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# Characterization of PI feedback controller gains for interconnected ODE - hyperbolic PDE systems

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**Abstract:** In this paper, we give explicit criteria to guarantee the robust stabilization of simple first-order transport systems coupled with finite-dimensional dynamics, using Proportional or Proportional Integral boundary output feedback controllers. Using a frequency-sweeping approach developed for Time-Delay Systems, we characterize the closed-loop stability intervals in the parameter space for several interconnected partial differential equations and ordinary differential equations configurations. The performances of the proposed boundary feedback controllers are compared with other classical approaches in simulation. The proposed methodology is a necessary step before a detailed comparison between proportional integral controllers with low computational complexity and other control approaches for infinite-dimensional systems.

*Keywords:* infinite dimensional systems; PI feedback controller; neutral time-delay systems.

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## 1. INTRODUCTION

Modern complex dynamical systems are highly interconnected and mutually interdependent, both physically and through a multitude of information and communication networks. By properly formulating them in terms of subsystems' interactions involving energy and mass transfer, their dynamical models can be derived from mass, energy, and information balance considerations. It generates dynamic states with values distributed over time and space. Such interconnections lead to increased complexity, emergence of new dynamics, whether or not feedback is present, environmental effects in a broad sense, with possibly the non-separability of these aspects.

In this context, interconnected Ordinary Differential Equations (ODEs) and hyperbolic Partial Differential Equations (PDEs) are one of the simplest structure that conserves the distributed nature of the underlying dynamics. These systems appear in many engineering applications, from electrical networks containing transmission lines Brayton (1968), or mechanical vibrations in drilling systems Aarsnes and Shor (2018) to thermo-acoustic oscillations in combustion dynamics de Andrade et al. (2018). The ODE part can efficiently model current/voltage in lumped elements, or loads or actuator dynamics, while the PDE part can model the dynamics of the lossless transmission lines in electrical networks or any propagation phenomena as vibrations.

Recently, several approaches have been developed for the control of coupled PDE-ODE systems (predictor-type feedback Krstic and Smyshlyaev (2008); Redaud et al. (2021), backstepping design Deutscher et al. (2018), flatness strategies Meurer and Kugi (2009)). Although these contributions enabled major breakthroughs, the analysis of the closed-loop behavior and robustness properties have

been neglected so far, due to the lack of experimental implementations or benchmarks. In particular, due to the complexity of their control architecture, infinite-dimensional controllers require know-how and computing power to be implemented on real systems. On the contrary, PID controllers are broadly applicable, and many experimental methods exist to tune their parameters. It explains why practitioners usually prefer using such standard controllers, despite the lack of theoretical results for ODE-PDE interconnections. To the best of the authors' knowledge, there do not exist explicit conditions guaranteeing the existence of appropriate gain values stabilizing such interconnected systems.

In this paper, we give explicit criteria to guarantee the robust stabilization of simple first-order transport systems coupled with finite-dimensional dynamics, using Proportional (P) or Proportional Integral (PI) boundary output feedback controllers. We first rewrite the system as a time-delay system of neutral type to which we apply the D-decomposition method Neimark (1949). Indeed, integrating along the characteristics lines, solutions of hyperbolic systems can be closely related to such time-delay systems Auriol and Di Meglio (2019). Using a frequency-domain representation, we study the location of the characteristic roots of the closed-loop systems, that characterizes the rate of exponential growth of their fundamental solution. We can therefore define proportional and integral gains to ensure that all the poles have negative real parts Gu et al. (2003). We determine the closed-loop stability intervals in the parameter space for one test case (transport equations with actuation dynamics). The main contribution of this paper is methodological and presents how frequency-domain approaches developed for delay-differential equations (DDEs) can be relevant for the analysis of coupled hyperbolic PDE-ODE systems. It should be mentioned

that the idea of reducing a class of nonlinear mixed initial boundary problems of such systems to a initial value problem of a system described by coupled continuous-time delay-difference with delay-differential equations goes back to the 60s Cooke and Krumme (1968). We propose a short comparison with state-of-the-art control approaches. It is a necessary step before a detailed comparison between industry-standard PI controllers and recently developed control approaches, that can cover more general cases by explicitly taking into account the delays and high-frequency dynamics at a cost of an increased complexity.

