Elementary Mathematics II

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INTRODUCTION

This book is a course given to L0 students at UFAZ during the second semester of the academic years 2019-2020, 2020-2021 and 2021-2022. The Frecnh-Azerbaijani University called this year the foundation year because it allows to fill the gap that between the French High School program and Azerbaijan High School's program. Its main objective is to give general backgrounds in mathematics to students who integrate UFAZ, in order to allow them to be able to follow first year's courses of the University of Strasbourg. In other words this year corresponds to the "Terminale" of french program. This book is concerned by the first Semester. In this book, we deal with *functions*, *itegration*, *first oder ordinary differential equations*, *second order ordinary differential equations* In each chapter we recall the general results of the topic and give examples that help to understand this subject. We point out that we took some exercises from the list of homework given by Javanshir Azizov (UFAZ) and Loic Célier (UFAZ).

ACKNOWLEDGE

I would like to thank my colleagues Ulviyya Abdulkarimova, Javanshir Azizov, Fuad Aliyev and Sabina Samadova for their help and support.

CONTENTS

CHAPTER 1

INTRODUCTION TO LIMITS AND CONTINUITY

Contents

Let f be a real function with domain D. Take two points x and y in D, which are close to each other. The images of these points by the function f are $f(x)$ and $f(y)$. We remind the points $f(x)$ and $f(y)$ belong to the set $f(D)$, which is the range of D by the function f. After this transformation, one may wounder if the image of x and y remain close to each other. There are two possible answers:

- the points $f(x)$ and $f(y)$ are close to each other
- the point $f(x)$ is far from $f(y)$.

In the first case we say the function f is continuous at the point x. In the second case the function f is said to be discontinuous at the point x. To say that a point x is near or far from a point y we need to measure the distance between these two points. We want to know from which distance we can say that the point x is close to y. To investigate all these questions we introduce the concept of limit and continuity. The notion of *Limit* is a key concept in analysis. We can say that it is the very *Basis* of the analysis. To give an idea of what is a limit, we consider the function $f(x) = \dfrac{1}{x+1}\cdot$ This function is defined if an only if $x + 1 \neq 0$. This means that the function f is defined if and only if $x \neq -1$. That is, $D_f = \mathbb{R}\backslash\{-1\}$.

Now we take $x_0 = 2$, we can see that when x approaches x_0 the value of the function $f(x)$ approaches $f(x_0)$. In other words when x goes to x_0, the value of $f(x)$ approaches $f(x_0) = \frac{1}{3}$. In this case we write

$$\lim_{x \to x_0} f(x) = f(x_0). \tag{1.0.1}$$

The expression (1.0.1) reads: limit when x goes to x_0 of $f(x)$ is $f(x_0)$.

Now we consider $x_0 = -1$, which is not in the domain of the function f. We can see that if $x = -0.999999$, we obtain $f(x) = 1000000$. We observe that, when x is as close as possible to x_0, the quantity $f(x)$ is as greater as possible. In this case we write

$$\lim_{x \to x_0} f(x) = +\infty.$$

May be another example could be helpful to better understand the concept of limit. Let f be the function defined by

$$f(x) = \begin{cases} x + 1, & x < 0 \\ 2, & x = 0 \\ 2x + 3, & x > 0. \end{cases}$$

We remark when x approaches 0 but remains less than 0 the value of $f(x)$ approaches the value 1. On the other hand when x approaches 0 but remains greater than 0 the value of the function $f(x)$ approaches 3. As we can see it the two limits are different to $f(0) = 2$. We see that the limits are finite but the function is not continuous at the point $x_0 = 0$.

The main objective of this chapter is to define rigorously limits and continuity by providing a method which will allows to measure efficiently the distance between x and x_0 in one part and $f(x)$ and the limit of f in the other part. This chapter is organized as follows: In section 1 we deal with limits at a point which is in the domain of a function f. In section 2, we define the limit at bounds of a domain of a function. The limits at such points lead usually to the concept of *asymptotes*. In section 3 we study continuity of functions.

1.1 Limit at a Point in the Domain of a Function

In this section we study limit at point on which a function is defined. Here we will only deal with functions which have their domains in $\mathbb{R}$. We define the limit of the function f at the point $x_0 \in D_f$, as follows:

Definition 1.1.1. *Let f be a real function with domain $D_f \subset \mathbb{R}$. We consider two real numbers ℓ and x_0 such that $x_0 \in D_f$. We say that the real number ℓ is the limit of f at x_0, if $f(x)$ approaches ℓ when x approaches x_0. In this case we write*

$$\lim_{x \to x_0} f(x) = \ell \tag{1.1.1}$$

Example 1.1.2. *We consider the function $f(x) = \frac{1}{x+3}$. This function is defined for all real numbers excepted -3. Then, the domain of f is $\mathbb{R}\backslash\{-3\}$. Since 1 and 2*

are in D_f, we have

$$\lim_{x\to 1} f(x) = \frac{1}{4} \quad \textit{and} \quad \lim_{x\to 2} f(x) = \frac{1}{5}.$$

Exercise 1.1.3. *Determine the following limits*

$$\lim_{x\to 0} x^3 + 2x^2 + 3x + 6, \quad \lim_{x\to 1} \frac{1}{x^2+3x+1}, \quad \lim_{x\to 2} x^2 + 1, \quad \lim_{x\to 6} \frac{x^2+2x+10}{x^3+1}.$$

Equation (1.1.1) needs some comments. Indeed, what we mean by (1.1.1) is when x is near to x_0, the value of $f(x)$ is approximately ℓ. That is, the quantity $|f(x) - \ell|$ is negligible. The error we make to replace $f(x)$ by ℓ is small enough. We specify from now on, each time we use ε or δ, we refer to strictly positive real numbers that are small enough. The positive real numbers ε and δ are as small as we want. By x is approximately equal to y, we means that, the positive real number $|x - y|$ is approximately 0. Then we introduce the following definition.

Definition 1.1.4. *Let x and y be two reals numbers. We say that x is approximately equals to y if there exists some real number $\varepsilon > 0$ small enough, such that,*

$$|y - x| < \varepsilon.$$

We recall that , $|y - x| < \varepsilon$ is equivalent to $-\varepsilon < y - x < \varepsilon$. This means that

$$x - \varepsilon < y < x + \varepsilon.$$

Hence $y \in \big(x - \varepsilon\,,\, x + \varepsilon\big)$. We can reformulate the previous definition as follows.

Definition 1.1.5. *Let x and y be two real numbers. We say y is approximately equal to x if there exist $\varepsilon > 0$ or ($\delta > 0$) small enough such that*

$$y \in \big(x - \varepsilon\,,\, x + \varepsilon\big) \quad \textit{or} \quad \big(y \in \big(x - \delta\,,\, x + \delta\big)\big)\,.$$

Example 1.1.6. *We take $x = 1$ and $y = 1 + 10^{-10}$. If we fix $\varepsilon = 10^{-10} + 10^{-11}$, we have*

$$|y - x| = 1 + 10^{-10} - 1 = 10^{-10} < \varepsilon.$$

Using ε and δ we can rewrite definition 1.1.1. Indeed, we mean by x tends to x_0, that x is approximately equal to x_0. That is, $|x - x_0|$ is small enough. Therefore, we can find some $\delta > 0$, such that, $|x - x_0| < \delta$. By $f(x)$ goes to ℓ we want to say that, we can make the distance between $f(x)$ and ℓ as small as we want. Then for all real number $\varepsilon > 0$, we can choose x near to x_0 so that $|f(x) - \ell| < \varepsilon$. We can summarize this in this way

$$\forall \varepsilon > 0, \exists \delta > 0,\ \forall x \in \mathbb{R},\ (\ |x - x_0| < \delta \Rightarrow\ |f(x) - \ell| < \varepsilon).$$

This sentence means that, for all positive real number $\varepsilon > 0$, there exists a positive real number $\delta > 0$, such that for any real $x \in \mathbb{R}$ that verifies $|x - x_0| < \delta$ implies $|f(x) - \ell| < \varepsilon$. We emphasize that the real number δ depends on ε. But for the sake of simplicity we will write δ instead of $\delta(\varepsilon)$ or δ_ε.

Then, definition 1.1.1 is equivalent to

Definition 1.1.7. *Let f be a real function with domain $D_f \subset \mathbb{R}$. Take $x_0 \in D_f$, and $\ell \in \mathbb{R}$. We say that ℓ is the limit of f when x approaches x_0, if*

$$\forall \varepsilon > 0, \exists \delta > 0,\ \forall x \in \mathbb{R},\ (\ |x - x_0| < \delta \Rightarrow\ |f(x) - \ell| < \varepsilon). \tag{1.1.2}$$

To illustrate this dependence we consider the following examples.

Example 1.1.8. *Let $f(x) = \dfrac{1}{x+3}\cdot$ We have established that*

$$\lim_{x\to 1} f(x) = \frac{1}{4}\cdot$$

Now we want to confirm this limit by using ε and δ. To this end we take some positive real number ε which is as small as we want. To find δ we proceed as follow. We consider our ε, such that $\left|f(x) - \dfrac{1}{4}\right| < \varepsilon$. This is equivalent to

$$\left|\frac{4 - x - 3}{4(x+3)}\right| < \varepsilon \Longrightarrow \left|\frac{-(x-1)}{4(x+3)}\right| < \varepsilon.$$

Since x tends to 1, we observe that x should be in the interval $\big(0\,,\,2\big)$. That is $x \leq 2$. Using this we obtain $4(x+3) \leq 4\times 5 = 20$. Consequently $\dfrac{1}{20} \leq \dfrac{1}{4(x+3)}\cdot$ This leads to

$$\frac{|x-1|}{20} \leq \frac{|x-1|}{4(x+3)} < \varepsilon.$$

From the inequality above we deduce $|x-1| < 20\,\varepsilon$. Setting $\delta := 20\,\varepsilon$, we obtain for every

$$x \in D_f, \quad |x-1| < 20\,\varepsilon \quad \text{we have} \quad \left|f(x) - \frac{1}{4}\right| < \varepsilon.$$

Therefore one obtains

$$\lim_{x\to 1} f(x) = \frac{1}{4}\cdot$$

Example 1.1.9. *We consider the function $g(x) = \sqrt{x-2}$. This function is defined in the interval $[2\,,\,+\infty)$. We can observe*

$$\lim_{x\to 6} g(x) = 2.$$

We want to prove this property using ε and δ. Let $\varepsilon > 0$ such that $\Big|g(x) - 2\Big|, \leq \varepsilon$. Then, we have $\left|\sqrt{x-2} - 2\right| \leq \varepsilon$. Now we multiply and divide the expression between the bars by $\sqrt{x-2}+2$. Here we can divide by $\sqrt{x-2}+2$, because $x \geq 2$. Thus, we obtain

$$\left|\frac{x-2-4}{\sqrt{x-2}+2}\right| = \left|\frac{x-6}{\sqrt{x-2}+2}\right| \leq \varepsilon.$$

Since x approaches 6, we have $x \leq 7$. This gives $x - 2 \leq 5$. Using the square root and adding 2 in both sides we have $\sqrt{x-2}+2 \leq \sqrt{5}+2$. This involves

$$\frac{1}{\sqrt{5}+2} \leq \frac{1}{\sqrt{x-2}+2}\cdot$$

Multiplying the inequality above by $|x-6|$ *in both sides we obtain*

$$\frac{|x-6|}{\sqrt{5}+2} \leq \frac{|x-6|}{\sqrt{x-2}+2} \leq \varepsilon.$$

This yields the following inequality

$$|x-6| \leq \left(\sqrt{5}+2\right)\varepsilon.$$

We define $\delta := \left(\sqrt{5}+2\right)\varepsilon$. *Then, for every* $x \in [2\,,\,+\infty[$ *satisfying* $|x-6| \leq \left(\sqrt{5}+2\right)\varepsilon$ *we have* $\left|g(x)-2\right| \leq \varepsilon$.

Exercise 1.1.10. *Using* ε *and* δ *prove the following statements*

$$\lim_{x\to 0}\sin(x) = 0,\ \lim_{x\to 1}\sqrt{x+3} = 2,\ \lim_{x\to 3}\frac{3x+5}{x+1} = \frac{7}{2}.$$

From now on, we define the limit of a function at a point x_0 which is in the domain. Let x_0 be a real number and f a real function with domain $]a\,,\,x_0[\cup]x_0\,,\,b[$. In the definition that will follow we define the limit of the function at the point x_0.

Definition 1.1.11. *Let* $(a\,,\,b)$ *be an interval of* $\mathbb{R}$. *We consider a real number* $x_0 \in (a\,,\,b)$ *and* $\ell \in \mathbb{R}$. *We consider the real function* f *with domain* $(a\,,\,b)\backslash\{x_0\}$. *We say that* ℓ *is the limit of* f, *as* x *approaches* x_0 *if*

$$\forall\varepsilon > 0,\ \exists\delta > 0,\ \forall x \in \mathbb{R},\ \Big(\ |x-x_0| < \delta \Rightarrow\ |f(x)-\ell| < \varepsilon\Big). \tag{1.1.3}$$

In this case we write

$$\lim_{x\to x_0} f(x) = \ell.$$

Remark 1.1.12. *From this definition we obtain the information that, the limit of a function does not depend on its value at* x_0. *Indeed, as we can see it here the function* f *is not defined at* x_0. *This means that the limit can be different to* $f(x_0)$. *Because* $f(x_0)$ *is not defined.*

1.1.1 Operations in the Set of Limits

In the first year of Bachelor degree,we will be able to prove that the set of all finite limits is an *algebra*. That is, we can do some operation on limits. Admitting this property, here we calculate sums and multiplications of limits. The first result of this subsection deals with sum of limits. Throughout this subsection we deal only with finite limits.

Theorem 1.1.13. *Let* f *and* g *be two real functions defined in some interval* I, *which contains* x_0. *We consider* ℓ_1 *and* ℓ_2 *two real numbers, such that,*

$$\lim_{x\to x_0} f(x) = \ell_1 \quad \textit{and} \quad \lim_{x\to x_0} g(x) = \ell_2.$$

Then, we have

$$\lim_{x\to x_0} f(x)+g(x) = \ell_1+\ell_2. \tag{1.1.4}$$

Proof. Let ℓ_1 and ℓ_2 be two real number, such that,

$$\lim_{x\to x_0} f(x) = \ell_1 \quad \text{and} \quad \lim_{x\to x_0} g(x) = \ell_2.$$

Then for all $\varepsilon > 0$ we can find two real numbers $\delta_1 > 0$ and $\delta_2 > 0$ such that, for all $x \in \mathbb{R}$,

$$\begin{aligned} |x - x_0| < \delta_1, \quad &\Longrightarrow \big|f(x) - \ell_1\big| < \frac{\varepsilon}{2} \\ |x - x_0| < \delta_2, \quad &\Longrightarrow \big|g(x) - \ell_2\big| < \frac{\varepsilon}{2}. \end{aligned}$$

Now we define $\delta = \min(\delta_1\,;\,\delta_2)$ and we consider $x \in \mathbb{R}$ such that

$$|x - x_0| < \delta. \tag{1.1.5}$$

On the other hand we know that,

$$\big|f(x) + g(x) - (\ell_1 + \ell_2)\big| = \big|(f(x) - \ell_1) - (g(x) - \ell_2)\big| \le \big|f(x) - \ell_1\big| + \big|g(x) - \ell_2\big|.$$

We remind that since $|x - x_0| < \delta$, then $|x - x_0| < \delta_1$ and $|x - x_0| < \delta_2$. Therefore $\big|f(x) - \ell_1\big| < \dfrac{\varepsilon}{2}$ and $\big|g(x) - \ell_2\big| < \dfrac{\varepsilon}{2}$. This involves,

$$\big|(f(x) + g(x)) - (\ell_1 + \ell_2)\big| < \varepsilon.$$

Therefore we have proved

$$\lim_{x\to x_0} f(x) + g(x) = \ell_1 + \ell_2.$$

□

Example 1.1.14. *Let $f(x) = 3\,x + 2$ and $g(x) = x^2 + 1$. We have*

$$\lim_{x\to 1} f(x) = 5 \quad \textit{and} \quad \lim_{x\to 1} g(x) = 3.$$

From this, we deduce

$$\lim_{x\to 1} \big(f + g\big)(x) = \lim_{x\to 1} f(x) + \lim_{x\to 1} g(x) = 8.$$

Exercise 1.1.15. *Find the limit of the following functions*

$$\lim_{x\to 1} x^3 + 4x^2 + 3x + 1,\ \lim_{x\to 2} \sqrt{x^2 + 3x + 1} + \sqrt{x^2 + 1},\ \lim_{x\to 3} \frac{1}{x+1} + x + 2,$$

$$\lim_{x\to -1} \frac{1}{x^2 + 3x + 1} + \frac{1}{x+3},\ \lim_{x\to -3} \frac{x+3}{x^2 + 10} + \sqrt{x^2 + 2x + 1}.$$

A similar result can be established for multiplication of functions.

Theorem 1.1.16. *Let f and g be two real functions defined in some interval I. We assume that x_0 belongs to I and we consider two reals numbers ℓ_1 and ℓ_2 such that,*

$$\lim_{x\to x_0} f(x) = \ell_1 \quad \textit{and} \quad \lim_{x\to x_0} g(x) = \ell_2.$$

Then,

$$\lim_{x\to x_0} f(x) \cdot g(x) = \ell_1 \cdot \ell_2. \tag{1.1.6}$$

Proof. Let f, g, x_0, ℓ_1 and ℓ_2 be as in the theorem. Then for any $\varepsilon > 0$, there are two real numbers $\delta_1 > 0$, and $\delta_2 > 0$, such that, for every $x \in \mathbb{R}$,

$$\begin{aligned} |x - x_0| < \delta_1, &\Rightarrow \big|f(x) - \ell_1\big| < \frac{\varepsilon}{2(1 + |\ell_2|)} \\ |x - x_0| < \delta_2, &\Rightarrow \big|g(x) - \ell_2\big| < \frac{\varepsilon}{2(1 + |\ell_1|)}. \end{aligned}$$

Let $x \in \mathbb{R}$, we have

$$\begin{aligned} f(x) \cdot g(x) - \ell_1 \cdot \ell_2 &= f(x) \cdot g(x) - g(x) \cdot \ell_1 + g(x) \cdot \ell_1 - \ell_1 \cdot \ell_2 \\ &= g(x)\Big(f(x) - \ell_1\Big) + \ell_1\Big(g(x) - \ell_2\Big). \end{aligned}$$

From this we deduce that $\Big|f(x) \cdot g(x) - \ell_1 \cdot \ell_2\Big| \leq \big|g(x)\big| \cdot \big|f(x) - \ell_1\big| + \big|\ell_1\big|\big|g(x) - \ell_2\big|$. We point out that $\big|g(x)\big| = \big|g(x) - \ell_2 + \ell_2\big| \leq \big|g(x) - \ell_2\big| + \big|\ell_2\big| \leq 1 + \big|\ell_2\big|$. This implies that

$$\begin{aligned} \big|f(x) \cdot g(x) - \ell_1 \cdot \ell_2\big| &< (1 + |\ell_1|) \cdot \frac{\varepsilon}{2(1 + |\ell_1|)} \\ &\quad + (1 + |\ell_2|) \cdot \frac{\varepsilon}{2(1 + |\ell_2|)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore

$$\lim_{x \to x_0} f(x) \cdot g(x) = \ell_1 \cdot \ell_2.$$

This completes the proof. □

Example 1.1.17. *We consider $f(x) = x + 3$ and $g(x) = x + 1$. We have*

$$\lim_{x \to 0} f(x) = 3 \quad \textit{and} \quad \lim_{x \to 0} g(x) = 1.$$

Therefore using the theorem above we obtain

$$\lim_{x \to 0} \big(f \cdot g\big)(x) = \lim_{x \to 0} f(x) \cdot \lim_{x \to 0} g(x) = 3.$$

Exercise 1.1.18. *Find the following limits*

$$\lim_{x \to 1}(3x + 2)(2x + 1), \ \lim_{x \to 3}(x^2 + 2x + 3)(5x + 8), \ \lim_{x \to 0}(x + 1)\sqrt{x^2 + 2x + 6}\,.$$

Lemma 1.1.19. *Let f be a real function defined in some interval I, which contains x_0. We consider a real number $\ell \neq 0$, and we assume*

$$\lim_{x \to x_0} f(x) = \ell.$$

Then,

$$\lim_{x \to x_0} \frac{1}{f(x)} = \frac{1}{\ell}. \tag{1.1.7}$$

Proof. Let f, x_0 and ℓ be as in the Lemma. We suppose

$$\lim_{x\to x_0} f(x) = \ell \neq 0.$$

Hence there exist $\delta > 0$, such that $|x - x_0| < \delta$, implies $\big|f(x) - \ell\big| < \frac{\varepsilon|\ell|^2}{2}\cdot$ One can also find $\delta_0 > 0$ such that $|x - x_0| < \delta_0$, implies $\big|f(x) - \ell\big| < \frac{|\ell|}{2}\cdot$ Defining $\delta_1 = \min(\delta\,;\,\delta_0)$. we have, for all $x \in \mathbb{R}$, such that $|x - x_0| < \delta_1$,

$$\big|f(x) - \ell\big| < \frac{\varepsilon|\ell|^2}{2} \quad \text{and} \quad \big|f(x) - \ell\big| < \frac{|\ell|}{2}\cdot$$

This implies that

$$|\ell| = \big|\ell - f(x) + f(x)\big| \leq \big|\ell - f(x)\big| + \big|f(x)\big| < \frac{|\ell|}{2} + \big|f(x)\big|.$$

From this, we deduce $\frac{|\ell|}{2} < \big|f(x)\big|$. This leads to the following inequality

$$\frac{1}{|f(x)|} < \frac{2}{|\ell|}\cdot$$

On the other part, we have

$$\left|\frac{1}{f(x)} - \frac{1}{\ell}\right| = \frac{\big|\ell - f(x)\big|}{|\ell|\,\big|f(x)\big|} < \frac{\varepsilon\,|\ell|^2}{2\,|\ell|}\cdot\frac{2}{|\ell|} = \varepsilon.$$

Therefore

$$\lim_{x\to x_0}\frac{1}{f(x)} = \frac{1}{\ell}\cdot$$

□

Theorem 1.1.20. *Let f and g be two real functions defined in some interval I, which contains x_0. We consider two real numbers ℓ_1 and ℓ_2, such that,*

$$\lim_{x\to x_0} f(x) = \ell_1 \quad \textit{and} \quad \lim_{x\to x_0} g(x) = \ell_2.$$

Moreover we assume $\ell_2 \neq 0$. Then

$$\lim_{x\to x_0}\frac{f(x)}{g(x)} = \frac{\ell_1}{\ell_2}\cdot$$

Proof. Let f, g, ℓ_1 and ℓ_2 be as in the theorem. In a neighborhood of x_0, we can write

$$\frac{f(x)}{g(x)} = f(x)\cdot\frac{1}{g(x)}\cdot$$

Therefore using Theorem 1.1.16 and Lemma 1.1.19, we obtain

$$\lim_{x\to x_0}\frac{f(x)}{g(x)} = \lim_{x\to x_0} f(x)\cdot\lim_{x\to x_0}\frac{1}{g(x)} = \ell_1\cdot\frac{1}{\ell_2} = \frac{\ell_1}{\ell_2}\cdot$$

□

Example 1.1.21. *Let $f(x) = x + 2$ and $g(x) = x^2 + 1$. Then, one observes that*

$$\lim_{x\to 1} f(x) = 3 \quad \textit{and} \quad \lim_{x\to 1} x^2 + 1 = 2.$$

This means that

$$\lim_{x\to 1} \frac{f(x)}{g(x)} = \frac{\lim_{x\to 1} f(x)}{\lim_{x\to 1} g(x)} = \frac{3}{2}.$$

Exercise 1.1.22. *Determine the following limits*

$$\lim_{x\to -1} \frac{x^2+3x+2}{x+3},\ \lim_{x\to 2} \frac{2x+3}{x+2},\ \lim_{x\to 0} \frac{x^3+2x+1}{x^2+3x+5}.$$

Sometimes it is more complicate to fine the limit of a function at some points. Because in some cases the limits are undefined. We will explain late on what we mean by undefined limits. Another problem can happen when we deal with oscillating function like cos or sin. In this case it is difficult to to obtain a unique limit. To treat these type of limits what we usually do is to bound the function that we want to find the limit by two other functions that have the same limit. Indeed let f be the function to which we cannot unfortunately find directly a limit. What we do is to use two other functions g and h such that $g \leq f \leq h$ and g and h have the same limit. From this we deduce the limit of f. But we have a little bit problem because we did not prove that if $f \leq h$, we have $\lim f \leq \lim h$. To establish this property, we proceed in three steps:

Step 1: we compare f to 0

Step 2: we compare f to a real number $M \neq 0$

Step 3: we compare f to a real function

Theorem 1.1.23. *Let f be a real function with domain D. We assume for every $x \in D$, $f(x) \geq 0$. Then, we have*

$$\lim_{x\to x_0} f(x) \geq 0.$$

Before doing the proof of this theorem, we remind the following logical properties . Proving the property (Proposition A $\Longrightarrow$ Proposition B) is equivalent to prove the following statement (nonProposition B $\Longrightarrow$ nonProposition A) .

Another method which is frequently used in demonstration of theorem is the proof by contradiction. Suppose we have to prove a Proposition A. We assume non-Proposition A holds and we establish that this leads to a contradiction or something which does not make sens.

We mean by nonProposition A the contrary of proposition A. For instance we define, Proposition A: I go to school. Then, nonProposition A : I don't go to school.

Proof. Here we will proceed by contradiction. Let f be a real function with domain D such that for every $x \in D$ we have $f(x) \geq 0$. We suppose $\lim_{x\to x_0} f(x) = \ell < 0$. Let $\varepsilon > 0$ be such that $\ell + \varepsilon < 0$. We can find $\delta > 0$, such that $|x - x_0| < \delta$ implies $\big|f(x) - \ell\big| \leq \varepsilon$. This is possible because $\lim_{x\to x_0} f(x) = \ell$. The inequality $\big|f(x) - \ell\big| \leq \varepsilon$ implies $f(x) \leq \ell + \varepsilon < 0$. Then, for every $x \in \big(-\delta + x_0\,,\,\delta + x_0\big)$, we

have $f(x) < 0$. This is in contradiction with the hypothesis that $f(x) \geq 0$ for all $x \in D$. Therefore $\ell \geq 0$ and we write

$$\lim_{x \to x_0} f(x) \geq 0.$$

□

Remark 1.1.24. *We draw the readers attention that strict inequalities do not pass to the limit. In other words if* $\forall\, x \in D$, $f(x) > 0$, *we do not necessarly have*

$$\lim_{x \to x_0} f(x) > 0.$$

As a counter example we define the function $f(x) = \frac{1}{x}$ for any $x \in \big(0\,,\, +\infty\big)$. Then,for every $x > 0$, we have $f(x) > 0$. But we know that

$$\lim_{x \to +\infty} f(x) = 0.$$

Theorem 1.1.25. *Let f be a real function with domain D. We consider a real number M and we assume for every $x \in D$, $f(x) \geq M$. Then,*

$$\lim_{x \to x_0} f(x) \geq M.$$

Proof. To prove this theorem, we define the function $g(x) := f(x) - M$. Then for all $x \in D$, we have $g(x) \geq 0$. Now we apply theorem 1.1.23 to the function g to obtain

$$\lim_{x \to x_0} g(x) \geq 0 \quad \Longrightarrow \quad \lim_{x \to x_0} f(x) \geq M.$$

□

Theorem 1.1.26. *Let f and g be two real functions defined in some interval I. We suppose that for every $x \in I$, we have $f(x) \geq g(x)$. Then*

$$\lim_{x \to x_0} f(x) \geq \lim_{x \to x_0} g(x).$$

Proof. Let $x \in I$,such that $f(x) = g(x)$. Then, one has

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x).$$

Now we suppose that for all $x \in I$, $f(x) > g(x)$. There exists a real number $\varepsilon_x > 0$ depending on x such that, $f(x) = g(x) + \varepsilon_x$. If we agree that $\lim_{x \to x_0} g(x) = \ell$, We have $\lim_{x \to x_0} g(x) + \varepsilon_x = \ell + \lim_{x \to x_0} \varepsilon_x \geq \ell$. Here we use an inequality large because maybe $\lim_{x \to x_0} \varepsilon_x = 0$. This involves $\lim_{x \to x_0} f(x) \geq \lim_{x \to x_0} g(x)$. □

The following theorem is helpful for dealing with limits of oscillating functions.

Theorem 1.1.27 (**Sandwich Theorem**). *Let f, g and h be three functions defined in an interval I, which contains the real number x_0. We suppose for any $x \in I$, $g(x) \leq f(x) \leq h(x)$ and*

$$\lim_{x \to x_0} g(x) = \lim_{x \to x_0} h(x) = \ell, \tag{1.1.8}$$

where ℓ is some real number. Then, we have

$$\lim_{x \to x_0} f(x) = \ell.$$

Proof. Let f, g, h, x_0 and ℓ be as in the theorem. Since $\lim_{x\to x_0} g(x) = \ell$, then for all $\varepsilon_1 > 0$, there exist $\delta_1 > 0$, such that for any $x \in I$, $|x - x_0| < \delta_1$ implies

$$\ell - \varepsilon_1 < g(x) < \ell + \varepsilon_1. \tag{1.1.9}$$

On the other hand, we know that for all $\varepsilon_2 > 0$, there exists $\delta_2 > 0$, such that, for all $x \in I$, $|x - x_0| < \delta_2$, implies

$$\ell - \varepsilon_2 < h(x) < \ell + \varepsilon_2. \tag{1.1.10}$$

Because we have $\lim_{x\to x_0} h(x) = \ell$.

Now we set $\delta := \min(\delta_1\,;\,\delta_2)$ and $\varepsilon := \max(\varepsilon_1\,;\,\varepsilon_2)$. Using (1.1.9) and (1.1.10) we obtain, for all $x \in I$, such that, $|x - x_0| < \delta$, $\ell - \varepsilon_1 < f(x) < \ell + \varepsilon_2$. This yields

$$-\varepsilon_1 < f(x) - \ell < \varepsilon_2.$$

Since $-\varepsilon \le -\varepsilon_1$ and $\varepsilon_2 \le \varepsilon$, we deduce, for $x \in I$, such that $|x - x_0| < \delta$, we have $-\varepsilon < f(x) - \ell < \varepsilon$. This means that,

$$\big|f(x) - \ell\big| \le \varepsilon.$$

Hence

$$\lim_{x\to x_0} f(x) = \ell.$$

□

Example 1.1.28. *We consider the function* $f(x) = x\sin\left(\frac{1}{x}\right)$. *This function is defined* $\mathbb{R}\backslash\{0\}$. *We know that for all* $x \in \mathbb{R}\backslash\{0\}$, $-1 \le \sin\left(\dfrac{1}{x}\right) \le 1$. *Multiplying by* x *in both sides , we obtain*

$$-x \le x\,\sin\left(\frac{1}{x}\right) \le x.$$

As we know that $\lim_{x\to 0} -x = \lim_{x\to 0} x = 0$. *We conclude that*

$$\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0.$$

Exercise 1.1.29. *Find the following limits*

$$\lim_{x\to 0} x^2\cos\left(\frac{1}{x}\right),\ \lim_{x\to 0} x\sin\left(\frac{1}{x^3}\right),\ \lim_{n\to+\infty} \frac{\cos(n^2)}{n}.$$

To end this subsection we define limits of function of the form $g \circ f$ where f and g are two real functions.

Theorem 1.1.30. *We consider three real numbers* x_0, y_0, *and* ℓ. *Let* f *be a real function defined in an interval* I *which contains* x_0 *into some interval* J *and* g *a real function defined in the interval* J, *which contains* y_0. *We assume*

$$\lim_{x\to x_0} f(x) = y_0 \quad \textit{and} \quad \lim_{y\to y_0} g(y) = \ell.$$

Then

$$\lim_{x\to x_0} g(f(x)) = \ell. \tag{1.1.11}$$

Proof. We remind that $\lim_{x\to x_0} g(f(x)) = \ell$ means :

$$\forall\, \varepsilon > 0,\ \exists\, \delta > 0,\ \forall\, x \in I,\ |x - x_0| < \delta \implies \big|g\big(f(x)\big) - \ell\big| < \varepsilon.$$

The assumption $\lim_{y\to y_0} g(y) = \ell$ implies for all $\varepsilon > 0$, there exists $\eta > 0$ depending on ε such that $|y - y_0| < \eta$ implies $\big|g(y) - \ell\big| < \varepsilon$.

On the other hand we know $\lim_{x\to x_0} f(x) = y_0$. Therefore, for any $\varepsilon_1 > 0$, there exists δ depending on ε_1, such that, $|x - x_0| < \delta$ implies $\big|f(x) - y_0\big| < \varepsilon_1$. Since this property holds for any positive real number ε_1, in particular it is true for $\varepsilon_1 = \eta$. In this case we can find a δ depending on η such that $|x - x_0| < \delta$ leads to $|f(x) - y_0| < \eta$. Since δ depends on η and η depends on ε, therefore δ depends on ε.

We have proved that for all $\varepsilon > 0$, there exists $\delta > 0$, such that $|x - x_0| < \delta$ gives

$$\big|g(f(x)) - \ell\big| < \varepsilon.$$

Consequently we have

$$\lim_{x\to x_0} g\big(f(x)\big) = \ell.$$

□

We draw the readers attention on the fact that some functions can have infinite limits. Indeed let $f(x) = \dfrac{1}{x}$ for any $x \neq 0$. We can see that

$$\lim_{x\to 0} f(x) = \pm\infty.$$

In the next section we study infinite limits of functions.

1.2 Convergence to Infinity

Let I be an interval which contains the real number x_0. We consider a real function f defined in I maybe excepted at x_0. We say that $f(x)$ converge to $\pm\infty$ ($\pm$ infinity), when x goes to x_0 if

$$\lim_{x\to x_0} f(x) = \pm\infty.$$

Example 1.2.1. *Consider the real function* $f(x) = \dfrac{1}{x}$. *We can easily check that*

$$\lim_{x\to 0} f(x) = \pm\infty.$$

Definition 1.2.2. *Let f be a real function defined in I, which contains x_0. We assume that f is not defined at x_0. We say that the line $x = x_0$ is a vertical asymptote for f, if*

$$\lim_{x\to x_0} f(x) = \pm\infty.$$

Example 1.2.3. *Let* $f(x) = \dfrac{1}{x+1}$. *The function f is defined in $\mathbb{R}\backslash\{-1\}$. We know that*

$$\lim_{x\to -1^+} f(x) = +\infty \quad \textit{and} \quad \lim_{x\to -1^-} f(x) = -\infty.$$

Then the line which has equation $x = -1$ is a vertical asymptote for the function f.

Example 1.2.4. *We consider the following function* $g(x) = \dfrac{3}{x-2}\cdot$ *We have* $D_g = \mathbb{R}\backslash\{2\}$ *and*

$$\lim_{x\to 2^+} f(x) = +\infty \quad \textit{and} \quad \lim_{x\to 2^-} f(x) = -\infty.$$

Therefore line with equation $x = 2$ *is a vertical asymptote for* f.

Remark 1.2.5. *Vertical asymptotes can only hold at a point on which the function is not defined.*

Exercise 1.2.6. *Define vertical asymtotes for the following functions.*

$$f(x) = \frac{x+1}{x^2-1},\ g(x) = \frac{1}{x^2+3x+2},\ h(x) = \frac{5}{x},\ i(x) = \frac{4}{x+5},\ j(x) = \frac{x+3}{x^2+2x+1}\cdot$$

Up to now we have only defined limits at points which are interior to some interval. In the next section, we define limits at the bounds of an open domain.

1.3 Limit of a Function at Infinity

In this section we consider function which are defined in intervals of the form $\big(-\infty\,,\,a\big)$ or $[a\,,\,+\infty)$, or $\big(-\infty\,,\,+\infty\big)$, where a is a real number. That is, we study limits functions at $\pm\infty$.

Definition 1.3.1. *Let* f *be a function defined in the interval* $[a\,,\,+\infty)$ *and* $\ell \in \mathbb{R}$. *We say that the real number* ℓ *is the limit of* f *at* $+\infty$ *if*

$$\lim_{x\to+\infty} f(x) = \ell.$$

Example 1.3.2. *We consider the function* $f(x) = \dfrac{1}{x^2}\cdot$ *The domain of the function* f *is* $D_f = (-\infty\,,\,0\,) \cup (\,0\,,\,+\infty\,)$. *We can see that*

$$\lim_{x\to+\infty} f(x) = 0.$$

Definition 1.3.3. *Let* f *be a real function defined in the interval* $(-\infty\,,\,a]$. *We that say the real number* ℓ *is the limit of* f *at* $-\infty$ *if*

$$\lim_{x\to-\infty} f(x) = \ell.$$

Example 1.3.4. *We take the function* $f(x) = \dfrac{1}{x+3}\cdot$ *The function* f *is defined in* $(-\infty\,,\,-3) \cup (-3\,,\,+\infty)$. *We can check that*

$$\lim_{x\to-\infty} f(x) = 0.$$

Finding limits at $\pm\infty$ can be a hard task. But sometimes the expression of the function leads to some simplifications. Let f be a polynomial function, we will prove that the limit of the function f at $\pm\infty$ is nothing but the limit of the term with the highest degree.

Theorem 1.3.5. *Let $f(x) = a_n\, x^n + \cdots + a_1\, x + a_0$, with $n \geq 1$ and $a_n \neq 0$ be a polynomial function of degree n. Then, we have*

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} a_n x^n.$$

Proof. To prove this theorem we define $f(x) = a_n\, x^n + \cdots + a_n\, x + a_0$, such that $a_n \neq 0$. Since x goes to $\pm\infty$ we can suppose $x \neq 0$. We write the function f in this way

$$f(x) = a_n x^n \left(1 + \frac{a_{n-1}}{a_n x} + \cdots \frac{a_1}{a_n x^{n-1}} + \frac{a_0}{a_n x^n}\right)\cdot$$

Using the fact that

$$\lim_{x\to\pm\infty} \frac{a_{n-1}}{a_n x} = \cdots = \lim_{x\to\pm\infty} \frac{a_0}{a_n x^n} = 0,$$

one deduces that,

$$\lim_{x\to\pm\infty} 1 + \frac{a_{n-1}}{a_n x} + \cdots + \frac{a_1}{a_n x^{n-1}} + \frac{a_0}{a_n x^n} = 1\cdot$$

We conclude that

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} a_n x^n.$$

□

Theorem 1.3.6. *We consider n and m in $\mathbb{N}^*$. Let $a_n \neq 0$ and $b_m \neq 0$ be two real numbers. We define the function $f(x) = \dfrac{a_n x^n + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + b_0}\cdot$ Then, the following holds*

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{a_n x^n}{b_m x^m}\cdot$$

Proof. Let $a_n \neq 0$ and $b_m \neq 0$ be two real numbers. Since x tends to $\pm\infty$ we can suppose it different to 0. Now we observe that

$$f(x) = \frac{a_n x^n}{b_m x^m} \cdot \frac{1 + \dfrac{a_{n-1}}{a_n x} + \dfrac{a_{n-2}}{a_n x^2} \cdots + \dfrac{a_0}{a_n x^n}}{1 + \dfrac{b_{m-1}}{b_m x} + \dfrac{b_{m-2}}{b_m x^2} \cdots + \dfrac{b_0}{b_m x^m}}\cdot$$

Remarking that

$$\lim_{x\to\pm\infty} 1 + \frac{a_{n-1}}{a_n x} + \cdots + \frac{a_0}{a_n x^n} = 1, \quad \text{and} \quad \lim_{x\to\pm\infty} 1 + \frac{b_{m-1}}{b_m x} + \cdots + \frac{b_0}{b_m x^m} = 1,$$

we obtain

$$\lim_{x\to\pm\infty} \frac{1 + \dfrac{a_{n-1}}{a_n x} + \dfrac{a_{n-2}}{a_n x^2} \cdots + \dfrac{a_0}{a_n x^n}}{1 + \dfrac{b_{m-1}}{b_m x} + \dfrac{b_{m-2}}{b_m x^2} \cdots + \dfrac{b_0}{b_m x^m}} = 1\cdot$$

Consequently, we have

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{a_n x^n}{b_m x^m}\cdot$$

□

Example 1.3.7. *We consider the function* $f(x) = \dfrac{x^3+3x+5}{x^2+6x+1}$. *Then, we have*

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x^3}{x^2} = \lim_{x\to+\infty} x = +\infty.$$

Example 1.3.8. *We define the function* $f(x) = \dfrac{2x^2+x+4}{x+1}$. *One has*

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{2x^2}{x} = \lim_{x\to-\infty} 2x = -\infty.$$

Exercise 1.3.9. *Define the limit of the following functions at* $+\infty$.

$$f(x) = \frac{x+3}{x^2+2x+1},\ \ g(x) = \frac{x+1}{x+2},\ \ h(x) = \frac{x^2+2x+3}{x^2+4x+1}.$$

Exercise 1.3.10. *Determine the limit of the following functions at* $-\infty$

$$f(x) = \frac{x+1}{x^3+2x+3},\ \ g(x) = \frac{x^2+1}{x^2+2x+1},\ \ h(x) = \frac{x^2}{x^2+3}.$$

Definition 1.3.11. *Let* f *be a real function defined in the interval* $[a\,,+\infty)$. *We say that the line with equation* $y = \ell_1$ *is a horizontal asymptote for* f *at* $+\infty$ *if*

$$\lim_{x\to+\infty} f(x) = \ell_1.$$

Definition 1.3.12. *Let* f *be a real function defined in the interval* $(-\infty\,,a]$. *We say that the line* $y = \ell_2$ *is a horizontal asymptote for* f *at* $-\infty$ *if*

$$\lim_{x\to-\infty} f(x) = \ell_2.$$

Example 1.3.13. *Let* $f(x) = \dfrac{x+3}{x+1}$. *Then, we have*

$$\lim_{x\to+\infty} f(x) = 1.$$

Hence the line $y = 1$, *is a horizontal asymptote for the function* f *at* $+\infty$.

Example 1.3.14. *We consider the function* $g(x) = \dfrac{1}{x+3}$. *We can see that*

$$\lim_{x\to-\infty} g(x) = 0.$$

Then, the line $y = 0$, *is a horizontal asymptote for the function* f *at* $-\infty$.

Exercise 1.3.15. *Say whether the following functions have horizontal asymptote*

$$f(x) = \frac{x+6}{x+1},\ g(x) = \frac{|x|}{x+1},\ h(x) = \frac{|x+3|}{x+9},\ i(x) = \frac{x^2+3x+7}{x^2+3},\ j(x) = \frac{x}{|x-1|}.$$

One may think that the limit of a function at $\pm\infty$ is always finite. Here we will specify that the limit of a function at $\pm\infty$ can be infinite. In truth, most of the time these limits are infinite. A simplest example can be given by the function $f(x) = x$. We see

$$\lim_{x\to+\infty} f(x) = +\infty \quad \text{and} \quad \lim_{x\to-\infty} f(x) = -\infty.$$

Before going further, it is important to explain what we mean by a function has limit $\pm\infty$.

