decomposition, at least one disjunct belongs to Γ because $(\varphi \vee \psi) \in \Gamma$ implies that either $\varphi \in \Gamma$ or $\psi \in \Gamma$ by Fact 1. Finally, after having decomposed each argument of \odot , we obtain that there is $(\pm'_1, \dots, \pm'_n) \in \{+, -\}^n$ such that $\odot((\boxtimes_1 \Gamma_1^m) \times_1 (\pm'_1 \varphi_1^m), \dots, (\boxtimes_n \Gamma_n^m) \times_n (\pm'_n \varphi_n^m)) \in \Gamma$.

Now, we prove that $\Gamma_1, \dots, \Gamma_n$ are $k_j\text{-}\mathcal{P}_C$ -consistent. To do so, it suffices to prove that for all $m \geq 0$, we have that Γ_j^m is $k_j\text{-}\mathcal{P}_C$ -consistent. We prove it by induction on m . The base case $m = 0$ holds by assumption. Let $m \geq 0$ be fixed and assume that Γ_j^m is $k_j\text{-}\mathcal{P}_C$ -consistent. Then, $\odot((\boxtimes_1 \Gamma_1^m) \times_1 (\pm'_1 \varphi_1^m), \dots, (\boxtimes_n \Gamma_n^m) \times_n (\pm'_n \varphi_n^m))$ is $k_j\text{-}\mathcal{P}_C$ -consistent because it belongs to the $k_j\text{-}\mathcal{P}_C$ -consistent set Γ_j^m . Thus, by Lemma 5, for all $j \in \llbracket 1; n \rrbracket$, if $\pm_j = +$ then $\bigwedge \Gamma_j^m \wedge \pm'_j \varphi_j^m$ is $k_j\text{-}\mathcal{P}_C$ -consistent and if $\pm_j = -$ then $\bigwedge \Gamma_j^m \wedge (\neg \pm'_j) \varphi_j^m$ is

Algorithm 1

Require: $(\varphi_1, \dots, \varphi_n) \in \mathcal{L}_C^{k_1} \times \dots \times \mathcal{L}_C^{k_n}$ and a maximal $k\text{-}\mathcal{P}_C$ -consistent set Γ such that $\odot(\varphi_1, \dots, \varphi_n) \in \Gamma$ with $\bullet = (k_1, \dots, k_{n+1}, \exists, \pm_1, \dots, \pm_n, \pm, \sigma)$.

Ensure: A n -tuple of maximal $k_j\text{-}\mathcal{P}_C$ -consistent sets $(\Gamma_1, \dots, \Gamma_n)$ such that $R_{\odot}^{\pm\sigma}\Gamma_1 \dots \Gamma_n \Gamma$ and $\pm_1\varphi_1 \in \Gamma_1, \dots, \pm_n\varphi_n \in \Gamma_n$.

Let $(\varphi_1^0, \dots, \varphi_n^0), \dots, (\varphi_1^m, \dots, \varphi_n^m), \dots$ be an enumeration of $\mathcal{L}_C^{k_1} \times \dots \times \mathcal{L}_C^{k_n}$;

$\Gamma_1^0 := \{\pm_1\varphi_1\}; \dots; \Gamma_n^0 := \{\pm_n\varphi_n\}$;

5:

for all $m \geq 0$ **do**

for all $(\pm'_1, \dots, \pm'_n) \in \{+, -\}^n$ **do**

if $\odot((\boxtimes_1\Gamma_1^m) \times_1 (\pm'_1\varphi_1^m), \dots, (\boxtimes_n\Gamma_n^m) \times_n (\pm'_n\varphi_n^m)) \in \Gamma$ **then**

$\Gamma_1^{m+1} := \Gamma_1^m \cup \{(\pm_1\pm'_1)\varphi_1^m\}$;

\vdots

10:

$\Gamma_n^{m+1} := \Gamma_n^m \cup \{(\pm_n\pm'_n)\varphi_n^m\}$;

end if

end for

end for

15:

$\Gamma_1 := \bigcup_{m \geq 0} \Gamma_1^m; \dots; \Gamma_n := \bigcup_{m \geq 0} \Gamma_n^m$;

where for all $\varphi \in \mathcal{L}_C^k$, $\pm\varphi \triangleq \begin{cases} \varphi & \text{if } \pm = + \\ \neg\varphi & \text{if } \pm = - \end{cases}$ for all $j \in \llbracket 1; n \rrbracket$, $\times_j \triangleq \begin{cases} \wedge & \text{if } \pm_j = + \\ \vee & \text{if } \pm_j = - \end{cases}$ and $\boxtimes_j \Gamma_j^m \triangleq \begin{cases} \wedge \{\varphi \mid \varphi \in \Gamma_j^m\} & \text{if } \pm_j = + \\ \vee \{\neg\varphi \mid \varphi \in \Gamma_j^m\} & \text{if } \pm_j = - \end{cases}$.

$k_j\text{-}\mathcal{P}_C$ -consistent. That is, in both cases, Γ_j^{m+1} is $k_j\text{-}\mathcal{P}_C$ -consistent. We have proven by induction that for all $m \geq 0$, Γ_j^m is $k_j\text{-}\mathcal{P}_C$ -consistent. Thus, $\Gamma_1, \dots, \Gamma_n$ are $k_j\text{-}\mathcal{P}_C$ -consistent. Moreover, for all $j \in \llbracket 1; n \rrbracket$, Γ_j are *maximally* $k_j\text{-}\mathcal{P}_C$ -consistent because by construction for all $\varphi \in \mathcal{L}_C^k$ either $\varphi \in \Gamma_j$ or $\neg\varphi \in \Gamma_j$.

Finally, we prove that $R_{\odot}^{\pm\sigma}\Gamma_1 \dots \Gamma_n \Gamma$, that is, we prove that for all $\psi_1, \dots, \psi_n \in \mathcal{L}_C^k$ if $\psi_1 \triangleleft_1 \Gamma_1$ and \dots and $\psi_n \triangleleft_n \Gamma_n$ then $\odot(\psi_1, \dots, \psi_n) \in \Gamma$, that is, since $\Gamma_1, \dots, \Gamma_n$ are maximally $k_j\text{-}\mathcal{P}_C$ -consistent sets, if $\pm_1\psi_1 \in \Gamma_1$ and \dots and $\pm_n\psi_n \in \Gamma_n$ then $\odot(\psi_1, \dots, \psi_n) \in \Gamma$. Assume that $\pm_1\psi_1 \in \Gamma_1$ and \dots and $\pm_n\psi_n \in \Gamma_n$, we are going to prove that $\odot(\psi_1, \dots, \psi_n) \in \Gamma$. Now $(\psi_1, \dots, \psi_n) \in \mathcal{L}_C^{k_1} \times \dots \times \mathcal{L}_C^{k_n}$, so there is $m_0 \geq 0$ such that $(\varphi_1^{m_0}, \dots, \varphi_n^{m_0}) = (\psi_1, \dots, \psi_n)$. Since $\Gamma_1^{m_0+1} \subseteq \Gamma_1$ and \dots and $\Gamma_n^{m_0+1} \subseteq \Gamma_n$, we have that the tuple (\pm'_1, \dots, \pm'_n) satisfying the condition of line 8 of Algorithm 1 is $(+, \dots, +)$, because of the way $\Gamma_1^{m_0+1}, \dots, \Gamma_n^{m_0+1}$ are defined. So, the condition of line 8, which is fulfilled, is $\odot((\boxtimes_1\Gamma_1^{m_0}) \times_1 \varphi_1^{m_0}, \dots, (\boxtimes_n\Gamma_n^{m_0}) \times_n \varphi_n^{m_0}) \in \Gamma$. Then, for all $j \in \llbracket 1; n \rrbracket$, if $\pm_j = +$ then $\vdash_{\mathcal{H}}(((\boxtimes_j\Gamma_j^{m_0}) \times_j \varphi_j^{m_0}) \rightarrow \varphi_j^{m_0})$ and if $\pm_j = -$ then $\vdash_{\mathcal{H}}(\varphi_j^{m_0} \rightarrow ((\boxtimes_j\Gamma_j^{m_0}) \times_j \varphi_j^{m_0}))$. Therefore, applying rules R5 and R6, we obtain that

$\vdash_{\mathcal{H}} \odot((\boxtimes_1 \Gamma_1^{m_0}) \times_1 \varphi_1^{m_0}, \dots, (\boxtimes_n \Gamma_n^{m_0}) \times_n \varphi_n^{m_0}) \rightarrow \odot(\varphi_1^{m_0}, \dots, \varphi_n^{m_0})$. Since we have proven that $R_{\odot}^{\pm\sigma}((\boxtimes_1 \Gamma_1^{m_0}) \times_1 \varphi_1^m, \dots, (\boxtimes_n \Gamma_n^{m_0}) \times_n \varphi_n^{m_0}) \in \Gamma$, we obtain by the cut lemma that $\odot(\varphi_1^{m_0}, \dots, \varphi_n^{m_0}) \in \Gamma$ as well, that is $\odot(\psi_1, \dots, \psi_n) \in \Gamma$.

Conversely, assume that $\mathcal{M}^c, \Gamma \Vdash \odot(\varphi_1, \dots, \varphi_n)$, we are going to show that $\odot(\varphi_1, \dots, \varphi_n) \in \Gamma$. By definition, we have that there are $\Gamma_1, \dots, \Gamma_n \in \mathcal{M}^c$ such that $R_{\odot}^{\pm\sigma} \Gamma_1 \dots \Gamma_n \Gamma$ and $\Gamma_1 \Vdash \llbracket \varphi_1 \rrbracket$ and \dots and $\Gamma_n \Vdash \llbracket \varphi_n \rrbracket$. By Induction Hypothesis, we have that $\varphi_1 \Vdash_1 \Gamma_1$ and \dots and $\varphi_n \Vdash_n \Gamma_n$. Then, by definition of $R_{\odot}^{\pm\sigma}$ in Definition 16, we have that $\odot(\varphi_1, \dots, \varphi_n) \in \Gamma$.

2. We deal with the subcase $\mathcal{A} = \forall$.

Assume that $\odot(\varphi_1, \dots, \varphi_n) \in \Gamma$. We have to show that $\mathcal{M}^c, \Gamma \Vdash \odot(\varphi_1, \dots, \varphi_n)$, *i.e.* for all $\Gamma_1, \dots, \Gamma_n \in \mathcal{M}^c$, $(\Gamma_1, \dots, \Gamma_n, \Gamma) \in R_{\odot}^{\pm\sigma}$ or $\Gamma_1 \Vdash_1 \llbracket \varphi_1 \rrbracket$ or \dots or $\Gamma_n \Vdash_n \llbracket \varphi_n \rrbracket$. Assume that $(\Gamma_1, \dots, \Gamma_n, \Gamma) \notin R_{\odot}^{\pm\sigma}$. Then, since $\odot(\varphi_1, \dots, \varphi_n) \in \Gamma$, we have by Definition 16 that $\varphi_1 \Vdash_1 \Gamma_1$ or \dots or $\varphi_n \Vdash_n \Gamma_n$. So, by Induction Hypothesis, we have that $\Gamma_1 \Vdash_1 \llbracket \varphi_1 \rrbracket$ or \dots or $\Gamma_n \Vdash_n \llbracket \varphi_n \rrbracket$.

Conversely, we reason by contraposition and we assume that $\odot(\varphi_1, \dots, \varphi_n) \notin \Gamma$. We are going to show that $\text{not } \mathcal{M}^c, \Gamma \Vdash \odot(\varphi_1, \dots, \varphi_n)$, *i.e.* there are $\Gamma_1, \dots, \Gamma_n \in \mathcal{M}^c$ such that $(\Gamma_1, \dots, \Gamma_n, \Gamma) \notin R_{\odot}^{\pm\sigma}$ and $\text{not } \varphi_1 \Vdash_1 \Gamma_1$ and \dots and $\text{not } \varphi_n \Vdash_n \Gamma_n$. That is, there are $\Gamma_1, \dots, \Gamma_n \in \mathcal{M}^c$ such that $\text{not } \varphi_1 \Vdash_1 \Gamma_1$ and \dots and $\text{not } \varphi_n \Vdash_n \Gamma_n$ and for all $\varphi_1^m \in \mathcal{L}_{\mathcal{C}}^{k_1}, \dots, \varphi_n^m \in \mathcal{L}_{\mathcal{C}}^{k_n}$, if $\odot(\varphi_1^m, \dots, \varphi_n^m) \in \Gamma_{n+1}$ then $\varphi_1^m \Vdash_1 \Gamma_1$ or \dots or $\varphi_n^m \Vdash_n \Gamma_n$. That is, there are $\Gamma_1, \dots, \Gamma_n \in \mathcal{M}^c$ such that $\text{not } \varphi_1 \Vdash_1 \Gamma_1$ and \dots and $\text{not } \varphi_n \Vdash_n \Gamma_n$ and for all $\varphi_1^m \in \mathcal{L}_{\mathcal{C}}^{k_1}, \dots, \varphi_n^m \in \mathcal{L}_{\mathcal{C}}^{k_n}$, if $\text{not } \varphi_1^m \Vdash_1 \Gamma_1$ and \dots and $\text{not } \varphi_n^m \Vdash_n \Gamma_n$ then $\odot(\varphi_1^m, \dots, \varphi_n^m) \notin \Gamma_{n+1}$. That is, there are $\Gamma_1, \dots, \Gamma_n \in \mathcal{M}^c$ such that $\text{not } \varphi_1 \Vdash_1 \Gamma_1$ and \dots and $\text{not } \varphi_n \Vdash_n \Gamma_n$ and for all $\varphi_1^m \in \mathcal{L}_{\mathcal{C}}^{k_1}, \dots, \varphi_n^m \in \mathcal{L}_{\mathcal{C}}^{k_n}$, if $\text{not } \varphi_1^m \Vdash_1 \Gamma_1$ and \dots and $\text{not } \varphi_n^m \Vdash_n \Gamma_n$ then $\odot(\varphi_1^m, \dots, \varphi_n^m) \notin \Gamma_{n+1}$.

