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Fractal Complex Dimensions and Cohomology of the Weierstrass Curve

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Abstract

In this survey article, we present the authors' main results concerning the Complex Dimensions of the Weierstrass Curve, along with their links with the associated fractal cohomology, as developed in our previous papers [DL22b], [DL22c]. Our results shed new light on the theory and the interpretation of Complex Fractal Dimensions, insofar as we envision the fractal Complex Dimensions as dynamical quantities, which evolve with the scales. Accordingly, we define the Complex Dimensions of the Weierstrass Curve as the set of the Complex Dimensions of the sequence of Weierstrass Iterated Fractal Drums which converge to the Curve. By means of *fractal tube formulas*, we then obtain the associated *Weierstrass fractal zeta functions*, whose poles yield the set of possible Complex Dimensions. In particular, we show that the Complex Dimensions are periodically distributed along countably many vertical lines, with the same oscillatory period. As expected, the Minkowski (or box-counting) dimension is the Complex Dimension with maximal real part, and zero imaginary part. We then show how those Dimensions are connected to the cohomological properties of the Curve: the elements of the cohomology groups related to the Curve are obtained, by induction, as sums indexed by the cohomological Complex Dimensions. We explicitly determine both the infinite sequence of prefractal cohomology spaces and the corresponding inductive limit, the fractal (or total) cohomology space of the Weierstrass Curve. In particular, we show that the elements of these cohomology spaces – viewed as suitable continuous functions on the Curve – admit a fractal power series expansion taken over the cohomological Complex Dimensions, that are akin to Taylor-like expansions.

MSC Classification: 11M41, 28A12, 28A75, 28A80.

Keywords: Weierstrass Curve, best Hölder exponent, discrete differential calculus, reverse Hölder continuity, Complex Dimensions of an iterated fractal drum, box-counting (or Minkowski) dimension, fractal tube formula, tube zeta function, (upper, lower and average) Minkowski contents, Minkowski

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non-measurability, Minkowski nondegeneracy, nowhere differentiability, differential operators, differentials, fractal cohomology, cohomology groups, topological invariants, fractal expansions, local and global zeta functions, cohomology infinitesimal, fractal complex.

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1 Introduction

The Weierstrass Function is mainly known as one of those so-called “pathological objects”, continuous everywhere, while nowhere differentiable. Given $\lambda \in]0, 1[$, and an odd integer b such that $\lambda b > 1 + \frac{3\pi}{2}$, it has been introduced by K. Weierstrass [Wei75], as the sum of the uniformly convergent trigonometric series

$$x \in \mathbb{R} \mapsto \sum_{n=0}^{\infty} \lambda^n \cos(\pi b^n x).$$

We refer to [Tit39] (pages 351–353) for an exposition of the original proof, which was then completed by G. H. Hardy [Har16], in the more general case, where b is any real number such that $\lambda b > 1$.

Yet, beyond this property of nowhere differentiability, the associated Curve is also of great interest, due to its self-similarity properties: the Curve is *a fractal*, as evoked, for instance, by Benoît Mandelbrot in his famous books [Man77], [Man83]. Impressively, Mandelbrot guessed, in a purely intuitive way, the exact value of the associated Hausdorff dimension, i.e., $D_{\mathcal{W}} = 2 + \frac{\ln \lambda}{\ln b}$, thereby joining the cohort of the great mathematical conjectures. “Comment certains mathématiciens sont parvenus à une conjecture, cela s’apparente à un saut dans le vide” – “How some mathematicians came to a conjecture is akin to a leap into the void”, as stated by the French mathematician Martin Andler [Cul20].

From this perspective, the main topic that interested the mathematical community has long been the determination of the box-counting dimension – or Minkowski dimension, of the Curve, as it can be found in the existing literature on the subject, from discussions in Falconer’s book [Fal86], or results by J.-L. Kaplan, J. Mallet-Paret and J. A. Yorke [KMPY84], involving Fourier analysis and tools from the theory of dynamical systems. One should also cite the works of F. Przytycki and M. Urbański [PU89], T.-Y. Hu and K.-S. Lau [HL93], followed, in the case of the Hausdorff dimension, by B. Hunt [Hun98], who also examined the random case. Further results concerning the Hausdorff dimension have been obtained by K. Barański, B. Bárány and J. Romanowska [BBR14], then, by W. Shen [She18] (in the general case), and G. Keller [Kel17].

The specific question of the box-counting dimension is of importance. It has been considered from a geometric point of view by the first author in [Dav18], in the case when $b = N_b$ is an integer, where, by contrast to existing work, it is proposed that it can be obtained in a simple way, without requiring

theoretical background in the theory of dynamical systems. More recently, Mandelbrot's conjecture – interpreted in terms of the box-counting dimension or, equivalently, of the Minkowski dimension of the Weierstrass Curve – has been rigorously established in [DL23b] (announced in [DL23a]), building on *loc. cit.* and using new methods relying on suitable polyhedral neighborhoods of the Curve. This is not all. It appears to be deeply linked to the non-differentiability property of the function. One might then want to go further, and question the operators at stake: maybe there is a way to obtain new ones, fitted to this singular object, that would make differentiation possible?

This is where the theory of Complex Dimensions comes into play. Developed for many years now by M. L. Lapidus and his collaborators, for example in [Lap91], [Lap92], [Lap93], [LP93], [LM95], [LP06], [Lap08], [LPV], [LvF13], [LRŽ17a], [LRŽ17b], [LRŽ18], [Lap19], [HL21], [Lap24], it makes the connection between the geometry of an object and its differentiability properties. This is done by means of *geometric (or fractal) zeta functions*, which stand for the trace of the differential operator at a complex order s . The Complex Dimensions are obtained as the poles of those fractal zeta functions. They account, in particular, for the maximal order of differentiation, which coincides with the Minkowski dimension of the compact set under study. Usually, the Minkowski (or box-counting) dimension) coincides with the maximum value of the associated real parts.

With regard to fractals, the determination of the Complex Dimensions of the Weierstrass Curve was, until now, an open problem (see [LRŽ17b], Problem 6.2.24, page 560), that we recently solved in [DL22b] (and further investigated from *a polyhedral point of view* in [DL23b]). The required geometric zeta functions are local and global fractal tube zeta functions, obtained by means of so-called *Weierstrass Iterated Fractal Drums*; i.e., tubular neighborhoods of prefractal polygonal approximations of the Curve. The main difficulty was to handle the nonlinear features at stake, since, contrary to classical fractals such as, for instance, the Koch Curve, the Weierstrass Curve is obtained by means of a suitable nonlinear and noncontractive iterated function system (i.f.s.); see [Dav18]. The i.f.s. is essential: fractals are usually obtained as limits of so-called *prefractals*, i.e., a sequence of finite graphs that converge towards them. In our case, nonlinearity makes the geometry especially complicated; in particular, one cannot obtain the exact values of the underlying elementary lengths and angles.

In order to deal with those difficulties, we had to rely on geometrical properties that had never been obtained previously:

- i.* Explicit lower and upper bounds for the elementary lengths, in relation with the Hölder and anti-Hölder (or reverse Hölder) properties of the Weierstrass function.
- ii.* Nonincreasing property for the sequence of geometric angles.
- iii.* Conditions under which there exist reentrant angles.

Regarding the Complex Dimensions, in the case when $b = N_b$ is an integer, we have obtained the following key results:

- i.* In Corollary 2.5, on page 9 and Theorem 2.7, on page 10 along with Corollary 2.3, on page 10, where we prove the sharp Hölder continuity, and sharp local reverse Hölder continuity, with optimal Hölder exponent

$$2 - D_{\mathcal{W}} = \alpha_{\mathcal{W}} = \frac{\ln \frac{1}{\lambda}}{\ln N_b} \in (0, 1)$$

for the Weierstrass function.

- ii. For a suitable sequence of very small values of the positive parameter ϵ , called *the cohomology infinitesimal* (see Definition 3.1, on page 11) the expression of the ϵ -neighborhood of the Curve – a Weierstrass Fractal Tube Formula, which involves an expansion of the form

$$\sum_{\alpha \text{ real part of a Complex Dimension}} \epsilon^{2-\alpha} G_{\alpha} \left(\ln_{N_b} \left(\frac{1}{\epsilon} \right) \right), \quad (\star)$$

where, for any real part α of a Complex Dimension, G_{α} denotes a continuous and periodic function. Furthermore, for $\alpha = \alpha_{max} = D_{\mathcal{W}}$, the Minkowski dimension of the Curve – i.e., for α being the maximal real part of the Complex Dimensions – $G_{\alpha_{max}}$ is nonconstant, as well as bounded away from zero and infinity. See Theorem 4.2, on page 19, and Theorem 4.5, on page 24.

- iii. The values of the possible Complex Dimensions of the Weierstrass Curve (or, rather, of the associated Weierstrass iterated fractal drum), defined as the poles of the associated tube zeta function. In particular, we show that the Complex Dimensions different from -2 are periodically distributed along countably many vertical lines, with abscissae $D_{\mathcal{W}} - k$ ($2 - D_{\mathcal{W}}$), $1 - 2k$, where k in \mathbb{N} is arbitrary. (Here, and hencefort, $\mathbb{N} = \{0, 1, 2, \dots\}$.) Furthermore, all of the Complex Dimensions (different from -2) have the same oscillatory period $\mathbf{p} = \frac{2\pi}{\ln N_b}$. Moreover, -2 is also a possible Complex Dimensions. See Theorem 4.4, on page 24.
- iv. The existence of the global effective tube zeta function and the determination of the corresponding *exact intrinsic Complex Dimensions* (Definition 4.3, on page 24, and Theorem 4.4, on page 24).
- v. The nondegeneracy, in the Minkowski sense (see [LRŽ17b], but still with a suitable adaptation here), of the Curve, which comes from the fact that the lower and upper Minkowski contents of the Curve are respectively positive and finite. See Theorem 4.6, on page 27, along with Corollary 4.7, on page 28.
- vi. As a corollary, the fact that the number $D_{\mathcal{W}}$ is a Complex Dimension of the Curve, and coincides with the Minkowski dimension (which then exists) of the Curve, and takes the expected value, as given by Mandelbrot's conjecture; namely, $D_{\mathcal{W}} = 2 - \frac{\ln \frac{1}{\lambda}}{\ln N_b} \in (1, 2)$, where $N_b = b \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$. See Corollary 4.7, on page 28.
- vii. The *fractality* of the Weierstrass Curve, in the sense of [LvF00], [LvF06], [LvF13], [LRŽ17b], [Lap19]; i.e., the existence of *nonreal* Complex Dimensions (with real part $D_{\mathcal{W}}$) giving rise to geometric oscillations, in the Fractal Tube Formula obtained in [DL22b]. In the terminology of [LvF13] and [LRŽ17b], the Weierstrass Curve is fractal in countably many dimensions d_k , with $d_k \rightarrow -\infty$, as $k \rightarrow \infty$.

Beyond these results, we have studied the links with *a fractal cohomology*, a natural feature, in so far as discrete differences are deeply connected with differentiation (see the paper on *h-Cohomology* [DL23d]). We place ourselves within a completely different framework from the one that can be found, for instance, in the work by Marius Ionescu, Luke G. Rogers and Alexander Teplyaev in [IRT12], and which is based on the construction of Fredholm modules. Along the lines of previous results obtained or conjectured about fractal cohomology by the second author and his collaborators in [LvF00], [LvF06], [Lap08],

[LvF13], [CL17], [Lap19] and [Lap24], we have obtained the following key results:

i. In Theorem 5.3, on page 32, where we prove that the cohomology groups associated to the Weierstrass Curve consist of continuous functions on the Weierstrass Curve, satisfying fractal expansions expressed as sums indexed by the underlying (cohomological) Complex Dimensions. Such a result had not been established (or even expected) before.

ii. In Theorem 5.4, on page 36 which gives the total cohomology of the Weierstrass Curve, in terms of Taylor-like expansions, again indexed by the underlying (cohomological) Complex Dimensions.

The aim of the present paper, which is a *survey article*, is to present those results in a summarized form. Hence, we will not give detailed proofs, which can be found in the original texts [DL22b], and [DL22c].

Section 2 is devoted to the geometric framework. The next one, Section 3, gives the expression of *the tubular neighborhood of the Curve*, from which one deduces, in Section 4, *the tube zeta function*, and the values of the possible Complex Dimensions. The *fractal cohomology* is discussed in Section 5, where we also show, as could be expected, that the Weierstrass function \mathcal{W} belongs to the total cohomology of the Weierstrass Curve.

2 Geometric Framework

Henceforth, we place ourselves in the Euclidean plane, equipped with a direct orthonormal frame. The usual Cartesian coordinates are denoted by (x, y) . The horizontal and vertical axes will be respectively referred to as $(x'x)$ and $(y'y)$.

Notation 1 (Set of all Nonnegative Integers, and Intervals). As in Bourbaki [Bou04] (Appendix E. 143), we denote by $\mathbb{N} = \{0, 1, 2, \dots\}$ the set of all nonnegative integers, and set $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$.

Given a, b with $-\infty \leq a \leq b \leq +\infty$, $]a, b[= (a, b)$ denotes an open interval, while, for example, $]a, b] = (a, b]$ denotes a half-open, half-closed interval.

Notation 2 (Wave Inequality Symbol). Given two positive-valued functions f and g , defined on a subset \mathcal{I} of \mathbb{R} , we use the following notation, for all $x \in \mathcal{I}$: $f(x) \lesssim g(x)$ when there exists a strictly positive constant C such that, for all $x \in \mathcal{I}$, $f(x) \leq C g(x)$.

Notation 3 (Weierstrass Parameters).

In the sequel, λ and N_b are two real numbers such that

$$0 < \lambda < 1 \quad , \quad N_b \in \mathbb{N}^* \quad \text{and} \quad \lambda N_b > 1 .$$

Note that this implies that $N_b > 1$ (i.e., $N_b \geq 2$).

Definition 2.1 (Weierstrass Function, Weierstrass Curve). We consider the *Weierstrass function* \mathcal{W} , defined, for any real number x , by

$$\mathcal{W}(x) = \sum_{n=0}^{\infty} \lambda^n \cos(2\pi N_b^n x) .$$

We call the associated graph the *Weierstrass Curve*.

Due to the periodic properties of the \mathcal{W} function, from now on, we restrict our study to the interval $[0, 1[= [0, 1)$ (or, equivalently, to $[0, 1]$). Accordingly, the Weierstrass Curve $\Gamma_{\mathcal{W}}$ is viewed as a compact subset of \mathbb{R}^2 .

Notation 4 (Logarithm). Given $y > 0$, $\ln y$ denotes the natural logarithm of y , while, given $a > 1$, $\ln_a y = \frac{\ln y}{\ln a}$ denotes the logarithm of y in base a ; so that, in particular, $\ln = \ln_e$.

Notation 5. For the parameters λ and N_b satisfying the indicated hypotheses earlier in this section, we denote by

$$D_{\mathcal{W}} = 2 + \frac{\ln \lambda}{\ln N_b} = 2 - \ln_{N_b} \frac{1}{\lambda} \in]1, 2[$$

the box-counting dimension (or Minkowski dimension) of the Weierstrass Curve $\Gamma_{\mathcal{W}}$, which happens to be equal to its Hausdorff dimension [KMPY84], [BBR14], [She18], [Kel17]. Our later results in [DL23b], [DL23a] also provide a direct and rigorous proof of the fact that $D_{\mathcal{W}}$, the Minkowski dimension (or box-counting dimension) of $\Gamma_{\mathcal{W}}$, exists and takes the above value.

Convention (The Weierstrass Curve as a Cyclic Curve)

In the sequel, we identify the points $(0, \mathcal{W}(0))$ and $(1, \mathcal{W}(1)) = (1, \mathcal{W}(0))$. This is justified by the fact that the Weierstrass function \mathcal{W} is 1-periodic, since N_b is an integer.

Remark 2.1 (Nonlinear and Noncontractive Iterated Function System (i.f.s.)). Following our previous work [Dav18], we approximate the restriction $\Gamma_{\mathcal{W}}$ to $[0, 1[\times \mathbb{R}$, of the Weierstrass Curve, by a sequence of finite graphs, built via an iterative process. For this purpose, we use the nonlinear iterated function system of the family of C^∞ maps from \mathbb{R}^2 to \mathbb{R}^2 denoted by

$$\mathcal{T}_{\mathcal{W}} = \{T_0, \dots, T_{N_b-1}\},$$

where, for any integer i belonging to $\{0, \dots, N_b - 1\}$ and any point (x, y) of \mathbb{R}^2 ,

$$T_i(x, y) = \left(\frac{x+i}{N_b}, \lambda y + \cos\left(2\pi \left(\frac{x+i}{N_b}\right)\right) \right).$$

We point out that those maps *are not contractions* (see [Dav19]). Yet, they correspond, in a sense, to the composition of a contraction of ratio r_x in the horizontal direction, and a dilatation of factor r_y in the vertical one, with

$$r_x r_y < 1.$$

Proposition 2.1 (Attractor of the i.f.s. [Dav18], [Dav19]).

The Weierstrass Curve is the attractor of the i.f.s. $\mathcal{T}_{\mathcal{W}}$:

$$\Gamma_{\mathcal{W}} = \bigcup_{i=0}^{N_b-1} T_i(\Gamma_{\mathcal{W}}).$$

Notation 6 (Fixed Points).

For any integer i belonging to $\{0, \dots, N_b - 1\}$, we denote by:

$$P_i = (x_i, y_i) = \left(\frac{i}{N_b - 1}, \frac{1}{1 - \lambda} \cos\left(\frac{2\pi i}{N_b - 1}\right) \right)$$

the unique fixed point of the map T_i ; see [Dav19].

Definition 2.2 (Sets of Vertices, Prefractals).

We denote by V_0 the ordered set (according to increasing abscissae), of the points

$$\{P_0, \dots, P_{N_b-1}\}.$$

The set of points V_0 – where, for any i of $\{0, \dots, N_b - 2\}$, the point P_i is linked to the point P_{i+1} – constitutes an oriented graph, according to increasing abscissae, that we will denote by $\Gamma_{\mathcal{W}_0}$. Then, V_0 is called the set of vertices of the graph $\Gamma_{\mathcal{W}_0}$.

For any positive integer m , i.e., for $m \in \mathbb{N}^*$, we set $V_m = \bigcup_{i=0}^{N_b-1} T_i(V_{m-1})$.

The set of points V_m , where two consecutive points are linked, is an oriented graph, according to increasing abscissae, called the m^{th} \mathcal{W} -**prefractal** (or m^{th} *prefractal approximation* to $\Gamma_{\mathcal{W}}$). Then, V_m is called the set of vertices of the m^{th} prefractal graph $\Gamma_{\mathcal{W}_m}$; see Figure 1, on page 44.

Definition 2.3 (Adjacent Vertices, Edge Relation). For any $m \in \mathbb{N}$, the m^{th} prefractal graph $\Gamma_{\mathcal{W}_m}$ is equipped with an edge relation \sim_m , as follows: two vertices X and Y of $\Gamma_{\mathcal{W}_m}$, i.e. two points belonging to V_m are said to be *adjacent* (i.e., *neighboring* or *junction points*) if and only if the line segment $[X, Y]$ is an edge of $\Gamma_{\mathcal{W}_m}$; we then write $X \sim_m Y$. Note that this edge relation depends on m , which means that points adjacent in V_m might not remain adjacent in V_{m+1} .

Proposition 2.2. [Dav18]

For any $m \in \mathbb{N}$, the following statements hold:

- i. $V_m \subset V_{m+1}$.
- ii. $\#V_m = (N_b - 1) N_b^m + 1$.
- iii. The m^{th} prefractal graph $\Gamma_{\mathcal{W}_m}$ has exactly $(N_b - 1) N_b^m$ edges.
- iv. The consecutive vertices of the prefractal graph $\Gamma_{\mathcal{W}_m}$ are the vertices of N_b^m simple polygons $\mathcal{P}_{m,k}$. For any integer $m \in \mathbb{N}^*$, the junction point between two consecutive polygons is the point

$$\mathcal{P}_{m,k} \cap \mathcal{P}_{m,k+1} = \left\{ \left(\frac{(N_b - 1)k}{(N_b - 1)N_b^m}, \mathcal{W} \left(\frac{(N_b - 1)k}{(N_b - 1)N_b^m} \right) \right) \right\}, \quad 1 \leq k \leq N_b^m - 1.$$

Hence, the total number of junction points is $N_b^m - 1$. For instance, in the case $N_b = 3$, one gets triangles; see Figure 2, on page 45.

In the sequel, we will denote by \mathcal{P}_0 **the initial polygon**, whose vertices are the fixed points of the maps T_i , $0 \leq i \leq N_b - 1$, introduced in Definition 2.2, on page 7, i.e., $\{P_0, \dots, P_{N_b-1}\}$.

Definition 2.4 (Vertices of the Prefractals, Elementary Lengths, Heights and Angles).

Given a strictly positive integer m , we denote by $(M_{j,m})_{0 \leq j \leq (N_b-1)N_b^m}$ **the set of vertices** of the prefractal graph $\Gamma_{\mathcal{W}_m}$. One thus has, for any integer j in $\{0, \dots, (N_b - 1) N_b^m\}$,

$$M_{j,m} = \left(\frac{j}{(N_b - 1) N_b^m}, \mathcal{W} \left(\frac{j}{(N_b - 1) N_b^m} \right) \right).$$

We also introduce, for any integer j in $\{0, \dots, (N_b - 1) N_b^m - 1\}$, the following quantities:

i. the elementary horizontal lengths:

$$L_m = \frac{1}{(N_b - 1) N_b^m};$$

ii. the elementary lengths:

$$l_{j,j+1,m} = d(M_{j,m}, M_{j+1,m}) = \sqrt{L_m^2 + h_{j,j+1,m}^2},$$

where $h_{j,j+1,m}$ is defined in iii. just below.

iii. the elementary heights:

$$h_{j-1,j,m} = \left| \mathcal{W}\left(\frac{j}{(N_b - 1) N_b^m}\right) - \mathcal{W}\left(\frac{j-1}{(N_b - 1) N_b^m}\right) \right|,$$

$$h_{j,j+1,m} = \left| \mathcal{W}\left(\frac{j+1}{(N_b - 1) N_b^m}\right) - \mathcal{W}\left(\frac{j}{(N_b - 1) N_b^m}\right) \right|;$$

iv. the minimal height:

$$h_m^{inf} = \inf_{0 \leq j \leq (N_b - 1) N_b^m - 1} h_{j,j+1,m}, \quad (1)$$

along with the maximal height:

$$h_m = \sup_{0 \leq j \leq (N_b - 1) N_b^m - 1} h_{j,j+1,m}, \quad (2)$$

v. the geometric angles:

$$\theta_{j-1,j,m} = ((y'y), (\widehat{M_{j-1,m} M_{j,m}})) \quad , \quad \theta_{j,j+1,m} = ((y'y), (\widehat{M_{j,m} M_{j+1,m}})),$$

where $(y'y)$ denotes the vertical axis, which yield **the following value of the geometric angle between consecutive edges**, namely, $[M_{j-1,m} M_{j,m}, M_{j,m} M_{j+1,m}]$, with $\arctan = \tan^{-1}$:

$$\theta_{j-1,j,m} + \theta_{j,j+1,m} = \arctan \frac{L_m}{h_{j-1,j,m}} + \arctan \frac{L_m}{h_{j,j+1,m}}.$$

(Note that, of course, $\theta_{j-1,j,m} = \arctan \frac{L_m}{h_{j-1,j,m}}$ and $\theta_{j,j+1,m} = \arctan \frac{L_m}{h_{j,j+1,m}}$.)