The layout of this paper is the following. In Section 2 we present the system under consideration and the control strategy. In Section 3, the D-decomposition method used for the stability analysis of scalar neutral-type equations is briefly introduced. Then, in Section 4, this strategy is applied to design stabilizing P and PI controllers for transport equations with actuation dynamics. We characterize the closed-loop stability intervals in the parameter space, and we give some illustrative simulation results.

*Notations* The Hilbert space of square integrable functions is denoted  $L^2([0, 1]; \mathbb{R}) \doteq L^2(0, 1)$ . We denote  $\chi = \mathbb{R} \times L^2([0, 1]; \mathbb{R}^2)$ . For all state  $(X, u, v) \in \chi$ , we define the  $\chi$ -norm as the sum of  $L^2$ -norms and Euclidian norm of the states  $\|(X, u, v)\|_\chi = (\|u\|_{L^2}^2 + \|v\|_{L^2}^2 + \|X\|^2)^{\frac{1}{2}}$ .

## 2. PROBLEM DESCRIPTION

### 2.1 Systems under consideration

In this paper we consider a scalar linear heterodirectional hyperbolic system (namely a system of two transport equations) coupled at one end with a scalar ODE. More precisely, we consider systems with the following structure

$$\partial_t u(t, x) + \lambda \partial_x u(t, x) = 0, \quad (1)$$

$$\partial_t v(t, x) - \mu \partial_x v(t, x) = 0, \quad (2)$$

$$\dot{X}(t) = aX(t) + bv(t, 0) + V(t), \quad (3)$$

with the boundary conditions

$$u(t, 0) = qv(t, 0) + cX(t), \quad (4)$$

$$v(t, 1) = \rho u(t, 1), \quad (5)$$

where  $t > 0$  and  $x \in [0, 1]$  are respectively the time and space variables,  $X \in \mathbb{R}$  is the ODE state,  $u(t, x) \in \mathbb{R}$  and  $v(t, x) \in \mathbb{R}$  are the PDE states, and  $V(t)$  is the control input. The transport velocities  $\lambda > 0$  and  $\mu > 0$ , and the coupling terms  $a, b, c, q, \rho$  are all real. The output of the system is denoted  $y(t)$  and defined by

$$y(t) = X(t). \quad (6)$$

Note that signal  $y(t)$  corresponds to collocated measurements<sup>1</sup>.

Such system is schematically illustrated in Figure 1. Here, the ODE corresponds to actuator dynamics acting on a wave-like propagation system. This case is motivated by industrial applications such as suppressing mechanical vibrations in drilling devices with the bit off-bottom. These systems are composed of long flexible strings subject to torsional vibrations that propagate upwards and downwards, represented by the PDE states  $(u, v)$  Aarsnes and Shor (2018).

<sup>1</sup> The case of non-collocated measurements could be covered using similar techniques.

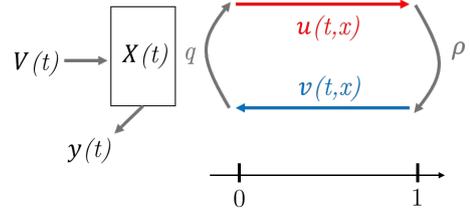


Fig. 1. Schematic representation of system (1)-(5)

Due to space restrictions, we did not present the case of load dynamics.

The initial condition is denoted  $(X(0), u(x, 0), v(x, 0)) = (X_0, u_0(x), v_0(x)) \in \chi$ . The open-loop system is well-posed (Bastin and Coron, 2016, Theorem A.6). In what follows, we denote  $\tau$  the transport delay induced by the PDE system  $\tau = \frac{1}{\lambda} + \frac{1}{\mu}$  and  $r = \rho q$ . Finally, we make the following assumption:

*Assumption 1.* The boundary couplings  $\rho$  and  $q$  verify  $|r| = |\rho q| < 1$ .

