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► **To cite this version:**

Claire David, Michel L Lapidus. Fractal Complex Dimensions and Cohomology of the Weierstrass Curve. 2022. hal-03797595v2

HAL Id: hal-03797595

<https://hal.science/hal-03797595v2>

Preprint submitted on 16 Dec 2022 (v2), last revised 17 Jan 2024 (v3)

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Fractal Complex Dimensions

and

Cohomology of the Weierstrass Curve

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November 8, 2022

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Abstract

In this survey article, we present the authors' main results concerning the Complex Dimensions of the Weierstrass Curve, along with their links with the associated fractal cohomology, as developed in our previous papers [DL22a], [DL22b]. Our results shed new light on the theory and the interpretation of Complex Fractal Dimensions, insofar as we envision the fractal Complex Dimensions as dynamical quantities, which evolve with the scales. Accordingly, we define the Complex Dimensions of the Weierstrass Curve as the set of the Complex Dimensions of the sequence of Weierstrass Relative Fractal Drums which converge to the Curve. By means of *fractal tube formulae*, we then obtain the associated *Weierstrass fractal zeta functions*, whose poles yield the set of Complex Dimensions. In particular, we show that the Complex Dimensions (apart from 0 and -2) are periodically distributed along countably many vertical lines, with the same oscillatory period. As expected, the Minkowski (or box-counting) dimension is the Complex Dimension with maximal real part, and zero imaginary part.

We then show how those Dimensions are connected to the cohomological properties of the Curve: the cohomological groups related to the Curve are obtained, by induction, as sums indexed by the cohomological Complex Dimensions. We determine explicitly both the infinite sequence of prefractal cohomology spaces and the corresponding inductive limit, the fractal (or total) cohomology space of the Weierstrass Curve. In particular, we show that the elements of these cohomology spaces – viewed as suitable continuous functions on the Curve – admit a fractal power series expansion taken over the cohomological Complex Dimensions, that are akin to Taylor-like expansions.

MSC Classification: 11M41, 28A12, 28A75, 28A80.

*The research of M. L. L. was supported by the Burton Jones Endowed Chair in Pure Mathematics, as well as by grants from the U. S. National Science Foundation.

Keywords: Weierstrass Curve, best Hölder exponent, reverse Hölder continuity, Complex Dimensions of an iterated fractal drum, box-counting (or Minkowski) dimension, fractal tube formula, tube zeta function, distance zeta function, (upper, lower and average) Minkowski contents, Minkowski non-measurability, Minkowski nondegeneracy, nowhere differentiability, differential operators, differentials, fractal cohomology, cohomology groups, topological invariants, fractal expansions, cohomology infinitesimal, fractal complex.

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1 Introduction

The Weierstrass Function is mainly known as one of those so-called “pathological objects”, continuous everywhere, while nowhere differentiable. Given $\lambda \in]0, 1[$, and an odd integer b such that $\lambda b > 1 + \frac{3\pi}{2}$, it has been introduced by K. Weierstrass [Wei75], as the sum of the uniformly convergent trigonometric series

$$x \in \mathbb{R} \mapsto \sum_{n=0}^{\infty} \lambda^n \cos(\pi b^n x).$$

We refer to [Tit39] (pages 351-353) for an exposition of the original proof, which was then completed by G. Hardy [Har16], in the more general case, where b is any real number such that $\lambda b > 1$.

Yet, beyond this property of nowhere differentiability, the associated Curve is also of great interest, due to its self-similarity properties: the Curve is *a fractal*, as evoked, for instance, by Benoît Mandelbrot in his famous books [Man77], [Man83]. Impressively, Mandelbrot guessed, in a purely intuitive way, the exact value of the associated Hausdorff dimension, i.e., $D_{\mathscr{W}} = 2 + \frac{\ln \lambda}{\ln b}$, thereby joining the cohort of the great mathematical conjectures. “Comment certains mathématiciens sont parvenus à une conjecture, cela s’apparente à un saut dans le vide” – “How some mathematicians came to a conjecture is akin to a leap into the void”, as stated by the French mathematician Martin Andler [Cul20].

From this perspective, the main topic that interested the mathematical community has long been the determination of the box-counting dimension – or Minkowski dimension, of the Curve, as it can be found in the existing literature on the subject, from discussions in Falconer’s book [Fal86], or results by J.-L. Kaplan, J. Mallet-Paret and J. A. Yorke [KMPY84], involving Fourier analysis and tools from the theory of dynamical systems. One should also cite the works of F. Przytycki and M. Urbański [PU89], T.-Y. Hu and K.-S. Lau [HL93], followed, in the case of the Hausdorff dimension, by B. Hunt [Hun98], who also examined the randomized case. Further results concerning the Hausdorff dimension have been obtained by K. Barański, B. Bárány and J. Romanowska [BBR14],

then, by W. Shen [She18] (in the general case), and G. Keller [Kel17].

The specific question of the box-counting dimension is of importance. It has been proved by the first author [Dav18], in the case where $b = N_b$ is an integer, and by contrast to existing work, that it can be obtained in a simple way, without requiring theoretical background in the theory of dynamical systems. This is not all. It appears to be deeply linked to the non-differentiability property of the function. One might then want to go further, and question the operators at stake: maybe there is a way to obtain new ones, fitted to this singular object, that would make differentiation possible?

This is where the theory of Complex Dimensions comes into play. Developed for many years now by M. L. Lapidus and his collaborators for example in [Lap91], [Lap92], [Lap93], [LP93], [LM95], [LP06], [Lap08], [LPW11], [LvF06], [LvF13], [LRŽ17a], [LRŽ17b], [LRŽ18], [Lap19], [HL21], [Lap22], it makes the connection between the geometry of an object and its differentiability properties. This is done by means of *geometric (or fractal) zeta functions*, which stand for the trace of the differential operator at a complex order s . The Complex Dimensions are obtained as the poles of those fractal zeta functions. They account, in particular, for the maximal order of differentiation, which coincides with the Minkowski dimension of the compact set under study. Usually, the Minkowski (or box-counting) dimension) coincides with the maximum value of the associated real parts.

With regard to fractals, the determination of the Complex Dimensions of the Weierstrass Curve was, until now, an open problem (see [LRŽ17b], Problem 6.2.24, page 560), that we recently solved in [DL22a]. The required geometric zeta functions are local and global fractal tube zeta functions, obtained by means of so-called *Weierstrass Iterated Fractal Drums*; i.e., tubular neighborhoods of prefractal polygonal approximations of the Curve. The main difficulty was to handle the nonlinear features at stake, since, contrary to classical fractals such as, for instance, the Koch Curve, the Weierstrass Curve is obtained by means of a suitable nonlinear and noncontractive iterated function system (i.f.s.); see [Dav18]. The i.f.s. is essential: fractals are usually obtained as limits of so-called *prefractals*, i.e., a sequence of finite graphs that converge towards them. In our case, nonlinearity makes the geometry especially complicated; in particular, one cannot obtain the exact values of the underlying elementary lengths and angles.

In order to deal with those difficulties, we had to rely on geometrical properties that had never been obtained previously:

- i.* Explicit lower and upper bounds for the elementary lengths, in relation with the Hölder and anti-Hölder (or reverse Hölder) properties of the Weierstrass function.
- ii.* Nonincreasing property for the sequence of geometric angles.
- iii.* Conditions under which there exist reentrant angles.

Regarding the Complex Dimensions, in the case where $b = N_b$ is an integer, we have obtained the following key results:

- i.* In Corollary 2.6, and Theorem 2.7, along with Corollary 2.2, where we prove the sharp Hölder continuity, and sharp local reverse Hölder continuity, with optimal Hölder exponent

$$2 - \mathcal{D}_{\mathcal{W}} = \alpha_{\mathcal{W}} = \frac{\ln \frac{1}{\lambda}}{\ln N_b} \in (0, 1) \text{ for the Weierstrass function.}$$

- ii. For very small values of the positive parameter ϵ , the expression of the ϵ -neighborhood of the Curve – a Weierstrass Fractal Tube Formula, which (apart from two terms associated with the Complex Dimensions 0 and -2) involves an expansion of the form

$$\sum_{\alpha \text{ real part of a Complex Dimension}} \epsilon^{2-\alpha} G_{\alpha} \left(\ln_{N_b} \left(\frac{1}{\epsilon} \right) \right), \quad (\star)$$

where, for any real part α of a Complex Dimension, G_{α} denotes a continuous and periodic function. Furthermore, for $\alpha = \alpha_{max} = D_{\mathcal{W}}$, the Minkowski dimension of the Curve – i.e., for α being the maximal real part of the Complex Dimensions – $G_{\alpha_{max}}$ is nonconstant, as well as bounded away from zero and infinity. See Theorem 4.2.

- iii. The values of the possible Complex Dimensions of the Curve, defined as the poles of the associated Tube Zeta Function. In particular, we show that the Complex Dimensions different from 0 and -2 are periodically distributed along countably many vertical lines, with abscissae $\mathcal{D}_{\mathcal{W}} - k(2 - \mathcal{D}_{\mathcal{W}})$, $1 - 2k$, where k in \mathbb{N} is arbitrary. Furthermore, all of the Complex Dimensions (different from 0 and -2) have the same oscillatory period $\mathbf{p} = \frac{2\pi}{\ln N_b}$. Moreover, zero and -2 are also Complex Dimensions. See Theorems 5.3 and 5.4, along with Corollary 4.4.

In addition, we know that the dimensions 0, 1, -2 and $\mathcal{D}_{\mathcal{W}} - k(2 - \mathcal{D}_{\mathcal{W}})$, where k in \mathbb{N} is arbitrary, are actual Complex Dimensions of the Curve, and they are simple; i.e., they are simple poles of the fractal zeta function of the Weierstrass Curve (or, rather, of the associated Weierstrass iterated fractal drum).

- iv. The nondegeneracy, in the Minkowski sense (see [LRŽ17b]), of the Curve, which comes from the fact that the lower and upper Minkowski contents of the Curve are respectively positive and finite. See Theorem 4.6 along with Corollary 4.7.

- v. As a corollary, the fact that the number $D_{\mathcal{W}}$ is a Complex Dimension of the Curve, and coincides with the Minkowski dimension (which then exists) of the Curve, and takes the expected value, as given by Mandelbrot's conjecture; namely, $D_{\mathcal{W}} = 2 - \frac{\ln \frac{1}{\lambda}}{\ln N_b} \in (1, 2)$.

- vi. The *fractality* of the Weierstrass Curve, in the sense of [LvF00], [LvF06], [LvF13], [LRŽ17b], [Lap19]; i.e., the existence of *nonreal* Complex Dimensions (with real part $D_{\mathcal{W}}$) giving rise to geometric oscillations, in the Fractal Tube Formula obtained in [DL22a]. In the terminology of [LvF13] and [LRŽ17b], the Weierstrass Curve is fractal in countably many dimensions d_k , with $d_k \rightarrow -\infty$, as $k \rightarrow \infty$.

Beyond these results, we have studied the links with *a fractal cohomogy*, a natural feature, in so far as discrete differences are deeply connected with differentiation (see the paper on *h-Cohomology* [DL21]). We place ourselves within a completely different framework from the one that can be found, for instance, in the work by Marius Ionescu, Luke G. Rogers and Alexander Teplyaev in [IRT12], and which is based on the construction of Fredholm modules. Along the lines of previous results obtained or conjectured by the second author and his collaborators in [LvF00], [LvF06], [Lap08], [LvF13], [CL17], [Lap19] and [Lap22], we have obtained the following key results:

- i.* In Theorem 5.3, where we prove that the cohomology groups associated to the Weierstrass Curve consist of continuous functions on the Weierstrass Curve, satisfying fractal expansions expressed as sums indexed by the underlying Complex Dimensions. Such a result had not been established (or even expected) before.
- ii.* In Theorem 5.4, which gives the total cohomology of the Weierstrass Curve, in terms of Taylor-like expansions, again indexed by the underlying Complex Dimensions.

The aim of the present paper, which is *a survey article*, is to present those results in a summarized form. Hence, we will not give detailed proofs, which can be found in the original texts [DL22a], and [DL22b].

Section 2 is devoted to the geometric framework. The next one, Section 3, gives the expression of *the tubular neighborhood of the Curve*, from which one deduces *the tube zeta function* (as well as *the distance zeta function*), and the values of the possible Complex Dimensions. The *fractal cohomology* is discussed in Section 5, where we also show, as could be expected, that the Weierstrass function \mathscr{W} belongs to the total cohomology of the Weierstrass Curve.

2 Geometric Framework

Henceforth, we place ourselves in the Euclidean plane of dimension 2, equipped with a direct orthonormal frame. The usual Cartesian coordinates are denoted by (x, y) . The horizontal and vertical axes will be respectively referred to as $(x'x)$ and $(y'y)$.

Notation 1 (Set of all Natural Numbers, and Intervals).

As in Bourbaki [Bou04] (Appendix E. 143), we denote by $\mathbb{N} = \{0, 1, 2, \dots\}$ the set of all natural numbers, and set $\mathbb{N}^\star = \mathbb{N} \setminus \{0\}$.

Given a, b with $-\infty \leq a \leq b \leq \infty$, $]a, b[= (a, b)$ denotes an open interval, while, for example, $]a, b] = (a, b]$ denotes a half-open, half-closed interval.

Notation 2 (Wave Inequality Symbol).

Given two positive numbers a and b , we will use the notation $a \lesssim b$ when there exists a strictly positive constant C such that $a \leq Cb$.

Notation 3 (Weierstrass Parameters).

In the sequel, λ and N_b are two real numbers such that

$$0 < \lambda < 1 \quad , \quad N_b \in \mathbb{N}^\star \quad \text{and} \quad \lambda N_b > 1 \quad . \quad (\clubsuit)$$

Note that this implies that $N_b > 1$ (i.e., $N_b \geq 2$).

Definition 2.1 (Weierstrass Function, Weierstrass Curve).

We consider the *Weierstrass function* \mathscr{W} , defined, for any real number x , by

$$\mathscr{W}(x) = \sum_{n=0}^{\infty} \lambda^n \cos(2\pi N_b^n x) .$$

We call the associated graph the *Weierstrass Curve*.

Due to the periodic properties of the \mathscr{W} function, from now on, we restrict our study to the interval $[0, 1[= [0, 1)$ (or, equivalently, to $[0, 1]$). Accordingly, the Weierstrass Curve $\Gamma_{\mathscr{W}}$ is viewed as a compact subset of \mathbb{R}^2 .

Notation 4 (Logarithm).

Given $y > 0$, $\ln y$ denotes the natural logarithm of y , while, given $a > 0$, $a \neq 1$, $\ln_a y = \frac{\ln y}{\ln a}$ denotes the logarithm of y in base a ; so that, in particular, $\ln = \ln_e$.

Notation 5. For the parameters λ and N_b satisfying condition (\clubsuit) (see Notation 3), we denote by

$$D_{\mathscr{W}} = 2 + \frac{\ln \lambda}{\ln N_b} = 2 - \ln_{N_b} \frac{1}{\lambda} \in]1, 2[$$

the box-counting dimension (or Minkowski dimension) of the Weierstrass Curve $\Gamma_{\mathscr{W}}$, which happens to be equal to its Hausdorff dimension [KMPY84], [BBR14], [She18], [Kel17]. Our results also provide a direct proof of the fact that $D_{\mathscr{W}}$, the Minkowski dimension (or box-counting dimension) of $\Gamma_{\mathscr{W}}$, exists and takes the above value.

Convention (The Weierstrass Curve as a Cyclic Curve).

In the sequel, we identify the points $(0, \mathscr{W}(0))$ and $(1, \mathscr{W}(1)) = (1, \mathscr{W}(0))$. This is justified by the fact that the Weierstrass function \mathscr{W} is 1-periodic, since N_b is an integer.

Proposition 2.1 (Nonlinear and Noncontractive Iterated Function System (i.f.s.)).

Following our previous work [Dav18], we approximate the restriction $\Gamma_{\mathscr{W}}$ to $[0, 1[\times \mathbb{R}$, of the Weierstrass Curve, by a sequence of finite graphs, built via an iterative process. For this purpose, we use the nonlinear iterated function system of the family of C^∞ maps from \mathbb{R}^2 to \mathbb{R}^2 denoted by

$$\mathcal{T}_{\mathscr{W}} = \{T_0, \dots, T_{N_b-1}\} ,$$

where, for any integer i belonging to $\{0, \dots, N_b - 1\}$ and any point (x, y) of \mathbb{R}^2 ,

$$T_i(x, y) = \left(\frac{x+i}{N_b}, \lambda y + \cos\left(2\pi \left(\frac{x+i}{N_b}\right)\right) \right) .$$

We point out that those maps are not contractions (see [Dav19]). Yet, they correspond, in a sense, to the composition of a contraction of ratio r_x in the horizontal direction, and a dilatation of factor r_y in the vertical one, with

$$r_x r_y < 1 .$$

Property 2.2 (Attractor of the i.f.s. [Dav18], [Dav19]).

The Weierstrass Curve is the attractor of the i.f.s. $\mathcal{T}_{\mathcal{W}}$:

$$\Gamma_{\mathcal{W}} = \bigcup_{i=0}^{N_b-1} T_i(\Gamma_{\mathcal{W}}).$$

Notation 6 (Fixed Points).

For any integer i belonging to $\{0, \dots, N_b - 1\}$, we denote by:

$$P_i = (x_i, y_i) = \left(\frac{i}{N_b - 1}, \frac{1}{1 - \lambda} \cos\left(\frac{2\pi i}{N_b - 1}\right) \right)$$

the unique fixed point of the map T_i ; see [Dav19].

Definition 2.2 (Sets of Vertices, Prefractals).

We denote by V_0 the ordered set (according to increasing abscissae), of the points

$$\{P_0, \dots, P_{N_b-1}\}.$$

The set of points V_0 – where, for any i of $\{0, \dots, N_b - 2\}$, the point P_i is linked to the point P_{i+1} – constitutes an oriented graph, according to increasing abscissa, that we will denote by $\Gamma_{\mathcal{W}_0}$. Then, V_0 is called the set of vertices of the graph $\Gamma_{\mathcal{W}_0}$.