By f tends to $+\infty$ when x approaches $+\infty$, we mean that the values of $f(x)$ can be as greater as possible in the condition that x be as greater as possible. In other words for any real number $A > 0$, we can find x in the domain of f such that the value $f(x)$ be greater than A. But this holds when x is greater enough. The expression x greater enough means there exists a big number x_0 such that $x \in [x_0\,, +\infty)$. This leads to the following definition.

Definition 1.3.16. *Let f be a real function. The function f has limit $+\infty$ when x goes to $+\infty$ if : $\forall\, A > 0,\ \exists\, x_0 > 0,\ \forall\, x \in D_f,\ x \geq x_0,\ \Longrightarrow f(x) > A$.*

In this case we write

$$\lim_{x\to+\infty} f(x) = -\infty.$$

Definition 1.3.17. *Let f be a real function. The function f has limit $-\infty$ at $+\infty$ if : $\forall\, M \in \mathbb{R},\ \exists\, x_0 > 0,\ \forall\, x \in D_f,\ x \geq x_0,\ \Longrightarrow f(x) < M$.*

The definitions of the following expressions

$$\lim_{x\to-\infty} f(x) = -\infty \quad \text{and} \quad \lim_{x\to-\infty} f(x) = +\infty$$

are stated below.

Definition 1.3.18. *Let f be a real function. We say the function f has limit $-\infty$ at $-\infty$ if : $\forall\, A > 0,\ \exists\, x_0 > 0,\ \forall\, x \in D_f,\ x \leq -x_0,\ \Longrightarrow f(x) < -A$.*

Definition 1.3.19. *Let f be a real function. We have $\lim_{x\to-\infty} f(x) = +\infty$ if : $\forall\, A > 0,\ \exists\, x_0 > 0,\ \forall\, x \in D_f,\ x \leq -x_0,\ \Longrightarrow f(x) > A$.*

In order to show how to deal with these types of limits we introduce the following section.

1.4 Infinite Limit of a Function at Infinity

This section is devoted to functions that have infinite limits, when x approaches $\pm\infty$. Some function has complicate expression, in these cases determining their limits can be so difficult. To find the limits of these types of functions, we try to bound them below or above by elementary functions that we can determine easily their limits.Thus we deduce their limits from limits of such elementary functions.To clarify what mean, let f and g be two functions defined in some interval $[a\,, +\infty)$. We assume the limit of the function g is known and $g(x) \leq f(x)$ for all $x \in [a\,, +\infty)$. Using the limit of the function g we can find the limit of f when x tends to $+\infty$. To do so we use a property called the comparison principle. This property is enunciated in the following theorems.

Theorem 1.4.1. *Let f and g be two real functions defined in the interval $[a\,,\,+\infty)$, such that for all $x \in [a\,;\,+\infty)$, $f(x) \geq g(x)$ and*

$$\lim_{x\to+\infty} g(x) = +\infty.$$

Then, one has

$$\lim_{x\to+\infty} f(x) = +\infty.$$

Since $f(x) \geq g(x)$ for all x in $[a\,,\,+\infty)$ we have

$$\lim_{x\to x_0} f(x) \geq \lim_{x\to x_0} g(x).$$

Taking here $x_0 = +\infty$, we obtain

$$+\infty = \lim_{x\to+\infty} g(x) \leq \lim_{x\to+\infty} f(x).$$

Therefore,

$$\lim_{x\to+\infty} f(x) = +\infty.$$

To rephrase this, we say if a function f is bounded below by a function g which increases to $+\infty$, then f must go to $+\infty$.

Theorem 1.4.2. *Let f and g be two real functions defined in $[a\,,\,+\infty)$, such that for all $x \in [a\,;\,+\infty)$, $f(x) \geq g(x)$ and*

$$\lim_{x\to+\infty} f(x) = -\infty.$$

Then,

$$\lim_{x\to+\infty} g(x) = -\infty.$$

The principle of the proof is the same. As we know for every $x \in [a\,,\,+\infty)$, $f(x) \geq g(x)$, then

$$\lim_{x\to x_0} f(x) \geq \lim_{x\to x_0} g(x).$$

Now we take $x_0 = +\infty$. Hence we obtain

$$\lim_{x\to+\infty} g(x) \leq \lim_{x\to+\infty} f(x) = -\infty.$$

We conclude that

$$\lim_{x\to+\infty} g(x) = -\infty.$$

Example 1.4.3. *Let $f(x) = x^2$ and $g(x) = x^2 + 1$. One has $f(x) \leq g(x)$ for all $x \in \mathbb{R}$. On the other hand, we know that*

$$\lim_{x\to+\infty} f(x) = +\infty.$$

Hence

$$\lim_{x\to+\infty} g(x) = +\infty.$$

As we mentioned it before in this section we are dealing with functions which have limits $\pm\infty$ at $\pm\infty$. Since we know these functions go to $\pm\infty$ when $x \to +\infty$, we need to know how fast they converge to $\pm\infty$. In other words we study their asymptotic behavior. To this end we compare them to affine functions. This comparison leads to the key concept of oblique asymptotes

Definition 1.4.4. *Let f be a real function with domain $D_f = [a\,, +\infty)$. We define the line d that has equation $y = a\,x + b$, where $a \neq 0$. We say that the line d is an oblique asymptote of f at $+\infty$ if*

$$\lim_{x\to+\infty} \Big(f(x) - ax - b\Big) = 0.$$

Remark 1.4.5. *What this definition says is, when x goes to $+\infty$ the behavior of the function f resembles to the behavior of the line d.*

Definition 1.4.6. *Let f be real function with domain $D_f = (-\infty\,, a]$. We define a line $d : y = a\,x + b$, where $a \neq 0$. We say the line d is an oblique asymptote of f at $-\infty$ if*

$$\lim_{x\to-\infty} \Big(f(x) - ax - b\Big) = 0.$$

The question now is, how to define the equation of the line d. This is equivalent to find a and b.

Since we know that $\lim_{x\to\pm\infty} f(x) = \pm\infty$, to determine the value of a we proceed in this way: We compute

$$\lim_{x\to\pm\infty} \big(f(x)/x\big).$$

If this limit exists (is finite) we define

$$a := \lim_{x\to\pm\infty} \frac{f(x)}{x}. \tag{1.4.1}$$

Now we consider the following limit:

$$\lim_{x\to\pm\infty} \big(f(x) - ax\big).$$

If this limit is finite we fix

$$b := \lim_{x\to\pm\infty} \left(f(x) - a\,x\right). \tag{1.4.2}$$

Example 1.4.7. *Let $f(x) = \dfrac{x^2 + 2x + 1}{x + 2}$. We can see that*

$$\lim_{x\to+\infty} f(x) = +\infty \quad \textit{and} \quad \lim_{x\to+\infty} \frac{f(x)}{x} = 1.$$

In this case we can take $a = 1$. On the other hand we have

$$\lim_{x\to+\infty} \left(f(x) - x\right) = \lim_{x\to+\infty} \frac{1}{x+1} = 0.$$

Therefore the line d with equation $y = x$ is an oblique asymptote for the function f at $+\infty$.

Example 1.4.8. *We consider the function* $g(x) = \frac{2x^3+3x^2+2x+1}{x^2+2x+2}$. *Wehave*

$$\lim_{x\to-\infty} g(x) = \lim_{x\to-\infty} \frac{2x^3}{x^2} = -\infty.$$

We can observe that

$$\lim_{x\to-\infty} \frac{g(x)}{x} = \lim_{x\to-\infty} \frac{2x^3+3x^2+2x+1}{x^3+2x^2+2x} = \lim_{x\to-\infty} \frac{2x^3}{x^3} = 2.$$

Therefore, we have $a = 2$. *We know that*

$$\begin{aligned}\lim_{x\to-\infty} \Big(g(x) - 2x\Big) &= \lim_{x\to-\infty} \frac{2x^3+3x^2+2x+1}{x^2+2x+2} - 2x \\ &= \lim_{x\to-\infty} \frac{2x^3+3x^2+2x+1-2x^3-4x^2-4x}{x^2+2x+2} \\ &= \lim_{x\to-\infty} \frac{-x^2-2x+1}{x^2+2x+2} = \lim_{x\to-\infty} \frac{-x^2}{x^2} = -1.\end{aligned}$$

We conclude that the line $y = 2x - 1$ *is an oblique asymptote for the function* g *at* $-\infty$.

Exercise 1.4.9. *Define oblique asymptotes for the following functions*

$$f(x) = \frac{x^2+6x+1}{x+1},\ g(x) = x^2+2x+13,\ h(x) = \frac{x^3+6x+3}{x^2+9}.$$

We recall the fact that a function is defined at some point x_0 does not mean that the function has a limit at this point x_0. As an counter example we consider the function

$$f(x) = \begin{cases} 3x+2, & \text{if} \quad x \geq 0 \\ \\ 4, & \text{if} \quad x < 0 \end{cases}$$

This function is defined at 0 but does not have a limit at 0. But a careful study of the two expressions which defined the function allows us to identify the notion of *left and right hand side limit.*

Definition 1.4.10. *We say* x *approaches* x_0 *to the right hand side, if* x *approaches* x_0 *and remains greater than* x_0. *In this case we write*

$$\lim_{x\to x_0^+} = \lim_{\substack{x\to x_0 \\ x>x_0}} = \lim_{x\downarrow x_0}.$$

The arrow is down because x goes to x_0 by decreasing.

Definition 1.4.11. *We say* x *approaches* x_0 *to the left hand side, if* x *approaches* x_0 *and remains less than* x_0. *In this case we write*

$$\lim_{x\to x_0^-} = \lim_{\substack{x\to x_0 \\ x<x_0}} = \lim_{x\uparrow x_0}.$$

The arrow is up, because x approaches x_0 by increasing.

Definition 1.4.12. *We say $\ell_r \in \mathbb{R}$ is the right hand side limit of f when x tends to x_0, if the value of $f(x)$ approaches ℓ_r when x approaches x_0 and remains grater than x_0. We write*

$$\lim_{x\to x_0^+} f(x) = \lim_{\substack{x\to x_0\\ x>x_0}} f(x) = \lim_{x\downarrow x_0} f(x) = \ell_r.$$

Definition 1.4.13. *We say that $\ell_l \in \mathbb{R}$ is the left hand side limit of f when x tends to x_0, if the value of $f(x)$ approaches ℓ_l when x approaches x_0 and remains less than x_0. We write*

$$\lim_{x\to x_0^-} f(x) = \lim_{\substack{x\to x_0\\ x<x_0}} f(x) = \lim_{x\uparrow x_0} f(x) = \ell_l.$$

Example 1.4.14. *We consider the function*

$$f(x) = \begin{cases} 3x+2, & \text{if} \quad x \geq 0 \\ \\ 4, & \text{if} \quad x < 0 \end{cases}$$

Then, one has

$$\lim_{x\to 0^-} f(x) = 4 \quad \text{and} \quad \lim_{x\to 0^+} f(x) = 2.$$

Exercise 1.4.15. *Find the left and the right hand side of the following functions at 1, 2 and -2*

$$f(x) = \begin{cases} x-1, & \text{if} \quad x \geq 1 \\ \\ \sqrt{x^2+2}, & \text{if} \quad x < 1, \end{cases} \qquad g(x) = \begin{cases} -x+2, & \text{if} \quad x \leq 2 \\ \\ -5x+3, & \text{if} \quad x > 2, \end{cases}$$

$$h(x) = \begin{cases} \dfrac{x^2+3x+1}{x+2}, & \text{if} \quad x \geq -2 \\ \\ \sqrt{-x+2}, & \text{if} \quad x < -2 \end{cases}$$

Theorem 1.4.16. *Let f be a real function with domain D which contains x_0. We assume $\lim_{x\to x_0^-} f(x)$ and $\lim_{x\to x_0^+} f(x)$ exist. Then, the function f has a limit at x_0 if and only if*

$$\lim_{x\to x_0^-} f(x) = \lim_{x\to x_0^+} f(x).$$

In this case we write

$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0^-} f(x) = \lim_{x\to x_0^+} f(x).$$

In some particular case the limit of the function f at x_0 is nothing but $f(x_0)$. When this situation holds we say the function f is continuous at x_0. In the next section we discuss continuity of functions.

1.5 Continuity of Functions

In this section deals with *continuous functions*. We start by defining the continuity of a function at a point. Choosing an arbitrary point we generalize this definition to interval. In the second paragraph , we introduce the IVT theorem, the mean value theorem and their corollaries. We end the section by establishing that every continuous function in a segment $[a\,;\,b]$ reaches its *maximum* and its *minimum*.

Definition 1.5.1. *Let f be a real function with domain D_f. We consider an interval $I \subset D_f$ and a real number $x_0 \in I$. We say that, the function f is continuous at x_0, if*

$$\lim_{x \to x_0} f(x) = f(x_0). \tag{1.5.1}$$

A possible interpretation of the definition above is the following one: When the real number x approaches x_0 the value of the function $f(x)$ approaches $f(x_0)$. This can be expressed in terms of ε and δ.

Definition 1.5.2. *Let f be a real function with domain D. We consider an interval $I \subset D$ which contains the real number x_0. We say the function f is continuous at x_0, if, $\forall\, \varepsilon > 0$, there exists a real number $\delta_\varepsilon(x_0) > 0$ such that, $\forall\, x \in D$*

$$|x - x_0| < \delta_\varepsilon(x_0) \quad \Longrightarrow \quad \Big|f(x) - f(x_0)\Big| < \varepsilon. \tag{1.5.2}$$

We have to specify here that, the real number δ depends on ε and x_0. In other words, f is continuous at x_0 if

$$\forall\, \varepsilon > 0, \exists\, \delta_\varepsilon(x_0) > 0, \forall\, x \in D, \quad |x - x_0| < \delta_\varepsilon(x_0) \quad \Longrightarrow \quad \Big|f(x) - f(x_0)\Big| < \varepsilon.$$

For a function which has a left hand side and right hand limit we can establish the following theorem

Theorem 1.5.3. *Let f be a real function. We consider $x_0 \in D_f$ and suppose f has left and right hand side limits. Then, the function f is continuous at x_0 if and only if*

$$\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = f(x_0).$$

To simplify notations, we write δ instead of $\delta_\varepsilon(x_0)$.

Example 1.5.4. *Let $f(x) = x^2$ and $x_0 = 3$. We have $f(x_0) = 9$ and*

$$\lim_{x \to 3} f(x) = f(x_0) = 9.$$

Therefore the function f is continuous at $x_0 = 3$.

Exercise 1.5.5. *Say whether the following function are continuous at x_0*

1. *$f(x) = 6x + 5$ and $x_0 = 1$*
2. *$g(x) = x^2 + 5x + 3$ and $x_0 = 3$*
3. *$h(x) = \dfrac{x+3}{x-1}$ and $x_0 = 1$*

4. $i(x) = \sqrt{x+3}$ *and* $x_0 = -2$

5. $j(x) = \dfrac{1}{x}$ *and* $x_0 = 0$.

Below we establish some properties for continuous functions. We will establish if f is a continuous function at $x_0 \in D_f$ and λ a real number the function λf is continuous at x_0.

Lemma 1.5.6. *Let f be a real function which is continuous at $x_0 \in D_f$. We consider a constant real number λ. Then the real function λf is a continuous at x_0.*

Proof. We consider a function f which is continuous at x_0. If $\lambda = 0$, the function $\lambda f \equiv 0$. We know the constant function 0 is continuous.

Now we take $\lambda \in \mathbb{R}^*$. Hence for all $\varepsilon > 0$, we can find $\delta > 0$, such that for all $x \in \mathbb{R}$, $|x - x_0| < \delta \implies \big|f(x) - f(x_0)\big| < \frac{\varepsilon}{|\lambda|}$.

Let $x \in \mathbb{R}$, such that $|x - x_0| < \delta$. Then,

$$\big|\lambda f(x) - \lambda f(x_0)\big| = \big|\lambda\big|\big|f(x) - f(x_0)\big| < |\lambda|\frac{\varepsilon}{|\lambda|} = \varepsilon.$$

Therefore the function λf is continuous at x_0. □

In this proof we choose $\delta > 0$ such that $|x - x_0| < \delta$ implies

$$\big|f(x) - f(x_0)\big| < \frac{\varepsilon}{|\lambda|}$$

in order to simplify notations.

But one can take δ such that

$$|x - x_0| < \delta \quad \text{implies} \quad \big|f(x) - f(x_0)\big| < \varepsilon. \tag{1.5.3}$$

In this case we obtain some $\varepsilon' = |\lambda|\,\varepsilon$. Indeed let us consider δ as in (1.5.3) then,

$$\big|\lambda\, f(x) - \lambda\, f(x_0)\big| = \big|\lambda\big|\,\big|f(x) - f(x_0)\big| < \big|\lambda\big|\,\varepsilon = \varepsilon'.$$

Let f and g be two functions defined in some interval I. We assume the functions f and g continuous at $x_0 \in I$, we can prove $f + g$ is continuous at x_0.

Lemma 1.5.7. *Let f and g be two real functions defined in an interval $I \subset D_f \cap D_g$. We consider a real number $x_0 \in I$. We assume that f and g are continuous at x_0. Then, the function $f + g$ is a continuous at x_0.*

Proof. Let $\varepsilon > 0$ and $x_0 \in I$. Since f and g are continuous at x_0, we can find $\delta > 0$, such that, for every $x \in I$, such that, $|x - x_0| < \delta$ implies

$$\Big|f(x) - f(x_0)\Big| < \frac{\varepsilon}{2} \quad \text{and} \quad \Big|g(x) - g(x_0)\Big| < \frac{\varepsilon}{2}.$$

Take $x \in I$ such that $|x - x_0| < \delta$. Therefore

$$\Big|(f+g)(x) - (f+g)(x_0)\Big| \leq \Big|f(x) - f(x_0)\Big| + \Big|g(x) - g(x_0)\Big| < \varepsilon.$$

Then the function $f + g$ is continuous at x_0. □

We can give a more simplest proof by pointing out that,

$$\lim_{x\to x_0}\big(f(x)+g(x)\big)=\lim_{x\to x_0}f(x)+\lim_{x\to x_0}g(x).$$

Proof 2. Let f and g be two continuous functions at x_0. Thus, we have

$$\lim_{x\to x_0}f(x)=f(x_0)\quad\text{and}\quad\lim_{x\to x_0}g(x)=g(x_0).$$

Using Theorem 1.1.4, we have

$$\lim_{x\to x_0}f(x)+g(x)=\lim_{x\to x_0}f(x)+\lim_{x\to x_0}g(x)=f(x_0)+g(x_0).$$

Then $f+g$ is a continuous function at x_0. □

Take two real functions f and g, which we suppose continuous at $x_0\in D_f\cap D_g$. One may wonder whether the function $f\cdot g$ is continuous at x_0. The answer to this question is given in the following lemma.

Lemma 1.5.8. *Let f and g be two real functions defined in an interval $I\subset D_f\cap D_g$. We consider a real number $x_0\in I$. We assume the function f and g continuous at x_0. Then, the real function $f\cdot g$ is a continuous at x_0.*

Proof. Let f and g be two continuous functions at x_0. Thus we have

$$\lim_{x\to x_0}f(x)=f(x_0)\quad\text{and}\quad\lim_{x\to x_0}g(x)=g(x_0).$$

Then by Theorem 1.1.6,

$$\lim_{x\to x_0}f(x)\cdot g(x)=\lim_{x\to x_0}f(x)\cdot\lim_{x\to x_0}g(x)=f(x_0)\cdot g(x_0)=\Big(f\cdot g\Big)(x_0).$$

The function $f\cdot g$ is continuous function at x_0. □

Using similar arguments we can prove that the division of continuous function at x_0 is also continuous at x_0.

Lemma 1.5.9. *Let f and g be two real functions defined in some interval $I\subset D_f\cap D_g$. We consider a real number $x_0\in I$. We assume that f and g are continuous at x_0 and $g(x_0)\neq 0$. Then the function $\dfrac{f}{g}$ is continuous at x_0.*

Proof. Let f and g be two continuous functions at x_0. By definition of continuity, we have

$$\lim_{x\to x_0}f(x)=f(x_0)\quad\text{and}\quad\lim_{x\to x_0}g(x)=g(x_0).$$

Since $g(x_0)\neq 0$, we deduce from Theorem 1.1.7, that

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=\frac{\lim_{x\to x_0}f(x)}{\lim_{x\to x_0}g(x)}=\frac{f(x_0)}{g(x_0)}.$$

The function $\dfrac{f}{g}$ continuous at x_0. □

Some functions are more regular. They can be continuous at each point of their domains. In this case we say the are continuous in their domains. We state the following definition.

Definition 1.5.10. *Let f be a real function defined in I. We say that f is continuous on I if it continuous at each point x_0 of I.*

Example 1.5.11. *Let $f(x) = x^2+2x+3$ and $I = [2\,,\,3]$. This function is continuous at each point of I. Then, the function f is continuous on I.*

Exercise 1.5.12. *Prove that the following functions are continuous in the given interval I*

1. *$f(x) = 3x + 2$ and $I = [0\,,\,3]$*
2. *$h(x) = \dfrac{1}{x+2}$ and $I = (-3\,,\,6]$*
3. *$i(x) = \sqrt{x+5}$ and $I = (-6\,,\,10]$.*

One can prove without any difficulties that the following functions are continuous in their domains.

Lemma 1.5.13. *The following statements hold:*

- *Polynomial functions are continuous*
- *The function $f(x) = \dfrac{1}{x}$ is continuous in its domain $D_f = \mathbb{R}\backslash\{0\}$*
- *The function $f(x) = \sqrt{x}$ is continuous in its domain.*

As in the case of functions which are continuous at some point x_0 we have following the following result.

Theorem 1.5.14. *Let f and g be two real functions defined in some interval $I \subset D_f \cap D_g$. We suppose f and g are continuous on I. Then,*

1. *the function $f + g$ is continuous on I*
2. *the function $f \cdot g$ is continuous on I*
3. *if $g(x) \neq 0$, for every $x \in I$, the function f/g is continuous on I.*

Example 1.5.15. *Let $f(x) = x + 1$ and $g(x) = x^2 + 1$. The functions f and g are continuous. Then using the theorem above, we obtain*

1. *$h(x) = f(x) + g(x) = x^2 + x + 2$ is continuous on $\mathbb{R}$*
2. *$p(x) = f(x) \cdot g(x) = x^3 + x^2 + x + 1$ is continuous on $\mathbb{R}$*
3. *$q(x) = \dfrac{x+1}{x^2+1}$ is continuous on $\mathbb{R}$.*

Exercise 1.5.16. *Find the domain of continuity of the following functions*

$$f(x) = \frac{x+3}{2x+3} + 3x^2 + 2x, \quad g(x) = x^2 + 4x + 3, \quad h(x) = \frac{x+6}{3x+5}.$$

We can observe that, all elementary functions are continuous in their domains. Combining this with theorem 1.5.14 and composition of functions, we obtain the following results.

Lemma 1.5.17. *Let f be a continuous function in its domain I. We define the subset $I_+ = \{x \in I \mid f(x) \geq 0\}$. Then,the function*

$$g(x) = \sqrt{f(x)},$$

is continuous on I_+

In other words the function g is continuous at any point x, such that, $f(x) \geq 0$.

Example 1.5.18. *We consider $f(x) = \sqrt{x^2+3x+2}$. We know the function f is defined if and only if $x^2+3x+2 \geq 0$. This correspond to the set*

$$D_f = (-\infty\,,\,-2] \cup [-1\,,\,+\infty).$$

Applying the previous theorem we conclude, the function f is continuous in its domain.

Exercise 1.5.19. *Find the domain of continuity of the following functions*

$$f(x) = \sqrt{2x^2+6x+2},\ \ g(x) = \sqrt{\frac{x+1}{3x+5}},\ \ h(x) = \sqrt{(x^2+6x+8)(x^2+3x+7)}\,.$$

Lemma 1.5.20. *Let f be a continuous function with domain I. We define the subset $I^* = \{x \in I \mid f(x) \neq 0\}$. Then, the function*

$$g(x) = \frac{1}{f(x)}$$

is continuous on I^*.

The function g is continuous at any x, such that $f(x) \neq 0$.

Example 1.5.21. *We take the function $f(x) = \frac{1}{x+2}$. This function is defined if and only if $x \neq -2$. This implies the function f is continuous on $\mathbb{R}\backslash\{-2\}$.*

Exercise 1.5.22. *Find the domain of continuity of the following functions*

$$f(x) = \frac{1}{x^2+3x+2},\ \ g(x) = \frac{3x+3}{5x+3},\ \ h(x) = \frac{x+3}{\sqrt{2x+7}},\ \ i(x) = \frac{\sqrt{x^2+4x+3}}{2x+3}.$$

We remind that for equations including polynomial function of degree three one can sometimes use Cardan' formulas to find solutions to the equation $f(x) = 0$. But for polynomial function with degree $n \geq 4$ or a general function we cannot most of the time find a formula which gives the solutions of this equation. In such a situation what we usually do is:

1. to prove that equation (1.5.4) has a solution which belongs in some interval
2. to give an approximation of solution of equation (1.5.4).

Below we will see that using the continuity of a function, we can say if whether equation : find $x \in \mathbb{R}$, such that,

$$f(x) = 0. \tag{1.5.4}$$

has a solution or not. We draw the readers attention that theorems which will be stated say just there exist solutions, but do not give any information on what such solutions look like.

Theorem 1.5.23 (Intermediate Value Theorem(IVT))**.** *Let $f : [a\,,\,b] \to \mathbb{R}$ be a continuous function on $[a\,,\,b]$. Then, for all $k \in f([a\,,\,b])$, there exist at least one $x_0 \in [a\,,\,b]$, such that,*

$$f(x_0) = k. \tag{1.5.5}$$

Corollary 1.5.24. *Let $f : [a\,,\,b] \to \mathbb{R}$ be a continuous function. We assume*

$$f(a) \cdot f(b) < 0.$$

Then, there exists at least one $x_0 \in (a\,,\,b)$, such that, $f(x_0) = 0$.

Some authors prefer to formulate the IVT theorem in this way. But you can see that the formulation below is just a corollary of the IVT theorem.

Theorem 1.5.25. *Let $f : [a\,,\,b] \to \mathbb{R}$ be a continuous function. Then, the set $f\big([a\,,\,b]\big)$ is an interval of $\mathbb{R}$.*

Example 1.5.26. *To prove that the equation $x^3 + x + 1 = 0$ has a solution in the interval $[-1\,,\,1]$, we proceed in this way:*

We define the function $f(x) = x^3 + x + 1$ which is continuous in $\mathbb{R}$. Then, its restriction in $[-1\,,\,1]$ is a continuous function.

We have $f(-1) = (-1)^3 + (-1) + 1 = -2 + 1 = -1 < 0$. On the other hand we have $f(1) = 3$. Then, one deduces

$$f(-1) \cdot f(1) = -3 < 0.$$

Using corollary 1.5.24, we conclude that, there exists $x_0 \in (-1\,,\,1)$, such that, $f(x_0) = 0$ Hence the equation $x^3 + x + 1 = 0$ has a solution in $[-1\,,\,1]$.

Exercise 1.5.27. *Say whether the following equations have solutions in the indicated interval.*

1. $x^5 + 3x + 1 = 0$ *and* $I = [-1\,,\,2]$
2. $x^6 - 3x^4 + 5x = 1$ *and* $I = [-2\,,\,1]$
3. $\sqrt{x^2 + 3x + 3} = 3$ *and* $I = [1\,,\,3]$.

Corollary 1.5.28. *Let $f : [a\,,\,b] \to \mathbb{R}$ be a continuous and increasing function. We assume $f(a) < 0$ and $f(b) > 0$. Then, there exists a unique $x_0 \in (a\,,\,b)$, such that,*

$$f(x_0) = 0.$$

Corollary 1.5.29. *Let $f : [a\,,\,b] \to \mathbb{R}$ be a continuous and decreasing function. We assume $f(a) > 0$ and $f(b) < 0$. Then, there exists a unique $x_0 \in (a\,,\,b)$, such that,*

$$f(x_0) = 0.$$

In the next chapter, we will define increasing and decreasing functions.

Until now, we never pay attention to the form of the interval we used. Now we need to make the different between open and closed interval. We have to specify what we mean by a segment.

Definition 1.5.30. *An interval I is open if it has one of the following form, $(-\infty\,,\,a)$ or $(a\,,\,b)$ or $(b\,,\,+\infty)$, a and b are real numbers such that $a < b$.*

Definition 1.5.31. *An interval I is closed if it has one of the following form, $I = (-\infty\,,\,a]$ or $[a\,,\,b]$ or $I = [b\,,\,+\infty)$, a and b are real numbers such that $a < b$.*

In some books we can find these notations $(-\infty\,,\,a) =]-\infty\,,\,a[$, $(a\,,\,b) =]a\,,\,b[$, $(b\,,\,+\infty) =]b\,,\,+\infty[$, $(-\infty\,,\,a] =]-\infty\,,\,a]$ and $[b\,,\,+\infty) = [b\,,\,+\infty[$.

Definition 1.5.32. *An interval I is called a segment if there exist two real numbers a and b, $a < b$, such that*

$$I = [a\,,\,b].$$

In the last theorem of this chapter, we show that a function which is continuous on a segment reaches its maximum and its minimum.

Theorem 1.5.33. *Let $f : [a\,,\,b] \to \mathbb{R}$ be a continuous function on $[a\,,\,b]$. Then there exist x_1 and x_2 in $[a\,,\,b]$, such that,*

$$f(x_1) = \max_{x\in[a\ b]} f(x) \quad \textit{and} \quad f(x_2) = \min_{x\in[a\,,\,b]} f(x).$$

Example 1.5.34. *We consider $f(x) = -x^2 + 2x - 1$ and $I = [0\,,\,2]$. The function f is continuous in I and reaches it maximum at 1. Then*

$$f(1) = \max_{x\in[0\,,\,2]} f(x) = 0.$$

Example 1.5.35. *Let $f(x) = x^2$ and $I = [-1\,,\,1]$. The function f is continuous in I, and has a minimum at 0.*

A Combination of the theorem above and IVT theorem gives the following lemma

Lemma 1.5.36. *Let f be a continuous function on I. We consider a segment $[a\,,\,b] \subset I$. Then,*

$$f\Big([a\,,\,b]\Big)$$

is a segment.

In other words, there exist two real numbers m and M, $m < M$, such that,

$$f\Big([a\,,\,b]\Big) = [m\,,\,M].$$

CHAPTER 2

DERIVABILITY

Contents

In the previous chapter, we study continuity of functions. At its end we introduce the notion of increasing and decreasing functions. We promised that we will come back to the definitions of these properties. Now we consider a function $f : [a\,,\,b] \to \mathbb{R}$ and two arbitrary elements of $[a\,,\,b]$ x and y such that, $a \leq x < y \leq b$. We say that f is an increasing function, if $f(x) \leq f(y)$. The function f is said to be decreasing, if $f(x) \geq f(y)$.

In other words, the function $f : [a\,;\,b] \to \mathbb{R}$ is increasing, if for all x and y in $[a\,,\,b]$, such that $a \leq x < y \leq b$ the following inequality holds $f(y) - f(x) \geq 0$. Since $y - x > 0$, this yields

$$\frac{f(y) - f(x)}{y - x} = \frac{f(x) - f(y)}{x - y} \geq 0.$$

The function $f : [a\,;\,b] \to \mathbb{R}$ is decreasing, if for all u and v in $[a\,,\,b]$, such that $a \leq u < v \leq b$, $f(u) - f(v) \geq 0$. As we know that $u - v < 0$, then,

$$\frac{f(u) - f(v)}{u - v} \leq 0.$$

The quantity

$$\Delta f(x) = \frac{f(x) - f(y)}{x - y} \tag{2.0.1}$$

is called the *rate of increase* of the function f. As we can observe it, the real number $\Delta f(x)$ seems to be the determinant element to characterize variation of functions. Using $\Delta f(x)$ we can easily check whether a function is increasing or decreasing in a given interval.

Lemma 2.0.1. *Let f be a real function defined on $[a\,,\,b]$. We consider two arbitrary elements x and y in $[a\,,\,b]$, such that, $a \leq x < y \leq b$. Then,*

- *the function f is increasing in $[a\,,\,b]$, if $\Delta f(x) \geq 0$,*
- *the function f is decreasing in $[a\,,\,b]$, if $\Delta f(x) \leq 0$.*

We precise that a function f is strictly increasing if $\Delta f(x) > 0$. It is strictly decreasing, if $\Delta f(x) < 0$.

The main purpose of this chapter is to define a method which allows us to study easily variations of functions. This chapter is organized as follows: in section 1, we define the derivative of a function. In the second section, we explain what we mean by the study of a function. In the last section we use differentiation to characterize some property of functions.

2.1 Derivative at some Point x_0

In this section, we define, the derivative of a function at some real point. Applying this definition to an arbitrary element of the domain of the domain of a function we establish that the function is differentiable in its domain. Using the derivatives we will be able tocharacterize increasing and decreasing functions.

We remind that the sets $(a\,,\,b)$ an $]a\,,\,b[$ represent the same open interval. We will use both notations in this chapter

Definition 2.1.1. *Let f be a real function which is defined in the interval $(\,a\,,\,b\,)$. We consider a fixed real number $x_0 \in (a\,,\,b)$. We say that the function f is derivable or differentiable at x_0, if the following limit exists:*

$$\lim_{\substack{x \to x_0 \\ x \neq x_0}} \frac{f(x) - f(x_0)}{x - x_0}. \tag{2.1.1}$$

We recall that the expression $x \to x_0$ means that x is as close as possible to x_0. That is, we can find a real number $h > 0$ as small as possible such that $x = x_0 + h$. The best way to represent this in our minds is to let h going to 0. Using this concept, we can reformulate Definition 2.1.1.

Definition 2.1.2. *We consider the function $f : (a\,,\,b) \to \mathbb{R}$ and a real number $x_0 \in (a\,,\,b)$. The function f is differentiable at x_0 if*

$$\lim_{\substack{h \to 0 \\ h \neq 0}} \frac{f(x_0 + h) - f(x_0)}{h} \quad \textit{exists.}$$

When the limit (2.1.1) exists, we denote it

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = f'(x_0) = \frac{df}{dx}(x_0). \tag{2.1.2}$$

In this case we call it the derivative or (differential) of the function f at x_0.

Example 2.1.3. *We take* $f(x) = 2\,x+1$ *and* $x_0 = 0$. *Then we have* $f(0+h) = 2\,h+1$ *and* $f(0) = 1$. *From this we deduce,*

$$\frac{f(0+h)-f(0)}{h} = 2. \quad \textit{Hence} \quad \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = 2.$$

We conclude that $f'(0) = 2$.

Exercise 2.1.4. *Compute* $f'(x_0)$

1. $f(x) = 3x + 2,\ x_0 = 1$, 2. $g(x) = x^2 + 6x + 1,\ x_0 = 2$, 3. $h(x) = \dfrac{1}{x},\ x_0 = 4$
4. $i(x) = \sqrt{x},\ x_0 = 1$, 5. $j(x) = 12,\ x_0 = 1$.

Let f be a real function, which is derivable at x_0. Therefore, $\lim_{h\to 0} \big(f(x_0 + h) - f(x_0)\big)/h$ is finite. We can find some real number k, such that,

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x)}{h} = k.$$

Now we take a real number $h \neq 0$. We write

$$f(x_0+h) - f(x_0) = \frac{f(x_0+h)-f(x_0)}{h} \cdot h.$$

Letting h to 0, we obtain

$$\begin{aligned}\lim_{h\to 0} \big(f(x_0+h) - f(x_0)\big) &= \lim_{h\to 0} \left(\frac{f(x_0+h)-f(x_0)}{h} \cdot h\right) \\ &= \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} \cdot \lim_{h\to 0} h = k \cdot 0 = 0.\end{aligned}$$

Hence $\lim_{h\to 0} f(x_0 + h) = f(x_0)$. This is equivalent to

$$\lim_{x\to x_0} f(x) = f(x_0)$$

when we define $x = x_0 + h$. This establish the function f is continuous at x_0. This leads to the following theorem.

Theorem 2.1.5. *Let* f *be a real function with domain* $(\,a\,,\,b\,)$. *We suppose the function* f *differentiable at* $x_0 \in (\,a\,,\,b\,)$. *Then, the function* f *is continuous at* x_0.

The reverse of this theorem is not true. A function can be continuous at some point x_0, without being differentiable at this point. As an example we consider the function $f(x) = |x|$. This function is continuous at 0. Indeed we have

$$\lim_{x\to 0} f(x) = f(0) = 0.$$

But the function $|x|$ is not differentiable at 0. Because

$$\lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = -1 \quad \text{and} \quad \lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^+} \frac{x}{x} = 1.$$

Since $\lim_{x\to 0^-}\big(|x|/x\big) \neq \lim_{x\to 0^+}\big(|x|/x\big)$, then, $\lim_{x\to 0}\big(f(x)/x\big)$ does not exists. The function $|x|$ is not derivable at 0.

This theorem proves that derivability or (differentiability) is stronger than continuity. In other words, a differentiable function at x_0 is more regular than a continuous function at x_0 in the following interval $(\, x_0 - \varepsilon\,,\, x_0 + \varepsilon\,)$.

Definition 2.1.6. *Let f be a function with domain D. We take an open interval $I \subset D$. We say that the function f is derivable in the interval I if it is derivable at each point x_0 in I.*

This means that f is differentiable in I if for every $x_0 \in I$,

$$f'(x_0) = \frac{df}{dx}(x_0) \text{ exists.}$$

Example 2.1.7. *To establish that the function $f(x) = x^2$ is derivable in $\mathbb{R}$, we consider an arbitrary real number x_0 . Let $h \neq 0$ be a fixed positive real number. One has $f(x_0) = x_0^2$ and $f(x_0 + h) = x_0^2 + 2x_0 \cdot h + h^2$. From this, we deduce that*

$$\lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h\to 0} \frac{2x_0\, h + h^2}{h} = 2x_0.$$

Therefore f is derivable at x_0. Since x_0 is an arbitrary real number, we conclude that the function f is derivable in $\mathbb{R}$.

Exercise 2.1.8. *Prove that the following functions are derivable in the given intervals.*

$$\text{1. } f(x) = 3\,x + 2, \quad I =]-\infty\,,\,0], \quad \text{2. } g(x) = \sqrt{x}, \quad I = [3\,,\,+\infty[,$$
$$\text{3. } h(x) = 3x^2 + 9x + 1, \quad I =]-3\,,\,5[, \quad \text{4. } i(x) = \frac{1}{x+1}, \quad I =]0\,,\,2[,$$
$$\text{5. } j(x) = \frac{2x-5}{x+1}, \quad I = [0\,,\,1], \quad \text{6. } k(x) = \big(x+2\big)^3 + 1, \quad I = [-1\,,\,7].$$

We know that elementary functions are the bricks that allow to build more complicated functions. That is why every new operation on the set of functions should be tested on them, before being generalized. In the following section we define the derivative of the elementary functions.

2.2 Derivative of Elementary Functions

In this section we define derivative of elementary functions. We start by polynomial functions. In the second time we define derivative of the inverse and square root function. We end the section by the derivatives of cosine and sine functions.

The first result of this section deals with constant functions which are the simplest polynomial functions.

Lemma 2.2.1. *Let $k \in \mathbb{R}$. We consider the constant real function $f(x) = k$, for every $x \in \mathbb{R}$. Then, the constant function is derivable in $\mathbb{R}$ and for all $x \in \mathbb{R}$, $f'(x) = 0$.*

Proof. We consider the constant function $f(x) = k$. Let $x_0 \in \mathbb{R}$ be an arbitrary real number and $h \neq 0$. We have $f(x_0) = f(x_0 + h) = k$. Therefore,

$$\lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h} = 0 = f'(x_0).$$

Since x_0 is an arbitrary real number the function $f(x) = k$ is differentiable in $\mathbb{R}$ and $\forall x \in \mathbb{R}$, $f'(x) = 0$. □

Example 2.2.2. *For all $x \in \mathbb{R}$, we define the function $f(x) = 2$. Then, the function f is differentiable in $\mathbb{R}$. The derivative of f is $f'(x) = 0$ for all $x \in \mathbb{R}$.*

Now we consider an affine function $f(x) = ax + b$, with $a \neq 0$. Let $x_0 \in \mathbb{R}$ be an arbitrary real number and $h \neq 0$ small enough. We have $f(x_0) = ax_0 + b$ and $f(x_0 + h) = ax_0 + ah + b$. This involves

$$\lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h} = a = \lim_{h\to 0} \frac{ax_0 + ah + b - ax_0 - b}{h} = f'(x_0).$$

Then for all $x \in \mathbb{R}$, $f'(x) = a$.

Lemma 2.2.3. *Let $f(x) = ax + b$, with $a \neq 0$ be an affine function. Then, the function f is derivable in $\mathbb{R}$ and for all $x \in \mathbb{R}$, $f'(x) = a$.*

Example 2.2.4. *We take the function $f(x) = 7x + 5$. The domain of f is $\mathbb{R}$. Since f is affine, it is derivable in $\mathbb{R}$ and for all $x \in \mathbb{R}$, $f'(x) = 7$.*

Exercise 2.2.5. *Find the derivative of the following function*

1. $f(x) = x + 2$, 2. $g(x) = 6x$, 3. $h(x) = 3x + 1$, 4. $i(x) = 5$, 5. $j(x) = 4x + 5$.

To deal with derivative of polynomial functions of degree $n \geq 2$, we introduce the following definitions.

We define the natural number $n!$ as follows:

$$\begin{aligned} 0! &= 1 \\ 1! &= 1 \\ n! &= 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n, \quad \forall\, n \geq 2. \end{aligned}$$

Example 2.2.6. *From the definition above one deduces*

$$\begin{aligned} 2! &= 1.2 = 2 \\ 3! &= 1.2.3 = 6 \\ 6! &= 1.2.3.4.5.6 = 720. \end{aligned}$$

Using the definition of $n!$ we introduce the following notations. The natural number $\binom{n}{k} = C_k^n$ is defined by

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}.$$

As examples we have

$$\binom{n}{0} = \binom{n}{n} = 1 \quad \text{and} \quad \binom{n}{n-1} = \binom{n}{1} = n.$$

Lemma 2.2.7. *Let $n \geq 1$ be a natural number. For any natural numbers k, such that, $0 \leq k < n$ the following holds*

$$\binom{n}{k+1} = \binom{n-1}{k} + \binom{n-1}{k+1}.$$

Proof. To prove this lemma we point out that

$$\binom{n-1}{k} = \frac{(n-1)!}{(n-k-1)!\,k!} \quad \text{and} \quad \binom{n-1}{k+1} = \frac{(n-1)!}{(n-k-2)!\,(k+1)!}.$$

This implies

$$\begin{aligned}\binom{n-1}{k} + \binom{n-1}{k+1} &= \frac{(n-1)!}{(n-(k+1))!\,(k+1)!}\Big[k+1+n-k-1\Big] \\ &= \frac{(n-1)!n}{(n-(k+1))!\,(k+1)!} = \frac{n!}{(n-(k+1))!(k+1)!} = \binom{n}{k+1}\end{aligned}$$

This completes the proof. □

Using this lemma above one can define Pascal's Triangle

$$\begin{array}{lccccccc} n=0, & 1 & & & & & & \\ n=1, & 1 & 1 & & & & & \\ n=2, & 1 & 2 & 1 & & & & \\ n=3, & 1 & 3 & 3 & 1 & & & \\ n=4, & 1 & 4 & 6 & 4 & 1 & & \\ n=5, & 1 & 5 & 10 & 10 & 5 & 1 & \\ \cdots & \cdots & \cdots & \cdots & \cdots & & & \end{array}$$

In High School we often use Pascal's triangle to define the following identities.

$$\begin{aligned}(x+y)^2 &= x^2 + 2xy + y^2 \\ (x+y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3 \\ (x+y)^4 &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.\end{aligned}$$

The generalization of the formulas above is given by the so called Newton's binomial Formula:

Lemma 2.2.8. *Let $n \geq 1$ be a natural number, then*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k}\, x^n\, y^{n-k}. \tag{2.2.1}$$

We remark that $(x+y)^0 = 1$ and the coefficient $\binom{n}{k}$ is sometimes denoted C_k^n.