Proposition 2.3 (Scaling Properties of the Weierstrass Function [DL22b]). *For any strictly positive integer m and any j in $\{0, \dots, \#V_m\}$,*

$$\mathcal{W}\left(\frac{j}{(N_b - 1) N_b^m}\right) = \lambda^m \mathcal{W}\left(\frac{j}{(N_b - 1)}\right) + \sum_{k=0}^{m-1} \lambda^k \cos\left(\frac{2\pi N_b^k j}{(N_b - 1) N_b^m}\right).$$

Proposition 2.4 (Explicit Lower and Upper Bounds for the Elementary Heights [DL22b]).

For any strictly positive integer m , and any j in $\{0, \dots, (N_b - 1) N_b^m\}$,

$$C_{inf} L_m^{2-D_W} \leq \underbrace{|\mathcal{W}((j+1)L_m) - \mathcal{W}(jL_m)|}_{h_{j,j+1,m}} \leq C_{sup} L_m^{2-D_W},$$

where the finite and positive constants C_{inf} and C_{sup} are given by

$$C_{inf} = (N_b - 1)^{2-D_{\mathcal{W}}} \min_{0 \leq j \leq N_b - 1, \mathcal{W}\left(\frac{j+1}{N_b-1}\right) \neq \mathcal{W}\left(\frac{j}{N_b-1}\right)} \left| \mathcal{W}\left(\frac{j+1}{N_b-1}\right) - \mathcal{W}\left(\frac{j}{N_b-1}\right) \right|$$

and

$$C_{sup} = (N_b - 1)^{2-D_{\mathcal{W}}} \left(\max_{0 \leq j \leq N_b - 1} \left| \mathcal{W}\left(\frac{j+1}{N_b-1}\right) - \mathcal{W}\left(\frac{j}{N_b-1}\right) \right| + \frac{2\pi}{(N_b - 1)(\lambda N_b - 1)} \right).$$

One should note, in addition, that these constants C_{inf} and C_{sup} depend on the initial polygon \mathcal{P}_0 .

Corollary 2.5 (of Proposition 2.4). For any strictly positive integer m and any integer j in $\{0, \dots, (N_b - 1)N_b^m\}$ one then has, for the elementary heights,

$$h_{j-1, j, m} = L_m^{2-D_{\mathcal{W}}} \mathcal{O}(1),$$

where $C_{inf} \leq \mathcal{O}(1) \leq C_{sup}$.

Proposition 2.6 ([DL22b]). Given any strictly positive integer m , we have the following properties:

i. For any j in $\{0, \dots, \#V_m\}$, the point

$$\left(\frac{j}{(N_b - 1)N_b^m}, \mathcal{W}\left(\frac{j}{(N_b - 1)N_b^m}\right) \right)$$

is the image of the point

$$\begin{aligned} & \left(\frac{j}{(N_b - 1)N_b^{m-1}} - i, \mathcal{W}\left(\frac{j}{(N_b - 1)N_b^{m-1}} - i\right) \right) = \\ & = \left(\frac{j - i(N_b - 1)N_b^{m-1}}{(N_b - 1)N_b^{m-1}}, \mathcal{W}\left(\frac{j - i(N_b - 1)N_b^{m-1}}{(N_b - 1)N_b^{m-1}}\right) \right) \end{aligned}$$

under the map T_i , for each integer i such that $0 \leq i \leq N_b - 1$.

Consequently, for $0 \leq j \leq N_b - 1$, the j^{th} vertex of the polygon $\mathcal{P}_{m, k}$, $0 \leq k \leq N_b^m - 1$, i.e., the point

$$\left(\frac{(N_b - 1)k + j}{(N_b - 1)N_b^m}, \mathcal{W}\left(\frac{(N_b - 1)k + j}{(N_b - 1)N_b^m}\right) \right),$$

is the image of the point

$$\left(\frac{(N_b - 1)(k - i(N_b - 1)N_b^{m-1}) + j}{(N_b - 1)N_b^{m-1}}, \mathcal{W}\left(\frac{(N_b - 1)(k - i(N_b - 1)N_b^{m-1}) + j}{(N_b - 1)N_b^{m-1}}\right) \right),$$

which is also the j^{th} vertex of the polygon $\mathcal{P}_{m-1, k-i(N_b-1)N_b^{m-1}}$. Therefore, there is an exact correspondance between vertices of the polygons at consecutive steps $m - 1$, m .

ii. Given j in $\{0, \dots, N_b - 2\}$ and k in $\{0, \dots, N_b^m - 1\}$, we have that

$$\begin{aligned} & \text{sgn} \left(\mathcal{W}\left(\frac{k(N_b - 1) + j + 1}{(N_b - 1)N_b^m}\right) - \mathcal{W}\left(\frac{k(N_b - 1) + j}{(N_b - 1)N_b^m}\right) \right) = \\ & = \text{sgn} \left(\mathcal{W}\left(\frac{j + 1}{N_b - 1}\right) - \mathcal{W}\left(\frac{j}{N_b - 1}\right) \right). \end{aligned}$$

Theorem 2.7 (Sharp Local Discrete Reverse Hölder Properties of the Weierstrass Function [DL22b]). For any natural integer m , let us consider a pair of real numbers (x, x') such that:

$$x = \frac{(N_b - 1)k + j}{(N_b - 1)N_b^m} = ((N_b - 1)k + j) L_m,$$

$$x' = \frac{(N_b - 1)k + j + \ell}{(N_b - 1)N_b^m} = ((N_b - 1)k + j + \ell) L_m,$$

where $0 \leq k \leq N_b^m - 1$, and

i. if the integer N_b is odd,

$$0 \leq j < \frac{N_b - 1}{2} \quad \text{and} \quad 0 < j + \ell \leq \frac{N_b - 1}{2}$$

or

$$\frac{N_b - 1}{2} \leq j < N_b - 1 \quad \text{and} \quad \frac{N_b - 1}{2} < j + \ell \leq N_b - 1;$$

ii. if the integer N_b is even,

$$0 \leq j < \frac{N_b}{2} \quad \text{and} \quad 0 < j + \ell \leq \frac{N_b}{2}$$

or

$$\frac{N_b}{2} + 1 \leq j < N_b - 1 \quad \text{and} \quad \frac{N_b}{2} + 1 < j + \ell \leq N_b - 1.$$

This means that the points $(x, \mathcal{W}(x))$ and $(x', \mathcal{W}(x'))$ are vertices of the polygon $\mathcal{P}_{m,k}$ (see Proposition 2.2, on page 7), both located on the left side of the polygon, or both located on the right side (see Figure 3, on page 45).

Then, one has the following reverse-Hölder inequality, with sharp Hölder exponent $-\frac{\ln \lambda}{\ln N_b} = 2 - D_{\mathcal{W}}$,

$$C_{inf} |x' - x|^{2-D_{\mathcal{W}}} \leq |\mathcal{W}(x') - \mathcal{W}(x)|.$$

Remark 2.2. It is clear that, for any natural integer m , and any pair $((x, \mathcal{W}(x)), (x', \mathcal{W}(x')))$ of adjacent vertices of the finite prefractal graph $\Gamma_{\mathcal{W}_m}$, the same following (discrete, local) Hölder and reverse-Hölder inequality, with sharp Hölder exponent $-\frac{\ln \lambda}{\ln N_b} = 2 - D_{\mathcal{W}}$, holds; i.e., still with C_{inf} and C_{sup} given as in Proposition 2.4, on page 8 above, we have that

$$C_{inf} |x' - x|^{2-D_{\mathcal{W}}} \leq |\mathcal{W}(x') - \mathcal{W}(x)| \leq C_{sup} |x' - x|^{2-D_{\mathcal{W}}}.$$

Remark 2.3. Thus far, no such reverse Hölder estimates had been obtained for the Weierstrass function. The fact that they are discrete ones is natural, since the Weierstrass Curve is approximated by a sequence of polygonal prefractal graphs.

Corollary 2.8 (Optimal Hölder Exponent for the Weierstrass Function [DL22b]). The local reverse Hölder property of Theorem 2.7, on page 10, in conjunction with the Hölder condition satisfied by the Weierstrass function (see [Zyg02], Chapter II, Theorem 4.9, page 47), shows that the Codimension $2 - D_{\mathcal{W}} = -\frac{\ln \lambda}{\ln N_b}$ is the best (i.e., optimal) Hölder exponent for the Weierstrass function.

Corollary 2.9 (Coming from Proposition 2.6, on page 9). *Thanks to Proposition 2.4, on page 8, and Corollary 2.5, on page 9, one may now write, for any strictly positive integer m and any integer j in $\{0, \dots, (N_b - 1)N_b^m - 1\}$:*

i. for the elementary heights:

$$h_{j-1,j,m} = L_m^{2-D_W} \mathcal{O}(1) ; \quad (3)$$

ii. for the elementary quotients:

$$\frac{h_{j-1,j,m}}{L_m} = L_m^{1-D_W} \mathcal{O}(1) , \quad (4)$$

and where

$$0 < C_{inf} \leq \mathcal{O}(1) \leq C_{sup} .$$

Corollary 2.10 (Nonincreasing Sequence of Geometric Angles). *For the **geometric angles** $\theta_{j-1,j,m}$, $0 \leq j \leq (N_b - 1)N_b^m$, $m \in \mathbb{N}$, we have the following property:*

$$\theta_{j-1,j,m} > \theta_{j-1,j,m+1} \quad \text{and} \quad \theta_{j-1,j,m+1} \lesssim L_m^{D_W-1} .$$

3 Iterated Fractal Drums and Tubular Neighborhood of the Weierstrass IFD

We introduce here the notion of an *iterated fractal drum* (IFD), which is well suited to the present setting – and, as we expect, to many other complicated examples of fractal curves and of their higher-dimensional analogs. As we will see, it is defined in terms of a suitably adapted sequence of tubular neighborhoods, which can be thought of as a sequence of scales.

Notation 7 (Euclidean Distance).

In the sequel, we denote by d the Euclidean distance on \mathbb{R}^2 .

Our results on fractal cohomology obtained in [DL22c] have highlighted the part played by specific threshold values for the number $\varepsilon > 0$ at any step $m \in \mathbb{N}$ of the prefractal graph approximation; namely, the m^{th} *cohomology infinitesimal* introduced in Definition 3.1, on page 11 just below.

Definition 3.1 (m^{th} Cohomology Infinitesimal [DL22b], [DL22c] and m^{th} Intrinsic Cohomology Infinitesimal [DL23b], [DL23a]). From now on, given any $m \in \mathbb{N}$, we will call m^{th} *cohomology infinitesimal* the number $\varepsilon_m^m > 0$ which also corresponds to the elementary horizontal length introduced in part *i.* in Definition 2.4, on page 7; i.e., $\varepsilon_m^m = (\varepsilon_m)^m = \frac{1}{N_b - 1} \frac{1}{N_b^m}$.

Observe that, clearly, ε_m itself – and not just ε_m^m – depends on m .

In addition, since $N_b > 1$, ε_m^m satisfies the following asymptotic behavior,

$$\varepsilon_m^m \rightarrow 0, \quad \text{as } m \rightarrow \infty,$$

which, naturally, results in the fact that the larger m , the smaller ε_m^m . It is for this reason that we call ε_m^m – or rather, the *sequence* $(\varepsilon_m^m)_{m=0}^\infty$ of positive numbers tending to zero as $m \rightarrow \infty$, with $\varepsilon_m^m = (\varepsilon_m)^m$, for each $m \in \mathbb{N}$ – an *infinitesimal*, more specifically, *the cohomology infinitesimal*. Note that this m^{th}

cohomology infinitesimal is the one naturally associated to the scaling relation of Proposition 2.3, on page 8.

In the sequel, it is also useful to keep in mind that the sequence of positive numbers $(\varepsilon_m)_{m=0}^\infty$ itself satisfies

$$\varepsilon_m \sim \frac{1}{N_b}, \text{ as } m \rightarrow \infty ;$$

i.e., $\varepsilon_m \rightarrow \frac{1}{N_b}$, as $m \rightarrow \infty$. In particular, $\varepsilon_m \not\rightarrow 0$, as $m \rightarrow \infty$, but, instead, ε_m tends to a strictly positive and finite limit.

We also introduce, given any $m \in \mathbb{N}$, the m^{th} *intrinsic cohomology infinitesimal*, denoted by $\varepsilon^m > 0$, such that

$$\varepsilon^m = \frac{1}{N_b^m},$$

where

$$\varepsilon = \frac{1}{N_b}.$$

We call ε *the intrinsic scale*, or *intrinsic subdivision scale*.

Note that

$$\varepsilon_m^m = \frac{\varepsilon^m}{N_b - 1}$$

and that the m^{th} intrinsic cohomology infinitesimal ε^m is asymptotic (when m tends to ∞) to the m^{th} cohomology infinitesimal ε_m^m .

Remark 3.1 (Addressing Numerical Estimates). From a practical point of view, an important question is the value of the ratio

$$\frac{\text{Cohomology infinitesimal}}{\text{Maximal height}} = \frac{\varepsilon_m^m}{h_m};$$

see relation (2), on page 8.

Thanks to the estimates given in relation (4), on page 11, we have that

$$\frac{\varepsilon_m^m}{h_m} = L_m^{1-D_{\mathcal{W}}} \mathcal{O}(1) = \varepsilon_m^{m(1-D_{\mathcal{W}})} \mathcal{O}(1),$$

with

$$0 < C_{inf} \leq \mathcal{O}(1) \leq C_{sup}.$$

Given $q \in \mathbb{N}^*$, we then have

$$\frac{1}{10^q} C_{inf} \leq \frac{\varepsilon_m^m}{h_m} \leq \frac{1}{10^q} C_{sup}$$

when

$$\frac{C_{inf}}{10^q} \leq e^{(1-D_{\mathcal{W}}) \ln L_m} \leq \frac{C_{sup}}{10^q},$$

or, equivalently, when

$$-\frac{1}{\ln N_b} \ln \left((N_b - 1) \left(\frac{C_{sup}}{10^q} \right)^{\frac{1}{1-D_{\mathcal{W}}}} \right) \leq m \leq -\frac{1}{\ln N_b} \ln \left((N_b - 1) \left(\frac{C_{inf}}{10^q} \right)^{\frac{1}{1-D_{\mathcal{W}}}} \right).$$

Numerical values for $N_b = 3$ and $\lambda = \frac{1}{2}$ yield:

- i. For $q = 1$: $2 \leq m \leq 3$.
- ii. For $q = 2$: $7 \leq m \leq 9$.
- iii. For $q = 3$: $13 \leq m \leq 15$.

Hence, when m increases, the ratio decreases, and tends to 0. This numerical – but very practical and explicit argument – also accounts for our forthcoming neighborhoods, of width equal to the cohomology infinitesimal.

Definition 3.2 (Iterated Fractal Drums (IFDs)).

Let us consider a fractal curve $\mathcal{F} \subset \mathbb{R}^2$, obtained by means of a suitable IFS $\mathcal{T}_{\mathcal{F}}$ (consisting, in particular, of a family of C^∞ maps from \mathbb{R}^2 to \mathbb{R}^2). For each $m \in \mathbb{N}$, we denote by \mathcal{F}_m the m^{th} prefractal approximation to the fractal \mathcal{F} . We restrict ourselves to the case when there exists a natural scaling relation associated to the sequence $(\Gamma_{\mathcal{F}_m})_{m \in \mathbb{N}}$, involving a sequence of elementary lengths (or cohomology infinitesimals) $(\varepsilon_{m,\mathcal{F}}^m)_{m \in \mathbb{N}}$, and, as in Definition 3.1, on page 11 above.

We then call *Iterated Fractal Drum* (in short, *IFD*), and denote by $\mathcal{F}^{\mathcal{I}}$, the sequence of ordered pairs $(\mathcal{F}_m, \varepsilon_{m,\mathcal{F}}^m)_{m \in \mathbb{N}}$, where, for each $m \in \mathbb{N}$, \mathcal{F}_m is the m^{th} prefractal (graph) approximation associated with the fractal \mathcal{F} .

Definition 3.3 (Weierstrass Iterated Fractal Drum (Weierstrass IFD)). We call *Weierstrass Iterated Fractal Drum* (in short, *Weierstrass IFD*), and denote by $\Gamma_{\mathcal{W}}^{\mathcal{I}}$, the sequence of ordered pairs $(\Gamma_{\mathcal{W}_m}, \varepsilon_m^m)_{m \in \mathbb{N}}$ where, for each $m \in \mathbb{N}$, $\Gamma_{\mathcal{W}_m}$ is the m^{th} prefractal approximation to the Weierstrass Curve $\Gamma_{\mathcal{W}}$, as introduced in Definition 2.2, on page 7, and where ε_m^m is the m^{th} cohomology infinitesimal, as introduced in Definition 3.1, on page 11 above. Note that the m^{th} prefractal graph approximation (viewed as an oriented curve) determines the m^{th} prefractal curve (viewed as an oriented polygonal curve), and conversely. Indeed, the line segments of which the latter polygonal curve is comprised are nothing but the edges of the former prefractal graph.

Notation 8 (Integer to Cohomology Infinitesimal Map). Given $m \in \mathbb{N}^*$, we hereafter introduce the map

$$\varepsilon_m^m \mapsto m(\varepsilon_m^m) = \left[-\ln_{N_b} \left((N_b - 1) \varepsilon_m^m \right) \right],$$

where $[.]$ denotes the integer part. Note that this map is only applied for the m^{th} cohomology infinitesimal $\varepsilon_m^m = (\varepsilon_m)^m = \frac{1}{N_b - 1} \frac{1}{N_b^m}$, introduced in Definition 3.1, on page 11.

For notational simplicity, we temporarily set $x = -\ln_{N_b} \left((N_b - 1) \varepsilon_m^m \right)$.

Proposition 3.1 ((m, ε_m^m) -Neighborhood [DL22b]).

Given a nonnegative integer m , and a point $M \in \mathbb{R}^2$, we denote by $d(M, \Gamma_{\mathcal{W}_m})$ the Euclidean distance from M to $\Gamma_{\mathcal{W}_m}$. Then, the (m, ε_m^m) -neighborhood of the m^{th} prefractal approximation $\Gamma_{\mathcal{W}_m}$ to the Weierstrass Curve,

$$\mathcal{D}(\Gamma_{\mathcal{W}_m}, \varepsilon_m^m) = \left\{ M = (x, y) \in \mathbb{R}^2, d(M, \Gamma_{\mathcal{W}_m}) \leq \varepsilon_m^m \right\}$$

(see Figure 4, on page 46), is obtained by means of:

- i. $(N_b - 1)N_b^m$ upper and lower **overlapping rectangles** (see Figure 5, on page 47), of length $\ell_{j-1,j,m}$, $1 \leq j \leq N_b^m - 1$, and height ε_m^m .

One of the difficulties is, then, to compute and subtract the area counted twice, which correspond to **parallelograms**, of height ε_m^m and basis

$$\varepsilon_m^m \cotan(\pi - \theta_{j_k-1, j_k, m} - \theta_{j_k, j_k+1, m}),$$

and **extra outer triangles**, the area of which is of the form

$$\frac{1}{2} \varepsilon_m^m (b_{j_\ell-1, j_\ell, m} + b_{j_\ell, j_\ell+1, m}),$$

and where, for the sake of simplicity, we have denoted by j_k and j_ℓ the corresponding indices.

- ii. **Upper and lower wedges**, the area of which are of the form

$$\frac{1}{2} (\pi - \theta_{j_l-1, j_l, m} - \theta_{j_l, j_l+1, m}) (\varepsilon_m^m)^2, \quad 1 \leq j_l \leq N_b^m - 2.$$

The number of wedges is determined by the shape of the initial polygon \mathcal{P}_0 , and the existence of reentrant angles.

- iii. Two **extreme wedges** (respectively located at the abscissae $x = 0$ and $x = 1$), each of area equal to $\frac{1}{2} \pi (\varepsilon_m^m)^2$.

The small positive number ε_m^m will be referred to as the width of the (m, ε_m^m) -neighborhood $\mathcal{D}(\Gamma_{\mathcal{W}_m}, \varepsilon_m^m)$.

Thus far, we have considered the sequence of tubular neighborhoods associated to the prefractal sequence of graphs $(\Gamma_{\mathcal{W}_m})_{m \in \mathbb{N}}$, which approximate the Weierstrass Curve $\Gamma_{\mathcal{W}}$. In Proposition 3.2, on page 14 just below, we explain the connections between the aforementioned sequence of tubular neighborhoods, and the tubular neighborhood of the Weierstrass Curve $\Gamma_{\mathcal{W}}$ itself. Such a result is all the more important when it comes to determining the poles of the tubular zeta function obtained by means of the aforementioned sequence of tubular neighborhoods associated to the prefractal sequence $(\Gamma_{\mathcal{W}_m})_{m \in \mathbb{N}}$. One intuitively understands that we cannot explicitly compute the volume of the tubular neighborhood of $\Gamma_{\mathcal{W}}$.

Proposition 3.2 (Tubular Neighborhood of the Weierstrass Curve).