For any nonnegative integer m , i.e., for $m \in \mathbb{N}$, we set

$$V_m = \bigcup_{i=0}^{N_b-1} T_i(V_{m-1}).$$

The set of points V_m , where two consecutive points are linked, is an oriented graph, according to increasing abscissa, called the **m^{th} -order \mathcal{W} -prefractal**. Then, V_m is called the set of vertices of the m^{th} prefractal graph $\Gamma_{\mathcal{W}_m}$; see Figure 1.

Definition 2.3 (Adjacent Vertices, Edge Relation).

For any $m \in \mathbb{N}$, the m^{th} prefractal graph $\Gamma_{\mathcal{W}_m}$ is equipped with an edge relation \sim_m , as follows: two vertices X and Y of $\Gamma_{\mathcal{W}_m}$, i.e. two points belonging to V_m will be said to be *adjacent* (i.e., *neighboring* or *junction points*) if and only if the line segment $[X, Y]$ is an edge of $\Gamma_{\mathcal{W}_m}$; we then write $X \sim_m Y$. Note that this edge relation depends on m , which means that points adjacent in V_m might not remain adjacent in V_{m+1} .

Property 2.3. [Dav18]

For any $m \in \mathbb{N}$, the following statements hold:

- i. $V_m \subset V_{m+1}$.

ii. $\#V_m = (N_b - 1) N_b^m + 1$.

iii. The m^{th} prefractal graph $\Gamma_{\mathcal{W}_m}$ has exactly $(N_b - 1) N_b^m$ edges.

iv. The consecutive vertices of the prefractal graph $\Gamma_{\mathcal{W}_m}$ are the vertices of N_b^m simple polygons $\mathcal{P}_{m,k}$ with N_b sides. For any integer $m \in \mathbb{N}^*$, the junction point between two consecutive polygons is the point

$$\mathcal{P}_{m,k} \cap \mathcal{P}_{m,k+1} = \left\{ \left(\frac{(N_b - 1)k}{(N_b - 1)N_b^m}, \mathcal{W} \left(\frac{(N_b - 1)k}{(N_b - 1)N_b^m} \right) \right) \right\}, \quad 1 \leq k \leq N_b^m - 1.$$

Hence, the total number of junction points is $N_b^m - 1$. For instance, in the case $N_b = 3$, one gets triangles.

In the sequel, we will denote by \mathcal{P}_0 the **initial polygon**, whose vertices are the fixed points of the maps T_i , $0 \leq i \leq N_b - 1$, introduced in Definition 2.2, i.e., $\{P_0, \dots, P_{N_b-1}\}$.

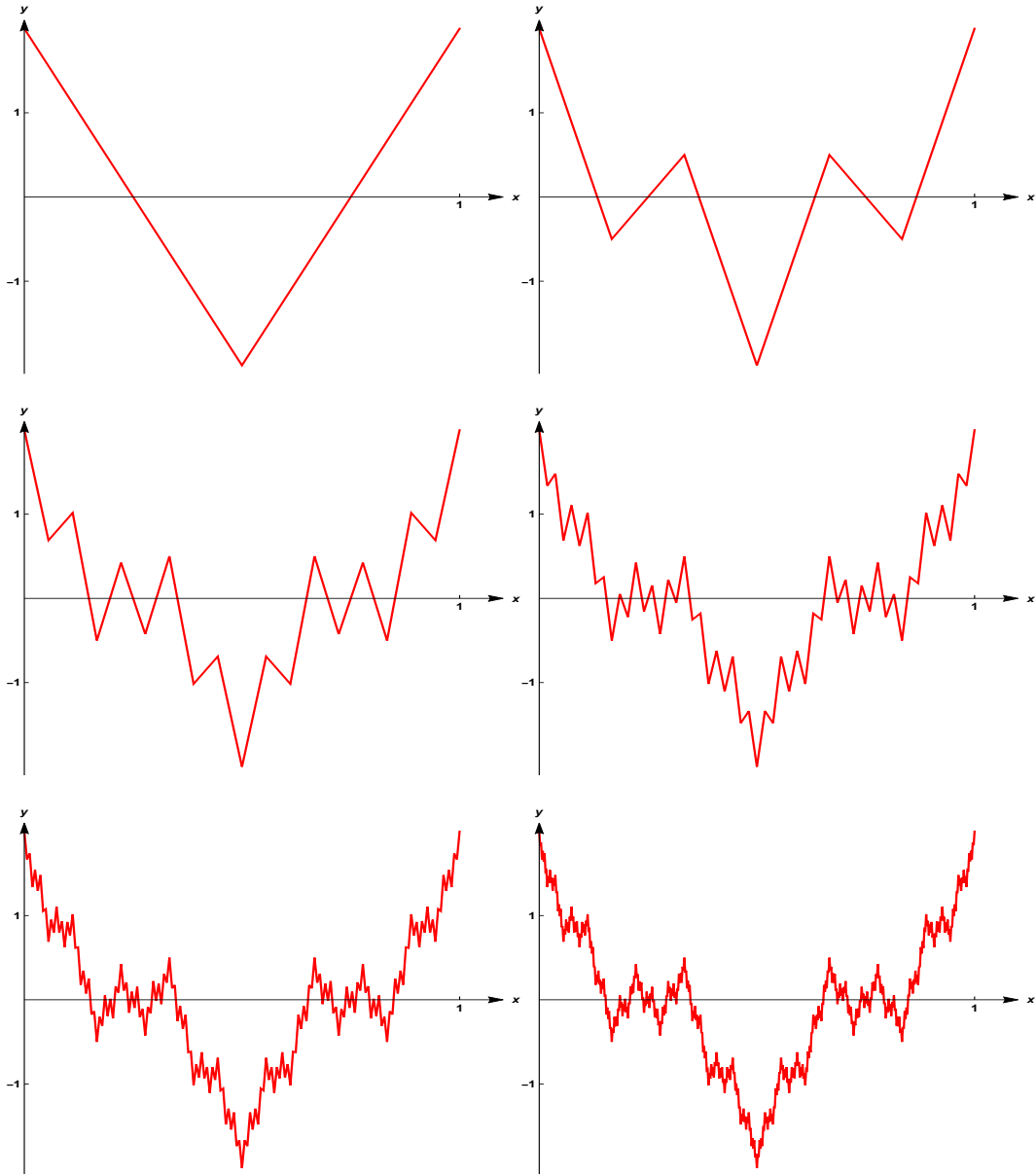


Figure 1: The prefractal graphs $\Gamma_{\mathcal{W}_0}, \Gamma_{\mathcal{W}_1}, \Gamma_{\mathcal{W}_2}, \Gamma_{\mathcal{W}_3}, \Gamma_{\mathcal{W}_4}, \Gamma_{\mathcal{W}_5}$, in the case where $\lambda = \frac{1}{2}$ and $N_b = 3$. For example, $\Gamma_{\mathcal{W}_1}$ is on the right side of the top row, while $\Gamma_{\mathcal{W}_4}$ is on the left side of the bottom row.

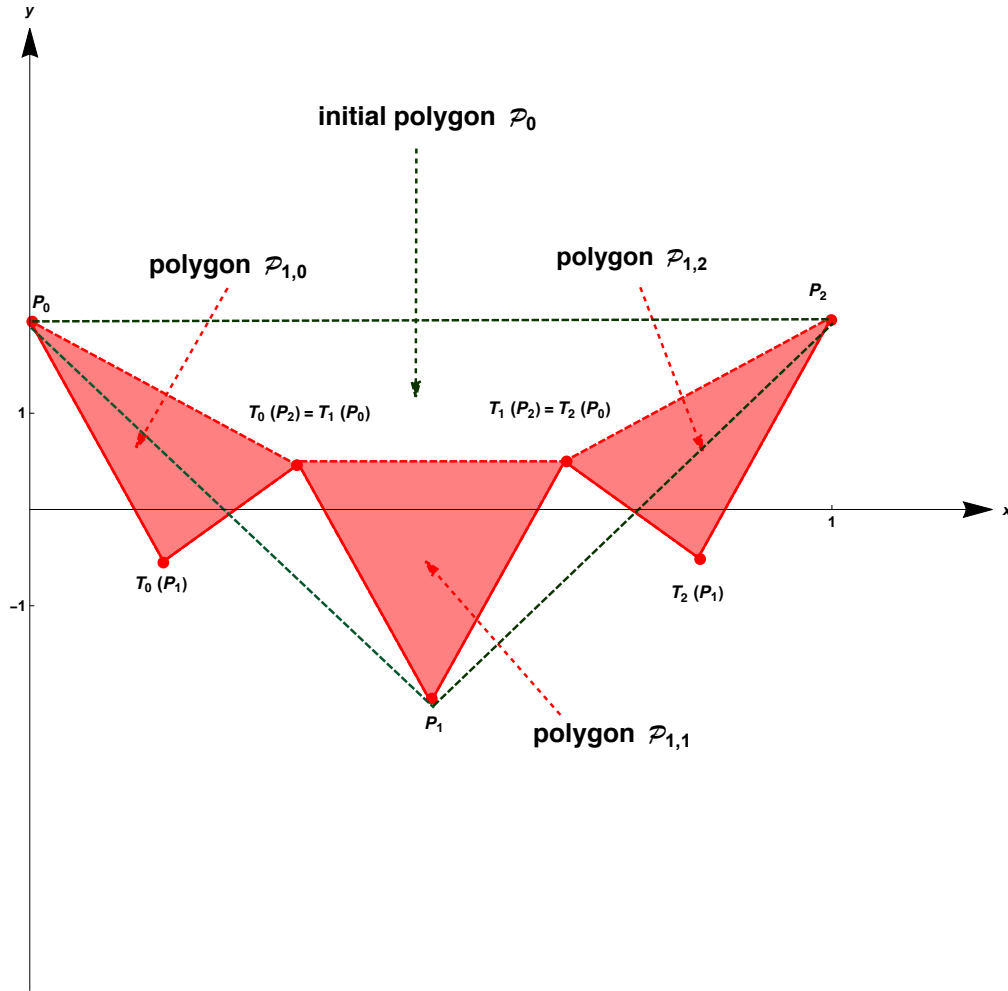


Figure 2: The initial polygon \mathcal{P}_0 , and the polygons $\mathcal{P}_{0,1}$, $\mathcal{P}_{1,1}$, $\mathcal{P}_{1,2}$, in the case where $\lambda = \frac{1}{2}$ and $N_b = 3$.

Definition 2.4 (Vertices of the Prefractals, Elementary Lengths, Heights and Angles).

Given a strictly positive integer m , we denote by $(M_{j,m})_{0 \leq j \leq (N_b-1)N_b^m}$ the set of vertices of the prefractal graph $\Gamma_{\mathcal{W}_m}$. One thus has, for any integer j in $\{0, \dots, (N_b-1)N_b^m\}$,

$$M_{j,m} = \left(\frac{j}{(N_b-1)N_b^m}, \mathcal{W} \left(\frac{j}{(N_b-1)N_b^m} \right) \right).$$

We also introduce, for any integer j in $\{0, \dots, (N_b-1)N_b^m - 1\}$, the following quantities:

i. the elementary horizontal lengths:

$$L_m = \frac{j}{(N_b-1)N_b^m};$$

ii. the elementary lengths:

$$\ell_{j,j+1,m} = d(M_{j,m}, M_{j+1,m}) = \sqrt{L_m^2 + h_{j,j+1,m}^2},$$

where $h_{j,j+1,m}$ is defined in *iii.* just below.

iii. the elementary heights:

$$h_{j-1,j,m} = \left| \mathscr{W} \left(\frac{j}{(N_b - 1) N_b^m} \right) - \mathscr{W} \left(\frac{j-1}{(N_b - 1) N_b^m} \right) \right|, \quad h_{j,j+1,m} = \left| \mathscr{W} \left(\frac{j+1}{(N_b - 1) N_b^m} \right) - \mathscr{W} \left(\frac{j}{(N_b - 1) N_b^m} \right) \right|;$$

iv. the geometric angles:

$$\theta_{j-1,j,m} = ((y^l y), (\widehat{M_{j-1,m} M_{j,m}})) \quad , \quad \theta_{j,j+1,m} = ((y^l y), (\widehat{M_{j,m} M_{j+1,m}})),$$

which yield **the value of the geometric angle between consecutive edges** $[M_{j-1,m} M_{j,m}, M_{j,m} M_{j+1,m}]$:

$$\theta_{j-1,j,m} + \theta_{j,j+1,m} = \arctan \frac{L_m}{|h_{j-1,j,m}|} + \arctan \frac{L_m}{|h_{j,j+1,m}|}.$$

Property 2.4 (Scaling Properties of the Weierstrass Function [DL22a]).

For any strictly positive integer m and any j in $\{0, \dots, \#V_m\}$,

$$\mathscr{W} \left(\frac{j}{(N_b - 1) N_b^m} \right) = \lambda^m \mathscr{W} \left(\frac{j}{(N_b - 1)} \right) + \sum_{k=0}^{m-1} \lambda^k \cos \left(\frac{2\pi N_b^k j}{(N_b - 1) N_b^m} \right).$$

Property 2.5 (Explicit Lower and Upper Bounds for the Elementary Heights [DL22a]).

For any strictly positive integer m , and any j in $\{0, \dots, (N_b - 1) N_b^m\}$,

$$C_{inf} L_m^{2-D_{\mathscr{W}}} \leq \underbrace{|\mathscr{W}((j+1)L_m) - \mathscr{W}(jL_m)|}_{h_{j,j+1,m}} \leq C_{sup} L_m^{2-D_{\mathscr{W}}},$$

where the finite and positive constants C_{inf} and C_{sup} are given by

$$C_{inf} = (N_b - 1)^{2-D_{\mathscr{W}}} \min_{0 \leq j \leq N_b - 1} \left| \mathscr{W} \left(\frac{j+1}{N_b - 1} \right) - \mathscr{W} \left(\frac{j}{N_b - 1} \right) \right|$$

and

$$C_{sup} = (N_b - 1)^{2-D_{\mathscr{W}}} \left(\max_{0 \leq j \leq N_b - 1} \left| \mathscr{W} \left(\frac{j+1}{N_b - 1} \right) - \mathscr{W} \left(\frac{j}{N_b - 1} \right) \right| + \frac{2\pi}{(N_b - 1)(\lambda N_b - 1)} \right).$$

One should note, in addition, that these constants C_{inf} and C_{sup} depend on the initial polygon \mathcal{P}_0 .

Corollary 2.6 (of Property 2.5).

For any strictly positive integer m and any integer j in $\{0, \dots, (N_b - 1) N_b^m - 1\}$, one then has, for the elementary heights,

$$h_{j-1,j,m} = L_m^{2-D_{\mathscr{W}}} \mathcal{O}(1),$$

where $C_{inf} \leq \mathcal{O}(1) \leq C_{sup}$.

Theorem 2.7 (Sharp Local Discrete Reverse Hölder Properties of the Weierstrass Function [DL22a]).

For any natural integer m , let us consider a pair of real numbers (x, x') such that:

$$x = \frac{(N_b - 1)k + j}{(N_b - 1)N_b^m} = ((N_b - 1)k + j) L_m \quad , \quad x' = \frac{(N_b - 1)k + j + \ell}{(N_b - 1)N_b^m} = ((N_b - 1)k + j + \ell) L_m$$

where $0 \leq k \leq N_b - 1^m - 1$, and

i. if the integer N_b is odd,

$$0 \leq j < \frac{N_b - 1}{2} \quad \text{and} \quad 0 < j + \ell \leq \frac{N_b - 1}{2}$$

or

$$\frac{N_b - 1}{2} \leq j < N_b - 1 \quad \text{and} \quad \frac{N_b - 1}{2} < j + \ell \leq N_b - 1;$$

ii. if the integer N_b is even,

$$0 \leq j < \frac{N_b}{2} \quad \text{and} \quad 0 < j + \ell \leq \frac{N_b}{2}$$

or

$$\frac{N_b}{2} + 1 \leq j < N_b - 1 \quad \text{and} \quad \frac{N_b}{2} + 1 < j + \ell \leq N_b - 1.$$

This means that the points $(x, \mathcal{W}(x))$ and $(x', \mathcal{W}(x'))$ are vertices of the polygon $\mathcal{P}_{m,k}$ (see Property 2.3), both located on the left-side of the polygon, or both located on the right-side (see Figure 3).

Then, one has the following reverse-Hölder inequality, with sharp Hölder exponent $-\frac{\ln \lambda}{\ln N_b} = 2 - D_{\mathcal{W}}$,

$$C_{inf} |x' - x|^{2-D_{\mathcal{W}}} \leq |\mathcal{W}(x') - \mathcal{W}(x)|.$$

Remark 2.1. It is clear that, for any natural integer m , and any pair $((x, \mathcal{W}(x)), (x', \mathcal{W}(x')))$ of adjacent vertices of the finite prefractal graph $\Gamma_{\mathcal{W}_m}$, the same following (discrete, local) Hölder and reverse-Hölder inequality, with sharp Hölder exponent $-\frac{\ln \lambda}{\ln N_b} = 2 - D_{\mathcal{W}}$, holds; i.e., still with C_{inf} and C_{sup} given as in Property 2.5 above, we have that

$$C_{inf} |x' - x|^{2-D_{\mathcal{W}}} \leq |\mathcal{W}(x') - \mathcal{W}(x)| \leq C_{sup} |x' - x|^{2-D_{\mathcal{W}}}.$$

Remark 2.2. Thus far, no such reverse Hölder estimates had been obtained for the Weierstrass function. The fact that they are discrete ones is natural, since the Weierstrass Curve is approximated by a sequence of polygonal prefractal graphs.

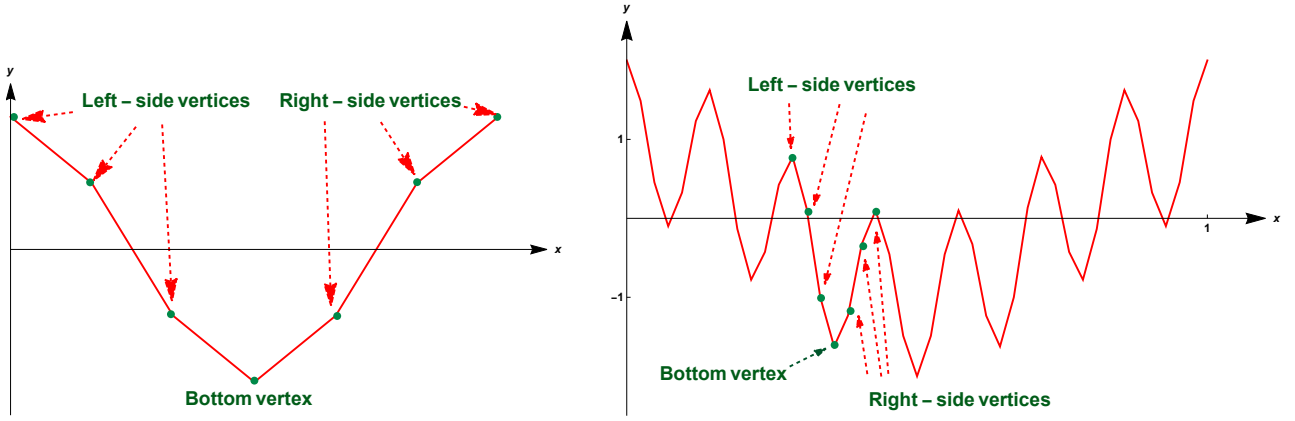


Figure 3: The left-side and right-side vertices.