Assumption 1 is slightly stronger than a necessary condition for delay-robust stabilization and prevents having an infinite number of poles on the Right-Half Plane (RHP) Logemann et al. (1996); Hale and Verduyn Lunel (2013); Auriol et al. (2018a,b).

### 2.2 Control objective

Even if the PDE and ODE systems can be separately stable, the coupling between them can be the source of instabilities. Recent contributions in the literature have considered the design of stabilizing controllers for the interconnected system (1)-(5) using a backstepping approach Redaud et al. (2021); Auriol et al. (2018b). However, the resulting output-feedback laws may be computationally expensive, as they require solving in real-time a PDE system (observer system) or performing a dynamical inversion of the ODE dynamics Bou Saba et al. (2019). Thus, it might be valuable to consider Proportional (P) or Proportional Integral (PI) controllers to lower the computational effort, which can be critical for practical implementations. However, one must be aware that the plant (1)-(5) may not always be stabilizable by P or PI controllers (1)-(5) Bastin et al. (2015), and the resulting output-feedback laws may induce lower closed-loop performance compared to the backstepping controllers. The objective of this paper is to derive general conditions under which it is possible to stabilize the plant (1)-(5) using P or PI controllers. More precisely, the control input is chosen as

$$V(t) = K_p y(t) + K_i \int_0^t y(\nu) d\nu, \quad (7)$$

the coefficient  $K_i$  being equal to zero for the proportional case. For each case, we give admissible bounds for the gains  $(K_p, K_i)$  so that the closed-loop is exponentially stabilized in the sense of the  $\chi$ -norm, that is to say there exist  $\nu > 0$ ,  $C_0 \geq 1$ , for all  $(X_0, u_0, v_0) \in \chi$ , such that

$$\|(X, u, v)\|_\chi \leq C_0 e^{-\nu t} \|(X_0, u_0, v_0)\|_\chi.$$

The performances of the proposed control laws will be compared with other classical approaches in simulation.

A complete analysis of the closed-loop quantitative performance is out of the scope of the paper but will be the purpose of future work.

### 2.3 Methodology

The methodology we follow relies on tools developed for time-delay systems:

- First, using the method of characteristics, we rewrite the original PDE-ODE system as a time-delay equation of neutral type;
- Next, applying the Laplace transform, we derive the transfer function of the system. We deduce the characteristic quasipolynomial equation  $\delta(s, \tau) = 0$  satisfied by its poles, in open-loop and in closed-loop with the control law (7);
- Following the *D-decomposition approach* Neimark (1949) and the methodology proposed in Niculescu (2001), we then determine stability regions on the parameter space for the closed-loop system, with appropriate conditions on the controller gain  $K_p$ ;
- Finally, we discuss the robustness properties of the closed-loop system regarding uncertainties in parameters. In particular, we study the impact of the integral term on the overall performance.

We give some Matlab simulation results to illustrate this approach. To guarantee consistency and ease interpretation, all simulations are made with the same initial conditions. Although the ODE state is assumed to be scalar, the method we present in this paper can be generalized to non-scalar ODEs. The case of non-scalar PDEs requires a deeper analysis as it induces several and possibly non-commensurate transport delays. However, it should be possible to follow a similar methodology.

## 3. PRELIMINARY RESULTS ON LINEAR TIME-DELAY SYSTEMS OF NEUTRAL TYPE

In this section, we recall some general results regarding the stability of time-delay systems of neutral type. In particular, we present the D-decomposition method and apply it to first-order neutral equations.