Given a small positive number δ , we define the δ -neighborhood of the Weierstrass Curve $\Gamma_{\mathcal{W}}$ by

$$\mathcal{D}(\Gamma_{\mathcal{W}}, \delta) = \{M = (x, y) \in \mathbb{R}^2, d(M, \Gamma_{\mathcal{W}}) \leq \delta\}.$$

Then, there exists an integer $m_0(\delta) \in \mathbb{N}$ such that,

$$\forall m \geq m_0(\delta) : \quad \mathcal{D}\left(\Gamma_{\mathcal{W}_m}, \frac{\delta}{2}\right) \subset \mathcal{D}(\Gamma_{\mathcal{W}}, \delta) \subset \mathcal{D}(\Gamma_{\mathcal{W}_m}, 2\delta).$$

Hence, the sequence of tubular neighborhoods associated to the prefractal sequence of graphs $(\Gamma_{\mathcal{W}_m})_{m \in \mathbb{N}}$ can also be interpreted as a sequence of tubular neighborhoods of $\Gamma_{\mathcal{W}}$, in the sense that, for m large enough, any tubular neighborhood of $\Gamma_{\mathcal{W}_m}$ contains a tubular neighborhood of $\Gamma_{\mathcal{W}}$. Even if the widths involved are not the same, we have a form of equivalence between a tubular neighborhood of $\Gamma_{\mathcal{W}_m}$, for $m \in \mathbb{N}$ sufficiently large, and a tubular neighborhood of $\Gamma_{\mathcal{W}}$. In our present setting, if we denote by $\mathcal{V}_m\left(\frac{\delta}{2}\right)$, $\mathcal{V}_m(2\delta)$ and $\mathcal{V}(\delta)$ the respective volumes (i.e., areas) of $\mathcal{D}\left(\Gamma_{\mathcal{W}_m}, \frac{\delta}{2}\right)$, $\mathcal{D}(\Gamma_{\mathcal{W}_m}, 2\delta)$ and $\mathcal{D}(\Gamma_{\mathcal{W}}, \delta)$, we can write

$$\mathcal{V}_m(2\delta) = \mathcal{V}(\delta) + \mathcal{R}_m(\delta) = \mathcal{V}_m\left(\frac{\delta}{2}\right) + \widetilde{\mathcal{R}}_m\left(\frac{\delta}{2}\right),$$

where $\mathcal{R}_m(\delta)$ and $\widetilde{\mathcal{R}}_m\left(\frac{\delta}{2}\right)$ denote strictly positive error terms, and are such that

$$\lim_{m \rightarrow \infty} \mathcal{R}_m(\delta) = 0^+ \quad \text{and} \quad \lim_{m \rightarrow \infty} \widetilde{\mathcal{R}}_m\left(\frac{\delta}{2}\right) = 0^+.$$

Proof. In this proof, for the sake of simplicity, we will write $M_{\star, m}$, $m \in \mathbb{N}$, for the points $M_{j, m}$, with $0 \leq j \leq \#V_m - 1$, of the prefractal graph $\Gamma_{\mathcal{W}_m}$.

i. Because of the density of the set $V^\star = \bigcup_{n \in \mathbb{N}} V_n$ in the Weierstrass Curve $\Gamma_{\mathcal{W}}$, there exists an integer $m_1(\delta) \in \mathbb{N}$ such that

$$\forall m \geq m_1(\delta) : \quad d(\Gamma_{\mathcal{W}_m}, \Gamma_{\mathcal{W}}) \leq \delta,$$

with

$$d(\Gamma_{\mathcal{W}_m}, \Gamma_{\mathcal{W}}) = \inf_{M_{\star, m} \in \Gamma_{\mathcal{W}_m}, M \in \Gamma_{\mathcal{W}}} d(M, M_{\star, m}) > 0.$$

Now, given a point $M \in \mathcal{D}(\Gamma_{\mathcal{W}}, \delta)$, for $m \geq m_1(\delta)$, we denote by $M_{\mathcal{W}} \in \Gamma_{\mathcal{W}}$ the closest point of M in $\Gamma_{\mathcal{W}}$. By the triangle inequality, we then have that

$$d(M, \Gamma_{\mathcal{W}_m}) \leq d(M, M_{\mathcal{W}}) + d(\Gamma_{\mathcal{W}_m}, \Gamma_{\mathcal{W}}) \leq 2\delta.$$

This implies that

$$M \in \mathcal{D}(\Gamma_{\mathcal{W}_m}, 2\delta),$$

from which we then deduce the desired result; i.e.,

$$\forall m \geq m_1(\delta) : \quad \mathcal{D}(\Gamma_{\mathcal{W}}, \delta) \subset \mathcal{D}(\Gamma_{\mathcal{W}_m}, 2\delta).$$

ii. At the same time, and also because of the density of the set $V^\star = \bigcup_{n \in \mathbb{N}} V_n$ in $\Gamma_{\mathcal{W}}$, there exists an integer $m_2(\delta) \in \mathbb{N}$ such that

$$\forall m \geq m_2(\delta) : \quad d(\Gamma_{\mathcal{W}_m}, \Gamma_{\mathcal{W}}) \leq \frac{\delta}{2}.$$

For $m \geq m_2(\delta)$, given a point $M \in \mathcal{D}\left(\Gamma_{\mathcal{W}_m}, \frac{\delta}{2}\right)$, we then have that

$$d(M, \Gamma_{\mathcal{W}}) \leq d(M, \Gamma_{\mathcal{W}_m}) + d(\Gamma_{\mathcal{W}_m}, \Gamma_{\mathcal{W}}) \leq \delta,$$

from which we then deduce the expected result, i.e.,

$$\forall m \geq m_2(\delta) : \quad \mathcal{D}\left(\Gamma_{\mathcal{W}_m}, \frac{\delta}{2}\right) \subset \mathcal{D}(\Gamma_{\mathcal{W}}, \delta).$$

iii. In order to conclude, we simply choose $m_0(\delta) = \max\{m_1(\delta), m_2(\delta)\}$.

□

Remark 3.2. Note that, in Proposition 3.2, on page 14, we have that

$$m_0 = m_0(\delta) \rightarrow \infty \quad , \quad \text{as } \delta \rightarrow 0^+.$$

Proposition 3.3 (Staggered Sequence of (m, ε_m^m) -Neighborhoods).

Given any integer $m \in \mathbb{N}$, there exists an integer $k_m \in \mathbb{N}$ such that, for each integer $k \geq k_m$, the $(m+k, \varepsilon_{m+k}^{m+k})$ -neighborhood of the Weierstrass Curve (where ε_{m+k}^{m+k} is the $(m+k)^{\text{th}}$ cohomology infinitesimal introduced in Definition 3.1, on page 11),

$$\mathcal{D}(\Gamma_{\mathcal{W}_{m+k}}, \varepsilon_{m+k}^{m+k}) = \left\{ M = (x, y) \in \mathbb{R}^2, d(M, \Gamma_{\mathcal{W}_{m+k}}) \leq \varepsilon_{m+k}^{m+k} \right\},$$

is contained in the (m, ε_m^m) -neighborhood of the Weierstrass Curve (with ε_m^m denoting, this time, the m^{th} cohomology infinitesimal),

$$\mathcal{D}(\Gamma_{\mathcal{W}_m}, \varepsilon_m^m) = \left\{ M = (x, y) \in \mathbb{R}^2, d(M, \Gamma_{\mathcal{W}_m}) \leq \varepsilon_m^m \right\};$$

namely,

$$\mathcal{D}(\Gamma_{\mathcal{W}_{m+k}}, \varepsilon_{m+k}^{m+k}) \subset \mathcal{D}(\Gamma_{\mathcal{W}_m}, \varepsilon_m^m).$$

Proof. This proof is based on the fact that the sequence of sets of vertices $(V_m)_{m \in \mathbb{N}}$ is increasing (see part *i.* of Proposition 2.2, on page 7), and that $V^* = \bigcup_{n \in \mathbb{N}} V_n$ is dense in the Weierstrass Curve $\Gamma_{\mathcal{W}}$, along with the fact that the prefractal graph sequence $(\Gamma_{\mathcal{W}_m})_{m \in \mathbb{N}}$ converges to the Weierstrass Curve $\Gamma_{\mathcal{W}}$ (for example, in the sense of the Hausdorff metric on \mathbb{R}^2).

Given a nonnegative integer $m \in \mathbb{N}$, there exists an integer $k_{0,m} \in \mathbb{N}$ such that, for each integer $k \geq k_{0,m}$, we have that

$$\begin{aligned} d(\Gamma_{\mathcal{W}_m}, \Gamma_{\mathcal{W}_{m+k}}) &= \inf_{\substack{0 \leq j \leq \#V_m - 1 \\ 0 \leq j' \leq \#V_{m+k} - 1}} \left\{ d(M_{j,m}, M_{j',m+k}), M_{j,m} \in V_m, M_{j',m+k} \in V_{m+k} \setminus V_m \right\} \\ &\leq \varepsilon_m^m. \end{aligned}$$

We then deduce that for all $k \geq k_{0,m}$,

$$\Gamma_{\mathcal{W}_{m+k}} \subset \mathcal{D}(\Gamma_{\mathcal{W}_m}, \varepsilon_m^m).$$

At the same time, since, for any $(m, k) \in \mathbb{N}^2$,

$$\varepsilon_{m+k}^{m+k} \leq \varepsilon_m^m,$$

along with the fact that, for any $m \in \mathbb{N}$,

$$\lim_{k \rightarrow \infty} \varepsilon_{m+k}^{m+k} = 0,$$

we can find another integer $k_{1,m} \in \mathbb{N}$ such that, for each integer $k \geq k_{1,m}$, we have that

$$\mathcal{D}(\Gamma_{\mathcal{W}_{m+k}}, \varepsilon_{m+k}^{m+k}) \subset \mathcal{D}(\Gamma_{\mathcal{W}_m}, \varepsilon_m^m).$$

The desired result is obtained by letting $k_m = \max \{k_{0,m}, k_{1,m}\}$. □

Remark 3.3 (Connection Between Fractality and the Cohomology Infinitesimal – Weierstrass Iterated Fractal Drums). As is mentioned in [DL22c], the cohomology infinitesimal (or, equivalently, the elementary length) – which obviously depends on the magnification scale (i.e., the chosen prefractal approximation) – can be seen as a *transition scale* between the fractal domain and the classical one. In fact, we could say that the system is fractal below this scale, and classical above

(for the level of magnification considered). In the limit when the integer m associated with the prefractal approximation tends to infinity, the system is fractal below the cohomology infinitesimal (i.e., at small scales) and is classical beyond (i.e., on large scales). Note that this is in perfect agreement with what is suggested by the French physicist Laurent Nottale in [Not98] about scale-relativity.

The Complex Dimensions of a fractal set characterize their intrinsic vibrational properties. Thus far, the values of the Complex Dimensions were obtained by studying the oscillations of a small neighborhood of the boundary, i.e., of a tubular neighborhood, where points are located within an epsilon distance from any edge; see, e.g., [LvF00], [LvF06], [LvF13], in the one-dimensional case (i.e., for fractal strings) and, in the higher-dimensional case, [LRŽ17a], [LRŽ17b], [LRŽ18]. In the present case of our fractal Weierstrass Curve $\Gamma_{\mathcal{W}}$, which is, also, the limit of the sequence of (polygonal) prefractal graphs $(\Gamma_{\mathcal{W}})_{m \in \mathbb{N}}$, it is natural – and consistent with the result of Proposition 3.3, on page 16, above – to envision the tubular neighborhood of $\Gamma_{\mathcal{W}}$ as the limit of the (obviously convergent) sequence $(\mathcal{D}(\Gamma_{\mathcal{W}_m}, \varepsilon_m^m))_{m \in \mathbb{N}}$ of ε_m^m -neighborhoods of $\Gamma_{\mathcal{W}_m}$, where, for each integer $m \in \mathbb{N}$, ε_m^m is the m^{th} cohomology infinitesimal introduced in Definition 3.1 above.

We note that, in a sense, this amounts to using a sequence of what we call *Weierstrass Iterated Fractal Drums* (in short, Weierstrass IFDs), by analogy with the *relative fractal drums* involved, for instance, in the case of the Cantor Staircase, in [LRŽ17b], Section 5.5.4, and in [LRŽ18]. In our present setting, the Weierstrass IFDs – i.e., the sets $\mathcal{D}(\Gamma_{\mathcal{W}_m}, \varepsilon_m^m)$, for all $m \in \mathbb{N}$ sufficiently large – contain the Weierstrass Curve $\Gamma_{\mathcal{W}}$, and are sufficiently close to $\Gamma_{\mathcal{W}}$, so that we can expect their Complex Dimensions to be the same.

Things change if we consider the fractal Complex Dimensions as dynamical quantities, which evolve with the scales: to each prefractal approximation $\Gamma_{\mathcal{W}_m}$ of $\Gamma_{\mathcal{W}}$, we can associate specific values of the Complex Dimensions, as is proved in [DL22c]. In this light, it is natural to define the Complex Dimensions of the Weierstrass Curve as the set of Complex Dimensions of $(\Gamma_{\mathcal{W}_m})_{m \in \mathbb{N}}$, obtained by means of the sequence of IFDs $(\mathcal{D}(\Gamma_{\mathcal{W}_m}, \varepsilon_m^m))_{m \in \mathbb{N}}$, as is done in Section 4.

4 Weierstrass Tube Zeta Function, and Associated Complex Dimensions

In this section, we discuss the results of [DL22b] in which we obtained the fractal tube formulas associated with the Weierstrass iterated fractal drums (Weierstrass IFDs, for short), and deduced from it the fractal (i.e., the tube and distance) zeta functions and hence, also, the associated Complex Dimensions (i.e., the poles of these fractal zeta functions). We also discuss the geometric consequences of these results for the Weierstrass IFDs and Curve – concerning, especially, the Minkowski nondegeneracy, the Minkowski measurability, the existence of the average Minkowski content, and the value of the Minkowski dimension (which is positive and is also the largest real part of the Complex Dimensions).

We first need the following useful, technical result.

Proposition 4.1 (Fourier Series Expansion of the 1-Periodic Map $x \mapsto N_b^{-\{x\}}$ [LvF06], [DL22b]).

The fractional part map $\{\cdot\}$ is one-periodic. Hence, it is also the case of the map $x \mapsto N_b^{-\{x\}}$, which admits, with respect to the real variable $x \in \mathbb{R} \setminus \mathbb{Z}$, the following Fourier Series expansion:

$$N_b^{-\{x\}} = \frac{N_b - 1}{N_b} \sum_{\ell \in \mathbb{Z}} \frac{e^{2i\pi m x}}{\ln N_b + 2i\ell\pi} = \frac{N_b - 1}{N_b} \sum_{\ell \in \mathbb{Z}} \frac{(N_b - 1)^{-i\ell\mathbf{p}} (\varepsilon_m^m)^{-i\ell\mathbf{p}}}{\ln N_b + 2i\ell\pi},$$

where the exponential Fourier coefficients c_ℓ have been obtained through

$$\begin{aligned} c_\ell &= \int_0^1 N_b^{-t} e^{-2i\pi\ell t} dt = \int_0^1 e^{-t \ln N_b} e^{-2i\pi\ell t} dt \\ &= -\frac{1}{\ln N_b + 2i\ell\pi} \left[e^{-t \ln N_b} e^{-2i\pi\ell t} \right]_0^1 \\ &= \frac{1}{\ln N_b + 2i\ell\pi} \left[1 - \frac{1}{N_b} \right] = \frac{N_b - 1}{N_b} \frac{1}{\ln N_b + 2i\ell\pi}. \end{aligned}$$

Thus, for any $x \in \mathbb{R} \setminus \mathbb{Z}$ and any integer $m \in \mathbb{N}$ sufficiently large,

$$N_b^{-\{x\}} = \frac{N_b - 1}{N_b} \sum_{\ell \in \mathbb{Z}} \frac{e^{2i\pi\ell x}}{\ln N_b + 2i\ell\pi}.$$

Since $x = -\ln_{N_b}((N_b - 1)\varepsilon_m^m)$, one has, for every $\ell \in \mathbb{Z}$,

$$e^{2i\pi\ell x} = e^{-2i\pi\ell \ln_{N_b}((N_b - 1)\varepsilon_m^m)} = e^{-2i\pi\ell \frac{\ln((N_b - 1)\varepsilon_m^m)}{\ln N_b}}.$$

Definition 4.1 (Oscillatory Period). Following [LvF00], [LvF06], [LvF13], [LRŽ17b], we introduce the *oscillatory period* of the Weierstrass Curve:

$$\mathbf{p} = \frac{2\pi}{\ln N_b}.$$

Definition 4.2 (Natural Tubular Volume Extension Formula – Effective Distance and Tube Zeta Functions Associated to an Arbitrary IFD of \mathbb{R}^2). Let $\mathcal{F}^{\mathcal{I}}$ be an iterated fractal drum of \mathbb{R}^2 ; i.e., given a cohomology infinitesimal $\varepsilon_{\mathcal{F}} = (\varepsilon_{m,\mathcal{F}}^m)_{m \in \mathbb{N}}$, as introduced in Definition 3.2, on page 13, and Definition 3.1, on page 11, $\mathcal{F}^{\mathcal{I}}$ is a sequence of ordered pairs $(\mathcal{F}_m, \varepsilon_{m,\mathcal{F}}^m)_{m \in \mathbb{N}}$, where, for each $m \in \mathbb{N}$, \mathcal{F}_m is the m^{th} prefractal approximation to a fractal curve \mathcal{F} .

We are assuming here that $(\varepsilon_{m,\mathcal{F}}^m)_{m \in \mathbb{N}}$ is a decreasing sequence of positive numbers tending to 0 as $m \rightarrow \infty$.

We hereafter consider the $\varepsilon_{m,\mathcal{F}}^m$ -neighborhood (or $\varepsilon_{m,\mathcal{F}}^m$ -tubular neighborhood) of \mathcal{F}_m ,

$$\mathcal{D}(\mathcal{F}_m, \varepsilon_{m,\mathcal{F}}^m) = \left\{ M \in \mathbb{R}^2, d(M, \mathcal{F}_m) \leq \varepsilon_{m,\mathcal{F}}^m \right\}, \quad (5)$$

of tubular volume (i.e., area) denoted $\mathcal{V}_{m,\mathcal{F}_m}(\varepsilon_{m,\mathcal{F}}^m)$.

We then introduce, for all sufficiently large $m \in \mathbb{N}^*$, $\tilde{\mathcal{V}}_{m,\mathcal{F}_m}$ as the continuous function defined for all $t \in [0, \varepsilon_{m,\mathcal{F}}^m]$ by substituting t for $\varepsilon_{m,\mathcal{F}}^m$ in the expression for $\mathcal{V}_{m,\mathcal{F}_m}(\varepsilon_{m,\mathcal{F}}^m)$ to be obtained in relation (10) of Theorem 4.2, on page 19. As is explained in [DL22b] (in the case of the ordinary Euclidean volume), one can think of $\tilde{\mathcal{V}}_{m,\mathcal{F}_m}(t)$ as being *the effective tubular volume* of the m^{th} prefractal approximation to the IFD. (Note that, for the Weierstrass IFD, the corresponding fractal power series is still convergent for all $t \in [0, \varepsilon_m^m]$.)

Indeed, in our present context, when it comes to obtaining the associated fractal tube zeta function, we cannot, as in the case of an arbitrary bounded subset of \mathbb{R}^2 (see [LRŽ17b], Definition 2.2.8, page 118), directly use an integral formula of the form

$$\tilde{\zeta}_{m,\mathcal{F}_m}(s) = \int_0^{\varepsilon_{m,\mathcal{F}}^m} t^{s-3} \mathcal{V}_{m,\mathcal{F}_m}(t) dt = \int_0^{\varepsilon_{m,\mathcal{F}}^m} t^{s-2} \mathcal{V}_{m,\mathcal{F}_m}(t) \frac{dt}{t}, \quad (6)$$

since the prefractal tube formulas that we will obtain in Theorem 4.2 below, on page 19, can only be expressed in an explicit way at a cohomology infinitesimal.

We can then obtain the resulting m^{th} *effective local tube zeta function* $\tilde{\zeta}_{m,\mathcal{F}}^e$ – a generalization to IFDs of the usual definition referred to just above – defined for all s in \mathbb{C} with sufficiently large real part (in fact, for $\text{Re}(s) > D_{m,\mathcal{F}_m}$, where D_{m,\mathcal{F}_m} is the abscissa of convergence of $\tilde{\zeta}_{m,\mathcal{F}_m}$), by

$$\tilde{\zeta}_{m,\mathcal{F}_m}^e(s) = \int_0^{\varepsilon_{m,\mathcal{F}}^m} t^{s-3} \tilde{\mathcal{V}}_{m,\mathcal{F}_m}(t) dt = \int_0^{\varepsilon_{m,\mathcal{F}}^m} t^{s-2} \tilde{\mathcal{V}}_{m,\mathcal{F}_m}(t) \frac{dt}{t}. \quad (7)$$

For the sake of simplicity, given $m \in \mathbb{N}$, we will from now on call *the m^{th} natural volume extension*, the volume extension function $\tilde{\mathcal{V}}_{m,\mathcal{F}_m}$ associated with $\mathcal{V}_{m,\mathcal{F}_m}$. Alternatively, as was mentioned earlier, $\tilde{\mathcal{V}}_{m,\mathcal{F}_m}$ will be called *the m^{th} effective tubular volume*.

Remark 4.1. As is explained in Remark 4.2 below, on page 19, in the present case of the Weierstrass IFDs, we stress the fact that $\tilde{\zeta}_{m,\mathcal{F}_m}^e$ does not coincide with the usual tube zeta function $\tilde{\zeta}_{\mathcal{F}_m}$ associated with the m^{th} polygonal prefractal approximation $\mathcal{F}_m \subset \mathbb{R}^2$ to the fractal curve \mathcal{F} , given, as in [LRŽ17b], for all $s \in \mathbb{C}$ with $\text{Re}(s)$ sufficiently large, by

$$\tilde{\zeta}_{\mathcal{F}_m}(s) = \int_0^{\varepsilon_{m,\mathcal{F}}^m} t^{s-3} \mathcal{V}_{m,\mathcal{F}_m}(t) dt = \int_0^{\varepsilon_{m,\mathcal{F}}^m} t^{s-2} \mathcal{V}_{m,\mathcal{F}_m}(t) \frac{dt}{t}.$$

This entire comment applies, in particular, to the Weierstrass IFD, which is the central object of this paper.

Remark 4.2. The reason why $\tilde{\zeta}_{m,\mathcal{F}_m}$ cannot coincide with $\tilde{\zeta}_{m,\mathcal{F}_m}^e$ – or, equivalently, why ζ_{m,\mathcal{F}_m} cannot coincide with $\zeta_{m,\mathcal{F}_m}^e$ – is that, otherwise, the abscissa of convergence of $\tilde{\zeta}_{m,\mathcal{F}_m}$ would coincide with that of $\tilde{\zeta}_{m,\mathcal{F}_m}^e$ (or, equivalently, for ζ_{m,\mathcal{F}_m} and $\zeta_{m,\mathcal{F}_m}^e$) (in the present case of the Weierstrass IFD) which would imply an impossible conclusion: in light of Theorem 4.2, on page 19, the Minkowski dimension of the m^{th} prefractal approximation – namely, 1, of course – must coincide with the Minkowski dimension (i.e., here, the common abscissa of convergence of $\tilde{\zeta}_{m,\mathcal{F}_m}^e$ and $\zeta_{m,\mathcal{F}_m}^e$) of the fractal curve $\Gamma_{\mathcal{W}}$, namely, $D_{\mathcal{W}} = 2 - \frac{\ln \frac{1}{\lambda}}{\ln N_b} > 1$, an obvious contradiction.