The maximal consistent sets $\Gamma_1, \dots, \Gamma_n$ are built up by the process described in Algorithm 2, which is the dual of Algorithm 1. Likewise, the proof that Algorithm 2 is sound is similar to the soundness proof of Algorithm 1. The only difference is that we use Axioms A3 and A4 (as well as the same Expressions (5), (6) of Lemma 4) for the decomposition phase and Rules R3 and R4 in Lemma 5 to prove maximal consistency. \square

Lemma 7. *Let $n \in \mathbb{N}^*$. For all $\odot, \odot' \in \text{ATM}$ such that $\mathcal{O}_{\alpha_n * \beta_n}(\odot) = \mathcal{O}_{\alpha_n * \beta_n}(\odot')$, we have that $R_{\odot} = R'_{\odot}$.*

Proof. We prove this lemma using Axiom A5. We prove that for all $\odot, \odot' \in \mathcal{C}$ for which there are $\tau_0, \dots, \tau_m \in \mathfrak{S}_{n+1}$ such that $\odot' = \tau_0 - \dots - \tau_m \odot$, we have that $R_{\odot} = R'_{\odot}$ (*). First, observe that if $\tau_i(n+1) = n+1$ then $-\tau_i - \bullet = \tau_i \bullet$. So, using the fact that by definition of the canonical model we have that $R_{-\odot} = R_{\odot}$, we can reduce the general case to the case whereby $\tau_i(n+1) \neq n+1$, for all $i \in \llbracket 0; m \rrbracket$. Hence, by definition of the group action, it suffices to prove (*) for permutations τ_i such that $\tau_i(n+1) \neq n+1$, for all $i \in \llbracket 0; m \rrbracket$. To be more precise, it even suffices to prove either that $R'_{\odot} \subseteq R_{\odot}$ or that $R_{\odot} \subseteq R'_{\odot}$ because the other inclusion is proven by the same reasoning, using the fact that $\odot = \tau_m^- - \dots - \tau_0^- \odot'$.

1. Assume that $\bullet = (\bar{k}, \forall, \pm_1, \dots, \pm_n, \pm, \sigma)$ and that $\bullet' = \tau_0 - \dots - \tau_m \bullet$ where $\tau_0, \dots, \tau_m \in \mathfrak{S}_{n+1}$ are such that $\tau_i(n+1) \neq n+1$ for all $i \in \llbracket 0; m \rrbracket$. Let $\bar{\tau} \triangleq \tau_0 \circ \dots \circ \tau_m$.

Algorithm 2

Require: $(\varphi_1, \dots, \varphi_n) \in \mathcal{L}_{\mathcal{C}}^{k_1} \times \dots \times \mathcal{L}_{\mathcal{C}}^{k_n}$ and a maximal k - $\mathcal{P}_{\mathcal{C}}$ -consistent set Γ such that $\neg \odot (\varphi_1, \dots, \varphi_n) \in \Gamma$ with $\bullet = (k_1, \dots, k_{n+1}, \forall, \pm_1, \dots, \pm_n, \pm, \sigma)$.

Ensure: A n -tuple of maximal k_j - $\mathcal{P}_{\mathcal{C}}$ -consistent sets $(\Gamma_1, \dots, \Gamma_n)$ such that $R_{\odot}^{\pm\sigma} \Gamma_1 \dots \Gamma_n \Gamma$ and $-\pm_1 \varphi_1 \in \Gamma_1, \dots, -\pm_n \varphi_n \in \Gamma_n$.

Let $(\varphi_1^0, \dots, \varphi_n^0), \dots, (\varphi_1^m, \dots, \varphi_n^m), \dots$ be an enumeration of $\mathcal{L}_{\mathcal{C}}^{k_1} \times \dots \times \mathcal{L}_{\mathcal{C}}^{k_n}$;

$\Gamma_1^0 := \{-\pm_1 \varphi_1\}; \dots; \Gamma_n^0 := \{-\pm_n \varphi_n\}$;

5:

for all $m \geq 0$ **do**

for all $(\pm'_1, \dots, \pm'_n) \in \{+, -\}^n$ **do**

if $\neg \odot ((\bowtie_1 \Gamma_1^m) \times_1 (\pm'_1 \varphi_1^m), \dots, (\bowtie_n \Gamma_n^m) \times_n (\pm'_n \varphi_n^m)) \in \Gamma$ **then**

$\Gamma_1^{m+1} := \Gamma_1^m \cup \{(-\pm_1 \pm'_1) \varphi_1^m\}$;

\vdots

10:

$\Gamma_n^{m+1} := \Gamma_n^m \cup \{(-\pm_n \pm'_n) \varphi_n^m\}$;

end if

end for

end for

15:

$\Gamma_1 := \bigcup_{m \geq 0} \Gamma_1^m; \dots; \Gamma_n := \bigcup_{m \geq 0} \Gamma_n^m$;

where for all $\varphi \in \mathcal{L}_{\mathcal{C}}^k$, $\pm\varphi \triangleq \begin{cases} \varphi & \text{if } \pm = + \\ \neg\varphi & \text{if } \pm = - \end{cases}$;

for all $j \in \llbracket 1; n \rrbracket$, $\times_j \triangleq \begin{cases} \wedge & \text{if } \pm_j = + \\ \vee & \text{if } \pm_j = - \end{cases}$ and

$\bowtie_j \Gamma_j^m \triangleq \begin{cases} \wedge \{ \varphi \mid \varphi \in \Gamma_j^m \} & \text{if } \pm_j = + \\ \vee \{ \neg\varphi \mid \varphi \in \Gamma_j^m \} & \text{if } \pm_j = - \end{cases}$.

(a) Assume that $\Delta^0 \Delta^{n+1} \pm_{\bar{\tau}-(n+1)} = -$, *i.e.* $\Delta^{n+1} \pm_{\bar{\tau}-(n+1)} = -^m$.

Then, by Theorem 2,

$$\bullet' = \left(\bar{\tau} \bar{k}, \exists, -^m \Delta^1 \pm_{\bar{\tau}-(1)}, \dots, -^m \Delta^n \pm_{\bar{\tau}-(n)}, -\pm, \bar{\tau} \circ \sigma \right). \quad (7)$$

Assume that $(\Gamma_1, \dots, \Gamma_n, \Gamma_{n+1}) \notin R_{\odot}^{\pm\sigma}$, we will show that $(\Gamma_1, \dots, \Gamma_n, \Gamma_{n+1}) \notin R_{\odot}^{\pm\sigma}$, *i.e.* $(\Gamma_1, \dots, \Gamma_n, \Gamma_{n+1}) \in R_{\odot}^{\prime\pm\sigma}$, *i.e.* $(\Gamma_{\bar{\tau}-(1)}, \dots, \Gamma_{\bar{\tau}-(n)}, \Gamma_{\bar{\tau}-(n+1)}) \in R_{\odot}^{\prime\pm\bar{\tau}\circ\sigma}$, *i.e.* for all $\varphi_1, \dots, \varphi_n \in \mathcal{L}_{\mathcal{C}}$, if $\varphi_1 \not\vdash'_1 \Gamma_{\bar{\tau}-(1)}$ and \dots and $\varphi_n \not\vdash'_n \Gamma_{\bar{\tau}-(n)}$ then $\odot'(\varphi_1, \dots, \varphi_n) \in \Gamma_{\bar{\tau}-(n+1)}$. So, assume that $\varphi_1 \not\vdash'_1 \Gamma_{\bar{\tau}-(1)}$ and \dots and $\varphi_n \not\vdash'_n \Gamma_{\bar{\tau}-(n)}$. We want to show that $\odot'(\varphi_1, \dots, \varphi_n) \in \Gamma_{\bar{\tau}-(n+1)}$. By Axiom A5, we have that

$$\models S \left(\odot, \pm_{\bar{\tau}-(1)} \varphi_{\bar{\tau}-(1)}, \dots, \pm^{n+1} \odot'(\varphi_1, \dots, \varphi_n), \dots, \pm_{\bar{\tau}-(n+1)} \varphi_{\bar{\tau}-(n+1)} \right).$$

That is,

$$\models \pm^{\bar{\tau}(n+1)} \varphi_{\bar{\tau}(n+1)} \rightarrow \odot \left(\pm^{\bar{\tau}(1)} \varphi_{\bar{\tau}(1)}, \dots, \pm^{n+1} \odot' (\varphi_1, \dots, \varphi_n), \dots, \pm^{\bar{\tau}(n)} \varphi_{\bar{\tau}(n)} \right)$$

where for all $j \in \llbracket 1; n+1 \rrbracket$, $\pm^j \triangleq \pm_1^j \pm_2^j \dots \pm_m^j$ and for all $i \in \llbracket 1; m \rrbracket$, $\pm_i^j \triangleq \begin{cases} - & \text{if } j = \tau_i \circ \tau_{i+1} \circ \dots \circ \tau_m(n+1) \\ + & \text{otherwise} \end{cases}$.

i. Assume that $\bar{\tau}(n+1) = n+1 (= \bar{\tau}^-(n+1))$.

Fact 2. *If $\bar{\tau}(n+1) = n+1$ and $\Delta^{n+1} = -^m$, then $\pm^{n+1} = -$.*

Proof. For all $i \in \llbracket 1; m \rrbracket$, $\pm_i^{n+1} = -\Delta_i^{n+1}$. So,

$$\begin{aligned} \pm^{n+1} &= \pm_1^{n+1} \pm_2^{n+1} \dots \pm_m^{n+1} \\ &= (-\Delta_1^{n+1})(-\Delta_2^{n+1}) \dots (-\Delta_m^{n+1}) \\ &= -^m \Delta_1^{n+1} \dots \Delta_m^{n+1} \\ &= -^m \Delta_0^{n+1} \Delta_1^{n+1} \dots \Delta_{m-1}^{n+1} [\Delta_m^{n+1} \Delta_0^{n+1}] \\ &= -^m \Delta^{n+1} \Delta_m^{n+1} \Delta_0^{n+1} \\ &= -^m -^m \Delta_m^{n+1} \Delta_0^{n+1} \text{ because } \Delta^{n+1} = -^m \pm_{\bar{\tau}(n+1)} = -^m \pm_{n+1} = -^m \\ &= \Delta_m^{n+1} \Delta_0^{n+1} \\ &= \Delta_m^{n+1} \text{ because } n+1 = \bar{\tau}^-(n+1) \\ &= - \text{ because } n+1 \neq \tau_m(n+1) \text{ by assumption.} \end{aligned}$$

□

So, since $\tau^-(n+1) = n+1$, we have that $\pm^{\bar{\tau}^-(n+1)} = -$. Hence, in that case, Axiom A5 rewrites

$$\models \neg \odot' (\varphi_1, \dots, \varphi_n) \rightarrow \odot \left(\pm^{\bar{\tau}(1)} \varphi_{\bar{\tau}(1)}, \dots, \pm^{\bar{\tau}(n)} \varphi_{\bar{\tau}(n)} \right) \quad (8)$$

Now, we have that $\varphi_1 \dashv'_1 \Gamma_{\bar{\tau}^-(1)}, \dots, \varphi_n \dashv'_n \Gamma_{\bar{\tau}^-(n)}$ by assumption. That is, $\varphi_{\bar{\tau}(1)} \dashv'_{\bar{\tau}(1)} \Gamma_1, \dots, \varphi_{\bar{\tau}(n)} \dashv'_{\bar{\tau}(n)} \Gamma_n$.

