Mathematical analysis of static and dynamic three-dimensional models of thermo-electro-magneto-elastic solids

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Abstract
This paper is devoted to the investigation of three-dimensional models of thermo-electro-magneto-elastic solids made of a multidomain inhomogeneous anisotropic material. General boundary and initial boundary value problems corresponding to the static and dynamic models are studied where, on certain parts of the boundary, mechanical displacement, electric and magnetic potentials and temperature vanish and, on the corresponding remaining parts of the boundary, the mechanical stress vector and components of the electric displacement, magnetic induction and heat flux along the outward normal vector of the boundary are given. Variational formulations of the boundary and initial boundary value problems are obtained and, applying them, existence and uniqueness results and the continuous dependence of solutions on given data, in suitable factor spaces of Sobolev spaces or spaces of vector-valued distributions, are proved.

Keywords
Thermo-electro-magneto-elastic solids, boundary and initial boundary value problems, existence and uniqueness of solution, variational methods, Sobolev spaces, vector-valued distributions

1. Introduction

Modern complex engineering constructions and technological processes are controlled by using sensors and actuators, which gather information and facilitate the adequate adjustment of constructions or processes. The need to construct sensors and actuators with the appropriate physical characteristics stimulates the analysis of interactions between various physical fields, such as elastic, thermal, electric and magnetic. Piezoelectric materials are the most popular materials currently being used in smart structures, owing to their direct and converse piezoelectric effects, which permit them to be utilized as both actuators and sensors. A wide area of their application is aerospace engineering, where most structures operate in changing thermal environments. Therefore, the investigation of multifield problems in continuum mechanics is important from a practical as

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well as a theoretical point of view; hence, owing to their numerous applications, the study of the mechanics of thermo-electro-magneto-elastic materials has attracted increasing attention in recent years.

One of the first rigorous theoretical models of piezoelectricity, describing the interaction between elastic, electric and thermal properties of a thermoelastic body, was constructed by Voigt [1]. Later on, Cady [2] treated the physical properties of piezoelectric crystals as well as their practical applications. A three-dimensional model accounting for the coupling of elastic, electric and thermal fields was derived by Mindlin [3], who applied a variational principle. Nowacki [4, 5] developed uniqueness and reciprocity theorems for thermo-piezoelectricity. Dhaliwal and Wang [6] proved a uniqueness theorem for linear three-dimensional thermo-piezoelectricity without restriction on the coupling constant between temperature and electric field, and positive definiteness of the elasticity tensor, which were used in [5]. Li [7] considered the coupling effects between elastic, electric, magnetic and thermal fields, and generalized the uniqueness result obtained in [6] and the reciprocity theorem of Nowacki [4], which were further strengthened by Aouadi [8], and the results were proved without positive definiteness assumptions on the conductivity tensor used in [7]. The existence, uniqueness and continuous dependence on the given data of a solution of an initial boundary value problem with the mixed boundary conditions for mechanical displacement, mechanical stress, electric potential and electric displacement corresponding to the three-dimensional model of an anisotropic inhomogeneous piezoelectric material with quasi-static equations for the electric field were proved in Sobolev spaces by Akamatsu and Nakamura [9]. Well-posedness results in specific function spaces for the three-dimensional model of thermo-piezoelectricity with inhomogeneous material parameters in the cases of homogeneous pure Dirichlet- or Neumann-type boundary conditions given on the entire boundary were obtained by Mulholland *et al.* [10]. Applying the potential method and theory of pseudodifferential equations, Natroshvili [11] studied static and pseudo-oscillation problems with basic, mixed and crack-type boundary conditions for homogeneous anisotropic thermo-electro-magneto-elasticity. For various classical and nonclassical models of piezoelectric solids, problems of propagation of waves, methods of solutions of corresponding initial and initial boundary value problems, applications of the obtained results and related topics are considered by many researchers (see [8–21] and the references given therein).

To the authors' knowledge, three-dimensional initial boundary value problems with general mixed boundary conditions for mechanical displacement, temperature, electric and magnetic potentials corresponding to the linear dynamic model with quasi-static equations for electric and magnetic fields for a multidomain inhomogeneous anisotropic thermo-electro-magneto-elastic body have not yet been investigated in Sobolev spaces. Only the boundary value problem with general mixed boundary conditions corresponding to the static model has been studied [22].

In this paper, we investigate static and dynamic models with quasi-static equations for electric and magnetic fields of a multidomain thermo-electro-magneto-elastic body, where each subdomain consists of an anisotropic inhomogeneous material, and obtain new existence, uniqueness and continuous dependence results in the corresponding function spaces. The dynamic model is studied by applying the variational approach and the method of successive approximations, which permits one to construct the algorithm for the numerical solution of the problem. One of the aims of the paper is to obtain a new well-posedness result in ordinary Sobolev spaces, which are widely used for the investigation of the initial boundary value problems corresponding to the dynamic models of the classical linear elasticity, without any additional structures of function spaces.

In Section 2, we consider the differential formulation of the initial boundary and boundary value problems corresponding to the linear dynamic and static three-dimensional models for a multidomain inhomogeneous anisotropic thermo-electro-magneto-elastic body with general mixed boundary conditions, where, on certain parts of the boundary, the surface force and components of the electric displacement, magnetic induction and heat flux along the outward normal vector are prescribed, and, on the remaining parts, the mechanical displacement, electric and magnetic potentials and temperature vanish. We obtain integral relations that require less regularity of the unknown functions than in the original problem and show that, in the space of twice continuously differentiable functions, the obtained integral relations are equivalent to the original differential equations.

In Section 3, on the basis of the integral relations obtained in Section 2, we consider the variational formulation in suitable Sobolev spaces of the boundary value problem corresponding to the static three-dimensional model of thermo-electro-magneto-elastic solids considered in Section 2. We define the structure of the set of solutions of the homogeneous boundary value problem and, applying it, we formulate results regarding the existence, uniqueness and continuous dependence of a solution of the three-dimensional boundary value problem on given data in suitable function spaces.

In Section 4, applying the integral relations obtained in Section 2, we obtain the variational formulation in suitable spaces of vector-valued distributions with respect to the time variable with values in Sobolev spaces of the initial boundary value problem corresponding to the dynamic three-dimensional model of thermo-electro-magneto-elastic solids considered in Section 2. We introduce suitable factor spaces and prove the existence and uniqueness theorem. The existence of a solution of the three-dimensional initial boundary value problem is proved by applying new a priori estimates and compactness arguments. Furthermore, an energy equality is obtained, which permits us to prove the uniqueness result and continuous dependence of a solution on given data in suitable function spaces. At the end of Section 4, we analyse the conditions of the theorem and obtain effective sufficient conditions for existence of a solution of the three-dimensional initial boundary value problem.

2. Formulation of the static and dynamic three-dimensional models

Throughout this paper, the indices i,j,p,q range in the set $\{1,2,3\}$ and the summation convention with respect to repeated indices is used; the indices or exponents $k,\overline{k}$ vary in the set $\{1,\ldots,K\}$, save when they are used for indexing sequences, where K is a natural number, and we do not use the summation convention with respect to $k,\overline{k}$. For any vectors $\boldsymbol{x},\boldsymbol{y}$ from the three-dimensional Euclidean space $\mathbb{R}^3$, let $\boldsymbol{x}\cdot\boldsymbol{y}$ and $\boldsymbol{x}\times\boldsymbol{y}$ denote the Euclidean inner product and the vector product of vectors $\boldsymbol{x},\boldsymbol{y}$ in the space $\mathbb{R}^3$, respectively, and we denote by $|\boldsymbol{x}|=\sqrt{\boldsymbol{x}\cdot\boldsymbol{x}}$ the Euclidean norm of $\boldsymbol{x}\in\mathbb{R}^3$. The Cartesian coordinates of a point $x\in\mathbb{R}^3$ are denoted x_i, and we let $\partial_i=\partial/\partial x_i$. The space of real 3×3 matrices we designate by $\mathbb{M}^3$ equipped with the matrix inner product $\boldsymbol{F}:\overline{\boldsymbol{F}}=F_{ij}\overline{F}_{ij}$, for all $\boldsymbol{F},\overline{\boldsymbol{F}}\in\mathbb{M}^3$. We denote by $\boldsymbol{H}*\boldsymbol{F}\in\mathbb{M}^3$, $\boldsymbol{R}*\boldsymbol{F}\in\mathbb{R}^3$ and $\boldsymbol{R}*\boldsymbol{x}\in\mathbb{M}^3$ the products of any fourth-order tensor $\boldsymbol{H}=(H_{ijpq})$, third-order tensor $\boldsymbol{R}=(R_{ipq})$, matrix $\boldsymbol{F}=(F_{pq})\in\mathbb{M}^3$ and vector $\boldsymbol{x}=(x_i)\in\mathbb{R}^3$, which are defined by $(\boldsymbol{H}*\boldsymbol{F})_{ij}=H_{ijpq}F_{pq}$, $(\boldsymbol{R}*\boldsymbol{F})_i=R_{ipq}F_{pq}$ and $(\boldsymbol{R}*\boldsymbol{x})_{pq}=R_{ipq}x_i$. Given a smooth enough vector field $\boldsymbol{v}=(v_i):\Omega\to\mathbb{R}^3$ and matrix field $\boldsymbol{\sigma}=(\sigma_{ij}):\Omega\to\mathbb{M}^3$, their divergences $\mathrm{div}\boldsymbol{v}:\Omega\to\mathbb{R}$ and $\mathbf{div}\,\boldsymbol{\sigma}=((\mathbf{div}\,\boldsymbol{\sigma})_i):\Omega\to\mathbb{R}^3$ are defined by $\mathrm{div}\boldsymbol{v}=\partial_i v_i$ and $(\mathbf{div}\,\boldsymbol{\sigma})_i=\partial_j\sigma_{ij}$, where $\Omega\subset\mathbb{R}^3$ is an open set. The gradients of a smooth enough scalar field $v:\Omega\to\mathbb{R}$ and vector field $\boldsymbol{v}=(v_i):\Omega\to\mathbb{R}^3$ are defined by $\mathbf{grad}\,v=((\mathbf{grad}\,v)_i):\Omega\to\mathbb{R}^3$, $(\mathbf{grad}\,v)_i=\partial_i v$ and $\nabla\boldsymbol{v}=((\nabla\boldsymbol{v})_{ij}):\Omega\to\mathbb{M}^3$, $(\nabla\boldsymbol{v})_{ij}=\partial_j v_i$. We denote by $\boldsymbol{n}=(n_i)$, $\boldsymbol{n}^k=(n_i^k)$ the unit outward normal vectors of the corresponding surfaces.

Let us consider a multidomain thermo-electro-magneto-elastic body with initial configuration $\overline{\Omega}=\cup_{k=1}^{K}\overline{\Omega_k}$, where each subdomain $\overline{\Omega_k}$ consists of a general inhomogeneous anisotropic material that is characterized by consistent spatially dependent parameters. The governing field equations of the dynamic linear three-dimensional model of the stress–strain state of each subdomain Ω_k of the thermo-electro-magneto-elastic body Ω in differential form with quasi-static equations for electric and magnetic fields, where the rate of the magnetic field is small, i.e., the electric field is curl free, and there is no electric current, i.e., the magnetic field is curl free, are of the following form [7, 11]:

$$\rho^k\frac{\partial^2\boldsymbol{u}^k}{\partial t^2}-\mathbf{div}\,\boldsymbol{\sigma}^k=\boldsymbol{f}^k \qquad \text{in } \Omega_k\times(0,T), \tag{1}$$

$$\mathrm{div}\boldsymbol{D}^k=f^{\varphi,k} \qquad \text{in } \Omega_k\times(0,T), \tag{2}$$

$$\mathrm{div}\boldsymbol{B}^k=0 \qquad \text{in } \Omega_k\times(0,T), \tag{3}$$

$$\varkappa^k\frac{\partial\theta^k}{\partial t}-\mathrm{div}\left(\boldsymbol{\eta}^k*\mathbf{grad}\;\theta^k\right)+\Theta_0\frac{\partial}{\partial t}\left(\boldsymbol{\lambda}^k:\boldsymbol{e}\left(\boldsymbol{u}^k\right)\right)-\Theta_0\frac{\partial}{\partial t}\left(\boldsymbol{\mu}^k\cdot\mathbf{grad}\,\varphi^k\right)-\Theta_0\frac{\partial}{\partial t}\left(\boldsymbol{m}^k\cdot\mathbf{grad}\,\psi^k\right)=f^{\theta,k}$$
$$\text{in } \Omega_k\times(0,T), \tag{4}$$

where $\boldsymbol{\sigma}^k=(\sigma_{ij}^k)$ is the mechanical stress tensor in the subdomain Ω_k, which is given by the following linear constitutive equation for a thermo-electro-magneto-elastic solid:

$$\boldsymbol{\sigma}^k=\boldsymbol{c}^k*\boldsymbol{e}\left(\boldsymbol{u}^k\right)+\boldsymbol{\varepsilon}^k*\mathbf{grad}\,\varphi^k+\boldsymbol{b}^k*\mathbf{grad}\,\psi^k-\boldsymbol{\lambda}^k\theta^k, \tag{5}$$

where $\boldsymbol{u}^k=(u_i^k):\overline{\Omega_k}\times[0,T]\to\mathbb{R}^3$ is the mechanical displacement vector-function; $\varphi^k:\overline{\Omega_k}\times[0,T]\to\mathbb{R}$ and $\psi^k:\overline{\Omega_k}\times[0,T]\to\mathbb{R}$ stand for the electric and magnetic potentials, such that electric and magnetic fields are $\boldsymbol{E}^k=-\mathbf{grad}\;\varphi^k$ and $\boldsymbol{H}^k=-\mathbf{grad}\;\psi^k$ and $\theta^k:\overline{\Omega_k}\times[0,T]\to\mathbb{R}$ is the temperature distribution. For smooth enough vector field, $\boldsymbol{v}=(v_i):\Omega\to\mathbb{R}^3$, $\boldsymbol{e}(\boldsymbol{v})=(e_{ij}(\boldsymbol{v}))$, $e_{ij}(\boldsymbol{v})=1/2\left(\partial_j v_i+\partial_i v_j\right)$ is the strain tensor; $\boldsymbol{c}^k=(c_{ijpq}^k(x))$ is the elasticity tensor; $\boldsymbol{\varepsilon}^k=(\varepsilon_{ipq}^k(x))$ and $\boldsymbol{b}^k=(b_{ipq}^k(x))$ are the piezoelectric and piezomagnetic

coefficients; $\boldsymbol{\lambda}^k = (\lambda_{ij}^k(x))$ is the stress–temperature tensor; ρ^k is the mass density in the reference configuration; $\boldsymbol{f}^k = (f_i^k) : \Omega_k \times (0,T) \to \mathbb{R}^3$ is the density of the applied body force; $\boldsymbol{D}^k = (D_i^k)$ is the electric displacement vector and $\boldsymbol{B}^k = (B_i^k)$ is the magnetic induction vector. The latter are given, respectively, by the following linear constitutive equations:

$$\boldsymbol{D}^k = \boldsymbol{\varepsilon}^k * \boldsymbol{e}\left(\boldsymbol{u}^k\right) - \boldsymbol{d}^k \mathbf{grad}\ \varphi^k - \boldsymbol{a}^k \mathbf{grad}\ \psi^k + \boldsymbol{\mu}^k \theta^k, \tag{6}$$

$$\boldsymbol{B}^k = \boldsymbol{b}^k * \boldsymbol{e}\left(\boldsymbol{u}^k\right) - \boldsymbol{a}^k \mathbf{grad}\ \varphi^k - \boldsymbol{\zeta}^k \mathbf{grad}\ \psi^k + \boldsymbol{m}^k \theta^k, \tag{7}$$

where $\boldsymbol{d}^k = (d_{ij}^k(x))$ and $\boldsymbol{\zeta}^k = (\zeta_{ij}^k(x))$ are the permittivity and permeability tensors; $\boldsymbol{a}^k = (a_{ij}^k(x))$ are the coupling coefficients connecting electric and magnetic fields; $\boldsymbol{\mu}^k = (\mu_i^k(x))$ and $\boldsymbol{m}^k = (m_i^k(x))$ are coefficients characterizing the relations between thermal and electric fields and between thermal and magnetic fields, respectively; $f^{\varphi,k} : \Omega_k \times (0,T) \to \mathbb{R}$ is the density of electric charges; $\varkappa^k$ is the thermal capacity; $\boldsymbol{\eta}^k = (\eta_{ij}^k(x))$ is the thermal conductivity tensor; $\Theta_0 > 0$ is the temperature of the thermo-electro-magneto-elastic body in the natural state of no deformation and electromagnetic fields, which is considered a reference temperature, and $f^{\theta,k} : \Omega_k \times (0,T) \to \mathbb{R}$ is the density of heat sources. Note that the mechanical displacement vector $\boldsymbol{u}$ of the entire thermo-electro-magneto-elastic body $\overline{\Omega}$ is equal to $\boldsymbol{u}^k$ on $\overline{\Omega_k}$, the electric potential φ of $\overline{\Omega}$ is equal to φ^k on $\overline{\Omega_k}$, the magnetic potential ψ of $\overline{\Omega}$ is equal to ψ^k on $\overline{\Omega_k}$ and the temperature θ of $\overline{\Omega}$ is equal to θ^k on $\overline{\Omega_k}$.

We consider classical boundary conditions on the boundary of the body Ω and on the interfaces between the subdomains Ω_k, which are widely used in applications. More precisely, we assume that the entire thermo-electro-magneto-elastic body Ω is clamped along a part $\Gamma_0 \subset \Gamma = \partial\Omega$ of the Lipschitz boundary Γ, and, on the remaining part $\Gamma_1 = \Gamma \backslash \Gamma_0$, an applied surface force vector, with density $\boldsymbol{g} = (g_i) : \Gamma_1 \times (0,T) \to \mathbb{R}^3$, is given, where $\Gamma = \Gamma_0 \cup \Gamma_1$ is a Lipschitz dissection [23] of Γ:

$$\boldsymbol{u}^k = \boldsymbol{0} \quad \text{on} \quad \Gamma_{0,k} = \Gamma_0 \cap \partial\Omega_k\,, \qquad \boldsymbol{\sigma}^k \boldsymbol{n}^k = \boldsymbol{g} \quad \text{on} \quad \Gamma_{1,k} = \Gamma_k \backslash \Gamma_{0,k}\,, \qquad \Gamma_k = \partial\Omega_k. \tag{8}$$

Along a part $\Gamma_0^\varphi \subset \Gamma$ of the boundary, the electric potential vanishes and, on the remaining part $\Gamma_1^\varphi = \Gamma \backslash \Gamma_0^\varphi$, the normal component of the electric displacement, with density $g^\varphi : \Gamma_1^\varphi \times (0,T) \to \mathbb{R}$, is given, where $\Gamma = \Gamma_0^\varphi \cup \Gamma_1^\varphi$ is a Lipschitz dissection of Γ:

$$\varphi^k = 0 \quad \text{on} \quad \Gamma_{0,k}^\varphi = \Gamma_0^\varphi \cap \partial\Omega_k\,, \qquad \boldsymbol{D}^k \cdot \boldsymbol{n}^k = g^\varphi \quad \text{on} \quad \Gamma_{1,k}^\varphi = \Gamma_k \backslash \Gamma_{0,k}^\varphi. \tag{9}$$

Along a part $\Gamma_0^\psi \subset \Gamma$, the magnetic potential vanishes and, on the remaining part $\Gamma_1^\psi = \Gamma \backslash \Gamma_0^\psi$, the normal component of the magnetic induction, with density $g^\psi : \Gamma_1^\psi \times (0,T) \to \mathbb{R}$, is given, where $\Gamma = \Gamma_0^\psi \cup \Gamma_1^\psi$ is a Lipschitz dissection of Γ:

$$\psi^k = 0 \quad \text{on} \quad \Gamma_{0,k}^\psi = \Gamma_0^\psi \cap \partial\Omega_k\,, \qquad \boldsymbol{B}^k \cdot \boldsymbol{n}^k = g^\psi \quad \text{on} \quad \Gamma_{1,k}^\psi = \Gamma_k \backslash \Gamma_{0,k}^\psi. \tag{10}$$

The temperature vanishes along a part $\Gamma_0^\theta \subset \Gamma$ of the boundary and the heat flux along the outward normal of Γ, with density $g^\theta : \Gamma_1^\theta \times (0,T) \to \mathbb{R}$, is given on $\Gamma_1^\theta = \Gamma \backslash \Gamma_0^\theta$, where $\Gamma = \Gamma_0^\theta \cup \Gamma_1^\theta$ is a Lipschitz dissection of Γ:

$$\theta^k = 0 \quad \text{on} \quad \Gamma_{0,k}^\theta = \Gamma_0^\theta \cap \partial\Omega_k\,, \qquad -\left(\boldsymbol{\eta}^k \mathbf{grad}\ \theta^k\right) \cdot \boldsymbol{n}^k = g^\theta \quad \text{on} \quad \Gamma_{1,k}^\theta = \Gamma_k \backslash \Gamma_{0,k}^\theta. \tag{11}$$

On the common interfaces $\Gamma_k \cap \Gamma_{\overline{k}}$, of the subdomains Ω_k and $\Omega_{\overline{k}}$, rigid contact conditions are assumed; i.e., the mechanical displacement vector, temperature, electric and magnetic potentials, mechanical stress vector and normal components of the heat flux, electric displacement and magnetic induction are continuous:

$$\boldsymbol{u}^k = \boldsymbol{u}^{\overline{k}}, \qquad \boldsymbol{\sigma}^k \boldsymbol{n} = \boldsymbol{\sigma}^{\overline{k}} \boldsymbol{n} \qquad \text{on } \Gamma_k \cap \Gamma_{\overline{k}}, \tag{12}$$

$$\varphi^k = \varphi^{\overline{k}}, \qquad \boldsymbol{D}^k \cdot \boldsymbol{n} = \boldsymbol{D}^{\overline{k}} \cdot \boldsymbol{n} \qquad \text{on } \Gamma_k \cap \Gamma_{\overline{k}}, \tag{13}$$

$$\psi^k = \psi^{\overline{k}}, \qquad \boldsymbol{B}^k \cdot \boldsymbol{n} = \boldsymbol{B}^{\overline{k}} \cdot \boldsymbol{n} \qquad \text{on } \Gamma_k \cap \Gamma_{\overline{k}}, \tag{14}$$

$$\theta^k = \theta^{\overline{k}}, \qquad \left(\boldsymbol{\eta}^k \mathbf{grad}\ \theta^k\right) \cdot \boldsymbol{n} = \left(\boldsymbol{\eta}^{\overline{k}} \mathbf{grad}\ \theta^{\overline{k}}\right) \cdot \boldsymbol{n} \qquad \text{on } \Gamma_k \cap \Gamma_{\overline{k}}. \tag{15}$$

At the initial moment of time, we have the following initial conditions for the mechanical displacement vector-function and temperature:

$$\boldsymbol{u}(x,0)=\boldsymbol{u}_0(x),\qquad \frac{\partial \boldsymbol{u}}{\partial t}(x,0)=\boldsymbol{u}_1(x),\qquad \theta(x,0)=\theta_0(x),\qquad x\in\Omega, \tag{16}$$

where $\boldsymbol{u}_0=(u_{0i})$ and $\boldsymbol{u}_1=(u_{1i})$ are the initial mechanical displacement and velocity vector-functions, respectively, and θ_0 is the initial distribution of temperature.

We assume that the elasticity tensors (c^k_{ijpq}), stress–temperature tensor (λ^k_{ij}), piezoelectric tensor (ε^k_{pij}) and piezomagnetic tensor (b^k_{pij}) satisfy the following symmetry conditions:

$$c^k_{ijpq}=c^k_{ijqp}=c^k_{jipq},\qquad \lambda^k_{ij}=\lambda^k_{ji},\qquad \varepsilon^k_{pij}=\varepsilon^k_{pji},\qquad b^k_{pij}=b^k_{pji}. \tag{17}$$

If $\boldsymbol{u}^k$, φ^k, ψ^k and θ^k are twice continuously differentiable, then by multiplying equation (1) by an arbitrary continuously differentiable function $\boldsymbol{v}^k:\overline{\Omega_k}\to\mathbb{R}^3$, which vanishes on $\Gamma_{0,k}$ and $\boldsymbol{v}^k=\boldsymbol{v}^{\overline{k}}$ on $\Gamma_k\cap\Gamma_{\overline{k}}$, multiplying equation (2) by a continuously differentiable function $\overline{\varphi}^k:\overline{\Omega_k}\to\mathbb{R}$, such that $\overline{\varphi}^k=0$ on $\Gamma^{\varphi}_{0,k}$ and $\overline{\varphi}^k=\overline{\varphi}^{\overline{k}}$ on $\Gamma_k\cap\Gamma_{\overline{k}}$, multiplying equation (3) by a continuously differentiable function $\overline{\psi}^k:\overline{\Omega_k}\to\mathbb{R}$, which vanishes on $\Gamma^{\psi}_{0,k}$ and $\overline{\psi}^k=\overline{\psi}^{\overline{k}}$ on $\Gamma_k\cap\Gamma_{\overline{k}}$, and multiplying equation (4) by a continuously differentiable function $\overline{\theta}^k:\overline{\Omega_k}\to\mathbb{R}$, such that $\overline{\theta}^k=0$ on $\Gamma^{\theta}_{0,k}$ and $\overline{\theta}^k=\overline{\theta}^{\overline{k}}$ on $\Gamma_k\cap\Gamma_{\overline{k}}$, by integrating on Ω_k and by using Green's formula, we obtain the following integral relations for a subdomain Ω_k:

$$\int\limits_{\Omega_k}\rho^k\frac{\partial^2\boldsymbol{u}^k}{\partial t^2}\cdot\boldsymbol{v}^k\mathrm{d}x-\int\limits_{\Gamma_k}\boldsymbol{\sigma}^k\boldsymbol{n}^k\cdot\boldsymbol{v}^k\mathrm{d}\Gamma_k+\int\limits_{\Omega_k}\boldsymbol{\sigma}^k:\nabla\boldsymbol{v}^k\mathrm{d}x=\int\limits_{\Omega_k}\boldsymbol{f}^k\cdot\boldsymbol{v}^k\mathrm{d}x, \tag{18}$$

$$\int\limits_{\Gamma_k}\boldsymbol{D}^k\cdot\boldsymbol{n}^k\overline{\varphi}^k\mathrm{d}\Gamma_k-\int\limits_{\Omega_k}\boldsymbol{D}^k\cdot\mathbf{grad}\,\overline{\varphi}^k\mathrm{d}x=\int\limits_{\Omega_k}f^{\varphi}\overline{\varphi}^k\mathrm{d}x, \tag{19}$$

$$\int\limits_{\Gamma_k}\boldsymbol{B}^k\cdot\boldsymbol{n}^k\overline{\psi}^k\mathrm{d}\Gamma_k-\int\limits_{\Omega_k}\boldsymbol{B}^k\cdot\mathbf{grad}\,\overline{\psi}^k\mathrm{d}x=0, \tag{20}$$

$$\begin{aligned}&\int\limits_{\Omega_k}\varkappa^k\frac{\partial\theta^k}{\partial t}\overline{\theta}^k\mathrm{d}x-\int\limits_{\Gamma_k}\left(\boldsymbol{\eta}^k\mathbf{grad}\;\theta^k\right)\cdot\boldsymbol{n}^k\overline{\theta}^k\mathrm{d}\Gamma+\int\limits_{\Omega_k}\left(\boldsymbol{\eta}^k\mathbf{grad}\;\theta^k\right)\cdot\mathbf{grad}\,\overline{\theta}^k\mathrm{d}x+\Theta_0\int\limits_{\Omega_k}\boldsymbol{\lambda}^k:\boldsymbol{e}\left(\frac{\partial\boldsymbol{u}^k}{\partial t}\right)\overline{\theta}^k\mathrm{d}x\\&-\Theta_0\int\limits_{\Omega_k}\left(\boldsymbol{\mu}^k\cdot\mathbf{grad}\,\frac{\partial\varphi^k}{\partial t}\right)\overline{\theta}^k\mathrm{d}x-\Theta_0\int\limits_{\Omega_k}\left(\boldsymbol{m}^k\cdot\mathbf{grad}\,\frac{\partial\psi^k}{\partial t}\right)\overline{\theta}^k\mathrm{d}x=\int\limits_{\Omega_k}f^{\theta,k}\overline{\theta}\mathrm{d}x.\end{aligned} \tag{21}$$

On the common interfaces $\Gamma_k\cap\Gamma_{\overline{k}}$, we have $\boldsymbol{v}^k=\boldsymbol{v}^{\overline{k}}$, $\overline{\varphi}^k=\overline{\varphi}^{\overline{k}}$, $\overline{\psi}^k=\overline{\psi}^{\overline{k}}$, $\overline{\theta}^k=\overline{\theta}^{\overline{k}}$ and $\boldsymbol{n}^k=-\boldsymbol{n}^{\overline{k}}$. Therefore, from the rigid contact conditions (12) to (15), after summation of equations (18) to (21) with respect to k, by applying the boundary conditions (8) to (11) and the constitutive equations (5) to (7), and taking into account the symmetry conditions (17), we obtain:

$$\begin{aligned}&\sum_{k=1}^{K}\int\limits_{\Omega_k}\rho^k\frac{\partial^2\boldsymbol{u}^k}{\partial t^2}\cdot\boldsymbol{v}^k\mathrm{d}x+\sum_{k=1}^{K}\int\limits_{\Omega_k}\left(\boldsymbol{c}^k*\boldsymbol{e}\left(\boldsymbol{u}^k\right)\right):\boldsymbol{e}\left(\boldsymbol{v}^k\right)\mathrm{d}x+\sum_{k=1}^{K}\int\limits_{\Omega_k}\left(\boldsymbol{\varepsilon}^k*\mathbf{grad}\,\varphi^k\right):\boldsymbol{e}\left(\boldsymbol{v}^k\right)\mathrm{d}x\\&+\sum_{k=1}^{K}\int\limits_{\Omega_k}\left(\boldsymbol{b}^k*\mathbf{grad}\,\psi^k\right):\boldsymbol{e}\left(\boldsymbol{v}^k\right)\mathrm{d}x-\sum_{k=1}^{K}\int\limits_{\Omega_k}\boldsymbol{\lambda}^k:\boldsymbol{e}\left(\boldsymbol{v}^k\right)\theta^k\mathrm{d}x=\int\limits_{\Omega}\boldsymbol{f}\cdot\boldsymbol{v}\mathrm{d}x+\int\limits_{\Gamma_1}\boldsymbol{g}\cdot\boldsymbol{v}\mathrm{d}\Gamma,\end{aligned} \tag{22}$$

$$-\sum_{k=1}^{K}\int_{\Omega_k}\left(\boldsymbol{\varepsilon}^k * \boldsymbol{e}\left(\boldsymbol{u}^k\right)\right)\cdot\mathbf{grad}\,\overline{\varphi}^k\mathrm{d}x+\sum_{k=1}^{K}\int_{\Omega_k}\left(\boldsymbol{d}^k\mathbf{grad}\,\varphi^k\right)\cdot\mathbf{grad}\,\overline{\varphi}^k\mathrm{d}x+\sum_{k=1}^{K}\int_{\Omega_k}\left(\boldsymbol{a}^k\mathbf{grad}\,\psi^k\right)\cdot\mathbf{grad}\,\overline{\varphi}^k\mathrm{d}x$$
$$-\sum_{k=1}^{K}\int_{\Omega_k}\boldsymbol{\mu}^k\cdot\mathbf{grad}\,\overline{\varphi}^k\theta^k\mathrm{d}x=\int_{\Omega}f^{\varphi}\overline{\varphi}\mathrm{d}x-\int_{\Gamma_1^{\varphi}}g^{\varphi}\overline{\varphi}\mathrm{d}\Gamma,\quad(23)$$

$$-\sum_{k=1}^{K}\int_{\Omega_k}\left(\boldsymbol{b}^k * \boldsymbol{e}\left(\boldsymbol{u}^k\right)\right)\cdot\mathbf{grad}\,\overline{\psi}^k\mathrm{d}x+\sum_{k=1}^{K}\int_{\Omega_k}\left(\boldsymbol{a}^k\mathbf{grad}\,\varphi^k\right)\cdot\mathbf{grad}\,\overline{\psi}^k\mathrm{d}x+\sum_{k=1}^{K}\int_{\Omega_k}\left(\boldsymbol{\zeta}^k\mathbf{grad}\,\psi^k\right)\cdot\mathbf{grad}\,\overline{\psi}^k\mathrm{d}x$$
$$-\sum_{k=1}^{K}\int_{\Omega_k}\left(\boldsymbol{m}^k\cdot\mathbf{grad}\,\overline{\psi}^k\right)\theta^k\mathrm{d}x=-\int_{\Gamma_1^{\psi}}g^{\psi}\overline{\psi}\mathrm{d}\Gamma,\quad(24)$$

$$\sum_{k=1}^{K}\int_{\Omega_k}\varkappa^k\frac{\partial\theta^k}{\partial t}\overline{\theta}^k\mathrm{d}x+\sum_{k=1}^{K}\int_{\Omega_k}\left(\boldsymbol{\eta}^k\mathbf{grad}\;\theta^k\right)\cdot\mathbf{grad}\,\overline{\theta}^k\mathrm{d}x+\Theta_0\sum_{k=1}^{K}\int_{\Omega_k}\boldsymbol{\lambda}^k:\boldsymbol{e}\left(\frac{\partial\boldsymbol{u}^k}{\partial t}\right)\overline{\theta}^k\mathrm{d}x$$
$$-\Theta_0\sum_{k=1}^{K}\int_{\Omega_k}\left(\boldsymbol{\mu}^k\cdot\mathbf{grad}\,\frac{\partial\varphi^k}{\partial t}\right)\overline{\theta}^k\mathrm{d}x-\Theta_0\sum_{k=1}^{K}\int_{\Omega_k}\left(\boldsymbol{m}^k\cdot\mathbf{grad}\,\frac{\partial\psi^k}{\partial t}\right)\overline{\theta}^k\mathrm{d}x=\int_{\Omega}f^{\theta}\overline{\theta}\mathrm{d}x-\int_{\Gamma_1^{\theta}}g^{\theta}\overline{\theta}\mathrm{d}\Gamma,\quad(25)$$

where $\boldsymbol{f}=\boldsymbol{f}^k, f^{\varphi}=f^{\varphi,k}$ and $f^{\theta}=f^{\theta,k}$ in Ω_k and $\boldsymbol{v}=\boldsymbol{v}^k$, $\overline{\varphi}=\overline{\varphi}^k$, $\overline{\psi}=\overline{\psi}^k$ and $\overline{\theta}=\overline{\theta}^k$ on $\overline{\Omega_k}$.