From now on, unless explicitly mentioned otherwise, we will only work with the Weierstrass IFD.

Theorem 4.2 (Prefractal Tube Formula for The Weierstrass Iterated Fractal Drums [DL22b]).

Given $m \in \mathbb{N}^*$ sufficiently large, the m^{th} total contribution to the tubular volume $\mathcal{V}_{m,\Gamma_{\mathcal{W}_m}}(\varepsilon_m^m)$, or two-dimensional Lebesgue measure of the ε_m^m -neighborhood of the m^{th} prefractal approximation $\Gamma_{\mathcal{W}_m}$,

$$\mathcal{D}(\varepsilon_m^m) = \left\{ M = (x, y) \in \mathbb{R}^2, d(M, \Gamma_{\mathcal{W}_m}) \leq \varepsilon_m^m \right\}, \quad (8)$$

where $\varepsilon = (\varepsilon_m^m)_{m \in \mathbb{N}}$ is the cohomology infinitesimal, as introduced in Definition 3.1, on page 11, is given by

$$\begin{aligned} \tilde{\mathcal{V}}_{m,\Gamma_{\mathcal{W}_m}}(\varepsilon_m^m) &= \\ &= \tilde{\mathcal{V}}_{m,\Gamma_{\mathcal{W}_m},\text{Rectangles}}(\varepsilon_m^m) + \tilde{\mathcal{V}}_{m,\Gamma_{\mathcal{W}_m},\text{wedges}}(\varepsilon_m^m) \\ &\quad + \tilde{\mathcal{V}}_{m,\Gamma_{\mathcal{W}_m},\text{extra outer triangles}}(\varepsilon_m^m) + \tilde{\mathcal{V}}_{m,\Gamma_{\mathcal{W}_m},\text{parallelograms}}(\varepsilon_m^m); \end{aligned} \quad (9)$$

i.e.,

$$\tilde{\mathcal{V}}_{m,\Gamma_{\mathcal{W}_m}}(\varepsilon_m^m) =$$

$$\begin{aligned}
&= C_{\text{Rectangles}} \sum_{k \in \mathbb{N}, \ell \in \mathbb{Z}} \left(\frac{1}{2} \right)^k \frac{N_b^{1-k(2-D_{\mathcal{W}})} - 1 (N_b - 1)^{-i\ell \mathbf{p}} (\varepsilon_m^m)^{2-D_{\mathcal{W}}+k(2-D_{\mathcal{W}})-i\ell \mathbf{p}}}{N_b^{1-k(2-D_{\mathcal{W}})} (1 - k(2 - D_{\mathcal{W}})) \ln N_b + 2i\ell \pi} \\
&+ C_{\text{wedges}}^1 \sum_{\ell \in \mathbb{Z}} \frac{(N_b - 1)^{-i\ell \mathbf{p}} (\varepsilon_m^m)^{3-i\ell \mathbf{p}}}{\ln N_b + 2i\ell \pi} + \pi (\varepsilon_m^m)^2 - \frac{\pi (\varepsilon_m^m)^4}{2} \\
&- C_{\text{wedges}}^2 \sum_{k \in \mathbb{N}, \ell \in \mathbb{Z}} \frac{(-1)^k}{2k+1} \frac{N_b^{((2k+1)D_{\mathcal{W}}-2k)} - 1}{N_b^{((2k+1)D_{\mathcal{W}}-2k)}} \frac{(N_b^{((2k+1)D_{\mathcal{W}}-2k)} - 1)^{-i\ell \mathbf{p}} (\varepsilon_m^m)^{2k+1-i\ell \mathbf{p}}}{((2k+1)D_{\mathcal{W}} - 2k) \ln N_b + 2i\ell \pi} \\
&+ C_{\text{wedges}}^3 \sum_{k \in \mathbb{N}, \ell \in \mathbb{Z}} \frac{(-1)^k}{2k+1} \frac{N_b^{(2k+1)(D_{\mathcal{W}}-1)} - 1}{N_b^{(2k+1)(D_{\mathcal{W}}-1)}} \frac{(N_b^{(2k+1)D_{\mathcal{W}}-2k+1} - 1)^{-i\ell \mathbf{p}} (\varepsilon_m^m)^{5+2k-i\ell \mathbf{p}}}{((2k+1)D_{\mathcal{W}} - 2k+1) \ln N_b + 2i\ell \pi} \\
&- (C_{\text{triangles}} + C_{\text{parallelograms}}) \sum_{\ell \in \mathbb{Z}} \frac{(N_b^{2-3D_{\mathcal{W}}} - 1)^{-i\ell \mathbf{p}}}{(2 - 3D_{\mathcal{W}}) \ln N_b + 2i\ell \pi} (\varepsilon_m^m)^{2-i\ell \mathbf{p}},
\end{aligned} \tag{10}$$

where $C_{\text{rectangles}}$, C_{wedges}^ℓ , $\ell = 1, 2, 3$, $C_{\text{triangles}}$, and $C_{\text{parallelograms}}$ denote strictly positive and finite constants respectively, which depend on m , but are uniformly bounded away from 0 and ∞ (in $m \in \mathbb{N}^*$ large enough); see [DL22b].

Also, recall from Definition 4.2, on page 18, that, by construction,

$$\tilde{\mathcal{V}}_{m, \Gamma_{\mathcal{W}_m}}(\varepsilon_m^m) = \mathcal{V}_{m, \Gamma_{\mathcal{W}_m}}(\varepsilon_m^m).$$

Actually, this identity follows from the corresponding identity for each of the terms on the right-hand side of relation (9), on page 19.

For the sake of clarity, and in order to highlight the role played by the one-periodic functions (with respect to the variable $\ln_{N_b}(\varepsilon_m^m)^{-1}$, see Proposition 4.1, on page 17), one can exchange the sums over k and m , which enables one to obtain an expression of the following form:

$$\begin{aligned}
&\tilde{\mathcal{V}}_{m, \Gamma_{\mathcal{W}_m}}(\varepsilon_m^m) = \\
&= \sum_{k \in \mathbb{N}, \ell \in \mathbb{Z}} f_{k, \ell, \text{Rectangles}}(\varepsilon_m^m)^{2-D_{\mathcal{W}}+k(2-D_{\mathcal{W}})-i\ell \mathbf{p}} \\
&+ \sum_{k \in \mathbb{N}, \ell \in \mathbb{Z}} \left(f_{k, \ell, \text{wedges}, 1}(\varepsilon_m^m)^{3-i\ell \mathbf{p}} + f_{k, \ell, \text{wedges}, 2}(\varepsilon_m^m)^{1+2k-i\ell \mathbf{p}} \right. \\
&\quad \left. + f_{k, \ell, \text{wedges}, 3}(\varepsilon_m^m)^{5+2k-i\ell \mathbf{p}} \right) \\
&+ \sum_{k \in \mathbb{N}, \ell \in \mathbb{Z}} f_{k, \ell, \text{triangles, parallelograms}}(\varepsilon_m^m)^{2-i\ell \mathbf{p}} + \pi (\varepsilon_m^m)^2 - \frac{\pi (\varepsilon_m^m)^4}{2},
\end{aligned} \tag{11}$$

where the notation $f_{k, \ell, \text{Rectangles}}$, $f_{k, \ell, \text{wedges}, \ell'}$, $1 \leq \ell' \leq 3$, and $f_{k, \ell, \text{triangles, parallelograms}}$, respectively account for the nonzero coefficients associated to the sums corresponding to the contribution of the rectangles, wedges, triangles and parallelograms, respectively given by:

$$f_{k,\ell,Rectangles} = C_{Rectangles} \binom{\frac{1}{2}}{k} \frac{N_b^{1-k(2-D_{\mathcal{W}})} - 1}{N_b^{1-k(2-D_{\mathcal{W}})}} \frac{(N_b - 1)^{-i\ell\mathbf{P}}}{(1 - k(2 - D_{\mathcal{W}})) \ln N_b + 2i\ell\pi}; \quad (12)$$

$$f_{k,\ell,wedges,1} = C_{wedges}^1 \frac{(N_b - 1)^{-i\ell\mathbf{P}}}{\ln N_b + 2i\ell\pi}; \quad (13)$$

$$f_{k,\ell,wedges,2} = -C_{wedges}^2 \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \frac{N_b^{((2k+1)D_{\mathcal{W}}-2k)} - 1}{N_b^{((2k+1)D_{\mathcal{W}}-2k)}} \frac{(N_b^{((2k+1)D_{\mathcal{W}}-2k)} - 1)^{-i\ell\mathbf{P}}}{((2k+1)D_{\mathcal{W}} - 2k) \ln N_b + 2i\ell\pi}; \quad (14)$$

$$f_{k,\ell,wedges,3} = C_{wedges}^3 \frac{(-1)^k}{2k+1} \frac{N_b^{(2k+1)(D_{\mathcal{W}}-1)} - 1}{N_b^{(2k+1)(D_{\mathcal{W}}-1)}} \frac{(N_b^{(2k+1)D_{\mathcal{W}}-2k+1} - 1)^{-i\ell\mathbf{P}}}{((2k+1)D_{\mathcal{W}} - 2k + 1) \ln N_b + 2i\ell\pi}; \quad (15)$$

$$f_{k,\ell,triangles, parallelograms} = -(C_{triangles} + C_{parallelograms}) \frac{(N_b^{2-3D_{\mathcal{W}}} - 1)^{-i\ell\mathbf{P}}}{(2 - 3D_{\mathcal{W}}) \ln N_b + 2i\ell\pi}. \quad (16)$$

Note that those coefficients do not depend on ε_m^m , and satisfy the following uniform estimates (independent of $m \in \mathbb{N}^*$ sufficiently large):

$$|f_{k,\ell,Rectangles}| \leq C_{Rectangles} \binom{\frac{1}{2}}{k} \frac{1}{2\ell\pi}; \quad (17)$$

$$|f_{k,\ell,wedges,1}| \leq \frac{C_{wedges}^1}{2\ell\pi}; \quad (18)$$

$$|f_{k,\ell,wedges,2}| \leq \frac{C_{wedges}^2}{2k+1} \frac{1}{2\ell\pi}; \quad (19)$$

$$|f_{k,\ell,wedges,3}| \leq \frac{C_{wedges}^3}{2k+1} \frac{1}{2\ell\pi}; \quad (20)$$

$$|f_{k,\ell,triangles, parallelograms}| \leq (C_{triangles} + C_{parallelograms}). \quad (21)$$

Finally, each of the double sums in formulae (10), on page 20, and (11), on page 20, is absolutely convergent (and hence, convergent).

Remark 4.3. Following (as well as adapting) [LRŽ17b], we define the tube zeta function of the sequence of Weierstrass IFDs associated to the cohomology infinitesimal by

$$\tilde{\zeta}_{\mathcal{W}_m}(s) = \int_0^{\varepsilon_m^m} t^{s-3} \mathcal{V}_m(t) dt = \int_0^{\varepsilon_m^m} t^{s-2} \mathcal{V}_m(t) \frac{dt}{t},$$

for all s in \mathbb{C} , with $\mathcal{R}e(s)$ sufficiently large, and, optimally, for $\mathcal{R}e(s) > D_{\mathcal{W}_m} = D_{\mathcal{W}}$.

Theorem 4.3 (Local and Global Tube Zeta Function for the Weierstrass Iterated Fractal Drums [DL22b], [DL23b]). Given $m \in \mathbb{N}^*$ sufficiently large, we denote by $\tilde{\zeta}_{m,\Gamma_{\mathcal{W}_m}}^{e,exact}$ the exact expression for the m^{th} local effective tube zeta function, which corresponds to the expression $\tilde{\mathcal{V}}_{m,\Gamma_{\mathcal{W}_m}}^{e,exact}(\varepsilon_m^m) = \tilde{\mathcal{V}}_{m,\Gamma_{\mathcal{W}_m}}^e(\varepsilon_m^m)$ where $\tilde{\mathcal{V}}_{m,\Gamma_{\mathcal{W}_m}}^e(\varepsilon_m^m)$ is given by relation (9), on page 19, and not by the approximate expression given in relation (10), on page 20. We then set, for any $s \in \mathbb{C}$:

$$\zeta_{m,\Gamma_{\mathcal{W}_m}}^{e,strict}(s) = \zeta_{m,\Gamma_{\mathcal{W}_m}}^{e,exact}(s) - \frac{\pi(\varepsilon_m^m)^s}{s} + \frac{\pi(\varepsilon_m^m)^{s+2}}{4(s+2)},$$

since the contribution of the m^{th} prefractal approximation $\Gamma_{\mathcal{W}_m}$ to $\zeta_{\mathcal{W}}^e$, the global effective tube zeta function of the Weierstrass IFD, is obtained by excluding the (artificial) terms $\frac{\pi(\varepsilon_m^m)^s}{s}$ and $-\frac{\pi(\varepsilon_m^m)^{s+2}}{4(s+2)}$ coming from the extreme wedges.

The global effective tube zeta function of the Weierstrass IFD admits a (necessarily unique) meromorphic continuation to all of \mathbb{C} , and is given, for any $s \in \mathbb{C}$, by the following expression (see [DL23b] and [DL23a] for the proof of the existence of the limit, which is locally uniform on \mathbb{C}):

$$\zeta_{\mathcal{W}}^e(s) = \lim_{m \rightarrow \infty} \zeta_{m,\Gamma_{\mathcal{W}_m}}^{e,strict}(s). \quad (22)$$

Note that, in light of Definition 4.2, on page 18, and for all integers $m \in \mathbb{N}^*$ sufficiently large, $\zeta_{m,\Gamma_{\mathcal{W}_m}}^{e,strict}$ is a (tamed) Dirichlet-type integral (in the sense of [LRŽ17b], Appendix A) and hence, admits an abscissa of (absolute) convergence. This fact can also be double-checked directly, by simply using the convergent fractal power series on the right-hand side of relation 10, on page 20, once the variable $t \in [0, \varepsilon_m^m]$ has been substituted for ε_m^m and $\zeta_{m,\Gamma_{\mathcal{W}_m}}^e$ calculated accordingly (as in [DL22b]).

Furthermore, the abscissa of convergence of $\zeta_{\mathcal{W}}^e$ is equal to

$$D_{\mathcal{W}} = 2 + \frac{\ln \lambda}{\ln b} = 2 - \ln_b \frac{1}{\lambda}.$$

Remark 4.4 (About the excluded poles 0 and -2). We do not include the (artificial) terms coming from the extreme wedges. Indeed, due to the periodicity of the Weierstrass function, we have restricted our study to the values of the abscissa $x \in [0, 1]$. In a sense, this amounts to *cut* the Curve, so that the poles arising because of this *cut* do not have to be taken into account. More precisely, given $m \in \mathbb{N}^*$ sufficiently large, the terms $-\frac{\pi(\varepsilon_m^m)^4}{2}$ and $\pi(\varepsilon_m^m)^2$ in the expression for $\tilde{\mathcal{V}}_{m,\Gamma_{\mathcal{W}_m}}(\varepsilon_m^m)$ given in relation (10), on page 20, are the terms which give rise to the terms $\frac{\pi(\varepsilon_m^m)^s}{s}$ and $-\frac{\pi(\varepsilon_m^m)^{s+2}}{2(s+2)}$ in the expression for the associated m^{th} local effective tube zeta function.

Sketch of the proof.

Note that, in this sketch of the proof, we will not use the approximate expression in relation (9), on page 19, which is not required here.

In [DL23b] (and since $\lim_{m \rightarrow \infty} \varepsilon_m^m = 0$), we have proved that, given $m \in \mathbb{N}^*$ sufficiently large, the m^{th} strict effective tubular volume

$$\tilde{\mathcal{V}}_{m,\Gamma_{\mathcal{W}_m}}^{e,strict}(\varepsilon_m^m) = \tilde{\mathcal{V}}_{m,\Gamma_{\mathcal{W}_m}}^{e,exact}(\varepsilon_m^m) + \frac{\pi(\varepsilon_m^m)^4}{2} - \pi(\varepsilon_m^m)^2$$

can be connected to the m^{th} effective polyhedral volume $\tilde{\mathcal{V}}_m^{\mathcal{P}}(\varepsilon_m^m)$ by means of the following relation:

$$\tilde{\mathcal{V}}_m^{\mathcal{P}}(\varepsilon_m^m) = \tilde{\mathcal{V}}_{m,\Gamma_{\mathcal{W}_m}}^{e,strict}(\varepsilon_m^m) + \tilde{\mathcal{R}}_m, \quad (23)$$

where $\tilde{\mathcal{R}}_m$ is a suitable error term such that the associated zeta function $\mathcal{Z}_{m,\mathcal{R}}$, given by

$$s \mapsto \int_0^{\varepsilon} t^{s-3} \tilde{\mathcal{R}}_m(t) dt,$$

(where $\varepsilon = \frac{1}{N_b}$ is the intrinsic scale, as introduced in Definition 3.1, on page 11) locally, uniformly on \mathbb{C} , converges to 0:

$$\lim_{m \rightarrow \infty} \int_0^\varepsilon t^{s-3} \widetilde{\mathcal{R}}_m(t) dt = 0.$$

We then obtain that (locally, uniformly on \mathbb{C})

$$\lim_{m \rightarrow \infty} \tilde{\zeta}_{m, \Gamma_{\mathcal{W}_m}}^{e, \text{strict}}(s) = \lim_{m \rightarrow \infty} \tilde{\zeta}_{m, \Gamma_{\mathcal{W}_m}}^{\mathcal{P}, e}(s) = \tilde{\zeta}_{\Gamma_{\mathcal{W}}}^e(s),$$

where, for all $m \in \mathbb{N}^\star$, $\tilde{\zeta}_{m, \Gamma_{\mathcal{W}_m}}^{\mathcal{P}, e}$ is the effective fractal zeta function associated with the sequence of polyhedral neighborhoods of the Weierstrass IFD (and called *the m^{th} local polyhedral effective zeta function*).

We then prove, in [DL23b], that the (uniform) limit fractal zeta function $\tilde{\zeta}_{\mathcal{W}}^e$, is meromorphic on all of \mathbb{C} .

We note that both sequences of local effective zeta functions $(\tilde{\zeta}_{m, \Gamma_{\mathcal{W}_m}}^{\mathcal{P}, e})_{m \in \mathbb{N}}$ and $(\tilde{\zeta}_{m, \Gamma_{\mathcal{W}_m}}^{e, \text{strict}})_{m \in \mathbb{N}}$ thus have the same limit, the global fractal zeta function $\tilde{\zeta}_{\mathcal{W}}^e$ of the Weierstrass IFD. As was mentioned above, an interesting comment to be made is that we do not pass to the limit in the expression for the strict tubular zeta function $\tilde{\zeta}_{m, \Gamma_{\mathcal{W}_m}}^{e, \text{strict}}$ defined in Theorem 4.3, on page 21 (and which cannot be obtained by using Theorem 4.2, on page 19, along with Definition 3.3, on page 13), for the following reasons:

- i. Given $m \in \mathbb{N}^\star$, we only have an unexplicit expression for the m^{th} effective tubular volume $\tilde{\mathcal{V}}_{m, \Gamma_{\mathcal{W}_m}}(\varepsilon_m^m)$.
- ii. The expected result only depends on the definition of the m^{th} effective tubular volume $\tilde{\mathcal{V}}_{m, \Gamma_{\mathcal{W}_m}}(\varepsilon_m^m)$, given in Theorem 4.2, on page 19, along with relation (23) above, on page 22.

Hereafter, a (complex-valued) meromorphic function f is viewed as a continuous function with values in the Riemann sphere (or complex projective line) $\mathbb{P}^1(\mathbb{C})$, equipped with the chordal metric, and such that, for any pole ω of f , $f(\omega)$ takes the value ∞ (for instance, as in [LvF13], Section 3.4 and Appendix C).

We can show that, for *the chordal metric*, defined, for all $(z_1, z_2) \in (\mathbb{P}^1(\mathbb{C}))^2$ by

$$\|z_1, z_2\| = \frac{|z_1 - z_2|}{\sqrt{1 + |z_1|^2} \sqrt{1 + |z_2|^2}}, \text{ if } z_1 \neq \infty \text{ and } z_2 \neq \infty,$$

and

$$\|z_1, \infty\| = \frac{1}{\sqrt{1 + |z_1|^2}}, \text{ if } z_1 \neq \infty,$$

we have, thanks to the local uniform convergence on \mathbb{C} ,

$$\lim_{m \rightarrow \infty} \|\tilde{\zeta}_m^{e, \text{strict}}, \tilde{\zeta}_{\mathcal{W}}^e\| = 0.$$

Indeed, for any $\eta > 0$ and any compact set $\mathcal{K} \subset \mathbb{C}$, we can choose $m_1 \in \mathbb{N}^\star$ such that, for all integers $m \geq m_1$ and all $s \in \mathcal{K}$, we have that

$$|\tilde{\zeta}_m^{e, \text{strict}}(s) - \tilde{\zeta}_{\mathcal{W}}^e(s)| \leq \eta,$$

and hence, still for $m \geq m_1$ and all $s \in \mathcal{K}$,

$$\|\tilde{\zeta}_m^{e, \text{strict}}(s), \tilde{\zeta}_{\mathcal{W}}^e(s)\| \leq |\tilde{\zeta}_m^{e, \text{strict}}(s) - \tilde{\zeta}_{\mathcal{W}}^e(s)| \leq \eta.$$

Remark 4.5. The fact that the global zeta function $\tilde{\zeta}_{\mathcal{W}}^e$ admits a meromorphic continuation to all of \mathbb{C} is proved in [DL23b], where $\tilde{\zeta}_{\mathcal{W}}^e$ is obtained as the (uniform) limit sum of a (locally) normally convergent series of functions.

Definition 4.3 (Intrinsic Complex Dimensions of the Weierstrass Curve).

The *intrinsic Complex Dimensions* of the Weierstrass Curve $\Gamma_{\mathcal{W}}$ (or, rather, of the Weierstrass IFD) are the poles of the global effective zeta function $\tilde{\zeta}_{\mathcal{W}}^e$, introduced in Theorem 4.3, on page 21.