Fact 3. *For all $j \in \llbracket 1; n \rrbracket$, $-^m \Delta^j = \Delta_0^j \pm^j$.*

Proof. Indeed,

$$\begin{aligned} -^m \Delta^j &= -^m \left(\Delta_0^j \Delta_1^j \dots \Delta_m^j \right) \\ &= -^m \Delta_0^j \left(\Delta_1^j \dots \Delta_m^j \right) \\ &= -^m \Delta_0^j \left(-\pm_1^j \dots -\pm_m^j \right) \\ &= -^m \Delta_0^j \left(-^m \pm^j \right) \\ &= \Delta_0^j \pm^j \end{aligned}$$

□

So, since $\bar{\tau}(n+1) = n+1$, we have that $\Delta_0^j = -$ for all $j \in \llbracket 1; n \rrbracket$ and so $-^m \Delta^j = -\pm^j$ by Fact 3. So, $\varphi_{\bar{\tau}(1)} \in \Gamma_1$ iff $-\pm^{\bar{\tau}(1)} \pm_1 = +, \dots, \varphi_{\bar{\tau}(n)} \in \Gamma_n$ iff $-\pm^{\bar{\tau}(n)} \pm_n = +$ by the tonicities of Expression (7). That is, $\pm^{\bar{\tau}(1)} \varphi_{\bar{\tau}(1)} \in \Gamma_1$ iff $\pm_1 = -, \dots, \pm^{\bar{\tau}(n)} \varphi_{\bar{\tau}(n)} \in \Gamma_n$ iff $\pm_n = -$.

Now, assume that $\pm^{\bar{\tau}(j)} \varphi_{\bar{\tau}(j)} \uparrow_j \Gamma_j$ for some $j \in \llbracket 1; n \rrbracket$. Then, $\pm^{\bar{\tau}(j)} \varphi_{\bar{\tau}(j)} \in \Gamma_j$ iff $\pm_j = +$. But we also have by assumption that $\pm^{\bar{\tau}(j)} \varphi_{\bar{\tau}(j)} \in \Gamma_j$ iff $\pm_j = -$. So, we have that $\pm_j = +$ iff $\pm_j = -$, which is impossible. Thus, for all $j \in \llbracket 1; n \rrbracket$, we do not have that

$$\pm^{\bar{\tau}(j)} \varphi_{\bar{\tau}(j)} \uparrow_j \Gamma_j.$$

Likewise, by assumption, we do not have that

$$(\Gamma_1, \dots, \Gamma_n, \Gamma_{n+1}) \in R_{\odot}^{\pm\sigma}.$$

Thus, by the Truth lemma, we have that

$$\mathcal{M}^c, \Gamma_{n+1} \not\models \odot \left(\pm^{\bar{\tau}(1)} \varphi_{\bar{\tau}(1)}, \dots, \pm^{\bar{\tau}(n)} \varphi_{\bar{\tau}(n)} \right).$$

So, by (8) and using contraposition, we have that

$$\mathcal{M}^c, \Gamma_n \models \odot' (\varphi_1, \dots, \varphi_n).$$

Hence, $\odot' (\varphi_1, \dots, \varphi_n) \in \Gamma_{n+1}$, *i.e.* $\odot' (\varphi_1, \dots, \varphi_n) \in \Gamma_{\bar{\tau}(n+1)}$.

- ii. Assume that $\bar{\tau}(n+1) \neq n+1$. Let $j_0 = \bar{\tau}(n+1)$, which is different from $n+1$. Then, by assumption, $\varphi_{j_0} \uparrow_{j_0}' \Gamma_{\bar{\tau}-(j_0)}$, *i.e.* $\varphi_{\bar{\tau}(n+1)} \uparrow_{\bar{\tau}(n+1)}' \Gamma_{n+1}$. That is, $\varphi_{\bar{\tau}(n+1)} \in \Gamma_{n+1}$ iff $-^m \Delta^{j_0} \pm_{\bar{\tau}-(j_0)} = +$. So, $\pm^{\bar{\tau}(n+1)} \varphi_{\bar{\tau}(n+1)} \in \Gamma_{n+1}$ iff $\pm^{j_0} \Delta_0^{j_0} \pm_{\bar{\tau}-(j_0)} = +$ because $\pm^{j_0} = \pm^{\bar{\tau}(n+1)}$ iff $\pm^{j_0} \pm_{\bar{\tau}-(j_0)} = +$ because $\Delta_0^{j_0} = +$ iff $\pm_{n+1} = +$ by definition of $\Delta_0^{j_0}$ since $j_0 = \bar{\tau}(n+1)$. and the latter is the case by definition. So, we do have that

$$\mathcal{M}^c, \Gamma_{n+1} \models \odot \left(\pm^{\bar{\tau}(1)} \varphi_{\bar{\tau}(1)}, \dots, \pm^{n+1} \odot' (\varphi_1, \dots, \varphi_n), \dots, \pm^{\bar{\tau}(n)} \varphi_{\bar{\tau}(n)} \right).$$

That is, for all $\Gamma'_1, \dots, \Gamma'_n \in \mathcal{M}^c$, either $(\Gamma'_1, \dots, \Gamma'_n, \Gamma_{n+1}) \in R_{\odot}^{\pm\sigma}$ or $\pm^{\bar{\tau}(1)} \varphi_{\bar{\tau}(1)} \uparrow_1 \Gamma'_1$ or \dots or $\pm^{n+1} \odot' (\varphi_1, \dots, \varphi_n) \uparrow_{\bar{\tau}(n+1)} \Gamma_{\bar{\tau}(n+1)}$ or \dots or $\pm^{\bar{\tau}(n)} \varphi_{\bar{\tau}(n)} \uparrow_n \Gamma'_n$. Now, if we take $\Gamma'_1 \triangleq \Gamma_1, \dots, \Gamma'_n \triangleq \Gamma_n$ then, since $(\Gamma_1, \dots, \Gamma_n, \Gamma_{n+1}) \notin R_{\odot}^{\pm\sigma}$ by assumption, we must have that $\pm^{\bar{\tau}(1)} \varphi_{\bar{\tau}(1)} \uparrow_1 \Gamma_1$ or \dots or $\pm^{n+1} \odot' (\varphi_1, \dots, \varphi_n) \uparrow_{\bar{\tau}(n+1)} \Gamma_{\bar{\tau}(n+1)}$ or \dots or $\pm^{\bar{\tau}(n)} \varphi_{\bar{\tau}(n)} \uparrow_n \Gamma_n$. Assume that there is some $j \in \llbracket 1; n \rrbracket$ such that $\pm^{\bar{\tau}(j)} \varphi_{\bar{\tau}(j)} \uparrow_j \Gamma_j$. That is,

$$\pm^{\bar{\tau}(j)} \varphi_{\bar{\tau}(j)} \in \Gamma_j \text{ iff } \pm_j = + \tag{9}$$

Now, by assumption, we have that $\varphi_{\bar{\tau}(j)} \uparrow_j' \Gamma_j$. That is,

$$\varphi_{\bar{\tau}(j)} \in \Gamma_j \text{ iff } -^m \Delta^{\bar{\tau}(j)} \pm_j = +$$

So, $\pm^{\bar{\tau}(j)} \varphi_{\bar{\tau}(j)} \in \Gamma_j$ iff $\pm^{\bar{\tau}(j)} -^m \Delta^{\bar{\tau}(j)} \pm_j = +$. But, by Fact 3, $-^m \Delta^{\bar{\tau}(j)} = \Delta_0^{\bar{\tau}(j)} \pm^{\bar{\tau}(j)}$. So,

$$\begin{aligned} \pm^{\bar{\tau}(j)} -^m \Delta^{\bar{\tau}(j)} \pm_j &= \pm^{\bar{\tau}(j)} \Delta_0^{\bar{\tau}(j)} \pm^{\bar{\tau}(j)} \pm_j \\ &= \Delta_0^{\bar{\tau}(j)} \pm_j \\ &= -\pm_j \text{ because } j \neq n+1 \text{ by definition of } \Delta_0^{\bar{\tau}(j)}. \end{aligned}$$

Hence, $\pm^{\bar{\tau}(j)}\varphi_{\bar{\tau}(j)} \in \Gamma_j$ iff $\pm_j = -$. Combining it with (9), we obtain that $\pm_j = +$ iff $\pm_j = -$, which is impossible. Therefore, for all $j \in \llbracket 1; n \rrbracket$ such that $j \neq \bar{\tau}^-(n+1)$, we have that it is not the case that $\pm^{\bar{\tau}(j)}\varphi_{\bar{\tau}(j)} \not\in \Gamma_j$. Thus, it must hold that

$$\pm^{n+1} \odot' (\varphi_1, \dots, \varphi_n) \not\in \Gamma_{\bar{\tau}(n+1)}$$

i.e. $\pm^{n+1} \odot' (\varphi_1, \dots, \varphi_n) \in \Gamma_{\bar{\tau}(n+1)}$ iff $\pm_{\bar{\tau}(n+1)} = +$
i.e.

$$\odot' (\varphi_1, \dots, \varphi_n) \in \Gamma_{n+1} \text{ iff } \pm^{n+1} \pm_{\bar{\tau}(n+1)} = +. \quad (10)$$

But by assumption, $\Delta^0 \Delta^{n+1} \pm_{\bar{\tau}^-(n+1)} = -$, *i.e.* $\Delta^{n+1} \pm_{\bar{\tau}^-(n+1)} = -^m$. Now,

$$\begin{aligned} \Delta^{n+1} &= \Delta_0^{n+1} \Delta_1^{n+1} \dots \Delta_{m-1}^{n+1} \\ &= \Delta_0^{n+1} (\Delta_1^{n+1} \dots \Delta_{m-1}^{n+1} \Delta_m^{n+1}) \Delta_m^{n+1} \\ &= \Delta_0^{n+1} (-\pm_1^{n+1}) \dots (-\pm_m^{n+1}) \Delta_m^{n+1} \\ &= \Delta_0^{n+1} -^m \pm^{n+1} \Delta_m^{n+1} \\ &= -^{m+1} \pm^{n+1} \Delta_m^{n+1} \text{ because } \Delta_0^{n+1} = - \text{ since } \bar{\tau}(n+1) \neq n+1 \\ &= -^{m+1} \pm^{n+1} - \text{ because } \Delta_m^{n+1} = - \text{ since } \bar{\tau}(n+1) \neq n+1 \\ &= -^m \pm^{n+1}. \end{aligned}$$

Hence, $\Delta^{n+1} \pm_{\bar{\tau}^-(n+1)} = -^m$
iff $-^m \pm^{n+1} \pm_{\bar{\tau}^-(n+1)} = -^m$
iff $\pm^{n+1} \pm_{\bar{\tau}^-(n+1)} = +$.

Thus, we obtain from (10) that $\odot' (\varphi_1, \dots, \varphi_n) \in \Gamma_{\bar{\tau}(n+1)}$.

(b) Assume that $\Delta^0 \Delta^{n+1} \pm_{\bar{\tau}^-(n+1)} = +$, *i.e.* $\Delta^{n+1} \pm_{\bar{\tau}^-(n+1)} = -^{m+1}$. Then,

$$\odot' = \left(\bar{\tau}^- \bar{k}, \forall, -^{m+1} \Delta^1 \pm_{\bar{\tau}^-(1)}, \dots, -^{m+1} \Delta^n \pm_{\bar{\tau}^-(n)}, \pm, \bar{\tau} \circ \sigma \right).$$

Assume that $(\Gamma_1, \dots, \Gamma_n, \Gamma_{n+1}) \notin R_{\odot}^{\pm\sigma}$, we will show that $(\Gamma_1, \dots, \Gamma_n, \Gamma_{n+1}) \notin R_{\odot}^{\pm\sigma}$,

i.e. $(\Gamma_{\bar{\tau}^-(1)}, \dots, \Gamma_{\bar{\tau}^-(n)}, \Gamma_{\bar{\tau}^-(n+1)}) \notin R_{\odot}^{\pm\bar{\tau}\circ\sigma}$

i.e. for all $\varphi_1, \dots, \varphi_n \in \mathcal{L}_{\mathcal{C}}$, if $\odot' (\varphi_1, \dots, \varphi_n) \in \Gamma_{\bar{\tau}^-(n+1)}$ then $\varphi_1 \not\in \Gamma_{\bar{\tau}^-(1)}$ or ... or $\varphi_n \not\in \Gamma_{\bar{\tau}^-(n)}$

where for all $j \in \llbracket 1; n \rrbracket$,

$$\varphi_j \not\in \Gamma_j \triangleq \begin{cases} \varphi_j \in \Gamma_j & \text{if } \pm_j' \triangleq -^{m+1} \Delta^j \pm_{\bar{\tau}^-(j)} = + \\ \varphi_j \notin \Gamma_j & \text{if } \pm_j' \triangleq -^{m+1} \Delta^j \pm_{\bar{\tau}^-(j)} = - \end{cases}.$$

So, assume that $\odot' (\varphi_1, \dots, \varphi_n) \in \Gamma_{\bar{\tau}^-(n+1)}$ and assume towards a contradiction that for all $j \in \llbracket 1; n \rrbracket$, we do *not* have that $\varphi_j \not\in \Gamma_{\bar{\tau}^-(j)}$. By Axiom A5, we have that

$$\models S \left(\odot, \pm^{\bar{\tau}(1)} \varphi_{\bar{\tau}(1)}, \dots, \pm^{n+1} \odot' (\varphi_1, \dots, \varphi_n), \dots, \pm^{\bar{\tau}(n+1)} \varphi_{\bar{\tau}(n+1)} \right)$$

where we recall that $\bar{\tau} = \tau_0 \circ \dots \circ \tau_m$. That is,

$$\models \pm^{\bar{\tau}(n+1)} \varphi_{\bar{\tau}(n+1)} \rightarrow \odot \left(\pm^{\bar{\tau}(1)} \varphi_{\bar{\tau}(1)}, \dots, \pm^{n+1} \odot' (\varphi_1, \dots, \varphi_n), \dots, \pm^{\bar{\tau}(n)} \varphi_{\bar{\tau}(n)} \right).$$

i. Assume that $\bar{\tau}(n+1) = n+1 (= \bar{\tau}^-(n+1))$.