Corollary 2.8 (Optimal Hölder Exponent for the Weierstrass Function [DL22a]).

The local reverse Hölder property of Theorem 2.7, in conjunction with the Hölder condition satisfied by the Weierstrass function (see [Zyg02], Chapter II, Theorem 4.9, page 47), shows that the Codimension $2 - D_{\mathcal{W}} = -\frac{\ln \lambda}{\ln N_b}$ is the best, i.e., optimal Hölder exponent for the Weierstrass function.

Corollary 2.9 (Nonincreasing Sequence of Geometric Angles).

For the *geometric angles* $\theta_{j-1,j,m}$, $0 \leq j \leq (N_b - 1) N_b^m$, $m \in \mathbb{N}$, we have the following property:

$$\theta_{j-1,j,m} > \theta_{j-1,j,m+1} \quad \text{and} \quad \theta_{j-1,j,m+1} \lesssim L_m^{D_{\mathcal{W}}-1}.$$

3 Iterated Fractal Drums and ϵ -Neighborhood of the \mathcal{W} -Curve

We introduce here the notion of an *iterated fractal drum* (IFD), which is well suited to the present setting – and, as we expect, to many other complicated examples of fractal curves and of their higher-dimensional analogs. As we will see, it is defined in terms of the ϵ -neighborhood of an appropriate infinitesimal sequence of lengths, $\epsilon = (\epsilon_m)_{m \in \mathbb{N}}$; intuitively, $(\epsilon_m)_{m \in \mathbb{N}}$, can be thought of as a sequence of scales.

Notation 7 (Euclidean Distance).

In the sequel, we denote by d the Euclidean distance on \mathbb{R}^2 .

Notation 8 (Small Parameter and Associated Map).

Given a natural integer m , we take ϵ_m in $\left(\frac{1}{(N_b - 1) N_b^{m+1}}, \frac{1}{(N_b - 1) N_b^m} \right]$, and introduce the map

$$\epsilon_m \mapsto m(\epsilon_m) = [-\ln_{N_b} ((N_b - 1) \epsilon_m)] = [m(\epsilon_m)] + \{m(\epsilon_m)\},$$

where $[\cdot]$ and $\{\cdot\}$ respectively denote the integer and fractional parts; so that $y = [y] + \{y\}$, with $[y] \in \mathbb{Z}$ and $0 \leq \{y\} < 1$.

For the sake of simplicity, we temporarily set $x = m(\epsilon_m) = -\ln_{N_b}((N_b - 1)\epsilon)$, which yields

$$N_b^m = N_b^{[x]} = N_b^{x-\{x\}} \quad \text{and} \quad \epsilon = \frac{1}{N_b - 1} N_b^{-([x]+\{x\})}.$$

Property 3.1 ((m, ϵ) -Neighborhood [DL22a]).

Given a natural integer m , and a point $M \in \mathbb{R}^2$, we denote by $d(M, \Gamma_{\mathcal{W}_m})$ the distance from M to $\Gamma_{\mathcal{W}_m}$. Then, for the small positive number ϵ_m previously introduced in Notation 8, the (m, ϵ_m) -Neighborhood of the Weierstrass Curve,

$$\mathcal{D}(\Gamma_{\mathcal{W}_m}, \epsilon_m) = \{M = (x, y) \in \mathbb{R}^2, d(M, \Gamma_{\mathcal{W}_m}) \leq \epsilon_m\}$$

(see Figure 4), is obtained by means of:

- i. $(N_b - 1) N_b^m$ upper and lower **overlapping rectangles** (see Figure 5), of length $\ell_{j-1, j, m}$, $1 \leq j \leq N_b^m - 1$, and height ϵ_m .

One of the difficulties is, then, to compute and subtract the area counted twice, which correspond to **parallelograms**, of height ϵ_m and basis $\epsilon_m \cotan(\pi - \theta_{j_k-1, j_k, m} - \theta_{j_k, j_k+1, m})$, and **extra outer triangles**, the area of which is of the form

$$\frac{1}{2} \epsilon_m (b_{j_\ell-1, j_\ell, m} + b_{j_\ell, j_\ell+1, m}),$$

and where, for the sake of simplicity, we have denoted by j_k and j_ℓ the corresponding indices.

- ii. **Upper and lower wedges**, the area of which are of the form

$$\frac{1}{2} (\pi - \theta_{j_l-1, j_l, m} - \theta_{j_l, j_l+1, m}) \epsilon_m^2 \quad , \quad 1 \leq j_l \leq N_b^m - 2.$$

The number of wedges is determined by the shape of the initial polygon \mathcal{P}_0 , and the existence of reentrant angles.

- iii. Two **extreme wedges** (respectively located at the abscissae $x = 0$ and $x = 1$), each of area equal to $\frac{1}{2} \pi \epsilon_m^2$.

The small positive number ϵ_m will be referred to as the width of the (m, ϵ_m) -neighborhood $\mathcal{D}(\Gamma_{\mathcal{W}_m}, \epsilon_m)$.

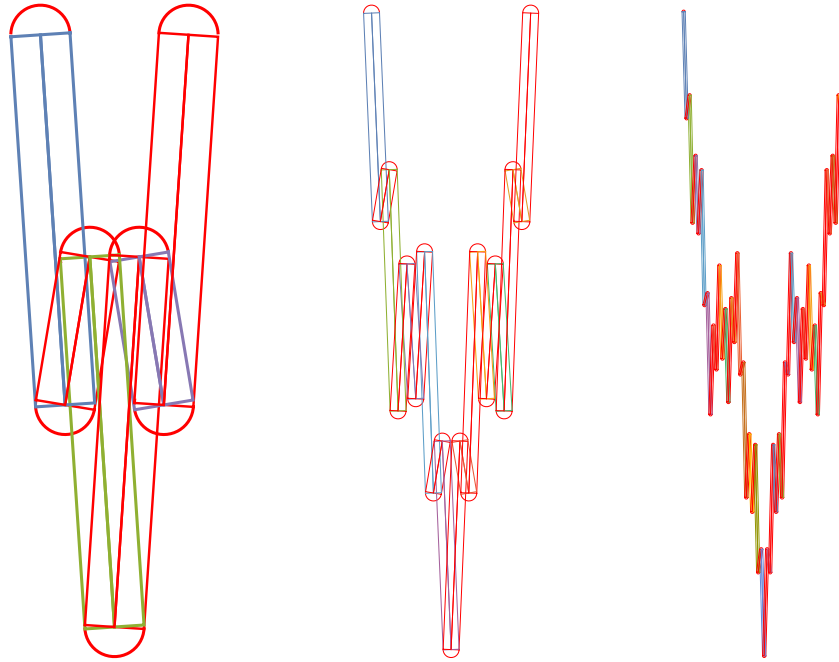


Figure 4: **The $(1, \epsilon_1)$, $(2, \epsilon_2)$ and $(3, \epsilon_3)$ -Neighborhoods, in the case where $\lambda = \frac{1}{2}$ and $N_b = 3$.**

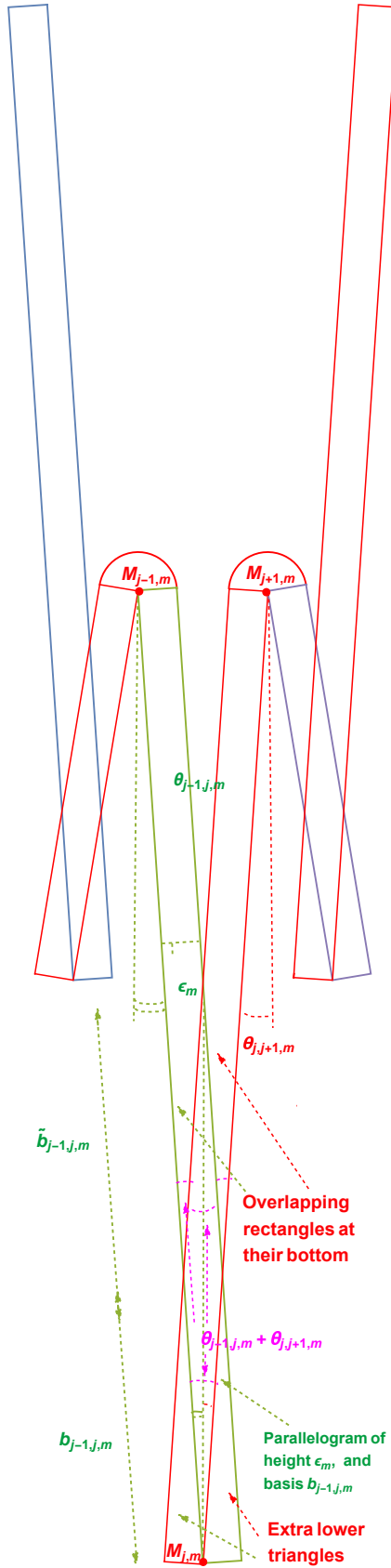


Figure 5: Two overlapping rectangles, in the case where $\lambda = \frac{1}{2}$ and $N_b = 3$.

Thus far, we have considered the sequence of tubular neighborhoods associated to the prefractal sequence of graphs $(\Gamma_{\mathcal{W}_m})_{m \in \mathbb{N}}$, which approximate the Weierstrass Curve $\Gamma_{\mathcal{W}}$. In Property 3.2 just below, we explain the connections between the aforementioned sequence of tubular neighborhoods, and the tubular neighborhood of the Weierstrass Curve $\Gamma_{\mathcal{W}}$ itself. Such a result is all the more important, when it comes to determine the poles of the tubular zeta function obtained by means of the aforementioned sequence of tubular neighborhoods associated to the prefractal sequence $(\Gamma_{\mathcal{W}_m})_{m \in \mathbb{N}}$. One intuitively understand that we cannot explicitly compute the volume of the tubular neighborhood of $\Gamma_{\mathcal{W}}$.

Property 3.2 (Tubular Neighborhood of the Weierstrass Curve).

Given a small positive number δ , we define the δ -Neighborhood of the Weierstrass Curve $\Gamma_{\mathcal{W}}$ by

$$\mathcal{D}(\Gamma_{\mathcal{W}}, \delta) = \{M = (x, y) \in \mathbb{R}^2, d(M, \Gamma_{\mathcal{W}}) \leq \delta\}.$$

Then, there exists an integer $m_0(\delta) \in \mathbb{N}$ such that,

$$\forall m \geq m_0(\delta) : \quad \mathcal{D}\left(\Gamma_{\mathcal{W}_m}, \frac{\delta}{2}\right) \subset \mathcal{D}(\Gamma_{\mathcal{W}}, \delta) \subset \mathcal{D}(\Gamma_{\mathcal{W}_m}, 2\delta).$$

Hence, the sequence of tubular neighborhoods associated to the prefractal sequence of graphs $(\Gamma_{\mathcal{W}_m})_{m \in \mathbb{N}}$ can also be interpreted as a sequence of tubular neighborhoods of $\Gamma_{\mathcal{W}}$, in the sense that, for m large enough, any tubular neighborhood of $\Gamma_{\mathcal{W}_m}$ contains a tubular neighborhoods of $\Gamma_{\mathcal{W}}$, while, at the same time, it also itself contains a tubular neighborhood of $\Gamma_{\mathcal{W}}$. Even if the widths involved are not the same, we have a form of equivalence between a tubular neighborhood of $\Gamma_{\mathcal{W}_m}$, for $m \in \mathbb{N}$ sufficiently large, and a tubular neighborhood of $\Gamma_{\mathcal{W}}$. In our present setting, if we denote by $\mathcal{V}_m\left(\frac{\delta}{2}\right)$, $\mathcal{V}_m(2\delta)$ and $\mathcal{V}(\delta)$ the respective volumes of $\mathcal{D}\left(\Gamma_{\mathcal{W}_m}, \frac{\delta}{2}\right)$, $\mathcal{D}(\Gamma_{\mathcal{W}_m}, 2\delta)$ and $\mathcal{D}(\Gamma_{\mathcal{W}}, \delta)$, we can write

$$\mathcal{V}_m(2\delta) = \mathcal{V}(\delta) + \mathcal{R}_m(\delta) = \mathcal{V}_m\left(\frac{\delta}{2}\right) + \mathcal{R}_m\left(\frac{\delta}{2}\right),$$

where $\mathcal{R}_m(\delta)$ and $\mathcal{R}_m\left(\frac{\delta}{2}\right)$ denote strictly positive error terms, and are such that

$$\lim_{m \rightarrow \infty} \mathcal{R}_m(\delta) = 0^+ \quad \text{and} \quad \lim_{m \rightarrow \infty} \mathcal{R}_m\left(\frac{\delta}{2}\right) = 0^+.$$

Proof. In this proof, for the sake of simplicity, we will write $M_{\star, m}$, $m \in \mathbb{N}$, for the points $M_{j, m}$, with $0 \leq j \leq \#V_m - 1$, of the prefractal graph $\Gamma_{\mathcal{W}_m}$.

i. Because of the density of the set $V^{\star} = \bigcup_{n \in \mathbb{N}} V_n$ in the Weierstrass Curve $\Gamma_{\mathcal{W}}$, there exists an integer $m_1(\delta) \in \mathbb{N}$ such that

$$\forall m \geq m_1(\delta) : \quad d(\Gamma_{\mathcal{W}_m}, \Gamma_{\mathcal{W}}) \leq \delta,$$

with

$$d(\Gamma_{\mathcal{W}_m}, \Gamma_{\mathcal{W}}) = \inf_{M_{\star, m} \in \Gamma_{\mathcal{W}_m}, M \in \Gamma_{\mathcal{W}}} d(M, M_{\star, m}) > 0.$$

Now, given a point $M \in \mathcal{D}(\Gamma_{\mathcal{W}}, \delta)$, for $m \geq m_1(\delta)$, we denote by $M_{\mathcal{W}} \in \Gamma_{\mathcal{W}}$ the closest point of M in $\Gamma_{\mathcal{W}}$. By the triangular inequality, we then have that

$$d(M, \Gamma_{\mathcal{W}_m}) \leq d(M, M_{\mathcal{W}}) + d(\Gamma_{\mathcal{W}_m}, \Gamma_{\mathcal{W}}) \leq 2\delta.$$

This implies that

$$M \in \mathcal{D}(\Gamma_{\mathcal{W}_m}, 2\delta),$$

from which we then deduce the desired result; i.e.,

$$\forall m \geq m_1(\delta) : \quad \mathcal{D}(\Gamma_{\mathcal{W}}, \delta) \subset \mathcal{D}(\Gamma_{\mathcal{W}_m}, 2\delta).$$

ii. At the same time, and also because of the density of the set $V^* = \bigcup_{n \in \mathbb{N}} V_n$ in $\Gamma_{\mathcal{W}}$, there exists an integer $m_2(\delta) \in \mathbb{N}$ such that

$$\forall m \geq m_2(\delta) : \quad d(\Gamma_{\mathcal{W}_m}, \Gamma_{\mathcal{W}}) \leq \frac{\delta}{2}.$$

For $m \geq m_2(\delta)$, given a point $M \in \mathcal{D}(\Gamma_{\mathcal{W}_m}, \frac{\delta}{2})$, we then have that

$$d(M, \Gamma_{\mathcal{W}}) \leq d(M, \Gamma_{\mathcal{W}_m}) + d(\Gamma_{\mathcal{W}_m}, \Gamma_{\mathcal{W}}) \leq \delta,$$

from which we then deduce the expected result, i.e.,

$$\forall m \geq m_2(\delta) : \quad \mathcal{D}(\Gamma_{\mathcal{W}_m}, \frac{\delta}{2}) \subset \mathcal{D}(\Gamma_{\mathcal{W}}, \delta).$$

iii. In order to conclude, we simply choose $m_0(\delta) = \max\{m_1(\delta), m_2(\delta)\}$. □

Remark 3.1. Note that, in Property 3.2, we have that

$$m_0 = m_0(\delta) \rightarrow \infty \quad , \quad \text{as } \delta \rightarrow 0^+.$$

Our results on fractal cohomology obtained in [DL22b] have highlighted the role played by specific threshold values for the number $\varepsilon > 0$ at any step $m \in \mathbb{N}$ of the prefractal graph approximation; namely, the m^{th} *cohomology infinitesimal* introduced in Definition 3.1 just below.

Definition 3.1 (m^{th} **Cohomology Infinitesimal and Infinitesimal Sequence**).

From now on, given any $m \in \mathbb{N}$, we will call m^{th} *cohomology infinitesimal* the number $\varepsilon^m > 0$ which, modulo a multiplicative constant equal to $\frac{1}{N_b - 1}$ (recall that $N_b > 1$), stands as the elementary horizontal length introduced in part *i.* of Definition 2.4, i.e.,

$$\frac{1}{N_b^m}.$$

Observe that, clearly, ε itself – and not just ε^m – depends on m ; hence, we should really write $\varepsilon^m = (\varepsilon_m)^m$, for all $m \in \mathbb{N}$.

Also, ε^m satisfies the following asymptotic behavior,

$$\varepsilon^m \sim \frac{1}{N_b^m}, \text{ as } m \rightarrow \infty,$$

which, naturally, results in the fact that the larger m , the smaller ε^m . It is for this reason that we call ε^m – or rather, the *infinitesimal sequence* $(\varepsilon^m)_{m=0}^\infty$ of positive numbers tending to zero as $m \rightarrow \infty$, with $\varepsilon^m = (\varepsilon_m)^m$, for each $m \in \mathbb{N}$ – an *infinitesimal*. Note that this m^{th} cohomology infinitesimal is the one naturally associated to the scaling relation of Property 2.4 above from [DL22a].