Theorem 4.4 (Intrinsic Complex Dimensions of the Weierstrass Curve [DL22b]). *The intrinsic Complex Dimensions of the Weierstrass Curve (or, equivalently, of the Weierstrass IFD) are all simple, exact, and given as follows:*

$$D_{\mathcal{W}} - k(2 - D_{\mathcal{W}}) + i\ell \mathbf{p} \quad , \quad \text{with } k \in \mathbb{N}, \ell \in \mathbb{Z}, \text{ both arbitrary.}$$

Consequently, the Weierstrass Curve is fractal, in the sense of the theory of Complex Dimensions developed in [LvF00], [LvF06], [LvF13], [LRŽ17b] and [Lap19], since it admits nonreal Complex Dimensions. In fact, in the terminology of [LRŽ17b], it is principally fractal because it is fractal in dimension $D_{\mathcal{W}}$ (i.e., it has nonreal Complex Dimensions with real part $D_{\mathcal{W}}$, the Minkowski dimension of $\Gamma_{\mathcal{W}}$).

Proof. This directly comes from the result obtained in [DL23b] for the global polyhedral effective zeta function, which is equal to the global effective zeta function $\tilde{\zeta}_{\mathcal{W}}^e$. □

Remark 4.6 (Possible Interpretation [DL22b]). Figure 6, on page 48, gives the distribution of the intrinsic Complex Dimensions of the Weierstrass IFD – and hence also, in practice, of the Weierstrass Curve itself.

Theorem 4.5 (Condensed Prefractal Tube Formula for the Weierstrass IFD (Corollary of Theorem 4.2, on page 19)). *Given $m \in \mathbb{N}$ sufficiently large, the tubular effective volume $\tilde{\mathcal{V}}_{m, \Gamma_{\mathcal{W}_m}}(\varepsilon_m^m)$ of the ε_m^m -neighborhood $\mathcal{D}(\varepsilon_m^m)$ of the Weierstrass IFD, can be expressed in the following manner:*

$$\begin{aligned} \tilde{\mathcal{V}}_{m, \Gamma_{\mathcal{W}_m}}(\varepsilon_m^m) = & \sum_{k=0}^{\infty} (\varepsilon_m^m)^{2-(D_{\mathcal{W}}-k(2-D_{\mathcal{W}}))} G_{k, D_{\mathcal{W}}} \left(\ln_{N_b} \left(\frac{1}{\varepsilon_m^m} \right) \right) \\ & + \sum_{k=0}^{\infty} (\varepsilon_m^m)^{2-(1-2k)} G_{k, 1} \left(\ln_{N_b} \left(\frac{1}{\varepsilon_m^m} \right) \right) + \pi (\varepsilon_m^m)^2 - \frac{\pi (\varepsilon_m^m)^4}{2}, \end{aligned} \tag{24}$$

where, for any fixed (but arbitrary) $k \in \mathbb{N}$, $G_{k, D_{\mathcal{W}}}$ and $G_{k, 1}$ denote, respectively, continuous one-periodic functions (with respect to the variable $\ln_{N_b}(\varepsilon_m^m)^{-1}$, see Proposition 4.1, on page 17) respectively associated to all of the Complex Dimensions of real parts $D_{\mathcal{W}} - k(2 - D_{\mathcal{W}})$ and the poles $1 - 2k$. Furthermore, all of the Fourier coefficients of the periodic functions $G_{k, D_{\mathcal{W}}}$ (for any $k \in \mathbb{N}$) and $G_{0, 1}$ are nonzero. In particular, these periodic functions are not constant. Moreover, the functions $G_{0, D_{\mathcal{W}}}$ and $G_{0, 1}$ are bounded away from zero and infinity.

This amounts to an expression of the form

$$\tilde{\mathcal{V}}_{m, \Gamma_{\mathcal{W}_m}}(\varepsilon_m^m) =$$

$$= \sum_{\substack{\alpha \text{ real part of a Complex Dimension} \\ \alpha \notin \{-2, 0\}}} (\varepsilon_m^m)^{2-\alpha} G_\alpha \left(\ln_{N_b} \left(\frac{1}{\varepsilon_m} \right) \right) + \pi (\varepsilon_m^m)^2 - \frac{\pi (\varepsilon_m^m)^4}{2}, \quad (25)$$

where, for any real part α of a Complex Dimension, G_α denotes a continuous and one-periodic function.

Naturally, it follows that the expression for $\tilde{\mathcal{V}}_{m, \Gamma_{\mathcal{W}_m}}(\varepsilon_m^m)$ is obtained by simply removing the last two terms on the right-hand side of relation 24, on page 24, or, equivalently, of relation 25, on page 25

In the spirit of the remainder of this section, the definition of (upper, lower) Minkowski contents and dimensions, for example, will be given in terms of the cohomology infinitesimal $(\varepsilon_m^m)_{m=0}^\infty$, viewed as a sequence of positive scales tending to zero, as $m \rightarrow \infty$. So will the notions of Minkowski nondegeneracy and Minkowski measurability, as well as that of effective average Minkowski content.

Definition 4.4 (Lower and Upper r -Dimensional Minkowski Contents – Lower and Upper Minkowski Dimensions, and Minkowski Dimension of an IFD). Let $\mathcal{F}^{\mathcal{I}}$ be an arbitrary iterated fractal drum of \mathbb{R}^2 ; see Definition 3.2, on page 13. More precisely, we hereafter consider the sequence of ordered pairs $(\mathcal{F}_m, \varepsilon_{\mathcal{F}, m}^m)_{m \in \mathbb{N}}$, where, for each $m \in \mathbb{N}$, \mathcal{F}_m is the m^{th} prefractal approximation to a fractal set \mathcal{F} , and where $\varepsilon_{\mathcal{F}, m}^m$ is the associated m^{th} cohomology infinitesimal.

Then, given $r \geq 0$, $m \in \mathbb{N}$, and the $\varepsilon_{\mathcal{F}, m}^m$ -neighborhood (or tubular neighborhood) of \mathcal{F}_m ,

$$\mathcal{D}_{\mathcal{F}_m}(\varepsilon_{\mathcal{F}, m}^m) = \{M \in \mathbb{R}^2, d(M, \mathcal{F}_m) \leq \varepsilon_{\mathcal{F}, m}^m\}, \quad (26)$$

of tubular volume $\mathcal{V}_{m, \mathcal{F}_m}(\varepsilon_{\mathcal{F}, m}^m)$, we define, much as in [LRŽ17b], the *lower r -dimensional Minkowski content* (resp., the *upper r -dimensional Minkowski content*) of the IFD as

$$\mathcal{M}_\star^r(\mathcal{F}^{\mathcal{I}}) = \liminf_{m \rightarrow \infty} \frac{\mathcal{V}_{m, \mathcal{F}_m}(\varepsilon_{\mathcal{F}, m}^m)}{(\varepsilon_{\mathcal{F}, m}^m)^{2-r}} \quad \left(\text{resp., } \mathcal{M}^{\star, r}(\mathcal{F}^{\mathcal{I}}) = \limsup_{m \rightarrow \infty} \frac{\mathcal{V}_{m, \mathcal{F}_m}(\varepsilon_{\mathcal{F}, m}^m)}{(\varepsilon_{\mathcal{F}, m}^m)^{2-r}} \right). \quad (27)$$

Recall that $\lim_{m \rightarrow \infty} \varepsilon_{\mathcal{F}, m}^m = 0$; see Definition 3.2, on page 13, along with Definition 3.1, on page 11, for the special case of the Weierstrass IFD, for which we also have (in the present notation),

$$\mathcal{V}_{m, \mathcal{F}_m}(\varepsilon_{\mathcal{F}, m}^m) = \tilde{\mathcal{V}}_{m, \mathcal{F}_m}(\varepsilon_{\mathcal{F}, m}^m),$$

for all $m \in \mathbb{N}$.

Note that, by definition, we have that

$$0 \leq \mathcal{M}_\star^r(\mathcal{F}^{\mathcal{I}}) \leq \mathcal{M}^{\star, r}(\mathcal{F}^{\mathcal{I}}) \leq \infty. \quad (28)$$

We then define the *lower Minkowski dimension* (resp., the *upper Minkowski dimension*) of the IFD by

$$\underline{D}(\mathcal{F}^{\mathcal{I}}) = \inf \{r > 0, \mathcal{M}_\star^r(\mathcal{F}^{\mathcal{I}}) < \infty\} \quad (29)$$

$$\left(\text{resp., } \overline{D}(\mathcal{F}^{\mathcal{I}}) = \inf \{r > 0, \mathcal{M}^{\star, r}(\mathcal{F}^{\mathcal{I}}) < \infty\} \right). \quad (30)$$

As usual, by definition, the Minkowski dimension $D_{\mathcal{F}^{\mathcal{I}}} = D(\mathcal{F}^{\mathcal{I}})$ of the IFD *exists* if

$$\underline{D}(\mathcal{F}^{\mathcal{I}}) = \overline{D}(\mathcal{F}^{\mathcal{I}}), \quad (31)$$

in which case, of course, we have that

$$D_{\mathcal{F}^{\mathcal{I}}} = D(\mathcal{F}^{\mathcal{I}}) = \underline{D}(\mathcal{F}^{\mathcal{I}}) = \overline{D}(\mathcal{F}^{\mathcal{I}}). \quad (32)$$

Definition 4.5 (Minkowski Nondegeneracy and Minkowski Measurability of an IFD).

Let $\mathcal{F}^{\mathcal{I}}$ be an arbitrary IFD. Assume that its Minkowski dimension $D_{\mathcal{F}^{\mathcal{I}}}$ exists, in the sense of Definition 4.4, on page 25 just above.

Then, with the same notation as in Definition 4.4, on page 25, the IFD $\mathcal{F}^{\mathcal{I}}$ is said to be *Minkowski nondegenerate* if the lower and upper Minkowski contents,

$$\mathcal{M}_{\star}^{D_{\mathcal{F}^{\mathcal{I}}}}(\mathcal{F}^{\mathcal{I}}) = \liminf_{m \rightarrow \infty} \frac{\mathcal{V}_{m, \mathcal{F}_m}(\varepsilon_{\mathcal{F}, m}^m)}{(\varepsilon_{\mathcal{F}, m}^m)^{2-D_{\mathcal{F}^{\mathcal{I}}}}}$$

and

$$\mathcal{M}^{\star, D_{\mathcal{F}^{\mathcal{I}}}}(\mathcal{F}^{\mathcal{I}}) = \limsup_{m \rightarrow \infty} \frac{\mathcal{V}_{m, \mathcal{F}_m}(\varepsilon_{\mathcal{F}, m}^m)}{(\varepsilon_{\mathcal{F}, m}^m)^{2-D_{\mathcal{F}^{\mathcal{I}}}},$$

are respectively positive and finite. Recall that the inequalities in (28), on page 25 always hold.

Finally, the IFD $\mathcal{F}^{\mathcal{I}}$ is said to be *Minkowski measurable* if it is Minkowski nondegenerate and

$$\mathcal{M}_{\star}^{D_{\mathcal{F}^{\mathcal{I}}}}(\mathcal{F}^{\mathcal{I}}) = \mathcal{M}^{\star, D_{\mathcal{F}^{\mathcal{I}}}}(\mathcal{F}^{\mathcal{I}}); \quad (33)$$

i.e., if the following limit exists in $]0, +\infty[$ (and necessarily equals this common value, denoted by $\mathcal{M}^{D_{\mathcal{F}^{\mathcal{I}}}}(\mathcal{F}^{\mathcal{I}})$):

$$\mathcal{M}^{D_{\mathcal{F}^{\mathcal{I}}}}(\mathcal{F}^{\mathcal{I}}) = \lim_{m \rightarrow \infty} \frac{\mathcal{V}_{m, \mathcal{F}_m}(\varepsilon_{\mathcal{F}, m}^m)}{(\varepsilon_{\mathcal{F}, m}^m)^{2-D_{\mathcal{F}^{\mathcal{I}}}}. \quad (34)$$

Then, $\mathcal{M}^{D_{\mathcal{F}^{\mathcal{I}}}}(\mathcal{F}^{\mathcal{I}})$ is called the *Minkowski content* of the IFD.

Definition 4.6 (Average Lower and Upper Minkowski Contents of an IFD).

We hereafter use the same notation as in Definition 4.4, on page 25, and in Definition 4.5, on page 26 just above, where $\mathcal{F}^{\mathcal{I}}$ denotes an arbitrary iterated fractal drum of \mathbb{R}^2 .

Then, by analogy with what can be found in [LRŽ17b], Definition 2.4.1, on page 178, we define, for all $m \in \mathbb{N}$ sufficiently large, the m^{th} *effective average lower-dimensional Minkowski content* (resp., the m^{th} *effective average upper-dimensional Minkowski content*) of \mathcal{F}_m as

$$\widetilde{\mathcal{M}}_{\star}^{D_m, e}(\mathcal{F}_m) = \liminf_{r \rightarrow +\infty} \frac{1}{\ln r} \int_{\frac{1}{r}}^{\varepsilon_{\mathcal{F}, m}^m} t^{D_m-3} \tilde{\mathcal{V}}_{m, \mathcal{F}_m}(t) dt \quad (35)$$

$$\left(\text{resp., } \widetilde{\mathcal{M}}^{\star, D_m, e}(\mathcal{F}_m) = \limsup_{r \rightarrow +\infty} \frac{1}{\ln r} \int_{\frac{1}{r}}^{\varepsilon_{\mathcal{F}, m}^m} t^{D_m-3} \tilde{\mathcal{V}}_{m, \mathcal{F}_m}(t) dt \right), \quad (36)$$

where $\tilde{\mathcal{V}}_{m, \mathcal{F}_m}$ is the natural volume extension of $\mathcal{F}^{\mathcal{I}}$ (or m^{th} effective tubular volume of \mathcal{F}_m ; see Definition 4.2, on page 18), and where D_m denotes the abscissa of convergence of the m^{th} local effective

tube zeta function $\tilde{\zeta}_{m, \mathcal{F}_m}^e$. Recall from Theorem 4.3, on page 21, that, for all sufficiently large m , we have that $D_m = D_{\mathcal{W}}$, in the case of the Weierstrass IFD.

In the case when both of these values coincide, their common value, denoted by $\widetilde{\mathcal{M}}^{D_m, e}(\mathcal{F}_m)$, is called the m^{th} local effective average Minkowski content of \mathcal{F}_m , which is then said to *exist*. Accordingly,

$$\widetilde{\mathcal{M}}^{D_m, e}(\mathcal{F}_m) = \lim_{r \rightarrow +\infty} \frac{1}{\ln r} \int_{\frac{1}{r}}^{\varepsilon_{\mathcal{F}_m}^m} t^{D_m-3} \tilde{\mathcal{V}}_{m, \mathcal{F}_m}(t)(t) dt. \quad (37)$$

Remark 4.7. Henceforth, in the case of the Weierstrass IFD associated with the sequence of polygonal prefractal approximation $(\Gamma_{\mathcal{W}_m})_{m \in \mathbb{N}}$ to $\Gamma_{\mathcal{W}}$, we use exactly the same terminology and definitions (adapted in the obvious way) as in Definitions 4.4–4.6 above.

Furthermore, we respectively denote by $D_{\mathcal{W}}$, $\mathcal{M}_{\star}^{D_{\mathcal{W}}}(\Gamma_{\mathcal{W}})$, $\mathcal{M}^{\star, D_{\mathcal{W}}}(\Gamma_{\mathcal{W}})$ and $\widetilde{\mathcal{M}}^{D_{\mathcal{W}}}(\Gamma_{\mathcal{W}})$ the associated Minkowski dimension, lower and upper Minkowski contents, as well as the average Minkowski content of the Weierstrass IFD.

Note that $D_{\mathcal{W}}$ also coincides with the Minkowski dimension of the Weierstrass Curve $\Gamma_{\mathcal{W}}$.

In some definite sense, the Weierstrass IFD (iterated fractal drum) is a natural geometric realization of – and substitute for – the Weierstrass Curve $\Gamma_{\mathcal{W}}$.

We can now state several new geometric consequences of our above results, especially, Theorem 4.2, on page 19.

Theorem 4.6 (Lower, Upper and Average $D_{\mathcal{W}}$ -Dimensional Minkowski Contents of the Weierstrass IFD [DL22b]). *For any $m \in \mathbb{N}$, let us denote by D_m the abscissa of convergence of the m^{th} local effective tube zeta function $\tilde{\zeta}_{m, \mathcal{W}}^e$. Then, the Minkowski dimension of the Weierstrass IFD $\Gamma_{\mathcal{W}}^{\mathcal{I}}$ exists and equals $D_m = D_{\mathcal{W}}$, for any sufficiently large $m \in \mathbb{N}^{\star}$, where $D_{\mathcal{W}} = 2 - \ln_{N_b} \frac{1}{\lambda} \in]1, 2[$ is the Minkowski dimension of the Weierstrass Curve; see Theorem 4.3, on page 21 above. Moreover, the lower and upper $D_{\mathcal{W}}$ -dimensional Minkowski contents of the Weierstrass IFD $\Gamma_{\mathcal{W}}^{\mathcal{I}}$, respectively*

$$\mathcal{M}_{\star}^{D_m}(\Gamma_{\mathcal{W}}^{\mathcal{I}}) = \mathcal{M}_{\star}^{D_{\mathcal{W}}}(\Gamma_{\mathcal{W}}^{\mathcal{I}}) \quad \text{and} \quad \mathcal{M}^{\star, D_m}(\Gamma_{\mathcal{W}}^{\mathcal{I}}) = \mathcal{M}^{\star, D_{\mathcal{W}}}(\Gamma_{\mathcal{W}}^{\mathcal{I}}),$$

take strictly positive and finite values; more specifically, they are such that

$$0 < \frac{C_{\text{Rectangles}}}{N_b} < \mathcal{M}_{\star}^{D_m}(\Gamma_{\mathcal{W}}^{\mathcal{I}}) < \mathcal{M}^{\star, D_m}(\Gamma_{\mathcal{W}}^{\mathcal{I}}) \leq C_{\text{Rectangles}} < \infty, \quad (38)$$

where $C_{\text{Rectangles}}$ denotes the strictly positive and finite constant involved in Theorem 4.2, on page 19.

Recall that $C_{\text{Rectangles}}$ may depend on $m \in \mathbb{N}^{\star}$, but is uniformly bounded away from 0 and infinity (with bounds independent of $m \in \mathbb{N}^{\star}$ large enough). Hence, the same is true of

$$\mathcal{M}_{\star}^{D_m}(\Gamma_{\mathcal{W}}^{\mathcal{I}}) = \mathcal{M}_{\star}^{D_{\mathcal{W}}}(\Gamma_{\mathcal{W}}^{\mathcal{I}}) \quad \text{and} \quad \mathcal{M}^{\star, D_m}(\Gamma_{\mathcal{W}}^{\mathcal{I}}) = \mathcal{M}^{\star, D_{\mathcal{W}}}(\Gamma_{\mathcal{W}}^{\mathcal{I}}),$$

where $D_m = D_{\mathcal{W}}$, for all sufficiently large $m \in \mathbb{N}^{\star}$.

In addition, the values of $\mathcal{M}_{\star}^{D_{\mathcal{W}}}(\Gamma_{\mathcal{W}}^{\mathcal{I}})$ and $\mathcal{M}^{\star, D_{\mathcal{W}}}(\Gamma_{\mathcal{W}}^{\mathcal{I}})$ are respectively equal to the minimum and maximum value of the one-periodic function $G_{D_{\mathcal{W}}} = G_{0, D_{\mathcal{W}}}$ introduced in Theorem 4.5, on page 24, associated to D_m in the expression of the fractal tube formula given in the same theorem (recall that the periodicity is with respect to the variable $\ln_{N_b}(\varepsilon_m^m)^{-1}$, see Proposition 4.1, on page 17).

Finally, for all sufficiently large $m \in \mathbb{N}^\star$, the m^{th} local effective average Minkowski content exists and is given by the mean value of the one-periodic function $G_{D_m} = G_{D_{\mathcal{W}}}$, as well as by the residues of $\tilde{\zeta}_{m, \Gamma_{\mathcal{W}_m}}^e$ at $s = D_m = D_{\mathcal{W}}$:

$$\widetilde{\mathcal{M}}^{D_m, e}(\Gamma_{\mathcal{W}_m}) = \int_0^1 G_{D_{\mathcal{W}}}(x) dx = \text{res}(\tilde{\zeta}_{m, \Gamma_{\mathcal{W}_m}}^e, D_m) = \frac{\text{res}(\zeta_{m, \Gamma_{\mathcal{W}_m}}^e, D_m)}{2 - D_m}, \quad (39)$$

where $\tilde{\zeta}_{m, \Gamma_{\mathcal{W}_m}}^e$ is the m^{th} effective local distance zeta function.

Hence, $\widetilde{\mathcal{M}}^{D_m, e}(\Gamma_{\mathcal{W}_m})$ is nontrivial; in fact,

$$0 < \mathcal{M}_\star^{D_m}(\Gamma_{\mathcal{W}}^{\mathcal{I}}) < \widetilde{\mathcal{M}}^{D_m, e}(\Gamma_{\mathcal{W}_m}) < \mathcal{M}^{\star, D_m}(\Gamma_{\mathcal{W}}^{\mathcal{I}}) < \infty.$$

More specifically, still for all m large enough and thus, with $D_m = D_{\mathcal{W}}$, the m^{th} local effective average Minkowski content $\widetilde{\mathcal{M}}^{D_m, e}(\Gamma_{\mathcal{W}_m})$ may depend on $m \in \mathbb{N}^\star$, but is uniformly bounded away from 0 and ∞ (with bounds independent of $m \in \mathbb{N}^\star$ large enough).

Corollary 4.7 ((of Theorem 4.6, on page 27) **Minkowski Dimension – Minkowski Non-degeneracy**). *The Weierstrass Curve $\Gamma_{\mathcal{W}}$ is Minkowski nondegenerate. Furthermore, the number $D_{\mathcal{W}} = 2 - \ln_{N_b} \frac{1}{\lambda}$ is a simple Complex Dimension of $\Gamma_{\mathcal{W}}$, and it coincides with the Minkowski Dimension of $\Gamma_{\mathcal{W}}$, which must also exist. Moreover, $\Gamma_{\mathcal{W}}$ is not Minkowski measurable. In addition, the same statement holds for the Weierstrass IFD.*

Remark 4.8 (Generalization to the Non-Integer Case [DL22b]). An interesting question is the generalization of our previous results to *the non-integer case*, i.e., when the Weierstrass function \mathcal{W} is defined, for any real number x , by

$$\mathcal{W}(x) = \sum_{n=0}^{\infty} \lambda^n \cos(2\pi b^n x),$$

where the real number $b > 1$ does not belong to the set of natural integers.