Exercise 2.2.9. *Using Newton's binomial Formula give the expression of the following functions*

$$1.\ f(x) = (x+3)^6, \quad 2.\ g(x) = (x+a)^7, \quad 3.\ h(x) = (x-1)^{13}.$$

We consider the function $f(x) = x^n$, with $n \geq 2$.Let $x_0 \in \mathbb{R}$ be an arbitrary real number and $h \neq 0$ small enough. One has $f(x_0) = x_0^n$ and $f(x_0 + h) = (x_0 + h)^n$. Using Newton's binomial formula, we obtain

$$(x_0 + h)^n = \sum_{k=0}^{n} \binom{n}{k} x_0^k \cdot h^{n-k} = x_0^n + \sum_{k=0}^{n-1} \binom{n}{k} x_0^k \cdot h^{n-k}.$$

From this we deduce that

$$f(x_0 + h) - f(x_0) = \binom{n}{n-1} x_0^{n-1}\, h + \sum_{k=0}^{n-2} \binom{n}{k} x_0^k \cdot h^{n-k}.$$

Since $n - k \geq 2$, and

$$\frac{1}{h} \sum_{k=0}^{n-2} \binom{n}{k} x_0^k \cdot h^{n-k} = \sum_{k=0}^{n-2} \binom{n}{k} x_0^k \cdot h^{n-k-1},$$

we have $\lim_{h\to 0} \sum_{k=0}^{n-2} \binom{n}{k} x_0^k \cdot h^{n-k-1} = 0$. This yields

$$\lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \binom{n}{n-1} x_0^{n-1}.$$

On the other hand we know that $\binom{n}{n-1} = n$. Therefore $f'(x) = nx^{n-1}$. This leads to the following lemma

Lemma 2.2.10. *Let $f(x) = x^n$, with $n \geq 2$ be a real function. Then, the function f is derivable in $\mathbb{R}$ and for every $x \in \mathbb{R}$, we have $f'(x) = nx^{n-1}$.*

Example 2.2.11. *We consider the following functions*

$$f(x) = x^2, \quad g(x) = x^6 \quad \text{and} \quad h(x) = x^7.$$

These functions are differentiable in $\mathbb{R}$. For all real number x we have

$$f'(x) = 2x, \quad g'(x) = 6x^5 \quad \text{and} \quad h'(x) = 7x^6.$$

Exercise 2.2.12. *Determine the derive of the following functions*

1. $f(x) = x^5 + 2x^9 + 5$, 2. $g(x) = x^3 + 3x^2 + 5x + 2$, 3. $i(x) = x^8 + x^7 + 2$
4. $h(x) = x^{10} + 2x^8 + 3x^6 + 2x + 3$, 5. $j(x) = x^5 + 4x^3 + x^2 + 2$.

Let $x \in \mathbb{R}^*$ and $f(x) = \dfrac{1}{x}$. For $x_0 \in \mathbb{R}^*$ and $h \neq 0$ be small enough, we have $f(x_0) = \dfrac{1}{x_0}$ and $f(x_0 + h) = \dfrac{1}{x_0 + h}$. Then,

$$\begin{aligned}\lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h} &= \lim_{h\to 0} \frac{\dfrac{1}{x_0 + h} - \dfrac{1}{x_0}}{h} = \lim_{h\to 0} \frac{x_0 - x_0 - h}{h(x_0 + h)x_0} \\ &= -\lim_{h\to 0} \frac{1}{x_0(x_0 + h)} = -\frac{1}{x_0^2}.\end{aligned}$$

We conclude that for all $x \in \mathbb{R}^*$, $f'(x) = -\dfrac{1}{x^2}$.

Lemma 2.2.13. *Let $f(x) = \frac{1}{x}\cdot$ Then, the function f is differentiable in $\mathbb{R}^*$ and for all $x \neq 0$, we have*

$$f'(x) = -\frac{1}{x^2}\cdot$$

Let x be a real number such that $x \neq 0$. We know that $\frac{1}{x^n} = x^{-n}$. From this we deduce that

$$f'(x) = -nx^{-n-1} = -\frac{n}{x^{n+1}}\cdot$$

This proves the following lemma

Lemma 2.2.14. *Let $n \geq 2$ be an integer. For any $x \neq 0$ we define the function $f(x) = \frac{1}{x^n}\cdot$ Then, the function f is differentiable in $\mathbb{R}\backslash\{0\}$ and for every $x \neq 0$, we have*

$$f'(x) = -n\frac{1}{x^{n+1}}\cdot$$

Example 2.2.15. *We consider the function $f(x) = \frac{1}{x^5}\cdot$ Then, the function f is differentiable in $\mathbb{R}^*$ and for all $x \neq 0$, we have*

$$f'(x) = -\frac{5}{x^6}\cdot$$

Exercise 2.2.16. *Find the derivative of the following functions*

$$1.\ f(x) = \frac{1}{x^3}, \quad 2.\ g(x) = \frac{1}{x^2}, \quad 3.\ h(x) = \frac{1}{x^{11}}\cdot$$

Let x belong to $[0\,;+\infty)$. We define the function $f(x) = \sqrt{x}$. We take an arbitrary real number x_0 in $]0\,;+\infty)$. For some real number $h \neq 0$ which is assumed to be small enough. We write $f(x_0+h) - f(x_0) = \sqrt{x_0+h} - \sqrt{x_0}$. Multiplying this expression by $(\sqrt{x_0+h} + \sqrt{x_0})/(\sqrt{x_0+h} + \sqrt{x_0})$. we obtain

$$\begin{aligned} f(x_0+h) - f(x_0) = \sqrt{x_0+h} - \sqrt{x_0} &= \frac{\left(\sqrt{x_0+h} - \sqrt{x_0}\right)\left(\sqrt{x_0+h} + \sqrt{x_0}\right)}{\sqrt{x_0+h} + \sqrt{x_0}} \\ &= \frac{h}{\sqrt{x_0+h} + \sqrt{x_0}}\cdot \end{aligned}$$

Therefore

$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h\to 0} \frac{h}{h(\sqrt{x_0+h} + \sqrt{x_0})} = \lim_{h\to 0} \frac{1}{\sqrt{x_0+h} + \sqrt{x_0}} = \frac{1}{2\sqrt{x_0}}\cdot$$

Lemma 2.2.17. *Let $f(x) = \sqrt{x}$ be a function defined on $[0\,,+\infty)$. Then the function is differentiable in $x \in (0\,;+\infty)$. For all $x > 0$,*

$$f'(x) = \frac{1}{2\sqrt{x}}\cdot$$

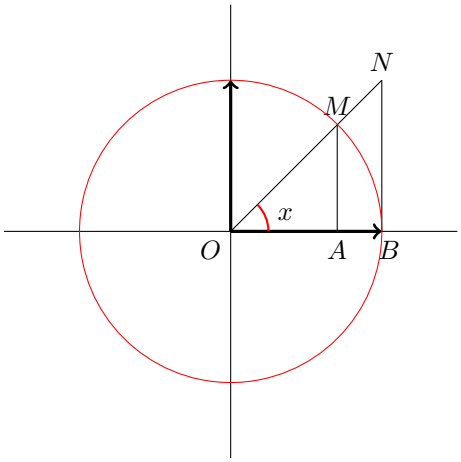
Remark 2.2.18. *We point out again that $f'(0)$ is not defined.*

Below we list some properties which allow to define easily the derivative of sine and cosine functions.

Lemma 2.2.19. *Let x be a positive real number, such that, $0 < x < \pi/2$. Then,*

$$\cos(x) \leq \frac{x}{sin(x)} \leq \frac{1}{\cos(x)}. \tag{2.2.2}$$

The proof of this lemma is based on the following geometrical observation. We can see in the figure below, the triangle OAM is included is the part of the disc delimited by the lines OB and OM. We denote this part of the disc OBM. This latter is included in the triangle OBN. As indicated below



Here the circle we consider is the trigonometric circle. That is, the distance $OB = OM = 1$. We know that the distance $OA = \cos(x)$ and $AM = \sin(x)$. Using Thales's theorem we find $BN = \tan(x)$.

On the other hand we have

$$\text{Area}(OAM) = \frac{\sin(x) \cdot \cos(x)}{2}, \quad \text{Area}(OBM) = \frac{x}{2\pi} \cdot \pi = \frac{x}{2},$$
$$\text{Area}(OBN) = \frac{\sin(x)}{2\cos(x)}.$$

Combining this, with the fact that the

$$\text{Area}(OAM) \leq \text{Area}(OBM) \leq \text{Area}(ABN),$$

we obtain

$$\frac{\sin(x) \cdot \cos(x)}{2} \leq \frac{x}{2} \leq \frac{\sin(x)}{2\cos(x)}.$$

Since $x \in (0\,, \pi/2)$, $\sin(x) \neq 0$, then we can divide by $\sin(x)$. Multiplying the previous inequalities by $\dfrac{2}{\sin(x)}$, we obtain $\cos(x) \leq \dfrac{x}{\sin(x)} \leq \dfrac{1}{\cos(x)}$.

Lemma 2.2.20. *Let x be a real number which is different to 0. Then, the following holds*

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1. \tag{2.2.3}$$

Proof. We consider $x \in (0\,,\,\pi/2)$. Using (2.2.2) we have

$$\cos(x) \leq \frac{x}{\sin(x)} \leq \frac{1}{\cos(x)}.$$

We know that $\cos(x) \neq 0$ and $\dfrac{\sin(x)}{x} \neq 0$. Because $0 < x < \pi/2$. We can calculate the inverse of each term in (2.2.2). This involves

$$\cos(x) \leq \frac{\sin(x)}{x} \leq \frac{1}{\cos(x)}.$$

Since $\lim_{x\to 0} \cos(x) = 1$, then, using Sandwich theorem we obtain

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1.$$

□

Lemma 2.2.21. *Let h be a real number which is $x \neq 0$. Then,*

$$\lim_{h\to 0} \frac{\cos(h) - 1}{h} = 0. \tag{2.2.4}$$

Proof. Let $x \neq 0$. On one hand we have

$$\begin{aligned}\frac{\cos(x) - 1}{x} &= \frac{\cos(x) - 1}{x} \cdot \frac{\cos(x) + 1}{\cos(x) + 1} = \frac{\cos^2(x) - 1}{x\,\big(\cos(x) + 1\big)} = \frac{-\sin^2(x)}{x\,\big(\cos(x) + 1\big)} \\ &= \frac{-\sin(x)}{x} \cdot \frac{\sin(x)}{\cos(x) + 1}.\end{aligned}$$

On the other hand we have

$$\lim_{x\to 0} \frac{-\sin(x)}{x} = 1 \quad \text{and} \quad \lim_{x\to 0} \frac{\sin(x)}{\cos(x) + 1} = 0.$$

Therefore, we have

$$\lim_{x\to 0} \frac{\cos(x) - 1}{x} = 0.$$

□

For the convenience of the readers we remind the following properties shared by the functions sine and cosine. Let a and b be two real numbers. Then, the following identities hold

$$\begin{aligned}\sin(a + b) &= \sin(a)\cos(b) + \cos(a)\sin(b), \\ \sin(a - b) &= \sin(a)\cos(b) - \cos(a)\sin(b), \\ \cos(a + b) &= \cos(a)\cos(b) - \sin(a)\sin(b), \\ \cos(a - b) &= \cos(a)\cos(b) + \sin(a)\sin(b).\end{aligned}$$

Now we consider the function $f(x) = \sin(x)$ and $x_0 \in \mathbb{R}$ an arbitrary real number. We know for all $h \in \mathbb{R}^*$,

$$\sin(x_0 + h) = \sin(x_0)\cos(h) + \cos(x_0)\sin(h).$$

Dividing by h we get

$$\frac{\sin(x_0+h)-\sin(x_0)}{h} = \sin(x_0)\frac{\cos(h)-1}{h} + \cos(x_0)\frac{\sin(h)}{h}.$$

Using (2.2.3) and (2.2.4), we obtain

$$\lim_{h\to 0}\frac{\sin(x_0+h)-\sin(x_0)}{h} = \cos(x_0).$$

Therefore for all $x \in \mathbb{R}$, $f'(x) = \cos(x)$.

Lemma 2.2.22. *Let $f(x) = \sin(x)$. Then, the function is differentiable in $\mathbb{R}$ and for all $x \in \mathbb{R}$,*

$$f'(x) = \cos(x).$$

We take $x \in \mathbb{R}$ and we define the function $f(x) = \cos(x)$. Let x_0 be an arbitrary real number and $h \neq 0$ a fixed real number. One knows that
$\cos(x_0+h) = \cos(x_0)\cos(h) - \sin(x_0)\sin(h)$. That is,

$$\frac{f(x_0+h)-f(x_0)}{h} = \cos(x_0)\frac{\cos(h)-1}{h} - \sin(x_0)\frac{\sin(h)}{h}.$$

Therefore,

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} = -\sin(x_0).$$

This leads to the following lemma

Lemma 2.2.23. *Let $f(x) = \cos(x)$. Then, the function f is differentiable in $\mathbb{R}$ and for all $x \in \mathbb{R}$,*

$$f'(x) = -\sin(x).$$

We can summarize all these properties in the following table.

$f(x)$	Domain of f	$f'(x)$	Domain of f'
a	$\mathbb{R}$	0	$\mathbb{R}$
$ax+b$	$\mathbb{R}$	a	$\mathbb{R}$
ax^2+bx+c	$\mathbb{R}$	$2ax+b$	$\mathbb{R}$
x^n	$\mathbb{R}$	nx^{n-1}	$\mathbb{R}$
$\frac{1}{x}$	$\mathbb{R}\backslash\{0\}$	$-\frac{1}{x^2}$	$\mathbb{R}\backslash\{0\}$
$\frac{1}{x^n},\ n\geq 2$	$\mathbb{R}\backslash\{0\}$	$-\frac{n}{x^{n+1}}$	$\mathbb{R}\backslash\{0\}$
$\sqrt{x}$	$[0\,;+\infty)$	$\frac{1}{2\sqrt{x}}$	$(0\,;+\infty)$
$\cos(x)$	$\mathbb{R}$	$-\sin(x)$	$\mathbb{R}$
$\sin(x)$	$\mathbb{R}$	$\cos(x)$	$\mathbb{R}$

We recall that general function are constructed from elementary ones by summing, multiplying or dividing etc. In the next section we deal with derivative of sum, multiplication, division, square root of such elementary functions.

2.3 Operation on the Differentiable Functions

This section is organized as follows. Firstly we define derivative of sum of functions. Secondly we study derivative of multiplication and division of functions. We end this section by dealing with square root an general composition of functions.

Let u and v be two differentiable functions in some open interval I. We consider the function $f(x) = u(x) + v(x)$. Take an arbitrary real number $x_0 \in I$. We fixed $h \neq 0$ a real number which is small enough. The definition of the function f gives

$$f(x_0) = u(x_0) + v(x_0) \quad \text{and} \quad f(x_0 + h) = u(x_0 + h) + v(x_0 + h).$$

Dividing by h and considering the limit when h goes to 0 we obtain

$$\begin{aligned}\lim_{h\to 0} \frac{u(x_0 + h) + v(x_0 + h) - u(x_0) - v(x_0)}{h} &= \lim_{h\to 0} \frac{u(x_0 + h) - u(x_0)}{h} \\ &+ \lim_{h\to 0} \frac{v(x_0 + h) - v(x_0)}{h} \\ &= u'(x) + v'(x)\cdot\end{aligned}$$

This proves the following lemma.

Lemma 2.3.1. *Let u and v be two differentiable functions in some open interval I. Then, the function $f(x) = u(x) + v(x)$ is differentiable in I and for all $x \in I$*

$$f'(x) = u'(x) + v'(x).$$

Example 2.3.2. *Let $f(x) = 2x^2 + 3x + 1 + \dfrac{1}{x}\cdot$ The domain of the function f is $\mathbb{R}^*$. From the previous section we know the function $u(x) = x^2 + 3x + 1$ is differentiable in $\mathbb{R}$ and the function $v(x) = \dfrac{1}{x}$ is differentiable in $\mathbb{R}^*$. Then, the function $f = u + v$ is differentiable in $\mathbb{R}\backslash\{0\}$ and we have,*

$$f'(x) = 4x + 3 - \frac{1}{x^2}\cdot$$

Exercise 2.3.3. *Define the derivatives of the following functions*

$$1.\ f(x) = 6x + \frac{1}{x}, \quad 2.\ g(x) = \frac{3}{x^3} + \frac{x^2 + x + 1}{6}, \quad 3.\ h(x) = \frac{1}{x^5} + \frac{2}{x^6} + \frac{1}{x^2}\cdot$$

We consider two functions u and v which are differentiable in an open interval I. We define the function $f(x) = u(x)\cdot v(x)$ for all $x \in I$. We take an arbitrary real number $x_0 \in I$, and fix $h \neq 0$ small enough. We have $f(x_0) = u(x_0)\cdot v(x_0)$ and $f(x_0 + h) = u(x_0 + h)\cdot v(x_0 + h)$. Thus dividing $f(x_0 + h) - f(x_0)$ by h, we obtain

$$\begin{aligned}\frac{f(x_0 + h) - f(x_0)}{h} &= \frac{u(x_0 + h)\cdot v(x_0 + h) - u(x_0)\cdot v(x_0)}{h} \\ &= \frac{u(x_0 + h)\cdot v(x_0 + h) - u(x_0)\cdot v(x_0 + h)}{h} \\ &+ \frac{u(x_0)\cdot v(x_0 + h) - u(x_0)\cdot v(x_0)}{h} \\ &= v(x_0 + h)\cdot \frac{u(x_0 + h) - u(x_0)}{h} + u(x_0)\cdot \frac{v(x_0 + h) - v(x_0)}{h}\cdot\end{aligned}$$

We know that $\lim_{h\to 0} v(x_0+h) = v(x_0)$. Using the definition of differentiablity at the point x_0, we have

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = u'(x_0)\cdot v(x_0) + u(x_0)\cdot v'(x_0).$$

That is, for every $x \in I$,

$$f'(x) = u'(x)\cdot v(x) + u(x)\cdot v'(x).$$

Lemma 2.3.4. *Let u and v be two differentiable functions in an open interval I. We define $f(x) = u(x)\cdot v(x)$. Then, the function f is derivable in I and for all $x \in I$,*

$$f'(x) = u'(x)\cdot v(x) + u(x)\cdot v'(x).$$

Example 2.3.5. *We define the function $f(x) = (3x+1)(x+1)$. We set $u(x) = 3x+1$ and $v(x) = x+1$. Then $u'(x) = 3$ and $v'(x) = 1$. We deduce*

$$f'(x) = 3(x+1) + (3x+1) = 3x+3+3x+1 = 6x+4.$$

Exercise 2.3.6. *Find the derivative of the following functions*

1 $f(x) = (x^2+4x+6)(x^3+5x+1)$, 2. $g(x) = (x^4+3x+5x+6)\cdot\dfrac{1}{x}$

3. $h(x) = (6x+2)\cdot\sqrt{x}$.

Let x and y be in $\mathbb{R}$, such that $x \neq 0$. We define $q = \dfrac{y}{x}$ and we define the sequence

$$\begin{cases} u_0 = 1 \\ u_{n+1} = q\,u_n,\ \forall n \geq 0. \end{cases}$$

The sequence $(u_n)_{n\geq 0}$ is geometric with common ratio $q = \dfrac{y}{x}$ and initial term $u_0 = 1$. Then, if $\frac{y}{x} \neq 1$, the sum of the n first terms of the the sequence (u_n) is

$$1 + \frac{y}{x} + \cdots + \left(\frac{y}{x}\right)^{n-1} = \frac{1-\dfrac{y^n}{x^n}}{1-\dfrac{y}{x}}\cdot$$

The expression above is equivalent to,

$$\left(1-\frac{y}{x}\right)\left(1+\frac{y}{x}+\cdots+\left(\frac{y}{x}\right)^{n-1}\right) = 1-\frac{y^n}{x^n}\cdot$$

Multiplying by x^n the identity above, we obtain

$$(x-y)\left(x^{n-1}+x^{n-2}y+\cdots+y^{n-1}\right) = (x-y)\sum_{k=0}^{n-1} x^{n-k-1}y^k = x^n - y^n.$$

Lemma 2.3.7. *Let x and y be two real numbers. Then, we have*

$$x^n - y^n = (x-y)\sum_{k=0}^{n-1} x^{n-k-1}y^k.$$

Lemma 2.3.8. *Let u be a differentiable function in an open interval I and n be a natural number greater than 2. We define the function $f(x) = \Big(u(x)\Big)^n$. Then f is differentiable in I and for all $x \in I$,*

$$f'(x) = n\,u'(x)\,\Big(u(x)\Big)^{n-1}.$$

Proof. Let u be a differentiable function in an open interval I. We denote $f(x) = (u(x))^n$. We consider an arbitrary real number $x_0 \in I$. For all real number $h \neq 0$ small enough, we have $f(x_0) = \big(u(x_0)\big)^n$ and $f(x_0+h) = \big(u(x_0+h)\big)^n$. From the lemma above we deduce

$$\frac{f(x_0+h)-f(x_0)}{h} = \frac{u^n(x_0+h)-u^n(x_0)}{h}.$$

We obtain

$$\frac{u^n(x_0+h)-u^n(x_0)}{h} = \frac{u(x_0+h)-u(x_0)}{h}. \quad \left(\sum_{k=0}^{n-1} u^{n-k-1}(x_0+h)u^k(x_0)\right).$$

For k, such that, $0 \leq k \leq n-1$, we have

$$\lim_{h\to 0} u^{n-k-1}(x_0+h) = u^{n-k-1}(x_0).$$

Hence

$$\begin{aligned}\lim_{h\to 0}\sum_{k=0}^{n-1} u^{n-k-1}(x_0+h)u^k(x_0) &= \sum_{k=0}^{n-1}\lim_{h\to 0} u^{n-k-1}(x_0+h)u^k(x_0) = \sum_{k=0}^{n-1} u^{n-1}(x_0)\\ &= nu^{n-1}(x_0).\end{aligned}$$

This involves

$$\begin{aligned}\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} &= \lim_{h\to 0}\frac{u^n(x_0+h)-u^n(x_0)}{h}\\ &= \lim_{h\to 0}\frac{u(x_0+h)-u(x_0)}{h}\cdot\lim_{h\to 0}\sum_{k=0}^{n-1} u^{n-k-1}(x_0+h)u^k(x_0)\\ &= nu'(x_0)\,u^{n-1}(x_0).\end{aligned}$$

That is, for all $x \in I$ $f'(x) = nu'(x)\,u^{n-1}(x)$. □

Example 2.3.9. *We define the function $f(x) = (x^3+3x^2+2x+7)^4$. It is differentiable in $\mathbb{R}$. Setting $u(x) = x^3+3x^2+2x+7$ we obtain $u'(x) = 3x^2+6x+2$. Therefore,*

$$f'(x) = 4\,(3x^2+6x+2)\,(x^3+3x^2+2x+7)^3.$$

Exercise 2.3.10. *Find the derivative of the following functions*

$$1.\ f(x) = (3x+5)^3, \quad 2.\ g(x) = (x^2+5x+1)^5, 3.\ h(x) = (2x+5)^{3/2}.$$

Now we consider a function u which is differentiable in an open interval I. Moreover we assume for every $x \in I$, $u(x) \neq 0$ and we define $f(x) = \dfrac{1}{u(x)}$. Let $x_0 \in \mathbb{R}$ be an arbitrary real number. We take $h \neq 0$ small enough. We have $f(x_0) = \dfrac{1}{u(x_0)}$ and $f(x_0+h) = \dfrac{1}{u(x_0+h)}$. Then, we have

$$\begin{aligned}\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} &= \lim_{h\to 0}\frac{\dfrac{1}{u(x_0+h)}-\dfrac{1}{u(x_0)}}{h} = \lim_{h\to 0}\frac{u(x_0)-u(x_0+h)}{h\cdot u(x_0+h)\cdot u(x_0)}\\ &= \lim_{h\to 0}\frac{1}{u(x_0+h)\cdot u(x_0)}\cdot\lim_{h\to 0}\frac{-(u(x_0+h)-u(x_0))}{h}.\end{aligned}$$

Since u is differentiable in I, it is continuous in I. Then, $\lim_{h\to 0} u(x_0+h) = u(x_0)$. We obtain

$$\lim_{h\to 0}\frac{1}{u(x_0+h)\cdot u(x_0)} = \frac{1}{u^2(x_0)} \quad \text{and} \quad \lim_{h\to 0}\frac{-(u(x_0+h)-u(x_0))}{h} = -u'(x_0).$$

Consequently we have

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} = -\frac{u'(x_0)}{u^2(x_0)}.$$

That is, for every $x \in I$, $f'(x) = -\dfrac{u'(x)}{u^2(x)}$.

This leads to the following lemma.

Lemma 2.3.11. *Let u be a differentiable function in an open interval I, such that, for all $x \in I$, $u(x) \neq 0$. We define $f(x) = \dfrac{1}{u(x)}$. Then, the function f is differentiable in I and for every $x \in I$,*

$$f'(x) = -\frac{u'(x)}{u^2(x)}.$$

Example 2.3.12. *We define the function $f(x) = \dfrac{1}{x+1}$. The function f is defined for all x in $\mathbb{R}\backslash\{-1\}$. We set $u(x) = x+1$. The function f is derivable in $\mathbb{R}^*$ and the derivative of the function u is $u'(x) = 1$. Using the previous lemma, we have*

$$f'(x) = -\frac{u'(x)}{u^2(x)} = -\frac{1}{(x+1)^2}.$$

Exercise 2.3.13. *Find the derivative of the following functions*

1. $f(x) = \dfrac{1}{5x+8}$, 2. $g(x) = \dfrac{1}{2x^2+5x+2}$, 3. $h(x) = \dfrac{1}{x-1}$.

The lemma above is a particular case of the the following one.

Lemma 2.3.14. *We consider a natural number $n \geq 2$. Let u be a differentiable function in an open interval I. We suppose that, for all $x \in I$, $u(x) \neq 0$ and we define the function $f(x) = \dfrac{1}{u^n(x)}$. Then, the function f is differentiable in I and for all $x \in I$,*

$$f'(x) = \frac{-nu'(x)}{u^{n+1}(x)}.$$

Proof. We consider u a differentiable function in the open interval I. We take a natural number $n \geq 2$. Remarking that $f(x) = \dfrac{1}{u^n(x)} = u^{-n}(x)$ we get

$$f'(x) = -n\,u'(x)\,u^{-n-1} = \frac{-n\,u'(x)}{u^{n+1}(x)}.$$

□

Example 2.3.15. *Let $f(x) = \frac{1}{(2x+1)^4}$. The function f is defined in $\mathbb{R}\backslash\{-1/2\}$. It is also differentiable in $\mathbb{R}\backslash\{-1/2\}$. To find the derivative of the function f we define $u(x) = 2x + 1$. The function u is differentiable in $\mathbb{R}$ and we have $u'(x) = 2$. Hence*

$$f'(x) = \frac{-4 \times 2}{(2x+1)^5} = \frac{-8}{(2x+1)^5}.$$

Exercise 2.3.16. *Find the derivative of the following functions*

1. $f(x) = \dfrac{1}{(x^2+5x+1)^3}$, 2. $g(x) = \dfrac{1}{(3x+4)^8}$, 3. $h(x) = \dfrac{1}{(x^3+6x^2+1)^5}$.

Let u and v be two differentiable functions in the open interval I, such that for all $x \in I$, $v(x) \neq 0$. We define $f(x) = \dfrac{u(x)}{v(x)}$. For any $x_0 \in I$ and $h \neq 0$, we have

$$\begin{aligned}\frac{f(x_0+h)-f(x_0)}{h} &= \frac{\dfrac{u(x_0+h)}{v(x_0+h)} - \dfrac{u(x_0)}{v(x_0)}}{h} = \frac{u(x_0+h)v(x_0) - u(x_0)v(x_0+h)}{h\,v(x_0+h)v(x_0)} \\ &= \frac{1}{v(x_0+h)v(x_0)} \cdot \frac{u(x_0+h)-u(x_0)}{h} v(x_0) \\ &\quad - \frac{1}{v(x_0+h)v(x_0)} \cdot \frac{v(x_0+h)-v(x_0)}{h} u(x_0).\end{aligned}$$

Since v is a continuous function, $\lim_{h\to 0} v(x_0+h) = v(x_0)$. This implies

$$\lim_{h\to 0} \frac{1}{v(x_0+h)v(x_0)} = \frac{1}{v^2(x_0)}, \quad \lim_{h\to 0} \frac{u(x_0+h)-u(x_0)}{h} v(x_0) = u'(x_0).v(x_0) \quad \text{and}$$
$$\lim_{h\to 0} \frac{v(x_0+h)-v(x_0)}{h} u(x_0) = u(x_0).v'(x_0).$$

Consequently we have

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = \frac{u'(x_0)v(x_0) - u(x_0)v'(x_0)}{v^2(x_0)}.$$

Since x_0 is an arbitrary element of I, we have the following lemma.

Lemma 2.3.17. *Let u and v be two differentiable functions in the open interval I, such that for all $x \in I$, $v(x) \neq 0$. We define $f(x) = \dfrac{u(x)}{v(x)}$. Then f is differentiable in I and for all $x \in I$,*

$$f'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{v^2(x)}.$$

Example 2.3.18. *We consider the function* $f(x) = \dfrac{x+3}{2x+1}$. *This function is differentiable in its domain* $\mathbb{R}\backslash\{-1/2\}$. *To find* f' *we consider* $u(x) = x+3$ *and* $v(x) = 2x+1$. *Then* $u'(x) = 1$ *and* $v'(x) = 2$. *This implies that for every* $x \in \mathbb{R}\backslash\{-1/2\}$,

$$f'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{v^2(x)} = \frac{2x+1-2(x+3)}{(2x+1)^2} = \frac{-5}{(2x+1)^2}.$$

Exercise 2.3.19. *Find the derivative of the following function*

$$1.\ f(x) = \frac{3x+5}{x^2+x+1}, \quad 2.\ g(x) = \frac{\sqrt{x}+x+1}{x+2}, \quad 3.\ h(x) = \frac{5x-6}{-4x+1}.$$

For the square root we have the following result.

Lemma 2.3.20. *Let* u *be a differentiable function in an open interval* I, *such that for every* $x \in I$, $u(x) > 0$. *Then the function* $f(x) = \sqrt{u(x)}$ *is differentiable in* I *and for every* $x \in I$

$$f'(x) = \frac{u'(x)}{2\sqrt{u(x)}}.$$

We can prove this lemma easily by observing that $f(x) = \big(u(x)\big)^{1/2}$. Then, the derivative of f is

$$f'(x) = \frac{1}{2}u'(x)\big(u(x)\big)^{-1/2} = \frac{u'(x)}{2\sqrt{u(x)}}.$$

But here we will give another proof.

Proof. Let u be as in the lemma. We set $f(x) = \sqrt{u(x)}$. We take $x_0 \in I$ and $h \neq 0$ small enough. We have $f(x_0+h) = \sqrt{u(x_0+h)}$ and $f(x_0) = \sqrt{u(x_0)}$. Therefore we have

$$\begin{aligned}\frac{f(x_0+h)-f(x_0)}{h} &= \frac{\sqrt{u(x_0+h)}-\sqrt{u(x_0)}}{h} = \frac{u(x_0+h)-u(x_0)}{h\left(\sqrt{u(x_0+h)}+\sqrt{u(x_0)}\right)} \\ &= \frac{u(x_0+h)-u(x_0)}{h} \cdot \frac{1}{\sqrt{u(x_0+h)}+\sqrt{u(x_0)}}.\end{aligned}$$

We have

$$\lim_{h\to 0} \frac{u(x_0+h)-u(x_0)}{h} = u'(x).$$

Since $\sqrt{u}$ is a continuous function, we obtain

$$\lim_{h\to 0} \frac{1}{\sqrt{u(x_0+h)}+\sqrt{u(x_0)}} = \frac{1}{2\sqrt{u(x_0)}}.$$

Then

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = \frac{u'(x_0)}{2\sqrt{u(x_0)}}.$$

□

Example 2.3.21. *Let $f(x) = \sqrt{2x+2}$. The function f is defined in $[-1\,;+\infty)$. It is differentiable in the interval $(-1\,,+\infty)$, for every $x \in (-1\,;+\infty)$, the derivative of f is defined by*

$$f'(x) = \frac{}{2\sqrt{2x+2}}.$$

Exercise 2.3.22. *Find the derivative of the following functions*

$$1.\ f(x) = \sqrt{3x+2}, \quad 2.\ g(x) = \sqrt{2x^2+3x+5}, \quad 3.\ h(x) = \sqrt{2x+3}.$$

We take a function u and an open interval I. We suppose u differentiable in I. Now we define the function $f(x) = \cos(u(x))$. For $x_0 \in I$ and $h \neq 0$, we calculate $(f(x_0+h) - f(x_0))/h$. Then, we obtain

$$\begin{aligned}\frac{f(x_0+h)-f(x_0)}{h} &= \frac{\cos(u(x_0+h)) - \cos(u(x_0)}{h} \\ &= \frac{\cos(u(x_0+h)) - \cos(u(x_0))}{h} \cdot \frac{u(x_0+h) - u(x)_0)}{u(x_0+h)-u(x_0)} \\ &= \frac{\cos(u(x_0+h)) - \cos(u(x_0))}{u(x_0+h)-u(x_0)} \cdot \frac{u(x_0+h) - u(x)_0)}{h}.\end{aligned}$$

Since u is differentiable in I it is continuous on I. Then, $\lim_{h\to 0} u(x_0+h) = u(x_0)$. This means that when $h \to 0$, $u(x_0+h) \to u(x_0)$. In other words if we set $y = u(x_0+h)$ and $y_0 = u(x_0)$, we obtain

$$\begin{aligned}\lim_{h\to 0} \frac{\cos(u(x_0+h)) - \cos(u(x_0))}{u(x_0+h)-u(x_0)} &= \lim_{y\to y_0} \frac{\cos(u(y)) - \cos(y_0))}{y-y_0} = -\sin(y_0) \\ &= -\sin(u(x_0)).\end{aligned}$$

Since

$$\lim_{h\to 0} \frac{u(x_0+h)-u(x_0)}{h} = u'(x_0),$$

we conclude that

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = -u'(x_0)\sin(u(x_0)).$$

Lemma 2.3.23. *Let u bea differentiable function in the interval I. Then, the function $f(x) = \cos(u(x))$ is differentiable in I and for all $x \in I$, we have*

$$f'(x) = -u'(x) \cdot \sin(u(x)).$$

Similarly we can establish the following result

Lemma 2.3.24. *Let u be differentiable function in an open interval I. Then, the function $f(x) = \sin(u(x))$ is differentiable in I and for all $x \in I$, we have*

$$f'(x) = u'(x) \cdot \cos(u(x)).$$

Example 2.3.25. *We define the following functions*

$$f(x) = \cos(3x+1) \quad \textit{and} \quad g(x) = \sin(5x+2).$$

Then, we have

$$f'(x) = -3\sin(3x+1) \quad \textit{and} \quad g'(x) = 5\cos(5x+2).$$

Exercise 2.3.26. *Give a proof of lemma 2.3.24.*

Exercise 2.3.27. *Find the derivative of the following functions*

1. $f(x) = \cos(4x + 6)$, 2. $g(x) = \sin(x^2 + x + 2)$, 3. $h(x) = \cos(\sqrt{x + 1})$,
4. $i(x) = \sin(7x + 3)$, 5. $j(x) = \cos(x - 1)$.

The last three lemmas we stated are just particular cases of the following general theorem.

Theorem 2.3.28. *Let u be a differentiable function in some open I and f a differentiable function in some open interval J. We suppose $u(I) \subset J$ and we define the function*

$$g(x) = f(u(x)).$$

Then, the function g is differentiable in I and for every $x \in I$, we have

$$g'(x) = u'(x) \cdot f'(u(x)). \tag{2.3.1}$$

Proof. We consider u and f as in the theorem. Let x_0 be an arbitrary element of I and $h \neq 0$. We have

$$\begin{aligned}
\lim_{h\to 0} \frac{g(x_0 + h) - g(x_0)}{h} &= \lim_{h\to 0} \frac{f(u(x_0 + h)) - f(u(x_0))}{h} \\
&= \lim_{h\to 0} \frac{f(u(x_0 + h)) - f(u(x_0))}{h} \cdot \frac{u(x_0 + h) - u(x_0)}{u(x_0 + h) - u(x_0)} \\
&= \lim_{h\to 0} \frac{f(u(x_0 + h)) - f(u(x_0))}{u(x_0 + h) - u(x_0)} \cdot \lim_{h\to 0} \frac{u(x_0 + h) - u(x_0)}{h}
\end{aligned}$$

When $h \to 0$, $y = u(x_0 + h) \to y_0 = u(x_0)$. This implies

$$\lim_{h\to 0} \frac{f(u(x_0 + h)) - f(u(x_0))}{u(x_0 + h) - u(x_0)} = \lim_{y\to y_0} \frac{f(y)) - f(y_0))}{y - y_0} = f'(y_0) = f'(u(x_0)).$$

On the other hand we have

$$\lim_{h\to 0} \frac{u(x_0 + h) - u(x_0)}{h} = u'(x_0).$$

Thus we conclude that

$$\lim_{h\to 0} \frac{g(x_0 + h) - g(x_0)}{h} = g'(x_0) = u'(x_0) \cdot f'(x_0).$$

□

All the results stated in section can be summarized in the following table. To make the presentation clear we will write u and u' instead of $u(x)$ and $u'(x)$. We denote by D_u the domain of the function u and $D_{u'}$ represent the domain of u'. We use also the following notations:

$$D_1 = \{x \in \mathbb{R} : u(x) \neq 0\},\ D_2 = \{x \in \mathbb{R} : u(x) \geq 0\} \quad \text{and}$$

$$D_3 = \{x \in \mathbb{R} : u(x) > 0\}.$$

$f(x)$	Domain of f	$f'(x)$	Domain of f'
u	D_u	u'	$D_{u'}$
u^2	D_u	$2u'.u$	$D_{u'}$
u^n	D_u	$nu'.u^{n-1}$	$D_{u'}$
$\frac{1}{u}$	D_1	$-\frac{u'}{u^2}$	D_1
$\frac{1}{u^n},\ n \geq 2$	D_1	$-\frac{nu'}{u^{n+1}}$	D_1
$\sqrt{u(x)}$	D_2	$\frac{u'}{2\sqrt{u}}$	D_3
$\cos(u)$	D_u	$-u'.\sin(u)$	D_u
$\sin(u)$	D_u	$u'.\cos(u)$	D_u

Let f and g be two functions. We aim to calculate this limit

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{g(x)-g(x_0)},$$

where we assume $g(x) \neq g(x_0)$. If both functions f and g go to 0 when $x \to x_0$, we will have

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{g(x)-g(x_0)} = \frac{0}{0}.$$

We know, this is undefined. The second case is when

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{g(x)-g(x_0)} = \frac{\pm\infty}{\pm\infty}.$$

To decide these two cases use the following result.

Theorem 2.3.29 (De Hopital's Rule)**.** *Let f and g be two differentiable functions at x_0. We suppose $g(x) \neq g(x_0)$ for every $x \neq x_0$. Then, we have*

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{g(x)-g(x_0)} = \frac{f'(x_0)}{g'(x_0)}.$$

Proof. The proof this is quite simple. Let us assume $g(x) \neq g(x_0)$ for every $x \neq x_0$. In this case we can write

$$\frac{f(x)-f(x_0)}{g(x)-g(x_0)} = \frac{\dfrac{f(x)-f(x_0)}{x-x_0}}{\dfrac{g(x)-g(x_0)}{x-x_0}}.$$

Since we assumed that the functions f and g are differentiable at x_0, we have

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = f'(x_0) \quad \text{and} \quad \lim_{x\to x_0} \frac{g(x)-(x_0)}{x-x_0} = g'(x_0).$$

Therefore we have,

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{g(x)-g(x_0)} = \frac{f'(x_0)}{g'(x_0)}.$$

□

Example 2.3.30. *We consider the following function* $f(x) = \dfrac{\sin(x)}{x}$. *We have*

$$\lim_{x\to 0} \frac{\sin(x)}{x} = \lim_{x\to 0} \frac{sin(x) - \sin(0)}{x - 0} = \cos(0) = 1.$$

Exercise 2.3.31. *Find the following limits*

$$1.\ \lim_{x\to 0} \frac{\cos(x) - 1}{x}, \quad 2.\ \lim_{x\to 0} \frac{\tan(x)}{x}, \quad 3.\ \lim_{x\to -1} \frac{x^3 + 1}{x + 1}.$$

In the previous sections we study derivatives of functions. In the next section we will see some applications of these derivatives.

2.4 Applications

In this section we enumerate some applications of the derivatives. In the first part we use derivative to study variations of functions. After this we show how derivatives can allow us to characterize a maximum and a minimum.

Theorem 2.4.1. *Let* f *be a differentiable function in an open interval* I. *Then,*

- *the function* f *is increasing in* I *if and only if, for every* $x \in I$, $f'(x) \geq 0$
- *the function* f *is decreasing in* I *if and only if, for every* $x \in I$, $f'(x) \leq 0$.

Remark 2.4.2. *We specify the following notations*

- *the function* f *is strictly increasing in* I *if and only if, for every* $x \in I$,

$$f'(x) > 0$$

- *the function* f *is strictly decreasing in* I *if and only if, for every* $x \in I$,

$$f'(x) < 0.$$

Example 2.4.3. *We consider the function* $f(x) = x^3 + 1$. *This function is polynomial then it is defined and derivable in* $\mathbb{R}$. *For every real number* x *the derivative of* f *is given by* $f'(x) = 3x^2$. *Then,* $f'(x) \geq 0$. *We conclude that the function* f *is increasing in* $\mathbb{R}$.

Example 2.4.4. *Let* $g(x) = -2x + 1$. *The function* g *is differentiable in* $\mathbb{R}$ *and for all real number* x *we have* $g'(x) = -2 < 0$. *Hence the function* g *is strictly decreasing in* $\mathbb{R}$.

Exercise 2.4.5. *Study the variations of the following functions*

$$1.\ f(x) = x^2 + 3x + 6, \quad 2.\ g(x) = \frac{x + 2}{x - 1}, \quad 3.\ h(x) = (x^2 + 2x + 4)(x + 5).$$

Below we remind the definition of a local maximum of a given function.

Definition 2.4.6. *Let* f *be a continuous function and* $I \subset D_f$ *an interval. We say the function* f *has local maximum at* $x_0 \in I$, *if for all* $x \in I$, $f(x_0) \geq f(x)$.