Fact 4. If $\bar{\tau}(n+1) = n+1$ and $\Delta^{n+1} = -^{m+1}$ then $\pm^{m+1} = +$.

Proof. Note that for all $i \in \llbracket 1; m \rrbracket$, $\pm_i^{n+1} = -\Delta_i^{n+1}$. So,

$$\begin{aligned}
\pm^{n+1} &= \pm_1^{n+1} \pm_2^{n+1} \dots \pm_m^{n+1} \\
&= (-\Delta_1^{n+1}) (-\Delta_2^{n+1}) \dots (-\Delta_m^{n+1}) \\
&= (-^m) \Delta_1^{n+1} \dots \Delta_m^{n+1} \\
&= (-^m) \Delta_0^{n+1} \Delta_1^{n+1} \dots \Delta_{m-1}^{n+1} [\Delta_m^{n+1} \Delta_0^{n+1}] \\
&= (-^m) \Delta^{n+1} \Delta_m^{n+1} \Delta_0^{n+1} \\
&= (-^m) (-^{m+1}) \Delta_m^{n+1} \Delta_0^{n+1} \\
&\text{because } \Delta^{n+1} = -^{m+1} \pm_{\bar{\tau}^-(n+1)} = -^{m+1} \pm_{n+1} = -^{m+1} \\
&= -\Delta_m^{n+1} \Delta_0^{n+1} \\
&= -(\Delta_m^{n+1}) \text{ because } n+1 = \bar{\tau}^-(n+1) \\
&= - - \text{ because } n+1 \neq \tau_m(n+1) \text{ by assumption} \\
&= +\square
\end{aligned}$$

So, since $\bar{\tau}(n+1) = n+1$, we have that $\pm^{\bar{\tau}(n+1)} = +$. Then, in that case, Axiom A5 rewrites

$$\models \odot'(\varphi_1, \dots, \varphi_n) \rightarrow \odot(\pm^{\bar{\tau}(1)} \varphi_{\bar{\tau}(1)}, \dots, \pm^{\bar{\tau}(n)} \varphi_{\bar{\tau}(n)}).$$

Now, $\mathcal{M}^c, \Gamma_{n+1} \models \odot'(\varphi_1, \dots, \varphi_n)$. So, $\mathcal{M}^c, \Gamma_{n+1} \models \odot(\pm^{\bar{\tau}(1)} \varphi_{\bar{\tau}(1)}, \dots, \pm^{\bar{\tau}(n)} \varphi_{\bar{\tau}(n)})$. That is, for all $\Gamma'_1, \Gamma'_n \in \mathcal{M}^c$, either $(\Gamma'_1, \dots, \Gamma'_n, \Gamma_{n+1}) \in R_{\odot}^{\pm\sigma}$ or $\pm^{\bar{\tau}(1)} \varphi_{\bar{\tau}(1)} \uparrow_1 \Gamma'_1$ or ... or $\pm^{\bar{\tau}(n)} \varphi_{\bar{\tau}(n)} \uparrow_n \Gamma'_n$. Now, take $\Gamma'_1 \triangleq \Gamma_1, \dots, \Gamma'_n \triangleq \Gamma_n$. Then, because by assumption $(\Gamma_1, \dots, \Gamma_n, \Gamma_{n+1}) \notin R_{\odot}^{\pm\sigma}$, we should have that

$$\pm^{\bar{\tau}(1)} \varphi_{\bar{\tau}(1)} \uparrow_1 \Gamma_1 \text{ or } \dots \text{ or } \pm^{\bar{\tau}(n)} \varphi_{\bar{\tau}(n)} \uparrow_n \Gamma_n \quad (11)$$

But by assumption, for all $j \in \llbracket 1; n \rrbracket$, we also do *not* have that $\varphi_j \uparrow_j \Gamma_{\bar{\tau}^-(j)}$. That is, we do *not* have that

$$\varphi_j \in \Gamma_{\bar{\tau}^-(j)} \text{ iff } -^{m+1} \Delta^j \pm_{\bar{\tau}^-(j)} = +.$$

But by Fact 3, $-^m \Delta^j = \Delta_0^j \pm^j = -\pm^j$ because $n+1 = \bar{\tau}(n+1)$ and so $\Delta_0^j = -$. So, $-^{m+1} \Delta^j = \pm^j$. Hence, we do *not* have that

$$\pm^j \varphi_j \in \Gamma_{\bar{\tau}^-(j)} \text{ iff } \pm_{\bar{\tau}^-(j)} = +.$$

So, for all $j \in \llbracket 1; n \rrbracket$, we do *not* have that

$$\pm^{\bar{\tau}(j)} \varphi_{\bar{\tau}(j)} \in \Gamma_j \text{ iff } \pm_j = +.$$

That is, we do *not* have that $\pm^{\bar{\tau}(j)} \varphi_{\bar{\tau}(j)} \uparrow_j \Gamma_j$. This contradicts (11).

- ii. Assume that $\bar{\tau}(n+1) \neq n+1$. Let $j_0 = \bar{\tau}(n+1)$, which is different from $n+1$. Then, by assumption, we do *not* have that

$$\begin{aligned}
\varphi_{j_0} \in \Gamma_{\bar{\tau}^-(j_0)} &\text{ iff } -^{m+1} \Delta^{j_0} \pm_{\bar{\tau}^-(j_0)} = + \\
&\text{ iff } -^{m+1} \Delta^{j_0} \pm_{n+1} = + \\
&\text{ iff } -^{m+1} \Delta^{j_0} = + \text{ because } \pm_{n+1} = +
\end{aligned}$$

But by Fact 3, $-^m \Delta^{j_0} = \Delta_0^{j_0} \pm^{j_0} = \pm^{j_0}$ because $j_0 = \bar{\tau}(n+1)$ by definition. So, we do *not* have that

$$\varphi_{j_0} \in \Gamma_{\bar{\tau}^-(j_0)} \text{ iff } \pm^{j_0} = -.$$

So, not $\pm^{j_0} \varphi_{j_0} \notin \Gamma_{\bar{\tau}^-(j_0)}$

i.e. $\pm^{\bar{\tau}(n+1)} \varphi_{\bar{\tau}(n+1)} \in \Gamma_{n+1}$

i.e. $\mathcal{M}^c, \Gamma_{n+1} \models \pm^{\bar{\tau}(n+1)} \varphi_{\bar{\tau}(n+1)}$.

So, by Axiom A5, we obtain that

$$\mathcal{M}^c, \Gamma_{n+1} \models \odot \left(\pm^{\bar{\tau}(1)} \varphi_{\bar{\tau}(1)}, \dots, \pm^{n+1} \odot' (\varphi_1, \dots, \varphi_n), \dots, \pm^{\bar{\tau}(n)} \varphi_{\bar{\tau}(n)} \right).$$

That is, for all $\Gamma'_1, \dots, \Gamma'_n \in \mathcal{M}^c$, either $(\Gamma'_1, \dots, \Gamma'_n, \Gamma_{n+1}) \in R_{\odot}^{\pm\sigma}$ or $\pm^{\bar{\tau}(1)} \varphi_{\bar{\tau}(1)} \not\vdash_1 \Gamma'_1$ or ... or $\pm^{n+1} \odot' (\varphi_1, \dots, \varphi_n) \not\vdash_{\bar{\tau}^-(n+1)} \Gamma'_{\bar{\tau}^-(n+1)}$ or ... or $\pm^{\bar{\tau}(n)} \varphi_{\bar{\tau}(n)} \not\vdash_n \Gamma'_n$.

Now, take $\Gamma'_1 \triangleq \Gamma_1, \dots, \Gamma'_n \triangleq \Gamma_n$. By assumption, we do *not* have that $\varphi_1 \not\vdash_1 \Gamma'_{\bar{\tau}^-(1)}$ nor ... nor $\varphi_n \not\vdash_n \Gamma'_{\bar{\tau}^-(n)}$.

That is, for all $j \in \llbracket 1; n \rrbracket - \{j_0\}$, we do not have that

$$\varphi_j \in \Gamma_{\bar{\tau}^-(j)} \text{ iff } -^{m+1} \Delta^j \pm_{\bar{\tau}^-(j)} = +.$$

But by Fact 3, $-^m \Delta^j = \Delta_0^j \pm^j = -\pm^j$ (because $j \neq j_0 = \bar{\tau}(n+1)$ and so $\Delta_0^j = -$).

So, $-^{m+1} \Delta^j = \pm^j$. Thus, we do not have that

$$\pm^j \varphi_j \in \Gamma_{\bar{\tau}^-(j)} \text{ iff } \pm_{\bar{\tau}^-(j)} = +$$

and so for all $j \in \llbracket 1; n \rrbracket - \{\bar{\tau}(n+1)\}$. So, for all $j \in \llbracket 1; n \rrbracket - \{\bar{\tau}(n+1)\}$, we do not have that

$$\pm^{\bar{\tau}(j)} \varphi_{\bar{\tau}(j)} \in \Gamma_j \text{ iff } \pm_j = +.$$

i.e. not $\pm^{\bar{\tau}(j)} \varphi_{\bar{\tau}(j)} \not\vdash_j \Gamma_j$. Therefore, we must have that

$$\pm^{n+1} \odot' (\varphi_1, \dots, \varphi_n) \not\vdash_{\bar{\tau}^-(n+1)} \Gamma_{\bar{\tau}^-(n+1)},$$

i.e. $\pm^{n+1} \odot' (\varphi_1, \dots, \varphi_n) \in \Gamma_{\bar{\tau}^-(n+1)}$ iff $\pm_{\bar{\tau}^-(n+1)} = +$

i.e. $\odot' (\varphi_1, \dots, \varphi_n) \in \Gamma_{\bar{\tau}^-(n+1)}$ iff $\pm^{n+1} \pm_{\bar{\tau}^-(n+1)} = +$.

But

$$\begin{aligned} \pm^{n+1} &= \pm_1^{n+1} \dots \pm_m^{n+1} \text{ by definition} \\ &= (-\Delta_1^{n+1}) \dots (-\Delta_m^{n+1}) \text{ because for all } i \in \llbracket 1; m \rrbracket, \pm_i^{n+1} = -\Delta_i^{n+1} \\ &= -^m \Delta_1^{n+1} \dots \Delta_m^{n+1} \\ &= -^m [\Delta_0^{n+1} \dots \Delta_{m-1}^{n+1}] \Delta_m^{n+1} \Delta_0^{n+1} \\ &= -^m \Delta^{n+1} \Delta_m^{n+1} \Delta_0^{n+1} \\ &= -^m \Delta^{n+1} - \Delta_0^{n+1} \text{ because } \Delta_m^{n+1} = - \text{ since } n+1 \neq \bar{\tau}(n+1) \\ &= -^m \Delta^{n+1} -- \text{ because } \Delta_0^{n+1} = - \text{ since } n+1 \neq \bar{\tau}(n+1) \\ &= -^m \Delta^{n+1} \end{aligned}$$

So,

$$\begin{aligned} \pm^{n+1} \pm_{\bar{\tau}^-(n+1)} &= [-^m \Delta^{n+1}] \pm_{\bar{\tau}^-(n+1)} \\ &= -^m [\Delta^{n+1} \pm_{\bar{\tau}^-(n+1)}] \\ &= -^m -^{m+1} \text{ by assumption (b)} \\ &= - \end{aligned}$$

Thus, $\odot' (\varphi_1, \dots, \varphi_n) \notin \Gamma_{\bar{\tau}(n+1)}$, which contradicts our assumption.

2. Assume that $\bullet = (\bar{k}, \forall, \pm_1, \dots, \pm_n, \pm, \sigma)$ and that $\bullet' = \tau_0 \dots \tau_m \bullet$ where $\tau_0, \dots, \tau_m \in \mathfrak{S}_{n+1}$ are such that $\tau_i(n+1) \neq n+1$ for all $i \in \llbracket 0; m \rrbracket$. Let $\bar{\tau} \triangleq \tau_0 \circ \dots \circ \tau_m$.