Therefore, if $\boldsymbol{u}^k$, φ^k, ψ^k and θ^k are solutions of equations (1) to (4) and satisfy the boundary conditions (8) to (11) and the rigid contact conditions (12) to (15), then $\boldsymbol{u}^k$, φ^k, ψ^k and θ^k are solutions of equations (22) to (25). Conversely, if $\boldsymbol{u}^k$, φ^k, ψ^k and θ^k are twice continuously differentiable solutions of equations (22) to (25), then by using Green's formula we obtain:

$$\sum_{k=1}^{K}\int_{\Omega_k}\rho^k\frac{\partial^2\boldsymbol{u}^k}{\partial t^2}\cdot\boldsymbol{v}^k\mathrm{d}x+\sum_{k=1}^{K}\int_{\Gamma_k}\boldsymbol{\sigma}^k\boldsymbol{n}^k\cdot\boldsymbol{v}^k\mathrm{d}\Gamma-\sum_{k=1}^{K}\int_{\Omega_k}\mathbf{div}\,\left(\boldsymbol{c}^k * \boldsymbol{e}\left(\boldsymbol{u}^k\right)+\boldsymbol{\varepsilon}^k * \mathbf{grad}\,\varphi^k+\boldsymbol{b}^k * \mathbf{grad}\,\psi^k-\boldsymbol{\lambda}^k\theta^k\right)\cdot\boldsymbol{v}^k\mathrm{d}x$$
$$=\int_{\Omega}\boldsymbol{f}\cdot\boldsymbol{v}\mathrm{d}x+\int_{\Gamma_1}\boldsymbol{g}\cdot\boldsymbol{v}\mathrm{d}\Gamma,\quad(26)$$

$$-\sum_{k=1}^{K}\int_{\Gamma_k}(\boldsymbol{D}^k\cdot\boldsymbol{n}^k)\overline{\varphi}^k\mathrm{d}\Gamma_k+\sum_{k=1}^{K}\int_{\Omega_k}\mathrm{div}\left(\boldsymbol{\varepsilon}^k * \boldsymbol{e}\left(\boldsymbol{u}^k\right)-\boldsymbol{d}^k\mathbf{grad}\,\varphi^k-\boldsymbol{a}^k\mathbf{grad}\,\psi^k+\boldsymbol{\mu}^k\theta^k\right)\overline{\varphi}^k\mathrm{d}x$$
$$=\int_{\Omega}f^{\varphi}\overline{\varphi}\mathrm{d}x-\int_{\Gamma_1^{\varphi}}g^{\varphi}\overline{\varphi}\mathrm{d}\Gamma,\quad(27)$$

$$-\sum_{k=1}^{K}\int_{\Gamma_k}\left(\boldsymbol{B}^k\cdot\boldsymbol{n}^k\right)\overline{\psi}^k\mathrm{d}\Gamma+\sum_{k=1}^{K}\int_{\Omega_k}\mathrm{div}\left(\boldsymbol{b}^k * \boldsymbol{e}\left(\boldsymbol{u}^k\right)-\boldsymbol{a}^k\mathbf{grad}\,\varphi^k-\boldsymbol{\zeta}^k\mathbf{grad}\,\psi^k+\boldsymbol{m}^k\theta^k\right)\overline{\psi}^k\mathrm{d}x=-\int_{\Gamma_1^{\psi}}g^{\psi}\overline{\psi}\mathrm{d}\Gamma,\quad(28)$$

$$\sum_{k=1}^{K}\int_{\Omega_k}\varkappa^k\frac{\partial\theta^k}{\partial t}\overline{\theta}^k\mathrm{d}x+\sum_{k=1}^{K}\int_{\Gamma_k}\left(\boldsymbol{\eta}^k\mathbf{grad}\,\theta^k\cdot\boldsymbol{n}^k\right)\overline{\theta}^k\mathrm{d}\Gamma-\sum_{k=1}^{K}\int_{\Omega_k}\mathrm{div}\left(\boldsymbol{\eta}^k\mathbf{grad}\,\theta^k\right)\overline{\theta}^k\mathrm{d}x+\Theta_0\sum_{k=1}^{K}\int_{\Omega_k}\boldsymbol{\lambda}^k:\boldsymbol{e}\left(\frac{\partial\boldsymbol{u}^k}{\partial t}\right)\overline{\theta}^k\mathrm{d}x$$
$$-\Theta_0\sum_{k=1}^{K}\int_{\Omega_k}\left(\boldsymbol{\mu}^k\cdot\mathbf{grad}\,\frac{\partial\varphi^k}{\partial t}\right)\overline{\theta}^k\mathrm{d}x-\Theta_0\sum_{k=1}^{K}\int_{\Omega_k}\left(\boldsymbol{m}^k\cdot\mathbf{grad}\,\frac{\partial\psi^k}{\partial t}\right)\overline{\theta}^k\mathrm{d}x=\int_{\Omega}f^{\theta}\overline{\theta}\mathrm{d}x-\int_{\Gamma_1^{\theta}}g^{\theta}\overline{\theta}\mathrm{d}\Gamma,\quad(29)$$

where $\boldsymbol{v}^k$, $\overline{\varphi}^k$, $\overline{\psi}^k$ and $\overline{\theta}^k$ are continuously differentiable functions on $\overline{\Omega_k}$, such that $\boldsymbol{v}^k=\boldsymbol{0}$ on $\Gamma_{0,k}$, $\overline{\varphi}^k=0$ on $\Gamma_{0,k}^{\varphi}$, $\overline{\psi}^k=0$ on $\Gamma_{0,k}^{\psi}$, $\overline{\theta}^k=0$ on $\Gamma_{0,k}^{\theta}$ and $\boldsymbol{v}^k=\boldsymbol{v}^{\bar{k}}$, $\overline{\varphi}^k=\overline{\varphi}^{\bar{k}}$, $\overline{\psi}^k=\overline{\psi}^{\bar{k}}$ and $\overline{\theta}^k=\overline{\theta}^{\bar{k}}$ on $\Gamma_k\cap\Gamma_{\bar{k}}$. By letting

$\boldsymbol{v}^k \in (C_0^1(\overline{\Omega_k}))^3$, $C_0^1(\overline{\Omega_k}) = \{v \in C^1(\overline{\Omega_k})\,|v = 0 \text{ on } \Gamma_k\}$, $\overline{\varphi}^k \in C_0^1(\overline{\Omega_k})$, $\overline{\psi}^k \in C_0^1(\overline{\Omega_k})$ and $\overline{\theta}^k \in C_0^1(\overline{\Omega_k})$, and by taking into account the density of $C_0^1(\overline{\Omega})$ in $L^2(\Omega)$, we obtain, from equations (26) to (29), that $\boldsymbol{u}^k$, φ^k, ψ^k and θ^k, $k = 1, \ldots, K$, satisfy equations (1) to (4). Now, if functions $\boldsymbol{v}^k$, $\overline{\varphi}^k$, $\overline{\psi}^k$ and $\overline{\theta}^k$ are arbitrary continuous functions on the surfaces $\Gamma_{1,k}$, $\Gamma_{1,k}^{\varphi}$, $\Gamma_{1,k}^{\psi}$ and $\Gamma_{1,k}^{\theta}$ and vanish on the remaining parts of the boundary Γ_k, then by applying equations (1) to (4) and the density of the sets of continuous functions on $\Gamma_{1,k}$, $\Gamma_{1,k}^{\varphi}$, $\Gamma_{1,k}^{\psi}$ and $\Gamma_{1,k}^{\theta}$, vanishing on the boundaries of the corresponding surfaces in spaces $L^2(\Gamma_{1,k})$, $L^2(\Gamma_{1,k}^{\varphi})$, $L^2(\Gamma_{1,k}^{\psi})$ and $L^2(\Gamma_{1,k}^{\theta})$, we infer, from equations (26) to (29), that $\boldsymbol{u}^k$, φ^k, ψ^k and θ^k satisfy the boundary conditions (8) to (11). Similarly, if the functions $\boldsymbol{v}^k$, $\overline{\varphi}^k$, $\overline{\psi}^k$ and $\overline{\theta}^k$ are arbitrary continuous functions on the interfaces $\Gamma_k \cap \Gamma_{\overline{k}}$ and vanish on the remaining parts of the boundaries Γ_k and $\Gamma_{\overline{k}}$, then, from equations (26) to (29), taking into account equations (1) to (4) and the density of the sets of continuous functions on $\Gamma_k \cap \Gamma_{\overline{k}}$, vanishing on the boundaries of the corresponding surfaces in space $L^2(\Gamma_k \cap \Gamma_{\overline{k}})$, we obtain that $\boldsymbol{u}^k$, φ^k, ψ^k and θ^k satisfy the rigid contact conditions (12) to (15).

Note that the differential linear three-dimensional model of the static equilibrium [22] of the thermo-electro-magneto-elastic body $\overline{\Omega}$ is given by the system of equations (1) to (4), together with the boundary conditions (8) to (11) and rigid contact conditions (12) to (15), where the functions $\boldsymbol{u}^k$, φ^k, ψ^k, θ^k, $\boldsymbol{f}^k$, $f^{\varphi,k}$, $f^{\theta,k}$, $\boldsymbol{g}$, g^{φ}, g^{ψ} and g^{θ} do not depend on the time variable t and the corresponding equations are fulfilled in Ω. Hence, in the static model, instead of equations (1) and (4), we have:

$$-\mathbf{div}\,\boldsymbol{\sigma}^k = \boldsymbol{f}^k \qquad \text{in } \Omega_k, \tag{30}$$

$$-\mathrm{div}\left(\boldsymbol{\eta}^k \mathbf{grad}\ \theta^k\right) = f^{\theta,k} \qquad \text{in } \Omega_k, \tag{31}$$

and, instead of equations (22) and (25), we have:

$$\sum_{k=1}^{K} \int\limits_{\Omega_k} \left(\boldsymbol{c}^k * \boldsymbol{e}\left(\boldsymbol{u}^k\right)\right) : \boldsymbol{e}(\boldsymbol{v}^k)\mathrm{d}x + \sum_{k=1}^{K} \int\limits_{\Omega_k} \left(\boldsymbol{\varepsilon}^k * \mathbf{grad}\,\varphi^k\right) : \boldsymbol{e}\left(\boldsymbol{v}^k\right)\mathrm{d}x$$

$$+ \sum_{k=1}^{K} \int\limits_{\Omega_k} \left(\boldsymbol{b}^k * \mathbf{grad}\,\psi^k\right) : \boldsymbol{e}\left(\boldsymbol{v}^k\right)\mathrm{d}x - \sum_{k=1}^{K} \int\limits_{\Omega_k} \left(\boldsymbol{\lambda}^k : \boldsymbol{e}\left(\boldsymbol{v}^k\right)\right)\theta^k \mathrm{d}x = \int\limits_{\Omega} \boldsymbol{f} \cdot \boldsymbol{v}\mathrm{d}x + \int\limits_{\Gamma_1} \boldsymbol{g} \cdot \boldsymbol{v}\mathrm{d}\Gamma, \tag{32}$$

$$\sum_{k=1}^{K} \int\limits_{\Omega_k} \boldsymbol{\eta}^k \mathbf{grad}\ \theta^k \cdot \mathbf{grad}\,\overline{\theta}^k \mathrm{d}x = \int\limits_{\Omega} f^{\theta}\overline{\theta}\mathrm{d}x - \int\limits_{\Gamma_1^{\theta}} g^{\theta}\overline{\theta}\mathrm{d}\Gamma. \tag{33}$$

Thus, the initial boundary value problem (1) to (4) and (8) to (16) corresponding to the dynamic three-dimensional model of a multidomain anisotropic inhomogeneous thermo-electro-magneto-elastic solid is equivalent to equations (22) to (25), together with the initial conditions (16) in spaces of twice continuously differentiable functions. The boundary value problem (2), (3), (8) to (15), (30) and (31) corresponding to the static three-dimensional model of the thermo-electro-magneto-elastic solid Ω is equivalent to equations (23), (24), (32) and (33) in spaces of twice continuously differentiable functions. On the basis of the latter, in the next sections we obtain the so-called weak or variational formulations of the static problem (2), (3), (8) to (15), (30) and (31) and the dynamic problem (1) to (4) and (8) to (16), and investigate the existence and uniqueness of the corresponding solutions in suitable function spaces.

3. Analysis of the static problem

Hereafter, for each real $s \geq 0$, $0 \leq \check{s} \leq 1$, we denote by $H^s(D)$ and $H^{\check{s}}(\check{\Gamma})$ the Sobolev spaces of real-valued functions based on $H^0(D) = L^2(D)$ and $H^0(\check{\Gamma}) = L^2(\check{\Gamma})$, respectively, where $D \subset \mathbb{R}^n$, $n \in \mathbb{N}$, is a bounded Lipschitz domain and $\check{\Gamma}$ is an element of a Lipschitz dissection of the boundary ∂D [23]; $H_0^s(D)$ denotes the closure of the set $\mathfrak{D}(D)$ of infinitely differentiable functions with compact support in D in the space $H^s(D)$. We denote the corresponding spaces of vector-valued functions by $\mathbf{H}^s(D) = [H^s(D)]^3$, $\mathbf{H}_0^s(D) = [H_0^s(D)]^3$, $s \geq 0$, $\mathbf{H}^{\check{s}}(\check{\Gamma}) = [H^{\check{s}}(\check{\Gamma})]^3$, $0 \leq \check{s} \leq 1$, $\mathbf{L}^{s_1}(\check{\Gamma}) = [L^{s_1}(\check{\Gamma})]^3$ and $s_1 \geq 1$ and the trace operators by $\mathrm{tr}_{\check{\Gamma}} : H^1(D) \to H^{1/2}(\check{\Gamma})$

and $\mathbf{tr}_{\check{\Gamma}} : \mathbf{H}^1(D) \to \mathbf{H}^{1/2}(\check{\Gamma})$. For any measurable set $D \subset \mathbb{R}^n$, $n \in \mathbb{N}$, $(.,.)_{\mathbf{L}^2(D)}$ and $(.,.)_{L^2(D)}$ are the classical scalar products in $\mathbf{L}^2(D)$ and $L^2(D)$, respectively.

Note that if functions v^k belong to $H^1(\Omega_k)$, and on the common interfaces $\Gamma_k \cap \Gamma_{\bar{k}}$ we have $\mathrm{tr}_{\Gamma_k \cap \Gamma_{\bar{k}}}(v^k) = \mathrm{tr}_{\Gamma_k \cap \Gamma_{\bar{k}}}(v^{\bar{k}})$, then there exists the function $v \in H^1(\Omega)$ such that $v = v^k$ in Ω_k. Therefore, from equations (23), (24), (32) and (33), we obtain the following variational formulation of the boundary value problem (2), (3), (8) to (15), (30) and (31):

Find $\boldsymbol{u} \in \mathbf{V}(\Omega) = \{\boldsymbol{v} \in \mathbf{H}^1(\Omega); \mathbf{tr}_\Gamma(\boldsymbol{v}) = \mathbf{0} \text{ on } \Gamma_0\}, \varphi \in V^\varphi(\Omega) = \{\overline{\varphi} \in H^1(\Omega); \mathrm{tr}_\Gamma(\overline{\varphi}) = 0 \text{ on } \Gamma_0^\varphi\}, \psi \in V^\psi(\Omega) = \{\overline{\psi} \in H^1(\Omega); \mathrm{tr}_\Gamma(\overline{\psi}) = 0 \text{ on } \Gamma_0^\psi\}, \theta \in V^\theta(\Omega) = \{\overline{\theta} \in H^1(\Omega); \mathrm{tr}_\Gamma(\overline{\theta}) = 0 \text{ on } \Gamma_0^\theta\}$ such that

$$c(\boldsymbol{u},\boldsymbol{v}) + \varepsilon(\varphi,\boldsymbol{v}) + b(\psi,\boldsymbol{v}) - \lambda(\theta,\boldsymbol{v}) = L^{\boldsymbol{u}}(\boldsymbol{v}), \qquad \forall \boldsymbol{v} \in \mathbf{V}(\Omega), \tag{34}$$

$$-\varepsilon(\overline{\varphi},\boldsymbol{u}) + d(\varphi,\overline{\varphi}) + a(\psi,\overline{\varphi}) - \mu(\theta,\overline{\varphi}) = L^{\varphi}(\overline{\varphi}), \qquad \forall \overline{\varphi} \in V^\varphi(\Omega), \tag{35}$$

$$-b(\overline{\psi},\boldsymbol{u}) + a(\varphi,\overline{\psi}) + \zeta(\psi,\overline{\psi}) - m(\theta,\overline{\psi}) = L^{\psi}(\overline{\psi}), \qquad \forall \overline{\psi} \in V^\psi(\Omega), \tag{36}$$

$$\eta(\theta,\overline{\theta}) = L^{\theta}(\overline{\theta}), \qquad \forall \overline{\theta} \in V^\theta(\Omega), \tag{37}$$

where

$$c(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega (\boldsymbol{c} * \boldsymbol{e}(\boldsymbol{u})) : \boldsymbol{e}(\boldsymbol{v})\mathrm{d}x, \qquad \varepsilon(\varphi,\boldsymbol{v}) = \int_\Omega (\boldsymbol{\varepsilon} * \mathbf{grad}\,\varphi) : \boldsymbol{e}(\boldsymbol{v})\mathrm{d}x,$$

$$b(\psi,\boldsymbol{v}) = \int_\Omega (\boldsymbol{b} * \mathbf{grad}\,\psi) : \boldsymbol{e}(\boldsymbol{v})\mathrm{d}x, \qquad d(\varphi,\overline{\varphi}) = \int_\Omega (\boldsymbol{d}\ \mathbf{grad}\,\varphi) \cdot \mathbf{grad}\,\overline{\varphi}\mathrm{d}x,$$

$$a(\psi,\overline{\varphi}) = \int_\Omega (\boldsymbol{a}\ \mathbf{grad}\,\psi) \cdot \mathbf{grad}\,\overline{\varphi}\mathrm{d}x, \qquad \zeta(\psi,\overline{\psi}) = \int_\Omega (\boldsymbol{\zeta}\ \mathbf{grad}\,\psi) \cdot \mathbf{grad}\,\overline{\psi}\mathrm{d}x,$$

$$\lambda(\theta,\boldsymbol{v}) = \int_\Omega (\boldsymbol{\lambda} : \boldsymbol{e}(\boldsymbol{v}))\,\theta\mathrm{d}x, \qquad \mu(\theta,\overline{\varphi}) = \int_\Omega (\boldsymbol{\mu} \cdot \mathbf{grad}\,\overline{\varphi})\,\theta\mathrm{d}x,$$

$$m(\theta,\overline{\psi}) = \int_\Omega \left(\boldsymbol{m} \cdot \mathbf{grad}\,\overline{\psi}\right)\theta\mathrm{d}x, \qquad \eta(\theta,\overline{\theta}) = \int_\Omega (\boldsymbol{\eta}\ \mathbf{grad}\ \theta) \cdot \mathbf{grad}\,\overline{\theta}\mathrm{d}x,$$

$$L^{\boldsymbol{u}}(\boldsymbol{v}) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\mathrm{d}x + \int_{\Gamma_1} \boldsymbol{g} \cdot \mathbf{tr}_{\Gamma_1}(\boldsymbol{v})\mathrm{d}\Gamma, \qquad L^{\varphi}(\overline{\varphi}) = \int_\Omega f^\varphi\overline{\varphi}\mathrm{d}x - \int_{\Gamma_1^\varphi} g^\varphi \mathrm{tr}_{\Gamma_1^\varphi}(\overline{\varphi})\mathrm{d}\Gamma,$$

$$L^{\psi}(\overline{\psi}) = -\int_{\Gamma_1^\psi} g^\psi \mathrm{tr}_{\Gamma_1^\psi}(\overline{\psi})\mathrm{d}\Gamma, \qquad L^{\theta}(\overline{\theta}) = \int_\Omega f^\theta\overline{\theta}\mathrm{d}x - \int_{\Gamma_1^\theta} g^\theta \mathrm{tr}_{\Gamma_1^\theta}(\overline{\theta})\mathrm{d}\Gamma,$$

and $\boldsymbol{u} = \boldsymbol{u}^k$, $\varphi = \varphi^k$, $\psi = \psi^k$, $\theta = \theta^k$, $\boldsymbol{c} = \boldsymbol{c}^k$, $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^k$, $\boldsymbol{b} = \boldsymbol{b}^k$, $\boldsymbol{\lambda} = \boldsymbol{\lambda}^k$, $\boldsymbol{d} = \boldsymbol{d}^k$, $\boldsymbol{a} = \boldsymbol{a}^k$, $\boldsymbol{\mu} = \boldsymbol{\mu}^k$, $\boldsymbol{\zeta} = \boldsymbol{\zeta}^k$, $\boldsymbol{m} = \boldsymbol{m}^k$ and $\boldsymbol{\eta} = \boldsymbol{\eta}^k$ in Ω_k.

Note that if the parts Γ_0, Γ_0^φ, Γ_0^ψ and Γ_0^θ of the boundary of Ω are empty sets, then the homogeneous problem (34) to (37), where $\boldsymbol{f} \equiv 0$, $\boldsymbol{g} \equiv 0$, $f^\varphi \equiv 0$, $g^\varphi \equiv 0$, $g^\psi \equiv 0$, $f^\theta \equiv 0$ and $g^\theta \equiv 0$, has non-trivial solutions. Hence, the solution of the problem (34) to (37) is not unique in the first-order Sobolev spaces mentioned in the variational formulation and it is necessary to introduce suitable factor spaces, where the solution of the problem (34) to (37) is unique.

As in the case of most practical applications, we assume that c_{ijpq}, ε_{pij}, b_{pij}, d_{ij}, ζ_{ij}, a_{ij}, λ_{ij}, μ_i, m_i and $\eta_{ij} \in L^\infty(\Omega)$ satisfy the following positive definiteness conditions:

$$(\boldsymbol{c}(x) * \boldsymbol{F}) : \boldsymbol{F} \geq \alpha_c \boldsymbol{F} : \boldsymbol{F}, \qquad \boldsymbol{\eta}(x)\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geq \alpha_\eta |\boldsymbol{\xi}|^2, \tag{38}$$

$$\boldsymbol{d}(x)\boldsymbol{\xi} \cdot \boldsymbol{\xi} + \boldsymbol{a}(x)\overline{\boldsymbol{\xi}} \cdot \boldsymbol{\xi} + \boldsymbol{a}(x)\boldsymbol{\xi} \cdot \overline{\boldsymbol{\xi}} + \boldsymbol{\zeta}(x)\overline{\boldsymbol{\xi}} \cdot \overline{\boldsymbol{\xi}} \geq \alpha(|\boldsymbol{\xi}|^2 + |\overline{\boldsymbol{\xi}}|^2), \tag{39}$$

for all $\boldsymbol{F} \in \mathbb{S}^3$, $\boldsymbol{\xi}, \overline{\boldsymbol{\xi}} \in \mathbb{R}^3$ and almost all $x \in \Omega$, where α_c, α_η and α are positive constants and $\mathbb{S}^3$ designates the set of all symmetric 3×3 matrices.

Let us denote by $\mathfrak{R}$ the set of solutions of the homogeneous problem (34) to (37), where $L^{\boldsymbol{u}}(\boldsymbol{v}) = 0$, $L^{\varphi}(\overline{\varphi}) = 0$, $L^{\psi}(\overline{\psi}) = 0$ and $L^{\theta}(\overline{\theta}) = 0$, for all $\boldsymbol{v} \in \mathbf{V}(\Omega)$, $\overline{\varphi} \in V^{\varphi}(\Omega)$, $\overline{\psi} \in V^{\psi}(\Omega)$ and $\overline{\theta} \in V^{\theta}(\Omega)$. The structure of the set $\mathfrak{R}$ is determined in [22] and $\mathfrak{R}$ is of the following form:

$$\mathfrak{R} = \Big\{ \Big(\boldsymbol{v}^{r\theta^r}, \overline{\varphi}^{r\theta^r}, \overline{\psi}^{r\theta^r}, \theta^r\Big) \in \mathbf{V}(\Omega) \times V^{\varphi}(\Omega) \times V^{\psi}(\Omega) \times V^{\theta}(\Omega);\ \boldsymbol{v}^{r\theta^r} = \boldsymbol{v}^r + \boldsymbol{u}^{r\theta^r},$$
$$\overline{\varphi}^{r\theta^r} = \overline{\varphi}^r + \varphi^{r\theta^r},\ \overline{\psi}^{r\theta^r} = \overline{\psi}^r + \psi^{r\theta^r}, \Big(\boldsymbol{v}^r, \overline{\varphi}^r, \overline{\psi}^r\Big) \in \mathfrak{R}_V,\ \theta^r \in \mathfrak{R}_{\theta}\Big\},$$

where

$$\mathfrak{R}_V = \Big\{ \Big(\boldsymbol{v}^r, \overline{\varphi}^r, \overline{\psi}^r\Big) \in \mathbf{V}(\Omega) \times V^{\varphi}(\Omega) \times V^{\psi}(\Omega); \boldsymbol{v}^r(x) = \boldsymbol{\alpha} + \boldsymbol{\beta} \times \boldsymbol{Ox}, \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^3, \boldsymbol{Ox} = (x_i), \overline{\varphi}^r = \alpha_{\varphi},$$
$$\alpha_{\varphi} \in \mathbb{R}, \overline{\psi}^r = \alpha_{\psi}, \alpha_{\psi} \in \mathbb{R}\Big\},$$

$$\mathfrak{R}_{\theta} = \big\{\theta^r \in V^{\theta}(\Omega); \theta^r = \alpha_{\theta},\ \alpha_{\theta} \in \mathbb{R}\big\}$$

and

$$\big(\boldsymbol{u}^{r\theta^r}, \varphi^{r\theta^r}, \psi^{r\theta^r}\big) \in \mathbf{V}(\Omega) \times V^{\varphi}(\Omega) \times V^{\psi}(\Omega)$$

is a solution of the homogeneous equations (34) to (36), when $\theta = \theta^r$.

Applying the set $\mathfrak{R}$, we can define the factor space $(\mathbf{V}(\Omega) \times V^{\varphi}(\Omega) \times V^{\psi}(\Omega) \times V^{\theta}(\Omega))/\mathfrak{R}$, which consists of the following elements:

$$\big(\boldsymbol{v}, \overline{\varphi}, \overline{\psi}, \overline{\theta}\big)^{\mathfrak{R}} = \Big\{\big(\boldsymbol{v}, \overline{\varphi}, \overline{\psi}, \overline{\theta}\big) + \Big(\boldsymbol{v}^{r\theta^r}, \overline{\varphi}^{r\theta^r}, \overline{\psi}^{r\theta^r}, \theta^r\Big); \Big(\boldsymbol{v}^{r\theta^r}, \overline{\varphi}^{r\theta^r}, \overline{\psi}^{r\theta^r}, \theta^r\Big) \in \mathfrak{R}\Big\},$$

for each $(\boldsymbol{v}, \overline{\varphi}, \overline{\psi}, \overline{\theta}) \in \mathbf{V}(\Omega) \times V^{\varphi}(\Omega) \times V^{\psi}(\Omega) \times V^{\theta}(\Omega)$, which is the Hilbert space equipped with the following norm:

$$\Big\|\big(\boldsymbol{v}, \overline{\varphi}, \overline{\psi}, \overline{\theta}\big)^{\mathfrak{R}}\Big\|_{*} = \inf\Big\{\Big\|\big(\boldsymbol{v}, \overline{\varphi}, \overline{\psi}, \overline{\theta}\big) + \Big(\boldsymbol{v}^{r\theta^r}, \overline{\varphi}^{r\theta^r}, \overline{\psi}^{r\theta^r}, \theta^r\Big)\Big\|_{(H^1(\Omega))^6}; \Big(\boldsymbol{v}^{r\theta^r}, \overline{\varphi}^{r\theta^r}, \overline{\psi}^{r\theta^r}, \theta^r\Big) \in \mathfrak{R}\Big\}.$$

By applying the set $\mathfrak{R}$, we can define the solution of the problem (34) to (37) in the factor space $(\mathbf{V}(\Omega) \times V^{\varphi}(\Omega) \times V^{\psi}(\Omega) \times V^{\theta}(\Omega))/\mathfrak{R}$. Indeed, for each solution $(\boldsymbol{u}, \varphi, \psi, \theta)$ of the problem (34) to (37), any vector-function $(\boldsymbol{u}, \varphi, \psi, \theta) + (\boldsymbol{v}^{r\theta^r}, \overline{\varphi}^{r\theta^r}, \overline{\psi}^{r\theta^r}, \theta^r)$, where $(\boldsymbol{v}^{r\theta^r}, \overline{\varphi}^{r\theta^r}, \overline{\psi}^{r\theta^r}, \theta^r) \in \mathfrak{R}$ is also a solution of equations (34) to (37). Therefore, we say that $(\boldsymbol{u}, \varphi, \psi, \theta)^{\mathfrak{R}} \in (\mathbf{V}(\Omega) \times V^{\varphi}(\Omega) \times V^{\psi}(\Omega) \times V^{\theta}(\Omega))/\mathfrak{R}$ is a solution of the problem (34) to (37), if any function from the equivalence class $(\boldsymbol{u}, \varphi, \psi, \theta)^{\mathfrak{R}}$ is a solution of the problem (34) to (37).

For the problem (34) to (37), which is equivalent to the boundary value problem (1) to (4) and (8) to (15) in the spaces of classical twice continuously differentiable functions, the following existence, uniqueness and continuous dependence theorem was proved in [22].