Theorem 2.4.7. *Let f be a differentiable function, a and b two real numbers in D_f, such that, $a < b$ and $[a\,,\,b] \subset D_f$. We consider $x_0 \in (a\,,\,b)$, such that the function f is increasing in $[a\,,\,x_0]$ and decreasing in $[x_0\,,\,b]$. Then, the function f has a local maximum at x_0. The value of this local maximum is $f(x_0)$.*

Proof. Let f, a and b satisfy the conditions in the theorem. The function f is increasing in $[a\,,\,x_0]$. Then $f'(x) \geq 0$ for all $x \in [a\,,\,x_0]$. From this we deduce that for all $x \in [a\,,\,x_0]$,

$$\frac{f(x_0) - f(x)}{x_0 - x} \geq 0$$

. Multiplying the inequality above by $x_0 - x > 0$, we obtain $f(x_0) - f(x) \geq 0$ This implies $f(x_0) \geq f(x)$, for every $x \in [a\,,\,x_0]$.

On the other hand we know the function f is decreasing in $[x_0\,,\,b]$. Hence $f'(x) \leq 0$, for all $x \in [x_0\,;\,b]$. That is, for every $x \in [x_0\,;\,b]$,

$$\frac{f(x) - f(x_0)}{x - x_0} \leq 0.$$

As we know that $x - x_0 \geq 0$, then, $f(x) - f(x_0) \leq 0$. Therefore for all $x \in [x_0\,,\,b]$ we have

$$f(x_0) \geq f(x).$$

We have proved that for every $x \in [a\,,\,b]$,

$$f(x_0) \geq f(x).$$

The function f has a local maximum at x_0. The value of this maximum is $f(x_0)$. □

Example 2.4.8. *We take the function $f(x) = -x^2 + 2x + 1$. The function f is derivable in $\mathbb{R}$ and we have $f'(x) = -2x + 2$. We can observe that $f'(x) \geq 0$ for all $x \in (-\infty\,,\,1]$. Therefore f is increasing in $(-\infty\,,\,1]$. The function $f'(x) \leq 0$ for all $x \in [1\,,\,+\infty)$. Then f is a decreasing in $[1\,,\,+\infty)$. Applying the theorem above we deduce that f has a maximum at $x_0 = 1$. The value of this maximum is $f(1) = -1^2 + 2 \cdot 1 + 1 = 2$.*

Theorem 2.4.9. *Let f be a differentiable function and $I \subset D_f$ an interval. We consider $x_0 \in I$ and we assume that f has a local maximum at x_0. Then $f'(x_0) = 0$.*

Proof. Let f be a differentiable function which has a local maximum at $x_0 \in I$. Then for all $x < x_0$, we have $\frac{f(x)-f(x_0)}{x-x_0} \geq 0$. From this we deduce that

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) \geq 0. \tag{2.4.1}$$

On the other hand for any $x \geq x_0$, we have $\frac{f(x)-f(x_0)}{x-x_0} \leq 0$. This implies

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) \leq 0. \tag{2.4.2}$$

From (2.4.1) and (2.4.2), we deduce that $0 \leq f'(x_0) \leq 0$. Therefore $f'(x_0) = 0$. □

Example 2.4.10. *We consider the function $f(x) = -x^2 + 2x + 1$. We have $f'(x) = -2x + 2$. We proved that the function f has a maximum at $x_0 = 1$. We can see that $f'(1) = -2.(1) + 2 = 0$.*

In the definition below, we recall the definition of a local minimum of a function.

Definition 2.4.11. *Let f be a continuous function and $I \subset D_f$ an interval. The function f has local minimum at $x_0 \in I$, if for every $x \in I$, $f(x) \geq f(x_0)$.*

Theorem 2.4.12. *Let f be a differentiable function a and b two real numbers in D_f, such that, $a < b$ and $[a\,,\,b] \subset D_f$. We consider $x_0 \in (a\,,\,b)$, such that f is decreasing in $[a\,,\,x_0]$ and increasing in $[x_0\,,\,b]$. Then the function f has a local minimum at x_0. The value of this local minimum is $f(x_0)$.*

Proof. Let f, a and b be as in the theorem. Since the function f is decreasing in $[a\,,\,x_0]$. Then $f'(x) \leq 0$. From this we deduce that for all $x \in [a\,,\,x_0]$,

$$\frac{f(x_0) - f(x)}{x_0 - x} \leq 0.$$

Since $x_0 - x > 0$, we have $f(x_0) - f(x) \leq 0$. This implies that $f(x_0) \leq f(x)$, for every $x \in [a\,,\,x_0]$.

On the other hand, the function f is increasing in $[x_0\,,\,b]$. Hence $f'(x) \geq 0$, for all $x \in [x_0\,,\,b]$. That is, for all $x \in [x_0\,,\,b]$, $\frac{f(x)-f(x_0)}{x-x_0} \geq 0$. The fact that $x - x_0 \geq 0$, involves $f(x) - f(x_0) \geq 0$. Therefore, for every $x \in [x_0\,,\,b]$, $f(x_0) \leq f(x)$. We have proved that for all $x \in [a\,,\,b]$ $f(x_0) \leq f(x)$. The function f has a local minimum at x_0. The value of this minimum is $f(x_0)$. □

Example 2.4.13. *We take the function $f(x) = x^2 - x + 1$. The derivative of f is $f'(x) = 2x - 1$. We can observe that $f'(x) \leq 0$, for all x in $(-\infty\,;\,1/2]$. Therefore f is decreasing in $(-\infty\,;\,1/2]$. The function f' is positive in $[1/2\,;\,+\infty[$. Then f is a increasing in $[1/2\,;\,+\infty[$. Applying the theorem above we deduce that f has a minimum at $x_0 = 1/2$ and we get $f'(1/2) = 2(1/2) - 1 = 0$.*

Theorem 2.4.14. *Let f be a differentiable function and $I \subset D_f$ which is an interval. We consider $x_0 \in I$ and we assume that f has a local minimum at x_0. Then*

$$f'(x_0) = 0.$$

Proof. Let f be a differentiable function which has a local manimum at x_0, then for all $x < x_0$, we have $\dfrac{f(x) - f(x_0)}{x - x_0} \leq 0$. Then,

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) \leq 0. \tag{2.4.3}$$

On the other hand, for any $x \geq x_0$, we have $\dfrac{f(x) - f(x_0)}{x - x_0} \geq 0$. This implies that

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) \geq 0. \tag{2.4.4}$$

From (2.4.3) and (2.4.4), we deduce that $0 \leq f'(x_0) \leq 0$. Therefore $f'(x_0) = 0$. □

Example 2.4.15. *Let $f(x) = x^2 - 2x + 1$. The function f is differentiable in $\mathbb{R}$ and $f'(x) = 2x - 2$. The function f has a minimum at $x_0 = 1$, and we have $f'(1) = 2.(1) - 2 = 0$.*

Exercise 2.4.16. *Say whether the following functions have local maximum or local minimum.*

$$1.\ f(x) = x^2 + 3x + 2, \quad 2.\ g(x) = -3x^2 + 5x + 1, \quad 3.\ h(x) = \frac{x+1}{x^2+3x+4}\cdot$$

A general case where $f'(x_0) = 0$ holds is in Rolle's theorem.

Theorem 2.4.17 (Rolle's Theoerem)**.** *Let f be a real function defined on $[a\,,\,b]$. We suppose*

- *the function f is continuous on $[a\,,\,b]$*
- *the function f is differentiable in $(a\,,\,b)$*
- *we have $f(a) = f(b)$.*

Then, there exist $x_0 \in\ (a\,,\,b)$, such that, $f'(x_0) = 0$

Proof. In this proof we will distinguish two cases.

Case 1: We assume for every $x \in [a\,,\,b]$, $f(x) = f(a)$.

In this case the function f is constant. Therefore, for all $x \in]a\,,\,b[$, $f'(x) = 0$.

Case 2: We assume the function f is not constant.

Since the function f is continuous on $[a\,,\,b]$, then the function f reaches its maximum and minimum. There are x_0 and x_1 in $[a\,,\,b]$, such that

$$m := \min_{x\in[a\,,\,b]} f(x) \quad \text{and} \quad M := \max_{x\in[a\,,\,b]} f(x).$$

In this case,we have one of the following situation either $m < f(a)$ or $M > f(a)$.

Here we suppose $m < f(a) = f(b)$. Then, there exists $x_0 \in]a\,,\,b[$, such that, $f(x_0) = m$. Since the function f has a local minimum at x_0, we have $f'(x_0) = 0$. □

We emphasize Rolle's Theorem is a particular case of the following theorem.

Theorem 2.4.18 (Mean Value Theorem)**.** *Let f be a real function defined on $[a\,,\,b]$. We suppose that*

- *the function f is continuous on $[a\,,\,b]$*
- *the function f is differentiable in $(a\,,\,b)$*

Then, there exists $c \in (a\,,\,b)$, such that,

$$f(b) - f(a) = (b-a)\,f'(c). \tag{2.4.5}$$

If we take $f(a) = f(b)$ in this theorem, we obtain Rolle's theorem. The proof of this theorem is simple.

Proof. Let $x \in [a\,,\,b]$. We define the function $g(x) = f(x) - f(a) - (x-a)\frac{f(b)-f(a)}{b-a}\cdot$ The function g satisfies these properties:

- the function g is continuous on $[a\,,\,b]$

- the function g is derivable in $(a\,,\,b)$ and $\quad g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$,
- we have $g(a) = g(b)$.

Then, applying Rolle's theorem to g, we find $c \in (a\,,\,b)$ such that $g'(c) = 0$. This involves

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

□

We remark that the identity (2.4.5) holds only in $\mathbb{R}$. When the dimension n of the space is greater than 1, (for example $n \geq 2$), we get an inequality..

Theorem 2.4.19. *Let f be a real function defined on $[a\,,\,b]$. We assume that*

- *the function f is continuous on $[a\,,\,b]$*
- *the function f is differentiable in $(a\,,\,b)$*
- *there exists a real number $M > 0$, such that, for every $x \in (a\,,\,b)$,*

$$|f'(x)| \leq M.$$

Then,

$$\big|f(b) - f(a)\big| \leq M\ |b - a|. \tag{2.4.6}$$

Proof. To prove this theorem we consider a function f satisfying the condition of the theorem. Then, there exists $c \in]a\,,\,b[$, such that

$$|f(b) - f(a)| = |f'(c)|\ |b - a|.$$

Since $c \in]a\,,\,b[$, we have $|f'(c)| \leq M$. Hence, we obtain

$$|f(b) - f(a)| \leq M|b - a|$$

□

To end this section we show how to use the derivative of a function to find a local linear approximation of a function. The most known of this type of approximation is the tangent line of the graph $\mathcal{C}$ of the function at the point x_0. Here we will have the privilege of showing when using second order derivative we can obtain more precise approximations of a function.

Definition 2.4.20. *Let f be a differentiable function. We consider $x_0 \in D_f$ and we suppose the function f differentiable at x_0. The equation of the tangent line of the curve of f at x_0 is defined by,*

$$(T_{x_0})\ :\ y = f'(x_0)(x - x_0) + f(x_0). \tag{2.4.7}$$

Sometimes we use the notation tangent line of f at x_0 instead of tangent line of the curve of f at x_0.

Example 2.4.21. *Let $f(x) = x^2 + 4x + 2$. Find the tangent line of f at $x_0 = 0$. We have $f'(x) = 2x + 4$. Then $f'(0) = 4$ and $f(0) = 2$. The equation of the tangent line of f at $x_0 = 0$ is given by*

$$T_0 : y = 4(x - 0) + 2 = 4x + 2.$$

Exercise 2.4.22. *Define the equation of the tangent line of the following functions at x_0.*

1. $f(x) = x^3 + 2x + 1,\ x_0 = 1$, 2. $g(x) = \sqrt{x+2},\ x_0 = 2$, 3. $h(x) = \dfrac{1}{x+1},\ x_0 = 1$

4. $i(x) = \dfrac{x+7}{x+2},\ x_0 = 3$, 5. $j(x) = (x+5)(x^2+5x+9),\ x_0 = 0$.

Convex functions share some nice properties with their tangent line. Below we will establish that the curve of a convex function lives above its tangent lines. To present this clearly, we need some preliminary results.

Lemma 2.4.23. *Let f be a convex and differentiable function in an open interval I. We consider three real numbers x_1, x_2 and x_3 in I, such that $x_1 < x_2 < x_3$. Then,*

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_1)}{x_3 - x_1} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2}. \tag{2.4.8}$$

Proof. We take f convex in I, x_1, x_2 and x_3 in I such that, $x_1 < x_2 < x_3$. Since the segment $[x_1\,,\,x_3]$ is convex, there is $t \in]0\,,\,1[$, such that, $x_2 = t\,x_1 + (1-t)\,x_3$. Using the definition of convex functions we have,

$$f(x_2) = f(tx_1 + (1-t)\,x_3) \leq tf(x_1) + (1-t)\,f(x_3).$$

This implies $f(x_2) - f(x_1) \leq (1-t)\big(f(x_3) - f(x_2)\big)$. Dividing by $x_2 - x_2 > 0$, we obtain

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq (1-t)\frac{f(x_3) - f(x_1)}{x_2 - x_1}.$$

From the definition of x_2, we observe that $x_3 - x_1 = (x_2 - x_1)/(1-t)$. This involves

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq (1-t)\frac{f(x_3) - f(x_1)}{x_2 - x_1} = (1-t)\frac{f(x_3) - f(x_1)}{(1-t)\dfrac{x_2 - x_1}{1-t}} = \frac{f(x_3) - f(x_1)}{x_3 - x_1}.$$

This establish the first part of (2.4.8).

To prove the second part of (2.4.8) we need remark that

$$t(f(x_3) - f(x_1) \leq f(x_3) - f(x_2).$$

Now we divide in booth sides this inequality by $x_3 - x_2 > 0$, then,

$$t\frac{(f(x_3) - f(x_1)}{x_3 - x_2} \leq \frac{(f(x_3) - f(x_2)}{x_3 - x_2}.$$

As we know that $x_3 - x_2 = t(x_3 - x_1)$ one has

$$t\frac{(f(x_3) - f(x_1)}{x_3 - x_2} = t\frac{f(x_3) - f(x_1)}{t\dfrac{x_3 - x_1}{t}} = \frac{f(x_3) - f(x_1)}{x_3 - x_1} \leq \frac{(f(x_3) - f(x_2)}{x_3 - x_2}.$$

This is the second part of inequality (2.4.8). Therefore we have (2.4.8). □

This lemma implies the following theorem.

Theorem 2.4.24. *Let f be a convex and differentiable function in an open interval I. Then, for all x and x_0 in I, we have*

$$f(x) \geq f'(x_0)(x - x_0) + f(x_0).$$

Proof. Let f be a convex function. We assume x_0, x. Then, for every $y \in [x_0\,,\, x]$, we

$$\frac{f(y) - f(x_0)}{y - x_0} \leq \frac{f(x) - f(x_0)}{x - x_0}.$$

Taking the limit when y goes to x_0, we obtain

$$\lim_{y \to x_0} \frac{f(y) - f(x_0)}{y - x_0} = f'(x_0) \leq \frac{f(x) - f(x_0)}{x - x_0}.$$

□

Remark 2.4.25. *The proof is the same when we assume $x_0 > x$.*

Considering again (2.4.8), we can see that if we let $x_2 \to x_1$ and $x_2 \to x_3$, we obtain the following theorem which mention that the derivative of a convex function is increasing.

Theorem 2.4.26. *Let f be convex and differentiable function in I. Then, the function f' is an increasing function.*

Proof. Let f be a convex function. From (2.4.8), we know that for any x_1, y and x_2 in I, such that $x_1 < y < x_2$, we have,

$$\frac{f(y) - f(x_1)}{y - x_1} \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x) - f(y)}{x_2 - y}.$$

First we consider the limit when $y \to x_1$. In this case we have

$$f'(x_1) \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1}. \tag{2.4.9}$$

Now we take the limit when $y \to x_2$. Thus, we get

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \lim_{y \to x_2} \frac{f(x) - f(y)}{x - y} = f'(x_2). \tag{2.4.10}$$

We conclude that the function f' is increasing. Because $f'(x_1) \leq f'(x_2)$. □

Remark 2.4.27. *The reciprocal of this theorem is also true. If f is a function such that, f' is increasing, then f is convex.*

The proof of this is based on the mean value theorem.

Let f be a convex function. The theorem above implies the function f' is an increasing function in the interval I. This means that if the function f' is differentiable in the interval I, then , its derivative f'' is positive for all $x \in I$.

Reversely if the function f'' is positive, then f' is an increasing function. Consequently the function f is convex.

This establish the following theorem.

Theorem 2.4.28. *Let f be a differentiable function in an interval I. We suppose f' differentiable on I. Then, the function f is convex if and only if, for all $x \in I$,*

$$f''(x) \geq 0.$$

Example 2.4.29. *Let $f(x) = x^2 + 3x + 2$. The function f is differentiable in $\mathbb{R}$. For every $x \in \mathbb{R}$,*

$$f'(x) = 2x + 3 \quad \textit{and} \quad f''(x) = 2 > 0.$$

We conclude that the function f is convex in $\mathbb{R}$.

Exercise 2.4.30. *Say whether the following functions are convex*

1. $f(x) = -3x^2 + 4x + 1$, 2. $g(x) = 5x^2 - 6$, 3. $h(x) = x^3 + 3x^2 + 2x + 4$.

With this exercise we close the parenthesis on convex function and go back to approximation of functions.

In general we can establish that any function can be approximated by a polynomial function. This holds in a neighborhood of some point x_0. To present this in a simple way we introduce the following notations

Definition 2.4.31. *Let f be a real function defined in the interval I.*

- *We say the function f is $\mathscr{C}^0$ in I if the function f is continuous in I*
- *the function f is $\mathscr{C}^1$ in I if f' is continuous in I*

We call the second derivative of the function f, the derivative of the function f'. We denote it f''. The third derivative of f is denoted f''' or $f^{(3)}$. The fourth derivative of f is denoted $f^{(4)}$. In general, we denote the nth derivative of f, $f^{(n)}$.

Definition 2.4.32. *Let f be a real function defined in the interval I.*

- *We say the function f is $\mathscr{C}^2$ in I, if f is two times differentiable and f'' is continuous in I*
- *The function f is a $\mathscr{C}^n$ function if the function f is n times differentiable and the function $f^{(n)}$ is continuous in I.*

For the sake of clarity we use the following notations. The set of all continuous functions on I is denoted $\mathscr{C}^0\big(I\big)$. For any $n \geq 1$ we use $\mathscr{C}^n\big(I\big)$ to mean the set of all $\mathscr{C}^n$ functions in I.

Definition 2.4.33. *We define the function $\varepsilon : \mathbb{R} \longrightarrow \mathbb{R}^+$, satisfying the following properties:*

1. *the function ε is continuous in $\mathbb{R}$*
2. *the function ε satisfies*

$$\lim_{x \to 0} \varepsilon(x) = 0 \quad \textit{and} \quad \lim_{x \to 0} \varepsilon(1) = 0$$

Definition 2.4.34. *Let f be a real function defined in I. We assume the function f admits first and second derivative at $x_0 \in I$. Let $\delta > 0$ be small enough. For any $x \in (x_0 - \delta\,,\, x_0 + \delta)$ we define the second order Taylor series of f at x_0 as follows*

$$f(x) = f(x_0) + f'(x_0) \cdot \frac{(x - x_0)}{1!} + f''(x_0) \cdot \frac{(x - x_0)^2}{2!} + (x - x_0)^2 \varepsilon((x - x_0)). \quad (2.4.11)$$

Remark 2.4.35. *When we take $x_0 = 0$ in (2.4.11), we obtain*

$$f(x) = f(0) + \frac{f'(0)}{1!} \cdot x + \frac{f''(0)}{2!} \cdot x^2 + x^2\varepsilon(x). \tag{2.4.12}$$

Another way to reformulate (2.4.11) is to define $x = x_0 + h$ for some small real number h. In this case we have $h = x - x_0$. Replacing in (2.4.11) one gets

$$f(x_0) = f(x_0) + f'(x_0) \cdot \frac{h}{1!} + f''(x_0) \cdot \frac{h^2}{2!} + h^2\varepsilon(h).$$

In truth we can define a n degree polynomial function which is an approximation of f in a neighborhood of a point. This polynomial function is called the nth order Taylor Series of f.

Definition 2.4.36. *Let f be a real function defined in I. We suppose f is n times derivable at $x_0 \in I$. Let $\delta > 0$ be a fixed real number. For any $x \in (x_0 - \delta\,,\, x_0 + \delta)$, we define the nth order Taylor Series of f at x_0 as follows*

$$f(x_0 + h) = f(x_0) + \sum_{k=1}^{n} f^{(k)}(x_0) \cdot \frac{h^k}{k!} + h^n \cdot \varepsilon(h), \tag{2.4.13}$$

where $h = x - x_0$.

Example 2.4.37. *We consider the function $f(x) = \cos(x)$. The function f is differentiable in $\mathbb{R}$. We have $f'(x) = -\sin(x)$ and $f''(x) = -\cos(x)$. Taking $x_0 = 0$ we have $f(0) = 1$, $f'(0) = \sin(0) = 0$ and $f''(0) = -1$. Therefore the second order Taylor Series of f is defined by*

$$\begin{aligned}\cos(x) &= f(0) + f'(0) \cdot x + f''(0) \cdot \frac{x^2}{2!} + x^2 \cdot \varepsilon(x) \\ &= 1 + 0 \cdot x - \frac{x^2}{2} + x^2 \cdot \varepsilon(x) = 1 - \frac{x^2}{2} + x^2 \cdot \varepsilon(x).\end{aligned}$$

Using this Taylor series we can easily find this limit $\lim_{x\to 0} \frac{\cos(x)-1}{x}$. Indeed, for $x \neq 0$, we have

$$\frac{\cos(x) - 1}{x} = -\frac{x}{2} + x \cdot \varepsilon(x).$$

Since

$$\lim_{x\to 0} \frac{\cos(x) - 1}{x} = -\lim_{x\to 0} \frac{x}{2} + \lim_{x\to 0} x \cdot \varepsilon(x) = 0$$

Example 2.4.38. *Let $f(x) = \sin(x)$. The function f is 3 times differentiable in $\mathbb{R}$. Therefore, the function f is 3 times differentiable at 0. We have $f'(x) = \cos(x)$, $f''(x) = -\sin(x)$ and $f'''(x) = -\cos(x)$. I this case we get $f(0) = \sin(0) = 0$, $f'(0) = \cos(0) = 1$, $f''(0) = \sin(0) = 0$ and $f'''(0) = -1$. This leads to the following third order Taylor series of f at 0*

$$\begin{aligned}f(x) &= f(0) + f'(0) \cdot x + f''(0) \cdot \frac{x^2}{2} + f'''(0) \cdot \frac{x^3}{3!} + x^3 \cdot \varepsilon(x) \\ &= 0 + x + 0 \cdot \frac{x^2}{2} - \frac{x^3}{6} + x^3 \cdot \varepsilon(x) = x - \frac{x^3}{6} + x^3 \cdot \varepsilon(x).\end{aligned}$$

From this Taylor series we deduce that , for any $x \neq 0$,

$$\frac{\sin(x) - x}{x} = -\frac{x^2}{6} + x^2 \cdot \varepsilon(x).$$

Hence

$$\lim_{x\to 0} \frac{\sin(x) - x}{x} = -\lim_{x\to 0} \frac{x^2}{6} + \lim_{x\to 0} x^2 \cdot \varepsilon(x) = 0.$$

We know what is a Taylor series, now we want to have criterion which will allows us to say whether a function has a Taylor series.

Theorem 2.4.39. *Let* I *be an open interval and* f *a real function. We suppose* $f \in \mathscr{C}^n\big(I\big)$. *Then, the function* f *admit a nth order Taylor Series at any point* $x_0 \in I$.

Theorem 2.4.40. *Let* f *be a polynomial function in* I. *Then, the function* f *is its own Taylor series.*

Exercise 2.4.41. *Find the 4th order Taylor series of the following functions at the indicated point*

1. $f(x) = \dfrac{1}{1-x}$, $x_0 = 1$, 2. $g(x) = \sqrt{1+x}$, $x_0 = -1$, 3. $h(x) = \dfrac{1}{1-x^2}$, $x_0 = 1$.

CHAPTER 3

STUDY OF REAL FUNCTIONS

Contents

In chapter 1 we defined limits of functions. We specified that the limits at the bounds of the domain of definition lead to the important concept of *asymptote*. Using limits again we defined continuity. We studied continuous functions and their properties in that chapter. In chapter 2 we defined derivative of one variable real functions. We showed several applications of derivatives.

The main purpose of this chapter is to gather all the properties studied in chapters 1 & 2. Studying together all these properties is called: *study of functions*. In short, studying a function consists of :

1. finding its *domain*

2. finding the *limits at the bound of the domain*
3. finding *asymptotes*
4. studying the *continuity* of the function
5. calculating *derivative* of the function
6. studying *variations* of the function
7. giving the table of *variations*
8. finding *tangent lines*
9. *sketching the graph of* f.

This chapter is organized as follows : in section 1, we study elementary functions. The section 2, is devoted to the study of second order polynomial functions. In the last section we deal with general functions.

3.1 Study of Elementary Functions

The aim of this section is to study elementary functions. We remind that elementary are bricks from which all complicated functions are constructed. Here we deal with the affine functions $ax + b$, the function x^2, the function $\sqrt{x}$ and the function $\frac{1}{x}$.

As we will see it, using algebraic operations and composition on these functions, we will be able to study more complicated functions. This will be done in section 2.

The most elementary functions are affine functions. That is why we start the study of functions with them. We recall that an affine function is a function in the form

$$f(x) = ax + b,$$

where a and b are constant real numbers such that $a \neq 0$.

3.1.1 Study of Affine Functions

Let $f(x) = ax + b$ be an affine function. We know from the previous chapters that the domain of f is $\mathbb{R}$. Now we undertake to study f.

We start by finding the limits at the bounds of the domain of f

Limits at the Bounds of the Domain

To compute correctly the limits of the function at the bounds of the domain, we have to distinguish two cases according to the sign of the real number a.

Case 1: $a > 0$ When the real number a is positive, the following limits hold

$$\lim_{x \to -\infty} f(x) = -\infty \quad \text{and} \quad \lim_{x \to +\infty} f(x) = +\infty.$$

Case 2: $a < 0$ If the real number a is negative, one has the following limits

$$\lim_{x \to -\infty} f(x) = +\infty \quad \text{and} \quad \lim_{x \to +\infty} f(x) = -\infty.$$

We observe that *affine* functions do not have horizontal asymptotes. Since affine functions are defined in $\mathbb{R}$ they do not have vertical asymptotes. The next step consists of finding the derivative of f.

Derivative of an Affine Function

We established that affine function are differentiable in $\mathbb{R}$ and for every real number x we have

$$f'(x) = a.$$

Here again we will use the sign of the real number a to determine the variations of f. There are two cases:

Case 1: $a > 0$. In this case the function f is strictly increasing in $\mathbb{R}$. Because for every $x \in \mathbb{R}$, we have $f'(x) > 0$.

Case 2: $a < 0$. Here $f'(x) < 0$ for every $x \in \mathbb{R}$. This implies the function f is strictly decreasing in $\mathbb{R}$.

Remark 3.1.1. *We point out that when $a = 0$, f is a constant function. Constant functions are a critical case. Because we can consider them either increasing or decreasing functions.*

Variational Table of an Affine Function

Below we define the variational table of f. This table was established in concordance with the sign of a.

We obtain the following table:

Case 1: $a > 0$. When the real number a is positive we obtain the following table

x	$-\infty$		$+\infty$
sign of f'		$+$	
variatons of f	$-\infty$	$\nearrow$	$+\infty$

This table shows that the function f is increasing from $-\infty$ to $+\infty$.

Case 2: $a < 0$. In this case the following variational table holds

x	$-\infty$		$-\infty$
sign of f'		$-$	
variatons of f	$+\infty$	$\searrow$	$-\infty$

From this table we deduce the function f is decreasing from $+\infty$ to $-\infty$.

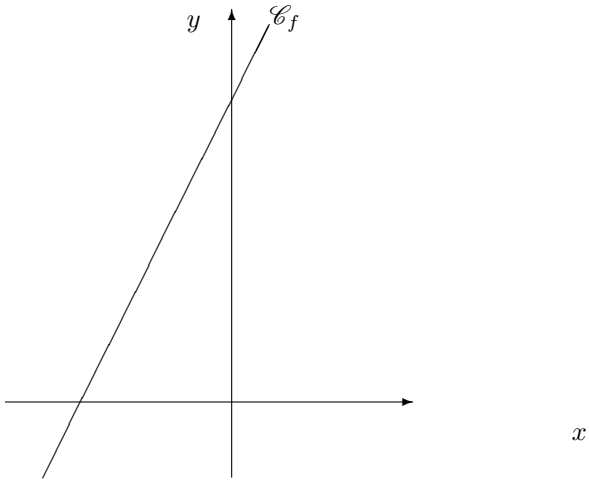
Example 3.1.2. *We consider the function $f(x) = 2x+4$. The function f is defined in $\mathbb{R}$. Since $a = 2 > 0$, we have*

$$\lim_{x \to -\infty} f(x) = -\infty \quad \textit{and} \quad \lim_{x \to +\infty} f(x) = +\infty.$$

We know the function f is differentiable in $\mathbb{R}$ and for every real number x $f'(x) = 2$. Therefore the function f is increasing in $\mathbb{R}$. Then, it follows that we have the following variational table

x	$-\infty$	$+\infty$
sign of f'	$+$	
variatons of f	$-\infty$ ↗	$+\infty$

The function f is represented by the following graph



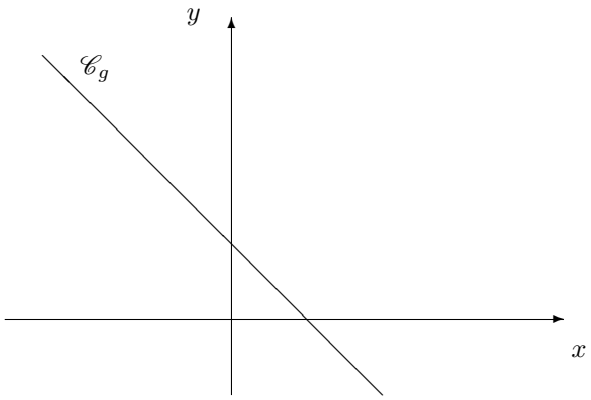
Example 3.1.3. *Let $g(x) = -x + 1$. The function g is defined in $\mathbb{R}$ and we have*

$$\lim_{x\to-\infty} g(x) = +\infty \quad \textit{and} \quad \lim_{x\to+\infty} g(x) = -\infty.$$

The function g is differentiable in $\mathbb{R}$ and for all $x \in \mathbb{R}$, we have $f(x) = -1 < 0$. Then g is a decreasing function. The following variational table holds

x	$-\infty$	$-\infty$
sign of g'	$-$	
variatons of g	$+\infty$ ↘	$-\infty$

Below we sketch the graph of g



Exercise 3.1.4. *Study the following affine functions*

1. $f(x) = -3x + 5$, 2. $g(x) = 5x + 7$, 3. $h(x) = -4x + 9$, 4. $i(x) = 11x + 33$.

in the following subsection we study the square function.

3.1.2 Study of the Square Function

Let $f(x) = x^2$. We have established that $D_f = \mathbb{R}$. The next step consist of finding its limits

Limits at the Bounds of the Domain

Since for any real number x, we have $x^2 \geq 0$, we deduce

$$\lim_{x\to-\infty} f(x) = +\infty \quad \text{and} \quad \lim_{x\to+\infty} f(x) = +\infty.$$

Hence the function $f(x) = x^2$ does not have horizontal asymptotes. The function f is defined in the whole set of real numbers $\mathbb{R}$. Therefore, the function does not admit vertical asymptotes.

On the other hand we have,

$$\lim_{x\to-\infty} \frac{f(x)}{x} = -\infty \quad \text{and} \quad \lim_{x\to+\infty} \frac{f(x)}{x} = +\infty.$$

This show that, we do not have oblique asymptotes at $\pm\infty$.

Derivative of the Square Function

From previous chapters, we know that the function f is differentiable in $\mathbb{R}$ and for every $x \in \mathbb{R}$, $f'(x) = 2x$.

To determine the variations of the function we need to study the sign of f'. Observing that f' is a linear function, we deduce that $f'(x) \leq 0$ for every $x \in \big(-\infty\,,\,0\big]$, and $f'(x) \geq 0$ for all $x \in \big[0\,,\,+\infty\big)$.

Variations of the Square Function

Using the sign of the function f' we conclude the function

- $f(x) = x^2$ is decreasing in $(-\infty\,,\,0]$
- f is increasing in $[0\,,\,+\infty)$.

Variational Table of the Square Function

The variations of the function f will allow to define the following table:

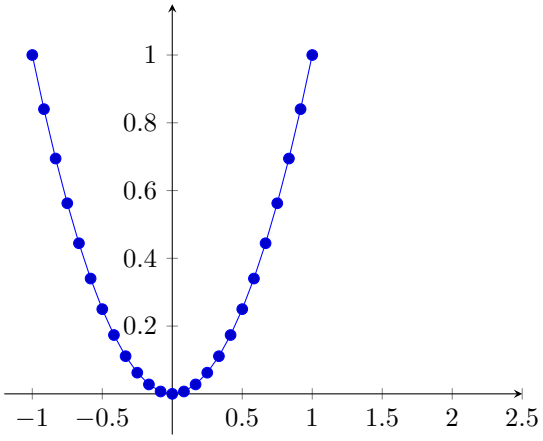
x	$-\infty$		0		$+\infty$
$f'(x)$		$-$	0	$+$	
$f(x)$	$+\infty$	$\searrow$	$f(0)$	$\nearrow$	$+\infty$

Tangent Line at 0

The tangent line of $f(x) = x^2$ at 0 is given by the following formula $(T_0) : y = f'(0)(x-0) + f(0) = 0$.

Since the function f is decreasing in $(-\infty\,,\,0]$ and increasing in $[0\,,\,+\infty)$. Then the function f has a minimum at $x_0 = 0$. This minimum is equal to $f(0) = 0$.

Below we represent the function $f(x) = x^2$.



Exercise 3.1.5. *Study the following functions*

$$1.\ f(x) = -x^2, \qquad 2.\ g(x) = x^2 + 3, \qquad 3.\ h(x) = -x^2 + 3.$$

3.1.3 Study of the Inverse Function

Now we consider the function $f(x) = \dfrac{1}{x}\cdot$ This function is defined in $\mathbb{R}\backslash\{0\}$.

Limits at $\pm\infty$

The limits of the function at $\pm\infty$ are

$$\lim_{x\to+\infty} f(x) = 0 \quad \text{and} \quad \lim_{x\to-\infty} f(x) = 0.$$

This involves the line $y = 0$ is a horizontal asymptote at $\pm\infty$.

Limits at 0

As it was shown in the chapter 1 we have

$$\lim_{x\to0^-} f(x) = -\infty \quad \text{and} \quad \lim_{x\to0^+} f(x) = +\infty.$$

This means that the line $x = 0$ is a vertical asymptote of f at the point $x_0 = 0$.

Derivative of the Inverse Function

The function f is differentiable in $\mathbb{R}\backslash\{0\}$. Let $x \in \mathbb{R}^*$ one has

$$f'(x) = -\frac{1}{x^2}.$$

This means that $f'(x) < 0$, for all $x \in \mathbb{R}\backslash\{0\}$. That is, f is a decreasing function in $\mathbb{R}\backslash\{0\}$.

Variational Table of f

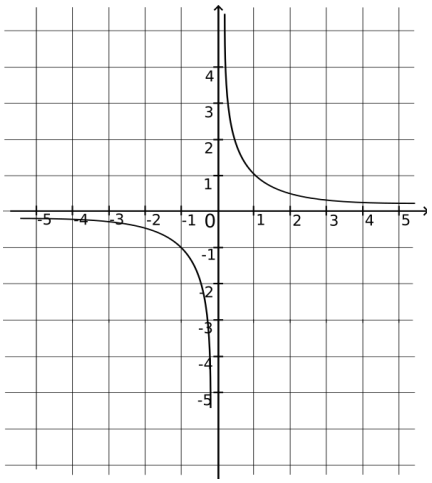
The variations of the function f lead to the following variational table

x	$-\infty$		0		$+\infty$
$f'(x)$	$-$		$\|\|$	$-$	
$f(x)$	0 ↘	$-\infty$	$\|\|$	$+\infty$ ↘	0

The function f does have not a global maximum neither a global minimum.

Remark 3.1.6. *We point out that the graph of a function do never cross its vertical asymptote. Vertical asymptotes hold only at point on which the function is not defined. In other words if the line $x = a$ is a vertical asymptote then the function f is not defined at a.*

The graph of the inverse function is given below.



As we proved it above the inverse function has a vertical asymptote at 0. We can observe that, here the graph of the inverse function do not cross the vertical line $x = 0$ below we sketch the graph of f.

3.1.4 Study of the Square Root Function

The aim of this subsection is to study the function $f(x) = \sqrt{x}$. We denote its domain $[0\,,\,+\infty)$. The function f is defined at 0 and $f(0) = 0$.

Limit at $+\infty$

The limit of the function square root at $+\infty$ is $+\infty$. This means that the function goes to $+\infty$ when x approaches $+\infty$. This involves

$$\lim_{x\to+\infty} f(x) = +\infty.$$

From this we deduce that the function does not have a horizontal asymptote at $+\infty$.

On the other hand, we have

$$\lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \frac{1}{\sqrt{x}} = 0\cdot$$

Therefore f does not have an oblique asymptote at $+\infty$. In the same way we observe that the curve of f does not have a vertical asymptote.

Derivative of the Square Root Function

The function f is differentiable in $(0\,,\,+\infty)$. For all real number x, such that, $x > 0$ we have

$$f'(x) = \frac{1}{2\sqrt{x}}\cdot$$

As we can observe it the function f' is not defined at 0. Hence the domain of the function f' is $(0\,,\,+\infty)$. For every x in the domain of f' we have $f'(x) > 0$. This yields the function f is increasing $[0\,,\,+\infty)$.

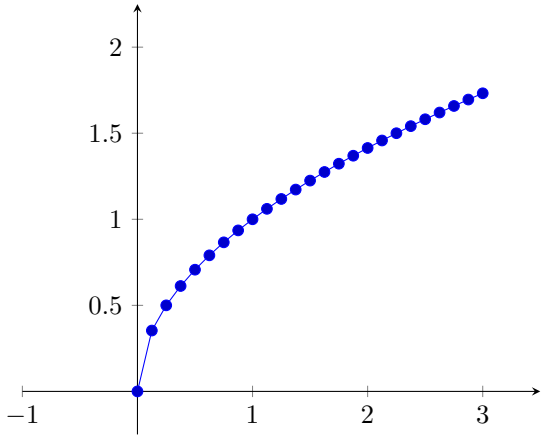
Variational Table of the Square Root Function

The previous step allows us to establish that the following variational table

x	0		$+\infty$
f'		$+$	
f	0	$\nearrow$	$+\infty$

Let $x \in [0\,;\,+\infty)$. Then one has $f(x) \geq f(0) = 0$. The function f has a global minimum at 0. The value of this minimum is $f(0) = 0$.

The graph of the square root function is given below.



In the next section we study second order polynomial functions. We remind that a second order polynomial functions is a function of the form $f(x) = ax^2 + bx + c$, where a, b and c are constant real numbers, such that $a \neq 0$.

3.2 Second Order Polynomial Functions

This section deals with variations of second order polynomial functions.In a second time, we study maximum and minimum of second order polynomial functions.

3.2.1 Variations of Second Order Polynomial Functions

In this subsection we study variations of polynomial functions of degree 2. Let $f(x) = ax^2 + bx + c$. Since f is a polynomial function, then it is differentiable in $\mathbb{R}$. For any real number x, $f'(x) = 2ax + b$. The derivative of f is an affine function.

On the other hand, the variations of the function f are determined by the sign of the function f'. Then, according to the sign of the real number a, we can establish the following theorems.

Theorem 3.2.1. *Let $f(x) = ax^2 + bx + c$, such that, $a > 0$. We define the real number $\alpha = \dfrac{-b}{2a}\cdot$ Then, the function f is decreasing in $(-\infty\,,\,\alpha]$ and increasing in $[\alpha\,,\,+\infty)$.*

Proof. We consider $f(x) = ax^2+bx+c$. Then, we have $f'(x) = 2ax+b$. The function f' is negative in $(-\infty\,,\,\alpha]$ and positive in $[\alpha\,,\,+\infty)$. Therefore, the function f is decreasing in $(-\infty\,,\,\alpha]$ and increasing in $[\alpha\,,\,+\infty)$. We obtain the following table:

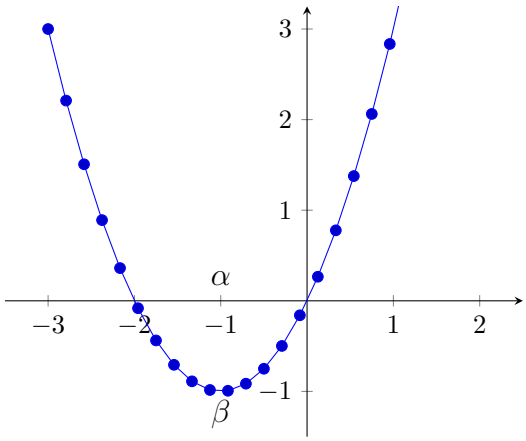
x	$-\infty$		α		$+\infty$
f'		$-$	0	$+$	
f	$+\infty$	$\searrow$	β	$\nearrow$	$+\infty$

□

As we can see it in the variational table above, the function f is decreasing in $(-\infty\,,\,\alpha]$ and increasing in $[\alpha\,,\,+\infty)$. Hence the function f has a minimum at $x_0 = \alpha$. The value of this minimum is $f(\alpha) = \beta$.

Lemma 3.2.2. *We consider the function $f(x) = ax^2 + bx + c$ such that $a > 0$. Then the function f has a minimum at α. The value of this minimum is*

$$f(\alpha) = \beta = -\frac{\Delta}{4a}\cdot$$



Example 3.2.3. *We consider the function $f(x) = 3x^2 + 2x + 1$. The function f is defined in $\mathbb{R}$. Now we define the real numbers α and β:*

$$\alpha = -\frac{b}{2a} = \frac{2}{6} = -\frac{1}{3} \quad \textit{and} \quad \beta = f(\alpha) = -\frac{\Delta}{4a} = -\frac{4-12}{12} = \frac{2}{3}\cdot$$

Thus the canonical form of f is $f(x) = 3\left(x + \dfrac{1}{3}\right)^2 + \dfrac{2}{3}\cdot$ Since $a = 3 > 0$, the function f is decreasing in $(-\infty\,,\,-1/3]$ and increasing in $[-1/3\,,\,+\infty)$. Then the function f has a minimum at $\alpha = -\dfrac{1}{3}$. The value of this minimum is equal to $f(\alpha) = \dfrac{2}{3}\cdot$

x	$-\infty$		$\frac{-1}{3}$		$+\infty$
f'		$-$	0	$+$	
f	$+\infty$	$\searrow$	$\frac{2}{3}$	$\nearrow$	$+\infty$

Lemma 3.2.4. *Let $f(x) = ax^2+bx+c$ be a second order polynomial function, such that, $a < 0$. We define the real number $\alpha = -\dfrac{b}{2a}$. Then the function f is increasing in $(-\infty\,,\,\alpha]$ and decreasing in $[\alpha\,,\,+\infty)$.*

Proof. Let $f(x) = ax^2 + bx + c$. The function f is differentiable in $\mathbb{R}$ and for every real number x, $f'(x) = 2ax + b$. Then, the function f' is positive in $(-\infty\,,\,\alpha]$ and negative in $[\alpha\,,\,+\infty)$. Thus, the function f is increasing in $(-\infty\,,\,\alpha]$ and decreasing in $[\alpha\,,\,+\infty)$.