### 3.1 General considerations

In what follows we consider a neutral equation of the form

$$\sum_{k=0}^M d_k z^{(k)}(t) + n_k z^{(k)}(t - \tau) = 0, \quad (8)$$

where  $M \in \mathbb{N}$ ,  $d_k$  and  $n_k$  are real coefficients such that  $d_M n_M \neq 0$ , and the state  $z$  is defined on  $[-\tau, 0]$  by an appropriate initial condition  $z_0 \in L^2([-\tau, 0], \mathbb{R})$ . The quasipolynomial characteristic equation associated to (8) is given in the Laplace domain by

$$\delta(s, \tau) = d(s) + n(s)e^{-\tau s} = 0, \quad (9)$$

with  $d(s) = \sum_{k=0}^M d_k s^k$ ,  $n(s) = \sum_{k=0}^M n_k s^k$ , and  $\deg(d) = \deg(n)$ . This is a major difference with retarded systems, for which  $\deg(d) > \deg(n)$ . The study of stability properties in the *neutral* case is much more complicated. Here, the real part of the rightmost root of the spectrum of the corresponding characteristic function is not necessarily

continuous, due to the presence of the *essential spectrum* Michiels and Niculescu (2014).

### 3.2 D-decomposition method

To analyze the stability of linear time-delay systems in the parameter-space, we consider that the delay  $\tau > 0$  is fixed (D-decomposition approach Neimark (1949)). The methodology used in the sequel can be resumed as follows Cooke and van den Driessche (1986); Niculescu (2001):

- (1) First, we consider the *delay-free* case  $\tau = 0$  to determine a first stability condition for the system without delay;
- (2) Then, we determine the frequency of characteristic roots crossing the imaginary axis, that can correspond to a switch in stability. If they exist, we define a stability interval  $[0, \tau_c)$  on which the system is stable.
- (3) If there are no crossing roots, since the change of stability can only occur through the imaginary axis, we derive conditions for stability of the closed-loop system independently of the intrinsic delay  $\tau$ .

### 3.3 Application to the first-order case

We show in the following that the interconnected system (1)-(5), in open-loop or subject to a proportional feedback controller, can be expressed as a first-order neutral equation yielding a quasi-polynomial of degree 3. We give first some general stability results applying the aforementioned approach.

Let us consider that two first-order polynomials  $d(s) = s + \alpha$  and  $n(s) = \gamma s + \beta$ , such that  $\delta(s, \tau)$  in (9) rewrites as:

$$\begin{aligned} \delta(s, \tau) &= s + \alpha + (\gamma s + \beta)e^{-\tau s} \\ &= s(1 + \gamma e^{-\tau s}) + \alpha + \beta e^{-\tau s}. \end{aligned} \quad (10)$$

As discussed in Hale and Verduyn Lunel (2013), the exponential stability of the trivial solution of the difference equation  $z(t) + \gamma z(t - \tau) = 0$  is a *necessary condition* for the exponential stability of the scalar neutral equation with the characteristic function  $\delta$ . In our case, this condition simply rewrites as  $|\gamma| < 1$ .

We first study the stability in the *delay-free* case, since, as discussed in Michiels and Niculescu (2014), increasing the delay in the closed-loop system induces instability. Without delay, the characteristic equation (10) rewrites as a first-order polynomial  $\delta(s, 0) = s(1 + \gamma) + \alpha + \beta$ , whose only root is given by  $s^* = -\frac{\alpha + \beta}{1 + \gamma}$ . It belongs to the LHP under the condition

$$\alpha + \beta > 0. \quad (11)$$

If this condition is not satisfied, the controller cannot even stabilize the delay-free plant. Next, consider the real delayed equation. Under condition (11), by a continuity argument, the system is stable for  $\tau \in [0, \tau_c)$ , where  $\tau_c$  corresponds to the frequency of the first imaginary axis crossing  $s = j\omega$ , with

$$(\alpha^2 + \omega^2) - (\gamma^2 \omega^2 + \beta^2) = 0 \implies (1 - \gamma^2)\omega^2 = \beta^2 - \alpha^2. \quad (12)$$