Theorem 1. *Suppose that $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz domain; the parameters c_{ijpq}, ε_{pij}, b_{pij}, d_{ij} , ζ_{ij}, a_{ij}, λ_{ij}, μ_i, m_i and $\eta_{ij} \in L^{\infty}(\Omega)$ satisfy the symmetry and positive definiteness conditions* (17), (38) *and* (39). *If $\boldsymbol{f} \in \mathbf{L}^{6/5}(\Omega)$, $\boldsymbol{g} \in \mathbf{L}^{4/3}(\Gamma_1)$, $f^{\varphi} \in L^{6/5}(\Omega)$, $g^{\varphi} \in L^{4/3}(\Gamma_1^{\varphi})$, $g^{\psi} \in L^{4/3}(\Gamma_1^{\psi})$, $f^{\theta} \in L^{6/5}(\Omega)$, $g^{\theta} \in L^{4/3}(\Gamma_1^{\theta})$, $L^{\theta}(\theta^r) = 0$ and $L^{\boldsymbol{u}}(\boldsymbol{v}^r) + L^{\varphi}(\overline{\varphi}^r) + L^{\psi}(\overline{\psi}^r) = 0$, for all $(\boldsymbol{v}^r, \overline{\varphi}^r, \overline{\psi}^r) \in \mathfrak{R}_V$, $\theta^r \in \mathfrak{R}_{\theta}$, then the problem* (34) *to* (37) *possesses a unique solution $(\boldsymbol{u}, \varphi, \psi, \theta)^{\mathfrak{R}} \in (V(\Omega) \times V^{\varphi}(\Omega) \times V^{\psi}(\Omega) \times V^{\theta}(\Omega))/\mathfrak{R}$, which continuously depends on the given data, i.e., the mapping $(\boldsymbol{f}, \boldsymbol{g}, f^{\varphi}, g^{\varphi}, g^{\psi}, f^{\theta}, g^{\theta}) \to (\boldsymbol{u}, \varphi, \psi, \theta)^{\mathfrak{R}}$ is linear and continuous from the space $\mathbf{L}^{6/5}(\Omega) \times \mathbf{L}^{4/3}(\Gamma_1) \times L^{6/5}(\Omega) \times L^{4/3}(\Gamma_1^{\varphi}) \times L^{4/3}(\Gamma_1^{\psi}) \times L^{6/5}(\Omega) \times L^{4/3}(\Gamma_1^{\theta})$ to the space $(\mathbf{V}(\Omega) \times V^{\varphi}(\Omega) \times V^{\psi}(\Omega) \times V^{\theta}(\Omega))/\mathfrak{R}$.*

Note that, if the areas of the surfaces Γ_0, Γ_0^{φ}, Γ_0^{ψ}, Γ_0^{θ} are positive, then the homogeneous equations (34) to (37) have only a trivial solution. Hence, $\mathfrak{R}_V = \{(\mathbf{0}, 0, 0)\}, \mathfrak{R}_{\theta} = \{0\}$, $\mathfrak{R} = \{(\mathbf{0}, 0, 0, 0)\}$, the factor space $(\mathbf{V}(\Omega) \times V^{\varphi}(\Omega) \times V^{\psi}(\Omega) \times V^{\theta}(\Omega))/\mathfrak{R}$ coincides with $\mathbf{V}(\Omega) \times V^{\varphi}(\Omega) \times V^{\psi}(\Omega) \times V^{\theta}(\Omega)$ and, from Theorem 1, we have the following theorem.

Theorem 2. *Suppose that $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz domain, the parameters c_{ijpq}, ε_{pij}, b_{pij}, d_{ij}, ζ_{ij}, a_{ij}, λ_{ij}, μ_i, m_i, $\eta_{ij} \in L^\infty(\Omega)$ satisfy the symmetry and positive definiteness conditions* (17), (38) *and* (39). *If $\Gamma_0 \neq \varnothing$, $\Gamma_0^\varphi \neq \varnothing$, $\Gamma_0^\psi \neq \varnothing$, $\Gamma_0^\theta \neq \varnothing$ and $\boldsymbol{f} \in \mathbf{L}^{6/5}(\Omega)$, $\boldsymbol{g} \in \mathbf{L}^{4/3}(\Gamma_1)$, $f^\varphi \in L^{6/5}(\Omega)$, $g^\varphi \in L^{4/3}(\Gamma_1^\varphi)$, $g^\psi \in L^{4/3}(\Gamma_1^\psi)$, $f^\theta \in L^{6/5}(\Omega)$, $g^\theta \in L^{4/3}(\Gamma_1^\theta)$, then the problem* (34) *to* (37) *possesses a unique solution $(\boldsymbol{u}, \varphi, \psi, \theta) \in \mathbf{V}(\Omega) \times V^\varphi(\Omega) \times V^\psi(\Omega) \times V^\theta(\Omega)$, and the mapping $(\boldsymbol{f}, \boldsymbol{g}, f^\varphi, g^\varphi, g^\psi, f^\theta, g^\theta) \to (\boldsymbol{u}, \varphi, \psi, \theta)$ is linear and continuous from the space $\mathbf{L}^{6/5}(\Omega) \times \mathbf{L}^{4/3}(\Gamma_1) \times L^{6/5}(\Omega) \times L^{4/3}(\Gamma_1^\varphi) \times L^{4/3}(\Gamma_1^\psi) \times L^{6/5}(\Omega) \times L^{4/3}(\Gamma_1^\theta)$ to the space $\mathbf{V}(\Omega) \times V^\varphi(\Omega) \times V^\psi(\Omega) \times V^\theta(\Omega)$.*

4. Analysis of the dynamic problem

In addition to the notation used in the previous section, we denote, here, by $C^{0,1}(\overline{D})$ the space of the Lipschitz continuous function on $\overline{D}$, where $D \subset \mathbb{R}^n$, $n \in \mathbb{N}$, is a bounded Lipschitz domain. For a Banach space X, we denote by $C([0,T];X)$ the space of continuous vector-functions on $[0,T]$ with values in X; $L^{s_1}(0,T;X)$, $1 \leq s_1 \leq \infty$, is the space of such measurable vector-functions $g : (0,T) \to X$, so that $\|g\|_X \in L^{s_1}(0,T)$ and the generalized derivative of g is denoted by $g' = \mathrm{d}g/\mathrm{d}t \in \mathfrak{D}'(0,T;X)$ [24]. If $g \in L^1(0,T;X)$ and X is a space of functions of variable $x \in D \subset \mathbb{R}^n$, $n \in \mathbb{N}$, we identify g with a function $g(x,t)$, and $g(t)$ denotes the function $g(t) : x \to g(x,t)$, for almost all $t \in (0,T)$. We identify the distributional derivative $\mathrm{d}g/\mathrm{d}t$ with the derivative $\partial g/\partial t$ of g in $\mathfrak{D}'(D \times (0,T))$. Hereafter, we use c_1, c_2, to denote generic constants that are independent of the main parameters involved, but whose values may differ from line to line and may change even within a single chain of estimates.

We identify the unknown vector-function $\boldsymbol{u}$ and the functions φ, ψ, θ with vector-functions defined on $[0,T]$ with values in suitable spaces of functions defined on Ω; by applying equations (22) to (25), we obtain the following variational formulation of the initial boundary value problem (1) to (4) and (8) to (16) in the spaces of vector-valued distributions:

Find $\boldsymbol{u} \in C([0,T];\mathbf{V}(\Omega))$, $\boldsymbol{u}' \in L^\infty(0,T;\mathbf{V}(\Omega))$, $\boldsymbol{u}'' \in L^\infty(0,T;\mathbf{L}^2(\Omega))$, $\varphi \in C([0,T];V^\varphi(\Omega))$, $\varphi' \in L^\infty(0,T;V^\varphi(\Omega))$, $\psi \in C([0,T];V^\psi(\Omega))$, $\psi' \in L^\infty(0,T;V^\psi(\Omega))$, $\theta \in C([0,T];V^\theta(\Omega))$, $\theta' \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;V^\theta(\Omega))$, which satisfy the following equations in the sense of distributions on $(0,T)$:

$$\left(\rho \boldsymbol{u}'', \boldsymbol{v}\right)_{\mathbf{L}^2(\Omega)} + c(\boldsymbol{u},\boldsymbol{v}) + \varepsilon(\varphi,\boldsymbol{v}) + b(\psi,\boldsymbol{v}) - \lambda(\theta,\boldsymbol{v}) = L^{\boldsymbol{u}}(\boldsymbol{v}), \qquad \forall \boldsymbol{v} \in \mathbf{V}(\Omega), \tag{40}$$

$$-\varepsilon\left(\overline{\varphi},\boldsymbol{u}\right) + d\left(\varphi,\overline{\varphi}\right) + a\left(\psi,\overline{\varphi}\right) - \mu\left(\theta,\overline{\varphi}\right) = L^\varphi\left(\overline{\varphi}\right), \qquad \forall \overline{\varphi} \in V^\varphi(\Omega), \tag{41}$$

$$-b\left(\overline{\psi},\boldsymbol{u}\right) + a\left(\varphi,\overline{\psi}\right) + \zeta\left(\psi,\overline{\psi}\right) - m\left(\theta,\overline{\psi}\right) = L^\psi\left(\overline{\psi}\right), \qquad \forall \overline{\psi} \in V^\psi(\Omega), \tag{42}$$

$$\left(\varkappa\theta', \overline{\theta}\right)_{L^2(\Omega)} + \eta\left(\theta,\overline{\theta}\right) + \Theta_0\lambda\left(\overline{\theta},\boldsymbol{u}'\right) - \Theta_0\mu\left(\overline{\theta},\varphi'\right) - \Theta_0 m\left(\overline{\theta},\psi'\right) = L^\theta\left(\overline{\theta}\right), \qquad \forall \overline{\theta} \in V^\theta(\Omega), \tag{43}$$

together with the initial conditions

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \qquad \boldsymbol{u}'(0) = \boldsymbol{u}_1, \qquad \theta(0) = \theta_0, \tag{44}$$

where $\rho = \rho^k$, $\varkappa = \varkappa^k$ in Ω_k.

Note that if the parts Γ_0^φ and Γ_0^ψ of the boundary of Ω are empty sets, then the homogeneous problem (40) to (44) has non-trivial solutions. Indeed, if the tensors $(d_{ij}(x))$, $(a_{ij}(x))$ and $(\zeta_{ij}(x))$ characterizing electric and magnetic fields satisfy the positive definiteness conditions (39) and $\boldsymbol{u} \equiv \boldsymbol{0}$, $\theta \equiv 0$, $f^\varphi \equiv 0$, $g^\varphi \equiv 0$, $g^\psi \equiv 0$, then the solutions φ and ψ are constants. Consequently, the solution of the problem (40) to (44) is not unique in the spaces mentioned in the variational formulation and it is necessary to introduce suitable factor spaces, where the solution of the problem (40) to (44) is unique.

Let us denote by $\mathfrak{R}_\varphi = \{\overline{\varphi}^r \in V^\varphi(\Omega); \overline{\varphi}^r = \alpha_\varphi,\ \alpha_\varphi \in \mathbb{R}\}$ and $\mathfrak{R}_\psi = \{\overline{\psi}^r \in V^\psi(\Omega); \overline{\psi}^r = \alpha_\psi,\ \alpha_\psi \in \mathbb{R}\}$ the sets of solutions φ and ψ of the homogeneous equations (41) and (42), when $\boldsymbol{u} \equiv \boldsymbol{0}$, $\theta \equiv 0$ and $f^\varphi \equiv 0$, $g^\varphi \equiv 0$, $g^\psi \equiv 0$. Applying them, we introduce the factor spaces $V^\varphi_{\mathfrak{R}_\varphi}(\Omega) = V^\varphi(\Omega)/\mathfrak{R}_\varphi$ and $V^\psi_{\mathfrak{R}_\psi}(\Omega) = V^\psi(\Omega)/\mathfrak{R}_\psi$, which consist of equivalence classes $\overline{\varphi}^{\mathfrak{R}_\varphi} = \{\overline{\varphi} + \overline{\varphi}^r; \overline{\varphi}^r \in \mathfrak{R}_\varphi\}$, for each $\overline{\varphi} \in V^\varphi(\Omega)$, and $\overline{\psi}^{\mathfrak{R}_\psi} = \{\overline{\psi} + \overline{\psi}^r; \overline{\psi}^r \in \mathfrak{R}_\psi\}$, for each $\overline{\psi} \in V^\psi(\Omega)$, respectively. The factor spaces $V^\varphi_{\mathfrak{R}_\varphi}(\Omega)$ and $V^\psi_{\mathfrak{R}_\psi}(\Omega)$ are the Hilbert spaces with respect to the norms $\|\overline{\varphi}^{\mathfrak{R}_\varphi}\|_{V^\varphi_{\mathfrak{R}_\varphi}(\Omega)} = \inf\{||\overline{\varphi} + \overline{\varphi}^r||_{H^1(\Omega)}; \overline{\varphi}^r \in \mathfrak{R}_\varphi\}$ and $\|\overline{\psi}^{\mathfrak{R}_\psi}\|_{V^\psi_{\mathfrak{R}_\psi}(\Omega)} = \inf\{||\overline{\psi} + \overline{\psi}^r||_{H^1(\Omega)}; \overline{\psi}^r \in \mathfrak{R}_\psi\}$.

The sets $\mathfrak{R}_\varphi$ and $\mathfrak{R}_\psi$ permit one to define a solution of the problem (40) to (44) in the spaces mentioned in the variational formulation, where $V^\varphi(\Omega)$ is replaced by $V^\varphi_{\mathfrak{R}_\varphi}(\Omega)$ and $V^\psi(\Omega)$ is replaced by $V^\psi_{\mathfrak{R}_\psi}(\Omega)$. Indeed, if $(\boldsymbol{u}, \varphi, \psi, \theta)$ is a solution of the problem (40) to (44), then for any $\overline{\varphi}^r \in \mathfrak{R}_\varphi$ and $\overline{\psi}^r \in \mathfrak{R}_\psi$, the vector-function $(\boldsymbol{u}, \varphi, \psi, \theta)+(\mathbf{0}, \overline{\varphi}^r, \overline{\psi}^r, 0)$ is also a solution of 40 to (44). Therefore, we say that $(\boldsymbol{u}, \varphi^{\mathfrak{R}_\varphi}, \psi^{\mathfrak{R}_\psi}, \theta)$ is a solution of the problem (40) to (44), if any vector-function $(\boldsymbol{u}, \varphi, \psi, \theta)$, where φ and ψ are functions from the equivalence classes $\varphi^{\mathfrak{R}_\varphi}$ and $\psi^{\mathfrak{R}_\psi}$, respectively, is a solution of the problem (40) to (44).

Note that if, in equations (41) and (42), $\overline{\varphi} = \overline{\varphi}^r \in \mathfrak{R}_\varphi$ and $\overline{\psi} = \overline{\psi}^r \in \mathfrak{R}_\psi$, then the left-hand sides vanish; hence, we obtain the following necessary conditions for the existence of a solution of the problem (40) to (44):

$$L^\varphi(\overline{\varphi}^r) = 0, \qquad L^\psi(\overline{\psi}^r) = 0, \qquad \forall \overline{\varphi}^r \in \mathfrak{R}_\varphi, \quad \overline{\psi}^r \in \mathfrak{R}_\psi. \tag{45}$$

Hereafter, we assume that the bilinear forms d, a, ζ, ε, b, μ, m are defined on the factor spaces $V^\varphi_{\mathfrak{R}_\varphi}(\Omega)$ and $V^\psi_{\mathfrak{R}_\psi}(\Omega)$,

$$d\left(\widehat{\varphi}^{\mathfrak{R}_\varphi}, \overline{\varphi}^{\mathfrak{R}_\varphi}\right) = d\left(\widehat{\varphi}, \overline{\varphi}\right), \qquad a\left(\overline{\varphi}^{\mathfrak{R}_\varphi}, \overline{\psi}^{\mathfrak{R}_\psi}\right) = a\left(\overline{\varphi}, \overline{\psi}\right), \qquad \zeta\left(\widehat{\psi}^{\mathfrak{R}_\psi}, \overline{\psi}^{\mathfrak{R}_\psi}\right) = \zeta\left(\widehat{\psi}, \overline{\psi}\right),$$
$$\varepsilon\left(\overline{\varphi}^{\mathfrak{R}_\varphi}, \boldsymbol{v}\right) = \varepsilon\left(\overline{\varphi}, \boldsymbol{v}\right), \qquad b\left(\overline{\psi}^{\mathfrak{R}_\psi}, \boldsymbol{v}\right) = b\left(\overline{\psi}, \boldsymbol{v}\right), \qquad \mu\left(\overline{\theta}, \overline{\varphi}^{\mathfrak{R}_\varphi}\right) = \mu\left(\overline{\theta}, \overline{\varphi}\right), \qquad m\left(\overline{\theta}, \overline{\psi}^{\mathfrak{R}_\psi}\right) = m\left(\overline{\theta}, \overline{\psi}\right),$$

and, taking into account the compatibility conditions (45), we also define the linear forms L^φ and L^ψ on the factor spaces

$$L^\varphi\left(\overline{\varphi}^{\mathfrak{R}_\varphi}\right) = \left(f^\varphi, \overline{\varphi}^{\mathfrak{R}_\varphi}\right)_{L^2(\Omega)} - \left(g^\varphi, \mathrm{tr}_{\Gamma_1^\varphi}\left(\overline{\varphi}^{\mathfrak{R}_\varphi}\right)\right)_{L^2\left(\Gamma_1^\varphi\right)} = \left(f^\varphi, \overline{\varphi}\right)_{L^2(\Omega)} - \left(g^\varphi, \mathrm{tr}_{\Gamma_1^\varphi}\left(\overline{\varphi}\right)\right)_{L^2\left(\Gamma_1^\varphi\right)},$$
$$L^\psi\left(\overline{\psi}^{\mathfrak{R}_\psi}\right) = -\left(g^\psi, \mathrm{tr}_{\Gamma_1^\psi}\left(\overline{\psi}^{\mathfrak{R}_\psi}\right)\right)_{L^2\left(\Gamma_1^\psi\right)} = -\left(g^\psi, \mathrm{tr}_{\Gamma_1^\psi}\left(\overline{\psi}\right)\right)_{L^2\left(\Gamma_1^\psi\right)},$$

where $\widehat{\varphi}, \overline{\varphi} \in V^\varphi(\Omega)$ and $\widehat{\psi}, \overline{\psi} \in V^\psi(\Omega)$ are arbitrary elements of the equivalence classes $\widehat{\varphi}^{\mathfrak{R}_\varphi}, \overline{\varphi}^{\mathfrak{R}_\varphi} \in V^\varphi_{\mathfrak{R}_\varphi}(\Omega)$ and $\widehat{\psi}^{\mathfrak{R}_\psi}, \overline{\psi}^{\mathfrak{R}_\psi} \in V^\psi_{\mathfrak{R}_\psi}(\Omega)$, respectively, and $\boldsymbol{v} \in \mathbf{V}(\Omega)$, $\overline{\theta} \in V^\theta(\Omega)$.

For the problem (40) to (44), which is equivalent to the initial boundary value problem (1) to (4) and (8) to (16) in the spaces of classical twice continuously differentiable functions, we prove the following existence, uniqueness and continuous dependence theorem.

Theorem 3. *Suppose that $\Omega, \Omega_1, \ldots, \Omega_K \subset \mathbb{R}^3$ are bounded Lipschitz domains, the parameters ρ, $\varkappa$, c_{ijpq}, ε_{pij}, b_{pij},d_{ij}, ζ_{ij}, a_{ij}, λ_{ij}, μ_i, $m_i \in L^\infty(\Omega)$, $\eta^k_{ij} \in C^{0,1}(\Omega_k)$, for almost all $x \in \Omega$, satisfy the symmetry conditions (17) and*

$$d_{ij}(x) = d_{ji}(x), \qquad \zeta_{ij}(x) = \zeta_{ji}(x), \qquad a_{ij}(x) = a_{ji}(x), \tag{46}$$

and positive definiteness conditions (38) *and*

$$\rho(x) > \alpha_\rho = \mathrm{const} > 0, \qquad \varkappa(x) > \alpha_\varkappa = \mathrm{const} > 0, \tag{47}$$

$$\boldsymbol{d}(x)\boldsymbol{\xi}\cdot\boldsymbol{\xi} + \boldsymbol{a}(x)\overline{\boldsymbol{\xi}}\cdot\boldsymbol{\xi} + \boldsymbol{a}(x)\boldsymbol{\xi}\cdot\overline{\boldsymbol{\xi}} + \boldsymbol{\zeta}(x)\overline{\boldsymbol{\xi}}\cdot\overline{\boldsymbol{\xi}} + \frac{1}{\Theta_0}\varkappa(x)\xi\xi - 2\xi\boldsymbol{\mu}(x)\cdot\boldsymbol{\xi} - 2\xi\boldsymbol{m}(x)\cdot\overline{\boldsymbol{\xi}} \geq \widetilde{\alpha}(|\boldsymbol{\xi}|^2 + |\overline{\boldsymbol{\xi}}|^2 + \xi^2), \tag{48}$$

for all $\boldsymbol{\xi}$, $\overline{\boldsymbol{\xi}} \in \mathbb{R}^3$, $\xi \in \mathbb{R}$ and for almost all $x \in \Omega$, where $\widetilde{\alpha}$ is a positive constant. If $\boldsymbol{f}$,$\boldsymbol{f}' \in L^2(0,T;\mathbf{L}^2(\Omega))$, $\boldsymbol{g}$,$\boldsymbol{g}'$,$\boldsymbol{g}'' \in L^2(0,T;\mathbf{L}^{4/3}(\Gamma_1))$, $f^\varphi, (f^\varphi)', (f^\varphi)'' \in L^2(0,T;L^{6/5}(\Omega))$, $g^\varphi, (g^\varphi)', (g^\varphi)'' \in L^2(0,T;L^{4/3}(\Gamma_1^\varphi))$, $g^\psi, (g^\psi)', (g^\psi)'' \in L^2(0,T;L^{4/3}(\Gamma_1^\psi))$, $f^\theta, (f^\theta)' \in L^2(0,T;L^2(\Omega))$, $g^\theta, (g^\theta)' \in L^2(0,T;L^{4/3}(\Gamma_1^\theta))$ and the initial conditions $\boldsymbol{u}_0 \in \mathbf{V}(\Omega)$, $\boldsymbol{u}_1 \in \mathbf{V}(\Omega)$, $\theta_0 \in V^\theta(\Omega)$, $\theta_0 = \theta_0^k$ in Ω_k, $\theta_0^k \in H^2(\Omega_k)$, satisfy the compatibility conditions (45) and

$$g^\theta(0) = -\mathrm{tr}_{\Gamma^\theta_{1,k}}\left(\boldsymbol{\eta}^k \mathbf{grad}\ \theta_0^k\right)\cdot\boldsymbol{n}^k \qquad \text{on} \quad \Gamma^\theta_{1,k}, \tag{49}$$

$$\mathrm{tr}_{\Gamma_k\cap\Gamma_{\overline{k}}}\left(\boldsymbol{\eta}^k \mathbf{grad}\ \theta_0^k\right)\cdot\boldsymbol{n} = \mathrm{tr}_{\Gamma_k\cap\Gamma_{\overline{k}}}\left(\boldsymbol{\eta}^{\overline{k}} \mathbf{grad}\ \theta_0^{\overline{k}}\right)\cdot\boldsymbol{n} \qquad \text{on} \quad \Gamma_k \cap \Gamma_{\overline{k}}, \tag{50}$$

and there exist $\boldsymbol{u}_2 \in \mathbf{L}^2(\Omega)$, $\varphi_0 \in V^{\varphi}(\Omega)$, $\psi_0 \in V^{\psi}(\Omega)$, *such that*

$$(\rho \boldsymbol{u}_2, \boldsymbol{v})_{\mathbf{L}^2(\Omega)} + c(\boldsymbol{u}_0, \boldsymbol{v} + \varepsilon(\varphi_0, \boldsymbol{v}) + b(\psi_0, \boldsymbol{v}) - \lambda(\theta_0, \boldsymbol{v}) = (\boldsymbol{f}(0), \boldsymbol{v})_{\mathbf{L}^2(\Omega)} + (\boldsymbol{g}(0), \mathbf{tr}_{\Gamma_1}(\boldsymbol{v}))_{\mathbf{L}^2(\Gamma_1)}, \quad \forall \boldsymbol{v} \in \mathbf{V}(\Omega), \quad (51)$$

$$-\varepsilon(\overline{\varphi}, \boldsymbol{u}_0) + d(\varphi_0, \overline{\varphi}) + a(\psi_0, \overline{\varphi}) - \mu(\theta_0, \overline{\varphi}) = (f^{\varphi}(0), \overline{\varphi})_{L^2(\Omega)} - (g^{\varphi}(0), \mathrm{tr}_{\Gamma_1^{\varphi}}(\overline{\varphi}))_{\Gamma_1^{\varphi}}, \quad \forall \overline{\varphi} \in V^{\varphi}(\Omega), \quad (52)$$

$$-b\left(\overline{\psi}, \boldsymbol{u}_0\right) + a\left(\varphi_0, \overline{\psi}\right) + \zeta\left(\psi_0, \overline{\psi}\right) - m\left(\theta_0, \overline{\psi}\right) = -\left(g^{\psi}(0), \mathrm{tr}_{\Gamma_1^{\psi}}\left(\overline{\psi}\right)\right)_{\Gamma_1^{\psi}}, \quad \forall \overline{\psi} \in V^{\psi}(\Omega), \quad (53)$$

then the problem (40) *to* (44) *possesses a unique solution* $(\boldsymbol{u}, \varphi^{\mathfrak{R}_{\varphi}}, \psi^{\mathfrak{R}_{\psi}}, \theta)$, *where* $\varphi^{\mathfrak{R}_{\varphi}} \in C([0,T]; V^{\varphi}_{\mathfrak{R}_{\varphi}}(\Omega))$, $(\varphi^{\mathfrak{R}_{\varphi}})' \in L^{\infty}(0,T; V^{\varphi}_{\mathfrak{R}_{\varphi}}(\Omega))$, $\psi^{\mathfrak{R}_{\psi}} \in C([0,T]; V^{\psi}_{\mathfrak{R}_{\psi}}(\Omega))$, $(\psi^{\mathfrak{R}_{\psi}})' \in L^{\infty}(0,T; V^{\psi}_{\mathfrak{R}_{\psi}}(\Omega))$, *which continuously depends on the given data, i.e., the mapping*

$$\left(\boldsymbol{u}_0, \boldsymbol{u}_1, \theta_0, \boldsymbol{f}, \boldsymbol{g}, \boldsymbol{g}', f^{\varphi}, (f^{\varphi})', g^{\varphi}, (g^{\varphi})', g^{\psi}, \left(g^{\psi}\right)', f^{\theta}, g^{\theta}\right) \to \left(\boldsymbol{u}, \boldsymbol{u}', \varphi^{\mathfrak{R}_{\varphi}}, \psi^{\mathfrak{R}_{\psi}}, \theta\right)$$

is linear and continuous from space

$$\begin{aligned}\mathbf{V}(\Omega) \times \mathbf{L}^2(\Omega) \times L^2(\Omega) \times L^2(0,T;\mathbf{L}^2(\Omega)) \times L^2(0,T;\mathbf{L}^{4/3}(\Gamma_1)) \times L^2(0,T;\mathbf{L}^{4/3}(\Gamma_1)) \\ \times L^2(0,T;L^{6/5}(\Omega)) \times L^2(0,T;L^{6/5}(\Omega)) \times L^2(0,T;L^{4/3}(\Gamma_1^{\varphi})) \times L^2(0,T;L^{4/3}(\Gamma_1^{\varphi})) \\ \times L^2(0,T;L^{4/3}(\Gamma_1^{\psi})) \times L^2(0,T;L^{4/3}(\Gamma_1^{\psi})) \times L^2(0,T;L^2(\Omega)) \times L^2(0,T;L^{4/3}(\Gamma_1^{\theta}))\end{aligned}$$

to space

$$\begin{aligned}C([0,T];\mathbf{V}(\Omega)) \times C([0,T];\mathbf{L}^2(\Omega)) \times C([0,T];V^{\varphi}_{\mathfrak{R}_{\varphi}}(\Omega)) \times C([0,T];V^{\psi}_{\mathfrak{R}_{\psi}}(\Omega)) \\ \times \left(C([0,T];L^2(\Omega)) \cap L^2(0,T;V^{\theta}(\Omega))\right),\end{aligned}$$

and the following energy equality is valid:

$$E(t) = E(0) + L(t), \qquad \forall t \in [0,T], \tag{54}$$

where

$$\begin{aligned}E(t) &= \left(\rho \boldsymbol{u}'(t), \boldsymbol{u}'(t)\right)_{\mathbf{L}^2(\Omega)} + c(\boldsymbol{u}(t), \boldsymbol{u}(t)) + \frac{1}{\Theta_0}(\varkappa\theta(t), \theta(t))_{L^2(\Omega)} + \frac{2}{\Theta_0}\int_0^t \eta(\theta,\theta)\mathrm{d}\tau + d(\varphi(t), \varphi(t)) \\ &\quad + 2a(\varphi(t), \psi(t)) + \zeta(\psi(t), \psi(t)) - 2\mu(\theta(t), \varphi(t)) - 2m(\theta(t), \psi(t)), \\ L(t) &= 2\int_0^t \left(\boldsymbol{f}(\tau), \boldsymbol{u}'(\tau)\right)_{\mathbf{L}^2(\Omega)} \mathrm{d}\tau + 2(\boldsymbol{g}(t), \mathbf{tr}_{\Gamma_1}(\boldsymbol{u}(t)))_{\mathbf{L}^2(\Gamma_1)} - 2(\boldsymbol{g}(0), \mathbf{tr}_{\Gamma_1}(\boldsymbol{u}(0)))_{\mathbf{L}^2(\Gamma_1)} \\ &\quad - 2\int_0^t \left(\boldsymbol{g}'(\tau), \mathbf{tr}_{\Gamma_1}(\boldsymbol{u}(\tau))\right)_{\mathbf{L}^2(\Gamma_1)} \mathrm{d}\tau + 2\int_0^t ((f^{\varphi})'(\tau), \varphi(\tau))_{L^2(\Omega)}\mathrm{d}\tau - 2\int_0^t ((g^{\varphi})'(\tau), \mathrm{tr}_{\Gamma_1^{\varphi}}(\varphi(\tau)))_{L^2(\Gamma_1^{\varphi})}\mathrm{d}\tau \\ &\quad - 2\int_0^t ((g^{\psi})'(\tau), \mathrm{tr}_{\Gamma_1^{\psi}}(\psi(\tau)))_{L^2(\Gamma_1^{\psi})}\mathrm{d}\tau + \frac{2}{\Theta_0}\int_0^t \left(f^{\theta}(\tau), \theta(\tau)\right)_{L^2(\Omega)} \mathrm{d}\tau - \frac{2}{\Theta_0}\int_0^t (g^{\theta}(\tau), \mathrm{tr}_{\Gamma_1^{\theta}}(\theta(\tau)))_{L^2(\Gamma_1^{\theta})}\mathrm{d}\tau,\end{aligned}$$

φ and ψ are any functions from the equivalence classes $\varphi^{\mathfrak{R}_{\varphi}}$ and $\psi^{\mathfrak{R}_{\psi}}$, respectively.