We plan to provide the details in a later work, but for now limit ourselves to a few comments.

From the geometric point of view, one cannot handle things in the same way. For instance, one cannot resort to a finite IFS, and the Weierstrass function, apart from its parity, has no periodicity property.

Yet, since the associated graph is the attractor of the infinite set of maps $(T_i)_{i \in \mathbb{Z}}$ such that, for any integer i and (x, y) in \mathbb{R}^2 ,

$$T_i(x, y) = \left(\frac{x+i}{b}, \lambda y + \cos\left(2\pi \left(\frac{x+i}{b}\right)\right) \right),$$

it is natural to consider the associated *infinite IFS (IIFS)*. As a consequence, the resulting prefractal graphs are infinite ones.

As for the tubular neighborhood, due to the polygonal approximation induced by the prefractals, it is still obtained by means of rectangles and wedges.

In the integer case, extra terms coming from overlapping rectangles vanished, thanks to the symmetry with respect to the vertical line $x = \frac{1}{2}$. In the non-integer case, one simply replaces this symmetry with the one with respect to the vertical axis $x = 0$, thanks to the parity of \mathcal{W} .

In this light, it is expected to lead to a fractal tube formula of the same type as the one obtained in the last part of Theorem 4.2, on page 19, where the powers of the corresponding cohomology infinitesimal ε_m^m are, respectively, and as previously,

$$\begin{aligned} & (\varepsilon_m^m)^{2-D_{\mathcal{W}}+k(2-D_{\mathcal{W}})-i\ell\mathbf{p}} , (\varepsilon_m^m)^{3-i\ell\mathbf{p}} , (\varepsilon_m^m)^{1+2k-i\ell\mathbf{p}} , \\ & (\varepsilon_m^m)^{5+2k-i\ell\mathbf{p}} , (\varepsilon_m^m)^{2-i\ell\mathbf{p}} , \end{aligned}$$

which would yield the same results concerning the possible Complex Dimensions, and the upper and lower Minkowski contents.

As in the integer case, the terms involving $(\varepsilon_m^m)^{2-D_{\mathcal{W}}+k(2-D_{\mathcal{W}})-i\ell\mathbf{p}}$ come from the contribution of the rectangles. The one-periodic functions (with respect to the variable $\ln_b (\varepsilon_m^m)^{-1}$ this time), respectively associated to the values $D_{\mathcal{W}} - k(2 - D_{\mathcal{W}})$, $k \in \mathbb{N}$, are thus nonconstant, with all of their Fourier coefficients being nonzero. Hence, as in Theorem 4.4, on page 24, for each $k \in \mathbb{N}$ and $\ell \in \mathbb{Z}$, $D_{\mathcal{W}} - k(2 - D_{\mathcal{W}}) + i\ell\mathbf{p}$, are all simple Complex Dimensions of the Weierstrass Curve (or, of the Weierstrass IFD); i.e., they are simple poles of the tube (or, equivalently, of the distance) zeta function.

5 Fractal Cohomology

We next discuss the fractal cohomology of the Weierstrass Curve $\Gamma_{\mathcal{W}}$ – or rather, perhaps, of the Weierstrass function \mathcal{W} – building on the ideas, definitions and results of [DL23d] – as well as providing a concrete geometric (and topological) realization of the number-theoretic and analytic work and conjectures about fractal cohomology in [LvF00], [LvF06], [Lap08], [LvF13], [CL17], [Lap19] and [Lap24].

Definition 5.1 ((m, p) -Fermion).

By analogy with particle physics, given a pair of positive integers (m, p) , we will call (m, p) -fermion on V_m , with values in \mathbb{C} , any antisymmetric map f from V_m^{p+1} to \mathbb{C} , where V_m^{p+1} denotes the $(p+1)^{th}$ fold Cartesian product space of V_m by itself. Note that these maps are not assumed to be multilinear.

A $(0, p)$ -fermion ($p \in \mathbb{N}^*$) on V_m (or a 0 -fermion, in short) is simply a map f from V_m to \mathbb{C} . We adopt the convention according to which a 0 -fermion on V_m is a 0 -antisymmetric map on V_m .

In the sequel, we will denote by $\mathcal{F}^p(V_m, \mathbb{C})$ the \mathbb{C} -module (i.e., the complex vector space) of (m, p) -fermions on V_m , with values in \mathbb{C} , which makes it an abelian group with respect to the addition, with an external law from $\mathbb{C} \times \mathcal{F}^p(V_m, \mathbb{C})$ to $\mathcal{F}^p(V_m, \mathbb{C})$.

In order to understand how things go, one may look at the initial polygon \mathcal{P}_0 : this polygon has exactly N_b vertices, which means, in terms of potential topological invariants (under the form of a complex-valued function f defined on the whole set of vertices $V^{\star} = \{M_{j,m} : 0 \leq j \leq \#V_m - 1, m \in \mathbb{N}\}$), a number which will – or not – be conserved when switching to $\Gamma_{\mathcal{W}_1}$.

This can be achieved by examining the following quantities (see Figure 7, on page 48):

$$f(P_j) - f(M_{1,1}) + f(M_{2,1}) \dots + (-1)^{N_b-1} f(P_{j+1}) , \text{ with } 0 \leq j \leq N_b ,$$

where the points P_j , for $0 \leq j \leq N_b - 1$, are the vertices of \mathcal{P}_0 .

One thus deals with alternate (i.e., antisymmetric) expressions with $N_b + 1$ terms.

When reaching the m^{th} prefractal graph, with $m \geq 1$, one will have to examine quantities of the following form (see Figure 8, on page 48):

$$f(M_{i,m-1}) - f(M_{j,m}) + f(M_{j+1,m}) \dots + (-1)^{N_b-1} f(M_{i+1,m-1}),$$

for $0 \leq i \leq \#V_{m-1} - 1$, and where

$$M_{i,m-1} \underset{m}{\sim} M_{i+1,m-1},$$

while, at the same time,

$$M_{i,m-1} \underset{m-1}{\sim} M_{j,m} \quad , \quad M_{j,m} \underset{m-1}{\sim} M_{j+1,m} \quad , \quad \dots ,$$

which means that the points $M_{i,m-1}$ and $M_{i+1,m-1}$ are consecutive vertices of V_{m-1} , and that the set of consecutive vertices of V_m located strictly between $M_{i,m-1}$ and $M_{i+1,m-1}$ consists of the points $M_{j,m}$, $M_{j+1,m}$; and so on.

Definition 5.2 (($m - 1, m$)-Path).

Given a strictly positive integer m , and two adjacent vertices $X_{m-1,k}$, $X_{m-1,k+1}$ in V_m , for $0 \leq k \leq \#V_{m-1}$, we call ($m - 1, m$)-*path* between $X_{m-1,k}$, $X_{m-1,k+1}$ the ordered set of vertices

$$\mathcal{P}_{m-1,m}(X_{m-1,k}, X_{m-1,k+1}) = \{X_{m,\ell}, 0 \leq \ell \leq N_b\},$$

where

$$X_{m,\ell} = X_{m-1,k} \quad \text{and} \quad X_{m,\ell+N_b} = X_{m-1,k+1};$$

see Figure 8, on page 49.

Definition 5.3 (($m - 1, m$)-Differential).

Given a strictly positive integer m , we define the $(m - 1, m)$ -differential $\delta_{m-1,m}$ from $\mathcal{F}^0(V_m, \mathbb{C})$ to $\mathcal{F}^{N_b+1}(V_m, \mathbb{C})$, for any f in $\mathcal{F}^0(\Gamma_{\mathcal{W}}, \mathbb{C})$ and any

$$(M_{i,m-1}, M_{i+1,m-1}, M_{j+1,m}, \dots, M_{j+N_b-2,m}) \in V_m^{N_b+1}$$

such that

$$M_{i,m-1} = M_{j,m} \quad \text{and} \quad M_{i+1,m-1} = M_{j+N_b,m},$$

by

$$\delta_{m-1,m}(f)(M_{i,m-1}, M_{i+1,m-1}, M_{j+1,m}, \dots, M_{j+N_b-1,m}) = c_{m-1,m} \left(\sum_{q=0}^{N_b} (-1)^q f(M_{j+q,m}) \right),$$

where $c_{m-1,m}$ denotes a suitable positive constant. Note that as one only handles differentials in this paper, one does not need to know – or fix – the values of this constant. It becomes of importance when operators involving the differentials, such as the Laplacian, are involved; see, for instance, Section 6 of [DL23d].

By induction, we can, equivalently, consider the $(0, 1)$ -differential $\delta_{0,1}$ from $\mathcal{F}^0(V_0, \mathbb{C})$ to $\mathcal{F}^{N_b+1}(V_1, \mathbb{C})$, then, the $(1, 2)$ -differential $\delta_{1,2}$ from $\mathcal{F}^{N_b+1}(V_1, \mathbb{C})$ to $\mathcal{F}^{N_b^2+1}(V_2, \mathbb{C})$, and so on, which means that the $(m - 1, m)$ -differential $\delta_{m-1,m}$ is defined from $\mathcal{F}^{N_b^{m-1}+1}(V_{m-1}, \mathbb{C})$ to $\mathcal{F}^{N_b^m+1}(V_m, \mathbb{C})$. In fact, at a given step $m \geq 0$, between two vertices of V_m , there are $N_b - 1$ consecutive vertices of $V_{m+1} \setminus V_m$. Hence, this amounts to $N_b - 1 + 2 = N_b + 1$ consecutive vertices of $V_{m+1} \supset V_m$. Among those $N_b + 1$ vertices, there are then N_b pairs of consecutive vertices of $V_{m+1} \supset V_m$, which themselves involve N_b^2 pairs of consecutive vertices of $V_{m+2} \supset V_{m+1}$, i.e., $N_b^2 + 1$ consecutive vertices of

$$V_{m+2} \supset V_{m+1} \supset V_m;$$

and so on, by induction.

Furthermore, because of the compactness of $\Gamma_{\mathcal{W}}$, along with the density of the set $V^\star = \bigcup_{n \in \mathbb{N}} V_n$ in the Weierstrass Curve $\Gamma_{\mathcal{W}}$, every continuous (and hence, uniformly continuous) function on $\Gamma_{\mathcal{W}}$ is uniquely determined by its restriction to V^\star (that is, to each V_m , for all $m \in \mathbb{N}$).

Proposition 5.1 (Fractal Complex). *Hereafter, the Complex involved, denoted by $(\mathcal{F}^\bullet(\Gamma_{\mathcal{W}}, \mathbb{C}), \delta^\bullet)$ is the algebraic structure, which consists in the sequence of abelian groups (of fermions) $(\mathcal{F}^{N_b^m+1}(V_m, \mathbb{C}))_{m \in \mathbb{N}^\star}$, where, for each integer $m \geq 2$, the group $\mathcal{F}^{N_b^{m-1}+1}(V_{m-1}, \mathbb{C})$ is connected to the group $\mathcal{F}^{N_b^m+1}(V_m, \mathbb{C})$ by means of the $(m - 1, m)$ -differentials $\delta_{m-1,m}$ introduced in Definition 5.3, on page 31, namely,*

$$\mathcal{F}^0(V_0, \mathbb{C}) \xrightarrow{\delta_{0,1}} \dots \mathcal{F}^{N_b^m+1}(V_m, \mathbb{C}) \xrightarrow{\delta_{m-1,m}} \mathcal{F}^{N_b^{m+1}+1}(V_{m+1}, \mathbb{C}) \xrightarrow{\delta_{m,m+1}} \dots$$

Because of the density of the set $V^\star = \bigcup_{n \in \mathbb{N}} V_n$ in the Weierstrass Curve $\Gamma_{\mathcal{W}}$, this complex can also be written in the following form,

$$\mathcal{F}^0(\Gamma_{\mathcal{W}}, \mathbb{C}) \xrightarrow{\delta_{0,1}} \dots \mathcal{F}^{N_b^m+1}(\Gamma_{\mathcal{W}}, \mathbb{C}) \xrightarrow{\delta_{m-1,m}} \mathcal{F}^{N_b^{m+1}+1}(\Gamma_{\mathcal{W}}, \mathbb{C}) \xrightarrow{\delta_{m,m+1}} \dots$$

Proposition 5.2 (Cohomology Groups). *In our present setting, with the differential introduced in Definition 5.3, on page 31, the cohomology groups are the quotient groups*

$$H_m = \ker \delta_{m-1,m} / \text{Im } \delta_{m-2,m-1} \quad , \quad \text{for } m \geq 0 \quad ,$$

with the additional convention that $\delta_{-2,-1} = 0$ and $\delta_{-1,0} = 0$, which ensures that $H_0 = \{0\}$.

Notation 9 (Argument of a Complex Number). Given a nonzero complex number z , we denote by $\arg(z)$ the *argument* of z ; i.e., the angle between the positive real axis and the line joining the origin and the point M with affix z . Implicitly, we always choose the same convention (e.g., $\arg(z) \in]-\pi, \pi]$) whenever evaluating $\arg(z)$ – and hence also, $\arg(z) - \arg(z')$, for $(z, z') \in \mathbb{C}^* \times \mathbb{C}^*$.

Definition 5.4 (Set of Functions of the Same Nature as the Weierstrass Function \mathcal{W}). *i.* We say that a continuous, complex-valued function f , defined on $\Gamma_{\mathcal{W}} \supset V^*$, is *of the same nature as the Weierstrass function \mathcal{W}* , if it satisfies local Hölder and reverse-Hölder properties analogous to those satisfied by the Weierstrass function \mathcal{W} ; i.e., for any pair of adjacent vertices (M, M') of respective affixes $(z, z') \in \mathbb{C}^2$ of the prefractal graph $\Gamma_{\mathcal{W}_m}$, with $m \in \mathbb{N}$ arbitrary (see Remark 2.2, on page 10),

$$\tilde{C}_{inf} |z' - z|^{2-D_{\mathcal{W}}} \leq |f(z') - f(z)| \leq \tilde{C}_{sup} |z' - z|^{2-D_{\mathcal{W}}} \quad ,$$

where \tilde{C}_{inf} and \tilde{C}_{sup} denote positive and finite constants (but not necessarily the same ones as for the Weierstrass function \mathcal{W} itself, in Proposition 2.4, on page 8). This can be written, equivalently, as

$$|z - z'|^{2-D_{\mathcal{W}}} \leq |f(z) - f(z')| \leq |z - z'|^{2-D_{\mathcal{W}}} \quad . \quad (40)$$

Hereafter, we will denote by $\mathcal{H}\ddot{o}\text{ld}(\Gamma_{\mathcal{W}})$ the set consisting of the continuous, complex-valued functions f , defined on $\Gamma_{\mathcal{W}} \supset V^*$ and satisfying relation (40), on page 32.

ii. Moreover, we will denote by $\mathcal{H}\ddot{o}\text{ld}_{geom}(\Gamma_{\mathcal{W}}) \subset \mathcal{H}\ddot{o}\text{ld}(\Gamma_{\mathcal{W}})$ the subset of $\mathcal{H}\ddot{o}\text{ld}(\Gamma_{\mathcal{W}})$ consisting of the functions f of $\mathcal{H}\ddot{o}\text{ld}(\Gamma_{\mathcal{W}})$ which satisfy the following *additional geometric condition* (41), again, for any pair of adjacent vertices (M, M') with respective affixes $(z, z') \in \mathbb{C}^2$ of the prefractal graph V_m , with $m \in \mathbb{N}$ arbitrary; we have that,

$$\left| \arg(f(z)) - \arg(f(z')) \right| \leq |z - z'|^{D_{\mathcal{W}}-1} \quad . \quad (41)$$

We can now state the following key result.

Theorem 5.3 (Cohomological Complex Dimensions Series Expansion and Characterization of the Prefractal Cohomology Groups H_m [DL22c]). *Let $m \in \mathbb{N}$ be arbitrary. Then:*

i. *Within the set $\mathcal{H}\ddot{o}\text{ld}(\Gamma_{\mathcal{W}})$ (see part *i.* of Definition 5.4, on page 32 just above), then, for any integer $m \geq 1$, and with the convention $H_0 = \text{Im } \delta_{-1,0} = \{0\}$, the cohomology groups*

$$H_m = \ker \delta_{m-1,m} / \text{Im } \delta_{m-2,m-1}$$

are comprised of the restrictions to V_m of $(m, N_b^m + 1)$ -fermions, i.e., the restrictions to (the Cartesian product space) $V_m^{N_b^m + 1}$ of antisymmetric maps on $\Gamma_{\mathcal{W}}$, with $N_b^m + 1$ variables (corresponding to the vertices of V_m), involving the restrictions to V_m of continuous functions f on $\Gamma_{\mathcal{W}}$, such that, for any vertex $M_{j,m} \in V_m$, the following Taylor-like expansion is satisfied,

$$f(M_{j,m}) = \sum_{k=0}^m c_k(f, M_{j,m}) \varepsilon_k^{k(2-D_{\mathcal{W}})} \quad , \quad M_{\star,m} \in V_m, c_k(f, M_{\star,m}) \in \mathbb{C} \quad , \quad (42)$$

where, for each integer k such that $0 \leq k \leq m$, the number $\varepsilon_k^k > 0$ is the k^{th} cohomology infinitesimal introduced in Definition 3.1, on page 11 above.

The coefficients $c_k(f, M_\star)$ are complex quantities, which only depend on the function f involved, and on the point M_\star at which they are evaluated. As will follow, these coefficients $c_k(f, M_\star)$ are the residues (at the possible Complex Dimensions $-k(2 - D_{\mathcal{W}})$) of a suitable global scaling zeta function, defined in [DL23b].

Note that, insofar as the functions f involved are, in a sense, determined, at any vertex $M_{j,m} \in V_m$, by the expansion given in relation (42), on page 32 above, it is natural to identify the elements of the cohomology groups H_m with those functions.

ii. If the functions f of part i. belong to $\text{Höld}_{\text{geom}}(\Gamma_{\mathcal{W}})$ (see part ii. of Definition 5.4, on page 32 above), then, for any strictly positive integer m , and again with the convention $H_0 = \text{Im } \delta_{-1,0} = \{0\}$, the cohomology groups

$$H_m = \ker \delta_{m-1,m} / \text{Im } \delta_{m-2,m-1}$$

are comprised of the restrictions to V_m of $(m, N_b^m + 1)$ -fermions, i.e., the restrictions to $V_m^{N_b^m + 1}$ of antisymmetric maps on $\Gamma_{\mathcal{W}}$, with $N_b^m + 1$ variables (corresponding to the vertices of V_m), involving the restrictions to V_m of continuous functions f on $\Gamma_{\mathcal{W}}$, such that, for any vertex $M_{j,m} \in V_m$,

$$\begin{aligned} f(M_{j,m}) &= \sum_{k=0}^m c_k(f, M_{j,m}) \varepsilon_k^{k(2-D_{\mathcal{W}})} \varepsilon_k^{ik \ell_{k,j,m} \mathbf{p}} \\ &= \sum_{k=0}^m c_k(f, M_{j,m}) \varepsilon_k^{k(2-D_{\mathcal{W}}) + i \ell_{k,j,m} \mathbf{p}}, M_{\star,m} \in V_m, \end{aligned} \tag{43}$$

where \mathbf{p} denotes the oscillatory period introduced in [DL22b],

$$\mathbf{p} = \frac{2\pi}{\ln N_b},$$

and where the coefficients $c_k(\star, \star)$ are complex numbers which still depend on the function f involved, and on the point at which they are evaluated. Here, in relation (43), for each integer k such that $0 \leq k \leq m$, $\ell_{k,j,m}$ denotes an integer (in \mathbb{Z}) satisfying the estimate

$$0 \leq \left\{ \ell_{k,j,m} \frac{\ln \varepsilon_k}{\ln N_b} \right\} \lesssim \frac{\varepsilon_k^{k(D_{\mathcal{W}}-1)}}{2\pi}, \tag{44}$$

where $\{.\}$ denote the fractional part.

Remark 5.1. Note that, since the fractional part map is one-periodic, this results in a kind of *periodicity* with respect to the integers $\ell_{k,j,m}$: the set $\{\ell_{k,j,m}, k \in \mathbb{N}\}$ is infinite and equal to \mathbb{Z} . In particular, $|\ell_{k,j,m}| \rightarrow +\infty$ as $k \rightarrow \infty$. We thus recover completely analogous results to those obtained in Theorem 4.4, on page 24 above (from [DL22b]).

Much as in part i., the coefficients $c_k(f, M_\star)$, for $0 \leq k \leq m$, depend only on the function f and on the point M_\star of V_m at which they are evaluated. Note that, obviously, the values of the constants c_k in part ii. are different from those obtained in part i. The specific topic of their possible extension to all $M_\star \in \Gamma_{\mathcal{W}}$ and their possible continuity in M_\star will be studied in more detail in a forthcoming work of the authors with M. Overduin [DLO23]. In particular, the use of the phrase *Taylor-like expansion* is partly justified by the results presented in [DLO23], where the coefficients $c_k(f, M_\star)$ are precisely determined in terms of the local differentials introduced in the first part of this section (see, especially, Definition 5.3 above, on page 31.)

Furthermore, these coefficients $c_k(f, M_\star)$ are the residues (at the possible Complex Dimensions $-(k(2 - D_{\mathcal{W}}))$) of a suitable global scaling zeta function, provided in [DL23b].

In both expansions (42), on page 32, and (43), on page 33, the coefficients $c_k(f, M_\star)$, for $0 \leq k \leq m$, reflect the dependence of the value taken by the map f at the vertex $M_{j,m}$ on the values taken by f at previous steps – vertices – of the m^{th} prefractal graph approximation, in conjunction with values taken by f at neighboring vertices of $M_{j,m}$ at the same level (m) of the prefractal sequence and with vertices which, in addition, strictly belong to the same polygon $\mathcal{P}_{m,k}$ introduced in Proposition 2.2, on page 7, with $1 \leq k \leq N_b^m - 1$ (by “strictly” here, we mean that the junction vertices are not included).

The expansion in part *ii.* (namely, relation (43), on page 33)) might be interpreted as a kind of generalized Taylor expansion with corresponding complex derivatives of orders

$$-\omega_k = k(2 - D_{\mathcal{W}}) + i \ell_{k,j,m} \mathbf{p},$$

where $k \in \mathbb{N}$ is arbitrary, the coefficients $c_k(f, M_\star)$ can thus be interpreted as (discrete) derivatives of complex order $-\omega_k$ of the function f , evaluated at the point M_\star of $V^\star \subset \Gamma_{\mathcal{W}}$. A similar comment can be made about the expansion in part *i.* (namely, (42), on page 32)), but now with the above value of $-\omega_k$ replaced by $k(2 - D_{\mathcal{W}})$.