If  $\alpha^2 > \beta^2$  (or equivalently  $\alpha > |\beta|$  using condition (11)), equation (12) has no solution  $\omega \in \mathbb{R}$ . Since zero is not a solution of  $\delta(s, \tau) = 0$ , the system is then stable with infinite delay margin under the condition

$$\alpha \geq |\beta|. \quad (13)$$

Else, (12) admits a unique solution  $\omega_c > 0$  corresponding to the frequency of the first crossing point on the imaginary axis. It is given by

$$\omega_c = \sqrt{\frac{\beta^2 - \alpha^2}{1 - \gamma^2}}. \quad (14)$$

We can now define the critical delay

$$\tau_c = \min\left\{\frac{\tau}{\omega} \mid s + \alpha + (\gamma s + \beta)z = 0, z = e^{-\tau s}\right\}.$$

Solving the equation  $z_c = e^{-j\omega_c\tau_c} = -\frac{\alpha + j\omega_c}{\beta + j\omega_c\gamma}$ , we obtain

$$\tau_c = \sqrt{\frac{1 - \gamma^2}{\beta^2 - \alpha^2}} \arctan\left(-\frac{\sqrt{(\beta^2 - \alpha^2)(1 - \gamma^2)}}{\alpha + \beta\gamma}\right). \quad (15)$$

In the case  $|\gamma| < 1$ , under condition (11), the system is stable for  $\tau \in [0, \tau_c)$ .

We now apply this methodology to derive conditions under which the system (1)-(5) can be stabilized by proportional feedback controllers. A similar analysis could be done to deal with the PI case. It is omitted here due to space restrictions, and is only briefly presented for the two considered test cases.

#### 4. APPLICATION TO THE PI STABILIZATION OF TRANSPORT EQUATIONS WITH ACTUATION DYNAMICS

##### 4.1 Neutral formulation and open-loop analysis

Let us denote  $z(t) = v(t, 0)$ . Using the method of characteristics, we immediately obtain

$$z(t) = \rho q z(t - \tau) + c p X(t - \tau). \quad (16)$$

We now take the Laplace transform of (3) and (16) neglecting their initial conditions. Indeed, initial conditions only impact the transient behavior and do not modify the stability analysis Hale and Verduyn Lunel (2013). We get

$$(s - a)X(s) = b z(s) + V(s), \quad (17)$$

$$z(s)(1 - \rho q e^{-\tau s}) = c p e^{-\tau s} X(s). \quad (18)$$

This yields  $X(s) = H_{XV}(s)V(s)$ , where the transfer function is given by

$$H_{XV}(s) = \frac{1 - \rho q e^{-\tau s}}{(s - a) - \rho(qs + bc - qa)e^{-\tau s}}. \quad (19)$$

*Open-loop stability* The poles of the open-loop system are the solutions of the characteristic equation

$$\delta_{OL}(s) = (s - a) - \rho(qs + bc - qa)e^{-\tau s} = 0. \quad (20)$$

It is a particular case of equation (10) with

$$\alpha = -a, \quad \beta = \rho(qa - bc), \quad \gamma = -\rho q. \quad (21)$$

Following the methodology given in Section 3.3, the first necessary stability condition (11) is given by  $a(1 - \rho q) + bc\rho < 0$ . Then, applying condition (13), the system is open-loop stable with infinite delay margin (i.e. independently of delay) if and only if  $-a \geq |\rho(qa - bc)|$ . Else, we must consider the crossing points of the imaginary axis to obtain a finite-delay margin. From (14)-(15) we get

$$\begin{cases} \omega_c = |a| + \sqrt{\frac{(\rho bc)^2 - 2abc\rho}{1 - r^2}} \\ \tau_c = \sqrt{\frac{1 - r^2}{(ra - bc\rho)^2 - a^2}} \operatorname{atan}\left(\frac{-bc\rho\sqrt{(1 - r^2)[(ra - bc\rho)^2 - a^2]}}{abc\rho - r(ra - bc\rho)^2}\right). \end{cases}$$

If the necessary stability condition is verified, and if the second stability condition is satisfied or if  $\tau < \tau_c$ , the system is already open-loop stable, so the boundary feedback controller can be used to fasten convergence rate.