Proof. First, let us prove the existence of a solution of the problem (40) to (44). Since $f^{\theta}, f^{\theta\prime} \in L^2(0,T;L^2(\Omega))$ and $g^{\theta}, g^{\theta\prime} \in L^2(0,T;L^{4/3}(\Gamma_1^{\theta}))$, it follows, from the embedding theorem [24], that $f^{\theta} \in C([0,T];L^2(\Omega))$ and

$g^\theta \in C([0,T];L^{4/3}(\Gamma_1^\theta))$. By applying Green's formula and by taking into account the compatibility conditions (49) and (50), we obtain:

$$\eta(\theta_0,\overline{\theta})+\left(g^\theta(0),\overline{\theta}\right)_{L^2(\Gamma_1^\theta)}=\sum_{k=1}^{K}\int\limits_{\Gamma_k}\mathbf{tr}_{\Gamma_k}\left(\boldsymbol{\eta}^k\mathbf{grad}\,\theta^k\right)\cdot\boldsymbol{n}^k\mathrm{tr}_{\Gamma_k}\left(\overline{\theta}^k\right)\mathrm{d}\Gamma_k-\sum_{k=1}^{K}\int\limits_{\Omega_k}\mathrm{div}\left(\boldsymbol{\eta}^k\mathbf{grad}\;\theta_0^k\right)\overline{\theta}^k\mathrm{d}x$$
$$+\sum_{k=1}^{K}\int\limits_{\Gamma_{1,k}^\theta}g^\theta(0)\mathrm{tr}_{\Gamma_k}\left(\overline{\theta}^k\right)\mathrm{d}\Gamma_k=-\left(\widetilde{\theta_0},\overline{\theta}\right)_{L^2(\Omega)},\qquad\forall\overline{\theta}\in V^\theta(\Omega),$$

where $\widetilde{\theta_0}=\mathrm{div}\left(\boldsymbol{\eta}^k\mathbf{grad}\;\theta_0^k\right)$ in Ω_k. Since the functions η_{ij}^k are Lipschitz continuous in bounded domains Ω_k, we have that $\eta_{ij}^k\in L^\infty(\Omega_k)$, and it follows from Rademacher's theorem [25] that they are differentiable almost everywhere in Ω_k and that their derivatives belong to $L^\infty(\Omega_k)$. Hence, by taking into account the fact that $\theta_0^k\in H^2(\Omega_k)$, we have $\widetilde{\theta_0}\in L^2(\Omega)$.

Since f^φ, $(f^\varphi)'$, $(f^\varphi)''\in L^2(0,T;L^{6/5}(\Omega))$, g^φ, $(g^\varphi)'$, $(g^\varphi)''\in L^2(0,T;L^{4/3}(\Gamma_1^\varphi))$, and $g^\psi,(g^\psi)',(g^\psi)''\in L^2(0,T;L^{4/3}(\Gamma_1^\psi))$, it follows from the embedding theorem [24] that $f^\varphi,(f^\varphi)'\in C([0,T];L^{6/5}(\Omega))$, $g^\varphi,(g^\varphi)'\in C([0,T];L^{4/3}(\Gamma_1^\varphi))$ and $g^\psi,(g^\psi)'\in C([0,T];L^{4/3}(\Gamma_1^\psi))$, and

$$||w||_{C([0,T];L^{6/5}(\Omega))}\le c_1\left(||w||_{L^2(0,T;L^{6/5}(\Omega))}+||w'||_{L^2(0,T;L^{6/5}(\Omega))}\right),\tag{55}$$
$$||\widehat{w}||_{C([0,T];L^{4/3}(\widehat{\Gamma}))}\le c_2\left(||\widehat{w}||_{L^2(0,T;L^{4/3}(\widehat{\Gamma}))}+||\widehat{w}'||_{L^2(0,T;L^{4/3}(\widehat{\Gamma}))}\right),\tag{56}$$

where $w=f^\varphi$ or $w=(f^\varphi)'$, $\widehat{\Gamma}=\Gamma_1^\varphi$ when $\widehat{w}=g^\varphi$ or $\widehat{w}=(g^\varphi)'$ and $\widehat{\Gamma}=\Gamma_1^\psi$ when $\widehat{w}=g^\psi$ or $\widehat{w}=(g^\psi)'$.

Therefore, we can consider the following problem:

Find $\theta_1\in L^2(\Omega)$, $\varphi_1^{\mathfrak{R}_\varphi}\in V_{\mathfrak{R}_\varphi}^\varphi(\Omega)$, $\psi_1^{\mathfrak{R}_\psi}\in V_{\mathfrak{R}_\psi}^\psi(\Omega)$ such that

$$-\varepsilon\left(\overline{\varphi}^{\mathfrak{R}_\varphi},\boldsymbol{u}_1\right)+d\left(\varphi_1^{\mathfrak{R}_\varphi},\overline{\varphi}^{\mathfrak{R}_\varphi}\right)+a\left(\psi_1^{\mathfrak{R}_\psi},\overline{\varphi}^{\mathfrak{R}_\varphi}\right)-\mu\left(\theta_1,\overline{\varphi}^{\mathfrak{R}_\varphi}\right)=\left((f^\varphi)'(0),\overline{\varphi}^{\mathfrak{R}_\varphi}\right)_{L^2(\Omega)}$$
$$-\left((g^\varphi)'(0),\mathrm{tr}_{\Gamma_1^\varphi}\left(\overline{\varphi}^{\mathfrak{R}_\varphi}\right)\right)_{L^2(\Gamma_1^\varphi)},\qquad\forall\overline{\varphi}^{\mathfrak{R}_\varphi}\in V_{\mathfrak{R}_\varphi}^\varphi(\Omega),\tag{57}$$

$$-b\left(\overline{\psi}^{\mathfrak{R}_\psi},\boldsymbol{u}_1\right)+a\left(\varphi_1^{\mathfrak{R}_\varphi},\overline{\psi}^{\mathfrak{R}_\psi}\right)+\zeta\left(\psi_1^{\mathfrak{R}_\psi},\overline{\psi}^{\mathfrak{R}_\psi}\right)-m\left(\theta_1,\overline{\psi}^{\mathfrak{R}_\psi}\right)=-\left((g^\psi)'(0),\mathrm{tr}_{\Gamma_1^\psi}\left(\overline{\psi}^{\mathfrak{R}_\psi}\right)\right)_{L^2(\Gamma_1^\psi)},$$
$$\forall\overline{\psi}^{\mathfrak{R}_\psi}\in V_{\mathfrak{R}_\psi}^\psi(\Omega),\tag{58}$$

$$(\varkappa\theta_1,\overline{\theta})_{L^2(\Omega)}+\Theta_0\lambda(\overline{\theta},\boldsymbol{u}_1)-\Theta_0\mu\left(\overline{\theta},\varphi_1^{\mathfrak{R}_\varphi}\right)-\Theta_0 m\left(\overline{\theta},\psi_1^{\mathfrak{R}_\psi}\right)=\left(f^\theta(0),\overline{\theta}\right)_{L^2(\Omega)}+\left(\widetilde{\theta_0},\overline{\theta}\right)_{L^2(\Omega)},$$
$$\forall\overline{\theta}\in V^\theta(\Omega).\tag{59}$$

Since $\overline{\varphi}^{\mathfrak{R}_\varphi}$, $\overline{\psi}^{\mathfrak{R}_\psi}$ and $\overline{\theta}$ are independent of each other, the problem (57) to (59) is equivalent to the following one:

Find

$$\left(\varphi_1^{\mathfrak{R}_\varphi},\psi_1^{\mathfrak{R}_\psi},\theta_1\right)\in W(\Omega)=V_{\mathfrak{R}_\varphi}^\varphi(\Omega)\times V_{\mathfrak{R}_\psi}^\psi(\Omega)\times L^2(\Omega)$$

such that

$$B\left(\left(\varphi_1^{\mathfrak{R}_\varphi},\psi_1^{\mathfrak{R}_\psi},\theta_1\right),\left(\overline{\varphi}^{\mathfrak{R}_\varphi},\overline{\psi}^{\mathfrak{R}_\psi},\overline{\theta}\right)\right)=F\left(\overline{\varphi}^{\mathfrak{R}_\varphi},\overline{\psi}^{\mathfrak{R}_\psi},\overline{\theta}\right),\qquad\forall\left(\overline{\varphi}^{\mathfrak{R}_\varphi},\overline{\psi}^{\mathfrak{R}_\psi},\overline{\theta}\right)\in W(\Omega),\tag{60}$$

where

$$
\begin{aligned}
B\left(\left(\varphi_1^{\mathfrak{R}_\varphi},\psi_1^{\mathfrak{R}_\psi},\theta_1\right),\left(\overline{\varphi}^{\mathfrak{R}_\varphi},\overline{\psi}^{\mathfrak{R}_\psi},\overline{\theta}\right)\right) &= d\left(\varphi_1^{\mathfrak{R}_\varphi},\overline{\varphi}^{\mathfrak{R}_\varphi}\right)+a\left(\psi_1^{\mathfrak{R}_\psi},\overline{\varphi}^{\mathfrak{R}_\varphi}\right)-\mu\left(\theta_1,\overline{\varphi}^{\mathfrak{R}_\varphi}\right)+a\left(\varphi_1^{\mathfrak{R}_\varphi},\overline{\psi}^{\mathfrak{R}_\psi}\right)\\
&\quad+\zeta\left(\psi_1^{\mathfrak{R}_\psi},\overline{\psi}^{\mathfrak{R}_\psi}\right)-m\left(\theta_1,\overline{\psi}^{\mathfrak{R}_\psi}\right)+\frac{1}{\Theta_0}\left(\varkappa\theta_1,\overline{\theta}\right)_{L^2(\Omega)}-\mu\left(\overline{\theta},\varphi_1^{\mathfrak{R}_\varphi}\right)\\
&\quad-m\left(\overline{\theta},\psi_1^{\mathfrak{R}_\psi}\right),\forall\left(\varphi_1^{\mathfrak{R}_\varphi},\psi_1^{\mathfrak{R}_\psi},\theta_1\right),\left(\overline{\varphi}^{\mathfrak{R}_\varphi},\overline{\psi}^{\mathfrak{R}_\psi},\overline{\theta}\right)\in W(\Omega),\\
F\left(\overline{\varphi}^{\mathfrak{R}_\varphi},\overline{\psi}^{\mathfrak{R}_\psi},\overline{\theta}\right) &= \varepsilon\left(\overline{\varphi}^{\mathfrak{R}_\varphi},\boldsymbol{u}_1\right)+\left(f^{\varphi}\right)'(0),\overline{\varphi}^{\mathfrak{R}_\varphi}\Big)_{L^2(\Omega)}-\left(\left(g^{\varphi}\right)'(0),\mathrm{tr}_{\Gamma_1^\varphi}\left(\overline{\varphi}^{\mathfrak{R}_\varphi}\right)\right)_{L^2(\Gamma_1^\varphi)}\\
&\quad+b\left(\overline{\psi}^{\mathfrak{R}_\psi},\boldsymbol{u}_1\right)-\left(\left(g^{\psi}\right)'(0),\mathrm{tr}_{\Gamma_1^\psi}\left(\overline{\psi}^{\mathfrak{R}_\psi}\right)\right)_{L^2(\Gamma_1^\psi)}-\lambda(\overline{\theta},\boldsymbol{u}_1)\\
&\quad+\frac{1}{\Theta_0}\left(f^{\theta}(0),\overline{\theta}\right)_{L^2(\Omega)}+\frac{1}{\Theta_0}\left(\widetilde{\theta}_0,\overline{\theta}\right)_{L^2(\Omega)},\forall\left(\overline{\varphi}^{\mathfrak{R}_\varphi},\overline{\psi}^{\mathfrak{R}_\psi},\overline{\theta}\right)\in W(\Omega).
\end{aligned}
$$

Since $d_{ij}, a_{ij}, \zeta_{ij}, \mu_i, m_i, \varepsilon_{pij}, b_{pij}, \lambda_{ij}, \varkappa \in L^\infty(\Omega),\ i,j,p,q=1,2,3$, we obtain, by applying the Cauchy–Schwarz inequality, the following estimates:

$$
\begin{aligned}
\left|d\left(\varphi_1^{\mathfrak{R}_\varphi},\overline{\varphi}^{\mathfrak{R}_\varphi}\right)\right| &\le \int_\Omega \left|(\boldsymbol{d}\,\mathbf{grad}\,\varphi_1)\cdot\mathbf{grad}\,\overline{\varphi}\right|\mathrm{d}x\\
&\le \max_{1\le i,j\le 3}\left\|d_{ij}\right\|_{L^\infty(\Omega)}\left\|\sum_{j=1}^{3}\left|\frac{\partial\varphi_1}{\partial x_j}\right|\right\|_{L^2(\Omega)}\left\|\sum_{i=1}^{3}\left|\frac{\partial\overline{\varphi}}{\partial x_i}\right|\right\|_{L^2(\Omega)}\\
&\le 3\max_{1\le i,j\le 3}\left\|d_{ij}\right\|_{L^\infty(\Omega)}\left\|\varphi_1^{\mathfrak{R}_\varphi}\right\|_{V^\varphi_{\mathfrak{R}_\varphi}(\Omega)}\left\|\overline{\varphi}^{\mathfrak{R}_\varphi}\right\|_{V^\varphi_{\mathfrak{R}_\varphi}(\Omega)}, \qquad (61)\\
\left|a\left(\varphi_1^{\mathfrak{R}_\varphi},\overline{\psi}^{\mathfrak{R}_\psi}\right)\right| &\le 3\max_{1\le i,j\le 3}\left\|a_{ij}\right\|_{L^\infty(\Omega)}\left\|\varphi_1^{\mathfrak{R}_\varphi}\right\|_{V^\varphi_{\mathfrak{R}_\varphi}(\Omega)}\left\|\overline{\psi}^{\mathfrak{R}_\psi}\right\|_{V^\psi_{\mathfrak{R}_\psi}(\Omega)}, \qquad (62)\\
\left|\zeta\left(\psi_1^{\mathfrak{R}_\psi},\overline{\psi}^{\mathfrak{R}_\psi}\right)\right| &\le 3\max_{1\le i,j\le 3}\left\|\zeta_{ij}\right\|_{L^\infty(\Omega)}\left\|\psi_1^{\mathfrak{R}_\psi}\right\|_{V^\psi_{\mathfrak{R}_\psi}(\Omega)}\left\|\overline{\psi}^{\mathfrak{R}_\psi}\right\|_{V^\psi_{\mathfrak{R}_\psi}(\Omega)}, \qquad (63)\\
\left|\mu\left(\overline{\theta},\varphi_1^{\mathfrak{R}_\varphi}\right)\right| &\le \int_\Omega\left|(\boldsymbol{\mu}\cdot\mathbf{grad}\,\varphi_1)\,\overline{\theta}\right|\mathrm{d}x\\
&\le \sqrt{3}\max_{1\le i\le 3}\|\mu_i\|_{L^\infty(\Omega)}\left\|\overline{\theta}\right\|_{L^2(\Omega)}\left\|\varphi_1^{\mathfrak{R}_\varphi}\right\|_{V^\varphi_{\mathfrak{R}_\varphi}(\Omega)}, \qquad (64)\\
\left|m\left(\overline{\theta},\psi_1^{\mathfrak{R}_\psi}\right)\right| &\le \int_\Omega\left|(\boldsymbol{m}\cdot\mathbf{grad}\,\psi_1)\,\overline{\theta}\right|\mathrm{d}x\\
&\le \sqrt{3}\max_{1\le i\le 3}\|m_i\|_{L^\infty(\Omega)}\left\|\overline{\theta}\right\|_{L^2(\Omega)}\left\|\psi_1^{\mathfrak{R}_\psi}\right\|_{V^\psi_{\mathfrak{R}_\psi}(\Omega)}, \qquad (65)\\
\left|\varepsilon\left(\overline{\varphi}^{\mathfrak{R}_\varphi},\boldsymbol{v}\right)\right| &\le \int_\Omega\left|(\boldsymbol{\varepsilon}*\mathbf{grad}\,\overline{\varphi}):\boldsymbol{e}(\boldsymbol{v})\right|\mathrm{d}x\\
&\le \max_{1\le p,i,j\le 3}\left\|\varepsilon_{pij}\right\|_{L^\infty(\Omega)}\left\|\sum_{p=1}^{3}\left|\frac{\partial\overline{\varphi}}{\partial x_p}\right|\right\|_{L^2(\Omega)}\left\|\sum_{i,j=1}^{3}\left|e_{ij}(\boldsymbol{v})\right|\right\|_{L^2(\Omega)}\\
&\le 3\sqrt{3}\max_{1\le p,i,j\le 3}\left\|\varepsilon_{pij}\right\|_{L^\infty(\Omega)}\left\|\overline{\varphi}^{\mathfrak{R}_\varphi}\right\|_{V^\varphi_{\mathfrak{R}_\varphi}(\Omega)}\|\boldsymbol{v}\|_{\mathbf{H}^1(\Omega)}, \qquad (66)
\end{aligned}
$$

$$\left|b\left(\overline{\psi}^{\mathfrak{R}_\psi},\boldsymbol{v}\right)\right| \leq 3\sqrt{3}\max_{1\leq p,i,j\leq 3}\left\|b_{pij}\right\|_{L^\infty(\Omega)}\left\|\overline{\psi}^{\mathfrak{R}_\psi}\right\|_{V^\psi_{\mathfrak{R}_\psi}(\Omega)}\|\boldsymbol{v}\|_{\mathbf{H}^1(\Omega)}, \tag{67}$$

$$\begin{aligned}\left|\lambda\left(\overline{\theta},\boldsymbol{v}\right)\right| &\leq \int_\Omega \left|\boldsymbol{\lambda}:\boldsymbol{e}(\boldsymbol{v})\overline{\theta}\right|\mathrm{d}x\\ &\leq 3\max_{1\leq i,j\leq 3}\left\|\lambda_{ij}\right\|_{L^\infty(\Omega)}\left\|\overline{\theta}\right\|_{L^2(\Omega)}\|\boldsymbol{v}\|_{\mathbf{H}^1(\Omega)},\end{aligned} \tag{68}$$

for all $\varphi_1^{\mathfrak{R}_\varphi}$, $\overline{\varphi}^{\mathfrak{R}_\varphi} \in V^\varphi_{\mathfrak{R}_\varphi}(\Omega)$, $\psi_1^{\mathfrak{R}_\psi}$, $\overline{\psi}^{\mathfrak{R}_\psi} \in V^\psi_{\mathfrak{R}_\psi}(\Omega)$, $\overline{\theta} \in L^2(\Omega)$, $\boldsymbol{v} \in \mathbf{H}^1(\Omega)$, where φ_1, $\overline{\varphi}$, ψ_1 are arbitrary elements of the equivalence classes $\varphi_1^{\mathfrak{R}_\varphi}$, $\overline{\varphi}^{\mathfrak{R}_\varphi}$, $\psi_1^{\mathfrak{R}_\psi}$, respectively. Hence, from estimates (61) to (65), we have that the bilinear form $B : W(\Omega)\times W(\Omega)\to\mathbb{R}$ is continuous.

Because $(f^\varphi)'(0)\in L^{6/5}(\Omega)$, $(g^\varphi)'(0)\in L^{4/3}(\Gamma_1^\varphi)$ and $(g^\psi)'(0)\in L^{4/3}(\Gamma_1^\psi)$, and by applying Hölder's inequality, the compatibility conditions (45), estimates (55), (56) for $w=(f^\varphi)'$, $\widehat{w}=(g^\varphi)'$ and $\widehat{w}=(g^\psi)'$, and continuity of the embedding $H^1(\Omega)\hookrightarrow L^6(\Omega)$ and the trace operator $\mathrm{tr}: H^1(\Omega)\to L^4(\Gamma)$ [26], we have:

$$\begin{aligned}&\left|\left((f^\varphi)'(0),\overline{\varphi}^{\mathfrak{R}_\varphi}\right)_{L^2(\Omega)}-\left((g^\varphi)'(0),\mathrm{tr}_{\Gamma_1^\varphi}\left(\overline{\varphi}^{\mathfrak{R}_\varphi}\right)\right)_{L^2(\Gamma_1^\varphi)}-\left((g^\psi)'(0),\mathrm{tr}_{\Gamma_1^\psi}\left(\overline{\psi}^{\mathfrak{R}_\psi}\right)\right)_{L^2(\Gamma_1^\psi)}\right|\\
&\quad\leq \inf_{\overline{\varphi}^r\in\mathfrak{R}_\varphi}\left|((f^\varphi)'(0),\overline{\varphi}+\overline{\varphi}^r)_{L^2(\Omega)}\right|+\inf_{\overline{\varphi}^r\in\mathfrak{R}_\varphi}\left|((g^\varphi)'(0),\mathrm{tr}_{\Gamma_1^\varphi}(\overline{\varphi}+\overline{\varphi}^r))_{L^2(\Gamma_1^\varphi)}\right|\\
&\qquad+\inf_{\overline{\psi}^r\in\mathfrak{R}_\psi}\left|((g^\psi)'(0),\mathrm{tr}_{\Gamma_1^\psi}(\overline{\psi}+\overline{\psi}^r))_{L^2(\Gamma_1^\psi)}\right|\\
&\quad\leq ||(f^\varphi)'(0)||_{L^{6/5}(\Omega)}\inf_{\overline{\varphi}^r\in\mathfrak{R}_\varphi}||\overline{\varphi}+\overline{\varphi}^r||_{L^6(\Omega)}+||(g^\varphi)'(0)||_{L^{4/3}(\Gamma_1^\varphi)}\inf_{\overline{\varphi}^r\in\mathfrak{R}_\varphi}||\mathrm{tr}_{\Gamma_1^\varphi}(\overline{\varphi}+\overline{\varphi}^r)||_{L^4(\Gamma_1^\varphi)}\\
&\qquad+||(g^\psi)'(0)||_{L^{4/3}(\Gamma_1^\psi)}\inf_{\overline{\psi}^r\in\mathfrak{R}_\psi}\left\|\mathrm{tr}_{\Gamma_1^\psi}\left(\overline{\psi}+\overline{\psi}^r\right)\right\|_{L^4(\Gamma_1^\psi)}\\
&\quad\leq c_1\left(\inf_{\overline{\varphi}^r\in\mathfrak{R}_\varphi}||\overline{\varphi}+\overline{\varphi}^r||_{H^1(\Omega)}+\inf_{\overline{\psi}^r\in\mathfrak{R}_\psi}\left\|\overline{\psi}+\overline{\psi}^r\right\|_{H^1(\Omega)}\right)\\
&\quad= c_1\left(\left\|\overline{\varphi}^{\mathfrak{R}_\varphi}\right\|_{V^\varphi_{\mathfrak{R}_\varphi}(\Omega)}+\left\|\overline{\psi}^{\mathfrak{R}_\psi}\right\|_{V^\psi_{\mathfrak{R}_\psi}(\Omega)}\right),\end{aligned}$$

where $\overline{\varphi}\in V^\varphi(\Omega)$ and $\overline{\psi}\in V^\psi(\Omega)$ are arbitrary elements of the equivalence classes $\overline{\varphi}^{\mathfrak{R}_\varphi}\in V^\varphi_{\mathfrak{R}_\varphi}(\Omega)$ and $\overline{\psi}^{\mathfrak{R}_\psi}\in V^\psi_{\mathfrak{R}_\psi}(\Omega)$, respectively. Therefore, since $f^\theta(0)\in L^2(\Omega)$, $\widetilde{\theta}_0\in L^2(\Omega)$, and taking into account estimates (66) to (68) and $\boldsymbol{u}_1\in\mathbf{V}(\Omega)$, we infer that the linear form $F: W(\Omega)\to\mathbb{R}$ is continuous.

From the positive definiteness conditions (47) and (48), we obtain:

$$\begin{aligned}B\left(\left(\overline{\varphi}^{\mathfrak{R}_\varphi},\overline{\psi}^{\mathfrak{R}_\psi},\overline{\theta}\right),\left(\overline{\varphi}^{\mathfrak{R}_\varphi},\overline{\psi}^{\mathfrak{R}_\psi},\overline{\theta}\right)\right) &= d(\overline{\varphi},\overline{\varphi})+a\left(\overline{\psi},\overline{\varphi}\right)+a\left(\overline{\varphi},\overline{\psi}\right)+\zeta\left(\overline{\psi},\overline{\psi}\right)+\frac{1}{\Theta_0}\left(\varkappa\overline{\theta},\overline{\theta}\right)_{L^2(\Omega)}\\
&\quad-2\mu\left(\overline{\theta},\overline{\varphi}\right)-2m\left(\overline{\theta},\overline{\psi}\right)\\
&\geq \min\left\{\widetilde{\alpha},\frac{\alpha_\varkappa}{\Theta_0}\right\}\int_\Omega\left(|\mathbf{grad}\,\overline{\varphi}|^2+|\mathbf{grad}\,\overline{\psi}|^2+\left(\overline{\theta}\right)^2\right)\mathrm{d}x,\end{aligned} \tag{69}$$

where $\overline{\varphi}\in V^\varphi(\Omega)$ and $\overline{\psi}\in V^\psi(\Omega)$ are arbitrary elements of the equivalence classes $\overline{\varphi}^{\mathfrak{R}_\varphi}\in V^\varphi_{\mathfrak{R}_\varphi}(\Omega)$ and $\overline{\psi}^{\mathfrak{R}_\psi}\in V^\psi_{\mathfrak{R}_\psi}(\Omega)$.

Applying the generalized Poincaré inequality [26], we have:

$$\int_\Omega|\mathbf{grad}\,v|^2\mathrm{d}x\geq c_1\left(\int_\Omega v^2\mathrm{d}x-\left|\int_\Omega v\mathrm{d}x\right|^2\right)=c_1\inf_{c_2\in\mathbb{R}}\|v+c_2\|^2_{L^2(\Omega)},$$

for all $v \in H^1(\Omega)$. Consequently, from (69), we infer that the bilinear form $B : W(\Omega) \times W(\Omega) \to \mathbb{R}$ satisfies the following inequality:

$$B\left(\left(\overline{\varphi}^{\mathfrak{R}_\varphi}, \overline{\psi}^{\mathfrak{R}_\psi}, \overline{\theta}\right), \left(\overline{\varphi}^{\mathfrak{R}_\varphi}, \overline{\psi}^{\mathfrak{R}_\psi}, \overline{\theta}\right)\right) \geq c_1 \left\|\left(\overline{\varphi}^{\mathfrak{R}_\varphi}, \overline{\psi}^{\mathfrak{R}_\psi}, \overline{\theta}\right)\right\|^2_{W(\Omega)}, \tag{70}$$

for all $(\overline{\varphi}^{\mathfrak{R}_\varphi}, \overline{\psi}^{\mathfrak{R}_\psi}, \overline{\theta}) \in W(\Omega)$. Thus, the bilinear form B is continuous and bounded below, and the linear form F is continuous, and from the Lax–Milgram theorem [23], we have that the problem (57) to (59) possesses a unique solution and

$$\left\|\left(\varphi_1^{\mathfrak{R}_\varphi}, \psi_1^{\mathfrak{R}_\psi}, \theta_1\right)\right\|_{W(\Omega)} \leq c_1 \left(||(f^\varphi)'(0)||_{L^{6/5}(\Omega)} + ||(g^\varphi)'(0)||_{L^{4/3}(\Gamma_1^\varphi)} + ||(g^\psi)'(0)||_{L^{4/3}(\Gamma_1^\psi)}\right).$$

To prove the existence of the solution, let us consider complete systems of linearly independent elements $\{\boldsymbol{v}_k\}_{k=1}^\infty$ in the space $\mathbf{V}(\Omega)$, $\{\overline{\varphi}_k^{\mathfrak{R}_\varphi}\}_{k=1}^\infty$ in the space $V^\varphi_{\mathfrak{R}_\varphi}(\Omega)$, $\{\overline{\psi}_k^{\mathfrak{R}_\psi}\}_{k=1}^\infty$ in the space $V^\psi_{\mathfrak{R}_\psi}(\Omega)$ and $\{\overline{\theta}_k\}_{k=1}^\infty$ in the space $V^\theta(\Omega)$. For each $n \in \mathbb{N}$, we denote by $\mathbf{V}_n(\Omega)$ the linear subspace of $\mathbf{V}(\Omega)$ defined by $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \ldots, \boldsymbol{v}_n\}$, by $V^\varphi_{\mathfrak{R}_\varphi n}(\Omega)$ the linear subspace of $V^\varphi_{\mathfrak{R}_\varphi}(\Omega)$ defined by $\{\overline{\varphi}_1^{\mathfrak{R}_\varphi}, \overline{\varphi}_2^{\mathfrak{R}_\varphi}, \overline{\varphi}_3^{\mathfrak{R}_\varphi}, \ldots, \overline{\varphi}_n^{\mathfrak{R}_\varphi}\}$, by $V^\psi_{\mathfrak{R}_\psi n}(\Omega)$ the linear subspace of $V^\psi_{\mathfrak{R}_\psi}(\Omega)$ defined by $\{\overline{\psi}_1^{\mathfrak{R}_\psi}, \overline{\psi}_2^{\mathfrak{R}_\psi}, \overline{\psi}_3^{\mathfrak{R}_\psi}, \ldots, \overline{\psi}_n^{\mathfrak{R}_\psi}\}$ and by $V_n^\theta(\Omega)$ the linear subspace of $V^\theta(\Omega)$ defined by $\{\overline{\theta}_1, \overline{\theta}_2, \overline{\theta}_3, \ldots, \overline{\theta}_n\}$. We consider a sequence of approximate solutions of the problem (40) to (44) given by

$$\widetilde{\boldsymbol{u}}_n = \sum_{k=1}^n u_{nk}(t)\boldsymbol{v}_k, \qquad \widetilde{\varphi}_n^{\mathfrak{R}_\varphi} = \sum_{k=1}^n \varphi_{nk}(t)\overline{\varphi}_k^{\mathfrak{R}_\varphi}, \qquad \widetilde{\psi}_n^{\mathfrak{R}_\psi} = \sum_{k=1}^n \psi_{nk}(t)\overline{\psi}_k^{\mathfrak{R}_\psi}, \qquad \widetilde{\theta}_n = \sum_{k=1}^n \theta_{nk}(t)\overline{\theta}_k,$$

which are solutions of the following problem:

Find $\widetilde{\boldsymbol{u}}_n, \widetilde{\boldsymbol{u}}_n', \widetilde{\boldsymbol{u}}_n'' \in C([0,T]; \mathbf{V}_n(\Omega))$, $\widetilde{\boldsymbol{u}}_n''' \in L^2(0,T; \mathbf{V}_n(\Omega))$, $\widetilde{\varphi}_n^{\mathfrak{R}_\varphi}, (\widetilde{\varphi}_n^{\mathfrak{R}_\varphi})' \in C([0,T]; V^\varphi_{\mathfrak{R}_\varphi n}(\Omega))$, $(\widetilde{\varphi}_n^{\mathfrak{R}_\varphi})'' \in L^2(0,T; V^\varphi_{\mathfrak{R}_\varphi n}(\Omega))$, $\widetilde{\psi}_n^{\mathfrak{R}_\psi}, (\widetilde{\psi}_n^{\mathfrak{R}_\psi})' \in C([0,T]; V^\psi_{\mathfrak{R}_\psi n}(\Omega))$, $(\widetilde{\psi}_n^{\mathfrak{R}_\psi})'' \in L^2(0,T; V^\psi_{\mathfrak{R}_\psi n}(\Omega))$, $\widetilde{\theta}_n, \widetilde{\theta}_n' \in C([0,T]; V_n^\theta(\Omega))$, $\widetilde{\theta}_n'' \in L^2(0,T; V_n^\theta(\Omega))$, which satisfy the following equations in the sense of distributions on $(0,T)$:

$$\left(\rho\widetilde{\boldsymbol{u}}_n''', \boldsymbol{v}_n\right)_{\mathbf{L}^2(\Omega)} + c\left(\widetilde{\boldsymbol{u}}_n', \boldsymbol{v}_n\right) + \varepsilon\left(\widetilde{\varphi}_n', \boldsymbol{v}_n\right) + b\left(\widetilde{\psi}_n', \boldsymbol{v}_n\right) - \lambda\left(\widetilde{\theta}_n', \boldsymbol{v}_n\right) = (\boldsymbol{f}', \boldsymbol{v}_n)_{\mathbf{L}^2(\Omega)} + \left(\boldsymbol{g}', \mathbf{tr}_{\Gamma_1}(\boldsymbol{v}_n)\right)_{\mathbf{L}^2(\Gamma_1)},$$
$$\forall \boldsymbol{v}_n \in \mathbf{V}_n(\Omega), \tag{71}$$

$$-\varepsilon\left(\overline{\varphi}_n^{\mathfrak{R}_\varphi}, \widetilde{\boldsymbol{u}}_n''\right) + d\left(\left(\widetilde{\varphi}_n^{\mathfrak{R}_\varphi}\right)'', \overline{\varphi}_n^{\mathfrak{R}_\varphi}\right) + a\left(\left(\widetilde{\psi}_n^{\mathfrak{R}_\psi}\right)'', \overline{\varphi}_n^{\mathfrak{R}_\varphi}\right) - \mu\left(\widetilde{\theta}_n'', \overline{\varphi}_n^{\mathfrak{R}_\varphi}\right)$$
$$= \left((f^\varphi)'', \overline{\varphi}_n^{\mathfrak{R}_\varphi}\right)_{L^2(\Omega)} - \left((g^\varphi)'', \mathrm{tr}_{\Gamma_1^\varphi}\left(\overline{\varphi}_n^{\mathfrak{R}_\varphi}\right)\right)_{L^2(\Gamma_1^\varphi)}, \forall \overline{\varphi}_n^{\mathfrak{R}_\varphi} \in V^\varphi_{\mathfrak{R}_\varphi n}(\Omega), \tag{72}$$

$$-b\left(\overline{\psi}_n^{\mathfrak{R}_\psi}, \widetilde{\boldsymbol{u}}_n''\right) + a\left(\left(\widetilde{\varphi}_n^{\mathfrak{R}_\varphi}\right)'', \overline{\psi}_n^{\mathfrak{R}_\psi}\right) + \zeta\left(\left(\widetilde{\psi}_n^{\mathfrak{R}_\psi}\right)'', \overline{\psi}_n^{\mathfrak{R}_\psi}\right) - m\left(\widetilde{\theta}_n'', \overline{\psi}_n^{\mathfrak{R}_\psi}\right) = -\left((g^\psi)'', \mathrm{tr}_{\Gamma_1^\psi}\left(\overline{\psi}_n^{\mathfrak{R}_\psi}\right)\right)_{L^2(\Gamma_1^\psi)},$$
$$\forall \overline{\psi}_n^{\mathfrak{R}_\psi} \in V^\psi_{\mathfrak{R}_\psi n}(\Omega), \tag{73}$$

$$\left(\varkappa\widetilde{\theta}_n'', \overline{\theta}_n\right)_{L^2(\Omega)} + \eta\left(\widetilde{\theta}_n', \overline{\theta}_n\right) + \Theta_0\lambda\left(\overline{\theta}_n, \widetilde{\boldsymbol{u}}_n''\right) - \Theta_0\mu\left(\overline{\theta}_n, \left(\widetilde{\varphi}_n^{\mathfrak{R}_\varphi}\right)''\right) - \Theta_0 m\left(\overline{\theta}_n, \left(\widetilde{\psi}_n^{\mathfrak{R}_\psi}\right)''\right)$$
$$= \left(f^{\theta\prime}, \overline{\theta}_n\right)_{L^2(\Omega)} - \left(g^{\theta\prime}, \mathrm{tr}_{\Gamma_1^\theta}(\overline{\theta}_n)\right)_{L^2(\Gamma_1^\theta)}, \forall \overline{\theta}_n \in V_n^\theta(\Omega), \tag{74}$$

together with the initial conditions

$$\widetilde{\boldsymbol{u}}_n(0)=\widetilde{\boldsymbol{u}}_{0n},\qquad \widetilde{\boldsymbol{u}}_n'(0)=\widetilde{\boldsymbol{u}}_{1n},\qquad \widetilde{\boldsymbol{u}}_n''(0)=\widetilde{\boldsymbol{u}}_{2n},\qquad \widetilde{\varphi}_n^{\mathfrak{R}_\varphi}(0)=\widetilde{\varphi}_{0n}^{\mathfrak{R}_\varphi},\qquad (\widetilde{\varphi}_n^{\mathfrak{R}_\varphi})'(0)=\widetilde{\varphi}_{1n}^{\mathfrak{R}_\varphi},$$
$$\widetilde{\psi}_n^{\mathfrak{R}_\psi}(0)=\widetilde{\psi}_{0n}^{\mathfrak{R}_\psi},\qquad \left(\widetilde{\psi}_n^{\mathfrak{R}_\psi}\right)'(0)=\widetilde{\psi}_{1n}^{\mathfrak{R}_\psi},\qquad \widetilde{\theta}_n(0)=\widetilde{\theta}_{0n},\qquad \widetilde{\theta}_n'(0)=\widetilde{\theta}_{1n}, \tag{75}$$

where $\widetilde{\boldsymbol{u}}_{0n},\widetilde{\boldsymbol{u}}_{1n},\widetilde{\boldsymbol{u}}_{2n}\in\mathbf{V}_n(\Omega)$, $\widetilde{\varphi}_{0n}^{\mathfrak{R}_\varphi},\widetilde{\varphi}_{1n}^{\mathfrak{R}_\varphi}\in V^{\varphi}_{\mathfrak{R}_\varphi n}(\Omega)$, $\widetilde{\psi}_{0n}^{\mathfrak{R}_\psi},\widetilde{\psi}_{1n}^{\mathfrak{R}_\psi}\in V^{\psi}_{\mathfrak{R}_\psi n}(\Omega)$, $\widetilde{\theta}_{0n},\widetilde{\theta}_{1n}\in V_n^{\theta}(\Omega)$. Note that, for each $n\in\mathbb{N}$, the problem (71) to (75) is an initial value one for a linear system of ordinary differential equations with respect to $(u_{nk})_{k=1}^n$, $(\varphi_{nk})_{k=1}^n$, $(\psi_{nk})_{k=1}^n$ and $(\theta_{nk})_{k=1}^n$, where the right-hand parts of the corresponding equations belong to $L^2(0,T)$. From the estimate (70) for $\overline{\theta}=0$, we infer that $((\varphi_{nk})'')_{k=1}^n$, $((\psi_{nk})'')_{k=1}^n$ can be expressed by invertible linear mapping of $((u_{nk})'')_{k=1}^n$ and $((\theta_{nk})'')_{k=1}^n$ from equations (72) and (73). Since the initial value problem for the system of ordinary differential equations possesses a unique solution, we have that the problem (71) to (75) has a unique solution.

Since the systems $\{\boldsymbol{v}_k\}_{k=1}^\infty$, $\{\overline{\varphi}_k^{\mathfrak{R}_\varphi}\}_{k=1}^\infty$, $\{\overline{\psi}_k^{\mathfrak{R}_\psi}\}_{k=1}^\infty$ and $\{\overline{\theta}_k\}_{k=1}^\infty$ are complete in the spaces $\mathbf{V}(\Omega)$, $V^{\varphi}_{\mathfrak{R}_\varphi}(\Omega)$, $V^{\psi}_{\mathfrak{R}_\psi}(\Omega)$ and $V^{\theta}(\Omega)$, respectively, the union of subspaces $\bigcup_{n\ge1}\mathbf{V}_n(\Omega)$ is dense in $\mathbf{V}(\Omega)$, the union $\bigcup_{n\ge1}V^{\varphi}_{\mathfrak{R}_\varphi n}(\Omega)$ is dense in $V^{\varphi}_{\mathfrak{R}_\varphi}(\Omega)$, the union $\bigcup_{n\ge1}V^{\psi}_{\mathfrak{R}_\psi n}(\Omega)$ is dense in $V^{\psi}_{\mathfrak{R}_\psi}(\Omega)$ and the union $\bigcup_{n\ge1}V_n^{\theta}(\Omega)$ is dense in $V^{\theta}(\Omega)$. Therefore, we can take $\widetilde{\boldsymbol{u}}_{0n}$, $\widetilde{\boldsymbol{u}}_{1n}$, $\widetilde{\boldsymbol{u}}_{2n}$, $\widetilde{\varphi}_{0n}^{\mathfrak{R}_\varphi}$, $\widetilde{\varphi}_{1n}^{\mathfrak{R}_\varphi}$, $\widetilde{\psi}_{0n}^{\mathfrak{R}_\psi}$, $\widetilde{\psi}_{1n}^{\mathfrak{R}_\psi}$, $\widetilde{\theta}_{0n}$ and $\widetilde{\theta}_{1n}$ such that

$$\widetilde{\boldsymbol{u}}_{0n}\to\boldsymbol{u}_0\ \text{ in }\mathbf{V}(\Omega),\qquad \widetilde{\boldsymbol{u}}_{1n}\to\boldsymbol{u}_1\ \text{ in }\mathbf{V}(\Omega),\qquad \widetilde{\boldsymbol{u}}_{2n}\to\boldsymbol{u}_2\ \text{ in }\mathbf{L}^2(\Omega),\qquad \text{as } n\to\infty, \tag{76}$$

$$\widetilde{\varphi}_{0n}^{\mathfrak{R}_\varphi}\to\varphi_0^{\mathfrak{R}_\varphi}\ \text{ in }V^{\varphi}_{\mathfrak{R}_\varphi}(\Omega),\qquad \widetilde{\varphi}_{1n}^{\mathfrak{R}_\varphi}\to\varphi_1^{\mathfrak{R}_\varphi}\ \text{ in }V^{\varphi}_{\mathfrak{R}_\varphi}(\Omega),\qquad \text{as } n\to\infty, \tag{77}$$

$$\widetilde{\psi}_{0n}^{\mathfrak{R}_\psi}\to\psi_0^{\mathfrak{R}_\psi}\ \text{ in }V^{\psi}_{\mathfrak{R}_\psi}(\Omega),\qquad \widetilde{\psi}_{1n}^{\mathfrak{R}_\psi}\to\psi_1^{\mathfrak{R}_\psi}\ \text{ in }V^{\psi}_{\mathfrak{R}_\psi}(\Omega),\qquad \text{as } n\to\infty, \tag{78}$$

$$\widetilde{\theta}_{0n}\to\theta_0\ \text{ in }V^{\theta}(\Omega),\qquad \widetilde{\theta}_{1n}\to\theta_1\ \text{ in }L^2(\Omega),\qquad \text{as } n\to\infty, \tag{79}$$

where $\varphi_0^{\mathfrak{R}_\varphi}=\left\{\varphi_0+\overline{\varphi}^r;\overline{\varphi}^r\in\mathfrak{R}_\varphi\right\}\in V^{\varphi}_{\mathfrak{R}_\varphi}(\Omega)$, $\psi_0^{\mathfrak{R}_\psi}=\left\{\psi_0+\overline{\psi}^r;\overline{\psi}^r\in\mathfrak{R}_\psi\right\}\in V^{\psi}_{\mathfrak{R}_\psi}(\Omega)$.

By letting $\boldsymbol{v}_n=\widetilde{\boldsymbol{u}}_n''$ in equation (71), $\overline{\varphi}_n^{\mathfrak{R}_\varphi}=(\widetilde{\varphi}_n^{\mathfrak{R}_\varphi})'$ in equation (72), $\overline{\psi}_n^{\mathfrak{R}_\psi}=(\widetilde{\psi}_n^{\mathfrak{R}_\psi})'$ in equation (73) and $\overline{\theta}_n=\widetilde{\theta}_n'/\Theta_0$ in equation (74), adding and integrating them on $(0,t)$, taking into account the symmetry conditions (46) and using the formula for integration by parts, we obtain:

$$\left(\rho\widetilde{\boldsymbol{u}}_n''(t),\widetilde{\boldsymbol{u}}_n''(t)\right)_{\mathbf{L}^2(\Omega)}+c\left(\widetilde{\boldsymbol{u}}_n'(t),\widetilde{\boldsymbol{u}}_n'(t)\right)+d\left(\left(\widetilde{\varphi}_n^{\mathfrak{R}_\varphi}\right)'(t),\left(\widetilde{\varphi}_n^{\mathfrak{R}_\varphi}\right)'(t)\right)+2a\left(\left(\widetilde{\varphi}_n^{\mathfrak{R}_\varphi}\right)'(t),\left(\widetilde{\psi}_n^{\mathfrak{R}_\psi}\right)'(t)\right)$$
$$+\zeta\left(\left(\widetilde{\psi}_n^{\mathfrak{R}_\psi}\right)'(t),\left(\widetilde{\psi}_n^{\mathfrak{R}_\psi}\right)'(t)\right)+\frac{1}{\Theta_0}\left(\varkappa\widetilde{\theta}_n'(t),\widetilde{\theta}_n'(t)\right)_{L^2(\Omega)}+\frac{2}{\Theta_0}\int_0^t\eta\left(\widetilde{\theta}_n',\widetilde{\theta}_n'\right)\mathrm{d}\tau-2\mu\left(\widetilde{\theta}_n'(t),\left(\widetilde{\varphi}_n^{\mathfrak{R}_\varphi}\right)'(t)\right)$$
$$-2m\left(\widetilde{\theta}_n'(t),\left(\widetilde{\psi}_n^{\mathfrak{R}_\psi}\right)'(t)\right)=(\rho\widetilde{\boldsymbol{u}}_{2n},\widetilde{\boldsymbol{u}}_{2n})_{\mathbf{L}^2(\Omega)}+c(\widetilde{\boldsymbol{u}}_{1n},\widetilde{\boldsymbol{u}}_{1n})+d\left(\widetilde{\varphi}_{1n}^{\mathfrak{R}_\varphi},\widetilde{\varphi}_{1n}^{\mathfrak{R}_\varphi}\right)+2a\left(\widetilde{\varphi}_{1n}^{\mathfrak{R}_\varphi},\widetilde{\psi}_{1n}^{\mathfrak{R}_\psi}\right)$$
$$+\zeta\left(\widetilde{\psi}_{1n}^{\mathfrak{R}_\psi},\widetilde{\psi}_{1n}^{\mathfrak{R}_\psi}\right)+\frac{1}{\Theta_0}\left(\varkappa\widetilde{\theta}_{1n},\widetilde{\theta}_{1n}\right)_{L^2(\Omega)}$$

$$
\begin{aligned}
&- 2\mu\left(\widetilde{\theta}_{1n}, \widetilde{\varphi}_{1n}^{\mathfrak{R}_\varphi}\right) - 2m\left(\widetilde{\theta}_{1n}, \widetilde{\psi}_{1n}^{\mathfrak{R}_\psi}\right) + 2\int_0^t (\boldsymbol{f}'(\tau), \widetilde{\boldsymbol{u}}_n''(\tau))_{\mathbf{L}^2(\Omega)}\mathrm{d}\tau + 2\left(\boldsymbol{g}'(t), \mathbf{tr}_{\Gamma_1}\left(\widetilde{\boldsymbol{u}}_n'(t)\right)\right)_{\mathbf{L}^2(\Gamma_1)} \\
&- 2\left(\boldsymbol{g}'(0), \mathbf{tr}_{\Gamma_1}(\widetilde{\boldsymbol{u}}_{1n})\right)_{\mathbf{L}^2(\Gamma_1)} - 2\int_0^t \left(\boldsymbol{g}''(\tau), \mathbf{tr}_{\Gamma_1}\left(\widetilde{\boldsymbol{u}}_n'(\tau)\right)\right)_{\mathbf{L}^2(\Gamma_1)}\mathrm{d}\tau + 2\int_0^t \left((f^\varphi)''(\tau), \left(\widetilde{\varphi}_n^{\mathfrak{R}_\varphi}\right)'(\tau)\right)_{L^2(\Omega)}\mathrm{d}\tau \\
&- 2\int_0^t \left((g^\varphi)''(\tau), \mathrm{tr}_{\Gamma_1^\varphi}\left(\left(\widetilde{\varphi}_n^{\mathfrak{R}_\varphi}\right)'(\tau)\right)\right)_{L^2(\Gamma_1^\varphi)}\mathrm{d}\tau + \frac{2}{\Theta_0}\int_0^t \left(\left(f^\theta\right)'(\tau), \widetilde{\theta}_n'(\tau)\right)_{L^2(\Omega)}\mathrm{d}\tau \\
&- 2\int_0^t \left((g^\psi)''(\tau), \mathrm{tr}_{\Gamma_1^\psi}\left(\left(\widetilde{\psi}_n^{\mathfrak{R}_\psi}\right)'(\tau)\right)\right)_{L^2(\Gamma_1^\psi)}\mathrm{d}\tau - \frac{2}{\Theta_0}\int_0^t \left(\left(g^\theta\right)'(\tau), \mathrm{tr}_{\Gamma_1^\theta}\left(\widetilde{\theta}_n'(\tau)\right)\right)_{L^2(\Gamma_1^\theta)}\mathrm{d}\tau, \forall t \in [0,T]. \quad (80)
\end{aligned}
$$

From the positive definiteness conditions (38) for tensors c_{ijpq} and η_{ij}, and by applying Korn's inequality [26], we have that the bilinear forms $c(.,.)$ and $\eta(.,.)$ are coercive; i.e., there exist positive constants $\widetilde{\alpha}_c, \widehat{\alpha}_c$ and $\widetilde{\alpha}_\eta, \widehat{\alpha}_\eta$ such that

$$
c(\boldsymbol{v}, \boldsymbol{v}) \geq \alpha_c \boldsymbol{e}(\boldsymbol{v}) : \boldsymbol{e}(\boldsymbol{v}) \geq \widetilde{\alpha}_c \|\boldsymbol{v}\|^2_{\mathbf{H}^1(\Omega)} - \widehat{\alpha}_c \|\boldsymbol{v}\|^2_{\mathbf{L}^2(\Omega)}, \qquad \forall \boldsymbol{v} \in \mathbf{H}^1(\Omega), \quad (81)
$$

$$
\eta(\overline{\theta}, \overline{\theta}) \geq \alpha_\eta \int_\Omega |\mathbf{grad}\,\overline{\theta}|^2 \mathrm{d}x \geq \widetilde{\alpha}_\eta \left\|\overline{\theta}\right\|^2_{H^1(\Omega)} - \widehat{\alpha}_\eta \left\|\overline{\theta}\right\|^2_{L^2(\Omega)}, \qquad \forall \overline{\theta} \in H^1(\Omega). \quad (82)
$$

Because $c_{ijpq} \in L^\infty(\Omega)$, and by applying the Cauchy–Schwarz inequality, we obtain:

$$
\begin{aligned}
c(\boldsymbol{v}, \overline{\boldsymbol{v}}) &\leq \max_{1\leq i,j,p,q\leq 3} \left\|c_{ijpq}\right\|_{L^\infty(\Omega)} \left\|\sum_{i,j=1}^3 \left|e_{ij}(\boldsymbol{v})\right|\right\|_{L^2(\Omega)} \left\|\sum_{p,q=1}^3 \left|e_{pq}(\overline{\boldsymbol{v}})\right|\right\|_{L^2(\Omega)} \\
&\leq 9 \max_{1\leq i,j,p,q\leq 3} \left\|c_{ijpq}\right\|_{L^\infty(\Omega)} \sum_{i,j=1}^3 \left\|\frac{\partial v_i}{\partial x_j}\right\|_{L^2(\Omega)} \sum_{p,q=1}^3 \left\|\frac{\partial \overline{v}_p}{\partial x_q}\right\|_{L^2(\Omega)} \leq c_1 \|\boldsymbol{v}\|_{\mathbf{H}^1(\Omega)} \|\overline{\boldsymbol{v}}\|_{\mathbf{H}^1(\Omega)}, \quad (83)
\end{aligned}
$$

where $\boldsymbol{v}, \overline{\boldsymbol{v}} \in \mathbf{H}^1(\Omega)$. Since $\boldsymbol{g}, \boldsymbol{g}', \boldsymbol{g}'' \in L^2(0,T;\mathbf{L}^{4/3}(\Gamma_1))$, it follows from the embedding theorem [24] that $\boldsymbol{g}, \boldsymbol{g}' \in C([0,T];\mathbf{L}^{4/3}(\Gamma_1))$. By applying Hölder's inequality, conditions (45), the continuity of the embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$ and the trace operator $\mathrm{tr} : H^1(\Omega) \to L^4(\Gamma)$ [26] and the ε-inequality

$$
|a_1 b_1| \leq \frac{1}{2\varepsilon_1} a_1^2 + \frac{\varepsilon_1}{2} b_1^2,
$$

for all $a_1, b_1 \in \mathbb{R}$, $\varepsilon_1 > 0$, we have:

$$
\begin{aligned}
\left|(\boldsymbol{g}'(t), \mathbf{tr}_{\Gamma_1}(v))_{\mathbf{L}^2(\Gamma_1)}\right| &\leq \left\|\boldsymbol{g}'(t)\right\|_{C([0,T];\mathbf{L}^{4/3}(\Gamma_1))} \left\|\mathbf{tr}_{\Gamma_1}(\boldsymbol{v})\right\|_{\mathbf{L}^4(\Gamma_1)} \\
&\leq c_1 \left\|\boldsymbol{g}'(t)\right\|_{C([0,T];\mathbf{L}^{4/3}(\Gamma_1))} \|\boldsymbol{v}\|_{\mathbf{H}^1(\Omega)} \\
&\leq c_1 \left(\frac{1}{2\varepsilon_1}\left\|\boldsymbol{g}'(t)\right\|^2_{C([0,T];\mathbf{L}^{4/3}(\Gamma_1))} + \frac{\varepsilon_1}{2}\|\boldsymbol{v}\|^2_{\mathbf{H}^1(\Omega)}\right) \\
&\leq \frac{c_2}{\varepsilon_1}\left(\left\|\boldsymbol{g}'\right\|^2_{L^2(0,T;\mathbf{L}^{4/3}(\Gamma_1))} + \left\|\boldsymbol{g}''\right\|^2_{L^2(0,T;\mathbf{L}^{4/3}(\Gamma_1))}\right) + \frac{c_1\varepsilon_1}{2}\|\boldsymbol{v}\|^2_{\mathbf{H}^1(\Omega)}, \\
&\qquad \forall \boldsymbol{v} \in \mathbf{H}^1(\Omega), t \in [0,T], \varepsilon_1 > 0, \quad (84)
\end{aligned}
$$

$$
\begin{aligned}
&\left|\left((f^{\varphi})''(\tau),\overline{\varphi}^{\mathfrak{R}_{\varphi}}\right)_{L^2(\Omega)}-\left((g^{\varphi})''(\tau),\mathrm{tr}_{\Gamma_1^{\varphi}}\left(\overline{\varphi}^{\mathfrak{R}_{\varphi}}\right)\right)_{L^2(\Gamma_1^{\varphi})}\right|\\
&\leq \left\|(f^{\varphi})''(\tau)\right\|_{L^{6/5}(\Omega)}\inf_{\overline{\varphi}^{r}\in\mathfrak{R}_{\varphi}}||\overline{\varphi}+\overline{\varphi}^{r}||_{L^6(\Omega)}+||(g^{\varphi})''(\tau)||_{L^{4/3}(\Gamma_1^{\varphi})}\inf_{\overline{\varphi}^{r}\in\mathfrak{R}_{\varphi}}||\mathrm{tr}_{\Gamma_1^{\varphi}}(\overline{\varphi}+\overline{\varphi}^{r})||_{L^4(\Gamma_1^{\varphi})}\\
&\leq \frac{1}{2}\left\|(f^{\varphi})''(\tau)\right\|^2_{L^{6/5}(\Omega)}+\frac{1}{2}\left\|(g^{\varphi})''(\tau)\right\|^2_{L^{4/3}(\Gamma_1^{\varphi})}+c_1\left\|\overline{\varphi}^{\mathfrak{R}_{\varphi}}\right\|^2_{V^{\varphi}_{\mathfrak{R}_{\varphi}}(\Omega)},\quad \forall\overline{\varphi}^{\mathfrak{R}_{\varphi}}\in V^{\varphi}_{\mathfrak{R}_{\varphi}}(\Omega),
\end{aligned}\tag{85}
$$

$$
\left|\left((g^{\psi})''(\tau),\mathrm{tr}_{\Gamma_1^{\psi}}\left(\overline{\psi}^{\mathfrak{R}_{\psi}}\right)\right)_{L^2(\Gamma_1^{\psi})}\right|\leq\frac{1}{2}\left\|(g^{\psi})''(\tau)\right\|^2_{L^{4/3}(\Gamma_1^{\psi})}+\frac{c_1}{2}\left\|\overline{\psi}^{\mathfrak{R}_{\psi}}\right\|^2_{V^{\psi}_{\mathfrak{R}_{\psi}}(\Omega)},\qquad\forall\overline{\psi}^{\mathfrak{R}_{\psi}}\in V^{\psi}_{\mathfrak{R}_{\psi}}(\Omega),\tag{86}
$$

for almost all $\tau\in(0,T)$, where $\overline{\varphi}\in V^{\varphi}(\Omega)$ and $\overline{\psi}\in V^{\psi}(\Omega)$ are arbitrary elements of the equivalence classes $\overline{\varphi}^{\mathfrak{R}_{\varphi}}\in V^{\varphi}_{\mathfrak{R}_{\varphi}}(\Omega)$ and $\overline{\psi}^{\mathfrak{R}_{\psi}}\in V^{\psi}_{\mathfrak{R}_{\psi}}(\Omega)$, respectively.

Similarly,

$$
\begin{aligned}
\left|\int_0^t\left(\boldsymbol{g}''(\tau),\mathbf{tr}_{\Gamma_1}\left(\widetilde{\boldsymbol{u}}_n'(\tau)\right)\right)_{\mathbf{L}^2(\Gamma_1)}\mathrm{d}\tau\right|&\leq\int_0^t\left\|\boldsymbol{g}''(\tau)\right\|_{\mathbf{L}^{4/3}(\Gamma_1)}\left\|\mathbf{tr}_{\Gamma_1}\left(\widetilde{\boldsymbol{u}}_n'(\tau)\right)\right\|_{\mathbf{L}^4(\Gamma_1)}\mathrm{d}\tau\\
&\leq\frac{1}{2}\left\|\boldsymbol{g}''\right\|^2_{L^2(0,T;\mathbf{L}^{4/3}(\Gamma_1))}+\frac{c_1}{2}\int_0^t\left\|\widetilde{\boldsymbol{u}}_n'(\tau)\right\|^2_{\mathbf{H}^1(\Omega)}\mathrm{d}\tau,
\end{aligned}\tag{87}
$$

$$
\begin{aligned}
\left|\int_0^t\left(\left(g^{\theta}\right)'(\tau),\mathrm{tr}_{\Gamma_1^{\theta}}\left(\widetilde{\theta}_n'(\tau)\right)\right)_{L^2(\Gamma_1^{\theta})}\mathrm{d}\tau\right|&\leq\int_0^t\left\|\left(g^{\theta}\right)'(\tau)\right\|_{L^{4/3}(\Gamma_1^{\theta})}\left\|\mathrm{tr}_{\Gamma_1^{\theta}}\left(\widetilde{\theta}_n'(\tau)\right)\right\|_{L^4(\Gamma_1^{\theta})}\mathrm{d}\tau\\
&\leq\frac{1}{2\varepsilon_1}\left\|\left(g^{\theta}\right)'\right\|^2_{L^2(0,T;L^{4/3}(\Gamma_1^{\theta}))}+\frac{c_1\varepsilon_1}{2}\int_0^t\left\|\widetilde{\theta}_n'(\tau)\right\|^2_{H^1(\Omega)}\mathrm{d}\tau,\\
&\qquad\forall t\in[0,T],\ \varepsilon_1>0.
\end{aligned}\tag{88}
$$

Hence, by using the conditions (47), the inequalities (61) to (65), (70) and (81) to (88) for small enough $\varepsilon_1>0$, and the Cauchy–Schwarz inequality, and from the equality (80), we obtain:

$$
\begin{aligned}
&\left\|\widetilde{\boldsymbol{u}}_n'(t)\right\|^2_{\mathbf{H}^1(\Omega)}+\left\|\widetilde{\boldsymbol{u}}_n''(t)\right\|^2_{\mathbf{L}^2(\Omega)}+\left\|\left(\widetilde{\varphi}_n^{\mathfrak{R}_{\varphi}}\right)'(t)\right\|^2_{V^{\varphi}_{\mathfrak{R}_{\varphi}}(\Omega)}+\left\|\left(\widetilde{\psi}_n^{\mathfrak{R}_{\psi}}\right)'(t)\right\|^2_{V^{\psi}_{\mathfrak{R}_{\psi}}(\Omega)}+\left\|\widetilde{\theta}_n'(t)\right\|^2_{L^2(\Omega)}+\int_0^t\left\|\widetilde{\theta}_n'(\tau)\right\|^2_{H^1(\Omega)}\mathrm{d}\tau\\
&\leq c_1\left(\left\|\widetilde{\boldsymbol{u}}_n'(t)\right\|^2_{\mathbf{L}^2(\Omega)}+\int_0^t\left\|\widetilde{\boldsymbol{u}}_n'(\tau)\right\|^2_{\mathbf{H}^1(\Omega)}\mathrm{d}\tau+\int_0^t\left\|\widetilde{\boldsymbol{u}}_n''(\tau)\right\|_{\mathbf{L}^2(\Omega)}\mathrm{d}\tau+\int_0^t\left\|\left(\widetilde{\varphi}_n^{\mathfrak{R}_{\varphi}}\right)'(\tau)\right\|^2_{V^{\varphi}_{\mathfrak{R}_{\varphi}}(\Omega)}\mathrm{d}\tau\right.\\
&\qquad\left.+\int_0^t\left\|\left(\widetilde{\psi}_n^{\mathfrak{R}_{\psi}}\right)'(\tau)\right\|^2_{V^{\psi}_{\mathfrak{R}_{\psi}}(\Omega)}\mathrm{d}\tau+\int_0^t\left\|\widetilde{\theta}_n'(\tau)\right\|^2_{L^2(\Omega)}\mathrm{d}\tau\right)\\
&\quad+c_2\left(\left\|\widetilde{\boldsymbol{u}}_{1n}\right\|^2_{\mathbf{H}^1(\Omega)}+\left\|\widetilde{\boldsymbol{u}}_{2n}\right\|^2_{\mathbf{L}^2(\Omega)}+\left\|\widetilde{\varphi}_{1n}^{\mathfrak{R}_{\varphi}}\right\|^2_{V^{\varphi}_{\mathfrak{R}_{\varphi}}(\Omega)}+\left\|\widetilde{\psi}_{1n}^{\mathfrak{R}_{\psi}}\right\|^2_{V^{\psi}_{\mathfrak{R}_{\psi}}(\Omega)}+\left\|\widetilde{\theta}_{1n}\right\|^2_{H^1(\Omega)}+\left\|\boldsymbol{f}'\right\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))}\right.\\
&\qquad+\left\|\boldsymbol{g}'\right\|^2_{L^2(0,T;\mathbf{L}^{4/3}(\Gamma_1))}+\left\|\boldsymbol{g}''\right\|^2_{L^2(0,T;\mathbf{L}^{4/3}(\Gamma_1))}+\left\|(f^{\varphi})''\right\|^2_{L^2(0,T;L^{6/5}(\Omega))}+\left\|(g^{\varphi})''\right\|^2_{L^2(0,T;L^{4/3}(\Gamma_1^{\varphi}))}\\
&\qquad\left.+\left\|(g^{\psi})''\right\|^2_{L^2(0,T;L^{4/3}(\Gamma_1^{\psi}))}+\left\|f^{\theta\prime}\right\|^2_{L^2(0,T;L^2(\Omega))}+\left\|g^{\theta\prime}\right\|^2_{L^2(0,T;L^{4/3}(\Gamma_1^{\theta}))}\right),\forall t\in[0,T].
\end{aligned}\tag{89}
$$

Since $\widetilde{\boldsymbol{u}}_n', \widetilde{\boldsymbol{u}}_n'' \in C([0,T];\mathbf{V}_n(\Omega))$, we have:

$$\left\|\widetilde{\boldsymbol{u}}_n'(t)\right\|^2_{\mathbf{L}^2(\Omega)} \leq 2\left\|\widetilde{\boldsymbol{u}}_{1n}\right\|^2_{\mathbf{L}^2(\Omega)} + 2t\int_0^t \left\|\widetilde{\boldsymbol{u}}_n''(\tau)\right\|^2_{\mathbf{L}^2(\Omega)}\mathrm{d}\tau, \qquad \forall t\in[0,T]. \tag{90}$$

