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# In-domain damping assignment of a Timoshenko-beam using state feedback boundary control

Jeanne Redaud<sup>1\*</sup>, Jean Aurio<sup>1</sup>, and Yann Le Gorrec<sup>2</sup>

**Abstract**—In this paper, we combine the backstepping methodology and the port Hamiltonian framework to design a boundary full state feedback that modifies the closed-loop in-domain damping of a Timoshenko beam. The beam under consideration is clamped in one end of its spatial domain and actuated at the opposite one. The port Hamiltonian formulation is used to derive several boundedly invertible transformations that map the original system into an exponentially stable closed-loop target system with additional in-domain damping terms. The proposed methodology allows the introduction of tuning parameters with clear physical interpretations for achievable closed-loop behavior. Simulations illustrate the performances of the controller.

## I. INTRODUCTION

Beams appear in various engineering applications and can be represented by different mathematical models. The most popular ones are the Euler Bernoulli and Timoshenko beam models. The latter [14] is usually used to represent compliant mechanical structures such as cantilevers, or flexible endoscopes [17]. It takes into account shear deformation and rotational inertia. Even though such systems are naturally stable, control strategies, as static boundary feedbacks [5], [11], have been used to improve their closed-loop performances (settling time and oscillations). They were implemented with equivalent passive dampers or more elaborate active controllers [2]. Later on, the backstepping approach proved to be an efficient way of designing controllers for hyperbolic Partial Differential Equations (PDEs) [6], and in particular, for wave equations and Timoshenko beams [7]. However, this approach must be adapted to each configuration.

In the meanwhile, the Port-Hamiltonian Systems (PHS) framework initially developed for finite-dimensional systems has been extended to infinite-dimensional systems [16]. Exploiting the natural passivity property of beams [10], it is particularly relevant for control design using damping assignment, or energy shaping [9] methods. It has been shown that physical properties can be advantageously used to find a well-posed exponentially stable target system [18] and to derive appropriate boundary controllers, at least in some simple cases. It is then interesting to investigate further on developing full-state feedback boundary controllers inspired by the backstepping methodology for systems represented in this framework [12]. We show in this paper that the two approaches can be advantageously combined in order

to overcome the traditional drawback of backstepping, *i.e.* choosing a physically meaningful target system and state transformations, and achieving interesting closed-loop performances using boundary controllers.

We propose an innovative approach for stabilizing a cantilever Timoshenko beam with one actuated end. In particular, we are not simply seeking to stabilize the system but rather to assign it a specified closed-loop behavior, which corresponds to a copy of the original dynamics with additional in-domain damping that has a clear energy interpretation. We first reformulate the original coupled hyperbolic PDEs in the PHS framework. Our objective is then to find an adequate controller that makes this system behave as the pre-determined target system. We introduce several boundedly invertible integral transforms to map the original system to this target system. These transformations mostly correspond to classical changes of variables and backstepping transformations [1], except one that corresponds to an original time-integral transform. The resulting control law guarantees that the closed-loop system has a behavior equivalent to that of the pre-determined target system.

The proposed methodology constitutes a significant extension of the backstepping approach since the PHS framework is now used to define a physically relevant class of target systems. Instead of simply stabilizing the system by choosing the simplest target system with the minimal number of couplings (as it is classically done with the backstepping methodology [6]), we here define a class of more complex target systems but with a predefinite closed-loop behavior. Thus, the PHS theory gives us a physical framework to apply the backstepping method while introducing natural tuning parameters and adding degrees of freedom to existing designs (e.g., the dissipation rate) with a clear energy interpretation. Although the analysis and design are here performed on a stable Timoshenko beam model, they can be extended to any wave-like equation, even with anti-damping terms. Adding such physical degrees of freedom could improve the closed-loop performance while reducing the control effort.

The organization of the paper is as follows. In Section II, we first present the Timoshenko beam model and its PHS formulation. We also introduce the target system and give some energy interpretation. Then, in Section III, we present a step-by-step approach guaranteeing the existence of an invertible and bounded transform mapping the original system into the desired target system. Finally, the performances of the controller are illustrated in Section IV. Some concluding remarks and perspectives end this paper in Section V.