##### 4.2 Boundary feedback stabilization

*Proportional feedback controller* We first aim to stabilize system (1)-(5) using a proportional feedback controller  $V(t) = K_p y(t)$ . The characteristic equation of the closed-loop system is now given by

$$\delta_{CL}(s) = s - (a + K_p) + (-\rho q s + \rho(q(a + K_p) - bc))e^{-\tau s} = 0.$$

It corresponds to equation (10) with

$$\alpha = -(a + K_p), \quad \beta = \rho[q(a + K_p) - bc], \quad \gamma = -\rho q. \quad (22)$$

Note that  $|\gamma| < 1$  under Assumption 1. Following the methodology given in Section 3.3, we obtain the first necessary stability condition

$$K_p < -a - \frac{bc\rho}{1 - \rho q}. \quad (23)$$

Next, we must determine the  $K_p$ -domain when the closed-loop system is stable with infinite delay margin, i.e. when  $(a + K_p)^2 - \rho^2(q(a + K_p) - bc)^2 \geq 0$ . This polynomial of discriminant  $(2\rho bc)^2 \geq 0$  admits one or two real solutions

$$K_p^\pm = -a - \frac{bc\rho(\rho q \pm 1)}{1 - (\rho q)^2}, \quad (24)$$

such that condition (13) is satisfied on  $(-\infty, K_p^-) \cup (K_p^+, +\infty)$ . If this condition is not verified, the system can still be stabilized for some  $K_p$ , if the system delay  $\tau$  is small enough. Finally, taking into account (23), we have the following result:

*Proposition 2.* Consider system (1)-(5) with the proportional feedback control law (7) ( $K_i = 0$ ) under Assumption 1. If the constant parameters of the system and the gain  $K_p$  satisfy either of the following set of inequalities

- (1)  $bc\rho \geq 0$ , and  $K_p < -a - \frac{bc\rho}{1 - \rho q}$ ;
- (2)  $bc\rho < 0$ , and  $K_p < -a + \frac{bc\rho}{1 + \rho q}$ ;
- (3)  $bc\rho < 0$ ,  $-a + \frac{bc\rho}{1 + \rho q} \leq K_p < -a - \frac{bc\rho}{1 - \rho q}$  and  $\tau < \tau_c$ ,

where

$$\begin{cases} \tau_c = \frac{1}{\omega_c} \arctan\left(-\frac{\omega_c bc\rho}{(a + K_p)(bc\rho - r(a + K_p)) - r\omega_c}\right), \\ \omega_c = \sqrt{\frac{(r(a + K_p) - bc\rho)^2 - (a + K_p)^2}{1 - r^2}}. \end{cases}$$

then, the closed-loop system (1)-(5) is exponentially stable in the  $\chi$ -norm.

**Proof.** Using the analysis presented above, we directly have that the state  $X$  is exponentially stable. Then, since  $|\rho q| < 1$ , we obtain the exponential stability of the state  $z(t)$ , which in turn implies the exponential stability of the system (1)-(5).

For these values of  $K_p$ , the closed-loop system is stabilized by the boundary feedback controller. Moreover, it is *robustly stabilized* as it is robust to uncertainties on the parameters Niculescu (2001). One can notice that it is always possible to choose  $K_p$  negative enough such that one of the two first conditions is satisfied. However, the

third condition allows choosing smaller absolute values of  $K_p$ , thus improving the robustness properties of the closed-loop system.

*PI feedback controller* We now consider the interest of adding an integral term in the feedback controller  $K_i \neq 0$ . In the Laplace domain, the control feedback becomes  $V(s) = (K_p + \frac{K_i}{s})X(s)$  and (17) rewrites

$$(s - a - K_p - \frac{K_i}{s})X(s) = bz(s). \quad (25)$$

Combining this equation with (18), the characteristic equation of the closed loop system rewrites as  $Q(s, \tau) + P(s, \tau)(K_p + \frac{K_i}{s}) = 0$ , with

$$\begin{cases} Q(s, \tau) = (s - a) - \rho[q(s - a) + bc]e^{-\tau s}, \\ P(s, \tau) = -1 + \rho q e^{-\tau s}. \end{cases} \quad (26)$$