From the convergence properties of the sequences $(\widetilde{\boldsymbol{u}}_{1n})_{n\geq 1}$, $(\widetilde{\boldsymbol{u}}_{2n})_{n\geq 1}$, $(\widetilde{\varphi}_{1n}^{\mathfrak{R}_\varphi})_{n\geq 1}$, $(\widetilde{\psi}_{1n}^{\mathfrak{R}_\psi})_{n\geq 1}$ and $(\widetilde{\theta}_{1n})_{n\geq 1}$, we infer that they are bounded in the spaces $\mathbf{H}^1(\Omega)$, $\mathbf{L}^2(\Omega)$, $V^{\varphi}_{\mathfrak{R}_\varphi}(\Omega)$, $V^{\psi}_{\mathfrak{R}_\psi}(\Omega)$ and $L^2(\Omega)$, respectively, and from the estimates (89) and (90), we have, for all $t\in[0,T]$,

$$\begin{aligned}
&\left\|\widetilde{\boldsymbol{u}}_n'(t)\right\|^2_{\mathbf{H}^1(\Omega)} + \left\|\widetilde{\boldsymbol{u}}_n''(t)\right\|^2_{\mathbf{L}^2(\Omega)} + \left\|\left(\widetilde{\varphi}_n^{\mathfrak{R}_\varphi}\right)'(t)\right\|^2_{V^{\varphi}_{\mathfrak{R}_\varphi}(\Omega)} + \left\|\left(\widetilde{\psi}_n^{\mathfrak{R}_\psi}\right)'(t)\right\|^2_{V^{\psi}_{\mathfrak{R}_\psi}(\Omega)} + \left\|\widetilde{\theta}_n'(t)\right\|^2_{L^2(\Omega)} + \int_0^t \left\|\widetilde{\theta}_n'(\tau)\right\|^2_{H^1(\Omega)}\mathrm{d}\tau \\
&\leq c_1\left(\int_0^t \left\|\widetilde{\boldsymbol{u}}_n'(\tau)\right\|^2_{\mathbf{H}^1(\Omega)}\mathrm{d}\tau + \int_0^t \left\|\widetilde{\boldsymbol{u}}_n''(\tau)\right\|_{\mathbf{L}^2(\Omega)}\mathrm{d}\tau + \int_0^t \left\|\left(\widetilde{\varphi}_n^{\mathfrak{R}_\varphi}\right)'(\tau)\right\|^2_{V^{\varphi}_{\mathfrak{R}_\varphi}(\Omega)}\mathrm{d}\tau + \int_0^t \left\|\left(\widetilde{\psi}_n^{\mathfrak{R}_\psi}\right)'(\tau)\right\|^2_{V^{\psi}_{\mathfrak{R}_\psi}(\Omega)}\mathrm{d}\tau \right.\\
&\qquad\left. + \int_0^t \left\|\widetilde{\theta}_n'(\tau)\right\|^2_{L^2(\Omega)}\mathrm{d}\tau\right) + c_2,
\end{aligned} \tag{91}$$

where c_1, c_2 are positive constants that do not depend on $\widetilde{\boldsymbol{u}}_n$, $\widetilde{\varphi}_n^{\mathfrak{R}_\varphi}$, $\widetilde{\psi}_n^{\mathfrak{R}_\psi}$ or $\widetilde{\theta}_n$. By applying Gronwall's lemma [24], we obtain:

$$\begin{aligned}
&\left\|\widetilde{\boldsymbol{u}}_n'(t)\right\|^2_{\mathbf{H}^1(\Omega)} + \left\|\widetilde{\boldsymbol{u}}_n''(t)\right\|^2_{\mathbf{L}^2(\Omega)} + \left\|\left(\widetilde{\varphi}_n^{\mathfrak{R}_\varphi}\right)'(t)\right\|^2_{V^{\varphi}_{\mathfrak{R}_\varphi}(\Omega)} + \left\|\left(\widetilde{\psi}_n^{\mathfrak{R}_\psi}\right)'(t)\right\|^2_{V^{\psi}_{\mathfrak{R}_\psi}(\Omega)} + \left\|\widetilde{\theta}_n'(t)\right\|^2_{L^2(\Omega)} + \int_0^t \left\|\widetilde{\theta}_n'(\tau)\right\|^2_{H^1(\Omega)}\mathrm{d}\tau \\
&\quad\leq c_1, \forall t\in[0,T].
\end{aligned}$$

Consequently, $\widetilde{\boldsymbol{u}}_n'$ is bounded in $L^\infty(0,T;\mathbf{V}(\Omega))$, $\widetilde{\boldsymbol{u}}_n''$ is bounded in $L^\infty(0,T;\mathbf{L}^2(\Omega))$, $(\widetilde{\varphi}_n^{\mathfrak{R}_\varphi})'$ is bounded in $L^\infty(0,T;V^{\varphi}_{\mathfrak{R}_\varphi}(\Omega))$, $(\widetilde{\psi}_n^{\mathfrak{R}_\psi})'$ is bounded in $L^\infty(0,T;V^{\psi}_{\mathfrak{R}_\psi}(\Omega))$, and $\widetilde{\theta}_n'$ is bounded in $L^\infty(0,T;L^2(\Omega))$ and $L^2(0,T;V^{\theta}(\Omega))$. Because

$$\widetilde{\boldsymbol{u}}_n(t) = \widetilde{\boldsymbol{u}}_{0n} + \int_0^t \widetilde{\boldsymbol{u}}_n'(\tau)\mathrm{d}\tau, \qquad \widetilde{\theta}_n(t) = \widetilde{\theta}_{0n} + \int_0^t \widetilde{\theta}_n'(\tau)\mathrm{d}\tau, \qquad \forall t\in[0,T],$$

$$\widetilde{\varphi}_n^{\mathfrak{R}_\varphi}(t) = \widetilde{\varphi}_{0n}^{\mathfrak{R}_\varphi} + \int_0^t \left(\widetilde{\varphi}_n^{\mathfrak{R}_\varphi}\right)'(\tau)\mathrm{d}\tau, \qquad \widetilde{\psi}_n^{\mathfrak{R}_\psi}(t) = \widetilde{\psi}_{0n}^{\mathfrak{R}_\psi} + \int_0^t \left(\widetilde{\psi}_n^{\mathfrak{R}_\psi}\right)'(\tau)\mathrm{d}\tau, \qquad \forall t\in[0,T],$$

we deduce that $\widetilde{\boldsymbol{u}}_n$ is bounded in $L^\infty(0,T;\mathbf{V}(\Omega))$, $\widetilde{\varphi}_n^{\mathfrak{R}_\varphi}$ is bounded in $L^\infty(0,T;V^{\varphi}_{\mathfrak{R}_\varphi}(\Omega))$, $\widetilde{\psi}_n^{\mathfrak{R}_\psi}$ is bounded in $L^\infty(0,T;V^{\psi}_{\mathfrak{R}_\psi}(\Omega))$ and $\widetilde{\theta}_n$ is bounded in $L^\infty(0,T;V^{\theta}(\Omega))$. Hence, by the weak-* compactness of a bounded set in non-reflexive Banach space and weak compactness of a bounded set in reflexive Banach space, there exists a subsequence $(\widetilde{\boldsymbol{u}}_{n_k})_{k\geq 1}$ of sequence $(\widetilde{\boldsymbol{u}}_n)_{n\geq 1}$, a subsequence $(\widetilde{\varphi}_{n_k}^{\mathfrak{R}_\varphi})_{k\geq 1}$ of $(\widetilde{\varphi}_n^{\mathfrak{R}_\varphi})_{n\geq 1}$, a subsequence $(\widetilde{\psi}_{n_k}^{\mathfrak{R}_\psi})_{k\geq 1}$ of

$(\widetilde{\psi}_n^{\mathfrak{R}_\psi})_{n\geq 1}$ and a subsequence $(\widetilde{\theta}_{n_k})_{k\geq 1}$ of $(\widetilde{\theta}_n)_{n\geq 1}$, which possess the following properties:

$$\begin{aligned}
&\widetilde{\boldsymbol{u}}_{n_k} \to \boldsymbol{u}, \widetilde{\boldsymbol{u}}'_{n_k} \to \boldsymbol{u}' \quad \text{weakly-* in } L^\infty(0,T;\mathbf{V}(\Omega)),\\
&\widetilde{\boldsymbol{u}}''_{n_k} \to \boldsymbol{u}'' \quad \text{weakly-* in } L^\infty(0,T;\mathbf{L}^2(\Omega)),\\
&\widetilde{\varphi}_{n_k}^{\mathfrak{R}_\varphi} \to \varphi^{\mathfrak{R}_\varphi}, (\widetilde{\varphi}_{n_k}^{\mathfrak{R}_\varphi})' \to (\varphi^{\mathfrak{R}_\varphi})' \quad \text{weakly-* in } L^\infty(0,T;V_{\mathfrak{R}_\varphi}^{\varphi}(\Omega)),\\
&\widetilde{\psi}_{n_k}^{\mathfrak{R}_\psi} \to \psi^{\mathfrak{R}_\psi}, (\widetilde{\psi}_{n_k}^{\mathfrak{R}_\psi})' \to (\psi^{\mathfrak{R}_\psi})' \quad \text{weakly-* in } L^\infty(0,T;V_{\mathfrak{R}_\psi}^{\psi}(\Omega)),\\
&\widetilde{\theta}_{n_k} \to \theta \quad \text{weakly-* in } L^\infty(0,T;V^\theta(\Omega)),\\
&\widetilde{\theta}'_{n_k} \to \theta' \quad \text{weakly-* in } L^\infty(0,T;L^2(\Omega)),\ \ \widetilde{\theta}'_{n_k} \to \theta' \ \text{ weakly in } L^2(0,T;V^\theta(\Omega)),\ \ \text{as } k\to\infty.
\end{aligned} \tag{92}$$

By integrating equations (71) and (74) from 0 to t, and by integrating equations (72) and (73) twice from 0 to t, for all $t\in[0,T]$ and for $\boldsymbol{v}_n \in \mathbf{V}_n(\Omega)$, $\overline{\varphi}_n^{\mathfrak{R}_\varphi} \in V_{\mathfrak{R}_\varphi n}^{\varphi}(\Omega)$, $\overline{\psi}_n^{\mathfrak{R}_\psi} \in V_{\mathfrak{R}_\psi n}^{\psi}(\Omega)$ and $\overline{\theta}_n \in V_n^\theta(\Omega)$, we obtain:

$$\begin{aligned}
&\left(\rho\widetilde{\boldsymbol{u}}''_n(t),\boldsymbol{v}_n\right)_{\mathbf{L}^2(\Omega)} + c\left(\widetilde{\boldsymbol{u}}_n(t),\boldsymbol{v}_n\right) + \varepsilon(\widetilde{\varphi}_n(t),\boldsymbol{v}_n) + b\left(\widetilde{\psi}_n(t),\boldsymbol{v}_n\right) - \lambda\left(\widetilde{\theta}_n(t),\boldsymbol{v}_n\right)\\
&\qquad - \left(\left(\rho\widetilde{\boldsymbol{u}}_{2n}(t),\boldsymbol{v}_n\right)_{\mathbf{L}^2(\Omega)} + c(\widetilde{\boldsymbol{u}}_{0n},\boldsymbol{v}_n) + \varepsilon(\widetilde{\varphi}_{0n},\boldsymbol{v}_n) + b\left(\widetilde{\psi}_{0n},\boldsymbol{v}_n\right) - \lambda\left(\widetilde{\theta}_{0n},\boldsymbol{v}_n\right)\right)\\
&\qquad\qquad = (\boldsymbol{f}(t),\boldsymbol{v}_n)_{\mathbf{L}^2(\Omega)} + (\boldsymbol{g}(t),\mathbf{tr}_{\Gamma_1}(\boldsymbol{v}_n))_{\mathbf{L}^2(\Gamma_1)} - (\boldsymbol{f}(0),\boldsymbol{v}_n)_{\mathbf{L}^2(\Omega)} - (\boldsymbol{g}(0),\mathbf{tr}_{\Gamma_1}(\boldsymbol{v}_n))_{\mathbf{L}^2(\Gamma_1)},
\end{aligned} \tag{93}$$

$$\begin{aligned}
&-\varepsilon\left(\overline{\varphi}_n^{\mathfrak{R}_\varphi},\widetilde{\boldsymbol{u}}_n(t)\right) + d\left(\widetilde{\varphi}_n^{\mathfrak{R}_\varphi}(t),\overline{\varphi}_n^{\mathfrak{R}_\varphi}\right) + a\left(\widetilde{\psi}_n^{\mathfrak{R}_\psi}(t),\overline{\varphi}_n^{\mathfrak{R}_\varphi}\right) - \mu\left(\widetilde{\theta}_n(t),\overline{\varphi}_n^{\mathfrak{R}_\varphi}\right)\\
&\quad - \left(-\varepsilon\left(\overline{\varphi}_n^{\mathfrak{R}_\varphi},\widetilde{\boldsymbol{u}}_{1n}\right) + d\left(\widetilde{\varphi}_{1n}^{\mathfrak{R}_\varphi},\overline{\varphi}_n^{\mathfrak{R}_\varphi}\right) + a\left(\widetilde{\psi}_{1n}^{\mathfrak{R}_\psi},\overline{\varphi}_n^{\mathfrak{R}_\varphi}\right) - \mu\left(\widetilde{\theta}_{1n},\overline{\varphi}_n^{\mathfrak{R}_\varphi}\right)\right)t\\
&\quad - \left(-\varepsilon\left(\overline{\varphi}_n^{\mathfrak{R}_\varphi},\widetilde{\boldsymbol{u}}_{0n}\right) + d\left(\widetilde{\varphi}_{0n}^{\mathfrak{R}_\varphi},\overline{\varphi}_n^{\mathfrak{R}_\varphi}\right) + a\left(\widetilde{\psi}_{0n}^{\mathfrak{R}_\psi},\overline{\varphi}_n^{\mathfrak{R}_\varphi}\right) - \mu\left(\widetilde{\theta}_{0n},\overline{\varphi}_n^{\mathfrak{R}_\varphi}\right)\right)\\
&= \left(f^\varphi(t),\overline{\varphi}_n^{\mathfrak{R}_\varphi}\right)_{L^2(\Omega)} - \left(g^\varphi(t),\mathrm{tr}_{\Gamma_1^\varphi}\left(\overline{\varphi}_n^{\mathfrak{R}_\varphi}\right)\right)_{L^2(\Gamma_1^\varphi)}\\
&\quad - \left(\left((f^\varphi)'(0),\overline{\varphi}_n^{\mathfrak{R}_\varphi}\right)_{L^2(\Omega)} - \left((g^\varphi)'(0),\mathrm{tr}_{\Gamma_1^\varphi}\left(\overline{\varphi}_n^{\mathfrak{R}_\varphi}\right)\right)_{L^2(\Gamma_1^\varphi)}\right)t - \left(\left(f^\varphi(0),\overline{\varphi}_n^{\mathfrak{R}_\varphi}\right)_{L^2(\Omega)} - \left(g^\varphi(0),\mathrm{tr}_{\Gamma_1^\varphi}\left(\overline{\varphi}_n^{\mathfrak{R}_\varphi}\right)\right)_{L^2(\Gamma_1^\varphi)}\right),
\end{aligned} \tag{94}$$

$$\begin{aligned}
&-b\left(\overline{\psi}_n^{\mathfrak{R}_\psi},\widetilde{\boldsymbol{u}}_n(t)\right) + a\left(\widetilde{\varphi}_n^{\mathfrak{R}_\varphi}(t),\overline{\psi}_n^{\mathfrak{R}_\psi}\right) + \zeta\left(\widetilde{\psi}_n^{\mathfrak{R}_\psi}(t),\overline{\psi}_n^{\mathfrak{R}_\psi}\right) - m\left(\widetilde{\theta}_n(t),\overline{\psi}_n^{\mathfrak{R}_\psi}\right)\\
&\quad - \left(-b\left(\overline{\psi}_n^{\mathfrak{R}_\psi},\widetilde{\boldsymbol{u}}_{1n}\right) + a\left(\widetilde{\varphi}_{1n}^{\mathfrak{R}_\varphi},\overline{\psi}_n^{\mathfrak{R}_\psi}\right) + \zeta\left(\widetilde{\psi}_{1n}^{\mathfrak{R}_\psi},\overline{\psi}_n^{\mathfrak{R}_\psi}\right) - m\left(\widetilde{\theta}_{1n},\overline{\psi}_n^{\mathfrak{R}_\psi}\right)\right)t\\
&\quad - \left(-b\left(\overline{\psi}_n^{\mathfrak{R}_\psi},\widetilde{\boldsymbol{u}}_{0n}\right) + a\left(\widetilde{\varphi}_{0n}^{\mathfrak{R}_\varphi},\overline{\psi}_n^{\mathfrak{R}_\psi}\right) + \zeta\left(\widetilde{\psi}_{0n}^{\mathfrak{R}_\psi},\overline{\psi}_n^{\mathfrak{R}_\psi}\right) - m\left(\widetilde{\theta}_{0n},\overline{\psi}_n^{\mathfrak{R}_\psi}\right)\right)\\
&= -\left(g^\psi(t),\mathrm{tr}_{\Gamma_1^\psi}\left(\overline{\psi}_n^{\mathfrak{R}_\psi}\right)\right)_{L^2(\Gamma_1^\psi)} + \left(\left(g^\psi\right)'(0),\mathrm{tr}_{\Gamma_1^\psi}\left(\overline{\psi}_n^{\mathfrak{R}_\psi}\right)\right)_{L^2(\Gamma_1^\psi)} t + \left(g^\psi(0),\mathrm{tr}_{\Gamma_1^\psi}\left(\overline{\psi}_n^{\mathfrak{R}_\psi}\right)\right)_{L^2(\Gamma_1^\psi)},
\end{aligned} \tag{95}$$

$$\begin{aligned}
&\left(\varkappa\widetilde{\theta}'_n(t),\overline{\theta}_n\right)_{L^2(\Omega)} + \eta\left(\widetilde{\theta}_n(t),\overline{\theta}_n\right) + \Theta_0\lambda\left(\overline{\theta}_n,\widetilde{\boldsymbol{u}}'_n(t)\right) - \Theta_0\mu\left(\overline{\theta}_n,\left(\widetilde{\varphi}_n^{\mathfrak{R}_\varphi}\right)'(t)\right) - \Theta_0 m\left(\overline{\theta}_n,\left(\widetilde{\psi}_n^{\mathfrak{R}_\psi}\right)'(t)\right)\\
&\quad - \left(\left(\varkappa\widetilde{\theta}_{1n},\overline{\theta}_n\right)_{L^2(\Omega)} + \eta\left(\widetilde{\theta}_{0n},\overline{\theta}_n\right) + \Theta_0\lambda\left(\overline{\theta}_n,\widetilde{\boldsymbol{u}}_{1n}\right) - \Theta_0\mu\left(\overline{\theta}_n,\widetilde{\varphi}_{1n}^{\mathfrak{R}_\varphi}\right) - \Theta_0 m\left(\overline{\theta}_n,\widetilde{\psi}_{1n}^{\mathfrak{R}_\psi}\right)\right)\\
&= \left(f^\theta(t),\overline{\theta}_n\right)_{L^2(\Omega)} - \left(g^\theta(t),\mathrm{tr}_{\Gamma_1^\theta}(\overline{\theta}_n)\right)_{L^2(\Gamma_1^\theta)} - \left(\left(f^\theta(0),\overline{\theta}_n\right)_{L^2(\Omega)} - \left(g^\theta(0),\mathrm{tr}_{\Gamma_1^\theta}\left(\overline{\theta}_n\right)\right)_{L^2(\Gamma_1^\theta)}\right).
\end{aligned} \tag{96}$$

From the density of the unions $\bigcup_{n\geq 1}\mathbf{V}_n(\Omega)$, $\bigcup_{n\geq 1}V_{\mathfrak{R}_\varphi n}^{\varphi}(\Omega)$, $\bigcup_{n\geq 1}V_{\mathfrak{R}_\psi n}^{\psi}(\Omega)$ and $\bigcup_{n\geq 1}V_n^\theta(\Omega)$ in $\mathbf{V}(\Omega)$, $V_{\mathfrak{R}_\varphi}^{\varphi}(\Omega)$, $V_{\mathfrak{R}_\psi}^{\psi}(\Omega)$ and $V^\theta(\Omega)$, respectively, there exist sequences $(\widehat{\boldsymbol{v}}_n)_{n\geq 1}$, $(\widehat{\varphi}_n^{\mathfrak{R}_\varphi})_{n\geq 1}$, $(\widehat{\psi}_n^{\mathfrak{R}_\psi})_{n\geq 1}$ and $(\widehat{\theta}_n)_{n\geq 1}$ such that $\widehat{\boldsymbol{v}}_n \to \boldsymbol{v}$ in $\mathbf{V}(\Omega)$, $\widehat{\varphi}_n^{\mathfrak{R}_\varphi} \to \overline{\varphi}^{\mathfrak{R}_\varphi}$ in $V_{\mathfrak{R}_\varphi}^{\varphi}(\Omega)$, $\widehat{\psi}_n^{\mathfrak{R}_\psi} \to \overline{\psi}^{\mathfrak{R}_\psi}$ in $V_{\mathfrak{R}_\psi}^{\psi}(\Omega)$ and $\widehat{\theta}_n \to \overline{\theta}$ in $V^\theta(\Omega)$, as $n\to\infty$. By taking $n = n_k$, $\boldsymbol{v}_{n_k} = \widehat{\boldsymbol{v}}_{n_k}\xi(t)$, $\xi\in\mathfrak{D}(0,T)$, $\overline{\varphi}_{n_k}^{\mathfrak{R}_\varphi} = \widehat{\varphi}_{n_k}^{\mathfrak{R}_\varphi}\xi(t)$, $\overline{\psi}_{n_k}^{\mathfrak{R}_\psi} = \widehat{\psi}_{n_k}^{\mathfrak{R}_\psi}\xi(t)$ and $\overline{\theta}_{n_k} = \widehat{\theta}_{n_k}\xi(t)$ in equations (93) to (96), by

integrating them on $(0,T)$ and passing to the limit as $k \to \infty$, and from the convergence properties (92), we infer:

$$\int_0^T \left((\rho \boldsymbol{u}''(t), \boldsymbol{v})_{\mathbf{L}^2(\Omega)} + c(\boldsymbol{u}(t), \boldsymbol{v}) + \varepsilon(\varphi(t), \boldsymbol{v}) + b(\psi(t), \boldsymbol{v}) - \lambda(\theta(t), \boldsymbol{v}) - (\boldsymbol{f}(t), \boldsymbol{v})_{\mathbf{L}^2(\Omega)} - (\boldsymbol{g}(t), \mathbf{tr}_{\Gamma_1}(\boldsymbol{v}))_{\mathbf{L}^2(\Gamma_1)}\right) \xi(t) \mathrm{d}t$$

$$= \int_0^T \left((\rho \boldsymbol{u}_2(t), \boldsymbol{v})_{\mathbf{L}^2(\Omega)} + c(\boldsymbol{u}_0, \boldsymbol{v}) + \varepsilon(\varphi_0, \boldsymbol{v}) + b(\psi_0, \boldsymbol{v}) - \lambda(\theta_0, \boldsymbol{v}) - (\boldsymbol{f}(0), \boldsymbol{v})_{\mathbf{L}^2(\Omega)} - (\boldsymbol{g}(0), \mathbf{tr}_{\Gamma_1}(\boldsymbol{v}))_{\mathbf{L}^2(\Gamma_1)}\right) \xi(t) \mathrm{d}t,$$

$$\forall \boldsymbol{v} \in \mathbf{V}(\Omega), \quad (97)$$

$$\begin{aligned}
&\int_0^T \Big(-\varepsilon\left(\overline{\varphi}^{\mathfrak{R}_\varphi}, \boldsymbol{u}(t)\right) + d\left(\varphi^{\mathfrak{R}_\varphi}(t), \overline{\varphi}^{\mathfrak{R}_\varphi}\right) + a\left(\psi^{\mathfrak{R}_\psi}(t), \overline{\varphi}^{\mathfrak{R}_\varphi}\right) - \mu\left(\theta(t), \overline{\varphi}^{\mathfrak{R}_\varphi}\right) - \left(f^\varphi(t), \overline{\varphi}^{\mathfrak{R}_\varphi}\right)_{L^2(\Omega)} \\
&\qquad + \left(g^\varphi(t), \mathrm{tr}_{\Gamma_1^\varphi}\left(\overline{\varphi}^{\mathfrak{R}_\varphi}\right)\right)_{L^2(\Gamma_1^\varphi)} \Big) \xi(t) \mathrm{d}t \\
&= \int_0^T \Big(-\varepsilon\left(\overline{\varphi}^{\mathfrak{R}_\varphi}, \boldsymbol{u}_1\right) + d\left(\varphi_1^{\mathfrak{R}_\varphi}, \overline{\varphi}^{\mathfrak{R}_\varphi}\right) + a\left(\psi_1^{\mathfrak{R}_\psi}, \overline{\varphi}^{\mathfrak{R}_\varphi}\right) - \mu\left(\theta_1, \overline{\varphi}^{\mathfrak{R}_\varphi}\right) - \left((f^\varphi)'(0), \overline{\varphi}^{\mathfrak{R}_\varphi}\right)_{L^2(\Omega)} \\
&\qquad + \left((g^\varphi)'(0), \mathrm{tr}_{\Gamma_1^\varphi}\left(\overline{\varphi}^{\mathfrak{R}_\varphi}\right)\right)_{L^2(\Gamma_1^\varphi)} \Big) t\xi(t) \mathrm{d}t \\
&\quad + \int_0^T (-\varepsilon(\overline{\varphi}^{\mathfrak{R}_\varphi}, \boldsymbol{u}_0) + d(\varphi_0^{\mathfrak{R}_\varphi}, \overline{\varphi}^{\mathfrak{R}_\varphi}) + a(\psi_0^{\mathfrak{R}_\psi}, \overline{\varphi}^{\mathfrak{R}_\varphi}) - \mu(\theta_0, \overline{\varphi}^{\mathfrak{R}_\varphi}) \\
&\qquad - (f^\varphi(0), \overline{\varphi}^{\mathfrak{R}_\varphi})_{L^2(\Omega)} + (g^\varphi(0), \mathrm{tr}_{\Gamma_1^\varphi}(\overline{\varphi}^{\mathfrak{R}_\varphi}))_{L^2(\Gamma_1^\varphi)}) \xi(t) \mathrm{d}t, \qquad \forall \overline{\varphi}^{\mathfrak{R}_\varphi} \in V_{\mathfrak{R}_\varphi}^\varphi(\Omega), \quad (98)
\end{aligned}$$

$$\begin{aligned}
&\int_0^T \Big(-b\left(\overline{\psi}^{\mathfrak{R}_\psi}, \boldsymbol{u}(t)\right) + a\left(\varphi^{\mathfrak{R}_\varphi}(t), \overline{\psi}^{\mathfrak{R}_\psi}\right) + \zeta\left(\psi^{\mathfrak{R}_\psi}(t), \overline{\psi}^{\mathfrak{R}_\psi}\right) - m\left(\theta(t), \overline{\psi}^{\mathfrak{R}_\psi}\right) \\
&\quad + \left(g^\psi(t), \mathrm{tr}_{\Gamma_1^\psi}\left(\overline{\psi}^{\mathfrak{R}_\psi}\right)\right)_{L^2(\Gamma_1^\psi)} \Big) \xi(t) \mathrm{d}t \\
&= \int_0^T \Big(-b\left(\overline{\psi}^{\mathfrak{R}_\psi}, \boldsymbol{u}_1\right) + a\left(\varphi_1^{\mathfrak{R}_\varphi}, \overline{\psi}^{\mathfrak{R}_\psi}\right) + \zeta\left(\psi_1^{\mathfrak{R}_\psi}, \overline{\psi}^{\mathfrak{R}_\psi}\right) - m\left(\theta_1, \overline{\psi}^{\mathfrak{R}_\psi}\right) \\
&\quad + \left((g^\psi)'(0), \mathrm{tr}_{\Gamma_1^\psi}\left(\overline{\psi}^{\mathfrak{R}_\psi}\right)\right)_{L^2(\Gamma_1^\psi)} \Big) t\xi(t) \mathrm{d}t \\
&\qquad + \int_0^T \Big(-b\left(\overline{\psi}^{\mathfrak{R}_\psi}, \boldsymbol{u}_0\right) + a\left(\varphi_0^{\mathfrak{R}_\varphi}, \overline{\psi}^{\mathfrak{R}_\psi}\right) + \zeta\left(\psi_0^{\mathfrak{R}_\psi}, \overline{\psi}^{\mathfrak{R}_\psi}\right) - m\left(\theta_0, \overline{\psi}^{\mathfrak{R}_\psi}\right) \\
&\qquad + \left(g^\psi(0), \mathrm{tr}_{\Gamma_1^\psi}\left(\overline{\psi}^{\mathfrak{R}_\psi}\right)\right)_{L^2(\Gamma_1^\psi)} \Big) \xi(t) \mathrm{d}t, \qquad \forall \overline{\psi}^{\mathfrak{R}_\psi} \in V_{\mathfrak{R}_\psi}^\psi(\Omega), \quad (99)
\end{aligned}$$

$$\int_0^T \Big(\big(\varkappa\theta'(t),\overline{\theta}\big)_{L^2(\Omega)} + \eta\left(\theta(t),\overline{\theta}\right) + \Theta_0\lambda\left(\overline{\theta},\boldsymbol{u}'(t)\right) - \Theta_0\mu\left(\overline{\theta},\left(\varphi^{\mathfrak{R}_\varphi}\right)'(t)\right)$$

$$-\Theta_0 m\left(\overline{\theta},\left(\psi^{\mathfrak{R}_\psi}\right)'(t)\right) - \left(f^\theta(t),\overline{\theta}\right)_{L^2(\Omega)} + \left(g^\theta(t),\mathrm{tr}_{\Gamma_1^\theta}(\overline{\theta})\right)_{L^2(\Gamma_1^\theta)}\Big)\,\xi(t)\mathrm{d}t$$

$$= \int_0^T \Big((\varkappa\theta_1,\overline{\theta})_{L^2(\Omega)} + \eta(\theta_0,\overline{\theta}) + \Theta_0\lambda(\overline{\theta},\boldsymbol{u}_1) - \Theta_0\mu\left(\overline{\theta},\varphi_1^{\mathfrak{R}_\varphi}\right)$$

$$-\Theta_0 m\left(\overline{\theta},\psi_1^{\mathfrak{R}_\psi}\right) - (f^\theta(0),\overline{\theta})_{L^2(\Omega)} + \left(g^\theta(0),\mathrm{tr}_{\Gamma_1^\theta}(\overline{\theta})\right)_{L^2(\Gamma_1^\theta)}\Big)\,\xi(t)\mathrm{d}t, \qquad \forall\overline{\theta}\in V^\theta(\Omega). \tag{100}$$

By taking into account equations (51) to (53) and equations (57) to (59), we deduce from equations (97) to (100) that $\boldsymbol{u}$, $\varphi^{\mathfrak{R}_\varphi}$, $\psi^{\mathfrak{R}_\psi}$ and θ satisfy equations (40) to (43) in the sense of distributions on $(0,T)$.

Furthermore, the limit functions $\boldsymbol{u}$ and θ satisfy the initial conditions (44). Indeed, for any continuously differentiable function $\xi\in C^1([0,T])$, such that $\xi(T)=0$, $\xi(0)\neq 0$ and $w_{n_k}, w'_{n_k}\in L^2(0,T;H)$, $\phi_{n_k}\in H$, we have [24]:

$$\int_0^T \left(w'_{n_k}(t),\phi_{n_k}\xi(t)\right)_H \mathrm{d}t = -(w_{n_k}(0),\phi_{n_k}\xi(0))_H - \int_0^T \left(w_{n_k}(t),\phi_{n_k}\xi'(t)\right)_H \mathrm{d}t, \tag{101}$$

where H is a Hilbert space and $(.,.)_H$ denotes the scalar product in H. We use equation (101) in three cases:

(a) $w_{n_k}=\widetilde{\boldsymbol{u}}_{n_k}$, $\phi_{n_k}=\widetilde{\boldsymbol{v}}_{n_k}$ and $H=\mathbf{L}^2(\Omega)$;
(b) $w_{n_k}=\widetilde{\boldsymbol{u}}'_{n_k}$, $\phi_{n_k}=\widetilde{\boldsymbol{v}}_{n_k}$ and $H=\mathbf{L}^2(\Omega)$;
(c) $w_{n_k}=\theta_{n_k}$, $\phi_{n_k}=\widetilde{\theta}_{n_k}$ and $H=L^2(\Omega)$.