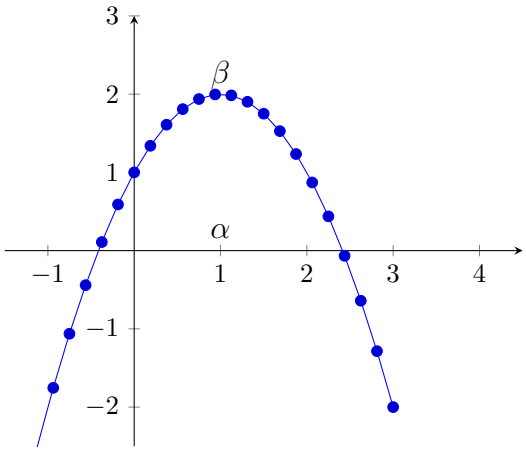
This yields the following variational table

x	$-\infty$		α		$+\infty$
f'		$+$	0	$-$	
f	$-\infty$	$\nearrow$	β	$\searrow$	$-\infty$

□

Since the function f is increasing in $(-\infty\,,\,\alpha]$ and decreasing in $[\alpha\,,\,+\infty)$, one deduces that, f has a maximum at α. This maximum is equal to $f(\alpha) = \beta$.

Lemma 3.2.5. *Let $f(x) = ax^2+bx+c$be a second order polynomial function, such that $a < 0$. Then, f has a maximum at α which is equal to $f(\alpha) = \beta$.*



Example 3.2.6. *Let $f(x) = -2x^2 + x + 1$. The domain of f is $\mathbb{R}$. The limits at the bounds of the domain of f are :*

$$\lim_{x\to-\infty} f(x) = -\infty \quad \textit{and} \quad \lim_{x\to+\infty} f(x) = -\infty.$$

We define the real numbers

$$\alpha = -\frac{b}{2a} = -\frac{1}{-4} \quad = \frac{1}{4} \quad \textit{and} \quad \beta = -\frac{1 - 4.(-2).1}{4.(-2)} = \frac{9}{8}.$$

Applying the previous theorem we deduce that the function

$$f(x) = -2\left(x - \frac{1}{4}\right)^2 + \frac{9}{8}.$$

is increasing in $(-\infty\,,\, 1/4]$*and decreasing in* $[1/4\,,\, +\infty)$.

This means that f *has a maximum at* $\alpha = \frac{1}{4}$ *which is equal to* $f(\alpha) = \frac{9}{8}$.

x	$-\infty$		$\frac{1}{4}$		$+\infty$
f'		$+$	0	$-$	
f	$-\infty$	$\nearrow$	$\frac{9}{8}$	$\searrow$	$-\infty$

Exercise 3.2.7. *Study the following functions*

1. $f(x) = 2x^2 + 3x + 1$, 2. $g(x) = -x^2 + 5x + 1$, 3. $h(x) = x^2 - 3x + 7$,

4. $i(x) = -3x^2 + 4x + 2$, 5. $j(x) = x^2 + 2x$.

Now we are interested in the study of general functions. In the next section we show how to deal with general functions

3.3 Study of General Functions

The main objective of this section, is not to study all general functions, but to give a kind of short user manual to deal with functions of the following form: $ax^3 + bx^2 + cx + d$, $u^n(x)$, $\sqrt{u(x)}$, $\dfrac{1}{u(x)}$, $\dfrac{1}{u^n(x)}$, $u(x) \cdot v(x)$ and $\dfrac{u(x)}{v(x)}$.

3.3.1 Study of Polynomial Function of Degree Three

This subsection is devoted to polynomial function of degree three. We recall a polynomial function of degree three is a function of the form

$$f(x) = ax^3 + bx^2 + bx + c,$$

with $a \neq 0$. The domain of the function f is $\mathbb{R}$. Thus, we remark the function f does not have vertical asymptote.

To find the limit of this function at the bounds of its domain, we distinguish two cases according to the sign of a.

Case 1: $a > 0$**.** When the real number a is strictly positive, we have

$$\lim_{x \to -\infty} f(x) = -\infty \quad \text{and} \quad \lim_{x \to +\infty} f(x) = +\infty.$$

An interpretation of these limits is, the graph of the function f does not have horizontal asymptotes. Calculating the limit of $f(x)/x$ we obtain

$$\lim_{x\to-\infty}\frac{f(x)}{x}=+\infty \quad \text{and} \quad \lim_{x\to+\infty}\frac{f(x)}{x}=+\infty.$$

Consequently the graph of the function f does not admit oblique asymptotes.

Case 2: $a<0$**.** In this case the following limits at the bounds of the domain hold

$$\lim_{x\to-\infty}f(x)=+\infty \quad \text{and} \quad \lim_{x\to+\infty}f(x)=-\infty.$$

This involves the graph of f does not have horizontal asymptotes. We observe that

$$\lim_{x\to\pm\infty}\frac{f(x)}{x}=-\infty.$$

This means that, we do not have oblique asymptotes.

The function f is differentiable in $\mathbb{R}$, because it a polynomial function. For every real number x the function f' is given by

$$f'(x)=3ax^2+2bx+c.$$

To determine the variations of the function f we need to study the sign of f'. Here the function f' is a second order polynomial function. To study the sign of a second order polynomial, we compute $\Delta=4b^2-4\times 3ac$. Since here we have $b'=2b$ we can calculate $\Delta'=b^2-3ac$ instead of Δ. Thus according to the sign of Δ and the sign of a one of the following cases holds

Case 1: $a>0$ **and** $\Delta>0$**.** As it was established in the previous chapters, when $\Delta>0$, the function $f'(x)=3ax^2+2bx+c$, has two real roots x_1 and x_2. Here we assume $x_1<x_2$. We know also that the sign of f' depends on the sign of a. Since $a>0$, then the following sign table holds

x	$-\infty$		x_1		x_2		$+\infty$
$f'(x)$		$+$	0	$-$	0	$+$	

Using the sign of f', we conclude that the function f is increasing in $(-\infty\,,\,x_1]\cup[x_2\,,\,+\infty)$ and decreasing in $[x_1\,,\,x_2]$. This leads to the following variational table.

x	$-\infty$		x_1		x_2		$+\infty$
f'		$+$	0	$-$	0	$+$	
f			$f(x_1)$				$+\infty$
		$\nearrow$		$\searrow$		$\nearrow$	
	$-\infty$				$f(x_2)$		

Example 3.3.1. *We consider the following function* $f(x)=x^3+2x^2+x+1$*. The domain of the function is* $\mathbb{R}$*. We have*

$$\lim_{x\to-\infty}f(x)=-\infty, \quad \lim_{x\to+\infty}f(x)=+\infty.$$

The polynomial function f is differentiable in $\mathbb{R}$ and for every real number x we have

$$f'(x) = 3x^2 + 4x + 1.$$

This implies $\Delta = 16 - 12 = 4$. Then, the function f' has two real roots

$$x_1 = \frac{-4-2}{6} = -1, \quad x_2 = \frac{-4+2}{6} = -\frac{1}{3}.$$

Applying the result above we conclude that the function f is increasing in $(-\infty\,,\,-1] \cup [-1/3\,,\,+\infty)$, and decreasing in $[-1\,,\,-1/3]$.

x	$-\infty$		-1		$-\frac{1}{3}$		$+\infty$
f'		$+$	0	$-$	0	$+$	
f			$f(-1)$				$+\infty$
		$\nearrow$		$\searrow$		$\nearrow$	
	$-\infty$				$f\left(-\frac{1}{3}\right)$		

Case 2: $a > 0$ **and** $\Delta = 0$**.** In this case the function f' has only one real root x_0 and f' is positive. We deduce that, f is increasing function $\mathbb{R}$. This allows to establish the following variational table

x	$-\infty$		x_0		$+\infty$
f'		$+$	0	$+$	
f					$+\infty$
			$\nearrow$		
	$-\infty$				

Example 3.3.2. *Let $f(x) = x^3 + 3x^2 + 3x + 7$. The function f is defined in the whole set of real number $\mathbb{R}$. The limits at the bounds of the domain are*

$$\lim_{x\to-\infty} f(x) = -\infty \quad \textit{and} \quad \lim_{x\to+\infty} f(x) = +\infty.$$

Since the function f is polynomial, it is differentiable in $\mathbb{R}$. For every real number x, $f'(x) = 3x^2 + 6x + 3$. In this example, $\Delta = 36 - 4 \times 9 = 36 - 36 = 0$. The function f' has only one real root $x_0 = -\frac{6}{6} = -1$. We conclude the function f is increasing in $\mathbb{R}$.

x	$-\infty$				$+\infty$
f'		$+$	0	$+$	
f					$+\infty$
			$\nearrow$		
	$-\infty$				

Case 3: $a > 0$ **and** $\Delta < 0$. In this case the function f' does not have real roots, and is strictly positive. Then, f is an increasing function.

x	$-\infty$				$+\infty$
f'		$+$		$+$	
f					$+\infty$
			$\nearrow$		
	$-\infty$				

Example 3.3.3. *We define the function* $f(x) = x^3 + x^2 + 3x + 5$. *This function is defined in* $\mathbb{R}$. *The limits at the bounds of the domainare*

$$\lim_{x\to-\infty} f(x) = -\infty \quad \textit{and} \quad \lim_{x\to+\infty} f(x) = +\infty.$$

Since f *is a polynomial function it is differentiable in* $\mathbb{R}$ *and for all* $x \in \mathbb{R}$

$$f'(x) = 3x^2 + 2x + 3.$$

We see that $\Delta = 4 - 4 \times 9 = -32$. *Therefore, the function* f' *does not have real roots and for every* $x \in \mathbb{R}$, $f'(x) > 0$. *Hence, the function* f *is increasing in* $\mathbb{R}$.

x	$-\infty$			$+\infty$
f'		$+$	$+$	
f	$-\infty$	$\nearrow$		$+\infty$

Case 4: $a < 0$ **and** $\Delta > 0$ **.** In this case the function f' has two real roots x_1 and x_2. We obtain the following sign table for f

x	$-\infty$		x_1		x_2		$+\infty$
f'		$-$	0	$+$	0	$-$	

This means that, the function f is decreasing in $(-\infty\,,\,x_1] \cup [x_2\,,\,+\infty)$ and increasing in $[x_1\,,\,x_2]$.

x	$-\infty$		x_1		x_2		$+\infty$
f'		$-$	0	$+$	0	$-$	
f	$+\infty$	$\searrow$	$f(x_1)$	$\nearrow$	$f(x_2)$	$\searrow$	$-\infty$

Example 3.3.4. *We consider the function* $f(x) = -x^3 + 2x^2 - x + 1$. *We know that* $D_f = \mathbb{R}$ *and*

$$\lim_{x\to-\infty} f(x) = +\infty, \quad \lim_{x\to+\infty} f(x) = -\infty.$$

The function f *is differentiable in* $\mathbb{R}$, *because it is a polynomial function. Let* $x \in \mathbb{R}$. *Then,* $f'(x) = -3x^2 + 4x - 1$. *From this we deduce* $\Delta = 16 - 12 = 4$. *Hence we have two real roots*

$$x_1 = \frac{-4+2}{-6} = \frac{1}{3}, \quad x_1 = \frac{-4-2}{-6} = 1\cdot$$

We can see the function f' *is negative in* $(-\infty\,,\,1/3] \cup [1\,,\,+\infty)$ *and* f' *is positive*

$[1/3\,,\,1]$. *The variations of* f *are given in the following table*

x	$-\infty$		$\frac{1}{3}$		1		$+\infty$
$f'(x)$		$-$	0	$+$	0	$-$	
f	$+\infty$	$\searrow$	$f\left(\frac{1}{3}\right)$	$\nearrow$	$f(1)$	$\searrow$	$-\infty$

Case 5: $a < 0$ **and** $\Delta = 0$**,** When $a < 0$ and $\Delta = 0$, the function f' has only one real root x_0. On the other hand we know the function f' and a have the same sign. That is, the function f' is negative in $\mathbb{R}$. Hence the function f is decreasing in $\mathbb{R}$.

x	$-\infty$		$+\infty$
$f(x)$	$+\infty$	$\searrow$	$-\infty$

Example 3.3.5. *Let* $f(x) = -x^3 + 3x^2 - 3x + 7$*. Then* $D_f = \mathbb{R}$ *and*

$$\lim_{x\to-\infty} f(x) = +\infty \quad \textit{and} \quad \lim_{x\to+\infty} f(x) = -\infty.$$

The function f *is differentiable in* $\mathbb{R}$*, and* $f'(x) = -3x^2 + 6x - 3$*, for any* $x \in \mathbb{R}$*. From this, we deduce that* $\Delta = 36 - 4 \times 9 = 36 - 36 = 0$*. The real root of* f' *is then given by* $x_0 = -\frac{6}{-6} = 1$ *. Since the function* f' *is negative in* $\mathbb{R}$*, one deduces the function* f *is decreasing* $\mathbb{R}$*.*

x	$-\infty$		0		$+\infty$
f'		$-$	0	$-$	
$f(x)$	$+\infty$		$\searrow$		$-\infty$

Case 6:$a < 0$ **and** $\Delta < 0$. When Δ is strictly negative, the function f' does not have real roots. Moreover, since $a < 0$, the function f' is strictly negative. This implies the function f is decreasing in $\mathbb{R}$.

x	$-\infty$			$+\infty$
f'		$-$	$-$	
f	$+\infty$		$\searrow$	$-\infty$

Example 3.3.6. *Let* $f(x) = -x^3 + x^2 - 3x + 5$*. Then* $D_f = \mathbb{R}$ *and we have*

$$\lim_{x\to-\infty} f(x) = +\infty \quad \textit{and} \quad \lim_{x\to+\infty} f(x) = -\infty.$$

We emphasize, there is no horizontal, no vertical and no oblique asymptotes. The function f *is differentiable in* $\mathbb{R}$*, because it is a polynomial function. The derivative*

of f is $f'(x) = -3x^2 + 2x - 3$. From this, we deduce $\Delta = 4 - 4 \times 9 = -32$. The function f' does not have real roots and for all $x \in \mathbb{R}$, $f'(x) < 0$. Therefore the function f decreasing in $\mathbb{R}$.

x	$-\infty$		$+\infty$
f'	$-$	$-$	
f	$+\infty$ $\searrow$		$-\infty$

Exercise 3.3.7. *Study the following functions*

1. $f(x) = x^3 + 3x^2 - 2x - 3$, 2. $g(x) = x^3 + 2x - 4$, 3. $h(x) = -2x^3 + 6x^2 + x + 1$,
4. $i(x) = -3x^3 + 2x^2 + x + 2$, 5. $j(x) = x^3 + x + 5$.

3.3.2 Study of the Function u^n

This subsection deals with function of the form $f(x) = u^n(x)$, where u is a real function. To show how to study such type of functions we proceed by examples.

Example 3.3.8. *Let $f(x) = \big(2x + 3\big)^2$. In this case $D_f = \mathbb{R}$ and the following limits hold*

$$\lim_{x \to -\infty} f(x) = +\infty \quad \textit{and} \quad \lim_{x \to +\infty} f(x) = +\infty.$$

The affine function $2x+3$ is differentiable in $\mathbb{R}$ and the function x^2 is also differentiable in $\mathbb{R}$. Since the function f is the composition of two diefferentiable functions, then f is differentiable. Here we set $u(x) = 2x+3$ and we observe that $f(x) = u^2(x)$. Therefore, for every real number x the derivative of f is defined by

$$f'(x) = 2u'(x) \cdot u(x) = 4(2x + 3).$$

We see that the sign of the function f' is given by the sign of the affine function $2x + 3$. Since $2x + 3$ is negative in the interval $(-\infty\,,\,-3/2]$ and positive in the interval $[-3/2\,,\,+\infty)$. We conclude that the function f is decreasing in $(-\infty\,,\,-3/2]$, and increasing in $[-3/2\,,\,+\infty)$.

x	$-\infty$		$-\dfrac{3}{2}$		$+\infty$
$f'(x)$		$-$	0	$+$	
$f(x)$	$+\infty$	$\searrow$	$f\left(-\dfrac{3}{2}\right)$	$\nearrow$	$+\infty$

Since the function f is decreasing in $(-\infty\,,\,-3/2]$, and increasing in $[-3/2\,,\,+\infty)$, it has a minimum at $x_0 = -3/2$. This minimum is given by$f(-3/2) = 0$.

Example 3.3.9. *We consider the function $f(x) = \big(-x^2 + 5x + 1\big)^3$. The domain of the function f is $\mathbb{R}$ and one has*

$$\lim_{x \to -\infty} f(x) = -\infty \quad \textit{and} \quad \lim_{x \to +\infty} f(x) = -\infty.$$

The function $f(x) = u^3(x)$, with $u(x) = -x^2 + 5x + 1$, is the composition of two differentiable functions in $\mathbb{R}$, then, f is differentiable in $\mathbb{R}$. For every $x \in \mathbb{R}$,

$$f'(x) = 3(-2x + 5)(-x^2 + 5x + 1)^2.$$

Since for all $x \in \mathbb{R}$, $3(-x^2 + 5x + 1)^2 \geq 0$, then the function f' and $-2x + 5$ have the same sign. Therefore we have the following table:

x	$-\infty$		$\frac{5}{2}$		$+\infty$
$f'(x)$		$+$	0	$-$	
$f(x)$	$-\infty$	$\nearrow$	$f\left(\frac{5}{2}\right)$	$\searrow$	$-\infty$

The function f is increasing in $(-\infty\,, 5/2]$, and decreasing in $[5/2\,, +\infty)$. Therefore it has a maximum at $x_0 = 5/2$.

Exercise 3.3.10. *Study the following function*

1. $f(x) = (x^3 + 3x + 1)^5$, 2. $g(x) = (3x + 6)^4$, 3. $h(x) = (2x^2 + 4x + 1)^3$,
4. $i(x) = (x + 1)^6$, 5. $j(x) = (2x + 4)^7$.

3.3.3 Study of the Function $\dfrac{1}{u}$

This section is devoted to functions of the form $f(x) = 1/u(x)$, where u is a real function. As in the previous subsection here we proceed again by examples.

Example 3.3.11. *Let $f(x) = \dfrac{1}{x+2}$. The function f is defined in $\mathbb{R}\backslash\{-2\}$. We obtain*

$$\lim_{x\to-\infty} f(x) = 0 \quad \textit{and} \quad \lim_{x\to+\infty} f(x) = 0.$$

Then, the line $y = 0$ is horizontal asymptote of $\mathscr{C}_f$ at $+\infty$ and at $-\infty$.

On the other hand we have,

$$\lim_{x\to-2^-} f(x) = -\infty \quad \textit{and} \quad \lim_{x\to-2^+} f(x) = +\infty.$$

This means, the line $x = -2$ is a vertical asymptote of $\mathscr{C}_f$. The function f is in the form $f(x) = 1/u(x)$, where $u(x) = x + 2$. For any $x \neq -2$, the denominator $u(x)$ is different to 0. This involves the function f is differentiable in $\mathbb{R}\backslash\{-2\}$ and for every $x \in \mathbb{R}\backslash\{-2\}$,

$$f'(x) = -\frac{u'(x)}{u^2(x)} = \frac{-1}{(x+2)^2}.$$

As we can see it the function f' is negative. Therefore the function f is a decreasing function in $\mathbb{R}\backslash\{-2\}$.

x	$-\infty$		-2		$+\infty$
f'		$-$	‖	$-$	
f	0	$\searrow$	‖	$\searrow$	0
		$-\infty$	‖	$+\infty$	

Exercise 3.3.12. *Study the following functions*

$$1.\ f(x)=\frac{1}{x^2+3x+1},\quad 2.\ g(x)=\frac{1}{x+5},\quad 3.\ h(x)=\frac{2}{x^2+6x+5},$$
$$4.\ i(x)=\frac{3}{x^3+5x^2+x+1},\quad 5.\ j(x)=\frac{1}{x+1}.$$

3.3.4 Study of the Function $\frac{1}{u^n}$

In this section, we study functions of the form $f(x)=1/u^n(x)$, where u is a real function and $n\geq 2$ a natural number. To explain the methodology we consider the following example.

Example 3.3.13. *We consider the function* $f(x)=\dfrac{1}{(x+1)^2}$. *The function* f *is defined for any* $x\neq -1$. *That is,* $D_f=\mathbb{R}\backslash\{-1\}$. *The limits at the bounds of the domain are defined by*

$$\lim_{x\to-\infty} f(x)=0 \quad \textit{and} \quad \lim_{x\to+\infty} f(x)=0.$$

We conclude the line $y=0$ *is a horizontal asymptote of* $\mathscr{C}_f$ *at* $+\infty$ *and* $-\infty$.

One knows that

$$\lim_{x\to-1^-} f(x)=+\infty \quad \textit{and} \quad \lim_{x\to-1^+} f(x)=+\infty.$$

This means that the line $x=-1$ *is a vertical asymptote of* $\mathscr{C}_f$. *The function* f *which is in the form* $1/u^2(x)$ *is differentiable at any* x *such that* $u(x)\neq 0$. *Then it is differentiable in* $\mathbb{R}\backslash\{-1\}$ *and for any* $x\neq -1$,

$$f'(x)=\frac{-u'(x)}{u^2(x)}=\frac{-2}{(x+1)^3}.$$

We deduce the following variational table

x	$-\infty$		-1		$+\infty$
$f'(x)$	$+$		‖	$-$	
$f(x)$	0	$\nearrow$	$+\infty$ ‖ $+\infty$	$\searrow$	0

Exercise 3.3.14. *Study the following functions*

$$1.\ f(x)=\frac{1}{(3x+2)^2},\quad 2.\ g(x)=\frac{1}{(x^2+2x+3)^4},\quad 3.\ h(x)=\frac{1}{(2x+6)^3},$$
$$4.\ i(x)=\frac{1}{(x^3+2x^2+5x+1)^5},\quad 5.\ j(x)=\frac{1}{(x+2)^8}.$$

3.3.5 Study of the Function $\sqrt{u}$

Let u be a real function. We consider the function $\sqrt{u}$. The aim of this subsection is to study the function $\sqrt{u}$. To study these type of functions we use the following example.

Example 3.3.15. *Let $u(x) = x + 3$ be a real function. We define the function $f(x) = \sqrt{x+3}$. The function f is defined if and only $x + 3 \geq 0$. The domain of f is $[-3\ +\infty)$. We have $f(-3) = 0$ and*

$$\lim_{x\to+\infty} f(x) = +\infty.$$

The function u is differentiable in $\mathbb{R}$ and the function $\sqrt{\cdot}$ is differentiable in $(0, +\infty)$ Since the function f is the composition of two differentiable functions, then it is differentiable. For every $x \geq 3$, we have

$$f'(x) = \frac{u'(x)}{2\sqrt{x+3}} = \frac{1}{2\sqrt{x+3}}.$$

We can observe that , f is not differentiable at -3. For all $x > -3$, $f'(x) > 0$. The function f is increasing in $[-3\,;\,+\infty)$. Then the following table holds

x	-3	$+\infty$
$f(x)$	0 ↗	$+\infty$

Exercise 3.3.16. *Study the following functions*

1. $f(x) = \sqrt{x^2 - 2x}$, 2. $g(x) = \sqrt{x^2 + 2x + 4}$, 3. $h(x) = \sqrt{x+2}$,
4. $i(x) = \sqrt{x^2 + 1}$, 5. $j(x) = \sqrt{x+1}$.

3.3.6 Study of the Function $u(x) \cdot v(x)$

As we pointed it out above, we will study functions by using examples. To study the function of the form $u \cdot v$ we consider the following function.

$$f(x) = (3x + 6)(x - 2).$$

The domain of f is $\mathbb{R}$. We have

$$\lim_{x\to-\infty} f(x) = +\infty \quad \text{and} \quad \lim_{x\to+\infty} f(x) = +\infty.$$

Since $u(x) = 3x + 6$ and $v(x) = x - 2$ are differentiables in $\mathbb{R}$, the function f is differentiable in $\mathbb{R}$ and for every $x \in \mathbb{R}$,

$$f'(x) = u'v + uv' = 3(x - 2) + (3x + 6) = 6x.$$

The function f' and x have the same sign. Hence the f' is positive in $[0\,;\,+\infty)$ and negative in $(-\infty\,;\,0]$. This leads to the following conclusions: the function f is decreasing in $(-\infty\,;\,0]$ and increasing in $[0\,;\,+\infty)$.

x	$-\infty$		0		$+\infty$
f'		$-$	0	$+$	
f	$+\infty$	↘	-12	↗	$+\infty$

Exercise 3.3.17. *Study the following functions*

1. $f(x) = (x^2 + 3x + 1)(2x + 1)$, 2. $g(x) = (x + 1)(3x + 9)$,
3. $h(x) = (x^2 + x + 1)(2x^2 + 1)$, 4. $i(x) = (\sqrt{x} + 1)\sqrt{x + 5}$,
5. $j(x) = (2x + 1)(3x + 2)$.

3.3.7 Study of the Function $\frac{u}{v}$

To study these type of functions, we consider the following example.

Example 3.3.18. *We define* $f(x) = \dfrac{3x+6}{-x+1}$. *The domain of the function* f *is* $\mathbb{R}\backslash\{1\}$. *We know*

$$\lim_{x\to-\infty} f(x) = -3.$$

The line $y = -3$ *is a horizontal asymptote of* $\mathscr{C}_f$ *at* $-\infty$. *We have*

$$\lim_{x\to+\infty} f(x) = -3.$$

This means the line $y = -3$ *is a horizontal asymptote of* $\mathscr{C}_f$ *at* $+\infty$.

On the other hand we have

$$\lim_{x\to1^-} f(x) = +\infty \quad \textit{and} \quad \lim_{x\to1^+} f(x) = -\infty.$$

This involves the line $x = 1$ *is a vertical asymptote of* $\mathscr{C}_f$.

The functions $u(x) = 3x + 6$ *and* $v(x) = -x + 1$ *are differentiable in* $\mathbb{R}$. *Then for any* $x \neq 1$ *the function* $u(x)/v(x)$ *is differentiable. Then, the function* f *is differentiable in* $\mathbb{R}\backslash\{1\}$ *and for all* $x \in \mathbb{R}\backslash\{1\}$

$$f'(x) = \frac{u'v - u\,v'}{v^2} = \frac{3(-x+1) + (3x+6)}{(-x+1)^2} = \frac{9}{(-x+1)^2}.$$

The function f' *is strictly positive. Then, the function* f *is increasing in* $\mathbb{R}\backslash\{-1\}$. *Therefore we have the following table*

x	$-\infty$		-1		$+\infty$
f'	$+$		‖	$+$	
f	-3	$\nearrow$	$+\infty$ ‖ $-\infty$	$\nearrow$	-3

Exercise 3.3.19. *Study the following functions*

1. $f(x) = \dfrac{x+1}{x^2+3x+1}$, 2. $g(x) = \dfrac{x^2+1}{x+3}$, 3. $h(x) = \dfrac{-x+5}{x+2}$,
4. $i(x) = \dfrac{x^2+4x+1}{x^2+2x+5}$, 5. $j(x) = \dfrac{\sqrt{x+1}}{x}$.

CHAPTER 4

EXPONENTIAL FUNCTION

Contents

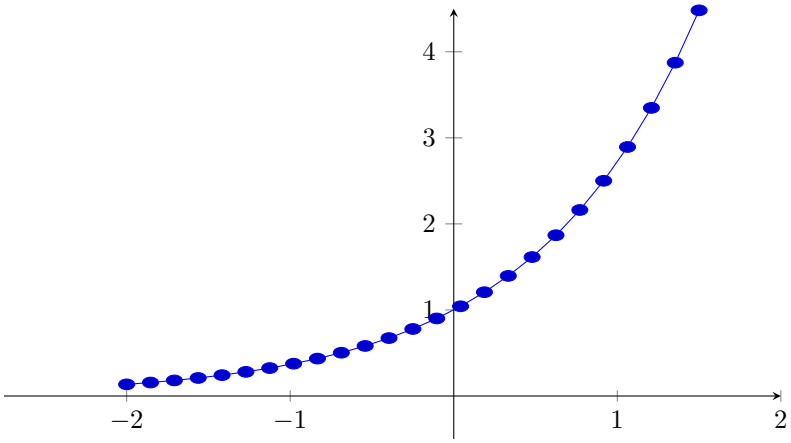
There exist several ways to introduce the exponential function. The method, which we use here to define this function is, to consider a functional equation. Indeed, we are looking for a continuous function $f : \mathbb{R} \to \mathbb{R}_+^*$, such that, for any x and y in $\mathbb{R}$

$$\begin{cases} f(x+y) = f(x) \cdot f(y) \\ f(0) = 1. \end{cases} \tag{4.0.1}$$

The solution of the functional equation (4.0.1) is called the exponential function. We denote this function

$$f(x) = \exp(x) = \mathrm{e}^x.$$

We point out that this function is one of the most important function in Mathematics. The graphic of the function exp is drawn below



We specify that when $x = 1$, we write e instead of e^1. We specify $\exp(0) = 1$.

The main objective of this chapter is to study the function exp and its properties. We organize this chapter as follows: In section 1 we define some properties of exp. In section 2 we solve equations using exp. Section 3 is devoted to the variations of exp. The last section deals with the following functions cosh, sinh and tanh.

4.1 Some Properties of Exponential

This section is devoted to the study of some properties of the function exp. We will establish that the function exp has some nice properties. Indeed let $x \in \mathbb{R}$, we have

$$\exp(x) \cdot \exp(-x) = \exp(x - x) = \exp(0) = 1.$$

Using (4.0.1), one deduces that, $\exp(x - x) = \exp(x) \cdot \exp(-x) = 1$. As we pointed it out above, $\exp(x) > 0$, for all $x \in \mathbb{R}$. Then, one can divide by $\exp(x)$ in the identity above. Therefore

$$\exp(-x) = \frac{1}{\exp(x)}\cdot \tag{4.1.1}$$

Lemma 4.1.1. *For any real number x, $\exp(-x)$ is defined by* (4.1.1).

From this lemma we deduce that for any real numbers a and b we have

$$\frac{\exp(a)}{\exp(b)} = \exp(a) \cdot \frac{1}{\exp(b)} = \exp(a - b). \tag{4.1.2}$$

Example 4.1.2. *Using* (4.1.1) (4.1.2) *we can establish the following identities*

$$1.\quad \frac{1}{\mathrm{e}^3} = \mathrm{e}^{-3}, \qquad 2.\quad \frac{1}{\mathrm{e}^{x^2}} = \mathrm{e}^{-x^2}, \qquad 3.\quad \frac{e^x}{e^3} = \mathrm{e}^{x-3}.$$

Example 4.1.3. *We consider the function $f(x) = \frac{\mathrm{e}^{2x}\cdot \mathrm{e}^{-2}}{\mathrm{e}^{x^2}}\cdot$ Using* (4.1.1) *and* (4.1.2) *we obtain*

$$f(x) = \mathrm{e}^{2x-2-x^2} = \mathrm{e}^{-x^2+2x-2}.$$

Exercise 4.1.4. *Simplify the expression of the following functions*

$$1.\ f(x)=\frac{\mathrm{e}^{3x}}{\mathrm{e}^{6x}},\qquad 2.\ g(x)=\mathrm{e}^{5}\cdot\mathrm{e}^{4x}\cdot\mathrm{e}^{x^2},\qquad 3.\ h(x)=\frac{e^{2x}\cdot\mathrm{e}^{6x}}{\mathrm{e}^{-7x}},$$
$$4.\ i(x)=\frac{\mathrm{e}^{2(x+1)^2}}{\mathrm{e}^{x^2+2x+1}},\qquad 5.\ j(x)=\frac{\mathrm{e}^{x^2+6x+1}}{\mathrm{e}^{x+1}\cdot\mathrm{e}^{5x+3}}.$$

Let $x\in\mathbb{R}$. We consider $n\in\mathbb{N}^*$ and we define $f(x)=\left(\mathrm{e}^x\right)^n$. From (4.0.1), we deduce that

$$f(x)=\underbrace{\mathrm{e}^x\cdot\cdots\cdot\mathrm{e}^x}_{n\text{ times}}=\mathrm{e}^{\overbrace{x+\cdots+x}^{n\ times}}=\mathrm{e}^{nx}.$$

This proves the following lemma

Lemma 4.1.5. *Let $x\in\mathbb{R}$ and $n\in\mathbb{N}^*$. Then, $(\exp(x))^n=\exp(n\,x)$.*

Remark 4.1.6. *Combining this lemma and* (4.1.1) *we establish that*

$$(\exp(x))^{-n}=\exp(-n\,x)=\frac{1}{\exp(n\,x)}$$

Example 4.1.7. *We consider the following function $f(x)=\left(\mathrm{e}^x\right)^6$. Then, we have*

$$f(x)=\mathrm{e}^{6x}.$$

Exercise 4.1.8. *Simplify the following expressions*

$$1.\ f(x)=\left(\mathrm{e}^x\right)^3\cdot\mathrm{e}^{-3x},\qquad 2.\ g(x)=\frac{\left(\mathrm{e}^3\right)^{-2}\cdot\left(\mathrm{e}^x\right)^5}{\left(\mathrm{e}^{2x}\right)^2},\qquad 3.\ h(x)=\frac{\left(\mathrm{e}^{-2x^2+1}\right)^2}{\left(\mathrm{e}^{-x+5}\right)^3},$$
$$4.\ i(x)=\frac{\left(\mathrm{e}^2\right)^{n^2}}{\mathrm{e}^{2\,n^2}},\qquad 5.\ j(x)=\frac{1}{\left(\mathrm{e}^{-x}\right)^n}.$$

Let a be a fixed real number. We are looking for $x\in\mathbb{R}$, such that, $\exp(x)=a$. In the next section we will study these types of equations.

4.2 Solutions of Some Equations

In this section we are interested in equations of the form: find $x\in$ such that $\exp(x)=a$ where a is a fixed real number. We can point out that, when $a\leq 0$, the equation does not have solutions. Because for any real number x, $\exp(x)>0$. In section (4.3.2) we will prove for every strictly positive real number b there exists a unique $y\in\mathbb{R}$, such that $\exp(y)=b$. This allows us to restrict our selves in the study of equation in the form $\exp(x)=\exp(a)$.

Let x and a be two real numbers, such that, $\exp(x)=\exp(a)$. Since for any real number a, $\exp(a)>0$, one can divide the equation above by $\exp(a)$. This holds

$$\frac{\exp(x)}{\exp(a)}=\exp\left(x-a\right)=\exp(0).$$

Therefore, $x-a=0$. This leads to the identity, $x=a$. We can summarize all these steps in the following lemma.

Lemma 4.2.1. *Let a be a fixed real numbers. Then, the unique solution of the equation $\exp(x) = \exp(a)$ is $x = a$.*

Example 4.2.2. *Find the solution of the following equations*

1. $\exp(x) = \exp(3)$, 2. $\exp(x+1) = \exp(5)$, 3. $\exp(3x+2) = \exp(x+6)$.

Solution:

1. Applying the lemma above we find that, the solution of the equation

$$\exp(x) = \exp(3) \quad is \quad x = 3.$$

2. The equation $\exp(x+1) = \exp(5)$ is equivalent to $x + 1 = 5$. This implies the solution is $x = 4$.

3. The equation $\exp(3x+2) = \exp(x+6)$ is equivalent to the following equation $3x + 2 = x + 6$. Then, we have $2x = 4$. This means that $x = 2$.

Exercise 4.2.3. *Find the solution of the following equations*

1. $\exp(x^2 + 5x + 1) = \exp(2x + 1)$, 2. $\exp(3x + 5) = \exp(x^2 + 4x + 4)$,
3. $\dfrac{1}{\exp(x^2 + 5x + 3)} = \exp(2)$, 4. $\dfrac{\exp(x^3 + 5x^2 - 3)}{\exp(x^3 + 2x^2)} = 1$,
5. $\exp(x + 1) = \exp(-x + 2), \cdot$

To simplify notations we will sometimes write e^x instead of $\exp(x)$. As we did it previously for other functions, here also we study continuity and variations of the function exponential.

4.3 Study of the Exponential Function

The aim of this section is to study the function exp. This section is organized as follows. In the first subsection we deal with limits at the bounds of the domain. Since the domain of exp is $\mathbb{R}$, we will compute the limits at $+\infty$ and $-\infty$. The second subsection is devoted to variations of the function exp. Using variations of the function we will be able to find solutions of inequations.

4.3.1 Limits at the Bounds of the Domain

In this subsection we define the limit of exp at $+\infty$ and at $-\infty$. In a second time, we compare exp to polynomial functions, when x goes to $\pm\infty$. A glance on its graph allows us to set the following definition.

Definition 4.3.1. *Let $f(x) = \exp(x)$. We have*

$$\lim_{x\to+\infty} e^x = +\infty \qquad and \qquad \lim_{x\to-\infty} e^x = 0. \tag{4.3.1}$$

We can remark from this definition that the line $y = 0$ is a horizontal asymptote of $\mathscr{C}_f$ at $-\infty$.

Since we defined $\lim_{x\to+\infty} f(x) = +\infty$, we want to know how fast the function exp goes to infinity. To this end we compare the function exp to polynomial functions at $+\infty$. Let $n \geq 1$ be a natural number. We fix $x \in (0\,;+\infty)$ and we define

the functions $f(x) = \exp(x)$ and $g(x) = x^n$. We want to know, between x^n and exp which one of them increases more fast than the other one. Here we will prove that the function e^x increases more fast than any power of x when $x > 0$. This is the content of the following lemma.

Lemma 4.3.2. *Let $n \geq 1$ be a natural number. We define $n! = 1 \times 2 \times \cdots \times n$. Then, for every x in $(0\,,\,+\infty)$*

$$\frac{x^n}{n!} \leq \mathrm{e}^x. \tag{4.3.2}$$

We agree that $0! = 1$.

Proof. We will prove (4.3.2) by induction. Let $x \in \mathbb{R}_+$, and $n \geq 0$ be a natural number.

Step 1: Since for $n = 0$, $x^0 = 1$ and $0! = 1$, we have $\dfrac{x^0}{0!} = 1 \leq \mathrm{e}^x$. Then, the property is true for $n = 0$.

Step 2: We assume that the property is true for some natural number k. This means that $\frac{x^k}{k!} \leq e^x$.

Step 3: Now we need to prove that the property holds for $k + 1$. In other words, we have to prove that

$$\frac{x^{k+1}}{(k+1)!} \leq \mathrm{e}^x.$$

To establish this inequality, we define the function $g(x) = \frac{x^{k+1}}{(k+1)!} - \mathrm{e}^x$. Here we admit the derivative of e^x is e^x. The function g is differentiable in $\mathbb{R}$ and for all $x \in \mathbb{R}$,

$$g'(x) = \frac{(k+1)x^k}{(k+1)!} - \mathrm{e}^x = \frac{x^k}{k!} - \mathrm{e}^x.$$

Since we assumed that $\dfrac{x^k}{k!} \leq \mathrm{e}^x$, then $g'(x) \leq 0$. Hence g is decreasing in $[0\,,\,+\infty)$. We know that the maximum of g in $[0\,,\,+\infty)$ is $g(0) = -1$. Therefore g is negative. Hence $\frac{x^{k+1}}{(k+1)!} \leq \mathrm{e}^x$.

We conclude that for all $x \geq 0$, $\dfrac{x^n}{n!} \leq \mathrm{e}^x$. □

Considering $n = 1$ in the lemma we obtain $x \leq \mathrm{e}^x$, for every $x > 0$. From this we get

$$\lim_{x\to+\infty} x \leq \lim_{x\to+\infty} \mathrm{e}^x.$$

This is a confirmation of definition (4.3.1).

Another consequence of the lemma above is the following result. Take $x > 0$. Let $n \geq 1$ be a natural number. From the lemma above, we know that $\frac{x^{n+1}}{(n+1)!} \leq \mathrm{e}^x$. As we have chosen $x > 0$, then $x^n \neq 0$. Hence we can divide by x^nin the inequality above. We get

$$\frac{x}{(n+1)!} \leq \frac{\mathrm{e}^x}{x^n}.$$

Since

$$\lim_{x\to+\infty} \frac{x}{(n+1)!} = +\infty,$$

we conclude that

$$\lim_{x\to+\infty}\frac{\mathrm{e}^x}{x^n}=+\infty.$$

This involves the following theorem.

Theorem 4.3.3. *Let $n\geq 1$ be a natural.Then,*

$$\lim_{x\to+\infty}\frac{\mathrm{e}^x}{x^n}=+\infty. \tag{4.3.3}$$

As we can see it, this theorem needs some explanations. Because if we split computations by considering $\lim_{x\to+\infty}\mathrm{e}^x=+\infty$ and $\lim_{x\to+\infty}x^n=+\infty$. Then, we obtain

$$\lim_{x\to+\infty}f(x)=\frac{+\infty}{+\infty}=\text{undefined !}$$

But what this theorem illustrates is, the fact that, the function e^x goes to $+\infty$ more fast than the function x^n. That is why, we have

$$\lim_{x\to+\infty}\frac{\mathrm{e}^x}{x^n}=+\infty\cdot$$

Example 4.3.4. *Let $f(x)=\dfrac{\mathrm{e}^x}{x^3}$. Then, we have $\lim_{x\to+\infty}f(x)=+\infty$.*

Example 4.3.5. *We consider the following function $g(x)=\dfrac{\mathrm{e}^x}{x+1}\cdot$ We set*

$$X:=x+1,$$

we can see that $\lim_{x\to+\infty}x+1=\lim_{x\to+\infty}X=+\infty$. Then,

$$f(x)=\frac{1}{\mathrm{e}}\cdot\frac{\mathrm{e}^X}{X}\cdot$$

Therefore,

$$\lim_{x\to+\infty}f(x)=\lim_{X\to+\infty}\frac{1}{\mathrm{e}}\cdot\frac{\mathrm{e}^X}{X}=+\infty\cdot$$

Exercise 4.3.6. *Determine the following limits*

$$1.\ \lim_{x\to+\infty}\frac{x^n+1}{\mathrm{e}^x},\quad 2.\ \lim_{x\to+\infty}\frac{\mathrm{e}^x}{x-1},\quad 3.\ \lim_{x\to+\infty}\frac{\mathrm{e}^{x+5}}{x^2}\cdot$$

Let $x>0$ be a real number. For any natural number $n\geq 0$ we define

$$f(x)=x^n\mathrm{e}^x.$$

We want to determine the limit of f at $-\infty$. We emphasize that, if we consider $u(x)=\mathrm{e}^x$ and $v(x)=x^n$ we will have

$$\lim_{x\to-\infty}u(x)=\lim_{x\to-\infty}\mathrm{e}^x=0\quad\text{and}\quad\lim_{x\to-\infty}v(x)=\lim_{x\to-\infty}x^n=\pm\infty.$$

Then,

$$\lim_{x\to-\infty}f(x)=\text{undefined !.}$$

However we know that the function exp goes to 0 more fast than x^n goes to $\pm\infty$.Hence, this limit exists and it is given in the following theorem

Theorem 4.3.7. *Let $n \geq 1$ be a natural number. Then, for every $x \in \mathbb{R}$*

$$\lim_{x\to-\infty} x^n \mathrm{e}^x = 0.$$

Proof. When $x = 0$, there is nothing to prove, because $x^n\,\mathrm{e}^x = 0$. When $n = 0$ we apply the definition (4.3.1).