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## Notations

We denote  $C^n([0, \ell])$  the space of  $n$ -differentiable functions defined on  $[0, \ell]$  with continuous  $n^{\text{th}}$ -derivative. We denote  $D_4^+$ , the space of diagonal matrices in  $\mathbb{R}^{4 \times 4}$  with positive coefficients. Let  $\chi \doteq L^2([0, \ell]; \mathbb{R}^4)$  be the state space equipped with the inner-product  $\langle u, v \rangle_\chi = \frac{1}{2} \int_0^\ell u(x)^T \mathcal{H}(x) v(x) dx$ , where  $\mathcal{H} \in D_4^+$ . We denote  $\|u\|_\chi$  the associated norm (that is equivalent to the standard  $L^2$ -norm). The notation  $I_n$  stands for the  $n \times n$  identity matrix (if the dimensions are not ambiguous, the subindex will be omitted). We denote  $\mathcal{S}$  as the square defined by  $\mathcal{S} = [0, \ell]^2$ . Its lower triangular part is denoted  $\mathcal{T}^- = \{(x, y) \in [0, \ell]^2 \mid 0 \leq y \leq x\}$ . We denote  $\mathcal{T}_1^+ = \{(x, y) \in [0, \ell] \times [0, \frac{\ell}{\mu}] \mid 0 \leq y \leq \frac{x}{\mu}\}$ ,  $\mathcal{T}_1^- = \{(x, y) \in [0, \ell] \times [0, \frac{\ell}{\mu}] \mid 0 \leq y \leq \frac{\ell}{\mu}(1-x)\}$  the two triangular subparts of  $[0, \ell] \times [0, \frac{\ell}{\mu}]$ , and  $\mathcal{P} = \{(x, y) \in [0, \ell] \times [0, 2\frac{\ell}{\mu}] \mid \frac{x}{\mu} \leq y \leq \frac{\ell+x}{\mu}\}$  the parallelogram domain. We denote  $C_{pw}$  the set of piecewise continuous functions. When there is no ambiguity, the time and/or space dependency of the different variables may be omitted.

## II. SYSTEM UNDER CONSIDERATION

### A. Timoshenko beam model

In this paper, we consider a clamped-actuated Timoshenko beam model. It stems from modeling an actuated cantilever of length  $\ell$ . We generally aim at considering weakly damped cantilevers and focus in this paper on the worst-case, *i.e.* the undamped beam. Its transverse normalized displacement (resp. rotation angle) is denoted  $w(t, x)$  (resp.  $\phi(t, x)$ ) and is defined on  $[0, \infty) \times [0, \ell]$ . These two states satisfy the following equations (obtained from the balance equations on the momenta)

$$\rho \frac{\partial^2 w}{\partial t^2}(t, x) = \frac{\partial}{\partial x} \left( K_s \left( \frac{\partial w}{\partial x}(t, x) - \phi(t, x) \right) \right), \quad (1)$$

$$I_\rho \frac{\partial^2 \phi}{\partial t^2}(t, x) = \frac{\partial}{\partial x} \left( EI \frac{\partial \phi}{\partial x} \right) + K_s \left( \frac{\partial w}{\partial x} - \phi \right). \quad (2)$$

For sake of simplicity, we assume that all physical parameters (mass per length unit  $\rho$ , rotary moment of inertia of a cross section  $I_\rho$ , Young's modulus of elasticity  $E$ , moment of inertia  $I$  and shear modulus  $K_s$ ) are space-independent. However, our results could be extended to spatially varying parameters (at the price of heavier computations). The first end of the beam ( $x = 0$ ) is clamped, such that no movement is allowed and  $\frac{\partial w}{\partial t}|_{x=0} = 0$ ,  $\frac{\partial \phi}{\partial t}|_{x=0} = 0$ . We consider here that the opposite end ( $x = \ell$ ) is fully actuated. We have  $K_s \left( \frac{\partial w}{\partial x}|_{x=\ell} - \phi(t, \ell) \right) = u_1(t)$ ,  $EI \frac{\partial \phi}{\partial x}|_{x=\ell} = u_2(t)$ . The initial position of the beam is given by  $w(x, 0) = w_0(x) \in C^1([0, \ell])$ ,  $\phi(x, 0) = \phi_0(x) \in C^1([0, \ell])$ .