From the convergence properties (76), (79) and (92), and by passing to the limit in equation (101) as $k\to\infty$, we obtain:

$$\int_0^T \left(w'(t),\phi\xi(t)\right)_H \mathrm{d}t = -(w_0,\phi\xi(0))_H - \int_0^T \left(w(t),\phi\xi'(t)\right)_H \mathrm{d}t, \tag{102}$$

where $w=\boldsymbol{u}$, $w_0=\boldsymbol{u}_0$ and $\phi=\boldsymbol{v}\in\mathbf{V}(\Omega)$ in case (a); $w=\boldsymbol{u}'$, $w_0=\boldsymbol{u}_1$ and $\phi=\boldsymbol{v}\in\mathbf{V}(\Omega)$ in case (b); $w=\theta$, $w_0=\theta_0$ and $\phi=\overline{\theta}\in V^\theta(\Omega)$ in case (c). On the other hand, by applying the formula for integration by parts for $w,w'\in L^2(0,T;H)$, $\phi\in H$, we have:

$$\int_0^T \left(w'(t),\phi\xi(t)\right)_H \mathrm{d}t = -(w(0),\phi\xi(0))_H - \int_0^T \left(w(t),\phi\xi'(t)\right)_H \mathrm{d}t. \tag{103}$$

Hence, from equations (102) and (103), we deduce that $(w_0,\phi)_H=(w(0),\phi)_H$ for all $\phi=\boldsymbol{v}\in\mathbf{V}(\Omega)$ or $\phi=\overline{\theta}\in V^\theta(\Omega)$. Note that, by the embedding theorem [24], $\boldsymbol{u}\in C([0,T];\mathbf{V}(\Omega))$, $\boldsymbol{u}'\in C([0,T];\mathbf{L}^2(\Omega))$ and $\theta\in C([0,T];V^\theta(\Omega))$, and we can let $w=\boldsymbol{u}$, $w=\boldsymbol{u}'$, or $w=\theta$ in equation (103). Consequently, by the density of $\mathfrak{D}(\Omega)$ in $L^2(\Omega)$, we get that $w(0)=w_0$, which is equivalent to the initial conditions (44). So, the problem (40) to (44) possesses a solution.

Now, let us prove the energy equality (54), uniqueness of the solution and continuous dependence of the solution on the given data. From the conditions $\eta_{ij}^k\in C^{0,1}(\Omega_k)$, we have $\eta_{ij}\in L^\infty(\Omega)$; hence, the following estimate is valid:

$$\begin{aligned}
\left|\eta\left(\widehat{\theta},\overline{\theta}\right)\right| &= \int_\Omega \left|\left(\boldsymbol{\eta}\,\mathbf{grad}\,\widehat{\theta}\right)\cdot\mathbf{grad}\,\overline{\theta}\right|\mathrm{d}x \\
&\le \max_{1\le i,j\le 3}\left\|\eta_{ij}\right\|_{L^\infty(\Omega)}\left\|\sum_{j=1}^3\left|\frac{\partial\widehat{\theta}}{\partial x_j}\right|\right\|_{L^2(\Omega)}\left\|\sum_{i=1}^3\left|\frac{\partial\overline{\theta}}{\partial x_i}\right|\right\|_{L^2(\Omega)} \\
&\le 3\max_{1\le i,j\le 3}\left\|\eta_{ij}\right\|_{L^\infty(\Omega)}\left\|\widehat{\theta}\right\|_{V^\theta(\Omega)}\left\|\overline{\theta}\right\|_{V^\theta(\Omega)}, \qquad \forall\widehat{\theta},\overline{\theta}\in V^\theta(\Omega).
\end{aligned} \tag{104}$$

It follows from the estimates (61) to (68), (83) and (104) that [23] there exist linear continuous operators $C : \mathbf{V}(\Omega) \to \mathbf{V}'(\Omega)$, $\mathcal{E} : V^{\varphi}_{\mathfrak{R}_\varphi}(\Omega) \to \mathbf{V}'(\Omega)$, $\overline{\mathcal{E}} : \mathbf{V}(\Omega) \to (V^{\varphi}_{\mathfrak{R}_\varphi}(\Omega))'$, $\mathcal{B} : V^{\psi}_{\mathfrak{R}_\psi}(\Omega) \to \mathbf{V}'(\Omega)$, $\overline{\mathcal{B}} : \mathbf{V}(\Omega) \to (V^{\psi}_{\mathfrak{R}_\psi}(\Omega))'$, $\Lambda : L^2(\Omega) \to \mathbf{V}'(\Omega)$, $\overline{\Lambda} : \mathbf{V}(\Omega) \to L^2(\Omega)$, $\mathcal{D} : V^{\varphi}_{\mathfrak{R}_\varphi}(\Omega) \to (V^{\varphi}_{\mathfrak{R}_\varphi}(\Omega))'$, $\mathcal{A} : V^{\varphi}_{\mathfrak{R}_\varphi}(\Omega) \to (V^{\psi}_{\mathfrak{R}_\psi}(\Omega))'$, $\overline{\mathcal{A}} : V^{\psi}_{\mathfrak{R}_\psi}(\Omega) \to (V^{\varphi}_{\mathfrak{R}_\varphi}(\Omega))'$, $\mathcal{Z} : V^{\psi}_{\mathfrak{R}_\psi}(\Omega) \to (V^{\psi}_{\mathfrak{R}_\psi}(\Omega))'$, $\mathcal{M} : L^2(\Omega) \to (V^{\varphi}_{\mathfrak{R}_\varphi}(\Omega))'$, $\overline{\mathcal{M}} : V^{\varphi}_{\mathfrak{R}_\varphi}(\Omega) \to L^2(\Omega)$, $\mathfrak{M} : L^2(\Omega) \to (V^{\psi}_{\mathfrak{R}_\psi}(\Omega))'$, $\overline{\mathfrak{M}} : V^{\psi}_{\mathfrak{R}_\psi}(\Omega) \to L^2(\Omega)$ and $\Xi : V^{\theta}(\Omega) \to (V^{\theta}(\Omega))'$, such that $\langle C\widehat{\boldsymbol{v}}, \boldsymbol{v}\rangle = c(\widehat{\boldsymbol{v}}, \boldsymbol{v})$, $\langle \mathcal{E}\overline{\varphi}^{\mathfrak{R}_\varphi}, \boldsymbol{v}\rangle = \varepsilon(\overline{\varphi}^{\mathfrak{R}_\varphi}, \boldsymbol{v})$, $\langle \overline{\mathcal{E}}\boldsymbol{v}, \overline{\varphi}^{\mathfrak{R}_\varphi}\rangle_\varphi = \varepsilon(\overline{\varphi}^{\mathfrak{R}_\varphi}, \boldsymbol{v})$, $\langle \mathcal{B}\overline{\psi}^{\mathfrak{R}_\psi}, \boldsymbol{v}\rangle = b(\overline{\psi}^{\mathfrak{R}_\psi}, \boldsymbol{v})$, $\langle \overline{\mathcal{B}}\boldsymbol{v}, \overline{\psi}^{\mathfrak{R}_\psi}\rangle_\psi = b(\overline{\psi}^{\mathfrak{R}_\psi}, \boldsymbol{v})$, $\langle \Lambda\widetilde{\theta}, \boldsymbol{v}\rangle = \lambda(\widetilde{\theta}, \boldsymbol{v})$, $(\overline{\Lambda}\boldsymbol{v}, \widetilde{\theta})_{L^2(\Omega)} = \lambda(\widetilde{\theta}, \boldsymbol{v})$, $\langle \mathcal{D}\widehat{\varphi}^{\mathfrak{R}_\varphi}, \overline{\varphi}^{\mathfrak{R}_\varphi}\rangle_\varphi = d(\widehat{\varphi}^{\mathfrak{R}_\varphi}, \overline{\varphi}^{\mathfrak{R}_\varphi})$, $\langle \mathcal{A}\overline{\varphi}^{\mathfrak{R}_\varphi}, \overline{\psi}^{\mathfrak{R}_\psi}\rangle_\psi = a(\overline{\varphi}^{\mathfrak{R}_\varphi}, \overline{\psi}^{\mathfrak{R}_\psi})$, $\langle \overline{\mathcal{A}\psi}^{\mathfrak{R}_\psi}, \overline{\varphi}^{\mathfrak{R}_\varphi}\rangle_\varphi = a(\overline{\varphi}^{\mathfrak{R}_\varphi}, \overline{\psi}^{\mathfrak{R}_\psi})$, $\langle \mathcal{Z}\widehat{\psi}^{\mathfrak{R}_\psi}, \overline{\psi}^{\mathfrak{R}_\psi}\rangle_\psi = \zeta(\widehat{\psi}^{\mathfrak{R}_\psi}, \overline{\psi}^{\mathfrak{R}_\psi})$, $\langle \mathcal{M}\widetilde{\theta}, \overline{\varphi}^{\mathfrak{R}_\varphi}\rangle_\varphi = \mu(\widetilde{\theta}, \overline{\varphi}^{\mathfrak{R}_\varphi})$, $(\overline{\mathcal{M}}\overline{\varphi}^{\mathfrak{R}_\varphi}, \widetilde{\theta})_{L^2(\Omega)} = \mu(\widetilde{\theta}, \overline{\varphi}^{\mathfrak{R}_\varphi})$, $\langle \mathfrak{M}\widetilde{\theta}, \overline{\psi}^{\mathfrak{R}_\psi}\rangle_\psi = m(\widetilde{\theta}, \overline{\psi}^{\mathfrak{R}_\psi})$, $(\overline{\mathfrak{M}}\,\overline{\psi}^{\mathfrak{R}_\psi}, \widetilde{\theta})_{L^2(\Omega)} = m(\widetilde{\theta}, \overline{\psi}^{\mathfrak{R}_\psi})$ and $\langle \Xi\widehat{\theta}, \overline{\theta}\rangle_\theta = \eta(\widehat{\theta}, \overline{\theta})$, for all $\boldsymbol{v}, \widehat{\boldsymbol{v}} \in \mathbf{V}(\Omega)$, $\widetilde{\theta} \in L^2(\Omega)$, $\widehat{\varphi}^{\mathfrak{R}_\varphi}, \overline{\varphi}^{\mathfrak{R}_\varphi} \in V^{\varphi}_{\mathfrak{R}_\varphi}(\Omega)$, $\widehat{\psi}^{\mathfrak{R}_\psi}, \overline{\psi}^{\mathfrak{R}_\psi} \in V^{\psi}_{\mathfrak{R}_\psi}(\Omega)$, $\widehat{\theta}, \overline{\theta} \in V^{\theta}(\Omega)$, where $\langle ., . \rangle$, $\langle ., . \rangle_\varphi$, $\langle ., . \rangle_\psi$ and $\langle ., . \rangle_\theta$ denote the duality relations between the spaces $\mathbf{V}(\Omega)$, $V^{\varphi}_{\mathfrak{R}_\varphi}(\Omega)$, $V^{\psi}_{\mathfrak{R}_\psi}(\Omega)$, $V^{\theta}(\Omega)$ and their duals $\mathbf{V}'(\Omega)$, $(V^{\varphi}_{\mathfrak{R}_\varphi}(\Omega))'$, $(V^{\psi}_{\mathfrak{R}_\psi}(\Omega))'$, $(V^{\theta}(\Omega))'$, respectively. Since the linear combinations of products $\xi\boldsymbol{v}$, $\xi\overline{\varphi}^{\mathfrak{R}_\varphi}$, $\xi\overline{\psi}^{\mathfrak{R}_\psi}$ and $\xi\overline{\theta}$, where $\xi \in \mathfrak{D}(\Omega)$, $\boldsymbol{v} \in \mathbf{V}(\Omega)$, $\overline{\varphi}^{\mathfrak{R}_\varphi} \in V^{\varphi}_{\mathfrak{R}_\varphi}(\Omega)$, $\overline{\psi}^{\mathfrak{R}_\psi} \in V^{\psi}_{\mathfrak{R}_\psi}(\Omega)$ and $\overline{\theta} \in V^{\theta}(\Omega)$, are dense in $L^2(0,T;\mathbf{V}(\Omega))$, $L^2(0,T;V^{\varphi}_{\mathfrak{R}_\varphi}(\Omega))$, $L^2(0,T;V^{\psi}_{\mathfrak{R}_\psi}(\Omega))$ and $L^2(0,T;V^{\theta}(\Omega))$, respectively, we infer that equations (40) to (43) are equivalent to the following ones:

$$\rho\frac{\mathrm{d}^2\boldsymbol{u}}{\mathrm{d}t^2} + C\boldsymbol{u} + \mathcal{E}\varphi^{\mathfrak{R}_\varphi} + \mathcal{B}\psi^{\mathfrak{R}_\psi} - \Lambda\theta = \boldsymbol{f} + \boldsymbol{G} \qquad \text{in} \quad L^2\left(0,T;\mathbf{V}'(\Omega)\right), \tag{105}$$

$$-\overline{\mathcal{E}}\boldsymbol{u} + \mathcal{D}\varphi^{\mathfrak{R}_\varphi} + \overline{\mathcal{A}}\psi^{\mathfrak{R}_\psi} - \mathcal{M}\theta = F^{\varphi} - G^{\varphi} \qquad \text{in} \quad L^2\left(0,T;\left(V^{\varphi}_{\mathfrak{R}_\varphi}(\Omega)\right)'\right), \tag{106}$$

$$-\overline{\mathcal{B}}\boldsymbol{u} + \mathcal{A}\varphi^{\mathfrak{R}_\varphi} + \mathcal{Z}\psi^{\mathfrak{R}_\psi} - \mathfrak{M}\theta = -G^{\psi} \qquad \text{in} \quad L^2\left(0,T;\left(V^{\psi}_{\mathfrak{R}_\psi}(\Omega)\right)'\right), \tag{107}$$

$$\varkappa\frac{\mathrm{d}\theta}{\mathrm{d}t} + \Xi\theta + \Theta_0\overline{\Lambda}\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} - \Theta_0\overline{\mathcal{M}}\frac{\mathrm{d}\varphi^{\mathfrak{R}_\varphi}}{\mathrm{d}t} - \Theta_0\overline{\mathfrak{M}}\frac{\mathrm{d}\psi^{\mathfrak{R}_\psi}}{\mathrm{d}t} = f^{\theta} - G^{\theta} \qquad \text{in} \quad L^2\left(0,T;\left(V^{\theta}(\Omega)\right)'\right), \tag{108}$$

where

$$\langle \boldsymbol{G}, \boldsymbol{v}\rangle = (\boldsymbol{g}, \mathbf{tr}_{\Gamma_1}(\boldsymbol{v}))_{\mathbf{L}^2(\Gamma_1)}, \quad \left\langle F^{\varphi} - G^{\varphi}, \overline{\varphi}^{\mathfrak{R}_\varphi}\right\rangle_\varphi = \left(f^{\varphi}, \overline{\varphi}^{\mathfrak{R}_\varphi}\right)_{L^2(\Omega)} - \left(g^{\varphi}, \mathrm{tr}_{\Gamma_1^{\varphi}}\left(\overline{\varphi}^{\mathfrak{R}_\varphi}\right)\right)_{L^2(\Gamma_1^{\varphi})},$$
$$\langle G^{\psi}, \overline{\psi}^{\mathfrak{R}_\psi}\rangle_\psi = \left(g^{\psi}, \mathrm{tr}_{\Gamma_1^{\psi}}\left(\overline{\psi}^{\mathfrak{R}_\psi}\right)\right)_{L^2(\Gamma_1^{\psi})}, \quad \left\langle G^{\theta}, \overline{\theta}\right\rangle_\theta = \left(g^{\theta}, \mathrm{tr}_{\Gamma_1^{\theta}}\left(\overline{\theta}\right)\right)_{L^2(\Gamma_1^{\theta})},$$

for all $\boldsymbol{v} \in \mathbf{V}(\Omega)$, $\overline{\varphi}^{\mathfrak{R}_\varphi} \in V^{\varphi}_{\mathfrak{R}_\varphi}(\Omega)$, $\overline{\psi}^{\mathfrak{R}_\psi} \in V^{\psi}_{\mathfrak{R}_\psi}(\Omega)$ and $\overline{\theta} \in V^{\theta}(\Omega)$. By applying continuity of the embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$ and the trace operator $\mathrm{tr} : H^1(\Omega) \to L^4(\Gamma)$, and taking into account condition (45), we have:

$$\left|\left(\frac{\mathrm{d}^{\alpha}\boldsymbol{g}}{\mathrm{d}t^{\alpha}}(t), \mathbf{tr}_{\Gamma_1}(\boldsymbol{v})\right)_{\mathbf{L}^2(\Gamma_1)}\right| \leq \left\|\frac{\mathrm{d}^{\alpha}\boldsymbol{g}}{\mathrm{d}t^{\alpha}}(t)\right\|_{\mathbf{L}^{4/3}(\Gamma_1)} \left\|\mathbf{tr}_{\Gamma_1}(\boldsymbol{v})\right\|_{\mathbf{L}^4(\Gamma_1)} \leq c_1 \left\|\frac{\mathrm{d}^{\alpha}\boldsymbol{g}}{\mathrm{d}t^{\alpha}}(t)\right\|_{\mathbf{L}^{4/3}(\Gamma_1)} \|\boldsymbol{v}\|_{\mathbf{H}^1(\Omega)}, \qquad \forall \boldsymbol{v} \in \mathbf{V}(\Omega),$$

$$\left|\left(\frac{\mathrm{d}^\alpha f^\varphi}{\mathrm{d}t^\alpha}(t),\overline{\varphi}^{\mathfrak{R}_\varphi}\right)_{L^2(\Omega)}-\left(\frac{\mathrm{d}^\alpha g^\varphi}{\mathrm{d}t^\alpha}(t),\mathrm{tr}_{\Gamma_1^\varphi}\left(\overline{\varphi}^{\mathfrak{R}_\varphi}\right)\right)_{L^2(\Gamma_1^\varphi)}\right|\le\left\|\frac{\mathrm{d}^\alpha f^\varphi}{\mathrm{d}t^\alpha}(t)\right\|_{L^{6/5}(\Omega)}\inf_{\overline{\varphi}^r\in\mathfrak{R}_\varphi}||\overline{\varphi}+\overline{\varphi}^r||_{L^6(\Omega)}$$

$$+\left\|\frac{\mathrm{d}^\alpha g^\varphi}{\mathrm{d}t^\alpha}(t)\right\|_{L^{4/3}(\Gamma_1^\varphi)}\inf_{\overline{\varphi}^r\in\mathfrak{R}_\varphi}||\mathrm{tr}_{\Gamma_1^\varphi}(\overline{\varphi}+\overline{\varphi}^r)||_{L^4(\Gamma_1^\varphi)}$$

$$\le c_1\left(\left\|\frac{\mathrm{d}^\alpha f^\varphi}{\mathrm{d}t^\alpha}(t)\right\|_{L^{6/5}(\Omega)}+\left\|\frac{\mathrm{d}^\alpha g^\varphi}{\mathrm{d}t^\alpha}(t)\right\|_{L^{4/3}(\Gamma_1^\varphi)}\right)\left\|\overline{\varphi}_n^{\mathfrak{R}_\varphi}\right\|_{V^\varphi_{\mathfrak{R}_\varphi}(\Omega)},\qquad\forall\overline{\varphi}^{\mathfrak{R}_\varphi}\in V^\varphi_{\mathfrak{R}_\varphi}(\Omega),$$

$$\left|\left(\frac{\mathrm{d}^\alpha g^\psi}{\mathrm{d}t^\alpha}(t),\mathrm{tr}_{\Gamma_1^\psi}\left(\overline{\psi}^{\mathfrak{R}_\psi}\right)\right)_{L^2(\Gamma_1^\psi)}\right|\le\left\|\frac{\mathrm{d}^\alpha g^\psi}{\mathrm{d}t^\alpha}(t)\right\|_{L^{4/3}(\Gamma_1^\psi)}\inf_{\overline{\psi}^r\in\mathfrak{R}_\psi}\left\|\mathrm{tr}_{\Gamma_1^\psi}\left(\overline{\psi}+\overline{\psi}^r\right)\right\|_{L^4(\Gamma_1^\psi)}$$

$$\le c_1\left\|\frac{\mathrm{d}^\alpha g^\psi}{\mathrm{d}t^\alpha}(t)\right\|_{L^{4/3}(\Gamma_1^\psi)}\left\|\overline{\psi}^{\mathfrak{R}_\psi}\right\|_{V^\psi_{\mathfrak{R}_\psi}(\Omega)},\quad\forall\overline{\psi}^{\mathfrak{R}_\psi}\in V^\psi_{\mathfrak{R}_\psi}(\Omega),$$

$$\left|\left(\frac{\mathrm{d}^\beta g^\theta}{\mathrm{d}t^\beta}(t),\mathrm{tr}_{\Gamma_1^\theta}(\overline{\theta})\right)_{L^2(\Gamma_1^\theta)}\right|\le\left\|\frac{\mathrm{d}^\beta g^\theta}{\mathrm{d}t^\beta}(t)\right\|_{L^{4/3}(\Gamma_1^\theta)}\left\|\mathrm{tr}_{\Gamma_1^\theta}(\overline{\theta})\right\|_{L^4(\Gamma_1^\theta)}$$

$$\le c_1\left\|\frac{\mathrm{d}^\beta g^\theta}{\mathrm{d}t^\beta}(t)\right\|_{L^{4/3}(\Gamma_1^\theta)}\left\|\overline{\theta}\right\|_{H^1(\Omega)},\qquad\forall\overline{\theta}\in V^\theta(\Omega),$$

for almost all $t\in(0,T)$, and, consequently, $\mathrm{d}^\alpha G/\mathrm{d}t^\alpha\in L^2(0,T;\mathbf{V}'(\Omega))$, $\mathrm{d}^\alpha(F^\varphi-G^\varphi)/\mathrm{d}t^\alpha\in L^2(0,T;(V^\varphi_{\mathfrak{R}_\varphi}(\Omega))')$, $\mathrm{d}^\alpha G^\psi/\mathrm{d}t^\alpha\in L^2(0,T;(V^\psi_{\mathfrak{R}_\psi}(\Omega))')$ and $\mathrm{d}^\beta G^\theta/\mathrm{d}t^\beta\in L^2(0,T;(V^\theta(\Omega))')$, $\alpha=0,1,2$, $\beta=0,1$. From the embedding theorem [24], we have that $G,G'\in C([0,T];\mathbf{V}'(\Omega))$, $F^\varphi-G^\varphi,(F^\varphi-G^\varphi)'\in C([0,T];(V^\varphi_{\mathfrak{R}_\varphi}(\Omega))')$, $G^\psi,(G^\psi)'\in C([0,T];(V^\psi_{\mathfrak{R}_\psi}(\Omega))')$ and $G^\theta\in C([0,T];(V^\theta(\Omega))')$.

Multiplying equation (105) by $\mathrm{d}\boldsymbol{u}/\mathrm{d}t$ in the space $L^2(0,T;\mathbf{V}'(\Omega))$, differentiating equations (106) and (107) and multiplying them by φ and ψ in the spaces $L^2(0,T;(V^\varphi_{\mathfrak{R}_\varphi}(\Omega))')$ and $L^2(0,T;(V^\psi_{\mathfrak{R}_\psi}(\Omega))')$, respectively, and multiplying equation (108) by θ/Θ_0 in the space $L^2(0,T;(V^\theta(\Omega))')$, integrating them on $(0,t)$, using the formula for integration by parts [24], and taking into account symmetry conditions (46), and adding the obtained equations, we get the energy equality (54).

By similar arguments, which have been used to obtain estimates (84) to (88), we infer:

$$\left|(\boldsymbol{g}(t),\mathbf{tr}_{\Gamma_1}(\boldsymbol{v}))_{\mathbf{L}^2(\Gamma_1)}\right|\le\frac{c_1}{\varepsilon_1}\left(\|\boldsymbol{g}\|^2_{L^2(0,T;\mathbf{L}^{4/3}(\Gamma_1))}+\|\boldsymbol{g}'\|^2_{L^2(0,T;\mathbf{L}^{4/3}(\Gamma_1))}\right)+\frac{c_2\varepsilon_1}{2}\|\boldsymbol{v}\|^2_{\mathbf{H}^1(\Omega)},\tag{109}$$

$$\left|\int_0^t(\boldsymbol{g}'(\tau),\mathbf{tr}_{\Gamma_1}(\boldsymbol{u}(\tau)))_{\mathbf{L}^2(\Gamma_1)}\mathrm{d}\tau\right|\le\frac{1}{2}\|\boldsymbol{g}'\|^2_{L^2(0,T;\mathbf{L}^{4/3}(\Gamma_1))}+\frac{c_1}{2}\int_0^t\|\boldsymbol{u}(\tau)\|^2_{\mathbf{H}^1(\Omega)}\,\mathrm{d}\tau,\tag{110}$$

$$\left|\int_0^t\left(\left((f^\varphi)'(\tau),\varphi^{\mathfrak{R}_\varphi}(\tau)\right)_{L^2(\Omega)}-\left((g^\varphi)'(\tau),\mathrm{tr}_{\Gamma_1^\varphi}\left(\varphi^{\mathfrak{R}_\varphi}(\tau)\right)\right)_{L^2(\Gamma_1^\varphi)}\right)\mathrm{d}\tau\right|$$

$$\le\frac{1}{2}\left\|(f^\varphi)'(\tau)\right\|^2_{L^2(0,T;L^{6/5}(\Omega))}+\frac{1}{2}\left\|(g^\varphi)'\right\|^2_{L^2(0,T;L^{4/3}(\Gamma_1^\varphi))}+c_1\int_0^t\left\|\varphi^{\mathfrak{R}_\varphi}(\tau)\right\|^2_{V^\varphi_{\mathfrak{R}_\varphi}(\Omega)}\mathrm{d}\tau,\tag{111}$$

$$\left|\int_0^t \left((g^\psi)'(\tau), \mathrm{tr}_{\Gamma_1^\psi}\left(\psi^{\mathfrak{R}_\psi}\right)(\tau)\right)_{L^2(\Gamma_1^\psi)} \mathrm{d}\tau\right| \leq \frac{1}{2}\left\|(g^\psi)'\right\|^2_{L^2(0,T;L^{4/3}(\Gamma_1^\psi))} + \frac{c_1}{2}\int_0^t \left\|\left(\psi^{\mathfrak{R}_\psi}\right)'(\tau)\right\|^2_{V^\psi_{\mathfrak{R}_\psi}(\Omega)} \mathrm{d}\tau, \quad (112)$$

$$\left|\int_0^t \left(g^\theta(\tau), \mathrm{tr}_{\Gamma_1^\theta}(\theta(\tau))\right)_{L^2(\Gamma_1^\theta)} \mathrm{d}\tau\right| \leq \frac{1}{2\varepsilon_1}\left\|g^\theta\right\|^2_{L^2(0,T;L^{4/3}(\Gamma_1^\theta))} + \frac{c_1\varepsilon_1}{2}\int_0^t \|\theta(\tau)\|^2_{H^1(\Omega)} \mathrm{d}\tau, \quad (113)$$

for all $t \in [0,T]$, $\varepsilon_1 > 0$, $\boldsymbol{v} \in \mathbf{H}^1(\Omega)$. By using the conditions (47), the inequalities (61) to (65), (70), (81), (82) and (109) to (113) for small enough $\varepsilon_1 > 0$, and the Cauchy–Schwarz inequality, and from the energy equality (54), we obtain:

$$\begin{aligned}
&\|\boldsymbol{u}(t)\|^2_{\mathbf{H}^1(\Omega)} + \left\|\boldsymbol{u}'(t)\right\|^2_{\mathbf{L}^2(\Omega)} + \left\|\varphi^{\mathfrak{R}_\varphi}(t)\right\|^2_{V^\varphi_{\mathfrak{R}_\varphi}(\Omega)} + \left\|\psi^{\mathfrak{R}_\psi}(t)\right\|^2_{V^\psi_{\mathfrak{R}_\psi}(\Omega)} + \|\theta(t)\|^2_{L^2(\Omega)} + \int_0^t \|\theta(\tau)\|^2_{H^1(\Omega)} \mathrm{d}\tau \\
&\leq c_1 \left(\|\boldsymbol{u}(t)\|^2_{\mathbf{L}^2(\Omega)} + \int_0^t \|\boldsymbol{u}(\tau)\|^2_{\mathbf{H}^1(\Omega)} \mathrm{d}\tau + \int_0^t \left\|\boldsymbol{u}'(\tau)\right\|_{\mathbf{L}^2(\Omega)} \mathrm{d}\tau + \int_0^t \left\|\varphi^{\mathfrak{R}_\varphi}(\tau)\right\|^2_{V^\varphi_{\mathfrak{R}_\varphi}(\Omega)} \mathrm{d}\tau \right. \\
&\qquad \left. + \int_0^t \left\|\psi^{\mathfrak{R}_\psi}(\tau)\right\|^2_{V^\psi_{\mathfrak{R}_\psi}(\Omega)} \mathrm{d}\tau + \int_0^t \|\theta(\tau)\|^2_{L^2(\Omega)} \mathrm{d}\tau \right) \\
&\quad + c_2 \left(\|\boldsymbol{u}_0\|^2_{\mathbf{H}^1(\Omega)} + \|\boldsymbol{u}_1\|^2_{\mathbf{L}^2(\Omega)} + \left\|\varphi_0^{\mathfrak{R}_\varphi}\right\|^2_{V^\varphi_{\mathfrak{R}_\varphi}(\Omega)} + \left\|\psi_0^{\mathfrak{R}_\psi}\right\|^2_{V^\psi_{\mathfrak{R}_\psi}(\Omega)} + \|\theta_0\|^2_{L^2(\Omega)} + \|\boldsymbol{f}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))} + \|\boldsymbol{g}\|^2_{L^2(0,T;\mathbf{L}^{4/3}(\Gamma_1))} \right. \\
&\qquad + \left\|\boldsymbol{g}'\right\|^2_{L^2(0,T;\mathbf{L}^{4/3}(\Gamma_1))} + \left\|(f^\varphi)'\right\|^2_{L^2(0,T;L^{6/5}(\Omega))} + \left\|(g^\varphi)'\right\|^2_{L^2(0,T;L^{4/3}(\Gamma_1^\varphi))} + \left\|(g^\psi)'\right\|^2_{L^2(0,T;L^{4/3}(\Gamma_1^\psi))} \\
&\qquad \left. + \left\|f^\theta\right\|^2_{L^2(0,T;L^2(\Omega))} + \left\|g^\theta\right\|^2_{L^2(0,T;L^{4/3}(\Gamma_1^\theta))} \right), \qquad \forall t \in [0,T]. \quad (114)
\end{aligned}$$

Since $\boldsymbol{u} \in C([0,T];\mathbf{V}(\Omega))$ and $\boldsymbol{u}' \in L^\infty(0,T;\mathbf{V}(\Omega))$, we have:

$$\|\boldsymbol{u}(t)\|^2_{\mathbf{L}^2(\Omega)} \leq 2\|\boldsymbol{u}_0\|^2_{\mathbf{L}^2(\Omega)} + 2t\int_0^t \left\|\boldsymbol{u}'(\tau)\right\|^2_{\mathbf{L}^2(\Omega)} \mathrm{d}\tau, \qquad \forall t \in [0,T]. \quad (115)$$