Now, we consider $x \in \mathbb{R}^*$, and a natural number $n \geq 1$. We define $f(x) = x^n\mathrm{e}^x$ and we set $t := -x$. Thus, we have $\lim_{x\to-\infty} t = +\infty$. On the other part, we observe that

$$f(x) = x^n\mathrm{e}^x = (-1)^n\, t^n\mathrm{e}^{-t} = \frac{(-1)^n t^n}{\mathrm{e}^t} = \frac{(-1)^n}{\dfrac{\mathrm{e}^t}{t^n}}\cdot$$

Since

$$\lim_{t\to+\infty} \frac{\mathrm{e}^t}{t^n} = +\infty, \quad \text{then,} \quad \lim_{x\to-\infty} f(x) = \lim_{t\to+\infty} \frac{(-1)^n}{\dfrac{\mathrm{e}^t}{t^n}} = 0\cdot$$

□

Example 4.3.8. *We consider the function $f(x) = x\mathrm{e}^x$. Then, we have*

$$\lim_{x\to-\infty} f(x) = 0$$

Exercise 4.3.9. *Determine the following limits*

$$1.\ \lim_{x\to-\infty} x^{10}\mathrm{e}^x, \quad 2.\ \lim_{x\to-\infty} x^2\mathrm{e}^x, \quad 3.\ \lim_{x\to-\infty} x^5\mathrm{e}^{-x}, \quad 4.\ \lim_{x\to+\infty} x^9\mathrm{e}^{-x^2}.$$

4.3.2 Variation of Exponential Function

In this subsection we study the variations of exp. We point out that, another way to introduce the exponential function is to use the relation between function exp and its derivative. The exponential function satisfies the following equation: Find a $\mathscr{C}^1$ function $f : \mathbb{R} \to (0\,, +\infty)$ that satisfies,

$$f'(x) = f(x) \quad \text{and} \quad f(0) = 1, \tag{4.3.4}$$

for every real number x. We admit the following theorem.

Theorem 4.3.10. *Let $f(x) = \mathrm{e}^x$. The function f is differentiable in $\mathbb{R}$ and for every $x \in \mathbb{R}$, we have*

$$f'(x) = \mathrm{e}^x.$$

From this definition, we can observe that $f'(x) = \mathrm{e}^x > 0$. Then the function f is increasing in $\mathbb{R}$.

Using the limits established in the previous subsection, we obtain the following variational table

x	$-\infty$		$+\infty$
			$+\infty$
f		$\nearrow$	
	0		

From this table we are able to state the following theorem.

Theorem 4.3.11. *Let $f(x) = \exp(x)$. Then the function f is increasing in $\mathbb{R}$.*

Now, we consider a real function u. Since the domain of exp is $\mathbb{R}$, we can always define $\exp(u(x))$, for all $x \in D_u$. We define $g(x) = \mathrm{e}^x$ and $f(x) = \mathrm{e}^{u(x)}$. We can observe that, $f(x) = g(u(x))$. We assume the function u is differentiable in its domain. Using the *chain rule property*, (or the derivative of composition of functions), we obtain

$$f'(x) = u'(x) \times g'(u(x)) = u'(x)\mathrm{e}^{u(x)}.$$

Theorem 4.3.12. *Let u be a differentiable real function in an open interval $I \subset D_u$. We define the function $f(x) = \mathrm{e}^{u(x)}$. Then the function f is differentiable in I, and for every $x \in I$,*

$$f'(x) = u'(x)\mathrm{e}^{u(x)}.$$

Consider $f(x) = \mathrm{e}^{u(x)}$, we know from the theorem above that $f'(x) = u'(x)\mathrm{e}^{u(x)}$. Since for all x in the domain of u , $\mathrm{e}^{u(x)} > 0$, therefore, f' and u' have the same sign. This means that the function u and $\exp(u)$ have the same variations.

Example 4.3.13. *We consider the function $f(x) = \mathrm{e}^{x+1}$. This function is defined in $\mathbb{R}$. Now we set $u(x) := x+1$. Since u and e^x are differentiable in $\mathbb{R}$, the function $f(x) = \mathrm{e}^{x+1}$ is diffenrentiable in $\mathbb{R}$. Let x be a real number. We have $u'(x) = 1$. Applying the theorem above we obtain*

$$f'(x) = u'(x) \cdot \mathrm{e}^{u(x)} = 1 \cdot \mathrm{e}^{x+1} = \mathrm{e}^{x+1} > 0.$$

We conclude that the function f is increasing in $\mathbb{R}$, because for every $x \in \mathbb{R}$, $\mathrm{e}^{x+1} > 0$.

On the other hand we have

$$\lim_{x\to-\infty} \mathrm{e}^{x+1} = 0 \quad \textit{and} \quad \lim_{x\to+\infty} f(x) = +\infty.$$

Indeed setting $X := x + 1$, we see that, $\lim_{x\to-\infty} X = -\infty$. Hence

$$\lim_{x\to-\infty} \mathrm{e}^{x+1} = \lim_{X\to-\infty} \mathrm{e}^{X} = 0.$$

This holds the following variational table

x	$-\infty$		$+\infty$
$f(x)$	0	$\nearrow$	$+\infty$

Example 4.3.14. *Let $g(x) = \mathrm{e}^{-2x+3}$. The function g is defined in $\mathbb{R}$. Now, we define $u(x) = -2x+3$. The functions u and e^x are differentiable in $\mathbb{R}$. Since e^{-2x+3} is the composition of the function* exp *and u, then e^{-2x+3} is differentiable in $\mathbb{R}$. The derivative of the function u is $u'(x) = -2$ for every $x \in \mathbb{R}$. Therefore*

$$g'(x) = -2\mathrm{e}^{-2x+3}.$$

As we know $\mathrm{e}^{-2x+3} > 0$, for every real number x, then $g'(x) = -2\mathrm{e}^{-2x+3} < 0$. The function f is decreasing in $\mathbb{R}$.

Now we introduce the following variable $X := -2x + 3$. *One can see that*

$$\lim_{x\to-\infty} X = +\infty \quad \textit{and} \quad \lim_{x\to+\infty} X = -\infty.$$

Therefore, the following limits hold

$$\lim_{x\to-\infty} \mathrm{e}^{-2x+3} = \lim_{X\to+\infty} \mathrm{e}^{X} = +\infty \quad \textit{and} \quad \lim_{x\to+\infty} \mathrm{e}^{-2x+3} = \lim_{X\to-\infty} \mathrm{e}^{X} = 0$$

Consequently we are able to establish this variational table.

x	$-\infty$		$+\infty$
g	$+\infty$	$\searrow$	0

Example 4.3.15. *In this example we take the following function* $h(x) = \mathrm{e}^{x^2+x+1}$. *This function is defined in* $\mathbb{R}$. *We define the function* $u(x) = x^2 + x + 1$ *for any real number* x. *The function* $\mathrm{e}^{u(x)}$ *is differentiable in* $\mathbb{R}$, *because it is the composition of two differentiable functions* u *and* e^x *in* $\mathbb{R}$. *Let* $x \in \mathbb{R}$, *the derivative of the function* u *is* $u'(x) = 2x + 1$. *This implies*

$$h'(x) = (2x + 1)\mathrm{e}^{x^2+x+1}.$$

Since $\exp\left(x^2+x+1\right) > 0$, *then the sign of* h' *depends on the sign of* $u'(x) = 2x+1$. *Hence we obtain the following variational table*

x	$-\infty$		$-1/2$		$+\infty$
h'		$-$	0	$+$	
h	$+\infty$	$\searrow$	$h\left(-1/2\right)$	$\nearrow$	$+\infty$

The function h *is decreasing in* $(-\infty\,;\,-1/2]$ *and increasing in* $[-1/2\,,\,+\infty)$.

Exercise 4.3.16. *Study the variation of the following functions*

$$1.\ f(x) = \mathrm{e}^{x^3+2x+1}, \quad 2.\ g(x) = \mathrm{e}^{3x+2}, \quad 3.\ h(x) = \mathrm{e}^{-x+6x+1}.$$

In the section above we proved the function exp is a strictly increasing function in $\mathbb{R}$. This means that for two real numbers a and b, such that $a \leq b$, we have $\exp(a) \leq \exp(b)$. In the following subsection we want establish the reverse implication. Let a and b be two real numbers such that $\exp(a) \leq \exp(b)$. Does it mean that $a \leq b$? To answer this question we introduce the following subsection.

4.3.3 Solutions of Some Inequations

Let a be a strictly positive real number. The function exp is continuous and strictly increasing from 0 to $+\infty$. Then, the IVT theorem implies there exist a unique $x_0 \in \mathbb{R}$ such that

$$\exp\left(x_0\right) = a.$$

In this subsection, we deal with inequations. This means that if we take a fixed real number b, we look for all real number x, such that

$$\exp(x) \leq b.$$

According to b two situations hold:

Situation 1: $b \leq 0$**.** When the real number b is negative or equal to 0, the inequation $\exp(x) \leq b$ does not have solutions. Indeed we know for every real number x, $\exp(x) > 0$. Then, there are no real number satisfying $\exp(x) \leq 0$. In this case we say the set of solutions is empty and we denote $S = \emptyset$

Situation 2: $b > 0$. In this case there is $x_0 \in \mathbb{R}$ such that

$$\exp(x) \leq \exp(x_0).$$

From this we remark that we can restrict our selves in inequations of the form find $x \in \mathbb{R}$ such that

$$\exp(x) \leq \exp(x_0) \quad \text{or} \quad \exp(x) \geq \exp(x_0).$$

Since the function exp is continuous and increasing there exist an increasing function denoted $f^{-1} : (0\,, +\infty) \to \mathbb{R}$, such that

$$\forall\, x \in \mathbb{R} \quad f^{-1}\left(\exp(x)\right) = x \quad \text{and} \quad \forall y \in (0\,, +\infty), \quad \exp\left(f^{-1}(y)\right) = y. \tag{4.3.5}$$

In general we can prove the following theorem

Theorem 4.3.17. *Let I and J be two intervals in $\mathbb{R}$ and $f : I \to J$. We assume that the function f continuous and strictly monotone. Then there exist a function $g : J \to I$, such that*

$$\forall\, x \in I \quad g\left(f(x)\right) = x \quad \textit{and} \quad \forall y \in J, \quad f\left(g(y)\right) = y.$$

We point out that the function g is continuous and strictly monotone. Moreover if the function f is increasing, then the function g is an increasing function. If the function f is decreasing, then the function g is also decreasing. The graph of g denoted $\mathscr{C}_g$ is symmetric to the graph of f denoted $\mathscr{C}_f$ with respect to the line $y = x$.

From (4.3.5) and theorem 4.3.17 we deduce there exist a strictly increasing function f^{-1} defined from $(0\,, +\infty)$ to $\mathbb{R}$, such that if $\exp(a) \leq \exp(b)$, we have

$$a = f^{-1}\left(\exp(a)\right) \leq b = f^{-1}\left(\exp(b)\right).$$

Using this function we can state the following lemma

Lemma 4.3.18. *Let a and b be two real numbers. Then, we have*

1. $\exp(a) \leq \exp(b)$ *is equivalent to* $a \leq b$
2. $\exp(a) \geq \exp(b)$ *is equivalent to* $a \geq b$

Let a be a fixed real number. We consider the inequation: find $x \in \mathbb{R}$, such that, $\exp(x) \geq \exp(a)$.

Applying the lemma above we obtain $x \geq a$. The real numbers which satisfy the inequation $\exp(x) \geq \exp(a)$ are the real numbers that belong to $[a\,, +\infty)$. In this case we write

$$S = [a\,, +\infty).$$

On the other hand if we consider the inequation $\exp(x) \leq \exp(a)$, we obtain

$$S = (-\infty\,, a].$$

Lemma 4.3.19. *Let a be a fixed real number. Then,*

1. *the solutions of the inequation* $\exp(x) \leq \exp(a)$ *are given by* $S = (-\infty\,,\, a]$
2. *the solutions of the inequation* $\exp(x) \geq \exp(a)$ *are given by* $S = [a\,,\, +\infty)$.

Example 4.3.20. *To find the solutions of the inequation* $\mathrm{e}^x \geq \mathrm{e}^3$ *we point out that this inequation is equivalent to* $x \geq 3$. *Hence* $S = [3\,;\, +\infty)$.

Example 4.3.21. *To define the solutions of the inequation* $\mathrm{e}^x \leq \mathrm{e}^3$, *we remark that It is equivalent to the inequality* $x \leq 3$. *Therefore* $S = (-\infty\,;\, 3]$.

Example 4.3.22. *Find the solutions of this inequation* $\mathrm{e}^{3x+1} > \mathrm{e}^{-2}$.
Since exp *is a continuous and increasing, this inequality is equivalent to*

$$3x + 1 > -2.$$

That is, $3x > 3$. *This implies that* $x > -1$. *Hence* $S = (-1\,;\, +\infty)$.

Example 4.3.23. *Find the solutions of* $\mathrm{e}^{x+1} < \mathrm{e}^2$.
The solutions are given by this inequality

$$x + 1 < 2.$$

This leads to $x < 1$. *Then* $S = (-\infty\,;\, 1)$.

Exercise 4.3.24. *Find the solutions of the following inequations*

1. $\mathrm{e}^{3x+1} > \mathrm{e}^{x^2+4x+1}$, 2. $\mathrm{e}^{x^2+2x+1} \leq 1$, 3. $\mathrm{e}^{-3x+1} \geq \mathrm{e}^{2x+4}$,
4. $\mathrm{e}^{x} \leq \mathrm{e}^{x^2+5x+6}$, 5. $\mathrm{e}^{x^2+6x+7} < \mathrm{e}^{2x^2+3x+5}$.

Using the function exp as an elementary function, we are able to construct new functions. In the next section we define *hyperbolic cosinus and sinus*

4.4 Hyperbolic Cosinus and Sinus

In this section, we will construct functions from exp which is considered here as an elementary function. We will also study variation of these new functions.To end the section we will give some properties of these functions by making an analogy with trigonometric functions.

Definition 4.4.1. *Let* $x \in \mathbb{R}$, *we define the hyperbolic cosinus of* x *denoted* $\cosh(x)$ *or* $\mathrm{ch}(x)$ *as follows:*

$$\cosh(x) = \mathrm{ch}(x) = \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2}\cdot$$

Definition 4.4.2. *Let* x *be a real number. We define the hyperbolic sinus of* x, *denoted* $\sinh(x)$ *or* $\mathrm{sh}(x)$ *by*

$$\sinh(x) = \mathrm{sh}(x) = \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2}\cdot$$

The definitions of these two functions show clearly that their domain is $\mathbb{R}$.

4.4.1 Study of Hyperbolic cosinus

Let us consider the following function $f(x) = (\exp(x) + \exp(-x))/2$. The function f is the sum of functions which are defined in $\mathbb{R}$. Then the domain of the function f is $\mathbb{R}$. Now we consider an arbitrary real number x, we have

$$f(-x) = \frac{\mathrm{e}^{-x} + \mathrm{e}^{-(-x)}}{2} = \frac{\mathrm{e}^{-x} + \mathrm{e}^{x}}{2} = \frac{\mathrm{e}^{x} + \mathrm{e}^{-x}}{2} = f(x).$$

We have proved that $\mathrm{ch}(x) = \mathrm{ch}(-x)$. In other words ch is an even function.

Lemma 4.4.3. *The function* ch *is an even function.*

A possible graphical interpretation of this lemma, is the following one: The graph of f denoted $\mathscr{C}_f$ is symmetric with respect to the $y-axis$.

Theorem 4.4.4. *Let* $f(x) = \cosh(x)$. *Then,*

$$\lim_{x\to-\infty} f(x) = +\infty \quad \textit{and} \quad \lim_{x\to+\infty} f(x) = +\infty.$$

Proof. To prove this theorem, we set $X := -x$. One knows, $\lim_{x\to-\infty} X = +\infty$. Thus we obtain,

$$\lim_{x\to-\infty} \frac{\mathrm{e}^{-x}}{2} = \lim_{X\to+\infty} \frac{\mathrm{e}^{X}}{2} = +\infty.$$

Since $\lim_{x\to-\infty} \dfrac{\mathrm{e}^{x}}{2} = 0$, then,

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{\mathrm{e}^{x}}{2} + \lim_{x\to-\infty} \frac{\mathrm{e}^{-x}}{2} = +\infty.$$

For the limit at $+\infty$, we need just to emphasize that, $\lim_{x\to+\infty} X = -\infty$. From this, we deduce that

$$\lim_{x\to+\infty} \frac{\mathrm{e}^{-x}}{2} = \lim_{X\to-\infty} \frac{\mathrm{e}^{X}}{2} = 0.$$

As we know, $\lim_{x\to+\infty} \dfrac{\mathrm{e}^{x}}{2} = +\infty$, therefore,

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{\mathrm{e}^{x}}{2} + \lim_{x\to-\infty} \frac{\mathrm{e}^{-x}}{2} = +\infty.$$

This completes the proof. □

Theorem 4.4.5. *Let* $f(x) = \cosh(x)$. *Then,*

1. *f is decreasing in $(-\infty\,,\,0]$ and increasing in $[0\,,\,+\infty)$.*
2. *The function f has a minimum at 0, which is equal to 1.*
3. *The equation of tangent line of $\mathscr{C}_f$ at 0 is given by*

$$(T_0) : y = 1.$$

What says the property 3 of the theorem above is the function f has a horizontal tangent line at $x_0 = 0$.

Proof. Let $f(x) = \mathrm{ch}(x)$.
proof of 1: To prove 1, we show at first that f is defferentiable in $\mathbb{R}$. Indeed the functions e^x and e^{-x} are differentiable in $\mathbb{R}$. This implies that $\mathrm{ch}(x)$ is differentiable in $\mathbb{R}$. For every real number x

$$f'(x) = \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2} = \mathrm{sh}(x)\cdot$$

To determine the variations of f we need to study the sign of f'. We know that, $f(x) \geq 0$ if and only if $\mathrm{e}^x - \mathrm{e}^{-x} \geq 0$. This is equivalent to $\mathrm{e}^x \geq \mathrm{e}^{-x}$. This leads to the following inequality $x \geq -x$. This means that $2x \geq 0$. That is, f' is negative in $(-\infty\,,\,0]$, and positive in $[0\,,\,+\infty)$. In other words, f is decreasing in $(-\infty\,,\,0]$, and increasing in $[0\,,\,+\infty)$.

x	$-\infty$		0		$+\infty$
$f'(x)$		$-$	0	$+$	
f	$+\infty$	$\searrow$	1	$\nearrow$	$+\infty$

proof of 2: To prove 2, we point out that, f is decreasing in $(-\infty\,,\,0]$ and increasing in $[0\,,\,+\infty)$. Then, the function f reaches its minimum at 0. The value of this minimum is

$$f(0) = \frac{\mathrm{e}^0 + \mathrm{e}^{-0}}{2} = 1.$$

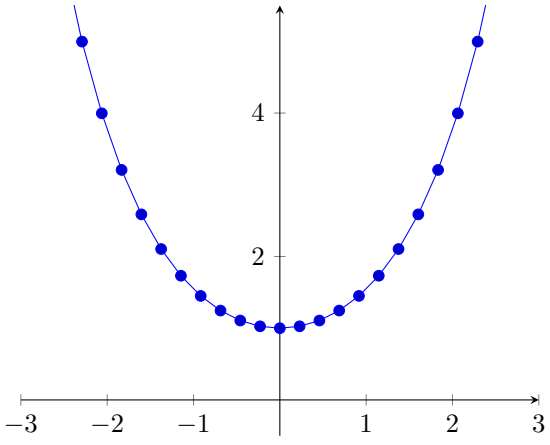
proof of 3: We recall that the tangent line of $\mathscr{C}_f$ at some point x_0 is defined by

$$(T_{x_0}) : \; y = f'(x_0)(x - x_0) + f(x_0).$$

Here we take $x_0 = 0$. We know $f'(0) = 0$ and $f(0) = 1$. Therefore,

$$(T_0) : \; y = f'(0)(x - 0) + f(0) = 1.$$

This proves the theorem. The graph of cosh is given below



□

In the following subsection we list some properties of the function sh.

4.4.2 Study of the Hyperbolic Sinus Function

Let $f(x) = \mathrm{sh}(x)$. The domain of f is $D_f = \mathbb{R}$. We consider an arbitrary real number x. we have

$$f(-x) = \frac{e^{-x} - \mathrm{e}^{-(-x)}}{2} = \frac{\mathrm{e}^{-x} - \mathrm{e}^{x}}{2} = -\frac{\mathrm{e}^{x} - \mathrm{e}^{-x}}{2} = -f(x).$$

Lemma 4.4.6. *Let $f(x) = \mathrm{sh}(x)$. Then, the function f is an odd function.*

Theorem 4.4.7. *Let $f(x) = \mathrm{sh}(x)$. Then, we have*

$$\lim_{x\to-\infty} f(x) = -\infty \quad \textit{and} \quad \lim_{x\to+\infty} f(x) = +\infty.$$

Proof. Let $x \in \mathbb{R}$. We define $X := -x$. We have $\lim_{x\to-\infty} X = +\infty$. From this, we deduce that

$$\lim_{x\to-\infty} \frac{\mathrm{e}^{-x}}{2} = \lim_{X\to+\infty} \frac{\mathrm{e}^{X}}{2} = +\infty.$$

Therefore,

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{\mathrm{e}^{x}}{2} - \lim_{x\to-\infty} \frac{\mathrm{e}^{-x}}{2} = -\infty.$$

On the other hand, we have $\lim_{x\to+\infty} X = -\infty$. Then,

$$\lim_{x\to+\infty} \frac{\mathrm{e}^{-x}}{2} = \lim_{X\to-\infty} \frac{\mathrm{e}^{X}}{2} = 0.$$

Hence

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{\mathrm{e}^{x}}{2} - \lim_{x\to+\infty} \frac{\mathrm{e}^{-x}}{2} = +\infty.$$

□

Theorem 4.4.8. *Let $f(x) = \mathrm{sh}(x)$. Then, the function f is increasing in $\mathbb{R}$.*

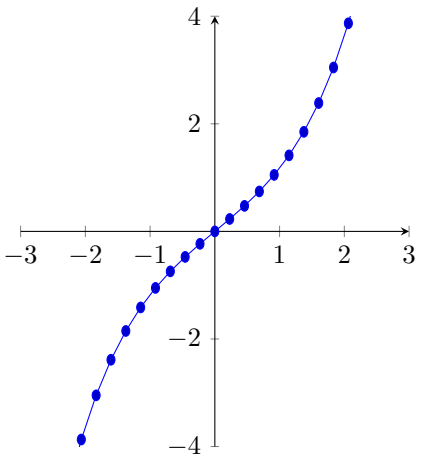
Proof. The function f is differentiable in $\mathbb{R}$, because it is the sum of two differentiable functions in $\mathbb{R}$. Indeed the functions $\mathrm{e}^x/2$ and $\mathrm{e}^{-x}/2$ are differentiable functions in $\mathbb{R}$. For every $x \in \mathbb{R}$,

$$f'(x) = \frac{\mathrm{e}^{x} + \mathrm{e}^{-x}}{2} = \mathrm{ch}(x).$$

Since for all $x \in \mathbb{R}$, $\mathrm{e}^x > 0$ and $\mathrm{e}^{-x} > 0$, we deduce the function $f'(x) = \mathrm{ch}(x) > 0$. Hence, the function f is increasing in $\mathbb{R}$

x	$-\infty$		$+\infty$
f	$-\infty$	$\nearrow$	$+\infty$

The graph of sinh is given below



□

Let x be a real number. We have

$$\mathrm{ch}(x) + \mathrm{sh}(x) = \mathrm{e}^{x} \quad \text{and} \quad \mathrm{ch}(x) - \mathrm{sh}(x) = \mathrm{e}^{-x}.$$

This involves

$$\mathrm{ch}^2(x) - \mathrm{sh}^2(x) = \big(\mathrm{ch}(x) - \mathrm{sh}(x)\big)\big(\mathrm{ch}(x) + \mathrm{sh}(x)\big) = \mathrm{e}^{x} \cdot \mathrm{e}^{-x} = 1.$$

Thus, we have proved the following lemma.

Lemma 4.4.9. *For any real number x we have*

$$\mathrm{ch}^2(x) - \mathrm{sh}^2(x) = 1. \tag{4.4.1}$$

In the following theorem we give some properties shared by cosh and sinh.

Theorem 4.4.10. *Let a and b be two real numbers. Then,*

1. $\mathrm{ch}(a+b) = \mathrm{ch}(a)\mathrm{ch}(b) + \mathrm{sh}(a)\mathrm{sh}(b)$,
2. $\mathrm{ch}(a-b) = \mathrm{ch}(a)\mathrm{ch}(b) - \mathrm{sh}(a)\mathrm{sh}(b)$,
3. $\mathrm{sh}(a+b) = \mathrm{sh}(a)\mathrm{ch}(b) + \mathrm{ch}(a)\mathrm{sh}(b)$,
4. $\mathrm{sh}(a-b) = \mathrm{sh}(a)\mathrm{ch}(b) - \mathrm{ch}(a)\mathrm{sh}(b)$.

Proof. We will just prove 1 and 3. Because the proof of 2 is similar to 1 and the proof of 4 is the same as 3.
proof of 1: We consider two real numbers a and b. We obtain

$$\begin{aligned}\mathrm{ch}(a)\mathrm{ch}(b) + \mathrm{sh}(a)\mathrm{sh}(b) &= \frac{\big(\mathrm{e}^{a} + \mathrm{e}^{-a}\big)\big(\mathrm{e}^{b} + \mathrm{e}^{-b}\big)}{4} + \frac{\big(\mathrm{e}^{a} - \mathrm{e}^{-a}\big)\big(\mathrm{e}^{b} - \mathrm{e}^{-b}\big)}{4} \\ &= \frac{2\mathrm{e}^{a+b} + 2\mathrm{e}^{-(a+b)}}{4} = \mathrm{ch}(a+b).\end{aligned}$$

proof of 3: We consider two real numbers a and b. Then, we have

$$\begin{aligned}\mathrm{sh}(a)\mathrm{ch}(b)+\mathrm{ch}(a)\mathrm{sh}(b) &= \frac{\left(\mathrm{e}^{a}-\mathrm{e}^{-a}\right)\left(\mathrm{e}^{b}+\mathrm{e}^{-b}\right)}{4}+\frac{\left(\mathrm{e}^{a}+\mathrm{e}^{-a}\right)\left(\mathrm{e}^{b}-\mathrm{e}^{-b}\right)}{4}\\ &= \frac{2\mathrm{e}^{a+b}-2\mathrm{e}^{-(a+b)}}{4}=\mathrm{sh}(a+b).\end{aligned}$$

This completes the proof of the theorem. □

Making a parallel with trigonometric function, we define for every $x \in \mathbb{R}$, the function

$$f(x)=\frac{\mathrm{sh}(x)}{\mathrm{ch}(x)}=\frac{\mathrm{e}^{x}-\mathrm{e}^{-x}}{\mathrm{e}^{x}+\mathrm{e}^{-x}}.$$

We can observe

$$f(x)=\frac{\mathrm{e}^{x}-\mathrm{e}^{-x}}{\mathrm{e}^{x}+\mathrm{e}^{-x}}=\frac{\mathrm{e}^{x}(1-\mathrm{e}^{-2x})}{\mathrm{e}^{x}(1+\mathrm{e}^{-2x})}=\frac{1-\mathrm{e}^{-2x}}{1+\mathrm{e}^{-2x}}.$$

The function f defined above is called the hyperbolic tangent function.It is denoted $f(x)=\tanh(x)=\mathrm{th(x)}$. In the following subsection we study the function tanh.

4.4.3 Study of Hyperbolic Tangent

Let $f(x)=\mathrm{th}(x)$. Since for all $x \in \mathbb{R}$, we have $\mathrm{ch}(x) \neq 0$. The function f is well defined in $\mathbb{R}$.

Now we take an arbitrary real number x and we consider the function $f(x)=\mathrm{th}(x)$. One has

$$\tanh(-x)=\frac{\mathrm{sh}(-x)}{\mathrm{ch}(-x)}=-\frac{\mathrm{sh}(x)}{\mathrm{ch}(x)}.$$

Lemma 4.4.11. *Let $f(x)=\tanh(x)$. Then, the function f is odd.*

Theorem 4.4.12. *Let $f(x)=\tanh(x)$. Then, the following limits hold*

$$\lim_{x\to-\infty} f(x)=-1 \quad \textit{and} \quad \lim_{x\to+\infty} f(x)=1.$$

Proof. We start by defining $X=-2x$. We have

$$\lim_{x\to+\infty} f(x)=\lim_{X\to-\infty}\frac{1-\mathrm{e}^{X}}{1+\mathrm{e}^{X}}=1.$$

On the other hand we have $\lim_{X\to+\infty}\dfrac{1}{\mathrm{e}^{X}}=0$. Therefore,

$$\lim_{x\to-\infty} f(x)=\lim_{X\to+\infty}\frac{1-\mathrm{e}^{X}}{1+\mathrm{e}^{X}}=\lim_{X\to+\infty}\frac{\dfrac{1}{\mathrm{e}^{X}}-1}{\dfrac{1}{\mathrm{e}^{X}}+1}=-1.$$

□

The meaning of these limits is

- The line $y=-1$ is a horizontal asymptote of $\mathscr{C}_f$ at $-\infty$.

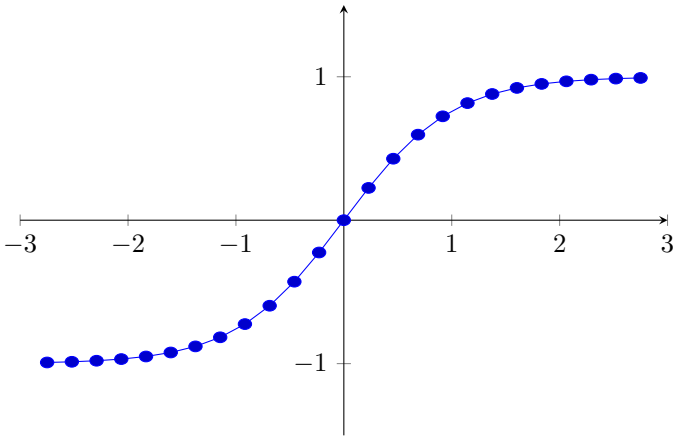
- The line $y = 1$ is a horizontal asymptote of $\mathscr{C}_f$ at $+\infty$.

Theorem 4.4.13. *Let $f(x) = \tanh(x)$. Then, the function f is increasing in $\mathbb{R}$.*

Proof. We define $f(x) = \tanh(x) = \sinh(x)/\cosh(x)$. The functions sinh and cosh are differentiable in $\mathbb{R}$ and for all $x \in \mathbb{R}$, $\cosh(x) \neq 0$. Then, the function f is differentiable in $\mathbb{R}$. For every $x \in \mathbb{R}$,

$$f'(x) = \frac{\cosh^2(x) - \sinh^2(x)}{\cosh^2(x)} = \frac{1}{\cosh^2(x)} > 0.$$

Hence, f is an increasing function in $\mathbb{R}$. Below we plot the graph of tanh



□

CHAPTER 5

LOGARITHMIC FUNCTIONS

Contents

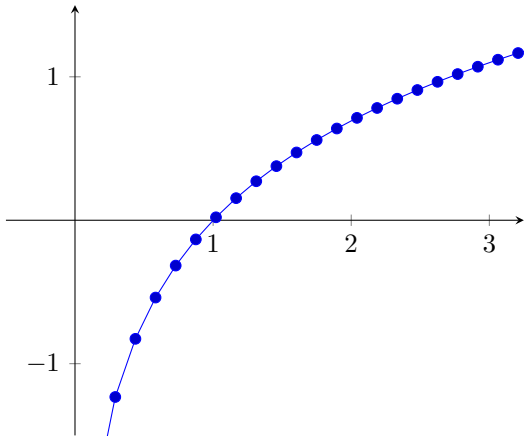
Let $x \in \mathbb{R}$. In the previous chapter we defined the function $\exp(x)$ and studied it. We established the function exp is continuous and strictly increasing. Then, there is a unique function $f:\ (0\,,\,+\infty) \to \mathbb{R}$, such that, $f(\exp(x)) = x, \quad \forall\, x \in \mathbb{R}$. In this case we know the function f satisfies

$$\begin{cases} f(\mathrm{e}^x) = x, \quad \text{for all} \quad x \in \mathbb{R}, \\ \\ \exp\left(f(x)\right) = x, \quad \text{for all} \quad x > 0, \\ \\ f(1) = 0. \end{cases} \tag{5.0.1}$$

This function f is called the *Naperian Logarithm* or the *Natural Logarithm* and we denote it by $\ln(x)$. Using the notation ln we can simplify (5.0.1), in the following way:

$$\begin{aligned} &\ln\left(\mathrm{e}^x\right) = x, \quad \forall\, x \in \mathbb{R}, \\ &\exp(\ln(x)) = x, \quad \forall\, x > 0, \\ &\ln(1) = 0. \end{aligned}$$

The graph of the function ln is drawn below



Let a be a strictly positive real number. We consider the equation, find x in $\mathbb{R}$, such that, $\exp(x) = a$. Using the function ln we obtain $\ln(\exp(x)) = \ln(a)$. Since $\ln(\exp(x)) = x$, we have $x = \ln(a)$.

The objective of this chapter is to study the properties of the function ln . We will see that the function ln has interesting properties.

To make the presentation clear, we organize this chapter as follows. In section 1 we give a list of properties satisfied by ln. In section 2, we deal with variations of ln. We will also solve equations using the function ln. The last section is devoted to the logarithm in basis $a > 0$ denoted Log_a.

5.1 Some Properties of the Natural Logarithm

Let $x > 0$ be a real number. We consider the function $f(x) = \ln(x)$. We know the function f is continuous and strictly monotone. We established also the function f is differentiable in $(0\,,\,+\infty)$. Here we prove some other properties satisfied by f. Before going further, we point out that we can define the function ln by using a functional equation : find a continuous function f that satisfies

$$f(x \cdot y) = f(x) + f(y) \quad \text{and} \quad f(1) = 0.$$

In section (5.2) we will see all there is another way to define the function ln by using its derivative. Indeed, we can seek for a differentiable function f defined in $(0\,,\,+\infty)$ that satisfies

$$f'(x) = \frac{1}{x}\cdot$$

In this section we establish at first elementary properties of the function ln .

Let a and b be in $\mathbb{R}_+^*$. We define $X := \ln(a\cdot b)$ and $Y = \ln(a)+\ln(b)$. Using (5.0.1), we have $\exp(X) = \exp(\ln(a \cdot b)) = a \cdot b$ and $\exp(Y) = \exp(\ln(a) + \ln(b))$. Using the properties of the function exp we obtain $\exp\big(\ln(a)\big) \cdot \exp\big(\ln(b)\big) = a \cdot b$. This means that $\exp\big(X\big) = \exp\big(Y\big)$. Therefore, $X = Y$. That is, $\ln(a \cdot b) = \ln(a) + \ln(b)$.

Proposition 5.1.1. *Let a and b be two strictly positive real numbers. Then,*

$$\ln(a \cdot b) = \ln(a) + \ln(b) \tag{5.1.1}$$

Example 5.1.2. *Using the proposition above, we have*

$$\ln(10) = \ln(2 \cdot 5) = \ln(2) + \ln(5) \quad \textit{and} \quad \ln(21) = \ln(3 \cdot 7) = \ln(3) + \ln(7).$$

Exercise 5.1.3. *Simplify the following expressions:*

(1). $\ln(6) + 2\ln(3) - \ln(2) = \cdots$, (2). $\ln(14) - 3\ln(7) + \ln(2) = \cdots$,
(3). $\ln(21) - 5\ln(3) + \ln(9) = \cdots$

We consider two real numbers a and b, which are strictly positive. We define $X := \ln\left(\frac{a}{b}\right)$ and $Y := \ln(a) - \ln(b)$. We have

$$\exp\left(X\right) = \exp\left(\ln\left(\frac{a}{b}\right)\right) = \frac{a}{b}, \quad \text{and} \quad \exp^Y = \exp\left(\ln(a) - \ln(b)\right) = \mathrm{e}^{\ln(a)} \cdot \mathrm{e}^{-\ln(b)}$$

Since, $\exp\left(-\ln(b)\right) = 1/\exp\left(\ln(b)\right) = 1/b$. This implies that, $\mathrm{e}^Y = a \cdot \frac{1}{b} = \frac{a}{b}$. Therefore, $\mathrm{e}^X = \mathrm{e}^Y$. This means that $X = Y$. That is,

$$\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b).$$

Proposition 5.1.4. *Let a and b be in $(0\,,+\infty)$. Then,*

$$\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b). \tag{5.1.2}$$

Remark 5.1.5. *If we fix $a = 1$ in Proposition 5.1.4, we obtain*

$$\ln\left(\frac{1}{b}\right) = \ln(1) - \ln(b).$$

On the other hand, we know by definition $\ln(1) = 0$. Hence $\ln\left(\frac{1}{b}\right) = -\ln(b)$.

Example 5.1.6. *We have*

$$\ln\left(\frac{3}{2}\right) = \ln(3) - \ln(2) \quad \textit{and} \quad \ln\left(\frac{1}{7}\right) = -\ln(7).$$

Exercise 5.1.7. *Simplify the following expressions*

(1). $\ln\left(\frac{5}{2}\right) - 3\ln(5) + 2\ln(2) = \cdots$, (2). $2\ln(3) + \ln\left(\frac{4}{3}\right) + 3\ln(4) = \cdots$,
(3). $\ln\left(\frac{6}{7}\right) - \ln(3) = \cdots$

Proposition 5.1.8. *Let $x \in (0\,,+\infty)$. We consider a natural number $n \geq 2$. Then,*

$$\ln\left(x^n\right) = n\ln(x). \tag{5.1.3}$$

Proof. We prove this proposition by induction. We take $x \in (0\,,+\infty)$. We fix a natural number $n \geq 2$.

Step 1: We consider $n = 2$, we have $\ln\left(x^2\right) = \ln(x \cdot x) = \ln(x) + \ln(x) = 2\ln(x)$. The property is true for $n = 2$.
Step 2: We assume the property true for some natural number k. This means that,

$$\ln\left(x^k\right) = k\ln(x).$$

Step 3: We need to prove that the property holds for $k+1$. In other words, we want to establish that,

$$\ln\left(x^{k+1}\right) = (k+1)\ln(x). \tag{5.1.4}$$

To prove (5.1.4), we set $X = x^k$. Then, we have

$$\ln\left(x^{k+1}\right) = \ln(X \cdot x) = \ln(X) + \ln(x).$$

By step 2, we know that $\ln(X) = \ln\left(x^k\right) = k\ln(x)$. Hence,

$$\ln\left(x^{k+1}\right) = k\ln(x) + \ln(x) = (k+1)\ln(x).$$

The property is true for $k+1$.
Conclusion: for every $x \in]0\,,+\infty[$, and $n \geq 2$, $\ln\left(x^n\right) = n\ln(x)$. □

Let $x \in (0\,,+\infty)$. We consider a natural number $n \geq 2$. We define $X = x^n$. Using (5.0.1), we have

$$\mathrm{e}^{\ln(X)} = \mathrm{e}^{\ln(x^n)} = \mathrm{e}^{n\ln(x)} = \left(\mathrm{e}^{\ln(x)}\right)^n = (x)^n = x^n.$$

This proves the following lemma

Lemma 5.1.9. *Let x be a real number in $(0\,,+\infty)$. Then, for any natural number $n \geq 2$,*

$$x^n = \exp\left(n\,\ln(x)\right).$$

Now, we consider, $x \in (0\,,+\infty)$. Let p and q be two natural numbers , such that, $q > 0$. We define $r = \dfrac{p}{q}\cdot$ The number r is called a rational number. For every $x \in (0\,,+\infty)$, we have

$$x^{p/q} = \left(x^p\right)^{1/q} = \left(\mathrm{e}^{p\ln(x)}\right)^{1/q} = \mathrm{e}^{(p\ln(x))/q} = \mathrm{e}^{r\ln(x)}.$$

Lemma 5.1.10. *Let $x \in (0\,,+\infty)$ and r be a positive rational number. Then, the following identity holds*

$$x^r = \exp\left(r\,\ln(x)\right).$$

Now we recall the following theorem which is admitted.

Theorem 5.1.11. *Let x be a real number. Then, there is a rational sequence $(r_n)_{n\geq 0}$, such that,*

$$x = \lim_{n\to+\infty} r_n.$$

The theorem 5.1.11 means that real numbers are obtained as limits of rational sequences.

Now we consider real number y. From theorem 5.1.11 we deduce there is a rational sequence $(r_n)_{n\geq 0}$, such that, $y = \lim_{n\to+\infty} r_n$. Let $x > 0$ be a real number. We have

$$\lim_{n\to+\infty} r_n \ln(x) = \ln(x) \lim_{n\to+\infty} r_n = y \ln(x).$$

Since the function exp is continuous, we obtain

$$\lim_{n\to+\infty} x^{r_n} = \lim_{n\to+\infty} \exp(r_n \ \ln(x)) = \exp\big(\lim_{n\to+\infty} r_n \ln(x)\big) = \exp\big(y \ln(x)\big).$$

This allows us to establish the following result.

Lemma 5.1.12. *Let x belongs to $(0\,,+\infty)$. Then, for every real number y, we have*

$$x^y = \mathrm{e}^{y\ln(x)}.$$

Example 5.1.13. *For all $x \in]0\,,+\infty[$, we have,*

$$1.\ x^3 = \exp\big(3\ln(x)\big), \quad 2.\ x^{1/5} = \exp\left(\ln(x)/5\right), \quad 3.\ \sqrt{x} = \exp\left(\ln(x)/2\right).$$

Exercise 5.1.14. *For any $x > 0$ define the following powers*

$$(1).\ x^z, \quad z \in \mathbb{R}, \quad (2).\ x^{1/n},\ \forall\, n \geq 1, \quad (3).\ n^{n^n},\ \forall\, n \geq 1.$$

In the next section we continue to study the properties of this function. Here we investigate continuity and variations of the function ln .

5.2 Study of the Logarithmic Function

The aim of this section is to study continuity and variations of ln. We will start by giving its limits at the bounds of its domain. In a second time, we define its derivative and specify its variations. At the end of this section, we will study functions of the form $\ln(u(x))$, where $u(x)$ is a real function such that, $u(x) > 0$, for all $x \in D_u$, and we deal with equations using ln and exp.

A look on the graph of ln allows us to set the following theorem.