### B. Port-Hamiltonian formulation

To rewrite the Timoshenko beam model (1)-(2) as a PHS, we first introduce the Hamiltonian density  $\mathcal{H}$  defined by

$$\mathcal{H} = \text{diag} \left( K_s, EI, \frac{1}{\rho}, \frac{1}{I_\rho} \right) \in D_4^+. \quad (3)$$

We then define the energy state variables  $X = (X_1, X_2, X_3, X_4)^T \in \chi$  by

$$\begin{cases} X_1(t, x) = \frac{\partial w}{\partial x}(t, x) - \phi(t, x) & : \text{shear displacement,} \\ X_2(t, x) = \frac{\partial \phi}{\partial x}(t, x) & : \text{angular displacement,} \\ X_3(t, x) = \rho \frac{\partial w}{\partial t}(t, x) & : \text{momentum,} \\ X_4(t, x) = I_\rho \frac{\partial \phi}{\partial t}(t, x) & : \text{angular momentum.} \end{cases}$$

In this formalism, the original system (1)-(2) rewrites

$$\frac{\partial X}{\partial t} = P_1 \frac{\partial}{\partial x} \mathcal{H} X(t, x) + P_0 \mathcal{H} X(t, x), \quad (4)$$

$$\text{with } P_1 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \in \mathbb{R}^{4 \times 4}, P_0 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{4 \times 4}.$$

The boundary conditions now read

$$X_4(t, 0) = X_3(t, 0) = 0, \quad (5)$$

$$K_s X_1(t, \ell) = u_1(t), \quad EI X_2(t, \ell) = u_2(t). \quad (6)$$

Using the Port-Hamiltonian formalism, we can show that for any control input  $U \in C^2([0, T]; \mathbb{R}^2)$ , for any initial conditions satisfying the *compatibility conditions* (see [4]), there exists a unique classical solution of (4)-(6) [4, Lemma 13.2.1]. The energy of the system is defined by  $\mathcal{E} = \|X\|_\chi^2$ . It verifies

$$\frac{d\mathcal{E}}{dt} = \frac{X_3(t, \ell)}{\rho} u_1(t) + \frac{X_4(t, \ell)}{I_\rho} u_2(t). \quad (7)$$

### C. Control objective

In open-loop, the system (4)-(6) is stable and its energy remains constant. In closed-loop, it can be asymptotically stabilized using a boundary feedback of form  $u_1(t) \propto -X_3(t, \ell)$ ,  $u_2(t) \propto -X_4(t, \ell)$  [5], which makes the energy strictly decreasing. In this paper, we do not simply want to stabilize the system, but we aim to impose a specific **closed-loop behavior**. In particular, we aim to impose a specific decay rate to the energy using *in-domain damping assignment*. More precisely, our objective is to design a control law  $U(t) = (u_1(t), u_2(t))$ , making the dynamics of the closed-loop equivalent to the ones of the target system

$$\frac{\partial}{\partial t} \bar{X} = P_1 \frac{\partial}{\partial x} (\mathcal{H} \bar{X}) + \bar{P}_0 (\mathcal{H} \bar{X}), \quad (8)$$

where  $\bar{X} \in \chi$  and where the distributed damping coefficients are positive ( $c_3, c_4 > 0$ ). The matrix  $\bar{P}_0$  is defined by  $\bar{P}_0 = P_0 + \text{diag}(0, 0, -c_3\rho, -c_4 I_\rho)$ . The boundary conditions are given by

$$\bar{X}_3(t, 0) = \bar{X}_4(t, 0) = 0, \quad \bar{X}_1(t, \ell) = \bar{X}_2(t, \ell) = 0. \quad (9)$$

In that case the energy now decays as

$$\frac{d\mathcal{E}}{dt} = - \int_0^\ell \left( \frac{c_3}{\rho} \bar{X}_3^2(t, x) + \frac{c_4}{I_\rho} \bar{X}_4^2(t, x) \right) dx. \quad (10)$$

### III. CONTROLLER DESIGN

#### A. Overall strategy

To fulfil our control objective, we follow the backstepping methodology. We aim to map the original system (4)-(6) to the target system (8)-(9) using a boundedly invertible transform. The proposed transformation will be decomposed into several successive elementary transforms (exponential changes of variables and classical integral coordinates changes). Although it should be possible to find a unique and global transformation, the *step-by-step* approach is intuitive, and could then be generalized to other systems. In particular, this could be of specific interest when defining suitable target systems for the backstepping approach. We give below the different steps of our approach:

- 1) We use a first change of variables to rewrite the system (4)-(6) in Riemann coordinates. The new state  $\xi$  now satisfies transport equations with in-domain couplings (Section III-B.1).
- 2) On the other hand, starting from the system (8)-(9), we use
  - a) the same change of variables to rewrite the target system in Riemann coordinates (state  $\underline{\xi}$ );
  - b) an exponential change of variables to suppress the diagonal coupling terms (due to  $c_3, c_4$ ). The new state is denoted  $\bar{\xi}$ . This is done in Section III-B.2.
- 3) We use classical backstepping Volterra transforms [3] ( $\mathcal{K}$  and  $\bar{\mathcal{K}}$ ) to map the system  $\xi$  (resp.  $\bar{\xi}$ ) to a simpler target system  $\gamma$  (resp.  $\bar{\gamma}$ ) for which most of the in-domain coupling terms have been moved at the actuated boundary, taking the form of integral couplings. This is done in Section III-C.1.
- 4) Finally, we use a specific invertible affine transform  $\mathcal{F}$  to map the system  $\gamma$  to the system  $\bar{\gamma}$  (Section III-D). It then becomes straightforward to design the corresponding stabilizing feedback law.

A schematic representation of the control strategy is given on Figure 1.

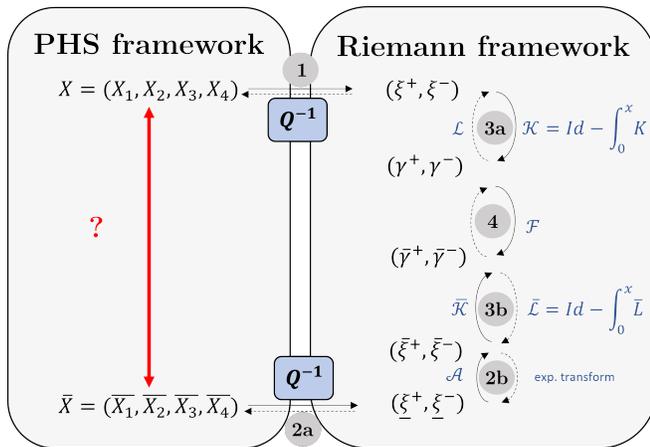


Fig. 1. Overall strategy

In what follows, we define the transport velocities  $\lambda =$

$\sqrt{\frac{K_s}{\rho}} > 0$ ,  $\mu = \sqrt{\frac{EI}{I_\rho}} > 0$  and the matrices  $\Lambda = \text{diag}(\lambda, \mu)$ ,  $R = \text{diag}(\frac{\lambda}{K_s}, \frac{1}{\mu I_\rho}) \in D_2^+$ . We consider the case  $\lambda > \mu$  (the proofs are analogous if  $\lambda \leq \mu$ ).

#### B. Riemann coordinates

The objective is to rewrite the two systems (4)-(6), and (8)-(9) in Riemann coordinates while avoiding the presence of diagonal coupling terms. The matrix  $P_1 \mathcal{H} \in \mathbb{R}^{4 \times 4}$  is invertible with four distinct real eigenvalues  $\{\pm\lambda, \pm\mu\}$ . It is diagonalizable, such that we have  $P_1 \mathcal{H} = Q \text{diag}(-\Lambda, \Lambda) Q^{-1}$  with

$$Q = \frac{1}{\sqrt{2}} \begin{pmatrix} -R & I_2 \\ I_2 & R^{-1} \end{pmatrix} \in \mathbb{R}^{4 \times 4}. \quad (11)$$

1) *Original system*: Let us define the new state  $\xi = Q^{-1}X$ , decomposed into  $\xi(t, x) = (\xi^+ \ \xi^-)^T \in L^2([0, \ell]; \mathbb{R}^4)$ . It verifies

$$\xi_t^+(t, x) + \Lambda \xi_x^+(t, x) = \Sigma^{++} \xi^+ + \Sigma^{+-} \xi^-, \quad (12)$$

$$\xi_t^-(t, x) - \Lambda \xi_x^-(t, x) = \Sigma^{-+} \xi^+ + \Sigma^{--} \xi^-, \quad (13)$$

with the boundary conditions

$$\xi^+(t, 0) = -R^{-1} \xi^-(t, 0), \quad (14)$$

$$\xi^-(t, \ell) = R \xi^+(t, \ell) + \sqrt{2} U(t), \quad (15)$$

where  $\Sigma^{++} = \frac{1}{2} \begin{pmatrix} 0 & \frac{K_s}{\lambda I_\rho} \\ -\lambda & 0 \end{pmatrix}$ ,  $\Sigma^{+-} = \frac{1}{2} \begin{pmatrix} 0 & \frac{\mu K_s}{\lambda} \\ K_s & 0 \end{pmatrix}$ ,

$$\Sigma^{-+} = \frac{1}{2} \begin{pmatrix} 0 & -\frac{1}{I_\rho} \\ -\frac{\lambda}{\mu I_\rho} & 0 \end{pmatrix}, \quad \Sigma^{--} = \frac{1}{2} \begin{pmatrix} 0 & -\mu \\ \frac{K_s}{\mu I_\rho} & 0 \end{pmatrix}.$$