(a) Assume that $\Delta^0 \Delta^{n+1} \pm_{\bar{\tau}^-(n+1)} = -$, *i.e.* $\Delta^{n+1} \pm_{\bar{\tau}^-(n+1)} = -^m$. Then,

$$\bullet' = (\bar{\tau}\bar{k}, \forall, -^m \Delta^1 \pm_{\bar{\tau}(1)}, \dots, -^m \Delta^n \pm_{\bar{\tau}(n)}, -\pm, \bar{\tau} \circ \sigma).$$

Assume that $(\Gamma_1, \dots, \Gamma_n, \Gamma_{n+1}) \in R_{\odot}^{\pm\sigma}$, we will show that $(\Gamma_1, \dots, \Gamma_n, \Gamma_{n+1}) \in R_{\odot}^{\pm\sigma}$, *i.e.* $(\Gamma_1, \dots, \Gamma_n, \Gamma_{n+1}) \notin R_{\odot}^{\prime\pm\sigma}$

i.e. $(\Gamma_{\bar{\tau}^-(1)}, \dots, \Gamma_{\bar{\tau}^-(n)}, \Gamma_{\bar{\tau}^-(n+1)}) \notin R_{\odot}^{\prime\pm\bar{\tau}\circ\sigma}$

i.e. for all $\varphi_1, \dots, \varphi_n \in \mathcal{L}_{\mathcal{C}}$, if $\odot'(\varphi_1, \dots, \varphi_n) \in \Gamma_{\bar{\tau}^-(n+1)}$ then $\varphi_1 \uparrow'_1 \Gamma_{\bar{\tau}^-(1)}$ or ... or $\varphi_n \uparrow'_n \Gamma_{\bar{\tau}^-(n)}$.

So, assume that $\odot'(\varphi_1, \dots, \varphi_n) \in \Gamma_{\bar{\tau}^-(n+1)}$. By Axiom A5, we have that

$$\models S \left(\odot, \pm^{\bar{\tau}(1)} \varphi_{\bar{\tau}(1)}, \dots, \pm^{n+1} \odot'(\varphi_1, \dots, \varphi_n), \dots, \pm^{\bar{\tau}(n+1)} \varphi_{\bar{\tau}(n+1)} \right)$$

where $\bar{\tau} = \tau_0 \circ \dots \circ \tau_m$. That is,

$$\begin{aligned} & \models \odot \left(\pm^{\bar{\tau}(1)} \varphi_{\bar{\tau}(1)}, \dots, \pm^{n+1} \odot(\varphi_1, \dots, \varphi_n), \dots, \pm^{\bar{\tau}(n)} \varphi_{\bar{\tau}(n)} \right) \\ & \rightarrow \pm^{\bar{\tau}(n+1)} \varphi_{\bar{\tau}(n+1)} \end{aligned} \quad (12)$$

i. Assume that $\bar{\tau}(n+1) = n+1 (= \bar{\tau}^-(n+1))$. By Fact 2, we have that $\pm^{\bar{\tau}^-(n+1)} = \pm^{n+1} = -$. So, we have that

$$\models \odot'(\varphi_1, \dots, \varphi_n) \rightarrow \neg \odot \left(\pm^{\bar{\tau}(1)} \varphi_{\bar{\tau}(1)}, \dots, \pm^{\bar{\tau}(n)} \varphi_{\bar{\tau}(n)} \right).$$

So, in particular,

$$\mathcal{M}^c, \Gamma_{n+1} \models \odot'(\varphi_1, \dots, \varphi_n) \rightarrow \neg \odot \left(\pm^{\bar{\tau}(1)} \varphi_{\bar{\tau}(1)}, \dots, \pm^{\bar{\tau}(n)} \varphi_{\bar{\tau}(n)} \right).$$

Therefore, since $n+1 = \bar{\tau}^-(n+1)$, we have $\Gamma_{\bar{\tau}^-(n+1)} = \Gamma_{n+1}$ and thus $\odot'(\varphi_1, \dots, \varphi_n) \in \Gamma_{n+1}$. So, we also have that

$$\mathcal{M}^c, \Gamma_{n+1} \models \neg \odot \left(\pm^{\bar{\tau}(1)} \varphi_{\bar{\tau}(1)}, \dots, \pm^{\bar{\tau}(n)} \varphi_{\bar{\tau}(n)} \right)$$

by Fact 3, since $\bar{\tau}(n+1) = n+1$, we have that $\Delta_0^j = -$ for all $j \in \llbracket 1; n \rrbracket$, and thus

$$-^m \Delta^j = -\pm^j \quad (13)$$

So, since $(\Gamma_1, \dots, \Gamma_n, \Gamma_{n+1}) \in R_{\odot}^{\pm\sigma}$, we must have that there is $j \in \llbracket 1; n \rrbracket$ such that $-\pm^{\bar{\tau}(j)} \varphi_{\bar{\tau}(j)} \uparrow_j \Gamma_j$, *i.e.*

$$-\pm^{\bar{\tau}(j)} \varphi_{\bar{\tau}(j)} \in \Gamma_j \text{ iff } \pm_j = +$$

i.e. there is $j \in \llbracket 1; n \rrbracket$ such that

$$-\pm^j \varphi_j \in \Gamma_{\bar{\tau}^-(j)} \text{ iff } \pm_{\bar{\tau}^-(j)} = +$$

because $\bar{\tau}(n+1) = n+1$, so $j \in \llbracket 1; n \rrbracket$ iff $\bar{\tau}^-(j) \in \llbracket 1; n \rrbracket$. That is, there is $j \in \llbracket 1; n \rrbracket$ such that

$$\begin{aligned} \varphi_j \in \Gamma_{\bar{\tau}^-(j)} &\text{ iff } -\pm^j \pm_{\bar{\tau}^-(j)} = + \\ \text{i.e. } \varphi_j \in \Gamma_{\bar{\tau}^-(j)} &\text{ iff } -^m \Delta^j \pm_{\bar{\tau}^-(j)} = + \end{aligned}$$

by Expression (13). So, there is $j \in \llbracket 1; n \rrbracket$ such that $\varphi_j \not\uparrow'_j \Gamma_{\bar{\tau}^-(j)}$, which is what we wanted to prove.

ii. Assume that $\bar{\tau}(n+1) \neq n+1$.

Assume towards a contradiction that we have neither that $\varphi_1 \not\uparrow'_1 \Gamma_{\bar{\tau}^-(1)}$ nor ... nor $\varphi_n \not\uparrow'_n \Gamma_{\bar{\tau}^-(n)}$. In particular, since $j_0 \triangleq \bar{\tau}(n+1) \neq n+1$, we do not have that $\varphi_{j_0} \not\uparrow'_{j_0} \Gamma_{\bar{\tau}^-(j_0)}$. Now,

$$\begin{aligned} -^m \Delta^{j_0} \pm_{\bar{\tau}^-(j_0)} &= -^m \Delta^{j_0} \pm_{n+1} \\ &= -^m \Delta^{j_0} \\ &= -^m \Delta_0^{j_0} \Delta_1^{j_0} \dots \Delta_m^{j_0} \\ &= -^m \Delta_0^{j_0} \left(-\pm_1^{j_0}\right) \dots \left(-\pm_m^{j_0}\right) \\ &= -^m -^m \Delta_0^{j_0} \left(\pm_1^{j_0} \dots \pm_m^{j_0}\right) \\ &= +\pm^{j_0} \\ &= \pm^{j_0} \end{aligned}$$

So, we have $\left(\varphi_{j_0} \in \Gamma_{\bar{\tau}^-(j_0)} \text{ iff } \pm^{j_0} = +\right)$ iff $\varphi_{j_0} \not\uparrow'_{j_0} \Gamma_{\bar{\tau}^-(j_0)}$. Therefore, we have that $\pm^{j_0} \in \Gamma_{\bar{\tau}^-(j_0)}$ iff $\varphi_{j_0} \not\uparrow'_{j_0} \Gamma_{\bar{\tau}^-(j_0)}$. Hence, we have that $\pm^{j_0} \varphi_{j_0} \notin \Gamma_{\bar{\tau}^-(j_0)}$, *i.e.* $\mathcal{M}^c, \Gamma_{n+1} \not\models \pm^{\bar{\tau}(n+1)} \varphi_{\bar{\tau}(n+1)}$. Hence, by Axiom A5, we have that

$$\mathcal{M}^c, \Gamma_{n+1} \not\models \odot \left(\pm^{\bar{\tau}(1)} \varphi_{\bar{\tau}(1)}, \dots, \pm^{n+1} \odot' (\varphi_1, \dots, \varphi_n), \dots, \pm^{\bar{\tau}(n)} \varphi_{\bar{\tau}(n)} \right).$$

So, either $(\Gamma_1, \dots, \Gamma_n, \Gamma_{n+1}) \notin R_{\odot}^{\pm\sigma}$ or not $\pm^{\bar{\tau}(1)} \varphi_{\bar{\tau}(1)} \not\uparrow_1 \Gamma_1$ or ... or not $\pm^{n+1} \odot' (\varphi_1, \dots, \varphi_n) \not\uparrow_{\bar{\tau}^-(n+1)} \Gamma_{\bar{\tau}^-(n+1)}$ or ... or not $\pm^{\bar{\tau}(n)} \varphi_{\bar{\tau}(n)} \not\uparrow_n \Gamma_n$. But $(\Gamma_1, \dots, \Gamma_n, \Gamma_{n+1}) \in R_{\odot}^{\pm\sigma}$, so there is $j \in \llbracket 1; n \rrbracket - \{\bar{\tau}^-(n+1)\}$ such that

$$\text{not } \pm^{\bar{\tau}(j)} \varphi_{\bar{\tau}(j)} \not\uparrow_j \Gamma_j \tag{14}$$

But $\varphi_{\bar{\tau}(j)} \not\uparrow'_{\bar{\tau}(j)} \Gamma_j$ holds iff

$$\begin{aligned} \varphi_{\bar{\tau}(j)} \in \Gamma_j &\text{ iff } -^m \Delta^{\bar{\tau}(j)} \pm_j = + \text{ by definition} \\ \varphi_{\bar{\tau}(j)} \in \Gamma_j &\text{ iff } \Delta_0^{\bar{\tau}(j_0)} \pm^{\bar{\tau}(j)} \pm_j = + \text{ by Fact 3} \\ \varphi_{\bar{\tau}(j)} \in \Gamma_j &\text{ iff } -\pm^{\bar{\tau}(j)} \pm_j = + \text{ by definition} \\ \varphi_{\bar{\tau}(j)} \in \Gamma_j &\text{ iff } \pm_j = - \end{aligned}$$

But we do *not* have that $\varphi_{\bar{\tau}(j)} \not\uparrow'_{\bar{\tau}(j)} \Gamma_j$. So, either $\left(\pm^{\bar{\tau}(j)} \varphi_{\bar{\tau}(j)} \in \Gamma_j \text{ and } \pm_j = +\right)$ or $\left(\pm^{\bar{\tau}(j)} \varphi_{\bar{\tau}(j)} \notin \Gamma_j \text{ and } \pm_j = -\right)$ *i.e.* $\left(\pm^{\bar{\tau}(j)} \varphi_{\bar{\tau}(j)} \in \Gamma_j \text{ iff } \pm_j = +\right)$

i.e. $\pm^{\bar{\tau}(j)}\varphi_{\bar{\tau}(j)} \not\vdash_j \Gamma_j$ which contradicts (14). Therefore, we must have that not $\pm^{n+1} \odot'(\varphi_1, \dots, \varphi_n) \not\vdash_{\bar{\tau}^-(n+1)} \Gamma_{\bar{\tau}^-(n+1)}$. Now, we use the same reasoning as in the proof of (1)(a)(ii), and we must have that $\odot'(\varphi_1, \dots, \varphi_n) \notin \Gamma_{\bar{\tau}^-(n+1)}$, which is also impossible.

(b) Assume that $\Delta^0 \Delta^{n+1} \pm_{\bar{\tau}^-(n+1)} = +$, *i.e.* $\Delta^{n+1} \pm_{\bar{\tau}^-(n+1)} = -^{m+1}$. Then,

$$\odot' = \left(\bar{\tau}^- \bar{k}, \exists, -^{m+1} \Delta^1 \pm_{\bar{\tau}^-(1)}, \dots, -^{m+1} \Delta^n \pm_{\bar{\tau}^-(n)}, \bar{\tau} \circ \sigma \right).$$

Assume that $(\Gamma_1, \dots, \Gamma_n, \Gamma_{n+1}) \in R_{\odot}^{\pm\sigma}$. We are going to show that $(\Gamma_1, \dots, \Gamma_n, \Gamma_{n+1}) \in R_{\odot}^{\pm\sigma}$

i.e. $(\Gamma_{\bar{\tau}^-(1)}, \dots, \Gamma_{\bar{\tau}^-(n)}, \Gamma_{\bar{\tau}^-(n+1)}) \in R_{\odot}^{\pm\bar{\tau}\circ\sigma}$

i.e. for all $\varphi_1, \dots, \varphi_n \in \mathcal{L}_{\mathcal{C}}$, if $\varphi_1 \not\vdash'_1 \Gamma_{\bar{\tau}^-(1)}$ and ... and $\varphi_n \not\vdash'_n \Gamma_{\bar{\tau}^-(n)}$, then $\odot'(\varphi_1, \dots, \varphi_n) \in \Gamma_{\bar{\tau}^-(n+1)}$.