Theorem 5.2.1. *Let $x \in\ (0\,;+\infty)$. We define the function $f(x) = \ln(x)$. Then,*

$$\lim_{x\to 0} f(x) = -\infty \quad \textit{and} \quad \lim_{x\to+\infty} f(x) = +\infty. \tag{5.2.1}$$

To establish the limits of the function at the bounds of the domain, we emphasize that the graph of ln and the graph of exp are symmetric with respect to the line $y = x$. Indeed, we know that if f is a continuous and strictly monotone then, f admits a reciprocal function f^{-1} ,such that, for every $x \in D_f$, we have $f^{-1}\big(f(x)\big) = x$ and for every y in the range of f, we have $f\big(f^{-1}(y)\big) = y$. Geometrically speaking, the graph of f, and the graph of the function f^{-1} are symmetric with respect to the line $y = x$.

From this we deduce, that

$$\lim_{x\to+\infty} \exp(x) = \lim_{x\to+\infty} \ln(x) = +\infty$$

Let $x \in \mathbb{R}$, we define the function $f(x) = \ln\big(\exp(x)\big)$. This means that $f(x) = x$. We consider the variable $X := \exp(x)$. We know that

$$\lim_{x\to-\infty} X = \lim_{x\to-\infty} \exp(x) = 0.$$

Using the limit above, we obtain

$$\lim_{X\to 0} \ln(X) = \lim_{x\to-\infty} x = -\infty.$$

5.2.1 Continuity of the Function ln

In this subsection we study the continuity of the function ln. The continuity of ln is a consequence of the continuity of the function exp. The following theorem is admitted

Theorem 5.2.2. *The function $f(x) = \ln(x)$ is continuous in the interval $(0\,,\,+\infty)$.*

In the following subsection we study the variations of the function ln

5.2.2 Variations of the Function ln

This subsection is devoted the the variations of the function ln. Indeed we want to know if the natural logarithm is whether increasing or decreasing.

Let us consider a real number x. We define the function $f(x) = \exp(x)$. We know from the definition of ln, that $\ln\big(\exp(x)\big) = x$.

From this we deduce the function $\ln\big(\exp(x)\big)$ is differentiable in $\mathbb{R}$. To simplify notations, we denote $g(x) = \ln(x)$ for any $x > 0$. Applying the theorem of derivative of composition of functions, we obtain

$$\Big(g\big(\exp(x)\big)\Big)' = g'\big(\exp(x)\big)\cdot\exp(x) = 1.$$

This involves

$$g'\big(\exp(x)\big) = \frac{1}{\exp(x)}\cdot$$

Setting $X := \exp(x)$, we obtain

$$\Big(\ln\big(X\big)\Big)' = \frac{1}{X}, \quad \text{for any} \quad X \in (0\,,\,+\infty).$$

We have established the following theoerem

Theorem 5.2.3. *Let $x \in (0\,;\,+\infty)$. We define $f(x) = \ln(x)$. Then the function f is differentiable in $(0\,;\,+\infty)$ and for every $x \in (0\,;\,+\infty)$, we have*

$$f'(x) = \frac{1}{x}\cdot$$

Since the real number x belongs to $(0\,,\,+\infty)$, we have $f'(x) > 0$. This implies the function f is increasing in $(0\,,\,+\infty)$.

Theorem 5.2.4. *Let $x \in (0\,;\,+\infty)$. Then, the function $f(x) = \ln(x)$ is increasing $(0\,,\,+\infty)$.*

Let x belong to $(0\,,\,+\infty)$. We define the function $f(x) = \ln(x)$. Using theorem 5.2.3 we have $f'(x) = \frac{1}{x} > 0$. The following variational table holds

x	0		$+\infty$
f	$-\infty$	$\nearrow$	$+\infty$

As we know $\ln(1) = 0$ and f is an increasing function, we are able to establish the following sign table

x	0		1		$+\infty$
$\ln(x)$		$-$	0	$+$	

Now, we consider a function $u : D_u \subset \mathbb{R} \to (0\,;\,+\infty)$. We know that the function $\ln : (0\,;\,+\infty) \to \mathbb{R}$. Since $u(D_u)$ is included in the domain of ln, we can define the function

$$f(x) = \ln(u(x)).$$

Hence the function $\ln(u(x))$ has domain D_u. Let us assume that the function u is differentiable in some open interval $I \subset D_u$. Then, the function f which is the composition of differentiable functions is differentiable in the open interval I. Denoting $g(x) = \ln(x)$ for every $x \in (0\,,\,+\infty)$ and using the *chain rule derivation*, we obtain

$$f'(x) = u'(x)g'(u(x)) = \frac{u'(x)}{u(x)}\cdot \tag{5.2.2}$$

This proves the following theorem.

Theorem 5.2.5. *Let $u : D_u \to (0\,,\,+\infty)$ be a real function and $I \subset D_u$ an open interval. We suppose that the function u is differentiable in I. Then, the function $f(x) = \ln(u(x))$ is differentiable in I, and for every $x \in I$, the derivative of the f is given by* (5.2.2)

We emphasize that this theorem means that the function $\ln(u(x))$ is differentiable in its domain.

Example 5.2.6. *As an example, we consider the function $f(x) = \ln(x^2 + 2)$. The domain of f is $D_f = \mathbb{R}$. The function $u(x) = x^2 + 2$ is differentiable in $\mathbb{R}$, and $u'(x) = 2x$. Then, the function f is differentiable in $\mathbb{R}$ and for every real number x, we have*

$$f'(x) = \frac{2x}{x^2 + 2}\cdot$$

Example 5.2.7. *We define the function $g(x) = \ln(x + 5)$. This function is defined in the interval $I = (-5\,,\,+\infty)$. The function $u(x) = x + 5$ is differentiable in $(-5\,,\,+\infty)$ and for every $x \in (-5\,,\,+\infty\,)$ we have $u'(x) = 1$. Therefore, the function is differentiable in I and for any $x \in I$, one has*

$$g'(x) = \frac{1}{x + 5}\cdot$$

Example 5.2.8. *In this example we consider the function* $f(x) = \ln(x^2 + 4x + 3)$. *We define the function* $u(x) = x^2+4x+3$. *In this case, we have* $\Delta = 16-12 = 4 > 0$. *We have two real roots,*

$$x_1 = \frac{-4-2}{2} = -3 \quad \textit{and} \quad x_2 = \frac{-4+2}{2} = -1.$$

We can establish the following sign table for u

x	$-\infty$	-3	-1	$+\infty$
x^2+4x+3	$+$	0	$-$ 0	$+$

This implies for any x *in* $(-\infty\,,\,-3) \cup (-1\,;\,+\infty)$, *we hvae* $u(x) > 0$. *Hence* $D_f = (-\infty\,,\,-3) \cup (-1\,;\,+\infty)$. *Let* $x \in D_f$, *the derivative of* u *is* $u'(x) = 2x + 4$. *This leads to the following identity*

$$f'(x) = \frac{2x+4}{x^2+4x+3}.$$

Exercise 5.2.9. *Find the derivative of the following functions*

1. $f(x) = \ln(3x+1)$, 2. $g(x) = \ln(x^2+5x+6)$, 3. $h(x) = \ln\left(\sqrt{x^2+4x+4}\right)$,
4. $i(x) = \ln\left(\dfrac{x+5}{2x+3}\right)$, 5. $j(x) = \ln((x+1)(x+4))$.

Let us consider again a real function u. We know that the following function $f(x) = \ln(u(x))$ is defined if and only if $u(x) > 0$. Hence the domain of the function f is $D = \{x \in \mathbb{R} \ : \ u(x) > 0\}$. Moreover, if we suppose that u is differentiable in D, we obtain

$$f'(x) = \frac{u'(x)}{u(x)}.$$

Since $u(x) > 0$ for every $x \in D$, we conclude that f' and u' have the same sign. This leads to the following theorem

Theorem 5.2.10. *Let* $u : D \longrightarrow (\,0\,,\,+\infty\,)$ *be a differentiable function. For every* $x \in D$, *we define the function* $f(x) = \ln(u(x))$. *Then, the functions* f *and* u *have the same variations in* D.

Example 5.2.11. *We consider the following function* $f(x) = \ln(2x+4)$ *and we set* $u(x) = 2x + 4$. *We can see that* $u(x) > 0$ *if and only if* $x \in (-2\,,\,+\infty\,)$. *The function* $\ln(2x+4)$ *is defined in the open interval* $]-2\,;\,+\infty\,[$. *In other words we have,* $D_f = (-2\,,\,+\infty\,)$.

Limits at the bounds:

We set $X = 2x + 4$. *We have*

$$\lim_{x\to+\infty} X = +\infty \quad \textit{and} \quad \lim_{x\to-2^+} 2x+4 = 0^+.$$

This implies that,

$$\lim_{x\to+\infty} \ln(2x+4) = \lim_{X\to+\infty} \ln(X) = +\infty.$$

$$\textit{and} \quad \lim_{x\to -2^+} \ln(2x+4) = \lim_{X\to 0^+} \ln(X) = -\infty.$$

We conclude that the function f does not have horizontal asymptote at $+\infty$ and the line $x=-2$ is a vertical asymptote of f.

Variations of f:

To study the variations of f we start by remarking that the function $u(x)=2x+4$ is differentiable in $(-2\,;+\infty)$. Therefore the function $f(x)=\ln(2x+4)$ is differentiable in the interval $]-2\,;+\infty[$. Since $u'(x)=2$ for all $x>-2$, we have

$$f'(x)=\frac{2}{2x+4}>0\cdot$$

From this we deduce that the function f is an increasing function in $(-2\,;+\infty)$ because the function u is increasing.

x	-2	$+\infty$
f	$-\infty$ ↗	$+\infty$

Example 5.2.12. *Let $f(x)=\ln\left(x^2+2x\right)$. We can easily check that the domain of this function is $D=(-\infty\,,-2\,)\cup(0\,,+\infty\,)$. Defining $X:=x^2+2x$, we establish that*

$$\lim_{x\to\pm\infty} X=+\infty, \quad \lim_{x\to -2^-} X=0 \quad \textit{and} \quad \lim_{x\to 0^+} X=0 \tag{5.2.3}$$

From (5.2.3) *one deduces*

$$\lim_{x\to\pm\infty} f(x)=\lim_{X\to+\infty}\ln(X)=+\infty, \quad \lim_{x\to -2^-} f(x)=\lim_{X\to 0^+}\ln(X)=-\infty$$
$$\textit{and} \quad \lim_{x\to 0^+} f(x)=\lim_{X\to 0^+}\ln(X)=-\infty.$$

These limits have the following interpretations:

- *the limits at $\pm\infty$ mean that we do not have horizontal asymptotes at $\pm\infty$*
- *the limit at -2 has the following meaning. The line $x=-2$ is a vertical asymptote of the curve of f*
- *the limit at 0 means that the line $x=0$ is a vertical asymptote of $\mathscr{C}_f$.*

Since the function f is the composition of differentiable functions in D, it is differentiable in D. Then, for every $x\in D$, we have

$$f'(x)=\frac{2x+2}{x^2+2x}\cdot$$

The sign of the function f' allows to establish the following variational table

x	$-\infty$	-2	0	$+\infty$
f	$+\infty$ ↘ $-\infty$	‖	‖ ↗ $-\infty$	$+\infty$

Exercise 5.2.13. *Study the following functions*

$$1.\ f(x) = \ln(5x+1),\ \ 2.\ g(x) = \ln(x^3-1),\ \ 3.\ h(x) = \ln\left(\frac{3x+5}{x+3}\right).$$

Let x be a strictly positive real number. We set $f(x) = \ln(x)$. The function f is continuous from $(0\,,\,+\infty)$ into $\mathbb{R}$. It is also strictly increasing. Hence we deduce from the IVT theorem and its corollaries , that, for all $b \in \mathbb{R}$, there exists a unique $x \in (0\,,\,+\infty)$, such that, $\ln(x) = b$. In the next section we show how to find solution of equations including the function ln. To make the presentation clear we will distinguish two type of equations
First Form: Let $a > 0$. Find $x \in (0\,,\,+\infty)$, such that, $\ln(x) = \ln(a)$.
Second Form: Let c be a fixed real number. Find $x \in (0\,,\,+\infty)$, such that,

$$\ln(x) = c.$$

5.2.3 Solutions of Some Equations

The objective of this subsection is to give solutions of equation of the first and second form.

Solutions of the First Form

Let $a > 0$ be a fixed real number. We aim to find a strictly positive real number x solution to the equation $\ln(x) = \ln(a)$. We remark that, this equation is equivalent to $\mathrm{e}^{\ln(x)} = \mathrm{e}^{\ln(a)}$. That is, $x = a$. This proves the following lemma

Lemma 5.2.14. *We consider fixed real number a which is strictly positive. Let x belongs to $(0\,,\,+\infty)$, such that, $\ln(x) = \ln(a)$. Then, $x = a$.*

This is a consequence of the fact that the function ln is continuous and strictly increasing, then, it is a bijection. In this case we have

$$\ln(x) = \ln(a) \Leftrightarrow x = a.$$

Example 5.2.15. *We consider the equation $\ln(x) = \ln(5)$. Using the lemma above we have, $x = 5$.*

Example 5.2.16. *We consider the following equation find $x \in \mathbb{R}$ such that, $\ln(x+1) = \ln(3)$. We know that the function $\ln(x+1)$ is defined if and only if $x \in (-1\,,\,+\infty)$. On the other hand the equation above is equivalent to $x + 1 = 3$. Hence, $x = 2$. Since $2 \in (-1\,,\,+\infty)$, the solution of the equation is $x = 2$.*

Exercise 5.2.17. *Find the solutions of the following equations:*

1. $\ln(3x+2) = \ln(4)$, 2. $\ln(x^2+5x+2) = \ln(2x+3)$, 3. $\ln(x+5) = \ln(x+6)$,
4. $\ln(x^2+2x+1) = \ln(x+1)$, 5. $\ln(2x+4) = \ln(2)$.

Solution of the Second Form

Let b be any fixed real number. We consider the following equation find x in $\mathbb{R}$, such that, $\ln(x) = b$. Using the function exp, one obtains $\exp\big(\ln(x)\big) = \exp(b)$. This leads to the identity $x = \mathrm{e}^b$.

Lemma 5.2.18. *We consider a fixed real number b. Let x belongs to $(0\,;+\infty)$, such that, $\ln(x)=b$. Then, $x=\mathrm{e}^b$.*

Example 5.2.19. *We consider the equation $\ln(x)=2$. Therefore, $x=\mathrm{e}^2$.*

Example 5.2.20. *To find solutions to the equation $\ln(x+3)=-1$ we proceed in this way. First we remind that, $\ln(x+3)$ is defined in $(-3\,;+\infty)$. The equation $\ln(x+3)=-1$ is equivalent to $x+3=\mathrm{e}^{-1}$. Thus, we obtain the identity $x=\mathrm{e}^{-1}-3$. We can easily check that $\mathrm{e}^{-1}-3\in\ (-3\,;+\infty)$. Therefore, the solution is $x=\mathrm{e}^{-1}-3$.*

Exercise 5.2.21. *Find the solutions to the following equations*

1. $\ln(x+1)=3$, 2. $\ln(3x+1)=5$, 3. $\ln(x+6)=1$, 4. $\ln(7x+2)=2$,
5. $\ln(2x+5)=-5$.

Using the previous subsection we can deal with more general equations including exponential and logarithmic functions. In the next subsection, we will solve equations of the following form: Find $x\in]0\,;+\infty[$

$$a\ln^2(x)+b\ln(x)+c=0, \tag{5.2.4}$$

where a, b and c are constant real numbers such that $a\neq 0$.

5.2.4 General Equations

To deal with (5.2.4), we introduce the following substitution. We define $X:=\ln(x)$. Thus the equation (5.2.4) becomes

$$aX^2+bX+c=0. \tag{5.2.5}$$

Equation (5.2.5) is more familiar to us. To find solutions to (5.2.5) we have to distinguish three cases according to the sign of $\Delta=b^2-4ac$.

Case 1: $\Delta>0$. In this case, we have two real solutions

$$X_1=\frac{-b-\sqrt{\Delta}}{2a}=\ln(x_1)\quad\text{and}\quad X_2=\frac{-b+\sqrt{\Delta}}{2a}=\ln(x_2).$$

This allows to state that (5.2.4) has two real solutions

$$x_1=\mathrm{e}^{X_1}\quad\text{and}\quad x_2=\mathrm{e}^{X_2}.$$

Example 5.2.22. *We consider the following equation : find a strictly positive real number x, such that,*

$$(*):\ \ln^2(x)+4\ln(x)+3=0.$$

Setting $X:=\ln(x)$, one obtains $X^2+4X+3=0$. Thus, we have

$$\Delta=16-12=4>0.$$

We have two real solutions

$$X_1=\frac{-4-2}{2}=-3\quad\textit{and}\quad X_1=\frac{-4+2}{2}-1.$$

This means $\ln(x_1)=-3$ and $\ln(x_2)=-1$. Therefore equation $(*)$ has two solutions

$$x_1=\mathrm{e}^{-3}\quad\textit{and}\quad x_2=\mathrm{e}^{-1}.$$

Case 2: $\Delta = 0$**.** In this case, we have only one real solution $X_0 = -\dfrac{b}{2a}\cdot$ In this case (5.2.4) has only one solution $x_0 = \mathrm{e}^{X_0}$.

Example 5.2.23. *To find a strictly positive real number x solution to the equation*

$$(**): \quad \ln^2(x) + 4\ln(x) + 4 = 0,$$

we define $X := \ln(x)$. Then we obtain $X^2+4X+4 = 0$. This implies, $\Delta = 16-16 = 0$. Therefore, we have only one solution $X_0 = -\dfrac{4}{2} = -2$. From this we deduce that, equation $(**)$ has only one solution $x_0 = \mathrm{e}^{-2}$.

Case 3: $\Delta < 0$**.** In this case there is no real solutions.

Example 5.2.24. *Let x belong to $(0\,,\,+\infty)$, such that,*

$$(***): \quad \ln^2(x) + \ln(x) + 2 = 0.$$

Setting $X := \ln(x)$, we have $X^2 + X + 2 = 0$. Thus, we get $\Delta = 1 - 8 = -7 < 0$. We conclude that $(***)$ does not have real solutions.

Exercise 5.2.25. *Find solutions to the following equations*

1) $\ln^2(x) + 3\ln(x) + 1 = 0$, 2) $2\ln^2(x) + 5\ln(x) + 2 = 0$,
3) $\ln^2(x) + 2\ln(x) + 1 = 0$, 4) $2\ln^2(x) + \ln(x) + 1 = 0$,
5) $\ln^2(x) + 2\ln(x) = 0$.

In chapter 4 we solve equations of the form $\exp(x) = \exp(a)$ for some real numbers x and a. We did not investigate equations of the form $\exp(x) = a$ for some real number a. The purpose of this paragraph is to use the function ln, to deal with this latter. Let a be a real number we consider the equation find $\exp(x) = a$. We know for any x in $\mathbb{R}$, we have $\exp(x) > 0$, then, when $a < 0$, the equation $\exp(x) = a$ does not have solutions.

Lemma 5.2.26. *Let a be a negative real number, then the equation $\exp(x) = a$, does not have real solutions. In this case we write*

$$S = \emptyset.$$

In the same way we can establish that the equation $\exp(x) = 0$ does not have solution. This means that when $a \leq 0$ the equation $\exp(x) = a$ does not have solutions. We say the set of real solutions is empty.

Now we consider a real number b which is strictly positive. We are looking for real numbers x, such that, $\exp(x) = b$. Using the function ln, and the fact that $\ln\big(\exp(x)\big) = x$, one obtains $\ln\big(\exp(x)\big) = \ln(b)$. This implies $x = \ln(b)$

Lemma 5.2.27. *Let $b \in \mathbb{R}$ be a strictly positive. Then, the solution of the equation $\exp(x) = b$ is given by*

$$x = \ln(b).$$

Example 5.2.28. *We consider the following equation $\exp(x) = 5$. Therefore , $x = \ln(5)$.*

Example 5.2.29. *Find a real number x, such that,* $\exp(3x+1)=7$. *We know the equation is equivalent to,* $3x+1=\ln(7)$. *Hence*

$$x=\frac{\ln(7)-1}{3}.$$

Example 5.2.30. *Find the solution of the following equation* $\exp(x)=-5$. *We know that for all* $x\in\mathbb{R}$, $\exp(x)>0$. *Then, the equation* $\mathrm{e}^x=-5$ *does not have solution. Hence we have*

$$S=\emptyset.$$

Exercise 5.2.31. *Find solutions to the following equations*

1. $\exp(x+2)=7$, 2. $\exp(6x+2)=-1$, 3. $\exp(2x+6)=3$,
4. $\exp(-x+5)=1$, 5. $\exp(x^2+2x+1)=2$, 6. $\exp(x^2+3x+2)=4$,
7. $\exp(-x+1)=-2$.

Now let a, b and c be three real numbers. We consider the following equation: find x in $\mathbb{R}$, such that,

$$a\,\mathrm{e}^{2x}+b\,\mathrm{e}^{x}+c=0. \tag{5.2.6}$$

We set $X=\exp(x)$. Then equation (5.2.6), becomes

$$(*):\ a\,X^2+b\,X+c=0.$$

We know that to find the solution of $(*)$ we have to compute $\Delta=b^2-4ac$. According to the sign of Δ we have to distinguish three cases.

Case 1: $\Delta>0$**.** We know that, when $\Delta>0$, the equation $(*)$ has two real solutions X_1 and X_2. In this case one of the following situation holds for equation (5.2.6):

1. $X_1<0$ and $X_2<0$: there is no solution.
2. $X_1>0$ and $X_2<0$: we have only one solution $x_1=\ln(X_1)$
3. $X_1<0$ and $X_2>0$: we have only one solution $x_2=\ln(X_2)$
4. $X_1>0$ and $X_2>0$: we have two solutions

$$x_1=\ln(X_1)\quad\text{and}\quad x_2=\ln(X_2).$$

Example 5.2.32. *We consider the following equation : find x in $\mathbb{R}$, such that,*

$$(1):\ \mathrm{e}^{2x}-3\mathrm{e}^{x}+2=0.$$

To find the solutions of (1) *we set* $X:=\exp(x)$. *Using the variable X we rewrite* (1) *in this way*

$$X^2-3X+2=0.$$

Thus we have $\Delta=9-8=1>0$. *This means that the equation* $X^2-3X+2=0$ *has two real solutions*

$$X_1=\frac{3-1}{2}=1\quad or\quad X_2=\frac{3+1}{2}=2.$$

Since $X_1 > 0$ and $X_2 > 0$ equation (1) has two real solutions which are

$$x_1 = \ln(X_1) = \ln(1) = 0 \quad \textit{and} \quad x_2 = \ln(X_2) = \ln(2).$$

In other words we obtain

$$S = \Big\{0\,;\, \ln(2)\Big\}.$$

Case 2: $\Delta = 0$. When $\Delta = 0$ the equation $aX^2 + bX + c = 0$ has only one solution which is $X_0 = -\dfrac{b}{2a}$. In this case two situations occur for (5.2.6)

1. $X_0 < 0$. Then $X_0 < 0$, equation (5.2.6) does not have solutions.
2. $X_0 > 0$. Then $x_0 > 0$, equation (5.2.6) has one solution $x_0 = \ln(X_0)$.

Example 5.2.33. *Find the solution of the following equation*

$$(2): \quad \mathrm{e}^{2x} - 4\mathrm{e}^{x} + 4 = 0.$$

We use the substitution $X = \exp(x)$ to obtain

$$(\star)\ X^2 - 4X + 4 = 0.$$

In this case we have $\Delta = 16 - 16 = 0$. Therefore equation $(\star)$ has only one solution which is defined by

$$X_0 = \frac{4}{2} = 2.$$

From this we deduce that the unique solution of (2) is $x_0 = \ln(2)$.

Case 3: $\Delta < 0$. In this case (5.2.6) does not have solution.

Exercise 5.2.34. *Determine solutions to the following equations*

1. $-\mathrm{e}^{2x} + 5\mathrm{e}^{x} + 4 = 0$, 2. $2\mathrm{e}^{2x} - \mathrm{e}^{x} + 1 = 0$, 3. $\mathrm{e}^{2x} + \mathrm{e}^{x} - 1 = 0$,
4. $\mathrm{e}^{2x} + 2\mathrm{e}^{x} + 1 = 0$, 5. $\mathrm{e}^{2x} + 6\mathrm{e}^{x} + 8 = 0$, 6. $-\mathrm{e}^{2x} + \mathrm{e}^{x} + 1 = 0$,
7. $3\mathrm{e}^{2x} + 5\mathrm{e}^{x} + 2 = 0$.

Let a and b be two real numbers which we want to compare between them. We consider one of the following inequalities:

$$\ln(a) \leq \ln(b), \quad \ln(a) \geq \ln(b), \quad \ln(a) \geq b.$$

Using only the continuity and the variations of ln, we will be able to treat the two first one. For the last one we need to use the function exp to define the set of solutions.

5.3 Solutions of Some Inequalities

We start this subsection by studying the last inequation. Let a be a fixed real number. We consider the following inequation $\ln(x) \geq a$. Using the function exp we obtain

$$\ln(x) \geq a \iff \exp\big(\ln(x)\big) \geq \exp(a) \iff x \geq \exp(a).$$

Hence the solutions are the real numbers that are gratter than $\exp(a)$.

Lemma 5.3.1. *Let a be a real number. Then, the solution of the inequation $\ln(x) \geq a$ is the set of all positive real numbers that are gratter than $\exp(a)$. We denote $S = [\,\mathrm{e}^a\,,\,+\infty\,)$.*

Example 5.3.2. *Find the solution of the following inequation $\ln(x) \geq 3$.*

We know that this is equivalent to $x \geq \exp(3)$. This implies $S = [\exp(3)\,,\,+\infty)$.

Example 5.3.3. *Find x, such that $\ln(x+5) \geq 7$.*

The function $\ln(x+5)$ is defined if and only if $x + 5 > 0$. This means that $x > -5$. In other words $\ln(x+5)$ is defined if and only if $x \in (-5\,,\,+\infty)$. On the other hand, we know that

$$\ln(x+5) \geq 7 \iff x+5 \geq \mathrm{e}^7 \iff x \geq \mathrm{e}^7 - 5.$$

Then,

$$S = (-5\,,\,+\infty) \cap [\exp(7) - 5\,,\,+\infty) = [\exp(7) - 5\,,\,+\infty[.$$

In the same way we can establish the following lemma

Lemma 5.3.4. *Let a be a real number. Then, the solution of this inequation $\ln(x) \leq a$ is the interval $(\,0\,,\,\mathrm{e}^a]$.*

Example 5.3.5. *Let x be a strictly positive real number such that*

$$\ln(x) \leq 2.$$

Using the function $\exp$ we obtain $x \leq \exp(2)$. Therefore we have $S = (\,0\,,\,\exp(2)\,]$.

Example 5.3.6. *Find the solution of the inequation $\ln(x) \leq 5$.*

Using the lemma above we can see that the solutions of this inequation is the set of all real numbers x, such that, $x \leq \exp(5)$. Hence, we have $S = (0\,,\exp(5)\,]$.

Example 5.3.7. *We Consider the following inequality $\ln(x+2) \leq 1$. The function $\ln(x+2)$ is defined in $(-2\,;\,+\infty)$. On the other hand, we have*

$$\ln(x+2) \leq 1 \iff x+2 \leq \mathrm{e}^1 \iff x \leq \mathrm{e} - 2.$$

This means that, $S = (-2\,;\,\mathrm{e}-2]$.

Exercise 5.3.8. *Determine the solutions of the following inequalities*

1. $\ln(x^2+x+1) \geq 0$, 2. $\ln(x+5) \geq 3$, 3. $\ln(2x+3) \geq -1$,
4. $\ln(2x+5) \leq 3$, 5. $\ln(x^2+3x+2) \leq 0$, 6. $\ln(x+5) \leq 1$.

In the next paragraph we focus on the inequalities of the form

$$\ln(X) \geq \ln(a) \quad \text{or} \quad \ln(x) \leq \ln(a),$$

for some positive real numbers X and a.

Now we fixed a positive real number a. We are looking for real numbers x such that, $\ln(x) \geq \ln(a)$. Since the function ln is continuous and strictly increasing the inequality $\ln(x) \geq \ln(a)$ is equivalent to $x \geq a$. The solutions of the inequation $\ln(x) \geq \ln(a)$ are the set of all real numbers x which are gratter than a. Hence $S = [\,a\,,\,+\infty\,[$.

Lemma 5.3.9. *Let a be a fixed positive real number. Then, the solutions are defined by the interval $[\,a\,,\,+\infty\,)$.*

Example 5.3.10. *We consider the inequation: find a real number x such that,* $\ln(x) \geq \ln(3)$.

The solutions of this inequation is gigen by the interval $[\,3\,,\,+\infty\,)$.

Example 5.3.11. *Find the solutions of the inequation* $\ln(x+1) \geq \ln(2)$.

Since ln *is an increasing function, we have $x+1 \geq 2$. Therefore $x \geq 1$. Hence*

$$S =]-1\,,\,+\infty\,) \cap [\,1\,,\,+\infty\,) = [\,1\,,\,+\infty\,).$$

Similar results can be established for the inequation $\ln(x) \leq \ln(a)$ for some fixed positive real number a.

Lemma 5.3.12. *If a is a fixed positive real number, then , the solutions of inequation* $\ln(x) \leq \ln(a)$ *are $S = (\,0\,,\,a\,]$.*

Example 5.3.13. *We consider the following inequation* $\ln(x) \leq \ln(5)$. *Then, the solution of the inequation are defined by $S = (\,0\,,\,5\,]$.*

Example 5.3.14. *Find x, such that,* $\ln(x+1) \leq \ln(2)$. *The function* $\ln(x+1)$ *is defined in* $(-1\,;\,+\infty)$. *Since*

$$\ln(x+1) \leq \ln(2) \iff x+1 \leq 2 \iff x \leq 1,$$

then,

$$S = (-1\,,\,+\infty\,) \cap (-\infty\,,\,1\,] = (-1\,,\,1\,].$$

Exercise 5.3.15. *Define the set of solutions for the following inequalities*

1. $\ln(3x+1) \leq \ln(x+1)$,
2. $\ln(x+3) \geq \ln(-x+2)$,
3. $\ln(x^2+6x+5) \geq \ln(2)$,
4. $\ln(x^2+4x+1) \leq \ln(3)$.

Let a and b be two fixed real numbers. We wounder if there exist x in $\mathbb{R}$ such that

$$\exp(x) \leq a \quad \text{or} \quad \exp(x) \geq b.$$

To make the presentation clear, we separate the two inequations and we start with the inequation, find x in $\mathbb{R}$, such that, $\exp(x) \leq a$, where a is a fixed real number. If the real number $a \leq 0$, then there is no solutions. Because for every $x \in \mathbb{R}$ we have $\exp(x) > 0$.

Lemma 5.3.16. *If a is a real number which is less than or equal to 0, then, the inequation* $\exp(x) \leq a$ *does not solutions.*

Now we suppose a strictly positive. In this case the inequality $\exp(x) \leq a$ is equivalent to

$$\ln(\mathrm{e}^x) \leq \ln(a) \iff x \leq \ln(a).$$

Consequently $S = (\,-\infty\,,\,\ln(a)\,]$. This proves the following lemma

Lemma 5.3.17. *If a is a strictly positive real number, then, the solutions of the inequation* $\exp(x) \leq a$ *are given by the interval* $(-\infty\,,\,\ln(a)\,]$.

Example 5.3.18. *Find x in $\mathbb{R}$ such that* $\exp(x+1) \leq -6$. *Since for all $x \in \mathbb{R}$,we have* $\exp(x+1) > 0$. *Therefore there is no solution. Hence $S = \emptyset$.*

Example 5.3.19. *Find $x \in \mathbb{R}$, such that,* $\exp(x+2) < 11$. *This inequality is equivalent to* $x+2 < \ln(11)$. *Therefore,* $x < \ln(11) - 2$. *We have* $S = (-\infty\,, \ln(11) - 2\,[$.

Exercise 5.3.20. *Define the set of solution for the following inequalities*

1. $\exp(2x+9) < 5$, 2. $\exp(x+4) \leq -2$, 3. $\exp(x^2+2x-1) \leq 3$,
4. $\exp(x+5) < 7$, 5. $\exp(x+1) \leq -3$.

Now we focus on the inequation $\exp(x) \geq b$, where b is a fixed real number. When the real number b is such that $b \leq 0$, then for every x in $\mathbb{R}$, we have $\exp(x) > 0 \geq b$. This means that any real number satisfies the inequality. Therefore $S = \mathbb{R}$.

When the real number b is strictly positive, the following chain of equivalence holds

$$\exp(x) \geq b \iff \ln\big(\exp(x)\big) \geq \ln(b) \iff x \geq \ln(b).$$

Therefore the set of solutions is equal to the set of all real numbers which are greater than $\ln(b)$. In this case we have $S = [\,\ln(b)\,, +\infty\,)$.

Example 5.3.21. *Find the solution of the following inequation* $\exp(x+1) \geq -2$.
Since $-2 < 0$, *then,* $S = \mathbb{R}$.

Example 5.3.22. *We consider the inequation* $\exp(x+2) \geq 6$.
This is equivalent to $x + 2 \geq \ln(6)$. *This leads to the following inequality* $x \leq \ln(6) - 2$. *We obtainS* $= [\ln(6) - 2\,, +\infty)$.

Exercise 5.3.23. *Find the solution of the following inequalities*

1. $\exp(x) \geq 0$, 2. $\exp(3x+5) > 8$, 3. $\exp(x+9) > 4$,
4. $\exp(x^2+2x+1) \geq 7$, 5. $\exp(-x+2) \geq -1$.

Let $x > 0$. We consider the functions $f(x) = \ln(x) - x$ and $g(x) = \ln(x)/x$. We aim to find $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to+\infty} g(x)$. If we calculate separately the limit of each term in f, we will get

$$\lim_{x\to+\infty} \ln(x) = +\infty \quad \text{and} \quad \lim_{x\to+\infty} -x = -\infty.$$

In this case we have

$$\lim_{x\to+\infty} f(x) = +\infty - \infty = \text{undefined}.$$

For the function g we point out that

$$\lim_{x\to+\infty} \ln(x) = +\infty \quad \text{and} \quad \lim_{x\to+\infty} x = +\infty.$$

Therefore

$$\lim_{x\to+\infty} \frac{\ln(x)}{x} = \frac{+\infty}{+\infty} = \text{undefined}.$$

To find the limits of these functions, we need to compare them to polynomial functions. This will be done in the next subsection.

5.3.1 Some Particular Limits

At the beginning of this section we defined the limit of the function ln. We said that it goes to $+\infty$ when x goes to $+\infty$. Now we want to know how fast it goes to $+\infty$. To this end we compare it to the function x^n. To make the presentation easy to read, we state the following lemmas.

Lemma 5.3.24. *For any $x \in (0, +\infty)$ we have $\ln(x) \leq x$.*

Proof. To prove this lemma, we define the function $f(x) = \ln(x) - x$. The function f is differentiable in $(0, +\infty)$ and we have

$$f'(x) = \frac{1}{x} - 1 = \frac{1-x}{x}.$$

Then, the following variational table holds

x	0		1		$+\infty$
$f'(x)$	$\|$	$+$	0	$-$	
f	$\|$	$\nearrow$	-1	$\searrow$	

Since for all $x \in (0, +\infty)$, $f(x) \leq -1$, we have,

$$\ln(x) - x \leq 0 \quad \Rightarrow \quad \ln(x) \leq x.$$

□

The previous lemma is a particular case of the following theorem.

Theorem 5.3.25. *If $x \in (0, +\infty)$ and $n \geq 1$ is a natural number, then,*

$$\ln(x) \leq x^n. \tag{5.3.1}$$

Proof. To prove this theorem, we define $g(x) = \ln(x) - x^n$. The function g is differentiable in $(0, +\infty)$ and for every $x > 0$, we have

$$g'(x) = \frac{1}{x} - nx^{n-1} = \frac{1 - nx^n}{x}.$$

We know that $g'(x) = 0$ is equivalent to this identity $nx^n - 1 = 0$. This leads to the following equation : find $x > 0$, such that

$$x^n - \frac{1}{n} = 0$$

Setting $y = \sqrt[n]{\frac{1}{n}}$, we obtain

$$x^n - \frac{1}{n} = x^n - y^n = (x-y)\Big(x^{n-1} + x^{n-2}\cdot y + \cdots + x\cdot y^{n-2} + y^{n-1}\Big) = 0.$$

For every $x > 0$, the factor $\Big(x^{n-1} + x^{n-2}\cdot y + \cdots + x\cdot y^{n-2} + y^{n-1}\Big) > 0$. Hence $nx^n - 1 = 0$ is equivalent to $x - y = 0$. This implies

$$x = \sqrt[n]{\frac{1}{n}}\,.$$

From this we deduce $x^n - y^n$ and $x - y$ have the same sign. Since

$$g\left(\sqrt[n]{\frac{1}{n}}\right) = \frac{1}{n}\ln\left(\frac{1}{n}\right) - \frac{1}{n} = -\frac{\ln(n)}{n} - \frac{1}{n} = -\frac{\ln(n)+1}{n} < 0.$$

The following variational table holds,

x	0		$\sqrt[n]{\frac{1}{n}}$		$+\infty$
$g'(x)$	$+$		0	$-$	
g		$\nearrow$	$-\frac{\ln(n)+1}{n}$	$\searrow$	

The variational table implies, for all $x \in]\,0\,,\,+\infty\,[$, we have $g(x) \leq -\dfrac{\ln(n)+1}{n}$. Therefore

$$\ln(x) - x^n \leq 0 \Longrightarrow \ln(x) \leq x^n.$$

This completes the proof. □

These two lemmas leads to the following theorem.

Theorem 5.3.26. *If $n \geq 1$ is a natural number, then, for any $x > 0$, we have,*

$$\lim_{x\to+\infty} \frac{\ln(x)}{x^n} = 0. \tag{5.3.2}$$

If we take $n = 1$ in this theorem above, we obtain

$$\lim_{x\to+\infty} \frac{\ln(x)}{x} = 0.$$

Proof. Let $x \in [\,1\,,\,+\infty\,)$. We consider a natural number $n \geq 1$. Then, the following inequalities hold $0 \leq \ln(x) \leq x$. Since $x \geq 1$, we can divide by x^n. Dividing by x^n one gets

$$0 \leq \frac{\ln(x)}{x^n} \leq \frac{1}{x^{n-1}}.$$

We know that,

$$\lim_{x\to+\infty} \frac{1}{x^{n-1}} = 0.$$

Applying Sandwich theorem, we conclude that

$$\lim_{x\to+\infty} \frac{\ln(x)}{x^n} = 0.$$

□

Example 5.3.27. *Find the limit of the following sequence:*

$$u_n = \sqrt[n]{n} = n^{1/n}, \quad \forall\, n \geq 1.$$

We know that, $u_n = \exp\left(\ln(n)/n\right)$ and

$$\lim_{n\to+\infty} \frac{\ln(n)}{n} = 0.$$

Consequently, we have

$$\lim_{n\to+\infty} \sqrt[n]{n} = \lim_{n\to+\infty} \mathrm{e}^{\ln(n)/n} = \mathrm{e}^0 = 1.$$

Exercise 5.3.28. *Determine the following limits*

$$1.\ \lim_{n\to+\infty} n^{1/2n}, \quad 2.\ \lim_{n\to+\infty} n^{2/n}, \quad 3.\ \lim_{n\to+\infty} n^{5/4n}, \quad 4.\ \lim_{x\to+\infty} \frac{\ln(x)}{x^4}.$$

Let $x \in (0\,,+\infty)$. We consider a natural number $n \geq 1$. Using the properties of the function ln we can write on one hand

$$\ln(x) = \frac{n}{n}\ln(x).$$

On the other hand, we have $\ln(x^n) = n\ln(x)$. This implies

$$\frac{n}{n}\ln(x) = \frac{1}{n}\ln(x^n).$$

Since, we know that $\ln(x) = -\ln\left(\dfrac{1}{x}\right)$. We deduce that,

$$\ln(x) = -\frac{1}{n}\ln\left(\frac{1}{x^n}\right)\cdot$$

This leads to the following identity

$$x^n\ln(x) = \frac{1}{n}x^n\ln(x^n) = -\frac{1}{n}x^n\ln\left(\frac{1}{x^n}\right)\cdot$$

Now, we define $X := \frac{1}{x^n}$, then, $\lim_{x\to 0} X = +\infty$. As we know that,

$$x^n\ln(x) = -\frac{1}{n}\frac{\ln(X)}{X}\cdot$$

Therefore,

$$\lim_{x\to 0} x^n\ln(x) = -\frac{1}{n}\lim_{X\to+\infty}\frac{\ln(X)}{X} = 0\cdot$$

This allows us to state the following theorem.

Theorem 5.3.29. *If $n \geq 1$ is a natural number and $x \in (0\,,+\infty)$, then,*

$$\lim_{x\to 0} x^n\ln(x) = 0.$$

Setting $n = 1$ in the theorem above, we are able to establish the following corollary

Corollary 5.3.30. *For any x in $(0\,,+\infty)$, we have*

$$\lim_{x\to 0} x\ln(x) = 0.$$

Example 5.3.31. *To determine the limit* $\lim_{x\to -1}(x+1)\ln(x+1)$*, we define* $X := x+1$*. Hence, one has*

$$\lim_{x\to -1} X = 0.$$

From this we deduce that

$$\lim_{x\to -1}(x+1)\ln(x+1) = \lim_{X\to 0} X\ln(X) = 0.$$

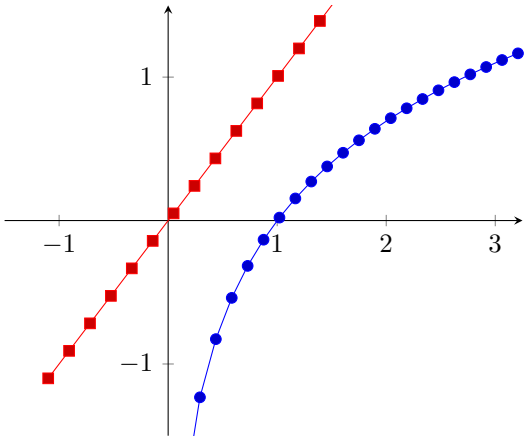
Exercise 5.3.32. *Specify the following limits*

1. $\lim_{x\to -2}(x+2)\ln(x+2)$, 2. $\lim_{x\to 1}(x^2-1)\ln(x-1)$, 3. $\lim_{x\to 3}(x-3)\ln(x-3)$.

Remark 5.3.33. *What this subsection established clearly is the following properties:*

1. *polynomial functions increase more fast than* ln,
2. *polynomial functions go to* 0 *more than* ln *goes to* $-\infty$.

To illustrate this, we represent below the graph of the functions $f(x) = x$ *and* $g(x) = \ln(x)$.



We remind that, we defined ln, such that, $\forall x \in \mathbb{R}$, $\ln(\mathrm{e}^x) = x$. This means that, $\ln(e) = \ln(e^1) = 1$. Now we introduce the following notations

$$\log_{\mathrm{e}}(x) = \frac{\ln(x)}{\ln(\mathrm{e})} = \ln(x).$$

We say that ln is the logarithm in basis e. Let $a > 0$ be a real number. The expression $\ln(a)$ make a sense. From now on we assume $a \neq 1$. In this case we can define the logarithm in basis a in this way

$$\log_a(x) = \frac{\ln(x)}{\ln(a)}. \tag{5.3.3}$$

In the next section we will study the function $\log_a$.