Under Assumption 1,  $P(j\omega, \tau) \neq 0$ ,  $\forall \omega \in \mathbb{R}$ , i.e.  $P$  has no roots on the imaginary axis. We obtain  $K_p + \frac{K_i}{s} = -\frac{Q(s, \tau)}{P(s, \tau)}$ . Looking for solutions of the characteristic equation corresponding to imaginary axis crossings, we consider  $s = j\omega \in j\mathbb{R}$  and identify  $K_p$  (resp.  $K_i$ ) with the real part (resp. imaginary part) of the second member. Indeed, we have

$$K_p = \Re\left(-\frac{Q(j\omega)}{P(j\omega)}\right), \quad K_i = \Im\left(\frac{Q(j\omega)}{P(j\omega)}\right). \quad (27)$$

In particular, in the considered case, we obtain:

$$\begin{cases} K_p = -\frac{a + \rho(2aq - bc) \cos(\omega\tau) + \rho^2 q(aq - bc)}{1 + (\rho q)^2 - 2\rho q \cos(\omega\tau)}, \\ K_i = -\omega \frac{\omega(1 + \rho^2 q^2) - 2\rho q \omega \cos(\omega\tau) + \rho bc \sin(\omega\tau)}{1 + (\rho q)^2 - 2\rho q \cos(\omega\tau)}. \end{cases} \quad (28)$$

By continuity, the *instability degree*, i.e number of poles in the RHP, changes when a new pole crosses the imaginary axis. We can define stable regions bounded by  $K_p(\omega), K_i(\omega)$  for which the instability degree is constant. We can then define the controller gains on compact intervals inside the domains corresponding to stability regions. Plotting (28) as a function of the frequency, we obtain Figure 2. Writing the characteristic equation of the closed-loop system in form (9), the necessary conditions for  $d(s, K_p, K_i) + n(s, K_p, K_i)$  to be Hurwitz is  $K_p < -a - \frac{bc\rho}{1-\rho q}$ ,  $K_i < 0$  (light blue region delimited by green lines). The dotted lines correspond to the first stability condition (11). In our case,  $bc\rho < 0$ , so the light green zone corresponds to a stability region, independently from the value of  $K_i$ . Finally, the light yellow regions correspond to stability regions with finite delay margin. For each  $(K_p, K_i)$ , the critical delay must be computed beforehand to ensure that  $\tau < \tau_c$ .

#### 4.3 Numerical simulations

We now illustrate our results with simulations. In what follows, the space domain  $[0, 1]$  is discretized with a mesh of 100 points. We simulate the PDE system (1)-(5) using a Godounov Scheme LeVeque (2002) ( $CFL = 1$ ) on the time interval  $[0, 50]$ s. We solve the ODE using the Matlab method `ode45`. The parameters are given in Table 1, and are chosen such that the open-loop system is unstable. The  $\chi$ -norm of the open-loop system is represented in Figure 4 (red curve).

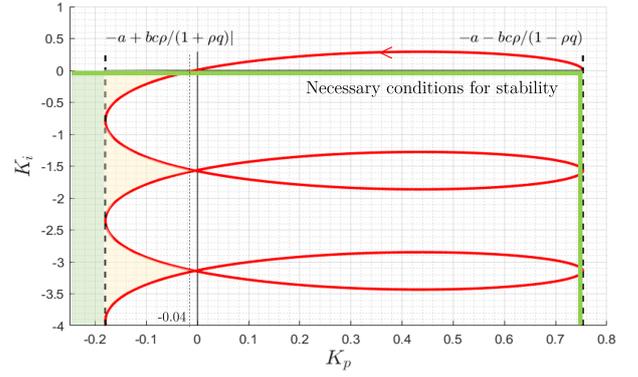


Fig. 2. Stability regions  $(K_p(\omega), K_i(\omega))$  ( $d = 1$ )

| Param. | Value | Param. | Value | Param. | Value |
|--------|-------|--------|-------|--------|-------|
| a      | -0.1  | b      | 0.7   | c      | -0.7  |
| q      | 0.5   | $\rho$ | 0.8   | $\tau$ | 4     |

Table 1. System parameters (Case 1)

We can select  $K_p$  so that the conditions of Prop. 2 are satisfied. Since  $bc\rho < 0$ , the closed-loop system is exponentially stable if  $K_p < -0.18$  (condition (2)). However, choosing  $K_p \in [-0.18, -0.04]$ , the third condition can also be satisfied, as illustrated in Figure 3. The evolution of the  $\chi$ -norm of the system is represented in Figure 4 for  $K_p = -0.1$ .