So, assume that $\varphi_1 \not\vdash'_1 \Gamma_{\bar{\tau}^-(1)}$ and ... and $\varphi_n \not\vdash'_n \Gamma_{\bar{\tau}^-(n)}$. We want to show that $\odot'(\varphi_1, \dots, \varphi_n) \in \Gamma_{\bar{\tau}^-(n+1)}$. By Axiom A5, we have that

$$\begin{aligned} & \models \odot \left(\pm^{\bar{\tau}(1)}\varphi_{\bar{\tau}(1)}, \dots, \pm^{n+1} \odot'(\varphi_1, \dots, \varphi_n), \dots, \pm^{\bar{\tau}(n)}\varphi_{\bar{\tau}(n)} \right) \\ & \rightarrow \pm^{\bar{\tau}(n+1)}\varphi_{\bar{\tau}(n+1)} \end{aligned} \quad (15)$$

i. Assume that $\bar{\tau}(n+1) = n+1 (= \bar{\tau}^-(n+1))$.

By Fact 4 and our assumptions, we have that $\pm^{\bar{\tau}^-(n+1)} = +$. So,

$$\models \odot \left(\pm^{\bar{\tau}(1)}\varphi_{\bar{\tau}(1)}, \dots, \pm^{\bar{\tau}(n)}\varphi_{\bar{\tau}(n)} \right) \rightarrow \odot'(\varphi_1, \dots, \varphi_n) \quad (16)$$

by Axiom A5. Like in the proof of (1)(b)(i), we can prove that we have $\varphi_1 \not\vdash'_1 \Gamma_{\bar{\tau}^-(1)}$ and ... and $\varphi_n \not\vdash'_n \Gamma_{\bar{\tau}^-(n)}$ iff $\pm^{\bar{\tau}(1)}\varphi_{\bar{\tau}(1)} \not\vdash_1 \Gamma_1$ and ... and $\pm^{\bar{\tau}(n)}\varphi_{\bar{\tau}(n)} \not\vdash_n \Gamma_n$. So, because we also have that $(\Gamma_1, \dots, \Gamma_n, \Gamma_{n+1}) \in R_{\odot}^{\pm\sigma}$, we have that $\mathcal{M}^c, \Gamma_{n+1} \models \odot \left(\pm^{\bar{\tau}(1)}\varphi_{\bar{\tau}(1)}, \dots, \pm^{\bar{\tau}(n)}\varphi_{\bar{\tau}(n)} \right)$. Hence, by (16), we obtain that $\odot'(\varphi_1, \dots, \varphi_n) \in \Gamma_{n+1} = \Gamma_{\bar{\tau}^-(n+1)}$.

ii. Assume that $\bar{\tau}^-(n+1) \neq n+1$.

Let $j_0 = \bar{\tau}(n+1)$, which is different from $n+1$. By assumption, we have that $\varphi_{j_0} \not\vdash' \Gamma_{\bar{\tau}(j_0)}$. Then, using the same reasoning as in the beginning of (1)(b)(ii), we can prove that $\mathcal{M}^c, \Gamma_{n+1} \not\models \pm^{\bar{\tau}(n+1)}\varphi_{\bar{\tau}(n+1)}$. So, by (15), we have that

$$\mathcal{M}^c, \Gamma_{n+1} \not\models \odot \left(\pm^{\bar{\tau}(1)}\varphi_{\bar{\tau}(1)}, \dots, \pm^{n+1} \odot'(\varphi_1, \dots, \varphi_n), \dots, \pm^{\bar{\tau}(n)}\varphi_{\bar{\tau}(n)} \right).$$

So, either $(\Gamma_1, \dots, \Gamma_n, \Gamma_{n+1}) \notin R_{\odot}^{\pm\sigma}$ or not $\pm^{\bar{\tau}(1)}\varphi_{\bar{\tau}(1)} \not\vdash_1 \Gamma_1$ or ... or not $\pm^{n+1} \odot'(\varphi_1, \dots, \varphi_n) \not\vdash_{\bar{\tau}^-(n+1)} \Gamma_{\bar{\tau}^-(n+1)}$ or ... or not $\pm^{\bar{\tau}(n)}\varphi_{\bar{\tau}(n)}$. By assumption, for all $j \in \llbracket 1; n \rrbracket - \{\bar{\tau}^-(n+1)\}$ we have that $\varphi_j \not\vdash' \Gamma_{\bar{\tau}^-(j)}$. Then, using the same reasoning as in (1)(b)(ii), we can prove that we have for all $j \in \llbracket 1; n \rrbracket - \{\bar{\tau}^-(n+1)\}$ that $\pm^{\bar{\tau}(j)}\varphi_{\bar{\tau}(j)} \not\vdash_j \Gamma_j$. Therefore, we must have that not $\pm^{n+1} \odot'(\varphi_1, \dots, \varphi_n) \not\vdash_{\bar{\tau}^-(n+1)} \Gamma_{\bar{\tau}^-(n+1)}$. Then, using the same reasoning as in the end of (1)(b)(ii), we must have that $\odot'(\varphi_1, \dots, \varphi_n) \in \Gamma_{\bar{\tau}^-(n+1)}$. \square

Lemma 8. *We recall that $\mathcal{M}^c = (W^c, \mathcal{R}^c)$ is the canonical model. There are a \mathcal{C} -model $M = (W, \mathcal{R})$ and bijection functions $f_k : W^k \rightarrow (W^c)^k$ for each dimension k of \mathcal{C} such that for all $\varphi \in \mathcal{L}_{\mathcal{C}}^k$ and all $(w_1, \dots, w_k) \in W^k$, we have that $M, (w_1, \dots, w_k) \Vdash \varphi$ iff $\mathcal{M}^c, f_k(w_1, \dots, w_k) \Vdash \varphi$.*

Proof. Let say the cardinality of the domain W^c of the canonical model \mathcal{M}^c is some infinite cardinal κ . Then, assuming the Axiom of Choice, for all $k \in \mathbb{N}^*$, the cardinality of $(W^c)^k$ is also κ . Therefore, there is a bijection between $(W^c)^k$ and W^c . Therefore, for each $k \in \mathbb{N}^*$, we can define a bijection f_k between the tuples of real numbers (w_1, \dots, w_k) of size k , that is \mathbb{R}^k , and the k - \mathcal{P}_C -consistent sets of the canonical model.

We define the C-model $M = (W, \mathcal{R})$ as follows. We first define $W \triangleq W^c$. For all dimensions k of C, a propositional letter $p \in C$ of dimension k holds in a tuple (w_1, \dots, w_k) of k states of W iff p belongs to $f_k(w_1, \dots, w_k)$ in W^c . For every $\odot \in C$ of skeleton $\bullet = (k_1, \dots, k_{n+1}, \mathbb{A}, \pm_1, \dots, \pm_n, \pm, \sigma)$, if R_\odot is the $k + k_1 + \dots + k_n$ -ary relation associated to \odot in M , then for all $w_1, \dots, w_{k+k_1+\dots+k_n} \in W$, we set $R_\odot^{\pm\sigma} w_1 \dots w_{k+k_1+\dots+k_n}$ iff $R_\odot^{\pm\sigma} f_{k_1}(w_1, \dots, w_{k_1}) \dots f_{k_n}(w_{k_1+\dots+k_{n-1}+1}, \dots, w_{k_1+\dots+k_{n-1}+k_n}) f_k(w_{k_1+\dots+k_n+1}, \dots, w_{k_1+\dots+k_n+k})$ (R_\odot is the relation of the canonical model associated to \odot). Then, by Lemma 7, M is a C-model and one can show by an easy induction on φ that for all $\varphi \in \mathcal{L}_C^k$ and all $(w_1, \dots, w_k) \in W^k$, we have that $M, (w_1, \dots, w_k) \Vdash \varphi$ iff $\mathcal{M}^c, f_k(w_1, \dots, w_k) \Vdash \varphi$. \square

Completeness proof. We prove that for all sets $\Gamma \subseteq \mathcal{L}_C^k$ and all $\varphi \in \mathcal{L}_C^k$, if $\Gamma \Vdash \varphi$ holds then φ is provable from Γ in \mathcal{P}_C . We reason by contraposition. Assume that φ is not provable from Γ in \mathcal{P}_C . That is, there is no proof of φ in \mathcal{P}_C from Γ . Hence, $\Gamma \cup \{\neg\varphi\}$ is k - \mathcal{P}_C -consistent. So, by Lemma 3, it can be extended into a maximal k - \mathcal{P}_C -consistent set Γ' such that $\{\neg\varphi\} \cup \Gamma \subseteq \Gamma'$. Now, Γ' is a state of the canonical model \mathcal{M}^c . Then, by the truth Lemma 6, we have that $(\mathcal{M}^c, \Gamma') \Vdash \Gamma \cup \{\neg\varphi\}$. Finally, by Lemma 8, we have that $(M, f_k^{-1}(\Gamma')) \Vdash \Gamma \cup \{\neg\varphi\}$, with $(M, f_k^{-1}(\Gamma'))$ a pointed C-model. Therefore, it is not the case that $\Gamma \Vdash \varphi$. \square

Proposition 4. *Let C be a set of atomic connectives. Then, for all $\odot, \odot' \in C$ of arity $n \in \mathbb{N}^*$ and all $\tau_0^1, \dots, \tau_{m_1}^1, \tau_0^2, \dots, \tau_{m_2}^2 \in \mathfrak{S}_{n+1}$ such that $\odot' = \tau_0^1 - \tau_1^1 \dots - \tau_{m_1}^1 \odot$ and $\odot' = \tau_0^2 - \tau_1^2 \dots - \tau_{m_2}^2 \odot$, we have for all $j \in \llbracket 1; n+1 \rrbracket$ that it holds that $\pm^{1,j} = \pm^{2,j}$, where $\pm^{1,j}$ is the \pm^j associated to $\tau_0^1, \dots, \tau_{m_1}^1$ and $\pm^{2,j}$ is the \pm^j associated to $\tau_0^2, \dots, \tau_{m_2}^2$ defined in Fig. 4.*

Proof. Let $((k_1, \dots, k_{n+1}, \mathbb{A}, \pm_1, \dots, \pm_n, \pm, \sigma), i)$ be an atomic connective. We have that $\odot = \tau_0^1 - \tau_1^1 - \dots - (\tau_{m_1}^1 \circ \tau_{m_2}^{2-}) - \tau_{m_2-1}^{2-} \dots - \tau_0^{2-} \odot$ because $\alpha_n * \beta_n$ is a group action. Moreover, by definition of $\alpha_n * \beta_n$, because the permutation of \odot stays the same through it, we also have necessarily that $\tau_0^1 \circ \tau_1^1 \circ \dots \circ \tau_{m_1}^1 \circ \tau_{m_2}^{2-} \circ \tau_{m_2-1}^{2-} \circ \dots \circ \tau_0^{2-} = \text{Id} (*)$. Thus, by soundness of Axiom A5, we have that for all $\varphi_1, \dots, \varphi_n \in \mathcal{L}_C$, $S(\odot, \pm^1 \varphi_1, \dots, \pm^n \varphi_n, \pm^{n+1} \odot(\varphi_1, \dots, \varphi_n))$. That is, if $\mathbb{A}(\odot) = \exists$, $\odot(\pm^1 \varphi_1, \dots, \pm^n \varphi_n) \rightarrow \pm^{n+1} \odot(\varphi_1, \dots, \varphi_n)$ is valid and, if $\mathbb{A}(\odot) = \forall$, $\pm^{n+1} \odot(\varphi_1, \dots, \varphi_n) \rightarrow \odot(\pm^1 \varphi_1, \dots, \pm^n \varphi_n)$ is valid. We are first going to prove that for all $j \in \llbracket 1; n+1 \rrbracket$, $\pm^j = +$. We will then obtain the result by showing that for all $j \in \llbracket 1; n+1 \rrbracket$, $\pm^{1,j} \pm^{2,j} = \pm^j$.