Again, as in part *i.*, it is natural to identify the cohomology groups H_m with the sets of functions satisfying, at any vertex $M_{j,m} \in V_m$, the expansion given in relation (43), on page 33.

Note that the result given in part *ii.* above is significantly stronger than the one given in part *i.* of this theorem. Indeed, the expansion (42) in part *i.*, on page 32, only involves (modulo a translation by $-D_{\mathcal{W}}$) the real parts of some of the Complex Dimensions listed in Theorem 4.4, on page 24. By contrast, in part *ii.*, we can also recover some of the imaginary parts of those same Complex Dimensions; note that this is possible only because the functions involved satisfy the geometric condition (41) introduced in Definition 5.4, on page 32 above.

It immediately follows from the expansion given in (43) of part *ii.* above, on page 33, as well as from the fact that the sequence of sets of vertices $(V_m)_{m \in \mathbb{N}}$ is increasing (see part *i.* of Proposition 2.2, on page 7), that the sequence of cohomology groups $(H_m)_{m \in \mathbb{N}}$ is increasing; i.e.,

$$\forall m \in \mathbb{N} : H_m \subset H_{m+1}.$$

Finally, in both expansions (42), on page 32, and (43), on page 33, since the m^{th} cohomology infinitesimal ε_m^m , in the sense of Definition 3.1, on page 11 above, depends on the geometry of the Curve, it can be interpreted as a (geometric) coefficient connecting the Weierstrass Curve and the complex-valued functions defined on the set $V^\star = \bigcup_{n \in \mathbb{N}} V_n$.

Finally, in both expansions (42) and (43), by contrast to classical Taylor expansions, we deal with a nonarbitrary sequence of infinitesimals – the cohomology infinitesimals – directly connected to the scaling properties of the Weierstrass Curve. This enables us to express the exact form of the associated scaling relationship (see Proposition 2.3, on page 8, along with Remark 5.2 just below). In other words, given $k \in \mathbb{N}$, each k^{th} cohomology infinitesimal ε_k^k , in the sense of Definition 3.1, on page 11 above – depends on the geometry of the Curve. In light of our expansions, it can be interpreted as a (geometric) coefficient connecting the Weierstrass Curve and the complex-valued functions defined on the set $V^\star = \bigcup_{n \in \mathbb{N}} V_n$.

Remark 5.2 (The Special Case of the Weierstrass Complexified Function [DL22c], [DL23a], [DL23b]). Among the continuous functions on $\Gamma_{\mathcal{W}}$ which possess, in the most natural manner, and, for any integer $m \in \mathbb{N}^\star$, an expansion of the form obtained in (43), on page 33, we have,

as could be expected, the *Weierstrass Complexified function* \mathcal{W}_{comp} , introduced in [DL22c], defined, for any real number x , by

$$\mathcal{W}_{comp}(x) = \sum_{n=0}^{\infty} \lambda^n e^{2i\pi N_b^n x}.$$

More precisely, as is shown in [DL22c], we have that, for any strictly positive integer m and any j in $\{0, \dots, \#V_m\}$, we have the following *exact* expansion, indexed by the Complex Codimensions $k(D_{\mathcal{W}} - 2) + i k \ell_{k,j,m}^{\mathcal{W}} \mathbf{P}$, with $0 \leq k \leq m$,

$$\begin{aligned} \mathcal{W}_{comp}(j \varepsilon_m^m) &= \mathcal{W}_{comp}\left(\frac{j \varepsilon^m}{N_b - 1}\right) \\ &= \varepsilon^{m(2-D_{\mathcal{W}})} \mathcal{W}_{comp}\left(\frac{j}{N_b - 1}\right) + \sum_{k=0}^{m-1} c_{k,j,m} \varepsilon^{k(2-D_{\mathcal{W}})} \varepsilon^{i \ell_{k,j,m} \mathbf{P}} \\ &= \sum_{k=0}^m c_{k,j,m} \varepsilon^{k(2-D_{\mathcal{W}})} \varepsilon^{i \ell_{k,j,m} \mathbf{P}}, \end{aligned} \quad (45)$$

where, for $0 \leq k \leq m$, ε^k is the k^{th} intrinsic cohomology infinitesimal, introduced in Definition 3.1, on page 11, with $\mathbf{P} = \frac{2\pi}{\ln N_b}$ denoting the oscillatory period of the Weierstrass Curve, as introduced in [DL22b] and where:

i. $\ell_{k,j,m} \in \mathbb{Z}$ is arbitrary.

ii. $c_{m,j,m} = \mathcal{W}_{comp}\left(\frac{j}{N_b - 1}\right)$ and, for $0 \leq k \leq m - 1$, $c_{k,j,m} \in \mathbb{C}$ is given by

$$c_{k,j,m} = \exp\left(\frac{2i\pi}{N_b - 1} j \varepsilon^{m-k}\right). \quad (\diamond\diamond) \quad (46)$$

For any $m \in \mathbb{N}$, the complex numbers $\{c_{0,j,m+1}, \dots, c_{m+1,j,m+1}\}$ and the integers $\{\ell_{0,j,m+1}, \dots, \ell_{m+1,j,m+1}\}$ respectively satisfy the following recurrence relations:

$$c_{m+1,j,m+1} = \mathcal{W}\left(\frac{j}{N_b - 1}\right) = c_{m,j,m} \quad (47)$$

and

$$\forall k \in \{1, \dots, m\} : c_{k,j,m+1} = c_{k,j,m}, \quad \varepsilon^{i \ell_{k,j,m+1} \mathbf{P}} = \varepsilon^{i \ell_{k,j,m} \mathbf{P}}. \quad (48)$$

In addition, since relation (45) is valid for any $m \in \mathbb{N}^*$ (and since, clearly, relation (46) implies that the coefficients $c_{k,j,m}$ are nonzero for $0 \leq k \leq m$), we deduce that the associated Complex Dimensions (i.e., in fact, the Complex Dimensions associated with the Weierstrass function) are

$$D_{\mathcal{W}} - k(2 - D_{\mathcal{W}}) + i \ell_{k,j,m} \mathbf{P}$$

where $0 \leq j \leq \#V_m - 1$, $0 \leq k \leq m$ and $\ell_{k,j,m} \in \mathbb{Z}$ is arbitrary.

This immediately ensures, for the Weierstrass function (i.e., the real part of the Weierstrass complexified function \mathcal{W}_{comp}), that, for any strictly positive integer m and for any j in $\{0, \dots, \#V_m\}$,

$$\begin{aligned}
\mathcal{W}(j \varepsilon_m^m) &= \varepsilon^{m(2-D_{\mathcal{W}})} \mathcal{W}_{comp} \left(\frac{j}{N_b - 1} \right) + \sum_{k=0}^{m-1} \varepsilon^{k(2-D_{\mathcal{W}})} \mathcal{R}e \left(c_{k,j,m} \varepsilon^{i \ell_{k,j,m} \mathbf{P}} \right) \\
&= \varepsilon^{m(2-D_{\mathcal{W}})} \mathcal{W}_{comp} \left(\frac{j}{N_b - 1} \right) + \frac{1}{2} \sum_{k=0}^{m-1} \varepsilon^{k(2-D_{\mathcal{W}})} \left(c_{k,j,m} \varepsilon^{i \ell_{k,j,m} \mathbf{P}} + \overline{c_{k,j,m}} \varepsilon^{-i \ell_{k,j,m} \mathbf{P}} \right) \\
&= \frac{1}{2} \sum_{k=0}^m \varepsilon^{k(2-D_{\mathcal{W}})} \left(c_{k,j,m} \varepsilon^{i \ell_{k,j,m} \mathbf{P}} + \overline{c_{k,j,m}} \varepsilon^{-i \ell_{k,j,m} \mathbf{P}} \right).
\end{aligned} \tag{49}$$

Remark 5.3 (About the Integers $\ell_{k,j,m}$ in part *ii.* of Theorem 5.3, on page 32, or in Theorem 5.4 below, on page 36). Note that since, the fractional part map is one-periodic, this results in a kind of *periodicity* with respect to the integers $\ell_{k,j,m}$.

Naturally, the set of integers $\{\ell_{k,j,m}, k \in \mathbb{N}\}$ (in part *ii.* of Theorem 5.3, on page 32) is infinite and equal to \mathbb{Z} . In particular, $|\ell_{k,j,m}| \rightarrow \infty$ as $k \rightarrow \infty$. We thus recover completely analogous results to those obtained in Theorem 4.4, on page 24 above (from [DL22b]).

An entirely similar comment can be made about the integers $\ell_{k,j,m}$ occurring in the statement of Theorem 5.4, on page 36 just below.

Furthermore, once $\ell_{k,j,m}$ has been chosen to be the smallest positive integer satisfying the required inequalities, for a given $k \in \mathbb{N}$, then we obtain a sequence of integers $(\ell_{k,j,mq})_{q \in \mathbb{Z}} = (\ell_{k,j,m} + q)_{q \in \mathbb{Z}}$ (isomorphic to \mathbb{Z}) satisfying the fractal series expansion (43), on page 33, where, for all $q \in \mathbb{Z}$,

$$0 \leq \left\{ (\ell_{k,j,m} + q) \frac{\ln \varepsilon_k^k}{\ln N_b} \right\} \lesssim \frac{\varepsilon_k^{k(D_{\mathcal{W}}-1)}}{2\pi}.$$

Theorem 5.4 (Fractal Cohomology of the Weierstrass Curve [DL22c]). *Within the set $\mathcal{H}old_{geom}(\Gamma_{\mathcal{W}})$ of continuous, complex-valued functions f , defined on the Weierstrass Curve $\Gamma_{\mathcal{W}} \supset V^{\star} = \bigcup_{m \in \mathbb{N}} V_m$ (see part *ii.* of Definition 5.4, on page 32 above), let us consider the Complex (which can be called the Fractal Complex of $\Gamma_{\mathcal{W}}$),*

$$H^{\star} = H^{\bullet}(\mathcal{F}^{\bullet}(\Gamma_{\mathcal{W}}, \mathbb{C}), \delta^{\bullet}) = \bigoplus_{m=0}^{\infty} H_m,$$

where, for any integer $m \geq 1$, and with the convention $H_0 = Im \delta_{-1,0} = \{0\}$, H_m is the cohomology group

$$H_m = \ker \delta_{m-1,m} / Im \delta_{m-2,m-1}.$$

Then, H^{\star} is the set consisting of functions f on $\Gamma_{\mathcal{W}}$, viewed as 0-fermions (in the sense of Definition 5.1, on page 29), and, for any integer $m \geq 1$, of the restrictions to V_m of $(m, N_b^m + 1)$ -fermions, i.e., the restrictions to (the Cartesian product space) $V_m^{N_b^m + 1}$ of antisymmetric maps on $\Gamma_{\mathcal{W}}$, with $N_b^m + 1$ variables (corresponding to the vertices of V_m), involving the restrictions to V_m of the continuous, complex-valued functions f on $\Gamma_{\mathcal{W}}$ – as, naturally, the aforementioned 0-fermions – satisfying the following convergent (and even, absolutely convergent) Taylor-like expansions (with $V^{\star} = \bigcup_{n \in \mathbb{N}} V_n$), for all $M_{\star, \star} \in V^{\star}$,

$$\begin{aligned}
f(M_{\star,\star}) &= \sum_{k=0}^{\infty} c_k(f, M_{\star,\star}) \varepsilon_k^{k(2-D_{\mathcal{W}})} \varepsilon_k^{i k \ell_{k,j,m} \mathbf{P}} \\
&= \sum_{k=0}^{\infty} c_k(f, M_{\star,\star}) \varepsilon_k^{k(2-D_{\mathcal{W}})+i \ell_{k,j,m} \mathbf{P}},
\end{aligned} \tag{50}$$

where, for each integer $k \geq 0$, the coefficient $c_k(\star, \star) = c_k(f, \star) \in \mathbb{C}$ is the same as in part ii. of Theorem 5.3, on page 32, the number $\varepsilon_k^k = (\varepsilon_k)^k > 0$ is the k^{th} component of the k^{th} cohomology infinitesimal introduced in Definition 3.1, on page 11, and where $\ell_{k,j,m}$ denotes an integer (in \mathbb{Z}) such that

$$0 \leq \left\{ \ell_{k,j,m} \frac{\ln \varepsilon_k^k}{\ln N_b} \right\} \lesssim \frac{\varepsilon_k^{k(D_{\mathcal{W}}-1)}}{2\pi}.$$

Note that since the functions f involved are uniformly continuous on the Weierstrass Curve $\Gamma_{\mathcal{W}} \supset V^{\star}$, and since the set V^{\star} is dense in $\Gamma_{\mathcal{W}}$, they are uniquely determined by their restriction to V^{\star} , as given by relation (50), on page 37. We caution the reader, however, that at this stage of our investigations, we do not know whether $f(M)$ is given by an expansion analogous to the one in relation (50), on page 37, for every $M \in \Gamma_{\mathcal{W}}$, rather than just for all $M \in V^{\star}$.

The convergence (or even, the absolute convergence) of the series

$$\sum_{k=0}^{\infty} c_k(f, M_{\star,\star}) \varepsilon_k^{k(2-D_{\mathcal{W}})+i \ell_{k,j,m} \mathbf{P}}$$

directly comes from the fact that the coefficients $c_k(\star, \star)$ are uniformly bounded and that, for any $k \in \mathbb{N}^{\star}$,

$$\left| \varepsilon_k^{k(2-D_{\mathcal{W}})+i \ell_{k,j,m} \mathbf{P}} \right| = \varepsilon_k^{k(2-D_{\mathcal{W}})} = \left(\varepsilon_k^{2-D_{\mathcal{W}}} \right)^k, \text{ with } 2 - D_{\mathcal{W}} > 0.$$

Finally, for each $M_{\star} = M_{\star,m} \in V^{\star}$, the coefficients $c_k(\star, \star)$ (for any $k \in \mathbb{N}$) are the residues at the possible Complex Dimensions $-k(2 - D_{\mathcal{W}}) + i \ell_{k,j,m} \mathbf{P}$ of a suitable global scaling zeta function $\zeta_{f, M_{\star,m}, \text{gl}}$ (see [DL22c]).

The group $H^{\star} = \bigoplus_{m=0}^{\infty} H_m$ is called the total fractal cohomology group of the Weierstrass Curve $\Gamma_{\mathcal{W}}$ (or else, of the Weierstrass function \mathcal{W}).

This amounts, for each $\varphi \in H^{\star}$, to

$$\varphi = (\varphi_m)_{m \in \mathbb{N}},$$

where, for each $m \in \mathbb{N}$, $\varphi_m \in H_m$, while, if we denote by $\pi : H_{m+1} \rightarrow H_m$ the projection from H_{m+1} onto H_m , we have that $\pi(\varphi_{m+1}) \in H_m$ coincides with φ_m .

The following statement is a corollary of both Theorems 5.3, on page 32, and Theorem 5.4, on page 36.

Corollary 5.5. For each $m \in \mathbb{N}$, the prefractal cohomology space H_m is different from $\{0\}$. Indeed, the Weierstrass function \mathcal{W} , viewed as an appropriate restriction of the identity map on the Weierstrass Curve $\Gamma_{\mathcal{W}}$, belongs to each H_m , for any $m \in \mathbb{N}$.

Similarly, and for exactly the same reason, the total fractal cohomology space H^{\star} , is not reduced to $\{0\}$ (because, in short, \mathcal{W} belongs to H^{\star}).

Therefore, in some definite sense, we can say that the Weierstrass function – or, alternatively, the Weierstrass Curve – belongs to its own cohomology.

Remark 5.4. These results are to be compared with previous ones of Michel L. Lapidus and collaborators (specifically, Machiel van Frankenhuijsen and Tim Cobler) obtained in [LvF00], [LvF06], [Lap08], [LvF13], [CL17] and [Lap24].

More precisely, in our present setting, and contrary to the classical cases of arithmetic or differentiable varieties, for which the decomposition of the total cohomology is indexed by integers, the total (fractal) cohomology H^\star is a sum of spaces indexed by the Complex Dimensions (as is expected in [LvF00], [LvF06], [LvF13] and [Lap08], and discussed in detail in [CL17], [Lap19] and [Lap24]), with an underlying quasiperiodicity property, induced by the estimate (on the imaginary parts of the complex dimensions) (44) in part *ii.* of Theorem 5.3, on page 32. We thus dispose of a quasiperiodic geometric property (reminiscent of, but not identical to, that established in Chapter 3 of [LvF13] for nonlattice self-similar strings), which can possibly be connected to the structure of a (generalized) quasicrystal (see [LvF13], Problem 3.22, page 89, and [Lap08], especially, Chapter 5 and Appendix F), especially in the case not considered in the present paper when N_b is not an integer, but for which we expect analogous results; see, in particular, Remark 4.8, on page 28 above.

Remark 5.5. Note that H^\star is to be understood in the sense of the *inductive limit* of the sequence of cohomology groups $(H_m)_{m \in \mathbb{N}}$; namely, for each fermion $\varphi \in H^\star$, and each $m \in \mathbb{N}$, the restriction $\varphi|_{V_m}$ of φ to the set of vertices V_m belongs to H_m ; the restriction $(\varphi|_{V_{m+1}})|_{V_m}$ to V_m of the restriction $\varphi|_{V_{m+1}}$ of φ to the set of vertices V_{m+1} (which is itself in H_{m+1}), coincides with the restriction $\varphi|_{V_m}$ of φ to V_m ; i.e.,

$$\forall m \in \mathbb{N} : \quad \varphi|_{V_m} \in H_m \quad \text{and} \quad (\varphi|_{V_{m+1}})|_{V_m} = \varphi|_{V_m} .$$

This amounts, for each $\varphi \in H^\star$, to

$$\varphi = (\varphi_m)_{m \in \mathbb{N}} ,$$

where, for each $m \in \mathbb{N}$, $\varphi_m \in H_m$, while, if we denote by $\pi : H_{m+1} \rightarrow H_m$ the natural projection from H_{m+1} onto H_m (recall from Remark 5.1, on page 33 that $H_m \subset H_{m+1}$), we have that $\pi(\varphi_{m+1}) \in H_m$ coincides with φ_m .

Remark 5.6. In some sense, the Complex Dimensions corresponding to the fractal cohomology – namely, $\omega_k = -k(2 - D_{\mathcal{W}}) + i\ell_{k,j,m} \mathbf{p}$, where $k \in \mathbb{N}$ is arbitrary – can be viewed as being associated to the Weierstrass function itself rather than with the Weierstrass Curve.

In [DL22c], we also provide a reformulation of Theorem 5.3, on page 32 and Theorem 5.4, on page 36 concerning, respectively, the m^{th} prefractal cohomology groups H_m ($m \in \mathbb{N}$) and the global (or total) fractal cohomology group H^\star . This reformulation – based on a pair of complex conjugate variables Z and \bar{Z} instead of $(x, y) \in \mathbb{R}^2$ – makes use of the natural symmetry of the Weierstrass Curve $\Gamma_{\mathcal{W}}$ and, correspondingly, enables us to obtain a natural version of *Poincaré Duality* for the associated cohomology spaces; see [DL22c] and Section 6, on page 38 below.

6 Concluding Comments and Perspectives

Up to now, the determination of the possible Complex Dimensions of the Weierstrass Curve had remained an open problem, that we have solved in [DL22b]. Contrary to classical fractals, the non-affine feature, coming from a nonlinear iterated function system, makes things a lot more complicated than for earlier examples studied in the literature (see, e.g., [LRŽ17a], [LRŽ17b], [LRŽ18], [Lap19]).

By considering the fractal Complex Dimensions as dynamical quantities, which evolve with the scales, we set up a broader and more general framework than the one which was envisioned before.

We can draw an analogy between this new framework and the theory of the French physicist Laurent Notale discussed, for instance, in [Not98]: in our context, the fractal dimension plays the role of the time variable; the length – seen as a function of the resolution, once replaced by its logarithm – plays the role of the position; as for the scale (or, in an additive way, its logarithm), it is the analogue of the speed. This explains why there is an insurmountable (minimum) resolution, directly depending on the geometry. (There is also a largest scale, but this is perhaps less surprising.)

One of the benefits of our study – insofar as we handled a more general setting than in the classical cases – will concern the extension to a large class of (possibly nowhere differentiable) fractal functions. Note that our results also provide a new and very direct proof of the fact that the value of the Minkowski (or box-counting) dimension of the Weierstrass Curve is $D_{\mathcal{W}} = 2 - \ln_{N_b} \frac{1}{\lambda}$. More precisely, this new complete proof is provided in our later work [DL23b], by using the polyhedral methods of that paper and of [DL22a].

The interest of considering such a fractal object is, besides its very rich geometrical properties – a self-shape similarity, coming from a prefractal polygonal sequence with vertices entirely contained and dense in the Curve – that one has a natural map defined on the Curve. Again, it is not usually the case (see, for instance, the Sierpiński Gasket, where one has to introduce specific functions). As is shown in [DL22c], it happens that the complexified Weierstrass function possesses a fractal power series expansion indexed by the cohomological Complex Dimensions; see Remark 5.2, on page 34. In this light, one could not bypass establishing a direct link with a fractal cohomology. As expected, the fractal expansion (or explicit formula) that enables us to express any given function belonging to the m^{th} cohomology group, at a given vertex of the associated m^{th} prefractal graph approximation, can exactly be written as a countably infinite sum indexed by the cohomological Complex Dimensions. Going further, this might be interpreted as a kind of *generalized Taylor expansion*, with fractional derivatives of underlying orders the corresponding Complex Dimensions. It appears that we have significantly extended and given a geometric meaning to the results (and conjectures) in [LvF00], [LvF06], [Lap08], [LvF13], along with the ones in [CL17] and [Lap24], where the authors suggested that there should exist a fractal cohomology theory having direct links with the theory of Complex Dimensions.

The natural Poincaré duality associated to the Weierstrass Curve (see [DL22c]) calls for further investigation. In our future works, we envision to lay out the foundations of a so-called *fractal differential calculus* along with a corresponding version of Hodge theory – the natural extension to fractal objects of the classic differential calculus. Specific topics include the conditions of existence of Taylor-like expansions, as well as of the explicit expressions of the corresponding (local) fractional derivatives, along with a proper definition, in the present context, of the notion of (a possibly higher-dimensional) *fractal manifold*, its associated Complex Dimensions, fractal curvatures and fractal cohomology, dual of a potential fractal homology which is yet to be constructed here.