In the sequel, it is also useful to keep in mind that the sequence of positive numbers $(\varepsilon_m)_{m=0}^\infty$ itself satisfies

$$\varepsilon_m \sim \frac{1}{N_b}, \text{ as } m \rightarrow \infty ;$$

i.e., $\varepsilon_m \rightarrow \frac{1}{N_b}$, as $m \rightarrow \infty$. In particular, $\varepsilon_m \not\rightarrow 0$, as $m \rightarrow \infty$, but, instead, ε_m decreases and tends to a strictly positive and finite limit.

In the sequel, an *infinitesimal* ϵ will refer to any sequence $(\epsilon_m)_{m \in \mathbb{N}}$ such that, for any $m \in \mathbb{N}$,

$$0 < \epsilon_m \leq \frac{1}{(N_b - 1) N_b^m} \cdot \left(\text{♯} \right)$$

Property 3.3 (Staggered Sequence of (m, ε^m) -Neighborhoods).

Given a natural integer $m \in \mathbb{N}$, there exists an integer $k_m \in \mathbb{N}$ such that, for each integer $k \geq k_m$, the $(m+k, \varepsilon^{m+k})$ -Neighborhood of the Weierstrass Curve (where ε^{m+k} is the $(m+k)^{\text{th}}$ cohomology infinitesimal introduced in Definition 3.1),

$$\mathcal{D}(\Gamma_{\mathcal{W}_{m+k}}, \varepsilon^{m+k}) = \{M = (x, y) \in \mathbb{R}^2, d(M, \Gamma_{\mathcal{W}_{m+k}}) \leq \varepsilon^{m+k}\},$$

is contained in the (m, ε^m) -Neighborhood of the Weierstrass Curve (with ε^m denoting, this time, the m^{th} cohomology infinitesimal),

$$\mathcal{D}(\Gamma_{\mathcal{W}_m}, \varepsilon^m) = \{M = (x, y) \in \mathbb{R}^2, d(M, \Gamma_{\mathcal{W}_m}) \leq \varepsilon^m\};$$

namely,

$$\mathcal{D}(\Gamma_{\mathcal{W}_{m+k}}, \varepsilon^{m+k}) \subset \mathcal{D}(\Gamma_{\mathcal{W}_m}, \varepsilon^m).$$

Proof. This proof is based on the fact that the sequence of sets of vertices $(V_m)_{m \in \mathbb{N}}$ is increasing (see part *i.* of Property 2.3), and that $V^\star = \bigcup_{n \in \mathbb{N}} V_n$ is dense in the Weierstrass Curve $\Gamma_{\mathcal{W}}$, along with the fact that the prefractal graph sequence $(\Gamma_{\mathcal{W}_m})_{m \in \mathbb{N}}$ converges to the Weierstrass Curve $\Gamma_{\mathcal{W}}$ (for example, in the sense of the Hausdorff metric on \mathbb{R}^2).

Given a natural integer $m \in \mathbb{N}$, there exists an integer $k_{0,m} \in \mathbb{N}$ such that, for each integer $k \geq k_{0,m}$, we have that

$$d(\Gamma_{\mathcal{W}_m}, \Gamma_{\mathcal{W}_{m+k}}) = \inf_{\substack{0 \leq j \leq \#V_m - 1 \\ 0 \leq j' \leq \#V_{m+k} - 1}} \{d(M_{j,m}, M_{j',m+k}), M_{j,m} \in V_m, M_{j',m+k} \in V_{m+k} \setminus V_m\} \leq \varepsilon^m.$$

We then deduce that for all $k \geq k_{0,m}$,

$$\Gamma_{\mathcal{W}_{m+k}} \subset \mathcal{D}(\Gamma_{\mathcal{W}_m}, \varepsilon^m).$$

At the same time, since, for any $(m, k) \in \mathbb{N}^2$,

$$\varepsilon^{m+k} \leq \varepsilon^m,$$

along with the fact that, for any $m \in \mathbb{N}$,

$$\lim_{k \rightarrow \infty} \varepsilon^{m+k} = 0,$$

we can find another integer $k_{1,m} \in \mathbb{N}$ such that, for each integer $k \geq k_{1,m}$, we have that

$$\mathcal{D}(\Gamma_{\mathcal{W}_{m+k}}, \varepsilon^{m+k}) \subset \mathcal{D}(\Gamma_{\mathcal{W}_m}, \varepsilon^m).$$

The desired result is obtained by letting $k_m = \max\{k_{0,m}, k_{1,m}\}$. □

Remark 3.2 (Connection Between Fractality and the Cohomology Infinitesimal – Weierstrass Iterated Fractal Drums).

As is mentioned in [DL22b], the cohomology infinitesimal (or, equivalently, the elementary length) – which obviously depends on the magnification scale (i.e., the chosen prefractal approximation) – can be seen as a *transition scale* between the fractal domain and the classical one. In fact, we could say that the system is fractal below this scale, and classical above (for the level of magnification considered). In the limit when the integer m associated with the prefractal approximation tends to infinity, the system is fractal below the cohomology infinitesimal (which is really an infinitesimal, in this case), i.e., at small scales, and is classical beyond, i.e., on a large scale. Note that this is in perfect agreement with what is evoked by the French physicist Laurent Nottale in [Not98] about scale-relativity.

The Complex Dimensions of a fractal set characterize their intrinsic vibrational properties. Thus far, the values of the Complex Dimensions were obtained by studying the oscillations of a small neighborhood of the boundary, i.e., of a tubular neighborhood, where points are located within an epsilon distance from any edge; see, e.g., [LRŽ17a], [LRŽ17b], [LRŽ18]. In the case of our fractal Weierstrass Curve $\Gamma_{\mathcal{W}}$, which is, also, the limit of the sequence of (polygonal) prefractal graphs $(\Gamma_{\mathcal{W}})_{m \in \mathbb{N}}$, it is natural – and consistent with the result of Property 3.3 above – to envision the tubular neighborhood of $\Gamma_{\mathcal{W}}$ as the limit of the (obviously convergent) sequence $(\mathcal{D}(\Gamma_{\mathcal{W}_m}, \varepsilon^m))_{m \in \mathbb{N}}$ of ε^m -neighborhoods of $\Gamma_{\mathcal{W}_m}$, where, for each integer $m \in \mathbb{N}$, ε^m is the m^{th} cohomology infinitesimal introduced in Definition 3.1 above.

We can note that, in a sense, this amounts to using a sequence of what we call *Weierstrass Iterated Fractal Drums* (in short, Weierstrass IFDs), by analogy with the *relative fractal drums* involved, for instance, in the case of the Cantor Staircase, in [LRŽ17b], Section 5.5.4, and in [LRŽ18]. In our present setting, the Weierstrass IFDs – i.e., the sets $\mathcal{D}(\Gamma_{\mathcal{W}_m}, \varepsilon^m)$, for $m \in \mathbb{N}$ sufficiently large – contain the Weierstrass Curve $\Gamma_{\mathcal{W}}$, and are sufficiently close to $\Gamma_{\mathcal{W}}$, so that we can expect their Complex

Dimensions to be the same.

Things change if we consider the fractal Complex Dimensions as dynamical quantities, which evolve with the scales: to each prefractal approximation $\Gamma_{\mathscr{W}_m}$ of $\Gamma_{\mathscr{W}}$, we can associate specific values of the Complex Dimensions, as is proved in [DL22b]. In this light, it is natural to define the Complex Dimensions of the Weierstrass Curve as the set of the Complex Dimensions of $(\Gamma_{\mathscr{W}_m})_{m \in \mathbb{N}}$, obtained by means of the sequence of IFDs $(\mathscr{D}(\Gamma_{\mathscr{W}_m}, \varepsilon^m))_{m \in \mathbb{N}}$, as is done in Section 4.

4 Weierstrass Tube Zeta Function, and Associated Complex Dimensions

In this section, we discuss the results of [DL22a] in which we obtained the fractal tube formulas associated with the Weierstrass iterated fractal drums (Weierstrass IFDs, for short), and deduced from it the fractal (i.e., the tube and distance) zeta functions and hence, also, the associated Complex Dimensions (i.e., the poles of these fractal zeta functions). We also discuss the geometric consequences of these results for the Weierstrass IFDs and Curve – concerning, especially, the Minkowski nondegeneracy, the Minkowski measurability, the existence of the average Minkowski content, and the value of the Minkowski dimension (which is positive and is also the largest real part of the Complex Dimensions).

We first need the following useful, technical result.

Property 4.1 (Fourier Series Expansion of the 1-Periodic Map $x \mapsto N_b^{-\{x\}}$ [LvF06], [DL22a]).

The fractional part map $\{\cdot\}$ is one-periodic. Hence, it is also the case of the map $x \mapsto N_b^{-\{x\}}$, which admits, with respect to the real variable x , the following Fourier Series expansion:

$$N_b^{-\{x\}} = \frac{N_b - 1}{N_b} \sum_{m \in \mathbb{Z}} \frac{e^{2i\pi m x}}{\ln N_b + 2i m \pi} = \frac{N_b - 1}{N_b} \sum_{m \in \mathbb{Z}} \frac{(N_b - 1)^{-i m \mathbf{p}} \epsilon_m^{-i m \mathbf{p}}}{\ln N_b + 2i m \pi},$$

where the exponential Fourier coefficients c_m have been obtained through

$$\begin{aligned} c_m &= \int_0^1 N_b^{-t} e^{-2i\pi m t} dt = \int_0^1 e^{-t \ln N_b} e^{-2i\pi m t} dt = -\frac{1}{\ln N_b + 2i m \pi} \left[e^{-t \ln N_b} e^{-2i\pi m t} \right]_0^1 \\ &= \frac{1}{\ln N_b + 2i m \pi} \left[1 - \frac{1}{N_b} \right] = \frac{N_b - 1}{N_b} \frac{1}{\ln N_b + 2i m \pi}. \end{aligned}$$

Thus, for any $x \in \mathbb{R}$ and any $\epsilon_m > 0$,

$$N_b^{-\{x\}} = \frac{N_b - 1}{N_b} \sum_{m \in \mathbb{Z}} \frac{e^{2i\pi m x}}{\ln N_b + 2i m \pi}.$$

Since $x = -\ln_{N_b}((N_b - 1)\epsilon_m)$, one has, for every $m \in \mathbb{Z}$,

$$e^{2i\pi m x} = e^{-2i\pi m \ln_{N_b}((N_b - 1)\epsilon_m)} = e^{-2i\pi m \frac{\ln((N_b - 1)\epsilon_m)}{\ln N_b}}.$$

Definition 4.1 (Oscillatory Period).

Following [LvF00], [LvF06], [LvF13], [LRŽ17b], we introduce the *oscillatory period* of the Weierstrass Curve:

$$\mathbf{p} = \frac{2\pi}{\ln N_b}.$$

Theorem 4.2 (Fractal Tube Formula for The Weierstrass Relative Fractal Drums [DL22a]).

Given an arbitrary integer $m \in \mathbb{N}$, and $\epsilon_m > 0$ sufficiently small (i.e., $0 < \epsilon_m \leq \epsilon^m$, where ϵ^m is the m^{th} cohomology infinitesimal introduced in Definition 3.1), the tubular volume $\mathcal{V}_m(\epsilon_m)$, or two-dimensional Lebesgue measure of the tubular neighborhood $\mathcal{D}(\Gamma_{\mathcal{W}_m}, \epsilon_m)$, is obtained by means of the respective contributions coming from rectangles, wedges, and triangles. It is given, for all $m \in \mathbb{N}$ sufficiently large, by the following fractal tube formula:

$$\begin{aligned} \mathcal{V}_m(\epsilon_m) = & \sum_{\ell \in \mathbb{Z}, k \in \mathbb{N}} f_{\ell, k, \text{Rectangles}} \epsilon_m^{2-D_{\mathcal{W}} + k(2-D_{\mathcal{W}}) - i\ell \mathbf{p}} \\ & + \sum_{\ell \in \mathbb{Z}, k \in \mathbb{N}} \left\{ f_{\ell, \text{wedges}, 1} \epsilon_m^{3-i\ell \mathbf{p}} + f_{\ell, k, \text{wedges}, 2} \epsilon_m^{1+2k-i\ell \mathbf{p}} + f_{\ell, k, \text{wedges}, 3} \epsilon_m^{5+2k-i\ell \mathbf{p}} \right\} \\ & + \sum_{\ell \in \mathbb{Z}, k \in \mathbb{N}} f_{\ell, k, \text{triangles, parallelograms}} \epsilon_m^{2-i\ell \mathbf{p}} + \pi \epsilon_m^2 - \frac{\pi \epsilon_m^4}{2}, \end{aligned} \quad \left(\text{🎵} \right)$$

where the notation $f_{\ell, k, \text{Rectangles}}$, $f_{\ell, k, \text{wedges}, j}$, $1 \leq j \leq 3$, and $f_{\ell, k, \text{triangles, parallelograms}}$, respectively account for the (bounded) coefficients associated to the sums corresponding to the contribution of the rectangles, wedges, triangles and parallelograms. Naturally, these coefficients depend on m as well, but for notational simplicity, we do not indicate it explicitly.

Note that each of the sums over the integers k and ℓ giving the tubular formula $\left(\text{🎵} \right)$ just above were, first, and as could be expected, sums taken over an integer k_m varying from 0 to m ; however, later manipulations involving asymptotic expansions of the elementary lengths yield infinite sums, as, for instance, in the case of a -strings (see, e.g., [LvF13], Subsection 6.5.1, Theorem 6.21).

Moreover, we have that the following limit exists and satisfies the following inequalities – with the obvious modifications and clearly associated to the entire sequence of Weierstrass IFDs, or, equivalently, to the sequence of prefractal polygonal approximations to the Weierstrass Curve $\Gamma_{\mathcal{W}}$:

$$0 < \underbrace{\lim_{m \rightarrow \infty} \mathcal{V}_m(\epsilon_m)}_{\mathcal{V}(\epsilon)} < \infty.$$

By necessity of concision, we do not repeat here the corresponding fractal tube formula for $\mathcal{V}(\epsilon)$, (with $\epsilon = (\epsilon_m)_{m \in \mathbb{N}}$ denoting the corresponding infinitesimal), which is simply obtained by taking the limit as $m \rightarrow \infty$ of the expression appearing on the right-hand side of $\left(\text{🎵} \right)$ above – and associated

to the entire sequence of Weierstrass IFDs, or, equivalently, to the sequence of prefractal polygonal approximations to the Weierstrass Curve $\Gamma_{\mathcal{W}}$.

Remark 4.1. Following (as well as adapting) [LRŽ17b], we define the tube zeta function of the sequence of Weierstrass IFDs associated to the cohomology infinitesimal (or, more generally, to any infinitesimal $\epsilon = (\epsilon_m)_{m \in \mathbb{N}}$, by

$$\tilde{\zeta}_{\mathcal{W}_m}(s) = \int_0^{\epsilon_m} t^{s-3} \mathcal{V}_m(t) dt = \int_0^{\epsilon_m} t^{s-2} \mathcal{V}_m(t) \frac{dt}{t},$$

for all s in \mathbb{C} , with $\Re(s)$ sufficiently large, and, optimally for $\Re(s) > D_{\mathcal{W}_m} = D_{\mathcal{W}}$.

Theorem 4.3 (Local and Global Tube Zeta Function for the Weierstrass Iterated Fractal Drums [DL22a]).

The global tube zeta function associated to the Weierstrass IFDs, $\tilde{\zeta}_{\mathcal{W}}$, defined by analogy with the work in [LRŽ17b], admits a meromorphic continuation to all of \mathbb{C} , and is given, for any complex number s , by the following expression:

$$\tilde{\zeta}_{\mathcal{W}}(s) = \lim_{m \rightarrow \infty} \tilde{\zeta}_{\mathcal{W}_m}(s),$$

where, for all $m \in \mathbb{N}$ sufficiently large, the local tube zeta function $\tilde{\zeta}_{\mathcal{W}_m}$ is given, for any complex number s , by

$$\begin{aligned} \tilde{\zeta}_{\mathcal{W}_m}(s) = & \sum_{\ell \in \mathbb{Z}, k \in \mathbb{N}} f_{\ell, k, \text{Rectangles}} \frac{\epsilon_m^{s - D_{\mathcal{W}} + k(2 - D_{\mathcal{W}}) - i\ell \mathbf{p}}}{s - D_{\mathcal{W}} + k(2 - D_{\mathcal{W}}) - i\ell \mathbf{p}} \\ & + \sum_{\ell \in \mathbb{Z}, k \in \mathbb{N}} \left\{ f_{\ell, k, \text{wedges}, 1} \frac{\epsilon_m^{s+1-i\ell \mathbf{p}}}{s+1-i\ell \mathbf{p}} + f_{\ell, k, \text{wedges}, 2} \frac{\epsilon_m^{s+2k-1-i\ell \mathbf{p}}}{s+2k-1-i\ell \mathbf{p}} + f_{\ell, k, \text{wedges}, 3} \frac{\epsilon_m^{s+3+2k-i\ell \mathbf{p}}}{s+3+2k-i\ell \mathbf{p}} \right\} \\ & + \sum_{\ell \in \mathbb{Z}, k \in \mathbb{N}} f_{\ell, k, \text{triangles, parallelograms}} \frac{\epsilon_m^{s-1-i\ell \mathbf{p}}}{s-1-i\ell \mathbf{p}} + \frac{\pi \epsilon_m^s}{s} - \frac{\pi \epsilon_m^{s+2}}{4(s+2)}, \end{aligned}$$

where, as already introduced in Theorem 4.2, the notation $f_{\ell, k, \text{Rectangles}}$, $f_{\ell, k, \text{wedges}, j}$, $1 \leq j \leq 3$, and $f_{\ell, k, \text{triangles, parallelograms}}$, respectively, depend on m , are uniformly bounded and account for the coefficients associated to the sums corresponding to the contribution of the rectangles, wedges, triangles and parallelograms.