We prove the first part. Assume first that $\mathbb{A}(\odot) = \exists$ and assume towards a contradiction that $\pm^{n+1} = -$. Then, $\odot(\pm^1 \varphi_1, \dots, \pm^n \varphi_n) \rightarrow \neg \odot(\varphi_1, \dots, \varphi_n)$ (**) is valid. Assume first that $\pm^1 = \dots = \pm^n = +$. Then, $\odot(\varphi_1, \dots, \varphi_n)$ is valid, for all $\varphi_1, \dots, \varphi_n \in \mathcal{L}_C$. This is impossible (it suffices to take a C-model \mathcal{M} where there exists $(\bar{w}_1, \dots, \bar{w}_{n+1}) \in R_\odot^{\pm\sigma} \neq \emptyset$ and some appropriate $\varphi_1, \dots, \varphi_n \in \mathcal{L}_C$ true (or false) at $\bar{w}_1, \dots, \bar{w}_n$). Now, assume that there is $i \in \llbracket 1; n \rrbracket$ such that $\pm^i = -$. W.l.o.g. and for better readability we can assume that this i is unique (the proof easily extends to the multiple case). Then, we can define a C-model \mathcal{M} where $(\bar{w}_1, \dots, \bar{w}_i, \dots, \bar{w}_{n+1}) \in R_\odot^{\pm\sigma}$ and $(\bar{w}_1, \dots, \bar{w}'_i, \dots, \bar{w}_{n+1}) \in R_\odot^{\pm\sigma}$ are such that $(\mathcal{M}, \bar{w}_i) \Vdash p$ but not $(\mathcal{M}, \bar{w}'_i) \Vdash p$. Then, there are $\varphi_1, \dots, \varphi_i, \dots, \varphi_n \in \mathcal{L}_C$ such that $(\mathcal{M}, \bar{w}_{n+1}) \Vdash \odot(\varphi_1, \dots, \varphi_i, \dots, \varphi_n)$ and $(\mathcal{M}, \bar{w}_{n+1}) \Vdash \odot(\varphi_1, \dots, \neg\varphi_i, \dots, \varphi_n)$: just take $\varphi_i = p$ and appropriate formulas for $\varphi_j, j \neq i$. This contradicts the validity of (**). Hence, necessarily, $\pm^{n+1} = +$. So, $\odot(\pm^1 \varphi_1, \dots, \pm^n \varphi_n) \rightarrow \odot(\varphi_1, \dots, \varphi_n)$ (***) is valid, for all $\varphi_1, \dots, \varphi_n \in \mathcal{L}_C$. Now, assume towards a contradiction that there is $i \in \llbracket 1; n \rrbracket$ such that $\pm^i = -$. Assume first that $\pm_i(\odot) = +$ and take a formula $\varphi_i \in \mathcal{L}_C$ such

that $\neg\varphi_i$ is valid. Then, $\neg\odot(\varphi_1, \dots, \varphi_i, \dots, \varphi_n)$ is valid by Rule R1. However, we can define a pointed C-model $(\mathcal{M}, \bar{w}_{n+1})$ such that $(\mathcal{M}, \bar{w}_{n+1}) \Vdash \odot(\pm^1\varphi_1, \dots, \neg\varphi_i, \dots, \pm^n\varphi_n)$ for some appropriately chosen φ_j for $j \neq i$ true (or false) at some $(\bar{w}_1, \dots, \bar{w}_{n+1}) \in R_{\odot}^{\pm\sigma}$. Therefore, $(***)$ cannot be valid. Assume now that $\pm_i(\odot) = -$ and take a formula $\varphi_i \in \mathcal{L}_C$ such that φ_i is valid. Then, $\neg\odot(\varphi_1, \dots, \varphi_i, \dots, \varphi_n)$ is valid by soundness of Rule R2. However, like in the previous subcase, we can define a pointed C-model (\mathcal{M}, \bar{w}) and some $\varphi_j \in \mathcal{L}_C$ for $j \neq i$ such that $(\mathcal{M}, \bar{w}) \Vdash \odot(\pm^1\varphi_1, \dots, \neg\varphi_i, \dots, \pm^n\varphi_n)$. Therefore, $(***)$ cannot be valid. So, we have proven that for all $j \in \llbracket 1; n \rrbracket$, $\pm^j = +$ in case $\mathcal{A}(\odot) = \exists$. Dually, we can prove that for all $j \in \llbracket 1; n \rrbracket$, $\pm^j = +$ in case $\mathcal{A}(\odot) = \forall$, using the soundness of Rules R3 and R4. Hence, we have proven that for all $j \in \llbracket 1; n+1 \rrbracket$, $\pm^j = +$.

Now, we prove the second part, namely that for all $j \in \llbracket 1; n+1 \rrbracket$ we have that $\pm^{1,j}\pm^{2,j} = \pm^j$. Because of $(*)$, we have that $\tau_0^1 \circ \tau_1^1 \circ \dots \circ \tau_{m_1}^1 = \tau_0^2 \circ \tau_1^2 \circ \dots \circ \tau_{m_2}^2$ and we denote this permutation $\bar{\tau}$. Let $j \in \llbracket 1; n+1 \rrbracket$. By definition (see Fig. 4), $\pm^j = \pm_1^j \pm_2^j \dots \pm_{m_1}^j \pm_{m_1+1}^j \dots \pm_{m_1+m_2}^j$. Now, for all $i \in \llbracket 1; m_1 \rrbracket$,

$$\begin{aligned} \pm_i^j &\triangleq \begin{cases} - & \text{if } \tau_i^1 \circ \tau_{i+1}^1 \circ \dots \circ \tau_{m_1}^1 \circ \tau_{m_2}^{2-} \circ \tau_{m_2-1}^{2-} \circ \dots \circ \tau_0^{2-}(j) = n+1 \\ + & \text{otherwise} \end{cases} \\ &= \begin{cases} - & \text{if } \tau_i^1 \circ \tau_{i+1}^1 \circ \dots \circ \tau_{m_1}^1(\bar{\tau}^-(j)) = n+1 \\ + & \text{otherwise} \end{cases} \\ &= \pm_i^{1, \bar{\tau}^-(j)} \end{aligned}$$

Likewise, for all $i \in \llbracket m_1+1; m_1+m_2 \rrbracket$,

$$\begin{aligned} \pm_i^j &\triangleq \begin{cases} - & \text{if } \tau_{m_1+m_2-i}^{2-} \circ \tau_{m_1+m_2-(i-1)}^{2-} \circ \dots \circ \tau_0^{2-}(j) = n+1 \\ + & \text{otherwise} \end{cases} \\ &= \begin{cases} - & \text{if } \tau_{m_1+m_2-i+1}^2 \circ \dots \circ \tau_{m_2-1}^2 \circ \tau_{m_2}^2(\bar{\tau}^-(j)) = n+1 \\ + & \text{otherwise} \end{cases} \\ &= \pm_{m_1+m_2+1-i}^{2, \bar{\tau}^-(j)} \end{aligned}$$

Therefore, for all $j \in \llbracket 1; n+1 \rrbracket$,

$$\begin{aligned} + = \pm^j &= \pm_1^j \pm_2^j \dots \pm_{m_1}^j \pm_{m_1+1}^j \dots \pm_{m_1+m_2}^j \\ &= \left(\pm_1^{1, \bar{\tau}^-(j)} \pm_2^{1, \bar{\tau}^-(j)} \dots \pm_{m_1}^{1, \bar{\tau}^-(j)} \right) \left(\pm_{m_2}^{2, \bar{\tau}^-(j)} \pm_{m_2-1}^{2, \bar{\tau}^-(j)} \dots \pm_1^{2, \bar{\tau}^-(j)} \right) \\ &= \pm^{1, \bar{\tau}^-(j)} \pm^{2, \bar{\tau}^-(j)} \end{aligned}$$

So, for all $j \in \llbracket 1; n+1 \rrbracket$, $\pm^{1, \bar{\tau}^-(j)} = \pm^{2, \bar{\tau}^-(j)}$. That is, for all $j \in \llbracket 1; n+1 \rrbracket$, $\pm^{1,j} = \pm^{2,j}$ (because $\bar{\tau}^-$ is a permutation of \mathfrak{S}_{n+1}). \square

Corollary 1. *Every basic atomic logic is compact.*

Proof. It follows from the soundness and *strong* completeness of \mathcal{P}_C w.r.t. basic atomic logics. We prove the contraposition of the compactness property. Let C be an arbitrary set of atomic connectives and let S be a set of formulas of \mathcal{L}_C which is unsatisfiable. So, we have that $S \models \perp$ (we extend the language and consider instead the Boolean completion of C). Then, by strong completeness, we have that $S \vdash \perp$ is provable in \mathcal{P}_C . Therefore, there is a *finite* subset $S_0 \subseteq S$ such that $S_0 \vdash \perp$ is provable in \mathcal{P}_C . So, by soundness of \mathcal{P}_C , there is a finite subset $S_0 \subseteq S$ such that S_0 is unsatisfiable. \square

D Proofs of Theorems 4, 5 and Corollary 2

Theorem 4. *Let C, C^+ be common Boolean sets of atomic connectives such that $C \subseteq C^+$ and such that C^+ is displayable enough. The calculus $\mathcal{P}_{C, C^+}^{\text{BOOL}}$ of Fig. 6 is sound and strongly complete for the basic atomic logic based on C . Moreover, it is a proper display calculus and enjoys cut elimination.*

Proof. The soundness is proven without particular difficulty and follows the same lines as the soundness proof in [2]. As for completeness, it suffices to prove that all the axioms and inference rules of \mathcal{P}_C of Fig. 4 are derivable in $\mathcal{P}_{C, C^+}^{\text{BOOL}}$ of Fig. 6. Again, this is proven without particular difficulty. First, one should observe that $\varphi \vdash \psi$ is provable in $\mathcal{P}_{C, C^+}^{\text{BOOL}}$ iff $\vdash \varphi \rightarrow \psi$ is also provable in $\mathcal{P}_{C, C^+}^{\text{BOOL}} (*)$. Then, we can prove all the axioms and inference rules of \mathcal{P}_C in $\mathcal{P}_{C, C^+}^{\text{BOOL}}$. We start with the axioms and we first prove Axiom A1. Let $\rho \triangleq \tau_0 - \dots - \tau_m \in \mathfrak{S}_{n+1} * \mathbb{Z}/2\mathbb{Z}$ be such that $\tau_0 \circ \dots \circ \tau_m(j) = n+1$ (it exists because C^+ is displayable enough). W.l.o.g. we assume that $\pm \bar{\tau}^{(n+1)} = +$ (if it was equal to $-$, we would use the rules of (dr2) to display φ_j as the sole antecedent below). Then,

$$\frac{\frac{\frac{\varphi_1 \vdash \varphi_1 \quad \dots \quad \varphi_n \vdash \varphi_n}{[\odot](\varphi_1, \dots, \varphi_j, \dots, \varphi_n) \vdash \odot(\varphi_1, \dots, \varphi_j, \dots, \varphi_n)} (\vdash \odot)}{[\odot](\varphi_1, \dots, \varphi_j, \dots, \varphi_n), * \odot(\varphi_1, \dots, \varphi'_j, \dots, \varphi_n) \vdash \odot(\varphi_1, \dots, \varphi_j, \dots, \varphi_n)} (\text{K} \vdash)}{\frac{[\odot](\varphi_1, \dots, \varphi_j, \dots, \varphi_n) \vdash \odot(\varphi_1, \dots, \varphi_j, \dots, \varphi_n), \odot(\varphi_1, \dots, \varphi'_j, \dots, \varphi_n)}{[\odot](\varphi_1, \dots, \varphi_j, \dots, \varphi_n) \vdash \odot(\varphi_1, \dots, \varphi_j, \dots, \varphi_n) \vee \odot(\varphi_1, \dots, \varphi'_j, \dots, \varphi_n)} (\vdash \vee)}{\varphi_j \vdash [\rho \odot](\pm \bar{\tau}^{(1)} \varphi_1, \dots, \pm^{n+1} \odot(\varphi_1, \dots, \varphi_j, \dots, \varphi_n) \vee \odot(\varphi_1, \dots, \varphi'_j, \dots, \varphi_n), \dots, \pm \bar{\tau}^{(n)} \varphi_n)} (\text{dr3})$$

Likewise, we can prove in $\mathcal{P}_{C, C^+}^{\text{BOOL}}$ that

$$\varphi'_j \vdash [\rho \odot](\pm \bar{\tau}^{(1)} \varphi_1, \dots, \pm^{n+1} \odot(\varphi_1, \dots, \varphi_j, \dots, \varphi_n) \vee \odot(\varphi_1, \dots, \varphi'_j, \dots, \varphi_n), \dots, \pm \bar{\tau}^{(n)} \varphi_n)$$

Then, we obtain from the contraction rule (WI \vdash) and rule ($\vee \vdash$) that

$$\frac{\varphi_j \vee \varphi'_j \vdash [\rho \odot](\pm \bar{\tau}^{(1)} \varphi_1, \dots, \pm^{n+1} \odot(\varphi_1, \dots, \varphi_j, \dots, \varphi_n) \vee \odot(\varphi_1, \dots, \varphi'_j, \dots, \varphi_n), \dots, \pm \bar{\tau}^{(n)} \varphi_n)}{\frac{[\odot](\varphi_1, \dots, \varphi_j \vee \varphi'_j, \dots, \varphi_n) \vdash \odot(\varphi_1, \dots, \varphi_j, \dots, \varphi_n) \vee \odot(\varphi_1, \dots, \varphi'_j, \dots, \varphi_n)}{\odot(\varphi_1, \dots, \varphi_j \vee \varphi'_j, \dots, \varphi_n) \vdash \odot(\varphi_1, \dots, \varphi_j, \dots, \varphi_n) \vee \odot(\varphi_1, \dots, \varphi'_j, \dots, \varphi_n)} (\odot \vdash)}{\vdash \odot(\varphi_1, \dots, \varphi_j \vee \varphi'_j, \dots, \varphi_n) \rightarrow \odot(\varphi_1, \dots, \varphi_j, \dots, \varphi_n) \vee \odot(\varphi_1, \dots, \varphi'_j, \dots, \varphi_n)} (*)} (\text{dr3})$$

Axioms A2-A4 are proven similarly. Axiom A5 is proven by instantiating Rule (dr3) with $X_1 = \varphi_1, \dots, X_n = \varphi_n$ and $X_{n+1} = \odot'(\varphi_1, \dots, \varphi_n)$.