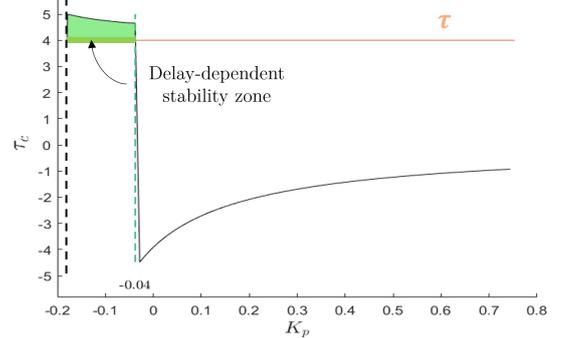


Fig. 3. Available values for  $K_p$  ( $d = 1$ ) under condition (3) of Prop. 2

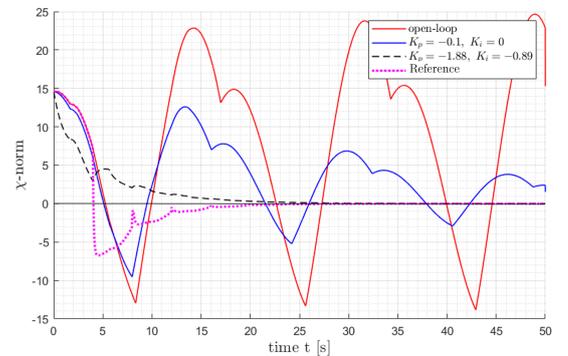


Fig. 4. Evolution of  $\|(X, u, v)\|_\chi$  in open-loop and closed-loop ( $d = 1$ ) for different control strategies

We now consider the introduction of an integral gain to improve the closed-loop performance. Using a gradient sampling algorithm Burke et al. (2005) accelerated by BFGS,

we can find values of  $K_p, K_i$  minimizing the spectral abscissa of the corresponding time-delay system in closed-loop. In this case, we obtain  $K_i \approx -0.89, K_p \approx -1.89$ , which belongs to the stability region as expected. As illustrated in Figure 4 (black dotted curve), the stabilization is faster when we add an integral term in the controller. Finally, for comparison purposes, we also plotted in Figure 4 the closed-loop behavior using the controller developed in Bou Saba et al. (2019) ("reference"). It has comparable performance to the ones of the well-tuned PI controller at the cost of higher complexity.

## 5. CONCLUSION AND PERSPECTIVES

In this paper, we presented a methodology to determine adequate gains for a proportional or PI feedback controller stabilizing an interconnected system of ODE and PDEs. Using a frequency-sweeping approach developed for Time-Delay Systems, we characterize the closed-loop stability intervals in the parameter space. In future work, we will pursue this study and develop quantitative criteria to quantify closed-loop performance and allows a fair comparison between PI controllers and more complex control strategies. This work also paves the way for the stability analysis of more complex networks of interconnected PDE-ODEs, for instance, a PDE-ODE-PDE interconnection known as the Rijke tube de Andrade et al. (2018). The performances of the PI controller proposed can then be compared with other full-state feedback controllers like the one obtained by the backstepping method in terms of performance, and numerical implementation de Andrade et al. (2020). Finally, the case of coupled hyperbolic systems is much more challenging as it induces distributed delay terms in the neutral equation. The characteristic equation is not a quasipolynomial anymore. However, extending the proposed methodology to such systems is necessary to compare PI controllers with infinite-dimensional controllers for such systems.

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