Remark 4.2. The fact that the global tube zeta function $\tilde{\zeta}_{\mathcal{W}}$ admits a meromorphic continuation to all of \mathbb{C} is obtained by applying the Weierstrass theorem for uniformly convergent sequences of holomorphic functions. First, we can note that, for all sufficiently large $m \in \mathbb{N}$, the set \mathcal{L} of possible poles of the local tube zeta function $\tilde{\zeta}_{\mathcal{W}_m}(s)$ (discrete, and thus closed) does not depend on m , and is given by Theorem 4.5 below. The local tube zeta function $\tilde{\zeta}_{\mathcal{W}_m}$ is then holomorphic on the connected open subset of \mathbb{C} given by $\mathbb{C} \setminus \mathcal{L}$. We can clearly see that the sequence of functions $(\tilde{\zeta}_{\mathcal{W}_m})_{m \in \mathbb{N}}$ converges normally (and hence, uniformly) in a connected open (and relatively compact) neighborhood of any given $\omega \in \mathbb{C} - \text{i.e., for } s = x + iy \in \mathbb{C} \text{ close to } \omega$. The Weierstrass theorem, applied once again, then ensures the holomorphicity of the limit $\tilde{\zeta}_{\mathcal{W}}$ on the domain $\mathbb{C} \setminus \mathcal{L}$. It follows that the global tube zeta function $\tilde{\zeta}_{\mathcal{W}}$ is meromorphic in all of \mathbb{C} , with possible set of poles given by \mathcal{L} .

Corollary 4.4 ((of Theorem 4.3) Local and Global Distance Zeta Function for the Weierstrass Iterated Fractal Drums).

According to the functional equation given in [LRŽ17b] (Theorem 2.2.1., page 112), along with Theorem 4.3 just above, given an infinitesimal ϵ (see Definition 3.1), the global distance zeta function $\zeta_{\mathcal{W}}$ is given, for any complex number s , by the following expression:

$$\zeta_{\mathcal{W}}(s) = \lim_{m \rightarrow \infty} \zeta_{\mathcal{W}_m}(s),$$

where, for all $m \in \mathbb{N}$ sufficiently large, the local distance zeta function $\zeta_{\mathcal{W}_m}$ is given, for any complex number s , by

$$\begin{aligned} \zeta_{\mathcal{W}_m}(s) &= \int_{M \in \mathcal{D}(\epsilon_m)} d(M, \Gamma_{\mathcal{W}_m})^{s-2} dM \\ &= \epsilon_m^{s-2} \mathcal{V}_m(\epsilon_m) + (2-s) \int_0^{\epsilon_m} t^{s-3} \mathcal{V}_m(t) dt \\ &= \epsilon_m^{s-2} \mathcal{V}_m(\epsilon_m) + (2-s) \tilde{\zeta}_{\mathcal{W}_m}(s), \end{aligned}$$

where \mathcal{V}_m denotes the tubular volume obtained in Theorem 4.2, and where $\tilde{\zeta}_{\mathcal{W}_m}(s)$ is given in Theorem 4.3. The first two equalities are valid for $\Re(s) > D_{\mathcal{W}_m} = D_{\mathcal{W}}$, while the last one is valid for all s in \mathbb{C} . Furthermore, still for all $m \in \mathbb{N}$ sufficiently large, the distance zeta function $\zeta_{\mathcal{W}_m}$ admits a meromorphic continuation to all of \mathbb{C} , given by the last equality just above, with $\tilde{\zeta}_{\mathcal{W}_m}$ given as in Theorem 4.3.

We recall from [LRŽ17b] that the Complex Dimensions are defined as the poles of the meromorphic continuation of the tube (or, equivalently, the distance) zeta function. In our present setting, the set of Complex Dimensions of the Weierstrass Curve is the set of Complex Dimensions of the sequence of Weierstrass IFDs introduced in Remark 3.2. Hence, those Complex Dimensions are the poles of the tube zeta function – or, equivalently, the distance zeta function – associated to those IFDs, respectively obtained in Theorem 4.3 and Corollary 4.4 above.

Theorem 4.5 (Complex Dimensions of the Weierstrass Curve [DL22a]).

The possible complex dimensions of the Weierstrass Curve are all simple, and given as follows:

$$D_{\mathcal{W}} - k(2 - D_{\mathcal{W}}) + i\ell \mathbf{p} \quad , \quad \text{with } k \in \mathbb{N}, \ell \in \mathbb{Z},$$

$$1 - 2k + i\ell \mathbf{p} \quad , \quad \text{with } k \in \mathbb{N}, \ell \in \mathbb{Z}, \text{ along with } 0 \text{ and } -2.$$

Furthermore, the one-periodic functions (with respect to the variable $\ln_{N_b} \epsilon_m^{-1}$, see Property 4.1 above, and Equation (★) in the introduction), respectively associated to the values $D_{\mathcal{W}} - k(2 - D_{\mathcal{W}})$, $k \in \mathbb{N}$, are nonconstant. See also [DL22a] for the exceptional cases, depending on the values of the parameters of \mathcal{W} .

In addition, all of the Fourier coefficients of those periodic functions are nonzero, which implies that there are infinitely many Complex Dimensions that are nonreal, including all of those with maximal real part $D_{\mathcal{W}}$, which are the principal Complex Dimensions, in the terminology of [LRŽ17b], and therefore give rise to geometric oscillations (or vibrations) with the largest amplitude, in the fractal tube formula obtained in Theorem 4.2 above.

Finally, for each $k \in \mathbb{N}$ and $\ell \in \mathbb{Z}$, $D_{\mathcal{W}} - k(2 - D_{\mathcal{W}}) + i\ell\mathbf{p}$, 1 and 0 are all simple and actual Complex Dimensions of the Weierstrass Curve; i.e., they are simple poles of the tube (or, equivalently, of the distance) zeta function.

Consequently, the Weierstrass Curve is fractal, in the sense of the theory of Complex Dimensions developed in [LvF00], [LvF06], [LvF13], [LRŽ17b] and [Lap19], since it admits nonreal Complex Dimensions.

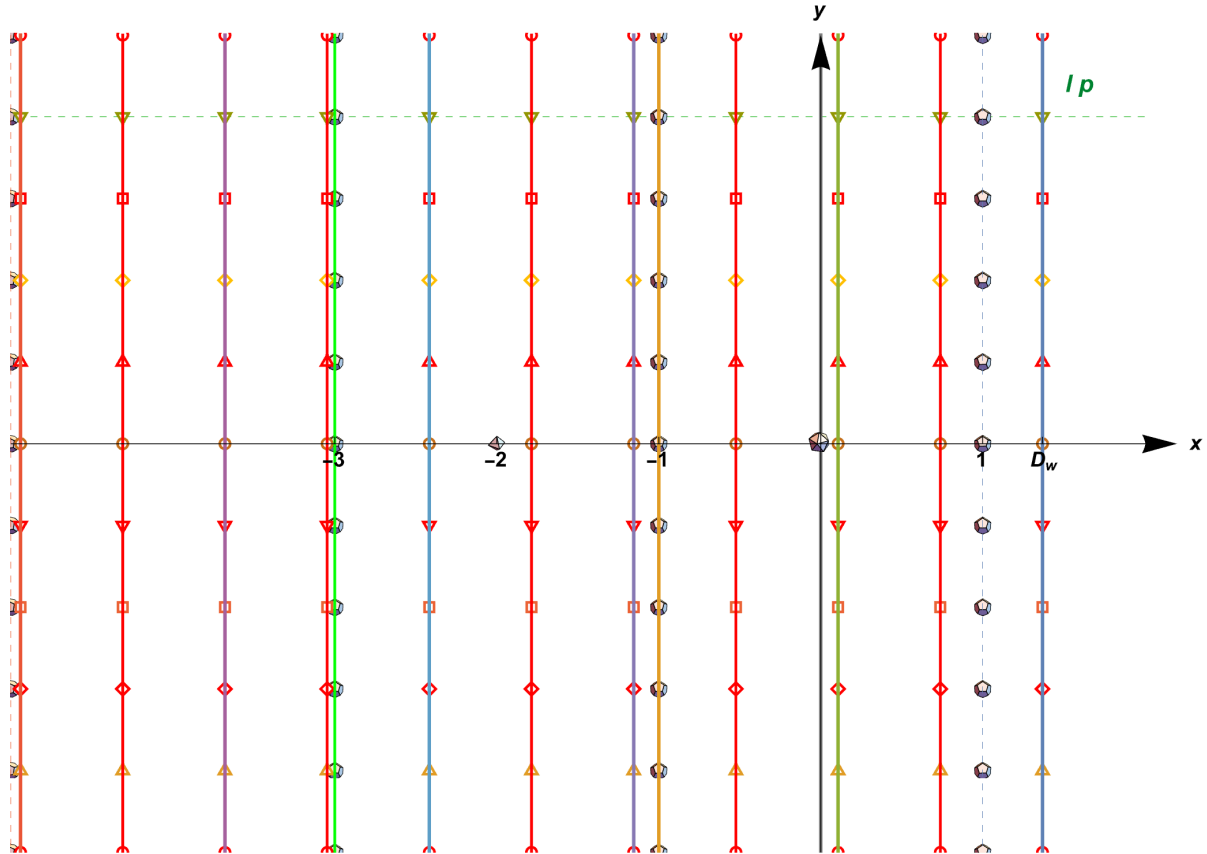


Figure 6: **The Complex Dimensions of the Weierstrass Curve.** The nonzero Complex Dimensions are periodically distributed (with the same period $\mathbf{p} = \frac{2\pi}{\ln N_b}$, the oscillatory period of $\Gamma_{\mathcal{W}}$) along countably many vertical lines, with abscissae $D_{\mathcal{W}} - k(2 - D_{\mathcal{W}})$ and $1 - 2k$, where $k \in \mathbb{N}$ is arbitrary. In addition, 0 and -2 are Complex Dimensions of $\Gamma_{\mathcal{W}}$.

Remark 4.3 (Possible Interpretation [DL22a]).

Figure 6 gives the *distribution of the possible Complex Dimensions* of the Weierstrass IFD – and hence also, in practise, of the Weierstrass Curve itself. We note that the vertical lines $x = 0$ and $x = 1$, respectively, correspond to oscillations coming from *points* and *lines*, which can be understood in an intuitive way, since, for any natural integer m , the prefractal graph $\Gamma_{\mathcal{W}_m}$ is constituted of points, connected between themselves by lines (or, rather, line segments).

Definition 4.2 (Lower and Upper r -Dimensional Minkowski Contents).

Given a bounded set A of \mathbb{R}^2 , and $\delta > 0$ sufficiently small, let us denote by $\mathcal{V}_A(\delta)$ the Lebesgue measure of the δ -neighborhood of A , $\mathcal{D}(A, \delta)$, defined as

$$\mathcal{D}(A, \delta) = \{M = (x, y) \in \mathbb{R}^2, d(M, A) \leq \delta\}.$$

As in [LRŽ17b], we define, for any nonnegative real number r :

i. The lower r -dimensional Minkowski content of the set A as

$$\mathcal{M}_\star^r(A) = \liminf_{\delta \rightarrow 0} \frac{\mathcal{V}_A(\delta)}{\delta^{2-r}}.$$

ii. The upper r -dimensional Minkowski content of the the set A as

$$\mathcal{M}^{\star,r}(A) = \limsup_{\delta \rightarrow 0} \frac{\mathcal{V}_A(\delta)}{\delta^{2-r}}.$$

Definition 4.3 (Minkowski Dimension).

If the lower Minkowski (or box) dimension of a bounded set A of \mathbb{R}^2 ,

$$\inf \{r \in \mathbb{R} : \mathcal{M}_\star^r(A) = 0\}$$

and its upper Minkowski (or box) dimension,

$$\inf \{r \in \mathbb{R} : \mathcal{M}^{\star,r}(A) = 0\}$$

coincide, then their common value, denoted by D_A , is the *Minkowski (or box) dimension* of the set A (which is said to exist in this case).

Definition 4.4 (Minkowski Nondegeneracy and Minkowski Measurability).

Given a bounded set A of \mathbb{R}^2 , and $\delta > 0$ sufficiently small, let us denote by $\mathcal{V}_A(\delta)$ the Lebesgue measure of the δ -neighborhood of A , $\mathcal{D}(A, \delta)$, defined as

$$\mathcal{D}(A, \delta) = \{M = (x, y) \in \mathbb{R}^2, d(M, A) \leq \delta\}.$$

As defined in [LRŽ17b], the set A of \mathbb{R}^2 is said to be *Minkowski nondegenerate*, if its lower and upper Minkowski contents, $\mathcal{M}_\star^{D_A}(A) = \liminf_{\delta \rightarrow 0^+} \frac{\mathcal{V}_A(\delta)}{\delta^{2-D_A}}$ and $\mathcal{M}^{\star D_A}(A) = \limsup_{\delta \rightarrow 0^+} \frac{\mathcal{V}_A(\delta)}{\delta^{2-D_A}}$ (where D_A has

been introduced in Definition 4.3), are respectively positive and finite. We then denote by $\mathcal{M}^{D_A}(A)$ their common value. Note that it follows from the assumption of the Minkowski nondegeneracy of A that D_A , the Minkowski dimension of A , exists.

Finally, the set A is said to be *Minkowski measurable* if it is Minkowski nondegenerate and

$$\mathcal{M}_\star^{D_A}(A) = \mathcal{M}^{\star, D_A}(A);$$

i.e., if the following limit exists in $]0, +\infty[$ (and necessarily equals this common value):

$$\mathcal{M}^{D_A}(A) = \lim_{\delta \rightarrow 0^+} \frac{\mathcal{V}_A(\delta)}{\delta^{2-D_A}}.$$

Then, $\mathcal{M}^{D_A}(A)$ is called the *Minkowski content* of A .

Definition 4.5 (Average Lower and Upper Minkowski Contents [DL22a]).

We hereafter use the same notation as in Definition 4.4 just above, where A denotes a bounded set \mathbb{R}^2 , and $\delta > 0$ a fixed number. Then, as can be found in [LRŽ17b], Definition 2.4.1, page 178, we define the *average lower-dimensional Minkowski content* (resp., *average upper-dimensional Minkowski content*) of A as

$$\widetilde{\mathcal{M}}_\star^{D_A}(A) = \liminf_{r \rightarrow +\infty} \frac{1}{\ln r} \int_{\frac{1}{r}}^\delta t^{D_A-3} \mathcal{V}_A(t) dt \quad \left(\text{resp., } \widetilde{\mathcal{M}}^{\star, D_A}(A) = \limsup_{r \rightarrow +\infty} \frac{1}{\ln r} \int_{\frac{1}{r}}^\delta t^{D_A-3} \mathcal{V}_A(t) dt \right).$$

In the case when both of these values coincide, their common value, denoted by $\widetilde{\mathcal{M}}^{D_A}(A)$, is called the *average Minkowski content* of A , which is then said to *exist*. Accordingly,

$$\widetilde{\mathcal{M}}^{D_A}(A) = \lim_{r \rightarrow +\infty} \frac{1}{\ln r} \int_{\frac{1}{r}}^\delta t^{D_A-3} \mathcal{V}_A(t) dt.$$

Without loss of generality, we may choose $\delta = 1$ in the present definition. Indeed, the value of $\widetilde{\mathcal{M}}^{D_A}(A)$ is independent of the choice of $\delta > 0$.

Remark 4.4. Henceforth, in the case of the Weierstrass IFD associated with the sequence of polygonal prefractal approximation $(\Gamma_{\mathcal{W}_m})_{m \in \mathbb{N}}$ to $\Gamma_{\mathcal{W}}$, we use exactly the same terminology and definitions (adapted in the obvious way) as in Definitions 4.2–4.5 above.

Furthermore, we denote by $D_{\mathcal{W}}$, $\mathcal{M}_\star^{D_{\mathcal{W}}}(\Gamma_{\mathcal{W}})$, $\mathcal{M}^{\star, D_{\mathcal{W}}}(\Gamma_{\mathcal{W}})$ and $\widetilde{\mathcal{M}}^{D_{\mathcal{W}}}(\Gamma_{\mathcal{W}})$ the associated Minkowski dimension, lower and upper Minkowski contents, as well as the average Minkowski content, respectively, of the Weierstrass IFD.

Note that $D_{\mathcal{W}}$ also coincides with the Minkowski dimension of the Weierstrass Curve $\Gamma_{\mathcal{W}}$.

In some definite sense, the Weierstrass IFD (iterated fractal drum) is a natural geometric realization of – and substitute for – the Weierstrass Curve $\Gamma_{\mathcal{W}}$.

We can now state several new geometric consequences of our above results, especially, Theorems 4.2.

Theorem 4.6 (Lower, Upper and Average $D_{\mathcal{W}}$ -Dimensional Minkowski Contents [DL22a]).

The lower and upper $D_{\mathcal{W}}$ -dimensional Minkowski contents, $\mathcal{M}_{\star}^{D_{\mathcal{W}}}(\Gamma_{\mathcal{W}})$ and $\mathcal{M}^{\star, D_{\mathcal{W}}}(\Gamma_{\mathcal{W}})$, of the Weierstrass Curve, obtained by means of the Weierstrass IFDs introduced in Remark 3.2, take strictly positive and finite values, and are such that

$$\frac{C_{\text{Rectangles}}}{N_b} \leq \mathcal{M}_{\star}^{D_{\mathcal{W}}}(\Gamma_{\mathcal{W}}) < \mathcal{M}^{\star, D_{\mathcal{W}}}(\Gamma_{\mathcal{W}}) \leq C_{\text{Rectangles}},$$

where $C_{\text{Rectangles}}$ denotes a strictly positive and finite constant (see [DL22a]).

Moreover, the values of $\mathcal{M}_{\star}^{D_{\mathcal{W}}}(\Gamma_{\mathcal{W}})$ and $\mathcal{M}^{\star, D_{\mathcal{W}}}(\Gamma_{\mathcal{W}})$ are respectively equal to the minimum and maximum value of a one-periodic function $G_{D_{\mathcal{W}}} = G_{0, D_{\mathcal{W}}}$ introduced in [DL22a], associated to $D_{\mathcal{W}}$ in the expression of the fractal tube formula given in the same theorem (the periodicity is with respect to the variable $\ln_{N_b} \delta^{-1}$, see Property 4.1). The function $G_{D_{\mathcal{W}}}$ is also nonconstant and bounded away from zero as well as from infinity.