2) *Target system*: We now perform the same change of variables on the system (8)-(9). We define  $\underline{\xi} = Q^{-1} \bar{X}$ , where  $\underline{\xi} = (\underline{\xi}^+ \ \underline{\xi}^-)^T \in L^2([0, \ell]; \mathbb{R}^4)$ . This new state verifies

$$\underline{\xi}_t^+(t, x) + \Lambda \underline{\xi}_x^+(t, x) = \underline{\Sigma}^{++} \underline{\xi}^+ + \underline{\Sigma}^{+-} \underline{\xi}^-, \quad (16)$$

$$\underline{\xi}_t^-(t, x) - \Lambda \underline{\xi}_x^-(t, x) = \underline{\Sigma}^{-+} \underline{\xi}^+ + \underline{\Sigma}^{--} \underline{\xi}^-, \quad (17)$$

where the coupling terms are defined by

$$\underline{\Sigma}^{++} = \frac{1}{2} \begin{pmatrix} -c_3 & \frac{K_s}{\lambda I_\rho} \\ -\lambda & -c_4 \end{pmatrix}, \quad \underline{\Sigma}^{+-} = \frac{1}{2} \begin{pmatrix} -\frac{K_s}{\lambda} c_3 & \frac{K_s \mu}{\lambda} \\ K_s & -\mu I_\rho c_4 \end{pmatrix},$$

$$\underline{\Sigma}^{-+} = \frac{1}{2} \begin{pmatrix} -\frac{\lambda}{K_s} c_3 & -\frac{1}{I_\rho} \\ -\frac{\lambda}{\mu I_\rho} & -\frac{c_4}{\mu I_\rho} \end{pmatrix}, \quad \underline{\Sigma}^{--} = \frac{1}{2} \begin{pmatrix} -c_3 & -\mu \\ \frac{K_s}{\mu I_\rho} & -c_4 \end{pmatrix}.$$

The boundary conditions read as

$$\underline{\xi}^+(t, 0) = -R^{-1} \underline{\xi}^-(t, 0), \quad \underline{\xi}^-(t, \ell) = R \underline{\xi}^+(t, \ell). \quad (18)$$

Due to the presence of the terms  $-c_3 \rho, -c_4 I_\rho$ , the diagonal coefficients of  $\underline{\Sigma} = \begin{pmatrix} \underline{\Sigma}^{++} & \underline{\Sigma}^{+-} \\ \underline{\Sigma}^{-+} & \underline{\Sigma}^{--} \end{pmatrix}$  are not equal to zero. Before using the backstepping methodology, we must get rid of these terms, that cannot be handled by Volterra integral transforms. To do so, we perform an exponential change of coordinates. We define  $\alpha \doteq \frac{c_3}{2\lambda}$ ,  $\beta \doteq \frac{c_4}{2\mu}$ , and introduce the bounded invertible exponential operator

$$\mathcal{A} : L^2([0, \ell]; \mathbb{R}^4) \longrightarrow L^2([0, \ell]; \mathbb{R}^4)$$

$$\begin{pmatrix} u \\ v \end{pmatrix} (\cdot) \mapsto \text{diag}(e^{\alpha \cdot}, e^{\beta \cdot}, e^{-\alpha \cdot}, e^{-\beta \cdot}) \begin{pmatrix} u \\ v \end{pmatrix} (\cdot).$$

We now consider the state  $\bar{\xi} = (\bar{\xi}^+ \ \bar{\xi}^-)^T = \mathcal{A}(\underline{\xi})$ . It verifies

$$\bar{\xi}_t^+(t, x) + \Lambda \bar{\xi}_x^+(t, x) = \sigma^{++}(x) \bar{\xi}^+ + \sigma^{+-}(x) \bar{\xi}^-, \quad (19)$$

$$\bar{\xi}_t^-(t, x) - \Lambda \bar{\xi}_x^-(t, x) = \sigma^{-+}(x) \bar{\xi}^+ + \sigma^{--}(x) \bar{\xi}^-, \quad (20)$$

with the boundary conditions

$$\bar{\xi}^+(t, 0) = -R^{-1} \bar{\xi}^-(t, 0), \quad \bar{\xi}^-(t, \ell) = \bar{R} \bar{\xi}^+(t, \ell), \quad (21)$$