It is worth pointing out that in [DL23b] – announced in [DL23a] – and building, in particular, on the methods introduced in [DL22b] and [DL22c] (and discussed in this paper) – we consider the Weierstrass IFD (or iterated fractal drum) defined by means of suitable polyhedral (i.e., here, polygonal) neighborhoods of the Weierstrass Curve and study in detail the associated local and global effective polyhedral zeta functions, associated respectively with the prefractal approximations $\Gamma_{\mathcal{W}_m}$ (for all $m \in \mathbb{N}^*$ sufficiently large) and the whole Weierstrass Curve $\Gamma_{\mathcal{W}}$. In particular, we determine the corresponding Complex Dimensions – called *the intrinsic Complex Dimensions* of $\Gamma_{\mathcal{W}}$ – and show that they are *exact* (i.e., actual) and simple poles of the global zeta function, as well as given by

$$D_{\mathcal{W}} - k (2 - D_{\mathcal{W}}) + i \ell \mathbf{p},$$

where $k \in \mathbb{N}$ and $\ell \in \mathbb{Z}$ are arbitrary and $\mathbf{p} = \frac{2\pi}{\ln N_b}$ is the same oscillatory period as in [DL22b], [DL22c] and in this paper.

These methods were further extended in [DL23c] to the prototypical case of the celebrated Koch Curve, another nowhere differentiable and fractal curve, for which we have now determined *the exact Complex Dimensions*, thereby significantly expanding the earlier work of the second author and Erin P. J. Pearse in [LP06].

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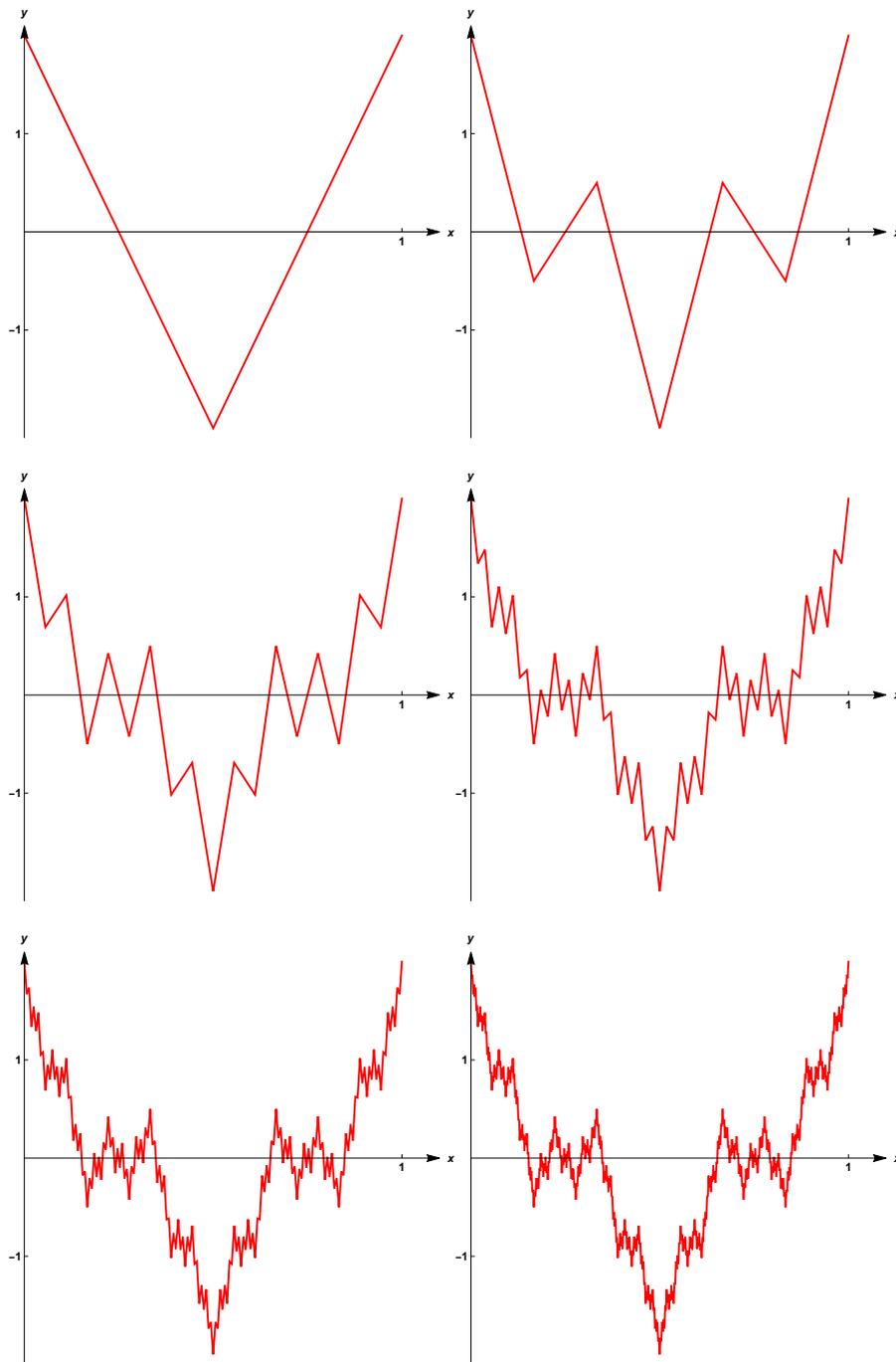


Figure 1: The prefractal graphs $\Gamma_{\mathcal{W}_0}, \Gamma_{\mathcal{W}_1}, \Gamma_{\mathcal{W}_2}, \Gamma_{\mathcal{W}_3}, \Gamma_{\mathcal{W}_4}, \Gamma_{\mathcal{W}_5}$, in the case when $\lambda = \frac{1}{2}$ and $N_b = 3$. For example, $\Gamma_{\mathcal{W}_1}$ is on the right side of the top row, while $\Gamma_{\mathcal{W}_4}$ is on the left side of the bottom row.

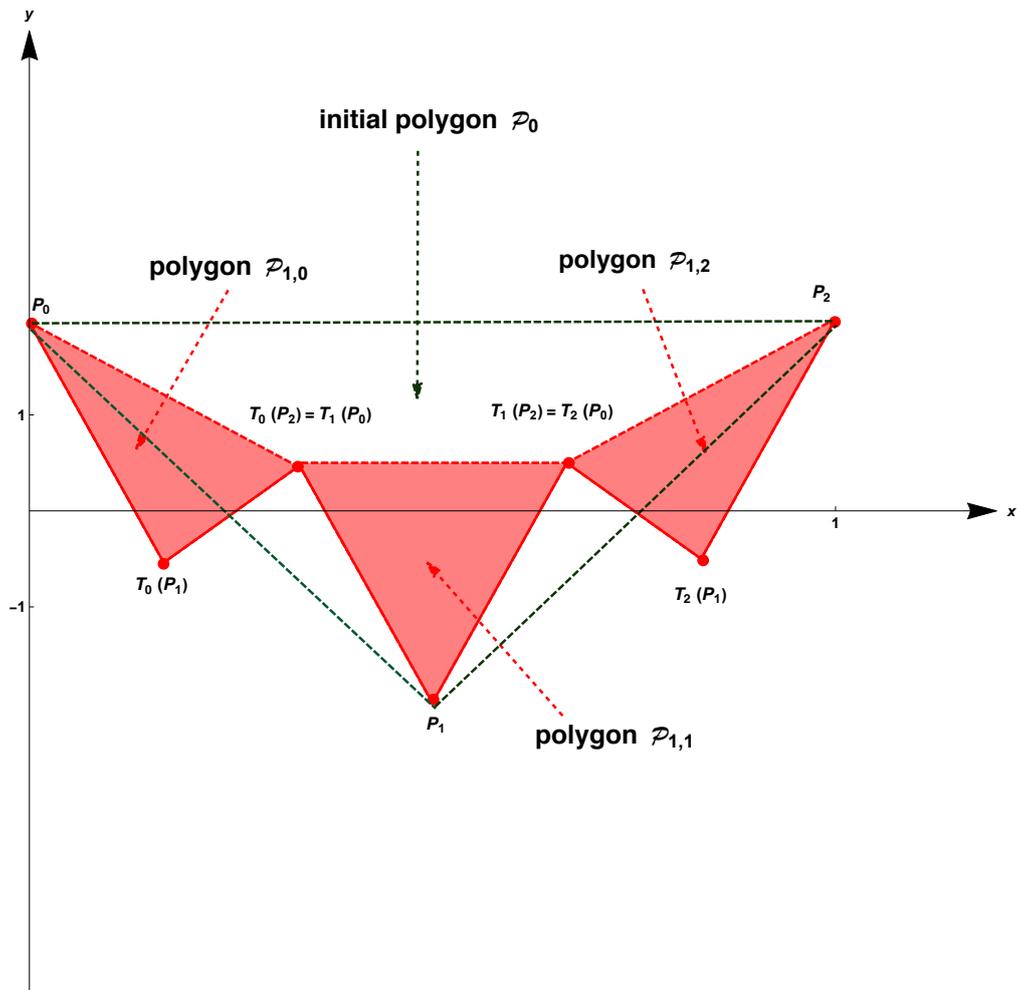


Figure 2: The initial polygon \mathcal{P}_0 , and the polygons $\mathcal{P}_{1,0}$, $\mathcal{P}_{1,1}$, $\mathcal{P}_{1,2}$, in the case when $\lambda = \frac{1}{2}$ and $N_b = 3$.

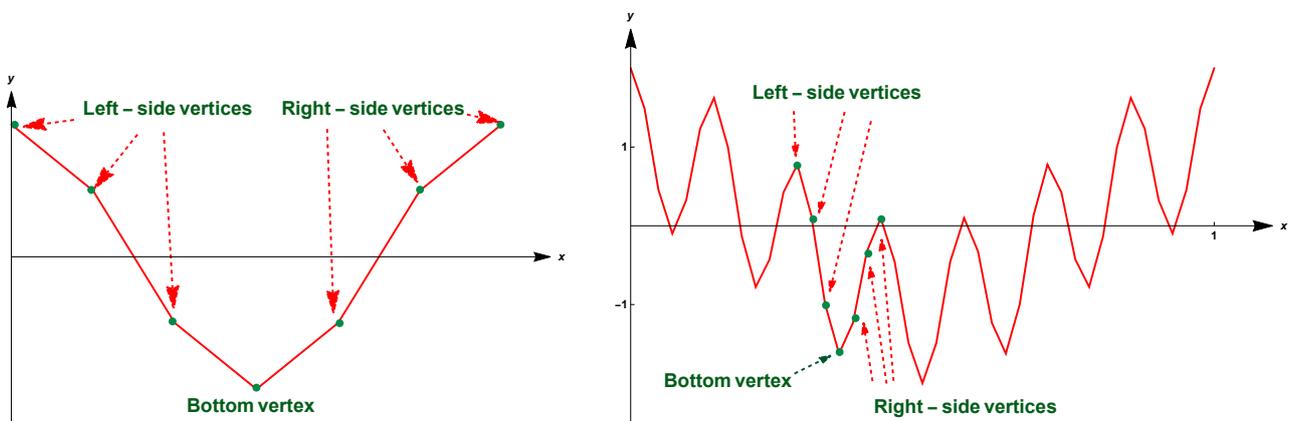


Figure 3: The left-side and right-side vertices.

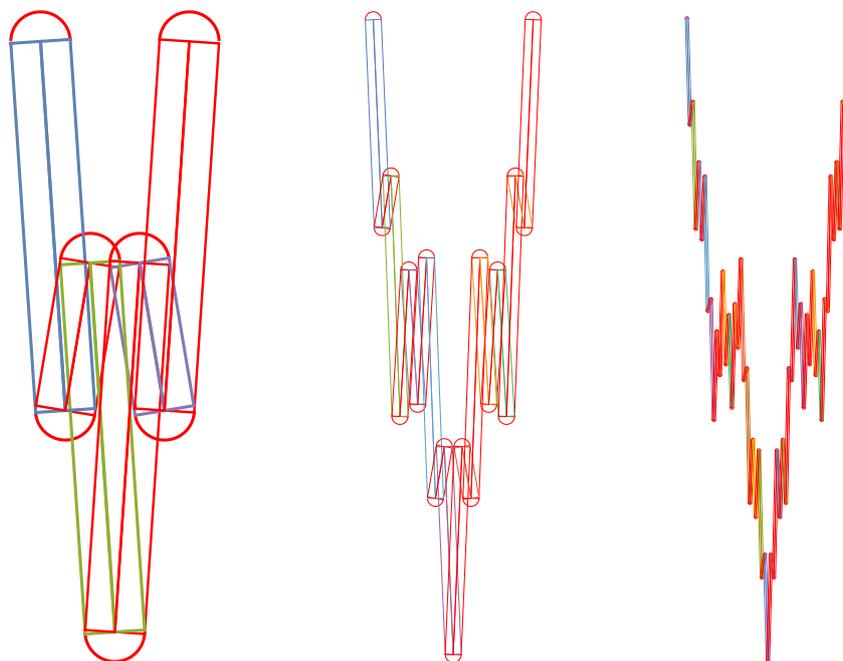


Figure 4: **The $(1, \epsilon_1)$, $(2, \epsilon_2)$ and $(3, \epsilon_3)$ -Neighborhoods, in the case when $\lambda = \frac{1}{2}$ and $N_b = 3$.**

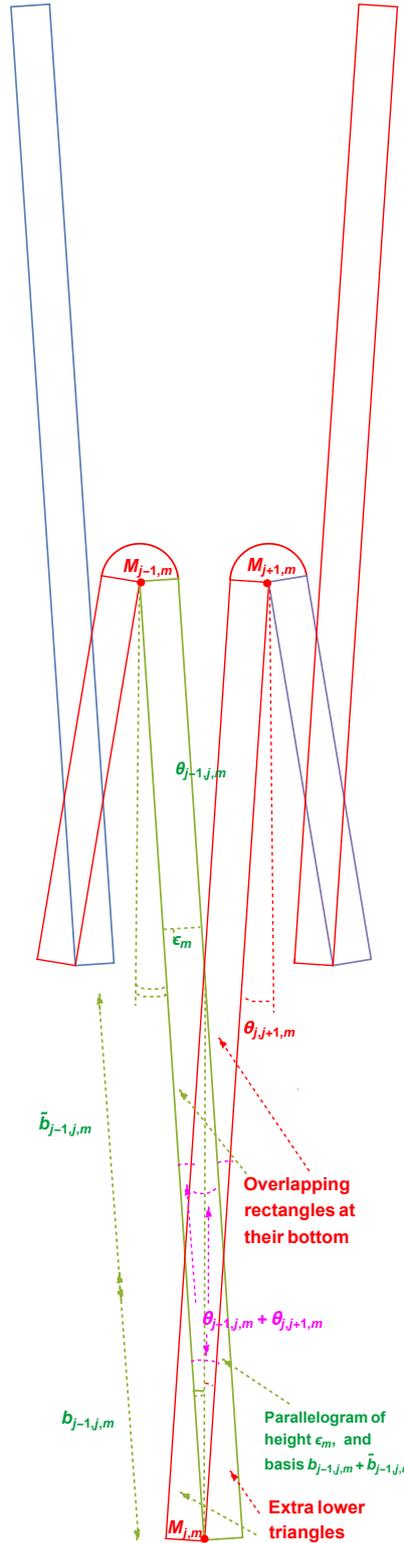


Figure 5: Two overlapping rectangles, coming from the upper tubular neighborhood, in the case when $\lambda = \frac{1}{2}$ and $N_b = 3$. The overlapping shape is a rhombus, because we deal with the upper tubular neighborhood, which means that the rectangles cannot be side by side, and thus, overlap.

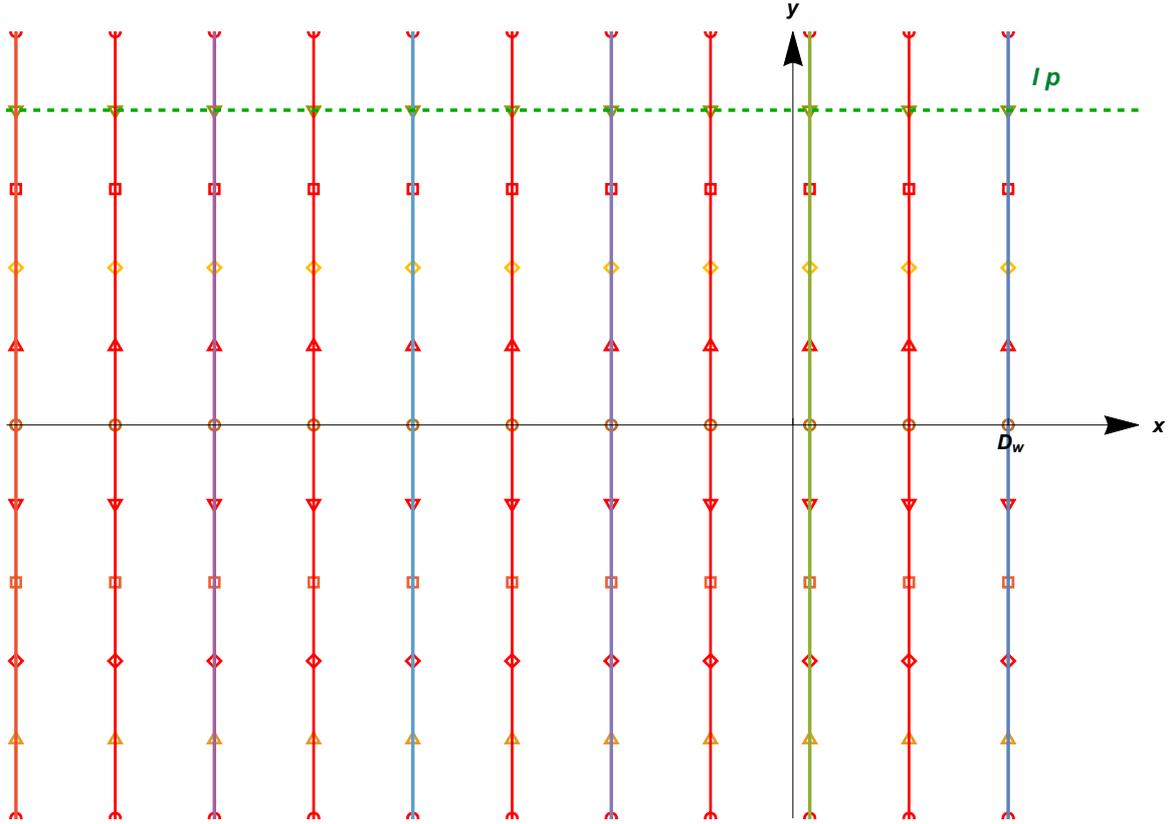


Figure 6: The intrinsic Complex Dimensions of the Weierstrass Curve. The Complex Dimensions are periodically distributed (with the same period $p = \frac{2\pi}{\ln N_b}$, the oscillatory period of $\Gamma_{\mathcal{W}}$) along countably many vertical lines, with abscissae $D_{\mathcal{W}} - k(2 - D_{\mathcal{W}})$, where $k \in \mathbb{N}$ is arbitrary.

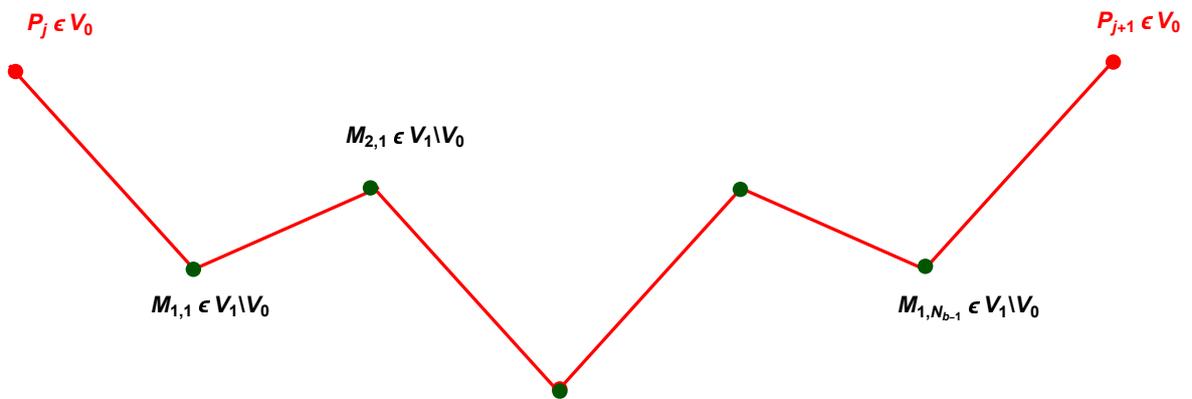


Figure 7: In search of invariants, when switching from the initial prefractal graph, to the first one.

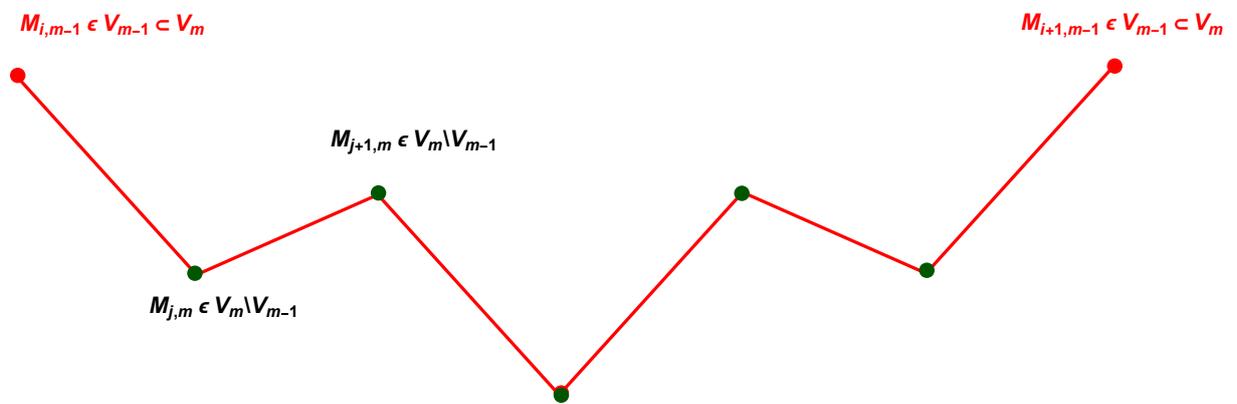


Figure 8: In search of invariants, when switching from the $(m - 1)^{th}$ prefractal graph, to the m^{th} one.