Finally, the average Minkowski content (which is independent of the choice of $\delta > 0$; see Definition 4.5) exists and is given by the mean value of the one-periodic function $G_{D_{\mathcal{W}}}$, as well as by the residues of $\zeta_{\mathcal{W}}$ at $s = D_{\mathcal{W}}$:

$$\widetilde{\mathcal{M}}^{D_{\mathcal{W}}}(\Gamma_{\mathcal{W}}) = \int_0^1 G_{D_{\mathcal{W}}}(x) dx = \text{res}(\zeta_{\mathcal{W}}, D_{\mathcal{W}}) = \frac{\text{res}(\zeta_{\mathcal{W}}, D_{\mathcal{W}})}{2 - D_{\mathcal{W}}}. \quad (\clubsuit\clubsuit)$$

Hence, $\widetilde{\mathcal{M}}^{D_{\mathcal{W}}}$ is nontrivial; in fact, $0 < \mathcal{M}_{\star}^{D_{\mathcal{W}}}(\Gamma_{\mathcal{W}}) < \widetilde{\mathcal{M}}^{D_{\mathcal{W}}}(\Gamma_{\mathcal{W}}) < \mathcal{M}^{\star, D_{\mathcal{W}}}(\Gamma_{\mathcal{W}}) < \infty$.

Corollary 4.7 ((of Theorem 4.6) Minkowski Dimension – Minkowski Nondegeneracy).

The Weierstrass Curve $\Gamma_{\mathcal{W}}$ is Minkowski nondegenerate. Furthermore, the number $D_{\mathcal{W}} = 2 - \ln_{N_b} \frac{1}{\lambda}$ is a simple Complex Dimension of $\Gamma_{\mathcal{W}}$, and it coincides with the Minkowski Dimension of $\Gamma_{\mathcal{W}}$, which must also exist. Moreover, $\Gamma_{\mathcal{W}}$ is not Minkowski measurable. In addition, the same statement holds for the Weierstrass IFD.

Proposition 4.8 (Generalization to the Non-Integer Case [DL22a]).

An interesting question is the generalization of our previous results to the non-integer case, i.e., when the Weierstrass function \mathcal{W} is defined, for any real number x , by

$$\mathcal{W}(x) = \sum_{n=0}^{\infty} \lambda^n \cos(2\pi b^n x),$$

where the real number b does not belong to the set of natural integers.

We plan to provide the details in a later work, but for now limit ourselves to a few comments.

From the geometric point of view, one cannot handle things in the same way. For instance, one cannot resort to a finite IFS, and the Weierstrass function, apart from its parity, has no periodicity property.

Yet, since the associated graph is the attractor of the infinite set of maps $(T_i)_{i \in \mathbb{Z}}$ such that, for any integer i and (x, y) in \mathbb{R}^2 ,

$$T_i(x, y) = \left(\frac{x+i}{b}, \lambda y + \cos \left(2\pi \left(\frac{x+i}{b} \right) \right) \right),$$

it is natural to consider the associated infinite IFS (IIFS). As a consequence, the resulting prefractal graphs are infinite ones.

As for the tubular neighborhood, due to the polygonal approximation induced by the prefractals, it is still obtained by means of rectangles and wedges.

In the integer case, extra terms coming from overlapping rectangles vanished, thanks to the symmetry with respect to the vertical line $x = \frac{1}{2}$. In the non-integer case, one simply replaces this symmetry with the one with respect to the vertical axis $x = 0$, thanks to the parity of \mathscr{W} .

In this light, it is expected to lead to a fractal tube formula of the same type as the one obtained in Theorem 4.2, where the powers of the small parameter ϵ are, respectively, and as previously,

$$\epsilon^{2-D_{\mathscr{W}}+k(2-D_{\mathscr{W}})-i\ell \mathbf{p}}, \quad \epsilon^{3-i\ell \mathbf{p}}, \quad \epsilon^{1+2k-i\ell \mathbf{p}}, \quad \epsilon^{5+2k-i\ell \mathbf{p}}, \quad \epsilon^{2-i\ell \mathbf{p}}, \quad \epsilon^2, \quad \epsilon^4,$$

which would yield the same results concerning the possible Complex Dimensions, and the upper and lower Minkowski contents.

As in the integer case, the terms involving $\epsilon^{2-D_{\mathscr{W}}+k(2-D_{\mathscr{W}})-i\ell \mathbf{p}}$ come from the contribution of the rectangles. The one-periodic functions (with respect to the variable $\ln_b \varepsilon^{-1}$ this time), respectively associated to the values $D_{\mathscr{W}} - k(2 - D_{\mathscr{W}})$, $k \in \mathbb{N}$, are thus nonconstant, with all of their Fourier coefficients being nonzero. Hence, as in Theorem 4.5, for each $k \in \mathbb{N}$ and $\ell \in \mathbb{Z}$, $D_{\mathscr{W}} - k(2 - D_{\mathscr{W}}) + i\ell \mathbf{p}$, are all simple Complex Dimensions of the Weierstrass Curve (or, of the Weierstrass IFD); i.e., they are simple poles of the tube (or, equivalently, of the distance) zeta function.

5 Fractal Cohomology

We next discuss the fractal cohomology of the Weierstrass Curve $\Gamma_{\mathscr{W}}$ – or rather, perhaps, of the Weierstrass function \mathscr{W} – building on the ideas, definitions and results of [DL21] – as well as providing a concrete geometric (and topological) realization of the number-theoretic and analytic work and conjectures about fractal cohomology in [LvF00], [LvF06], [Lap08], [LvF13], [CL17], [Lap19] and [Lap22].

Definition 5.1 ((m, p)-Fermion).

By analogy with particle physics, given a pair of positive integers (m, p) , we will call (m, p) -fermion on V_m , with values in \mathbb{C} , any antisymmetric map f from V_m^{p+1} to \mathbb{C} , where V_m^{p+1} denotes the $(p+1)^{th}$ fold Cartesian product space of V_m by itself. Note that these maps are not assumed to be multilinear.

A $(0, p)$ -fermion ($p \in \mathbb{N}^*$) on V_m (or a 0-fermion, in short) is simply a map f from V_m to \mathbb{C} . We adopt the convention according to which a 0-fermion on V_m is a 0-antisymmetric map on V_m .

In the sequel, we will denote by $\mathcal{F}^p(V_m, \mathbb{C})$ the \mathbb{C} -module (i.e., the complex vector space) of (m, p) -fermions on V_m , with values in \mathbb{C} , which makes it an abelian group with respect to the addition, with an external law from $\mathbb{C} \times \mathcal{F}^p(V_m, \mathbb{C})$ to $\mathcal{F}^p(V_m, \mathbb{C})$.

In order to understand how things go, one may look at the initial polygon \mathcal{P}_0 : this polygon has exactly N_b vertices, which means, in terms of potential topological invariants (under the form of a complex-valued function f defined on the whole set of vertices $V^* = \{M_{j,m}, j \in \#V_m, m \in \mathbb{N}\}$), a number which will – or not – be conserved when switching to $\Gamma_{\mathcal{H}_1}$.

This can be achieved by examining the following quantities (see Figure 7):

$$f(P_j) - f(M_{1,1}) + f(M_{2,1}) \cdots - f(P_{j+1}), \text{ with } 0 \leq j \leq N_b,$$

where the points P_j , for $0 \leq j \leq N_b - 1$, are the vertices of \mathcal{P}_0 .

One thus deals with alternate (i.e., antisymmetric) expressions with $N_b + 1$ terms.

When reaching the m^{th} prefractal graph, with $m \geq 1$, one will have to examine quantities of the following form (see Figure 8):

$$f(M_{i,m-1}) - f(M_{j,m}) + f(M_{j+1,m}) \cdots - f(M_{i+1,m-1}),$$

for $0 \leq i \leq \#V_{m-1} - 1$, and where

$$M_{i,m-1} \underset{m}{\sim} M_{i+1,m-1},$$

and, at the same time,

$$M_{i,m-1} \underset{m-1}{\sim} M_{j,m} \quad , \quad M_{j,m} \underset{m-1}{\sim} M_{j+1,m} \quad , \quad \cdots,$$

which means that the points $M_{i,m-1}$ and $M_{i+1,m-1}$ are consecutive vertices of V_{m-1} , and that the set of consecutive vertices of V_m located strictly between $M_{i,m-1}$ and $M_{i+1,m-1}$ consists of the points $M_{j,m}, M_{j+1,m}$; and so on.

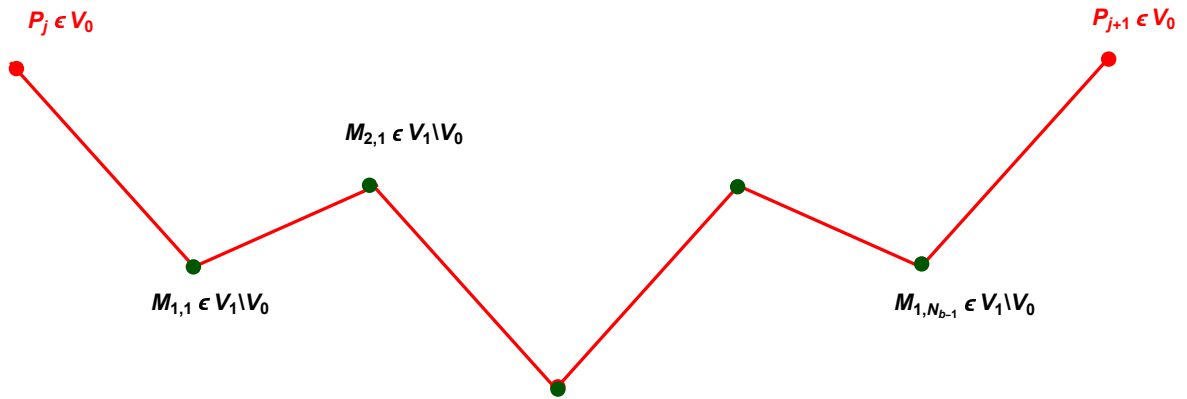


Figure 7: In search of invariants, when switching from the initial prefractal graph, to the first one.

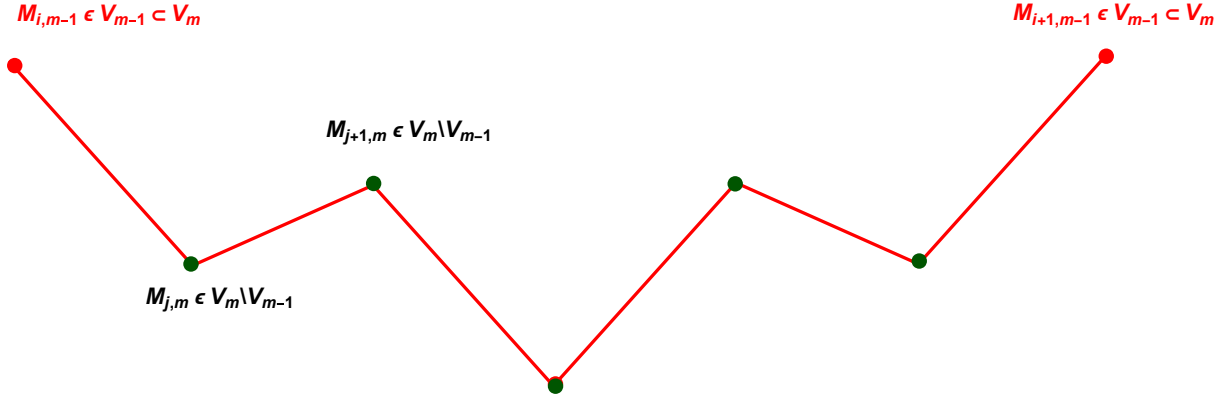


Figure 8: In search of invariants, when switching from the $(m-1)^{th}$ prefractal graph, to the m^{th} one.

Definition 5.2 ($(m-1, m)$ -Path).

Given a strictly positive integer m , and two adjacent vertices $X_{m-1,k}, X_{m-1,k+1}$ in V_m , for $0 \leq k \leq \#V_{m-1}$, we call $(m-1, m)$ -path between $X_{m-1,k}, X_{m-1,k+1}$ the ordered set of vertices

$$\mathcal{P}_{m-1,m}(X_{m-1,k}, X_{m-1,k+1}) = \{X_{m,\ell}, 0 \leq \ell \leq N_b\},$$

where

$$X_{m,\ell} = X_{m-1,k} \quad \text{and} \quad X_{m,\ell+N_b} = X_{m-1,k+1};$$

see Figure 8.

Definition 5.3 ($(m-1, m)$ -Differential).

Given a strictly positive integer m , we define the $(m-1, m)$ -differential $\delta_{m-1,m}$ from $\mathcal{F}^0(V_m, \mathbb{C})$ to $\mathcal{F}^{N_b+1}(V_m, \mathbb{C})$, for any f in $\mathcal{F}^0(\Gamma_{\mathcal{W}}, \mathbb{C})$ and any $(M_{i,m-1}, M_{i+1,m-1}, M_{j+1,m}, \dots, M_{j+N_b-2,m}) \in V_m^{N_b+1}$ such that

$$M_{i,m-1} = M_{j,m} \quad \text{and} \quad M_{i+1,m-1} = M_{j+N_b,m},$$

by

$$\delta_{m-1,m}(f)(M_{i,m-1}, M_{i+1,m-1}, M_{j+1,m}, \dots, M_{j+N_b-1,m}) = c_{m-1,m} \left(\sum_{q=0}^{N_b} (-1)^q f(M_{j+q,m}) \right),$$

where $c_{m-1,m}$ denotes a suitable positive constant. Note that as one handles differentials, one does not need to know – or fix – the values of this constant. It becomes of importance when operators involving the differentials, such as the Laplacian, are involved; see, for instance, Section 6 of [DL21].

By induction, we can, equivalently, consider the $(0, 1)$ -differential $\delta_{0,1}$ from $\mathcal{F}^0(V_0, \mathbb{C})$ to $\mathcal{F}^{N_b+1}(V_1, \mathbb{C})$, then, the $(1, 2)$ -differential $\delta_{1,2}$ from $\mathcal{F}^{N_b+1}(V_1, \mathbb{C})$ to $\mathcal{F}^{N_b^2+1}(V_2, \mathbb{C})$, and so on, which means that the $(m-1, m)$ -differential $\delta_{m-1,m}$ is defined from $\mathcal{F}^{N_b^{m-1}+1}(V_{m-1}, \mathbb{C})$ to $\mathcal{F}^{N_b^m+1}(V_m, \mathbb{C})$. In fact, at a given step $m \geq 0$, between two vertices of V_m , there are $N_b - 1$ consecutive vertices

of $V_{m+1} \setminus V_m$. Hence, this amounts to $N_b - 1 + 2 = N_b + 1$ consecutive vertices of $V_{m+1} \supset V_m$. Among those $N_b + 1$ vertices, there are then N_b pairs of consecutive vertices of $V_{m+1} \supset V_m$, which themselves involve N_b^2 pairs of consecutive vertices of $V_{m+2} \supset V_{m+1}$, i.e., $N_b^2 + 1$ consecutive vertices of $V_{m+2} \supset V_{m+1} \supset V_m$; and so on, by induction.

Furthermore, because of the compactness of $\Gamma_{\mathscr{W}}$, along with the density of the set $V^\star = \bigcup_{n \in \mathbb{N}} V_n$ in the Weierstrass Curve $\Gamma_{\mathscr{W}}$, every continuous (and hence, uniformly continuous) function on $\Gamma_{\mathscr{W}}$ is uniquely determined by its restriction to V^\star (that is, to each V_m , for all $m \in \mathbb{N}$).

Proposition 5.1 (Fractal Complex).

Hereafter, the Complex involved, denoted by $(\mathcal{F}^\bullet(\Gamma_{\mathscr{W}}, \mathbb{C}), \delta^\bullet)$ is the algebraic structure, which consists in the sequence of abelian groups (of fermions) $(\mathcal{F}^{N_b^m+1}(V_m, \mathbb{C}))_{m \in \mathbb{N}^\star}$, where, for each integer $m \geq 2$, the group $\mathcal{F}^{N_b^{m-1}+1}(V_{m-1}, \mathbb{C})$ is connected to the group $\mathcal{F}^{N_b^m+1}(V_m, \mathbb{C})$ by means of the $(m-1, m)$ -differentials $\delta_{m-1, m}$ introduced in Definition 5.3, namely,

$$\mathcal{F}^0(V_0, \mathbb{C}) \xrightarrow{\delta_{0,1}} \dots \mathcal{F}^{N_b^m+1}(V_m, \mathbb{C}) \xrightarrow{\delta_{m-1,m}} \mathcal{F}^{N_b^{m+1}+1}(V_{m+1}, \mathbb{C}) \xrightarrow{\delta_{m,m+1}} \dots$$

Because of the density of the set $V^\star = \bigcup_{n \in \mathbb{N}} V_n$ in the Weierstrass Curve $\Gamma_{\mathscr{W}}$, this complex can also be written in the following form,

$$\mathcal{F}^0(\Gamma_{\mathscr{W}}, \mathbb{C}) \xrightarrow{\delta_{0,1}} \dots \mathcal{F}^{N_b^m+1}(\Gamma_{\mathscr{W}}, \mathbb{C}) \xrightarrow{\delta_{m-1,m}} \mathcal{F}^{N_b^{m+1}+1}(\Gamma_{\mathscr{W}}, \mathbb{C}) \xrightarrow{\delta_{m,m+1}} \dots$$

Proposition 5.2 (Cohomology Groups).

In our present setting, with the differential introduced in Definition 5.3, the cohomology groups are the quotient groups

$$H_m = \ker \delta_{m-1,m} / \text{Im } \delta_{m-2,m-1} \quad , \quad \text{for } m \geq 1 \quad ,$$

with the additional convention that $H_0 = \text{Im } \delta_{-1,0} = \{0\}$.

Notation 9 (Argument of a Complex Number).

Given a nonzero complex number z , we denote by $\arg(z)$ the *argument* of z ; i.e., the angle between the positive real axis and the line joining the origin and the point M with affix z . Implicitly, we always choose the same convention (e.g., $\arg(z) \in]-\pi, \pi]$) whenever evaluating $\arg(z)$ – and hence also, $\arg(z) - \arg(z')$, for $(z, z') \in \mathbb{C}^\star \times \mathbb{C}^\star$.