where  $\bar{R} \doteq \text{diag}(e^{-2\alpha\ell}, e^{-2\beta\ell})R$ . The coupling matrix  $\sigma(x) = \begin{pmatrix} \sigma^{++} & \sigma^{+-} \\ \sigma^{-+} & \sigma^{--} \end{pmatrix}(x)$  is now space-dependent but does not have any diagonal terms. Its components are defined by

$$\begin{aligned} \sigma^{++}(x) &= \frac{1}{2} \begin{pmatrix} 0 & \frac{K_s}{\lambda I_\rho} e^{(\alpha-\beta)x} \\ -\lambda e^{-(\alpha-\beta)x} & 0 \end{pmatrix}, \\ \sigma^{+-}(x) &= \frac{1}{2} \begin{pmatrix} -\frac{K_s}{\lambda} c_3 e^{2\alpha x} & \frac{K_s \mu}{\lambda} e^{(\alpha+\beta)x} \\ K_s e^{(\alpha+\beta)x} & -\mu I_\rho c_4 e^{2\beta x} \end{pmatrix}, \\ \sigma^{-+}(x) &= \frac{1}{2} \begin{pmatrix} -\frac{\lambda}{K_s} c_3 e^{-2\alpha x} & -\frac{1}{I_\rho} e^{-(\alpha+\beta)x} \\ -\frac{\lambda}{\mu I_\rho} e^{-(\alpha+\beta)x} & -\frac{c_4}{\mu I_\rho} e^{-2\beta x} \end{pmatrix}, \\ \sigma^{--}(x) &= \frac{1}{2} \begin{pmatrix} 0 & -\mu e^{-(\alpha-\beta)x} \\ \frac{K_s}{\mu I_\rho} e^{(\alpha-\beta)x} & 0 \end{pmatrix}. \end{aligned}$$

### C. Classical backstepping transformations

Our next objective is to map systems (12)-(15) and (19)-(21) into simpler target systems, where most of the coupling terms have been moved to the boundary.

1) *Target system  $\gamma$* : Inspired by [1], [3], we define the Volterra integral transform  $\mathcal{K} : L^2([0, \ell]; \mathbb{R}^4) \rightarrow L^2([0, \ell]; \mathbb{R}^4)$  by

$$\mathcal{K}(\xi) = \xi(x) - \int_0^x K(x, y) \xi(y) dy, \quad (22)$$

where the kernels  $K = \begin{pmatrix} K^{++}(x, y) & K^{+-}(x, y) \\ K^{-+}(x, y) & K^{--}(x, y) \end{pmatrix} \in C_{pw}^1(\mathcal{T}^-; \mathbb{R}^{4 \times 4})$  satisfy the set of equations

$$\Lambda K_x^{++} + K_y^{++} \Lambda = -K^{++} \Sigma^{++} - K^{+-} \Sigma^{+-}, \quad (23)$$

$$\Lambda K_x^{+-} - K_y^{+-} \Lambda = -K^{++} \Sigma^{+-} - K^{+-} \Sigma^{--}, \quad (24)$$

$$\Lambda K_x^{-+} - K_y^{-+} \Lambda = K^{-+} \Sigma^{++} + K^{--} \Sigma^{-+}, \quad (25)$$

$$\Lambda K_x^{--} + K_y^{--} \Lambda = K^{-+} \Sigma^{+-} + K^{--} \Sigma^{--}, \quad (26)$$

with the boundary conditions in  $x = y$ ,

$$\Lambda K^{++}(x, x) - K^{++}(x, x) \Lambda = \Sigma^{++}, \quad (27)$$

$$\Lambda K^{+-}(x, x) + K^{+-}(x, x) \Lambda = \Sigma^{+-}, \quad (28)$$

$$\Lambda K^{-+}(x, x) + K^{-+}(x, x) \Lambda = -\Sigma^{-+}, \quad (29)$$

$$\Lambda K^{--}(x, x) - K^{--}(x, x) \Lambda = -\Sigma^{--}. \quad (30)$$

We have also boundary conditions in  $y = 0$ ,

$$K_{ij}^{++}(x, 0) = -(K^{+-}(x, 0) \Lambda R \Lambda^{-1})_{ij}, \quad i \leq j, \quad (31)$$

$$K_{ij}^{-+}(x, 0) = -(K^{-+}(x, 0) \Lambda R^{-1} \Lambda^{-1})_{ij}, \quad i \leq j. \quad (32)$$