From inequalities (114) and (115), we obtain:

$$\begin{aligned}
&\|\boldsymbol{u}(t)\|^2_{\mathbf{H}^1(\Omega)} + \left\|\boldsymbol{u}'(t)\right\|^2_{\mathbf{L}^2(\Omega)} + \left\|\varphi^{\mathfrak{R}_\varphi}(t)\right\|^2_{V^\varphi_{\mathfrak{R}_\varphi}(\Omega)} + \left\|\psi^{\mathfrak{R}_\psi}(t)\right\|^2_{V^\psi_{\mathfrak{R}_\psi}(\Omega)} + \|\theta(t)\|^2_{L^2(\Omega)} + \int_0^t \|\theta(\tau)\|^2_{H^1(\Omega)} \mathrm{d}\tau \\
&\leq c_1 \left(\int_0^t \|\boldsymbol{u}(\tau)\|^2_{\mathbf{H}^1(\Omega)} \mathrm{d}\tau + \int_0^t \left\|\boldsymbol{u}'(\tau)\right\|_{\mathbf{L}^2(\Omega)} \mathrm{d}\tau + \int_0^t \left\|\varphi^{\mathfrak{R}_\varphi}(\tau)\right\|^2_{V^\varphi_{\mathfrak{R}_\varphi}(\Omega)} \mathrm{d}\tau + \int_0^t \left\|\psi^{\mathfrak{R}_\psi}(\tau)\right\|^2_{V^\psi_{\mathfrak{R}_\psi}(\Omega)} \mathrm{d}\tau + \int_0^t \|\theta(\tau)\|^2_{L^2(\Omega)} \mathrm{d}\tau \right) \\
&+ c_2 \left(\|\boldsymbol{u}_0\|^2_{\mathbf{H}^1(\Omega)} + \|\boldsymbol{u}_1\|^2_{\mathbf{L}^2(\Omega)} + \left\|\varphi_0^{\mathfrak{R}_\varphi}\right\|^2_{V^\varphi_{\mathfrak{R}_\varphi}(\Omega)} + \left\|\psi_0^{\mathfrak{R}_\psi}\right\|^2_{V^\psi_{\mathfrak{R}_\psi}(\Omega)} + \|\theta_0\|^2_{L^2(\Omega)} + \|\boldsymbol{f}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))} + \|\boldsymbol{g}\|^2_{L^2(0,T;\mathbf{L}^{4/3}(\Gamma_1))} \right. \\
&\qquad + \left\|\boldsymbol{g}'\right\|^2_{L^2(0,T;\mathbf{L}^{4/3}(\Gamma_1))} + \left\|(f^\varphi)'\right\|^2_{L^2(0,T;L^{6/5}(\Omega))} + \left\|(g^\varphi)'\right\|^2_{L^2(0,T;L^{4/3}(\Gamma_1^\varphi))} + \left\|(g^\psi)'\right\|^2_{L^2(0,T;L^{4/3}(\Gamma_1^\psi))} \\
&\qquad \left. + \left\|f^\theta\right\|^2_{L^2(0,T;L^2(\Omega))} + \left\|g^\theta\right\|^2_{L^2(0,T;L^{4/3}(\Gamma_1^\theta))} \right), \qquad \forall t \in [0,T], \quad (116)
\end{aligned}$$

where c_1 and c_2 are positive constants that do not depend on $\boldsymbol{u}$, $\varphi^{\mathfrak{R}_\varphi}$, $\psi^{\mathfrak{R}_\psi}$ or θ. By applying Gronwall's lemma [24], and from inequality (116), we have:

$$\begin{aligned}
&\|\boldsymbol{u}(t)\|^2_{\mathbf{H}^1(\Omega)} + \left\|\boldsymbol{u}'(t)\right\|^2_{\mathbf{L}^2(\Omega)} + \left\|\varphi^{\mathfrak{R}_\varphi}(t)\right\|^2_{V^\varphi_{\mathfrak{R}_\varphi}(\Omega)} + \left\|\psi^{\mathfrak{R}_\psi}(t)\right\|^2_{V^\psi_{\mathfrak{R}_\psi}(\Omega)} + \|\theta(t)\|^2_{L^2(\Omega)} + \int_0^t \|\theta(\tau)\|^2_{H^1(\Omega)}\,\mathrm{d}\tau \\
&\leq Te^{c_1 T}c_2\left(\|\boldsymbol{u}_0\|^2_{\mathbf{H}^1(\Omega)} + \|\boldsymbol{u}_1\|^2_{\mathbf{L}^2(\Omega)} + \left\|\varphi_0^{\mathfrak{R}_\varphi}\right\|^2_{V^\varphi_{\mathfrak{R}_\varphi}(\Omega)} + \left\|\psi_0^{\mathfrak{R}_\psi}\right\|^2_{V^\psi_{\mathfrak{R}_\psi}(\Omega)} + \|\theta_0\|^2_{L^2(\Omega)} + \|\boldsymbol{f}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))}\right. \\
&\qquad + \|\boldsymbol{g}\|^2_{L^2(0,T;\mathbf{L}^{4/3}(\Gamma_1))} + \left\|\boldsymbol{g}'\right\|^2_{L^2(0,T;\mathbf{L}^{4/3}(\Gamma_1))} + \left\|(f^\varphi)'\right\|^2_{L^2(0,T;L^{6/5}(\Omega))} + \left\|(g^\varphi)'\right\|^2_{L^2(0,T;L^{4/3}(\Gamma_1^\varphi))} \\
&\qquad \left. + \left\|(g^\psi)'\right\|^2_{L^2(0,T;L^{4/3}(\Gamma_1^\psi))} + \left\|f^\theta\right\|^2_{L^2(0,T;L^2(\Omega))} + \left\|g^\theta\right\|^2_{L^2(0,T;L^{4/3}(\Gamma_1^\theta))}\right), \quad \forall t\in[0,T]. \qquad (117)
\end{aligned}$$

To obtain estimate for $\varphi_0^{\mathfrak{R}_\varphi}$ and $\psi_0^{\mathfrak{R}_\psi}$, note that, since $\overline{\varphi}^{\mathfrak{R}_\varphi}$ and $\overline{\psi}^{\mathfrak{R}_\psi}$ are independent of each other, the system of equations (52) and (53) can be written as follows:

$$\widetilde{B}\left(\left(\varphi_0^{\mathfrak{R}_\varphi},\psi_0^{\mathfrak{R}_\psi}\right),\left(\overline{\varphi}^{\mathfrak{R}_\varphi},\overline{\psi}^{\mathfrak{R}_\psi}\right)\right) = \widetilde{F}\left(\overline{\varphi}^{\mathfrak{R}_\varphi},\overline{\psi}^{\mathfrak{R}_\psi}\right), \qquad \forall\left(\overline{\varphi}^{\mathfrak{R}_\varphi},\overline{\psi}^{\mathfrak{R}_\psi}\right)\in\widetilde{W}(\Omega), \qquad (118)$$

where $\widetilde{W}(\Omega) = V^\varphi_{\mathfrak{R}_\varphi}(\Omega)\times V^\psi_{\mathfrak{R}_\psi}(\Omega)$ and $\varphi_0^{\mathfrak{R}_\varphi}$ and $\psi_0^{\mathfrak{R}_\psi}$ are equivalence classes of φ_0 and ψ_0, respectively,

$$\begin{aligned}
\widetilde{B}\left(\left(\varphi_0^{\mathfrak{R}_\varphi},\psi_0^{\mathfrak{R}_\psi}\right),\left(\overline{\varphi}^{\mathfrak{R}_\varphi},\overline{\psi}^{\mathfrak{R}_\psi}\right)\right) &= d\left(\varphi_0^{\mathfrak{R}_\varphi},\overline{\varphi}^{\mathfrak{R}_\varphi}\right) + a\left(\psi_0^{\mathfrak{R}_\psi},\overline{\varphi}^{\mathfrak{R}_\varphi}\right) + a\left(\varphi_0^{\mathfrak{R}_\varphi},\overline{\psi}^{\mathfrak{R}_\psi}\right) + \zeta\left(\psi_0^{\mathfrak{R}_\psi},\overline{\psi}^{\mathfrak{R}_\psi}\right), \\
&\qquad \forall(\overline{\varphi}^{\mathfrak{R}_\varphi},\overline{\psi}^{\mathfrak{R}_\psi})\in\widetilde{W}(\Omega), \\
\widetilde{F}\left(\overline{\varphi}^{\mathfrak{R}_\varphi},\overline{\psi}^{\mathfrak{R}_\psi}\right) &= \varepsilon\left(\overline{\varphi}^{\mathfrak{R}_\varphi},\boldsymbol{u}_0\right) + \mu\left(\theta_0,\overline{\varphi}^{\mathfrak{R}_\varphi}\right) + \left(f^\varphi(0),\overline{\varphi}^{\mathfrak{R}_\varphi}\right)_{L^2(\Omega)} - \left(g^\varphi(0),\mathrm{tr}_{\Gamma_1^\varphi}\left(\overline{\varphi}^{\mathfrak{R}_\varphi}\right)\right)_{L^2(\Gamma_1^\varphi)} \\
&\quad + b\left(\overline{\psi}^{\mathfrak{R}_\psi},\boldsymbol{u}_0\right) + m\left(\theta_0,\overline{\psi}^{\mathfrak{R}_\psi}\right) - \left(g^\psi(0),\mathrm{tr}_{\Gamma_1^\psi}\left(\overline{\psi}^{\mathfrak{R}_\psi}\right)\right)_{L^2(\Gamma_1^\psi)}, \forall\left(\overline{\varphi}^{\mathfrak{R}_\varphi},\overline{\psi}^{\mathfrak{R}_\psi}\right)\in\widetilde{W}(\Omega).
\end{aligned}$$

Since $f^\varphi(0)\in L^{6/5}(\Omega)$, $g^\varphi(0)\in L^{4/3}(\Gamma_1^\varphi)$ and $g^\psi(0)\in L^{4/3}(\Gamma_1^\psi)$, and by applying Hölder's inequality, the compatibility conditions (45), estimates (55) and (56) for $w=f^\varphi$, $\widehat{w}=g^\varphi$ and $\widehat{w}=g^\psi$, and the continuity of the embedding $H^1(\Omega)\hookrightarrow L^6(\Omega)$ and the trace operator $\mathrm{tr}: H^1(\Omega)\to L^4(\Gamma)$ [26], we have:

$$\begin{aligned}
&\left|\left(f^\varphi(0),\overline{\varphi}^{\mathfrak{R}_\varphi}\right)_{L^2(\Omega)} - \left(g^\varphi(0),\mathrm{tr}_{\Gamma_1^\varphi}\left(\overline{\varphi}^{\mathfrak{R}_\varphi}\right)\right)_{L^2(\Gamma_1^\varphi)} - \left(g^\psi(0),\mathrm{tr}_{\Gamma_1^\psi}\left(\overline{\psi}^{\mathfrak{R}_\psi}\right)\right)_{L^2(\Gamma_1^\psi)}\right| \\
&\leq ||f^\varphi(0)||_{L^{6/5}(\Omega)}\inf_{\overline{\varphi}^r\in\mathfrak{R}_\varphi}||\overline{\varphi}+\overline{\varphi}^r||_{L^6(\Omega)} + ||g^\varphi(0)||_{L^{4/3}(\Gamma_1^\varphi)}\inf_{\overline{\varphi}^r\in\mathfrak{R}_\varphi}||\mathrm{tr}_{\Gamma_1^\varphi}(\overline{\varphi}+\overline{\varphi}^r)||_{L^4(\Gamma_1^\varphi)} \\
&\quad + ||g^\psi(0)||_{L^{4/3}(\Gamma_1^\psi)}\inf_{\overline{\psi}^r\in\mathfrak{R}_\psi}\left\|\mathrm{tr}_{\Gamma_1^\psi}\left(\overline{\psi}+\overline{\psi}^r\right)\right\|_{L^4(\Gamma_1^\psi)} \\
&\leq c_1\Big(||f^\varphi||_{L^2(0,T;L^{6/5}(\Omega))} + ||(f^\varphi)'||_{L^2(0,T;L^{6/5}(\Omega))} + ||g^\varphi||_{L^2(0,T;L^{4/3}(\Gamma_1^\varphi))} + ||(g^\varphi)'||_{L^2(0,T;L^{4/3}(\Gamma_1^\varphi))} \\
&\qquad + ||g^\psi||_{L^2(0,T;L^{4/3}(\Gamma_1^\psi))} + ||(g^\psi)'||_{L^2(0,T;L^{4/3}(\Gamma_1^\psi))}\Big)\left(\left\|\overline{\varphi}^{\mathfrak{R}_\varphi}\right\|_{V^\varphi_{\mathfrak{R}_\varphi}(\Omega)} + \left\|\overline{\psi}^{\mathfrak{R}_\psi}\right\|_{V^\psi_{\mathfrak{R}_\psi}(\Omega)}\right),
\end{aligned}$$

where $\overline{\varphi}\in V^\varphi(\Omega)$ and $\overline{\psi}\in V^\psi(\Omega)$ are arbitrary elements of the equivalence classes $\overline{\varphi}^{\mathfrak{R}_\varphi}\in V^\varphi_{\mathfrak{R}_\varphi}(\Omega)$ and $\overline{\psi}^{\mathfrak{R}_\psi}\in V^\psi_{\mathfrak{R}_\psi}(\Omega)$. Therefore, by using estimates (64) to (67) for $\overline{\theta}=\theta_0$, $\boldsymbol{v}=\boldsymbol{u}_0$, the inequality $|a_1|+|b_1|\leq\sqrt{2}(a_1^2+b_1^2)^{1/2}$, $a_1,b_1\in\mathbb{R}$, and taking into account that $\theta_0\in V^\theta(\Omega)\subset L^2(\Omega)$, we infer that the linear form

$\widetilde{F} : \widetilde{W}(\Omega) \to \mathbb{R}$ is continuous and

$$\left|\widetilde{F}\left(\overline{\varphi}^{\mathfrak{R}_\varphi}, \overline{\psi}^{\mathfrak{R}_\psi}\right)\right| \leq \sqrt{2}c_2\Bigg(\|\boldsymbol{u}_0\|_{\mathbf{H}^1(\Omega)} + \|\theta_0\|_{L^2(\Omega)} + ||f^\varphi||_{L^2(0,T;L^{6/5}(\Omega))} + ||(f^\varphi)'||_{L^2(0,T;L^{6/5}(\Omega))} + ||g^\varphi||_{L^2(0,T;L^{4/3}(\Gamma_1^\varphi))}$$

$$+ ||(g^\varphi)'||_{L^2(0,T;L^{4/3}(\Gamma_1^\varphi))} + ||g^\psi||_{L^2(0,T;L^{4/3}(\Gamma_1^\psi))} + ||(g^\psi)'||_{L^2(0,T;L^{4/3}(\Gamma_1^\psi))}\Bigg) \left(\left\| \overline{\varphi}^{\mathfrak{R}_\varphi} \right\|^2_{V^\varphi_{\mathfrak{R}_\varphi}(\Omega)} + \left\| \overline{\psi}^{\mathfrak{R}_\psi} \right\|^2_{V^\psi_{\mathfrak{R}_\psi}(\Omega)} \right)^{1/2},$$

$$\forall \left(\overline{\varphi}^{\mathfrak{R}_\varphi}, \overline{\psi}^{\mathfrak{R}_\psi}\right) \in \widetilde{W}(\Omega). \quad (119)$$

Note that

$$\widetilde{B}\left(\left(\varphi_0^{\mathfrak{R}_\varphi}, \psi_0^{\mathfrak{R}_\psi}\right), \left(\overline{\varphi}^{\mathfrak{R}_\varphi}, \overline{\psi}^{\mathfrak{R}_\psi}\right)\right) = B\left(\left(\varphi_0^{\mathfrak{R}_\varphi}, \psi_0^{\mathfrak{R}_\psi}, 0\right), \left(\overline{\varphi}^{\mathfrak{R}_\varphi}, \overline{\psi}^{\mathfrak{R}_\psi}, 0\right)\right).$$

Hence, applying inequality (70), we have:

$$B\left(\left(\varphi_0^{\mathfrak{R}_\varphi}, \psi_0^{\mathfrak{R}_\psi}\right), \left(\varphi_0^{\mathfrak{R}_\varphi}, \psi_0^{\mathfrak{R}_\psi}\right)\right) \geq c_1 \left(\left\| \varphi_0^{\mathfrak{R}_\varphi} \right\|^2_{V^\varphi_{\mathfrak{R}_\varphi}(\Omega)} + \left\| \psi_0^{\mathfrak{R}_\psi} \right\|^2_{V^\psi_{\mathfrak{R}_\psi}(\Omega)} \right),$$

and, from equation (118) and inequality (119), for $(\overline{\varphi}^{\mathfrak{R}_\varphi}, \overline{\psi}^{\mathfrak{R}_\psi}) = (\varphi_0^{\mathfrak{R}_\varphi}, \psi_0^{\mathfrak{R}_\psi})$, using the inequality

$$|a_1| + \cdots + |a_n| \leq \sqrt{n}\left(a_1^2 + \cdots + a_n^2\right)^{1/2}, \qquad a_1, \ldots, a_n \in \mathbb{R}, \quad n \in \mathbb{N},$$

we obtain:

$$\left\| \overline{\varphi}_0^{\mathfrak{R}_\varphi} \right\|^2_{V^\varphi_{\mathfrak{R}_\varphi}(\Omega)} + \left\| \overline{\psi}_0^{\mathfrak{R}_\psi} \right\|^2_{V^\psi_{\mathfrak{R}_\psi}(\Omega)}$$

$$\leq \left(\frac{4c_2}{c_1}\right)^2 \Big(\|\boldsymbol{u}_0\|^2_{\mathbf{H}^1(\Omega)} + \|\theta_0\|^2_{L^2(\Omega)} + ||f^\varphi||^2_{L^2(0,T;L^{6/5}(\Omega))} + ||(f^\varphi)'||^2_{L^2(0,T;L^{6/5}(\Omega))}$$

$$+ ||g^\varphi||^2_{L^2(0,T;L^{4/3}(\Gamma_1^\varphi))} + ||(g^\varphi)'||^2_{L^2(0,T;L^{4/3}(\Gamma_1^\varphi))} + ||g^\psi||^2_{L^2(0,T;L^{4/3}(\Gamma_1^\psi))} + ||(g^\psi)'||^2_{L^2(0,T;L^{4/3}(\Gamma_1^\psi))} \Big).$$

Consequently, from inequality (117), we infer the following estimate:

$$\|\boldsymbol{u}(t)\|^2_{\mathbf{H}^1(\Omega)} + \left\|\boldsymbol{u}'(t)\right\|^2_{\mathbf{L}^2(\Omega)} + \left\| \varphi^{\mathfrak{R}_\varphi}(t) \right\|^2_{V^\varphi_{\mathfrak{R}_\varphi}(\Omega)} + \left\| \psi^{\mathfrak{R}_\psi}(t) \right\|^2_{V^\psi_{\mathfrak{R}_\psi}(\Omega)} + \|\theta(t)\|^2_{L^2(\Omega)} + \int_0^t \|\theta(\tau)\|^2_{H^1(\Omega)}\, \mathrm{d}\tau$$

$$\leq c_1 \Big(\|\boldsymbol{u}_0\|^2_{\mathbf{H}^1(\Omega)} + \|\boldsymbol{u}_1\|^2_{\mathbf{L}^2(\Omega)} + \|\theta_0\|^2_{L^2(\Omega)} + \|\boldsymbol{f}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))} + \|\boldsymbol{g}\|^2_{L^2(0,T;\mathbf{L}^{4/3}(\Gamma_1))} + \left\|\boldsymbol{g}'\right\|^2_{L^2(0,T;\mathbf{L}^{4/3}(\Gamma_1))}$$

$$+ ||f^\varphi||^2_{L^2(0,T;L^{6/5}(\Omega))} + \left\|(f^\varphi)'\right\|^2_{L^2(0,T;L^{6/5}(\Omega))} + ||g^\varphi||^2_{L^2(0,T;L^{4/3}(\Gamma_1^\varphi))} + \left\|(g^\varphi)'\right\|^2_{L^2(0,T;L^{4/3}(\Gamma_1^\varphi))}$$

$$+ ||g^\psi||^2_{L^2(0,T;L^{4/3}(\Gamma_1^\psi))} + \left\|(g^\psi)'\right\|^2_{L^2(0,T;L^{4/3}(\Gamma_1^\psi))} + \left\|f^\theta\right\|^2_{L^2(0,T;L^2(\Omega))} + \left\|g^\theta\right\|^2_{L^2(0,T;L^{4/3}(\Gamma_1^\theta))} \Big), \quad \forall t \in [0,T],$$

which implies the uniqueness of the solution and continuous dependence on the given data. □

Remark 1. If Γ_0^φ and Γ_0^ψ are not empty sets, then the homogeneous equations (41) and (42), when $\boldsymbol{u} \equiv \boldsymbol{0}$, $\theta \equiv 0$ and $f^\varphi \equiv 0$, $g^\varphi \equiv 0$, $g^\psi \equiv 0$, have only trivial solutions. Therefore, $\mathfrak{R}_\varphi = \mathfrak{R}_\psi = \{0\}$, the factor spaces $V^\varphi_{\mathfrak{R}_\varphi}(\Omega)$ and $V^\psi_{\mathfrak{R}_\psi}(\Omega)$ coincide with $V^\varphi(\Omega)$ and $V^\psi(\Omega)$, respectively, and Theorem 3 is valid, when $V^\varphi_{\mathfrak{R}_\varphi}(\Omega)$ is replaced by $V^\varphi(\Omega)$ and $V^\psi_{\mathfrak{R}_\psi}(\Omega)$ is replaced by $V^\psi(\Omega)$.

Remark 2. The conditions (51) and (53) of Theorem 3 for given $\boldsymbol{u}_2 \in \mathbf{L}^2(\Omega)$ and $\theta_0 \in V^\theta(\Omega)$ constitute a system of variational equations with respect to $\boldsymbol{u}_0$, φ_0, ψ_0. By subtracting $(\rho\boldsymbol{u}_2, \boldsymbol{v})_{\mathbf{L}^2(\Omega)}$ from both sides of equation (51), we infer that the system (51) to (53) coincides with equations (34) to (36) of the static problem, where instead of $\boldsymbol{f}$ we have $\boldsymbol{f}(0) - \rho\boldsymbol{u}_2$. Consequently, it follows from the results of the analysis of the static problem [22] that, for solvability of equations (51) to (53), $\boldsymbol{u}_2 \in \mathbf{L}^2(\Omega)$ should satisfy the following condition:

$$(\boldsymbol{f}(0) - \rho\boldsymbol{u}_2, \boldsymbol{v}^r)_{\mathbf{L}^2(\Omega)} + (\boldsymbol{g}(0), \mathbf{tr}_{\Gamma_1}(\boldsymbol{v}^r))_{\mathbf{L}^2(\Gamma_1)} = 0, \tag{120}$$

for all $\boldsymbol{v}^r \in \mathbf{V}(\Omega)$, $\boldsymbol{v}^r(x) = \boldsymbol{\alpha} + \boldsymbol{\beta} \times \boldsymbol{Ox}$, $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^3$, $\boldsymbol{Ox} = (x_i)$, and then there exist $\boldsymbol{u}_0 \in \mathbf{V}(\Omega)$, $\varphi_0 \in V^\varphi(\Omega)$, $\psi_0 \in V^\psi(\Omega)$ satisfying equations (51) to (53). Note that, if Γ_0 is not an empty set, then $\boldsymbol{\alpha} = \boldsymbol{\beta} = \mathbf{0}$; hence, for any $\boldsymbol{u}_2 \in \mathbf{L}^2(\Omega)$, there exists a unique $(\boldsymbol{u}_0, \varphi_0, \psi_0) \in \mathbf{V}(\Omega) \times V^\varphi(\Omega) \times V^\psi(\Omega)$ that satisfies equations (51) to (53).

Remark 3. Sufficient conditions for the existence of $\boldsymbol{u}_2 \in \mathbf{L}^2(\Omega)$, $\varphi_0 \in V^\varphi(\Omega)$ and $\psi_0 \in V^\psi(\Omega)$ satisfying equations (51) to (53) can be obtained by applying regularity results for strongly elliptic systems. To simplify the notations, we assume that the thermo-electro-magneto-elastic solid consists of a single domain, i.e., $K = 1$. In addition to the conditions of Theorem 3, let us assume that Ω is a bounded $C^{1,1}$ domain [23], $\overline{\Gamma_0^\varphi} \cap \overline{\Gamma_1^\varphi} = \varnothing$, $\overline{\Gamma_0^\psi} \cap \overline{\Gamma_1^\psi} = \varnothing$ and the given functions satisfy the following additional regularity properties: c_{ijpq}, ε_{pij}, b_{pij}, d_{ij}, ζ_{ij}, a_{ij}, λ_{ij}, μ_i, $m_i \in C^{0,1}(\overline{\Omega})$, $\boldsymbol{u}_0 \in \mathbf{V}(\Omega) \cap \mathbf{H}^2(\Omega)$, $f^\varphi(0) \in L^2(\Omega)$, $g^\varphi(0) \in H^{1/2}(\Gamma_1^\varphi)$ and $g^\psi(0) \in H^{1/2}(\Gamma_1^\psi)$. Hence, from Rademacher's theorem [25], we have that the functions c_{ijpq}, ε_{pij} , b_{pij}, d_{ij}, ζ_{ij}, a_{ij}, λ_{ij}, μ_i, m_i are differentiable almost everywhere in Ω and that their derivatives belong to $L^\infty(\Omega)$. Applying Green's formula, equations (52) and (53) can be written as

$$\begin{aligned} d(\varphi_0, \overline{\varphi}) + a(\psi_0, \overline{\varphi}) &= \int_{\Gamma_1^\varphi} \mathrm{tr}_{\Gamma_1^\varphi}(\overline{\varphi}) \mathrm{tr}_{\Gamma_1^\varphi}(\boldsymbol{\varepsilon} * \boldsymbol{e}(\boldsymbol{u}_0)) \cdot \boldsymbol{n} \mathrm{d}\Gamma_1^\varphi - \int_\Omega \overline{\varphi} \mathrm{div}\,(\boldsymbol{\varepsilon} * \boldsymbol{e}(\boldsymbol{u}_0))\,\mathrm{d}x \\ &+ \int_{\Gamma_1^\varphi} \mathrm{tr}_{\Gamma_1^\varphi}(\overline{\varphi}) \mathrm{tr}_{\Gamma_1^\varphi}(\boldsymbol{\mu}\theta_0) \cdot \boldsymbol{n} \mathrm{d}\Gamma_1^\varphi - \int_\Omega \overline{\varphi} \mathrm{div}(\boldsymbol{\mu}\theta_0)\mathrm{d}x + (f^\varphi(0), \overline{\varphi})_{L^2(\Omega)} - (g^\varphi(0), \mathrm{tr}_{\Gamma_1^\varphi}(\overline{\varphi}))_{L^2(\Gamma_1^\varphi)}, \end{aligned} \tag{121}$$

$$\begin{aligned} a(\varphi_0, \overline{\psi}) + \zeta(\psi_0, \overline{\psi}) &= \int_{\Gamma_1^\psi} \mathrm{tr}_{\Gamma_1^\psi}(\overline{\psi}) \mathrm{tr}_{\Gamma_1^\psi}(\boldsymbol{b} * \boldsymbol{e}(\boldsymbol{u}_0)) \cdot \boldsymbol{n} \mathrm{d}\Gamma_1^\psi - \int_\Omega \overline{\psi} \mathrm{div}(\boldsymbol{b} * \boldsymbol{e}(\boldsymbol{u}_0))\mathrm{d}x \\ &+ \int_{\Gamma_1^\psi} \mathrm{tr}_{\Gamma_1^\psi}(\overline{\psi}) \mathrm{tr}_{\Gamma_1^\psi}(\boldsymbol{m}\theta_0) \cdot \boldsymbol{n} \mathrm{d}\Gamma_1^\psi - \int_\Omega \overline{\psi} \mathrm{div}(\boldsymbol{m}\theta_0)\mathrm{d}x - (g^\psi(0), \mathrm{tr}_{\Gamma_1^\psi}(\overline{\psi}))_{L^2(\Gamma_1^\psi)}. \end{aligned} \tag{122}$$

It follows, from the positive definiteness condition (48), when $\xi = 0$, that equations (121) and (122) constitute a boundary value problem for a strongly elliptic system of second-order partial differential equations with respect to $\varphi_0 \in V^\varphi(\Omega)$ and $\psi_0 \in V^\psi(\Omega)$ [23], which is solvable since the compatibility conditions (45) are satisfied. From the regularity properties of the given functions, we have:

$$\begin{aligned} \mathrm{tr}_{\Gamma_1^\varphi}(\boldsymbol{\varepsilon} * \boldsymbol{e}(\boldsymbol{u}_0)) \cdot \boldsymbol{n} + \mathrm{tr}_{\Gamma_1^\varphi}(\boldsymbol{\mu}\theta_0) \cdot \boldsymbol{n} - g^\varphi(0) &\in H^{1/2}(\Gamma_1^\varphi), \\ -\mathrm{div}\,(\boldsymbol{\varepsilon} * \boldsymbol{e}(\boldsymbol{u}_0)) - \mathrm{div}(\boldsymbol{\mu}\theta_0) + f^\varphi(0) &\in L^2(\Omega), \\ \mathrm{tr}_{\Gamma_1^\psi}(\boldsymbol{b} * \boldsymbol{e}(\boldsymbol{u}_0)) \cdot \boldsymbol{n} + \mathrm{tr}_{\Gamma_1^\psi}(\boldsymbol{m}\theta_0) \cdot \boldsymbol{n} - g^\psi(0) &\in H^{1/2}(\Gamma_1^\psi), \\ -\mathrm{div}(\boldsymbol{b} * \boldsymbol{e}(\boldsymbol{u}_0)) - \mathrm{div}(\boldsymbol{m}\theta_0) &\in L^2(\Omega), \end{aligned}$$

hence, applying the regularity theorem [23], we infer that the solutions φ_0 and ψ_0 of equations (121) and (122) belong to $H^2(\Omega)$.

Assume now that the following compatibility condition is satisfied:

$$\boldsymbol{g}(0) = \mathrm{tr}_{\Gamma_1}\,(\boldsymbol{c} * \boldsymbol{e}(\boldsymbol{u}_0) + \boldsymbol{\varepsilon} * \mathbf{grad}\,\varphi_0 + \boldsymbol{b} * \mathbf{grad}\,\psi_0 - \boldsymbol{\lambda}\theta_0)\,\boldsymbol{n} \qquad \text{on } \Gamma_1. \tag{123}$$

Applying Green's formula and equation (123), we obtain:

$$c(\boldsymbol{u}_0,\boldsymbol{v})+\varepsilon(\varphi_0,\boldsymbol{v})+b(\psi_0,\boldsymbol{v})-\lambda(\theta_0,\boldsymbol{v})-(\boldsymbol{g}(0),\mathbf{tr}_{\Gamma_1}(\boldsymbol{v}))_{\mathbf{L}^2(\Gamma_1)}$$
$$=-\int_\Omega \mathbf{div}\,(\boldsymbol{c}*\boldsymbol{e}(\boldsymbol{u}_0)+\boldsymbol{\varepsilon}*\mathbf{grad}\,\varphi_0+\boldsymbol{b}*\mathbf{grad}\,\psi_0-\boldsymbol{\lambda}\theta_0)\cdot\boldsymbol{v}\mathrm{d}x,\qquad \forall\boldsymbol{v}\in\mathbf{V}(\Omega).$$

From equation (47), we infer that $1/\rho\in L^\infty(\Omega)$ and, taking

$$\boldsymbol{u}_2=\frac{1}{\rho}\left(\boldsymbol{f}(0)+\mathbf{div}\,(\boldsymbol{c}*\boldsymbol{e}(\boldsymbol{u}_0)+\boldsymbol{\varepsilon}*\mathbf{grad}\,\varphi_0+\boldsymbol{b}*\mathbf{grad}\,\psi_0-\boldsymbol{\lambda}\theta_0)\right),$$

we have $\boldsymbol{u}_2\in\mathbf{L}^2(\Omega)$ and $\boldsymbol{u}_2$, φ_0, ψ_0 satisfy equations (51) to (53). Thus, if the boundary Γ and the given functions satisfy the aforementioned additional regularity conditions and the compatibility condition (123), there exist $\boldsymbol{u}_2\in\mathbf{L}^2(\Omega)$, $\varphi_0\in V^\varphi(\Omega)$ and $\psi_0\in V^\psi(\Omega)$ that satisfy equations (51) to (53).

5. Conclusions

In this work, we investigated boundary and initial boundary value problems with general mixed boundary conditions for mechanical displacement, electric and magnetic potentials and temperature, corresponding to the linear static and dynamic three-dimensional models of multidomain inhomogeneous anisotropic thermo-electro-magneto-elastic bodies. We obtained the variational formulations of the three-dimensional problems in corresponding Sobolev spaces or spaces of vector-valued distributions with respect to the time variable with values in Sobolev spaces that are equivalent to the original differential formulations of boundary and initial boundary value problems in spaces of twice continuously differentiable functions. Also, we obtained new results on the existence and uniqueness of solutions of the three-dimensional boundary and initial boundary value problems in corresponding factor spaces of Sobolev spaces and, to prove the existence, we approximated the solution of the dynamic three-dimensional problem by solutions of finite dimensional systems of ordinary differential equations. Moreover, for the dynamic problem, we obtained an energy equality and, by applying it, we proved the continuous dependence of a solution on given data in suitable function spaces. The methodology presented here can be used to investigate various continuum mechanics models in Sobolev spaces and construction algorithms of their solutions.

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