Definition 5.4 (Set of Functions of the Same Nature as the Weierstrass Function \mathscr{W}).

i. We say that a continuous, complex-valued function f , defined on $\Gamma_{\mathscr{W}} \supset V^\star$, is of the same nature as the Weierstrass function \mathscr{W} , if it satisfies local Hölder and reverse-Hölder properties analogous

to those satisfied by the Weierstrass function \mathscr{W} ; i.e., for any pair of adjacent vertices (M, M') of respective affixes $(z, z') \in \mathbb{C}^2$ of the prefractal graph $\Gamma_{\mathscr{W}_m}$, with $m \in \mathbb{N}$ arbitrary (see Remark 2.1),

$$\tilde{C}_{inf} |z' - z|^{2-D_{\mathscr{W}}} \leq |f(z') - f(z)| \leq \tilde{C}_{sup} |z' - z|^{2-D_{\mathscr{W}}},$$

where \tilde{C}_{inf} and \tilde{C}_{sup} denote strictly positive and finite constants (but not necessarily the same ones as for the Weierstrass function \mathscr{W} itself, in Property 2.5). Using Notation 2, this can be written, equivalently, as

$$|z - z'|^{2-D_{\mathscr{W}}} \lesssim |f(z) - f(z')| \lesssim |z - z'|^{2-D_{\mathscr{W}}}. \quad (\blacklozenge)$$

Hereafter, we will denote by $\mathscr{H}\ddot{o}ld(\Gamma_{\mathscr{W}})$ the set consisting of the continuous, complex-valued functions f , defined on $\Gamma_{\mathscr{W}} \supset V^*$ and satisfying (\blacklozenge) .

ii. Moreover, we will denote by $\mathscr{H}\ddot{o}ld_{geom}(\Gamma_{\mathscr{W}}) \subset \mathscr{H}\ddot{o}ld(\Gamma_{\mathscr{W}})$ the subset of $\mathscr{H}\ddot{o}ld(\Gamma_{\mathscr{W}})$ consisting of the functions f in $\mathscr{H}\ddot{o}ld(\Gamma_{\mathscr{W}})$ which satisfy the following *additional geometric condition* (\blackspade) , again, for any pair of adjacent vertices (M, M') with respective affixes $(z, z') \in \mathbb{C}^2$ of the prefractal graph V_m , with $m \in \mathbb{N}$ arbitrary; namely,

$$|\arg(f(z)) - \arg(f(z'))| \lesssim |z - z'|. \quad (\blackspade)$$

We can now state the following key result.

Theorem 5.3 (Cohomological Complex Dimensions Series Expansion and Characterization of the Prefractal Cohomology Groups H_m [DL22b]).

Let $m \in \mathbb{N}$ be arbitrary. Then:

i. Within the set $\mathscr{H}\ddot{o}ld(\Gamma_{\mathscr{W}})$ (see part i. of Definition 5.4 just above), then, for any integer $m \geq 1$, and with the convention $H_0 = \text{Im } \delta_{-1,0} = \{0\}$, the cohomology groups

$$H_m = \ker \delta_{m-1,m} / \text{Im } \delta_{m-2,m-1}$$

are comprised of the restrictions to V_m of $(m, N_b^m + 1)$ -fermions, i.e., the restrictions to (the Cartesian product space) $V_m^{N_b^m + 1}$ of antisymmetric maps on $\Gamma_{\mathscr{W}}$, with $N_b^m + 1$ variables (corresponding to the vertices of V_m), involving the restrictions to V_m of continuous functions f on $\Gamma_{\mathscr{W}}$, such that, for any vertex $M_{j,m} \in V_m$, the following Taylor-like expansion is satisfied,

$$f(M_{j,m}) = \sum_{k=0}^m c_k(f, M_{j,m}) \varepsilon^{k(2-D_{\mathscr{W}})} \quad , \quad M_{\star,m} \in V_m, c_k(f, M_{\star,m}) \in \mathbb{C}, \quad (\blacklozenge\blacklozenge)$$

where, for each integer k such that $0 \leq k \leq m$, the number $\varepsilon^k > 0$ is the k^{th} cohomology infinitesimal introduced in Definition 3.1 above.

The coefficients $c_k(f, M_{\star})$ are complex quantities, which only depend on the function f involved, and on the point M_{\star} at which they are evaluated. As is explained in [DL22b], these coefficients $c_k(f, M_{\star})$ are the residues (at the possible cohomological Complex Dimensions $-k(2 - D_{\mathscr{W}})$) of a suitable local scaling zeta function evaluated at M_{\star} .

Note that, insofar as the functions f involved are, in a sense, determined, at any vertex $M_{j,m} \in V_m$, by the expansion given in ($\blacklozenge\blacklozenge$), it is natural to identify the cohomology groups H_m with those functions.

ii. If the functions f of part i. belong to $\mathcal{H}\ddot{o}ld_{geom}(\Gamma_{\mathcal{W}})$ (see part ii. of Definition 5.4 above), then, for any strictly positive integer m , and again with the convention $H_0 = Im \delta_{-1,0} = \{0\}$, the cohomology groups

$$H_m = \ker \delta_{m-1,m} / Im \delta_{m-2,m-1}$$

are comprised of the restrictions to V_m of $(m, N_b^m + 1)$ -fermions, i.e., the restrictions to $V_m^{N_b^m + 1}$ of antisymmetric maps on $\Gamma_{\mathcal{W}}$, with $N_b^m + 1$ variables (corresponding to the vertices of V_m), involving the restrictions to V_m of continuous functions f on $\Gamma_{\mathcal{W}}$, such that, for any vertex $M_{j,m} \in V_m$,

$$f(M_{j,m}) = \sum_{k=0}^m c_k(f, M_{j,m}) \varepsilon^{k(2-D_{\mathcal{W}})} \varepsilon^{i \ell_k \mathbf{p}} = \sum_{k=0}^m c_k(f, M_{j,m}) \varepsilon^{k(2-D_{\mathcal{W}}) + i \ell_k \mathbf{p}} \quad , \quad M_{\star,m} \in V_m \quad (\spadesuit\spadesuit)$$

where \mathbf{p} denotes the oscillatory period introduced in [DL22a],

$$\mathbf{p} = \frac{2\pi}{\ln N_b}$$

and where the coefficients $c_k(\star, \star)$ are complex numbers which still depend on the function f involved, and on the point at which they are evaluated. Here, in ($\spadesuit\spadesuit$), for each integer k such that $0 \leq k \leq m$, ℓ_k denotes an integer (in \mathbb{Z}) satisfying the estimate

$$\left| \left\{ \ell_k \frac{\ln \varepsilon}{\ln N_b} \right\} \right| \leq \frac{\varepsilon^{k(D_{\mathcal{W}}-1)}}{2\pi} \cdot \quad (\heartsuit)$$

Much as in part i., the coefficients $c_k(f, M_{\star})$, for $0 \leq k \leq m$, depend only on the function f and on the point M_{\star} of V_m at which they are evaluated. Note that obviously, the values of the constants c_k in part ii. are different from those obtained in part i. The specific topic of their possible extension to all $M_{\star} \in \Gamma_{\mathcal{W}}$ and their possible continuity in M_{\star} will be studied in more detail in a forthcoming work, [DLO23].

Furthermore, these coefficients $c_k(f, M_{\star})$ are the residues (at the possible cohomological Complex Dimensions $-(k(2 - D_{\mathcal{W}}) + i \ell_k \mathbf{p})$) of a suitable local scaling zeta function (evaluated at M_{\star}); see [DL22b].

In both expansions ($\blacklozenge\blacklozenge$) and ($\spadesuit\spadesuit$), the coefficients $c_k(f, M_{\star})$, for $0 \leq k \leq m$, reflect the dependence of the value taken by the map f at the vertex $M_{j,m}$ on the values taken by f at previous steps – vertices – of the m^{th} prefractal graph approximation, in conjunction with values taken by f at neighboring vertices of $M_{j,m}$ at the same level (m) of the prefractal sequence and with vertices which, in addition, strictly belong to the same polygon $\mathcal{P}_{m,k}$ introduced in part iv. of Property 2.3, with $1 \leq k \leq N_b^m - 1$ (by “strictly” here, we mean that the junction vertices are not concerned).

The expansion in part ii. (namely, ($\spadesuit\spadesuit$)) might be interpreted as a kind of generalized Taylor expansion with corresponding complex derivatives of orders $-\omega_k = k(2 - D_{\mathcal{W}}) + i \ell_k \mathbf{p}$, where $k \in \mathbb{N}$ is arbitrary, the coefficients $c_k(f, M_{\star})$ can thus be interpreted as (discrete) derivatives of complex order $-\omega_k$ of the function f , evaluated at the point M_{\star} of $V^{\star} \subset \Gamma_{\mathcal{W}}$. A similar comment can be made about the expansion in part i. (namely, ($\blacklozenge\blacklozenge$)), but now with the above value of $-\omega_k$ replaced by $k(2 - D_{\mathcal{W}})$.

Again, as in part *i.*, it is natural to identify the cohomology groups H_m with the sets of functions satisfying, at any vertex $M_{j,m} \in V_m$, the expansion given in ($\spadesuit\spadesuit$).

Remark 5.1. Note that the result given in part *ii.* above is significantly stronger than the one given in part *i.* of this theorem. Indeed, the expansion ($\diamond\diamond$) in part *i.* only involves (modulo a translation by $-D_{\mathcal{W}}$) the real parts of some of the Complex Dimensions listed in Theorem 4.5; namely, the Complex Dimensions of the Weierstrass Curve $\Gamma_{\mathcal{W}}$ with real parts different from -2 , 0 , and 1 . By contrast, in part *ii.*, we can also recover the imaginary parts of those same Complex Dimensions; note that this is possible only because the functions involved satisfy the geometric condition (\spadesuit) introduced in part *i.* of Definition 5.4 above.

It immediately follows from the expansion given in ($\spadesuit\spadesuit$) of part *ii.* of Theorem 5.3 above, as well as from the fact that the sequence of sets of vertices $(V_m)_{m \in \mathbb{N}}$ is increasing (see part *i.* of Property 2.3), that the sequence of cohomology groups $(H_m)_{m \in \mathbb{N}}$ is increasing; i.e.,

$$\forall m \in \mathbb{N} : H_m \subset H_{m+1}.$$

Finally, in both expansions ($\diamond\diamond$) and ($\spadesuit\spadesuit$), the number ε is not an arbitrary one. It is directly connected to the scaling properties of the Weierstrass Curve, and enables us to express the exact form of the associated scaling relationship (see Property 2.4, along with Remark 5.3 just below). In other words, the number ε – or, rather, the corresponding m^{th} cohomology infinitesimal ε^m , in the sense of Definition 3.1 above – depends on the geometry of the Curve. In light of our expansions, it can be interpreted as a (geometric) coefficient connecting the Weierstrass Curve and the complex-valued functions defined on the set $V^* = \bigcup_{n \in \mathbb{N}} V_n$.

Remark 5.2. We stress that the expansions ($\diamond\diamond$) and ($\spadesuit\spadesuit$) in Theorem 5.3 are also valid for any infinitesimal ε (see Definition 3.1, inequality (\mathfrak{A}) and the text surrounding it).

Remark 5.3 (The Special Case of the Weierstrass Complexified Function).

Among the continuous functions on $\Gamma_{\mathcal{W}}$ which possess, in the most natural manner, and, for any integer $m \in \mathbb{N}^*$, an expansion of the form obtained in ($\spadesuit\spadesuit$), we have, as could be expected, the *Weierstrass Complexified function* \mathcal{W}_{comp} , introduced in [DL22b], defined, for any real number x , by

$$\mathcal{W}_{comp}(x) = \sum_{n=0}^{\infty} \lambda^n e^{2i\pi N_b^n x}.$$

More precisely, as is shown in [DL22b], we have that, for any strictly positive integer m and any j in $\{0, \dots, \#V_m\}$, we have the following *exact* expansion, indexed by the Complex Codimensions $k(D_{\mathcal{W}} - 2) + i\ell_k \mathbf{p}$, with $0 \leq k \leq m$,

$$\mathcal{W}_{comp}(j\varepsilon^m) = (N_b - 1)\varepsilon^{m(2-D_{\mathcal{W}})} \mathcal{W}_{comp}\left(\frac{j}{N_b - 1}\right) + (N_b - 1) \sum_{k=0}^{m-1} c_k \varepsilon^{k(2-D_{\mathcal{W}})} \varepsilon^{i\ell_k \mathbf{p}},$$

where, for any integer k in $\{0, \dots, m-1\}$, $c_k \in \mathbb{C}$ and $\ell_k \in \mathbb{Z}$ are such that

$$(N_b - 1) e^{i\frac{2\pi}{N_b-1} j \varepsilon^{m-k}} = c_k \varepsilon^{i\ell_k \mathbf{p}}. \quad (\diamond)$$

Remark 5.4 (About the Integers ℓ_k in part ii. of Theorem 5.3, or in Theorem 5.4 below).

Note that since, the fractional part map is one-periodic, this results in a kind of *periodicity* with respect to the integers ℓ_k .

Naturally, the set $\{\ell_k, k \in \mathbb{N}\}$ is infinite and equal to \mathbb{Z} . In particular, $|\ell_k| \rightarrow \infty$ as $k \rightarrow \infty$. We thus recover completely analogous results to those obtained in Theorem 4.5 above (from [DL22a]).

An entirely similar comment can be made about the integers ℓ_k occurring in the statement of Theorem 5.4 just below.

Furthermore, once ℓ_k has been chosen to be the smallest positive integer satisfying the required inequalities, for a given $k \in \mathbb{N}$, then we obtain a sequence of integers $(\ell_{k,q})_{q \in \mathbb{Z}} = (\ell_k + q)_{q \in \mathbb{Z}}$ (isomorphic to \mathbb{Z}) satisfying the fractal series expansion ($\spadesuit\spadesuit$), where

$$\forall q \in \mathbb{Z} : \quad \left| \left\{ (\ell_k + q) \frac{\ln \varepsilon}{\ln N_b} \right\} \right| = \left| \left\{ \ell_k \frac{\ln \varepsilon}{\ln N_b} \right\} \right|.$$

This sequence also gives rise to the imaginary parts of the cohomological Complex Dimensions corresponding to the involved scale, since

$$\varepsilon^{i \ell_{k,q} \mathbf{P}} = e^{i(\ell_k + q) \mathbf{P} \ln \varepsilon} = e^{i[(\ell_k + q) \mathbf{P} \ln \varepsilon] + i\{(\ell_k + q) \mathbf{P} \ln \varepsilon\}} = e^{i 2\pi [(\ell_k + q) \frac{\ln \varepsilon}{\ln N_b}] + i\{(\ell_k + q) \frac{\ln \varepsilon}{\ln N_b}\}} = e^{i\{\ell_k \frac{\ln \varepsilon}{\ln N_b}\}} = \varepsilon^{i \ell_k \mathbf{P}}.$$

Theorem 5.4 (Fractal Cohomology of the Weierstrass Curve [DL22b]).

Within the set $\mathcal{H}\ddot{o}ld_{geom}(\Gamma_{\mathcal{W}})$ of continuous, complex-valued functions f , defined on the Weierstrass Curve $\Gamma_{\mathcal{W}} \supset V^{\star} = \bigcup_{n \in \mathbb{N}} V_n$ (see part ii. of Definition 5.4 above), let us consider the Complex (which can be called the Fractal Complex of $\Gamma_{\mathcal{W}}$),

$$H^{\star} = H^{\bullet}(\mathcal{F}^{\bullet}(\Gamma_{\mathcal{W}}, \mathbb{C}), \delta^{\bullet}) = \bigoplus_{m=0}^{\infty} H_m,$$

where, for any integer $m \geq 1$, and with the convention $H_0 = \text{Im } \delta_{-1,0} = \{0\}$, H_m is the cohomology group

$$H_m = \ker \delta_{m-1,m} / \text{Im } \delta_{m-2,m-1}.$$

Then, H^{\star} is the set consisting of functions f on $\Gamma_{\mathcal{W}}$, viewed as 0-fermions (in the sense of Definition 5.1), and, for any integer $m \geq 1$, of the restrictions to V_m of $(m, N_b^m + 1)$ -fermions, i.e., the restrictions to (the Cartesian product space) $V_m^{N_b^m + 1}$ of antisymmetric maps on $\Gamma_{\mathcal{W}}$, with $N_b^m + 1$ variables (corresponding to the vertices of V_m), involving the restrictions to V_m of the continuous, complex-valued functions f on $\Gamma_{\mathcal{W}}$ – as, naturally, the aforementioned 0-fermions – satisfying the following convergent (and even, absolutely convergent) Taylor-like expansions (with $V^{\star} = \bigcup_{n \in \mathbb{N}} V_n$),

$$\forall M_{\star, \star} \in V^{\star} : \quad f(M_{\star, \star}) = \sum_{k=0}^{\infty} c_k(f, M_{\star, \star}) \varepsilon^{k(2-D_{\mathcal{W}})} \varepsilon^{i \ell_k \mathbf{P}} = \sum_{k=0}^{\infty} c_k(f, M_{\star, \star}) \varepsilon^{k(2-D_{\mathcal{W}}) + i \ell_k \mathbf{P}}, (\spadesuit\spadesuit\spadesuit)$$

where, for each integer $k \geq 0$, the coefficient $c_k(\star, \star) = c_k(f, \star) \in \mathbb{C}$ is the same as in part ii. of Theorem 5.3, the number $\varepsilon^k > 0$ is the k^{th} component of the k^{th} cohomology infinitesimal introduced in Definition 3.1, and where ℓ_k denotes an integer (in \mathbb{Z}) such that

$$\left| \left\{ \ell_k \frac{\ln \varepsilon}{\ln N_b} \right\} \right| \leq \frac{\varepsilon^{k(D_{\mathcal{W}}-1)}}{2\pi}.$$

Note that since the functions f involved are uniformly continuous on the Weierstrass Curve $\Gamma_{\mathcal{W}} \supset V^{\star}$, and since the set V^{\star} is dense in $\Gamma_{\mathcal{W}}$, they are uniquely determined by their restriction to V^{\star} , as given by ($\spadesuit\spadesuit\spadesuit$). We caution the reader, however, that at this stage of our investigations, we do not know whether $f(M)$ is given by an expansion analogous to the one in ($\spadesuit\spadesuit\spadesuit$), for every $M \in \Gamma_{\mathcal{W}}$, rather than just for all $M \in V^{\star}$.