To have a well-posed system, we add two extra boundary conditions on  $x = \ell$  for  $K_{21}^{\pm\pm}$ :

$$K_{21}^{++}(\ell, y) = \frac{\lambda}{2(\lambda - \mu)}, \quad K_{21}^{--}(\ell, y) = \frac{K_s}{2\mu I_\rho(\lambda - \mu)}. \quad (33)$$

We have the following lemma

*Lemma 1*: [3] The system (23)-(33) admits a unique solution in  $C_{pw}^1(\mathcal{T}^-; \mathbb{R}^{4 \times 4})$ . The Volterra integral transform  $\mathcal{K}$  defined by (22) is boundedly invertible. Its inverse transform  $\mathcal{L} = \mathcal{K}^{-1}$  is also a Volterra integral transform. The kernel  $L \in C_{pw}^1(\mathcal{T}^-; \mathbb{R}^{4 \times 4})$  satisfies

$$L(x, y) = -K(x, y) + \int_y^x K(x, s) L(s, y) ds. \quad (34)$$

*Proof 1*: The proof can be adjusted from [3, Theorem A.1]. The kernels' regularity derives from the one of  $\Sigma$ .

We now define  $\gamma = \mathcal{K}(\xi)$ . Following the backstepping methodology and differentiating equation (22) with respect to time and space, the state  $\gamma = (\gamma^+ \ \gamma^-)^T$  verifies

$$\gamma_t^+(t, x) + \Lambda \gamma_x^+(t, x) = G_1(x) \gamma^-(t, 0), \quad (35)$$

$$\gamma_t^-(t, x) - \Lambda \gamma_x^-(t, x) = G_2(x) \gamma^-(t, 0), \quad (36)$$

with the boundary conditions

$$\gamma^+(t, 0) = -R^{-1} \gamma^-(t, 0), \quad (37)$$

$$\gamma^-(t, \ell) = R \gamma^+(t, \ell) + \sqrt{2} U(t) + I_1(t), \quad (38)$$

where  $G_1, G_2$  are strictly lower triangular matrices  $G_i(x) = \begin{pmatrix} 0 & 0 \\ g_i(x) & 0 \end{pmatrix}$ ,  $i \in \{1, 2\}$  defined by

$$\begin{aligned} g_1(x) &= K_s K_{21}^{++}(x, 0) + \lambda K_{21}^{+-}(x, 0), \\ g_2(x) &= K_s K_{21}^{-+}(x, 0) + \lambda K_{21}^{--}(x, 0). \end{aligned} \quad (39)$$

The integral term  $I_1(t)$  is expressed using the inverse kernels

$$\begin{aligned} I_1(t) &= \int_0^\ell (L^{-+}(\ell, y) - R L^{++}(\ell, y)) \gamma^+(t, y) \\ &\quad + (L^{--}(\ell, y) - R L^{+-}(\ell, y)) \gamma^-(t, y) dy. \end{aligned} \quad (40)$$

2) *Target system  $\bar{\gamma}$* : We can adjust the transformation (22) to simplify the system (19)-(21). Define the integral operator  $\bar{\mathcal{K}} : L^2([0, \ell]; \mathbb{R}^4) \rightarrow L^2([0, \ell]; \mathbb{R}^4)$  by

$$\bar{\mathcal{K}}(\bar{\xi}) = \bar{\xi}(x) - \int_0^x \bar{K}(x, y) \bar{\xi}(y) dy, \quad (41)$$

where the kernels  $\bar{K} = \begin{pmatrix} \bar{K}^{++}(x, y) & \bar{K}^{+-}(x, y) \\ \bar{K}^{-+}(x, y) & \bar{K}^{--}(x, y) \end{pmatrix} \in C_{pw}^1(\mathcal{T}^-; \mathbb{R}^{4 \times 4})$  satisfy analogous equations to the ones given by equations (23)-(32) (except that terms  $\Sigma^{\cdot\cdot}$  are replaced by functions  $\sigma^{\cdot\cdot}(y)$  in the PDEs, and in the boundary condition  $x = y$ ). To have a well-posed system, and guarantee continuity along the characteristic curve starting from  $(\ell, \ell)$ , we impose

$$\bar{K}_{21}^{++}(y, \ell) = \frac{\lambda e^{(\beta-\alpha)\ell}}{2(\lambda - \mu)}, \quad \bar{K}_{21}^{--}(y, \ell) = \frac{K_s e^{(\alpha-\beta)\ell}}{2\mu I_\rho(\lambda - \mu)}. \quad (42)$$

Transformation (41) is boundedly invertible and the inverse transformation is denoted  $\bar{\mathcal{L}}$  (the corresponding kernels being denoted  $\bar{L}(x, y)$ ).