Now, we prove that the rules of \mathcal{P}_C are derivable in $\mathcal{P}_{C, C^+}^{\text{BOOL}}$. Rule R3 is a direct application of Rule ($\vdash \odot$) with premises $\varphi_1 \vdash \varphi_1, \dots, \vdash \varphi_j, \dots, \varphi_n \vdash \varphi_n$ and conclusion $\vdash [\odot](\varphi_1, \dots, \varphi_j, \dots, \varphi_n)$ (recall the convention about empty structures for Rule ($\vdash \odot$)), which entails by ($\odot \vdash$) that $\vdash \odot(\varphi_1, \dots, \varphi_j, \dots, \varphi_n)$. As for Rule R4, if we assume that $\vdash \neg \varphi$ is provable in $\mathcal{P}_{C, C^+}^{\text{BOOL}}$, then one can easily prove that by induction on the length of the proof that $\vdash * \varphi$ is also provable in $\mathcal{P}_{C, C^+}^{\text{BOOL}}$ (using as induction hypothesis “for all structures U , we have that $U \vdash * \varphi$ iff $U \vdash \neg \varphi$ ”). So, by Rule (dr2), we obtain that $\varphi_j \vdash$ is provable in $\mathcal{P}_{C, C^+}^{\text{BOOL}}$. Then, applying Rule ($\vdash \odot$), we obtain that $[\odot](\varphi_1, \dots, \varphi_j, \dots, \varphi_n) \vdash$ is provable in $\mathcal{P}_{C, C^+}^{\text{BOOL}}$ and therefore also $\vdash \neg \odot(\varphi_1, \dots, \varphi_j, \dots, \varphi_n)$. Rules R1 and R2 are proven similarly. Finally, Rules R5 and R6 follow straightforwardly from Rule ($\vdash \odot$) and (*) above. \square

Theorem 5. *Let (C, C^+) be a pair of common sets of atomic connectives such that C is without Boolean connectives, $C \subseteq C^+$ and C^+ is displayable enough. The calculus \mathcal{P}_{C, C^+} is sound and strongly complete for the basic atomic logic based on C .*

Proof. It follows the same method as the proof of [2, Theorem 53] (except that rule (dr1) has to be replaced by rule (dr3) here). We proceed by showing how the introduction and structural rules for \neg, \wedge, \vee and \rightarrow can be eliminated from any proof in \mathcal{P}_{C, C^+} and be replaced by the Rules (dr2') and the third block of rules of (dr2). However, there is a slight difference w.r.t. [2, Theorem 53] because our structural connective $*$ in the present article is defined differently from the way it was defined in [2]. This has an impact in Stages A and B of the corresponding proof. In Stage A, we can end up with consecutions of the form $[\odot_1](X_1, \dots, X_m) \vdash * \dots * \odot_2(\psi_1, \dots, \psi_n)$ or $* \dots * \odot_2(\psi_1, \dots, \psi_n) \vdash [\odot_1](X_1, \dots, X_m)$ with none or several structural negations $*$. Here, we only consider like in [2, Theorem 53] the case $[\odot_1](X_1, \dots, X_m) \vdash * \dots * \odot_2(\psi_1, \dots, \psi_n)$ (the case $* \dots * \odot_2(\psi_1, \dots, \psi_n) \vdash [\odot_1](X_1, \dots, X_m)$ is proven similarly using the third block of rules of (dr2)). We assume moreover that there is an even number of structural negations $*$ in the consequent (the case when this number is an odd number is proven similarly). The consecutions that we can obtain at the end of Stage B can now be of the form:

1. $\vdash \odot_2(\psi_1, \dots, \psi_n)$
2. $[\odot_1](X_1, \dots, X_m) \vdash$
3. $[\odot_1](X_1, \dots, X_m) \vdash * \dots * \odot_2(\psi_1, \dots, \psi_n)$.

where there is an even number of $*$ in the third consecution. This last consecution can be obtained from $[\odot_1](X_1, \dots, X_m) \vdash \odot_2(\psi_1, \dots, \psi_n)$ in \mathcal{P}_{C, C^+} using repeatedly the rule (dr2) (third block). \square

Corollary 2. *Let C, C^+ be common sets of atomic connectives such that $C \subseteq C^+$. Then, the logics $(\mathcal{S}_{C^+}, \mathcal{C}_{C^+}, \Vdash)$ and $(\mathcal{L}_{C^+}, \mathcal{C}_{C^+}, \Vdash)$ are conservative extensions of $(\mathcal{S}_C, \mathcal{C}_C, \Vdash)$ and $(\mathcal{L}_C, \mathcal{C}_C, \Vdash)$ respectively.*

Proof. Let C'' be a common and displayable enough set of atomic connectives such that $C \subseteq C^+ \subseteq C''$. Then, because $\text{GGL}_{C, C'}^{\mathcal{D}}$ enjoys cut elimination by Theorem 4 and has the subformula property, every proof in $\text{GGL}_{C', C'}^{\mathcal{D}}$ of a formula of \mathcal{L}_C or a sequent of \mathcal{S}_C involving only atomic connectives of C can be translated into a proof of the same formula or sequent in $\text{GGL}_{C, C'}^{\mathcal{D}}$. So, if a formula of \mathcal{L}_C or a sequent of \mathcal{S}_C is valid in $(\mathcal{S}_{C^+}, \mathcal{C}_{C^+}, \Vdash)$ or $(\mathcal{L}_{C^+}, \mathcal{C}_{C^+}, \Vdash)$ respectively, then, using the completeness of $\text{GGL}_{C', C'}^{\mathcal{D}}$ w.r.t. $(\mathcal{S}_{C^+}, \mathcal{C}_{C^+}, \Vdash)$ and $(\mathcal{L}_{C^+}, \mathcal{C}_{C^+}, \Vdash)$ (as proven in Theorem 4), it is provable in $\text{GGL}_{C', C'}^{\mathcal{D}}$ and thus also in $\text{GGL}_{C, C'}^{\mathcal{D}}$. Hence, by soundness of $\text{GGL}_{C, C'}^{\mathcal{D}}$, it is also valid in $(\mathcal{S}_C, \mathcal{C}_C, \Vdash)$ or $(\mathcal{L}_C, \mathcal{C}_C, \Vdash)$. \square

E Proof of Theorem 6

Theorem 6. *Let C be a purely displayable set of atomic connectives without Boolean connectives. The sequent calculus $\mathcal{P}_{C, C}$ of Fig. 8 is sound and strongly complete for the basic atomic logic based on C .*

Proof. The sequent calculus $\mathcal{P}_{C, C}$ of Fig. 8 is obtained from the display calculus \mathcal{P}_{C, C^+} by taking $C^+ = C$ after performing several simplifications. First, because C is purely displayable, this entails that Rule (dr3) boils down to Rule (dr3'). Second, since $C = C^+$, this entails that we can confuse atomic connectives with their structural copies. This implies in turn that Rule $(\odot \vdash)$ is

no longer needed. Third, since no structural negation $*$ appears *via* the Rule (dr3'), this implies that the Rules (dr2') are also no longer needed in any proof (see the proof of Corollary 5 for details). Hence, we obtain the sequent calculus $\mathcal{P}_{C,C}$. \square

F Case study: modal logic and the axiom K

In this section, we are going to prove that the classical Axiom K from modal logic ($\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$) is derivable in the calculus \mathcal{P}_C for modal logic of Fig. 5.

Lemma 9 (Replacement lemma). *Let C be a set of atomic connectives. If $\psi \in \mathcal{L}_C$ is a subformula of $\varphi \in \mathcal{L}_C$ and φ' is the result of replacing 0 or more occurrences of ψ in φ by a formula $\chi \in \mathcal{L}_C$, then $\psi \leftrightarrow \chi$ is provable in \mathcal{P}_C implies that $\varphi \leftrightarrow \varphi'$ is provable in \mathcal{P}_C .*

Proof. By induction on the number of connectives of φ . The proof is similar to the proof of [34, Proposition 2.9]. \square

Lemma 10. *Let C be a set of atomic connectives. The following rule called Generalized Modus Ponens is admissible in \mathcal{P}_C : from $\varphi \rightarrow \psi$ and $\psi \rightarrow \chi$, infer $\varphi \rightarrow \chi$.*

Proof. It follows from the deduction theorem for propositional logic [34, Proposition 1.9]. We have the following derivation, which uses only rules from propositional logic:

1. $\varphi \rightarrow \psi$ Hyp (abbreviation for "Hypothesis")
2. $\psi \rightarrow \chi$ Hyp
3. φ Hyp
4. ψ 1,3, MP
5. χ 2,4, MP

Thus, $\varphi \rightarrow \psi, \psi \rightarrow \chi, \varphi \vdash \chi$. So, by the deduction theorem for propositional logic, from $\varphi \rightarrow \psi$ and $\psi \rightarrow \chi$ we can infer $\varphi \rightarrow \chi$ in \mathcal{P}_C . \square

In the rest of this section, C is the set of atomic connectives of modal logic of Example 3. We prove another lemma in \mathcal{P}_C , namely $\Box\chi \leftrightarrow \neg\Diamond\neg\chi$:

1. $\neg\neg\chi \rightarrow \chi$ \mathcal{P}_{PL}
2. $\Box\neg\neg\chi \rightarrow \Box\chi$ 1, Rule R5
3. $\neg\Diamond\neg\chi \rightarrow \Box\neg\neg\chi$ Axiom A5
4. $\neg\Diamond\neg\chi \rightarrow \Box\chi$ 2, 3, Generalized MP
5. $\Diamond\neg\chi \rightarrow \neg\Box\chi$ Axiom A5
6. $(\Diamond\neg\chi \rightarrow \neg\Box\chi) \rightarrow (\neg\neg\Box\chi \rightarrow \neg\Diamond\neg\chi)$ \mathcal{P}_{PL}
7. $\neg\neg\Box\chi \rightarrow \neg\Diamond\neg\chi$ 5, 6, MP
8. $\Box\chi \rightarrow \neg\neg\Box\chi$ \mathcal{P}_{PL}
9. $\Box\chi \rightarrow \neg\Diamond\neg\chi$ 7, 8, Generalized MP
10. $(\Box\chi \rightarrow \neg\Diamond\neg\chi) \wedge (\neg\Diamond\neg\chi \rightarrow \Box\chi)$ 4, 9, \mathcal{P}_{PL}
11. $\Box\chi \leftrightarrow \neg\Diamond\neg\chi$ 10, Rewriting.

Now, we prove Axiom K in \mathcal{P}_C :

- | | |
|---|---|
| <ol style="list-style-type: none"> 1. $\neg\psi \rightarrow (\neg(\varphi \rightarrow \psi) \vee \neg\varphi)$ 2. $\diamond\neg\psi \rightarrow \diamond(\neg(\varphi \rightarrow \psi) \vee \neg\varphi)$ 3. $\diamond(\neg(\varphi \rightarrow \psi) \vee \neg\varphi) \rightarrow \diamond(\varphi \vee \neg\psi) \vee \diamond\neg\varphi$ 4. $\diamond\neg\psi \rightarrow \diamond\neg(\varphi \rightarrow \psi) \vee \diamond\neg\varphi$ 5. $[\diamond\neg\psi \rightarrow \diamond\neg(\varphi \rightarrow \psi) \vee \diamond\neg\varphi] \rightarrow$
 $[\neg(\diamond\neg(\varphi \rightarrow \psi) \vee \diamond\neg\varphi) \rightarrow \neg\diamond\neg\psi]$ 6. $\neg(\diamond\neg(\varphi \rightarrow \psi) \vee \diamond\neg\varphi) \rightarrow \neg\diamond\neg\psi$ 7. $[\neg\diamond\neg(\varphi \rightarrow \psi) \wedge \neg\diamond\neg\varphi] \rightarrow$
 $[\neg(\diamond\neg(\varphi \rightarrow \psi) \vee \diamond\neg\varphi)]$ 8. $\neg\diamond\neg(\varphi \rightarrow \psi) \wedge \neg\diamond\neg\varphi \rightarrow \neg\diamond\neg\psi$ 9. $\Box\chi \leftrightarrow \neg\diamond\neg\chi$ 10. $\Box(\varphi \rightarrow \psi) \wedge \Box\varphi \rightarrow \Box\psi$ 11. $(\Box(\varphi \rightarrow \psi) \wedge \Box\varphi \rightarrow \Box\psi) \rightarrow$
 $(\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi))$ 12. $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$ | <p>theorem of \mathcal{P}_{PL}
 1, Rule R5
 Axiom A1
 2, 3, Generalized MP</p> <p>theorem of \mathcal{P}_{PL}
 4, 5, MP
 theorem of \mathcal{P}_{PL}</p> <p>6, 7, Generalized MP
 Lemma
 8, 9, Lemma 9
 theorem of \mathcal{P}_{PL}</p> <p>10, 11, MP</p> |
|---|---|