The convergence (or even, the absolute convergence) of the series $\sum_{k=0}^{\infty} c_k(f, M_{\star, \star}) \varepsilon^{k(2-D_{\mathcal{W}})+i\ell_k \mathbf{p}}$ directly comes from the fact that the coefficients $c_k(\star, \star)$ are uniformly bounded (see the inequalities ($\star\star$) in the proof of Theorem 5.3), and that, for any $k \in \mathbb{N}^{\star}$,

$$\left| \varepsilon^{k(2-D_{\mathcal{W}})+i\ell_k \mathbf{p}} \right| = \varepsilon^{k(2-D_{\mathcal{W}})} = \left(\varepsilon^{2-D_{\mathcal{W}}} \right)^k, \text{ with } 0 < \varepsilon < 1 \text{ and } 2 - D_{\mathcal{W}} > 0.$$

Finally, for each $M_{\star} = M_{\star, m} \in V^{\star}$, as is shown in [DL22b], the coefficients $c_k(\star, \star)$ (for any $k \in \mathbb{N}$) are the residues at the possible cohomological Complex Dimensions $-(k(2-D_{\mathcal{W}})+i\ell_k \mathbf{p})$ of a suitable global scaling zeta function evaluated at M_{\star} .

The group $H^{\star} = \bigoplus_{m=0}^{\infty} H_m$ is called the total fractal cohomology group of the Weierstrass Curve $\Gamma_{\mathcal{W}}$ (or else, of the Weierstrass function \mathcal{W}).

The following statement is a corollary of both Theorems 5.3 and 5.4.

Corollary 5.5. For each $m \in \mathbb{N}$, the prefractal cohomology space H_m is different from $\{0\}$. Indeed, the Weierstrass function \mathcal{W} , viewed as an appropriate restriction of the identity map on the Weierstrass Curve $\Gamma_{\mathcal{W}}$, belongs to each H_m , for any $m \in \mathbb{N}$.

Similarly, and for exactly the same reason, the fractal cohomology space H^{\star} , is not reduced to $\{0\}$ (because, in short, \mathcal{W} belongs to H^{\star}).

Therefore, in some definite sense, we can say that the Weierstrass function – or, alternatively, the Weierstrass Curve – belongs to its own cohomology.

Remark 5.5. These results are to be compared with previous ones of Michel L. Lapidus and collaborators obtained in [LvF00], [LvF06], [Lap08], [LvF13], [CL17], [Lap19] and [Lap22].

More precisely, in our present setting, and contrary to the classical cases of arithmetic or differentiable varieties, for which the decomposition of the total cohomology is indexed by integers, the total (fractal) cohomology H^{\star} is a sum of spaces indexed by the complex dimensions (as is expected in [LvF00], [LvF06], [LvF13] and [Lap08], and discussed in detail in [CL17], [Lap19] and [Lap22]),

with an underlying quasiperiodicity property, induced by the estimate (on the imaginary parts of the complex dimensions) (\clubsuit) in part *ii.* of Theorem 5.3. We thus dispose of a quasiperiodic geometric property (reminiscent of, but not identical to, that established in Chapter 3 of [LvF13] for nonlattice self-similar strings), which can possibly be connected to the structure of a (generalized) quasicrystal (see [LvF13], Problem 3.22, page 89, and [Lap08], especially, Chapter 5 and Appendix F), especially in the case not considered in the present paper when N_b is not an integer, but for which we expect analogous results; see, in particular, Proposition 4.8 above.

Remark 5.6. Note that H^\star is to be understood in the sense of *the inductive limit* of the sequence of cohomology groups $(H_m)_{m \in \mathbb{N}}$; namely, for each fermion $\varphi \in H^\star$, and each $m \in \mathbb{N}$, the restriction $\varphi|_{V_m}$ of φ to the set of vertices V_m belongs to H_m ; the restriction $(\varphi|_{V_{m+1}})|_{V_m}$ to V_m of the restriction $\varphi|_{V_{m+1}}$ of φ to the set of vertices V_{m+1} (which is itself in H_{m+1}), coincides with the restriction $\varphi|_{V_m}$ of φ to V_m ; i.e.,

$$\forall m \in \mathbb{N} : \quad \varphi|_{V_m} \in H_m \quad \text{and} \quad (\varphi|_{V_{m+1}})|_{V_m} = \varphi|_{V_m}.$$

This amounts, for each $\varphi \in H^\star$, to

$$\varphi = (\varphi_m)_{m \in \mathbb{N}},$$

where, for each $m \in \mathbb{N}$, $\varphi_m \in H_m$, while, if we denote by $\pi : H_{m+1} \rightarrow H_m$ the natural projection from H_{m+1} to H_m (recall from Remark 5.1 that $H_m \subset H_{m+1}$), we have that $\pi(\varphi_{m+1}) \in H_m$ coincides with φ_m .

Remark 5.7. In some sense, the Complex Dimensions corresponding to the fractal cohomology – namely, $\omega_k = -k(2 - D_{\mathscr{W}}) - i \ell_k \mathbf{p}$, where $k \in \mathbb{N}$ is arbitrary – can be viewed as being associated to the Weierstrass function itself rather than with the Weierstrass Curve.

In [DL22b], we also provide a reformulation of Theorems 5.3 and 5.4 concerning, respectively, the m^{th} prefractal cohomology groups H_m ($m \in \mathbb{N}$) and the global (or total) fractal cohomology group H^\star . This reformulation – based on a pair of complex conjugate variables Z and \bar{Z} instead of $(x, y) \in \mathbb{R}^2$ – makes use of the natural symmetry of the Weierstrass Curve $\Gamma_{\mathscr{W}}$ and, correspondingly, enables us to obtain a natural version of *Poincaré Duality* for the associated cohomology spaces; see [DL22b] and Section 6 below.

6 Concluding Comments and Perspectives

Up to now, the determination of the Complex Dimensions of the Weierstrass Curve had remained an open problem, that we have completely solved in [DL22a]. Contrary to classical fractals, the non-affine feature, coming from a nonlinear iterated function system, makes things a lot more complicated than for earlier examples studied in the literature (see, e.g., [LRŽ17a], [LRŽ17b], [LRŽ18], [Lap19]).

By considering the fractal Complex Dimensions as dynamical quantities, which evolve with the scales, we set up a broader and more general framework than the one which was envisioned before. We can draw an analogy between this new framework and the theory of the French physicist Laurent Notale discussed, for instance, in [Not98]: in our context, the fractal dimension plays the role of the time variable; the length – seen as a function of the resolution, once replaced by its logarithm – plays the role of the position; as for the scale (or, in an additive way, its logarithm), it is the analogue of the speed. This explains why there is an insurmountable (minimum) resolution, directly depending

on the geometry. (There is also a largest scale, but this is perhaps less surprising).

One of the benefits of our study – insofar as we handled a more general setting than in the classical cases – will concern the extension to a large class of (possibly nowhere differentiable) fractal functions. Note that our results also provide a new and very direct proof of the fact that the value of the Minkowski (or box-counting) dimension of the Weierstrass Curve is $D_{\mathcal{W}} = 2 - \ln_{N_b} \frac{1}{\lambda}$.

The interest of considering such a fractal object is, besides its very rich geometrical properties – a self-shape similarity, coming from a prefractal polygonal sequence with vertices entirely contained and dense in the Curve – that one has a natural map defined on the Curve. Again, it is not usually the case (see, for instance, the Sierpiński Gasket, where one has to introduce specific functions). As is shown in [DL22b], it happens that the complexified Weierstrass function possesses a series expansion indexed by the cohomological Complex Dimensions; see Remark 5.3. Under this light, one could not bypass establishing a direct link with a fractal cohomology. As expected, the fractal expansion (or explicit formula) that enables us to express any given function belonging to the m^{th} cohomology group, at a given vertex of the associated m^{th} prefractal graph approximation, can exactly be written as a countably infinite sum indexed by the cohomological Complex Dimensions. Going further, this might be interpreted as a kind of *generalized Taylor expansion*, with fractional derivatives of underlying orders the corresponding Complex Dimensions. It appears that we have significantly extended and given a geometric meaning to the results (and conjectures) in [LvF00], [LvF06], [Lap08], [LvF13], along with the ones in [CL17] and [Lap22], where the authors suggested that there should exist a fractal cohomology theory having direct links with the theory of Complex Dimensions.

The natural Poincaré duality associated to the Weierstrass Curve (see [DL22b]) calls for further investigation. In our future works, we envision to lay out the foundations of a so-called *fractal differential calculus* along with a corresponding version of Hodge theory – the natural extension to fractal objects of the classic differential calculus. Specific topics include the conditions of existence of Taylor-like expansions, as well as of the explicit expressions of the corresponding (local) fractional derivatives, along with a proper definition, in the present context, of the notion of (possibly higher-dimensional) *fractal manifold*, its associated Complex Dimensions, fractal curvatures and fractal cohomology, dual of a potential fractal homology which is yet to be constructed here.

References

- [BBR14] Krzysztof Barański, Balázs Bárány, and Julia Romanowska. On the dimension of the graph of the classical Weierstrass function. *Advances in Mathematics*, 265:791–800, 2014.
- [Bou04] Nicolas Bourbaki. *Theory of Sets*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2004. Reprint of the 1968 English translation [Hermann, Paris; MR0237342].
- [CL17] Tim Cobler and Michel L. Lapidus. Towards a fractal cohomology: Spectra of Polya–Hilbert operators, regularized determinants and Riemann zeros. In *Exploring the Riemann Zeta Function: 190 Years from Riemann’s Birth* (H. Montgomery, A. Nikeghbali and M. Rassias, eds.), pages 35–65. Springer, Basel, Berlin and New York, 2017. URL: <https://arxiv.org/abs/1705.06222>.
- [Cul20] Anais Culot. À l’assaut des grandes conjectures mathématiques. <https://lejournal.cnrs.fr/articles/a-lassaut-des-grandes-conjectures-mathematiques>, March 2020.

- [Dav18] Claire David. Bypassing dynamical systems: A simple way to get the box-counting dimension of the graph of the Weierstrass function. *Proceedings of the International Geometry Center*, 11(2):1–16, 2018. URL: <https://journals.onaft.edu.ua/index.php/geometry/article/view/1028>.
- [Dav19] Claire David. On fractal properties of Weierstrass-type functions. *Proceedings of the International Geometry Center*, 12(2):43–61, 2019. URL: <https://journals.onaft.edu.ua/index.php/geometry/article/view/1485>.
- [DL21] Claire David and Gilles Lebeau. h -Laplacians on singular sets. Preprint, 2021. URL: <https://hal.sorbonne-universite.fr/hal-03698953/document>.
- [DL22a] Claire David and Michel L. Lapidus. Weierstrass fractal drums - I - A glimpse of complex dimensions, April 2022. Preprint. URL: <https://hal.sorbonne-universite.fr/hal-03642326>.
- [DL22b] Claire David and Michel L. Lapidus. Weierstrass fractal drums - II - Towards a fractal cohomology, 2022. URL: <https://hal.archives-ouvertes.fr/hal-03758820/document>.
- [DLO23] Claire David, Michel L. Lapidus, and Matthew Overduin. Fractional Taylor series: Conditions of existence, and explicit formulas (tentative title), in preparation, 2023.
- [Fal86] Kenneth Falconer. *The Geometry of Fractal Sets*. Cambridge University Press, Cambridge, 1986.
- [Har16] Godfrey Harold Hardy. Weierstrass’s Non-Differentiable Function. *Transactions of the American Mathematical Society*, 17(3):301–325, 1916. URL: <https://www.ams.org/journals/tran/1916-017-03/S0002-9947-1916-1501044-1/S0002-9947-1916-1501044-1.pdf>.
- [HL93] Tian-You Hu and Ka-Sing Lau. Fractal dimensions and singularities of the Weierstrass type functions. *Transactions of the American Mathematical Society*, 335(2):649–665, 1993.
- [HL21] Hafedh Herichi and Michel L. Lapidus. *Quantized Number Theory, Fractal Strings and the Riemann Hypothesis: From Spectral Operators to Phase Transitions and Universality*. World Scientific Publishing, Singapore and London, 2021.
- [Hun98] Brian Hunt. The Hausdorff dimension of graphs of Weierstrass functions. *Proceedings of the American Mathematical Society*, 12(1):791–800, 1998.
- [IRT12] Marius Ionescu, Luke G. Rogers, and Alexander Teplyaev. Derivations and Dirichlet forms on fractals. *Journal of Functional Analysis*, 263(8):2141–2169, 2012.
- [Kel17] Gerhard Keller. A simpler proof for the dimension of the graph of the classical Weierstrass function. *Annales de l’Institut Henri Poincaré – Probabilités et Statistiques*, 53(1):169–181, 2017.
- [KMPY84] James L. Kaplan, John Mallet-Paret, and James A. Yorke. The Lyapunov dimension of a nowhere differentiable attracting torus. *Ergodic Theory and Dynamical Systems*, 4:261–281, 1984.
- [Lap91] Michel L. Lapidus. Fractal drum, inverse spectral problems for elliptic operators and a partial resolution of the Weyl-Berry conjecture. *Transactions of the American Mathematical Society*, 325:465–529, 1991.
- [Lap92] Michel L. Lapidus. Spectral and fractal geometry: From the Weyl-Berry conjecture for the vibrations of fractal drums to the Riemann zeta-function. In *Differential Equations and Mathematical Physics (Birmingham, AL, 1990)*, volume 186 of *Math. Sci. Engrg.*, pages 151–181. Academic Press, Boston, MA, 1992.

- [Lap93] Michel L. Lapidus. Vibrations of fractal drums, the Riemann hypothesis, waves in fractal media and the Weyl-Berry conjecture. In *Ordinary and Partial Differential Equations, Vol. IV (Dundee, 1992)*, volume 289 of *Pitman Research Notes Mathematical Series*, pages 126–209. Longman Sci. Tech., Harlow, 1993.
- [Lap08] Michel L. Lapidus. *In Search of the Riemann Zeros: Strings, Fractal Membranes and Noncommutative Spacetimes*. American Mathematical Society, Providence, RI, 2008.
- [Lap19] Michel L. Lapidus. An overview of complex fractal dimensions: From fractal strings to fractal drums, and back. In *Horizons of Fractal Geometry and Complex Dimensions* (R. G. Niemeyer, E. P. J. Pearse, J. A. Rock and T. Samuel, eds.), volume 731 of *Contemporary Mathematics*, pages 143–265. American Mathematical Society, Providence, RI, 2019. URL: <https://arxiv.org/abs/1803.10399>.
- [Lap22] Michel L. Lapidus. *From Complex Fractal Dimensions and Quantized Number Theory To Fractal Cohomology: A Tale of Oscillations, Unreality and Fractality*, book in preparation. World Scientific Publishing, Singapore and London, 2022.
- [LM95] Michel L. Lapidus and Helmut Maier. The Riemann hypothesis and inverse spectral problems for fractal strings. *Journal of the London Mathematical Society. Second Series*, 52(1):15–34, 1995.
- [LP93] Michel L. Lapidus and Carl Pomerance. The Riemann zeta-function and the one-dimensional Weyl-Berry conjecture for fractal drums. *Proceedings of the London Mathematical Society. Third Series*, 66(1):41–69, 1993.
- [LP06] Michel L. Lapidus and Erin P. J. Pearse. A tube formula for the Koch snowflake curve, with applications to complex dimensions. *Journal of the London Mathematical Society. Second Series*, 74(2):397–414, 2006.
- [LPW11] Michel L. Lapidus, Erin P. J. Pearse, and Steffen Winter. Pointwise tube formulas for fractal sprays and self-similar tilings with arbitrary generators. *Advances in Mathematics*, 227(4):1349–1398, 2011.
- [LRŽ17a] Michel L. Lapidus, Goran Radunović, and Darko Žubrinić. Distance and tube zeta functions of fractals and arbitrary compact sets. *Advances in Mathematics*, 307:1215–1267, 2017.
- [LRŽ17b] Michel L. Lapidus, Goran Radunović, and Darko Žubrinić. *Fractal Zeta Functions and Fractal Drums: Higher-Dimensional Theory of Complex Dimensions*. Springer Monographs in Mathematics. Springer, New York, 2017.
- [LRŽ18] Michel L. Lapidus, Goran Radunović, and Darko Žubrinić. Fractal tube formulas for compact sets and relative fractal drums: Oscillations, complex dimensions and fractality. *Journal of Fractal Geometry*, 5(1):1–119, 2018.
- [LvF00] Michel L. Lapidus and Machiel van Frankenhuysen. *Fractal Geometry and Number Theory: Complex Dimensions of Fractal Strings and Zeros of Zeta Functions*. Birkhäuser Boston, Inc., Boston, MA, 2000.
- [LvF06] Michel L. Lapidus and Machiel van Frankenhuysen. *Fractal Geometry, Complex Dimensions and Zeta Functions: Geometry and Spectra of Fractal Strings*. Springer Monographs in Mathematics. Springer, New York, 2006.
- [LvF13] Michel L. Lapidus and Machiel van Frankenhuysen. *Fractal Geometry, Complex Dimensions and Zeta Functions: Geometry and Spectra of Fractal Strings*. Springer Monographs in Mathematics. Springer, New York, second revised and enlarged edition (of the 2006 edition), 2013.

- [Man77] Benoît B. Mandelbrot. *Fractals: Form, Chance, and Dimension*. W. H. Freeman & Co Ltd, San Francisco, 1977.
- [Man83] Benoît B. Mandelbrot. *The Fractal Geometry of Nature*. English translation, revised and enlarged edition (of the 1977 edition). W. H. Freeman & Co, New York, 1983.
- [Not98] Laurent Nottale. *La relativité dans tous ses états*. Hachette Littératures; Hachette édition, Paris, 1998.
- [PU89] Feliks Przytycki and Mariusz Urbański. On the Hausdorff dimension of some fractal sets. *Studia Mathematica*, 93(2):155–186, 1989.
- [She18] Weixiao Shen. Hausdorff dimension of the graphs of the classical Weierstrass functions. *Mathematische Zeitschrift*, 289:223–266, 2018.
- [Tit39] Edward Charles Titchmarsh. *The Theory of Functions*. Oxford University Press, second edition, 1939.
- [Wei75] Karl Weierstrass. Über continuirliche Funktionen eines reellen Arguments, die für keinen Werth des letzteren einen bestimmten Differential quotienten besitzen. *Journal für die reine und angewandte Mathematik*, 79:29–31, 1875.
- [Zyg02] Antoni Zygmund. *Trigonometric Series. Vols. I, II*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, third edition, 2002. With a foreword by Robert A. Fefferman.