We now define the state  $\bar{\gamma} = \bar{\mathcal{K}}(\bar{\xi})$ . It verifies

$$\bar{\gamma}_t^+(t, x) + \Lambda \bar{\gamma}_x^+(t, x) = \bar{G}_1(x) \bar{\gamma}^-(t, 0), \quad (43)$$

$$\bar{\gamma}_t^-(t, x) - \Lambda \bar{\gamma}_x^-(t, x) = \bar{G}_2(x) \bar{\gamma}^-(t, 0), \quad (44)$$



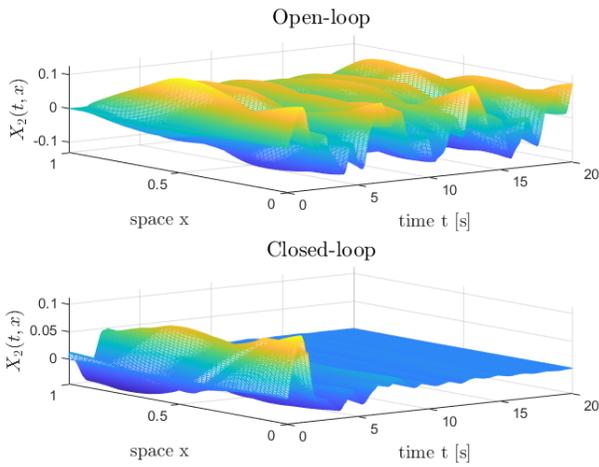


Fig. 2. Evolution of  $X_2(t, x)$  in Open-Loop and Closed-Loop (bottom)

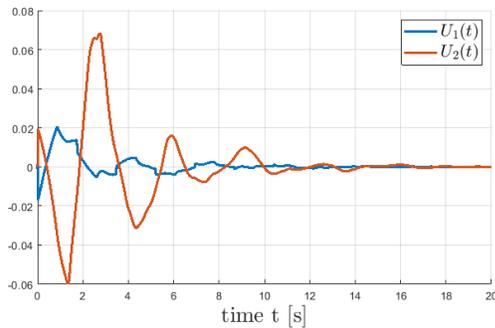


Fig. 3. Evolution of the control effort  $U(t)$

## V. CONCLUSION

In this paper, we presented an innovative full-state feedback boundary controller inspired by the backstepping methodology for stabilizing a Timoshenko beam with predefined closed-loop properties. We used the Port-Hamiltonian formalism to determine a target system of strictly decreasing energy. The controller was designed such that the closed-loop original system behaves like this target system. As the implementation of this full-state feedback controller requires the knowledge of all the states, the proposed controller should be coupled with a state observer. This could be done following the backstepping methodology, as in [7]. The PHS theory gave us a physical framework to apply the backstepping method while introducing natural tuning parameters and adding degrees of freedom (e.g., the dissipation rate) with a clear energy interpretation. This is an important step for the practical implementation of backstepping controllers as it allows the design of easily parametrizable (and attainable) target systems for which the closed-loop behaviors are perfectly known and match performance specifications. The proposed methodology can be extended to naturally unstable systems, as anti-damped wave equations [13] or to a general class of hyperbolic systems. In future works, we would like to take advantage of the intrinsic modularity of the Port-Hamiltonian framework to adapt this approach

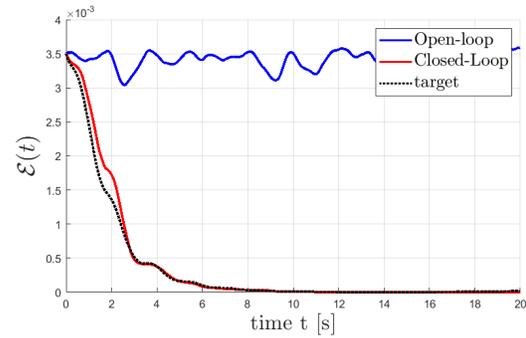


Fig. 4. Evolution of the energy  $\mathcal{E}(t)$  in open-loop and closed-loop

to larger networks of interconnected systems or to more complex models [15]. We will also consider the robustness aspects neglected for the moment.

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