5.4 Logarithm in Basis a

We start this section by reminding the definition of the logarithm in basis a.

Definition 5.4.1. *Let $a \in (0\,,\,1) \cup (1\,,\,+\infty)$ be a fixed real number. For any $x \in (0\,,\,+\infty)$ we define the logarithm in basis a by*

$$\log_a(x) = \frac{\ln(x)}{\ln(a)}\cdot$$

We specify the following notations

- when $a = \mathrm{e}$, we use the notation $\log_{\mathrm{e}}(x) = \ln(x)$
- when $a = 10$, we denote $\log_{10}(x) = \log(x)$

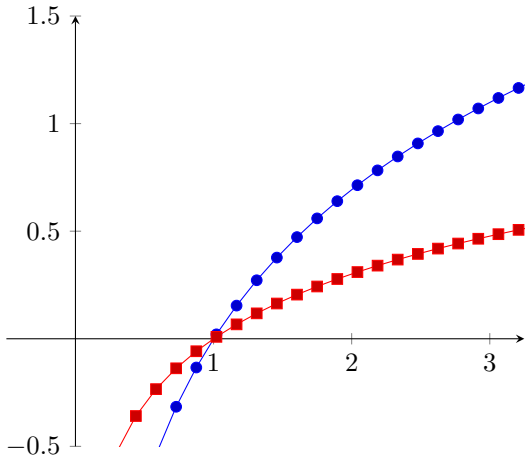
We remark that for all $a \in (0\,,\,1) \cup (1\,,\,+\infty)$, we have

$$\log_a(a) = \frac{\ln(a)}{\ln(a)} = 1\cdot$$

Example 5.4.2. *As an illustration we give the following examples*

$$1.\ f(x) = \log(x) = \frac{\ln(x)}{\ln(10)}, \quad 2.\ g(x) = \log_3(x) = \frac{\ln(x)}{\ln(3)},$$
$$3.\ h(x) = \log_{100}(x) = \frac{\ln(x)}{\ln(100)}\cdot$$

Below we represent the graph of the function ln in blue and log in red



Theorem 5.4.3. *If $a \in (0\,,\,1) \cup (1\,,\,+\infty)$ and for all x and y in $(0\,,\,+\infty)$,then, the function $\log_a$ satisfies the following properties*

1. $\log_a(x \cdot y) = \log_a(x) + \log_a(y)$
2. $\log_a\left(\dfrac{x}{y}\right) = \log_a(x) - \log_a(y)$

3. $\log_a(x^n) = n\log_a(x)$.

Proof. Let x and y be in $(0\,;\,,+\infty)$. Then, we have

$$\log_a(x\cdot y) = \frac{\ln(x\cdot y)}{\ln(a)} = \frac{\ln(x)+\ln(y)}{\ln(a)} = \log_a(x)+\log_a(y).$$

To prove 2, we point out that

$$\log_a\left(\frac{x}{y}\right) = \frac{\ln\left(\dfrac{x}{y}\right)}{\ln(a)} = \frac{\ln(x)-\ln(y)}{\ln(a)} = \log_a(x)-\log_a(y).$$

To prove 3, we observe that,

$$\log_a(x^n) = \frac{\ln(x^n)}{\ln(a)} = \frac{n\ln(x)}{\ln(a)} = n\log_a(x).$$

□

Proposition 5.4.4. *Let $a \in (0\,,\,1)\cup(1\,,\,+\infty)$. Then, for any natural number $n \geq 1$ we have*

$$\log_a(a^n) = n.$$

The proof of this proposition is simple. We need just to point out that,

$$\log_a(a^n) = \frac{\ln(a^n)}{\ln(a)} = \frac{n\ln(a)}{\ln(a)} = n\cdot$$

Proposition 5.4.5. *Let $a \in (0\,,\,1)\cup(1\,,\,+\infty)$. Then, for any , $x \in (\,0\,,\,+\infty\,)$*

$$a^{\log_a(x)} = x$$

The proof of this proposition is based on the fact that

$$a^{\log_a(x)} = \exp\left(\log_a(x)\ln(a)\right).$$

Since $\log_a(x)\ln(a) = \dfrac{\ln(x)}{\ln(a)}\ln(a) = \ln(x)$. Then, $a^{\log_a(x)} = \exp\left(\ln(x)\right) = x$.

Example 5.4.6. *For instance we have*

$$\log(100) = \log(10^2) = 2 \quad \textit{and} \quad 10^{\log(2.5)} = 2.5.$$

Let $a > 0$ be a real number which we assume different to 1. We define the function $\log_a(x)$ for any x in $(\,0\,,\,+\infty\,)$. The next step consist of studying such type of functions

5.5 Study of Functions $\log_a$

Let$a \in (0\,,\,1)\cup(1\,,\,+\infty)$. For any real number x which is strictly positive we define the function

$$f(x) = \log_a(x) = \frac{\ln(x)}{\ln(a)}\cdot$$

The function f is differentiable in its domain. For every real number x in $(0, +\infty)$, the derivative of f is defined by

$$f'(x) = \frac{1}{x \ln(a)}.$$

Since the real number x belongs to $(0, +\infty)$, we observe that the sign of f' depends on $\ln(a)$. Hence we have to distinguish two cases

Case 1: $a \in (0, 1)$. In this case we see that $\ln(a) < 0$. From this we deduce that $f(x) < 0$, for every $x \in (0, +\infty)$. Therefore the function f is decreasing in $(0, +\infty)$

Case 2: $a \in (1, +\infty)$. In this case, the real number $\ln(a)$ is strictly positive. Then, we deduce that $f'(x) > 0$, for every $x \in (0, +\infty)$. This means the function f is increasing in $(0, +\infty)$.

Theorem 5.5.1. *If $a \in (0, 1)$, the function $f(x) = \log_a(x)$ is decreasing in $(0, +\infty)$.*

Theorem 5.5.2. *If $a \in (1, +\infty)$, the function $f(x) = \log_a(x)$ is increasing in $(0, +\infty)$.*

Before ending this chapter, we specify some definitions. Indeed, in the previous chapter we studied the function exp. We called it exponential. To be rigourous we should called it the Naperian exponential in order to show that it is the reciprocal function to the naperian logarithm.

Using the function $\log_a$ for some positive real number a different to 1, we can define general exponentials.

5.6 General Exponentials

Let $a > 0$ be a real number. If $a = 1$, we have $a^x = 1$ for any real number x. Now assume that $a \neq 1$, we have $\log_a(a^x) = x$ for any real number x. On the other hand if we take a strictly positive real number x we will have $a^{\log_a(x)} = x$.

Now we take $x \in (0, +\infty)$, and we define $f(x) = \log_a(x)$. The function f is continuous and strictly monotone in $(0, +\infty)$, then it admits a reciprocal function g, such that,

$$\begin{cases} \log_a(g(y)) = y, & \forall\, y \in \mathbb{R} \\ \\ g(\log_a(x)) = x, & \forall\, x > 0. \end{cases}$$

As we know that the reciprocal function of $\log_a$ is uniqe, then, the function g should correspond to a^x. Hence we define

$$g(x) = a^x, \quad \forall\, x \in \mathbb{R}.$$

This function is called the general exponential .

Definition 5.6.1. *Let a be a strictly positive real number which is different to 1. For any real number x we call exponential of x the function*

$$f(x) = a^x,$$

defined, such that, $\log_a\left(a^x\right) = x$.

Thus if x and y are two real and a is as in the definition, we have

$$\log_a(a^x \cdot a^y) = \log_a(a^x) + \log_a(a^y) = x + y. \tag{5.6.1}$$

On the other part, we kow that if x and y are two real numbers, then,

$$x + y = \log_a(a^{x+y}). \tag{5.6.2}$$

From (5.6.1) and (5.6.2), we deduce that

$$a^x \cdot a^y = a^{x+y},$$

for any real number x and y. This proves the following proposition

Proposition 5.6.2. *Let $a \in (0\,,\,1) \cup (1\,,\,+\infty)$. Then, for any real numbers x and y, we have*

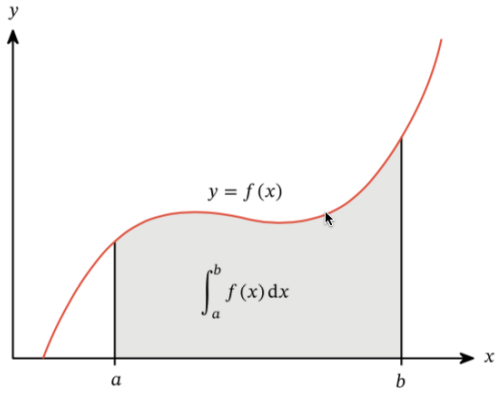
$$a^x \cdot a^y = a^{x+y}.$$

CHAPTER 6

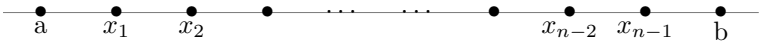
INTRODUCTION TO INTEGRATION

Contents

To introduce this chapter, we consider a real function f defined in some interval I. We represent this function by its graph. Let $[\,a\,,\,b\,]$ be a subinterval of I. Our main goal now is to calculate the area of the domain $\mathscr{D}$, [the part of the real plan] which is between the vertical lines $x = a$, $x = b$, the $\mathrm{x}-$axis, and the curve $\mathscr{C}_f$ of f. In other words we aim to find the colored area of the figure below.



Since from elementary school we know how to find the area of a rectangle, the first idea which comes up to our mind is to subdivide the domain in many small rectangles in this way: Let $[\,a\,,\,b\,]$ be an interval. We divide $[\,a\,,\,b\,]$ to n subinterval $[\,x_i\,,\,x_{i+1}\,]$, where $x_0 = a$, $x_n = b$ and for $0 \leq i \leq n-1$ we have $x_{i+1} = x_i + \dfrac{b-a}{n}$.



Since $x_i = x_{i-1} + \dfrac{b-a}{n}$ we obtain after i iterations

$$x_i = x_0 + i\,\frac{b-a}{n}. \tag{6.0.1}$$

The elements of the set $\{\,x_0,\,x_1,\cdots,x_{n-1},x_n\}$ satisfy the following property

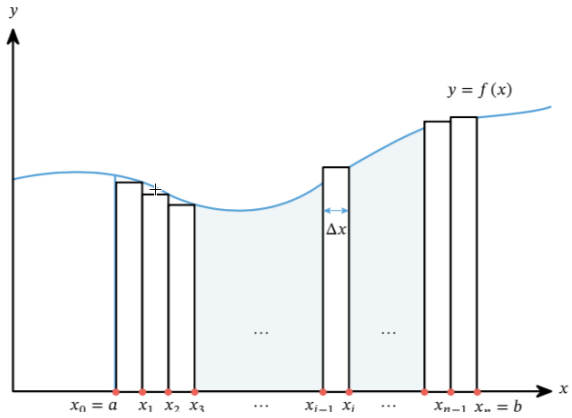
$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

The finite family $\{\,x_0,\,x_1,\cdots,x_{n-1},x_n\}$ is called a subdivision of the interval $[\,a\,,\,b\,]$. The real number

$$\Delta x = x_{i+1} - x_i = \frac{b-a}{n} \tag{6.0.2}$$

is the step of the subdivision.

Now we decompose the domain $\mathscr{D}$ into n small rectangles with lenght Δx [where Δx is a small real number] and height $f(x_i)$ for some real number x_i. The vertices of the small rectangle number $i+1$ $(0 \leq i \leq n)$ are the points $(x_i\,,\,0)$, $(x_{i+1}\,,\,0)$, $(x_i\,,\,f(x_{i+1})\,)$ and $(x_{i+1}\,,\,f(x_{i+1})\,)$. Another choce is possible we can take $(x_i\,,\,0)$, $(x_{i+1}\,,\,0)$, $(x_i\,,\,f(x_i)\,)$ and $(x_{i+1}\,,\,f(x_i)\,)$. as the vertices of the rectangle number $i+1$



Then, the area of the small rectangle number $i+1$ is $\mathcal{R}_{i+1} = \Delta x \cdot f(x_{i+1})$. To find the area of the whole domain, we need just to sum the area of all small rectangles. In this case we obtain

$$\text{Area of all Small Rectangles} = \sum_{i=0}^{n-1} \Delta x \cdot f(x_{i+1}) = \sum_{i=0}^{n-1} \mathcal{R}_{i+1}. \tag{6.0.3}$$

As we can observe it , the area of all small rectangle is just an approximation of the area of the domain. This approximation is as good as the the step Δx is small enough. This suggests us to let Δx go to 0. But the problem here is, the relation between Δx and n. The real number Δx goes to 0 means that n should go to $+\infty$. This leads to an infinite sum, which we do not know how to deal with. When the infinite sum has a mathematical meaning [when $\sum \mathcal{R}_i$ converges], we write

$$\text{Area of } \mathscr{D} = \lim_{n \to +\infty} \sum_{i=1}^{n} \mathcal{R}_i = \sum_{i=1}^{\infty} \mathcal{R}_i \tag{6.0.4}$$

If this limit exists we take it as the exact value of the area of the domain $\mathscr{D}$. We denote it the *integral* of the function f in the interval $[\,a\,,\,b\,]$. We write

$$\text{Area of } \mathscr{D} = \int_a^b f(x)\,dx. \tag{6.0.5}$$

We mean by (6.0.5): Area of $\mathscr{D}$ is equal to the integral from a to b of f. The symbol $\int$ is the letter S. We use it to mean we are summing from a to b.

In this chapter, we will only deal with elementary theory of integrations. To define the value of an *integral* we introduce the notion of primitive of a function. In a second time, we will study some properties of *integrals*. To end this chapter, we calculate *integral* of some elementary functions.

6.1 Primitive of Functions

This section is devoted to primitive of functions. After elementary definitions, we study primitives of elementary functions. In the second subsection we deal with primitives of general functions.

Definition 6.1.1. *Let $f : I \to \mathbb{R}$ be a continuous function. We consider a differentiable function $F : I \to \mathbb{R}$. We say that the function F is a primitive of f in I, if for every real number x in I,*

$$F'(x) = f(x). \tag{6.1.1}$$

Remark 6.1.2. *We draw the readers attention that primitive and integral have similar meanings. But here we reserved the term integral for another use.*

Example 6.1.3. *To help readers to better understand the notion of primitives, we introduce the following examples*

1. Let a be any real number. We define the function $F(x) = a$. The function F is differentiable and for every real number x, $F'(x) = 0$. Therefore the function F is a primitive of $f(x) = 0$.

2. Let a and b be two real numbers. We define the affine function $F(x) = ax+b$. It is differentiable in $\mathbb{R}$ and for every real number x we have $F'(x) = a$. Then, the function F is a primitive of the function $f(x) = a$.

3. Let x be a real number. We consider the function $F(x) = \dfrac{x^{n+1}}{n+1}\cdot$ Since the function F is a monomial, it is differentiable and for every real number x we have $F'(x) = x^n$. This implies the function F is a primitive of $f(x) = x^n$.

From definition 6.1.1 we know what is a primitive. Now we need to define a condition which allows us to have a primitive. A necessary condition is given in the lemma below.

Lemma 6.1.4. *Let f be a continuous function in I. Then the function f has a primitive defined in the interval I.*

We deduce from this lemma that, a continuous function has several primitives. Now we want to know what type of relations can exist between these primitives.

Theorem 6.1.5. *Let f defined in I be a continuous function. We consider F and G two primitives of the function f in I. Then, there exists a constant real number C, such that,*

$$G(x) = F(x) + C, \quad \forall\, x \in I. \tag{6.1.2}$$

Proof. Let F and G be two primitives of f in I. They are differentiable in I and for every real number x in I, we have $F'(x) = f(x) \quad \text{and} \quad G'(x) = f(x)$. For every

x in I we define the function $H(x) = G(x) - F(x)$, Since the functions F and G are differentiable in I, the function H is differentiable in I. In this case we have

$$H'(x) = G'(x) - F'(x) = f(x) - f(x) = 0.$$

From this, we deduce that H is a constant function. Consequently there exists a real number C, such that, $H(x) = G(x) - F(x) = C$. This means that $G(x) = F(x) + C$. This completes the proof of the theorem. □

Example 6.1.6. *We consider the function $f(x) = 2x + 1$. Then, the function $F(x) = x^2 + x$ is a primitive of f. We can find another primitive of f by adding any real number to F. For example the functions $G(x) = x^2+x+3$ and $H(x) = x^2+x+5$ are primitives of the functions f. Generally speaking, for any real number C the function $P(x) = x^2 + x + C$ is a primitive of f.*

We learn from the theorem 6.1.5

1. a primitive of a function is defined up to a constant
2. a functions has infinite number of primitives

Because of these two remarks, we cannot write the function F is *the* primitive of the function f. But we must say the function F is *a* primitive of f

At this level we know what is the definition of a primitive. We define also the relation between two primitives of a function. The next step will consist of finding primitives of some elementary functions .

6.1.1 Primitive of Elementary Functions

As we pointed it out in the previous chapters elementary functions are the bricks used to build a large part of functions. That is why we start with them here in order to show how our theory runs. In this section we will define primitives of polynomial functions, square root functions, inverse, ln and exp.

Let C be any constant real number. For every x in $\mathbb{R}$ we define the function $F(x) = C$. We emphasize the function F is differentiable in $\mathbb{R}$ and for every real number x the derivative of the function F is defined by $F'(x) = 0$. Hence we have proved the following lemma.

Lemma 6.1.7. *Let x be a real number. We define the function $f(x) = 0$. Then, the function $F(x) = C$, where C is any constant real number is a primitive of the function f.*

Example 6.1.8. *For every real number x, we consider the constant function $F(x) = 13$. It is a primitive of the function $f(x) = 0$, for every $x \in \mathbb{R}$.*

Now we take two real numbers a and b, such that, $a \neq 0$. For x in $\mathbb{R}$ we define the function $F(x) = ax + b$. Since the function F is affine, then it is differentiable in $\mathbb{R}$ and for every $x \in \mathbb{R}$, we have $F'(x) = a$. This leads to the following lemma.

Lemma 6.1.9. *Let a be a constant real numbers which is different to 0. For all $x \in \mathbb{R}$, we define the constant function $f(x) = a$. Then, the function $F(x) = ax+C$, where C is a constant real number, is a primitive of the function f .*

Example 6.1.10. *We consider the function $f(x) = 7$. Then, the affine function $F(x) = 7x + 1$ is a primitive of f. Another primitive of the function f can be given by $G(x) = 7x + C$, where C is any constant real number.*

Exercise 6.1.11. *Define primitives to the following functions*

1. $f(x) = 11$, 2. $g(x) = 3$, 3. $h(x) = 5$, 4. $u(x) = C$, 5. $v(x) = 2$.

Let x be a real number . We consider the function $F(x) = \frac{x^2}{2}\cdot$ This function is differentiable in $\mathbb{R}$ and for all $x \in \mathbb{R}$, we have $F'(x) = x$. This establishes the following lemma.

Lemma 6.1.12. *Let x belong to $\mathbb{R}$ and $f(x) = x$. Then, the function $F(x) = \frac{x^2}{2}$ is a primitive of f.*

We consider a natural number $n \geq 2$. For every real number x we define the $F(x) = \frac{x^{n+1}}{n+1}\cdot$ Then, the function F is differentiable and for all $x \in \mathbb{R}$, $F'(x) = x^n$. This leads to the following lemma.

Lemma 6.1.13. *Let $n \geq 2$ be a natural number. For x in $\mathbb{R}$ we consider the function $f(x) = x^n$. Then, the function $F(x) = \frac{x^{n+1}}{n+1}$ is a primitive of f.*

Example 6.1.14. *We consider the function $f(x) = x^3$. A primitive of f can be defined by $F(x) = \frac{x^{3+1}}{3+1} = \frac{x^4}{4}\cdot$*

Exercise 6.1.15. *Define a primitive for each of the following function*

1. $f(x) = x^5$, 2. $g(x) = x^7$, 3. $h(x) = x^{11}$, 4. $u(x) = 3x^2$, 5. $v(x) = x^{k_0}$.

Let $x \neq 0$ be a real number. We define the function $F(x) = \frac{1}{x}\cdot$ We know that F is differentiable in $\mathbb{R}^*$ and for every real number $x \neq 0$, we have $F'(x) = -\frac{1}{x^2}\cdot$ Consequently we can state the following result.

Lemma 6.1.16. *Let $x \neq 0$ be a real number. For any $x \in \mathbb{R}^*$ we define the function $f(x) = \frac{1}{x^2}\cdot$ Then, the function $F(x) = -\frac{1}{x}$ is a primitive of f.*

Let $n \geq 2$ be a natural number and x belong to $\mathbb{R}\backslash\{0\}$. We define the function $F(x) = -\frac{1}{(n-1)x^{n-1}}\cdot$ This function is differentiable in its domain and for all real number $x \neq 0$ we have $F'(x) = \frac{1}{x^n}\cdot$

Lemma 6.1.17. *Let $n \geq 2$ be a natural number. For every real number $x \neq 0$ we define the function $f(x) = \frac{1}{x^n}\cdot$ Then, the function $F(x) = -\frac{1}{(n-1)x^{n-1}}$ is a primitive of f.*

Example 6.1.18. *For every real number $x \neq 0$ we consider the function $f(x) = \frac{1}{x^6}\cdot$ A primitive of this function f is defined by $F(x) = -\frac{1}{5x^5}\cdot$*

Exercise 6.1.19. *Determine a primitive to each of the following functions*

$$1.\ f(x)=\frac{1}{x^5}\ ,\quad 2.\ g(x)=\frac{1}{x^9}\ ,\quad 3.\ h(x)=-\frac{7}{x^8}\ ,\quad 4.\ u(x)=\frac{1}{x^7}\cdot$$

Let $x \geq 0$ be a real number. We define the function $F(x)=2\sqrt{x}$. This function is differentiable in $[\,0\,,\,+\infty\,)$. For every $x>0$ we have $F'(x)=\dfrac{1}{\sqrt{x}}\cdot$ Therefore, we have the following result.

Lemma 6.1.20. *Let $x>0$ be a real number. We define the function $f(x)=\dfrac{1}{\sqrt{x}}\cdot$ Then, the function $F(x)=2\sqrt{x}$ is a primitive of f.*

We consider the two following functions $F(x)=-\cos(x)$ and $G(x)=\sin(x)$. These two functions are differentiable in $\mathbb{R}$. For $x\in\mathbb{R}$, we have $F'(x)=\sin(x)$ and $G'(x)=\cos(x)$.

Lemma 6.1.21. *Let $f(x)=\sin(x)$. Then, $F(x)=-\cos(x)$ is a primitive of f.*

Lemma 6.1.22. *Let $g(x)=\cos(x)$. Then, $G(x)=\sin(x)$ is a primitive of g.*

Let $x>0$ be a real number. We consider the function $F(x)=\ln(x)$. The function F is differentiable in $(\,0\,,\,+\infty\,)$, and for every $x>0$, we have $F'(x)=\dfrac{1}{x}\cdot$ Then, we can state the following lemma

Lemma 6.1.23. *Let $x>0$ be a real number. We define the function $f(x)=\dfrac{1}{x}\cdot$ Then, the function $F(x)=\ln(x)$ is a primitive of f.*

Let $x\in\mathbb{R}$. We define $F(x)=\mathrm{e}^x$. This function is differentiable and we have $F'(x)=\mathrm{e}^x$.

Lemma 6.1.24. *Let $f(x)=\mathrm{e}^x$. Then, the function $F(x)=\mathrm{e}^x$ is a primitive of f.*

In the following table we sum up some primitives of elementary functions.

$f(x)$	Domain of f	$F(x)$	Domain of F
0	$\mathbb{R}$	C	$\mathbb{R}$
a	$\mathbb{R}$	$ax+C$	$\mathbb{R}$
x	$\mathbb{R}$	$\dfrac{x^2}{2}+C$	$\mathbb{R}$
x^n	$\mathbb{R}$	$\dfrac{x^{n+1}}{n+1}+C$	$\mathbb{R}$
$\dfrac{1}{x^2}$	$\mathbb{R}\backslash\{0\}$	$-\dfrac{1}{x}+C$	$\mathbb{R}\backslash\{0\}$
$\dfrac{1}{x^n},\ n\geq 2$	$\mathbb{R}\backslash\{0\}$	$-\dfrac{1}{n-1}\cdot\dfrac{1}{x^{n-1}}+C$	$\mathbb{R}\backslash\{0\}$
$\dfrac{1}{\sqrt{x}}$	$(\,0\,,\,+\infty\,)$	$2\sqrt{x}+C$	$[\,0\,,\,+\infty\,)$
$\cos(x)$	$\mathbb{R}$	$\sin(x)+C$	$\mathbb{R}$
$\sin(x)$	$\mathbb{R}$	$-\cos(x)+C$	$\mathbb{R}$
$\exp(x)$	$\mathbb{R}$	$\exp(x)+C$	$\mathbb{R}$
$\dfrac{1}{x}$	$(\,0\,,\,+\infty\,)$	$\ln(x)+C$	$(\,0\,,\,+\infty\,)$

Example 6.1.25. *We consider $f(x) = x^5$. Using the table above, we set $n = 5$. A primitive of f is given by $F(x) = \frac{x^{5+1}}{5+1} + C = \dfrac{x^6}{6} + C$, where C is a constant.*

Example 6.1.26. *For every real number $x \neq 0$ we define the function $g(x) = \dfrac{1}{x^3}$. Setting $n = 3$, we have $G(x) = -\dfrac{1}{3-1} \cdot \dfrac{1}{x^{3-1}} + C = -\dfrac{1}{2\,x^2} + C$.*

Exercise 6.1.27. *Define a primitives for each of the following functions*

$$1.\ f(x) = x^7, \qquad 2.\ g(x) = 0, \qquad 3.\ h(x) = 3, \qquad 4.\ i(x) = \frac{1}{x^3}$$

$$5.\ j(x) = \sin(x), \qquad 6.\ u(x) = \frac{1}{x^{11}} \qquad 7.\ \ell(x) = x^4.$$

Theorem 6.1.28. *Let f and g be two continuous functions in some interval I with respective primitives F and G. We consider a real number k and the functions*

$$h(x) = k \cdot f(x) \quad \textit{and} \quad \ell(x) = f(x) + g(x).$$

Then,

- *the function $H(x) = k \cdot F(x) + C$, is a primitive of h*
- *the function $L(x) = F(x) + G(x) + C$ is a primitive of ℓ,*

where C is a constant real number.

Proof. Let f, g, F, G and k be as in the theorem. We define $H(x) = k \cdot F(x) + C$. Then, H is differentiable and for every $x \in I$, we have $H'(x) = k \cdot F'(x) = k \cdot f(x)$. This implies the function H is a primitive of $k \cdot f$.

Now we set $L(x) = F(x) + G(x) + C$. The function L is a differentiable and for every $x \in I$, we have $L'(x) = F'(x) + G'(x) = f(x) + g(x)$. This involves the function $F + G$ is a primitive of $f + g$. □

Example 6.1.29. *Find a primitive of the following function $f(x) = 2x^2 + 3x + 1$. We know that a primitive of x^2 is given by $\dfrac{x^3}{3}$. The function $\dfrac{x^2}{2}$ is a primitive of the function x and the function x is a primitive of 1. Therefore,*

$$F(x) = \frac{2x^3}{3} + \frac{3x^2}{2} + x$$

is a primitive of f. In this example we have taken $C = 0$.

Example 6.1.30. *We consider the following function for every real number in $(\,0\,,\,+\infty\,)$,*

$$g(x) = \cos(x) + 4x^3 + 5x + 2 + \frac{1}{x} + \frac{1}{x^2} + \mathrm{e}^x,$$

Since the function g is continuous in $(\,0\,,\,+\infty\,)$, g has primitives. A primitive of g can be defined by

$$G(x) = \sin(x) + x^4 + \frac{5}{2}x^2 + 2x + \ln(x) - \frac{1}{x} + \mathrm{e}^x + C.$$

Exercise 6.1.31. *Find the domain of the following functions and define a primitive for each them in its domain*

1. $f(x) = x^7 + x^6 + 2x^5 + 3x^4$, 2. $f_1(x) = 4x^3 + 3x^2 + \frac{1}{2}x + 7$,
3. $f_2(x) = 2\cos(x)$, 4. $g(x) = \cosh(x) + \frac{1}{x^2} + \sin(x)$, 5. $g_1(x) = \frac{1}{x^3} - \frac{5}{x} + x$,
6. $g_2(x) = \sin(x) + \sinh(x) + 4\mathrm{e}^x + \frac{2}{x^5} + x^2$, 7. $h(x) = 4x^3 - 3x + 5 + \frac{2}{x^2} + \mathrm{e}^x$.

An easy way to construct general functions is to use composition of elementary functions. In the next section, we will define primitives of general functions from primitives of elementary functions.

6.1.2 Primitive of General Functions

This section deals with primitives of general functions. Indeed here we study primitives of function of the forms u^n, $\sqrt{u}$, $\frac{1}{u}$, $\cos(u)$, $\sin(u)$. We specify, here u is a real function. These functions are obtained by using composition of elementary functions and u.

We consider a differentiable function $u : \mathbb{R} \longrightarrow \mathbb{R}$. For any natural number $n \geq 1$, we define the function $F(x) = \dfrac{u^{n+1}(x)}{n+1}\cdot$ We know from previous chapters, the function F is differentiable in D_u. Because it is a composition of differentiable functions. Let $x \in D_u$, then, we have, $F'(x) = u'(x)\, u^n(x)\cdot$ This leads to the following theorem.

Theorem 6.1.32. *Let $u : \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function . For any natural number $n \geq 1$ and $x \in D_u$, we define the function $f(x) = u'(x)u^n(x)$. Then, the function $F(x) = \dfrac{u^{n+1}(x)}{n+1}$ is a primitive of f.*

If we take $n = 1$ in the theorem above we obtain the following corollary.

Corollary 6.1.33. *Let $u : \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function. For every $x \in D_u$ we define the function $f(x) = u'(x)u(x)$. Then, the function $F(x) = \dfrac{u^2(x)}{2}$ is a primitive of f.*

Example 6.1.34. *For every real number x we define the function $f(x) = 3(3x+1)^5$.*

To determine a primitive of f we define the function $u(x) = 3x+1$. The function u is differentiable in $\mathbb{R}$. Because, it is an affine function. For all $x \in \mathbb{R}$, we get $u'(x) = 3$. Setting $n := 5$, we observe that f is in the form $f(x) = u'(x)u^5(x)$. Therefore, the function $F(x) = \dfrac{u^6(x)}{6} = \dfrac{(3x+1)^6}{6}$ is a primitive of f.

Example 6.1.35. *Let f be the function defined by $f(x) = (2x+2)(x^2+2x+3)$. To find a primitive to the function f we define the real function $u(x) = x^2 + 2x + 3$. This function is differentiable in $\mathbb{R}$ and for all $x \in \mathbb{R}$, we have $u'(x) = 2x + 2$. This involves $f(x) = u'(x)\, u(x)$. Therefore the function $F(x) = \dfrac{u^2(x)}{2} = \dfrac{(x^2+2x+3)^2}{2}$ is a primitive of f.*

Example 6.1.36. *We consider the function* $f(x) = (2x+5)^{11}$. *To define a primitive to the function* f *we set* $u(x) = 2x + 5$. *Then the function* u *is differentiable in* $\mathbb{R}$ *and* $u'(x) = 2$. *From this, we deduce that,*

$$f(x) = u^{11} = \frac{1}{2} \cdot 2\,(2x+5)^{11} = \frac{1}{2} \cdot u'(x)u^{11}(x).$$

Hence a primitive of f *is defined by* $F(x) = \dfrac{1}{2} \cdot \dfrac{u^{12}(x)}{12} = \dfrac{(2x+5)^{12}}{24}$.

Remark 6.1.37. *In general if the function* f *is in the form* $f(x) = (ax+b)^n$, *with a real number* $a \neq 0$. *A primitive of the function* f *is*

$$F(x) = \frac{1}{a} \cdot \frac{u^{n+1}(x)}{n+1} + C,$$

where C *is a real number.*

Exercise 6.1.38. *Define a primitive for each of the following functions.*

1. $f(x) = 2(2x+3)^3$, 2. $g(x) = (3x^2+2x+1)(x^3+x^2+x+11)^4$,
3. $h(x) = (x+3)(x^2+6x+2)$ 4. $u(x) = (5x+3)^2$, 5. $v(x) = (4x+9)$.

Let $u : I \longrightarrow \mathbb{R}^*$ be a differentiable function. For every $x \in I$ we define the function $F(x) = \dfrac{1}{u(x)}$. Then, the function F is differentiable in I and we have

$$F'(x) = -\frac{u'(x)}{u^2(x)}.$$

Lemma 6.1.39. *Let* $u : I \longrightarrow \mathbb{R}^*$ *be a differentiable function. For every* $x \in I$, *we consider the function* $f(x) = \dfrac{u'(x)}{u^2(x)}$. *Then, the function* $F(x) = -\dfrac{1}{u(x)}$ *is a primitive of* f.

Example 6.1.40. *We consider the function* $f(x) = \dfrac{2x+3}{(x^2+3x+4)^2}$. *To find a primitive to* f, *we define the function* $u(x) = x^2 + 3x + 4$. *The function* u *is differentiable in* $\mathbb{R}$ *and* $u'(x) = 2x + 3$. *This means,* $f(x) = \dfrac{u'(x)}{u^2(x)}$. *Therefore, the following function* $F(x) = -\dfrac{1}{x^2+3x+4}$ *is a primitive of* f.

Exercise 6.1.41. *Find a primitive to each of the following function*

1. $f(x) = \dfrac{5}{(5x+6)^2}$, 2. $g(x) = \dfrac{6x^2+2x+1}{(2x^3+x^2+x)^2}$, 3. $h(x) = \dfrac{1}{(4x+3)^2}$.

Now we take a natural $n \geq 2$ and a function $u : I \longrightarrow \mathbb{R}^*$ which is differentiable differentiable in I. We define the function $F(x) = \dfrac{-1}{(n-1)u^{n-1}(x)}$. Since the function F is a composition of differentiable functions it is differentiable in I. For every real number x, which belongs to I, we can establish

$$F'(x) = \frac{u'(x)}{u^n(x)}.$$

Theorem 6.1.42. *Let $u : I \longrightarrow \mathbb{R}^*$ be a differentiable function. For any natural number $n \geq 2$ and $x \in I$, we define the function $f(x) = \dfrac{u'(x)}{u^n(x)}\cdot$ Then, the function $F(x) = \dfrac{-1}{(n-1)u^{n-1}(x)}$ is a primitive of f.*

Example 6.1.43. *Let $f(x) = \dfrac{6}{(6x+1)^4}\cdot$ We know the domain of f is $D = \mathbb{R}\backslash\{-1/6\}$. To determine a primitive to the function f, we set $u(x) = 6x + 1$. The function u is differentiable in $\mathbb{R}$, then,it is differentiable in $\mathbb{R}\backslash\{-1/6\}$ and $u'(x) = 6$. From this , we deduce that, for all $x \neq \dfrac{-1}{6}$, $f(x) = \dfrac{u'(x)}{u^4(x)}\cdot$ Applying the theorem above, we obtain $F(x) = \dfrac{-1}{3(6x+1)^3}\cdot$*

Exercise 6.1.44. *Find a primitive to each of the following function*

1. $f(x) = \dfrac{4}{(4x+3)^3}$, 2. $g(x) = \dfrac{3x^2+10x+2}{(x^3+5x^2+2x+1)^7}$, 3. $h(x) = \dfrac{1}{(3x+5)^5}\cdot$

Let $u : I \longrightarrow (0\,;+\infty)$ be a differentiable functions . We consider the function $F(x) = \ln(u(x))$. We know the function F is differentiable as a composition of differentiable functions. For all $x \in I$, the derivative of F is

$$F'(x) = \frac{u'(x)}{u(x)}\cdot$$

Theorem 6.1.45. *Let $u : I \longrightarrow (\,0\,,+\infty\,)$ be a differentiable function. We define $f(x) = \dfrac{u'(x)}{u(x)}\cdot$ Then, the function $F(x) = \ln(u(x))$ is a primitive of f.*

Example 6.1.46. *Let $x \in (\,2\,,+\infty\,)$ and $f(x) = \dfrac{2}{2x-4}\cdot$ We define the function $u(x) = 2x - 4$. The function u is differentiable and for every $x \in (\,2\,,+\infty\,)$, we get $u'(x) = 2$. Hence the function f is in the form $f(x) = \dfrac{u'(x)}{u(x)}$. Therefore the real function $F(x) = \ln(2x-4)$ is a primitive of f.*

Exercise 6.1.47. *Determine a primitive to each of the following functions.*

1. $f(x) = \dfrac{2x}{(x^2+1)}$, 2. $g(x) = \dfrac{2}{(2x+5}$, 3. $h(x) = \dfrac{1}{6x+5}$, 4. $u(x) = \dfrac{x^2+1}{x^3+3x}$,
5. $v(x) = \dfrac{1}{7x}$, 6. $w(x) = \dfrac{2x}{3x^2+2}$, 7. $z(x) = \dfrac{5}{3x-2}\cdot$

Now we consider the function $F(x) = \exp(u(x))$ where $u : I \longrightarrow \mathbb{R}$ is any differentiable real function. The function F is differentiable in I and for every real number x in I, $F'(x) = u'(x)\exp(u(x))$.

Theorem 6.1.48. *Let $u : I \longrightarrow \mathbb{R}$ be a differentiable function. We define the function $f(x) = u'(x)\exp(u(x))$. Then, the function $F(x) = \exp(u(x))$ a primitive of f.*

Example 6.1.49. *We consider the function* $f(x) = 2\exp(2x+5)$ *and we define* $u(x) = 2x+5$. *The function* u *is differentiable and for every real number* x *we have* $u'(x) = 2$. *Consequently we obtain* $f(x) = u'(x)\exp(u(x))$. *This implies the function* $F(x) = \exp(2x+5)$ *is a primitive of* f.

Exercise 6.1.50. *Define a primitive to each of the following function*

$$1.\ f(x) = x\mathrm{e}^{x^2+1}, \quad 2.\ g(x) = 4\mathrm{e}^{4x+6}, \quad 3.\ h(x) = \mathrm{e}^{6x+1},$$
$$4.\ u(x) = x^3\mathrm{e}^{x^4+3}, \quad 5.\ v(x) = (x^2+2x+1)\mathrm{e}^{x^3+3x^2+3x+1}.$$

Let us consider a differentiable function $u : I \longrightarrow (\,0\,,+\infty\,)$. For every x in the interval I we introduce the function $F(x) = 2\sqrt{u(x)}$. Since u is differentible in I and the square root function is differentible in $(\,0\,,+\infty\,)$, the function F is differentiable in I.

Let x belong to I, then, we obtain

$$F'(x) = \frac{u'(x)}{\sqrt{u(x)}}\cdot$$

Theorem 6.1.51. *Let* $u : I \longrightarrow (\,0\,,+\infty\,)$ *be a differentiable function. For every* $x \in I$ *we define* $f(x) = \dfrac{u'(x)}{\sqrt{u(x)}}\cdot$ *Then, the function* $F(x) = 2\sqrt{u(x)}$ *is a primitive of* f.

Example 6.1.52. *We consider the following function* $f(x) = \dfrac{2}{\sqrt{2x+4}}$, *which is defined in* $(\,-2\,,+\infty\,)$. *Setting* $u(x) = 2x+4$, *one has* $u'(x) = 2$. *The function* f *is in the following form:* $f(x) = \dfrac{u'(x)}{\sqrt{u(x)}}$. *Since the function* f *is continuous it has primitive and a primitive of* f *is defined by* $F(x) = 2\sqrt{2x+4}$.

Example 6.1.53. *Let* $f(x) = \dfrac{2x+1}{\sqrt{x^2+x+1}}\cdot$ *The function* f *is defined in* $\mathbb{R}$. *Now we take* $u(x) = x^2+x+1$, *the derivative of* u *is* $u'(x) = 2x+1$. *Hence the function* $f(x) = \dfrac{u'(x)}{\sqrt{u(x)}}\cdot$ *As the function* f *is continuous in* I *it has primitives. The function* $F(x) = 2\sqrt{x^2+x+1}$ *is a primitive of* f.

Exercise 6.1.54. *Define a primitive to each of the following functions*

$$1.\ f(x) = \frac{1}{\sqrt{3x+2}}\,, \quad 2.\ g(x) = \frac{2x+3}{\sqrt{x^2+5x+1}}\,, \quad 3.\ h(x) = \frac{5}{\sqrt{7x+1}}\cdot$$

Let $u : I \longrightarrow \mathbb{R}$ be a differentiable function. For every real number x in I, we define $F(x) = \cos(u(x))$. Since u is differentiable in I and cos is differentiable in $\mathbb{R}$, the function $\cos(u(x))$ is differentiable in I. For any $x \in I$, we have $F'(x) = -u'(x)\sin(u(x))$. Then the following theorem holds.

Theorem 6.1.55. *We consider a differentiable function* $u : I \longrightarrow \mathbb{R}$. *For* $x \in I$, *we define the function* $f(x) = u'(x)\sin(u(x))$. *Then, the function* $F(x) = -\cos(u(x))$ *is a primitive of* f.

Example 6.1.56. *Let* $f(x) = 5\sin(5x+3$. *We set* $u(x) = 5x+3$. *The function* u *is differentiable in* $\mathbb{R}$ *and* $u'(x) = 5$. *We obtain* $f(x) = u'(x)\sin(u(x))$. *Since* f *is a continuous in* $\mathbb{R}$, *it has primitives. A primitive of* f *is defined by the function* $F(x) = -\cos(5x+3)$.

Example 6.1.57. *We consider the following function* $f(x) = \sin(3x+2)$. *We define* $u(x) = 3x+2$. *Then,* $u'(x) = 3$. *From this, we define*

$$f(x) = \frac{1}{3}\cdot 3\,\sin(3x+2) = \frac{1}{3}u'(x)\sin(u(x)).$$

Since f *is a continuous function, it admits primitives. A primitive of* f *is given by* $F(x) = -\dfrac{1}{3}\cos(3x+2)$.

Exercise 6.1.58. *Find a primitive to each of the following functions*

1. $f(x) = (2x+1)\sin(x^2+x+3)$, 2. $g(x) = \sin(6x+1)$, 3. $h(x) = -\sin(3x)$.

Now, we consider a differentiable function $u : I \to \mathbb{R}$. For any $x \in I$, we define the function $F(x) = \sin(u(x))$. Since u is differentiable in I and sin is differentiable in $\mathbb{R}$, then, $\sin(u(x))$ is differentiable in I. We have $F'(x) = u'(x)\cos(u(x))$. This proves the following theorem

Theorem 6.1.59. *Let* $u : I \longrightarrow \mathbb{R}$ *be a differentiable function. We consider the function* $f(x) = u'(x)\cos(u(x))$. *Then, the function* $F(x) = \sin(u(x))$ *is a primitive of* f.

Example 6.1.60. *We define* $f(x) = 6\cos(6x+1)$. *The function* f *is continuous in* $\mathbb{R}$. *We define* $u(x) = 6x+1$. *Therefore* $u'(x) = 6$. *The function* f *is in the form* $f(x) = u'(x)\cos(u(x))$. *Hence* $F(x) = \sin(6x+1)$ *is a primitive of* f.

Exercise 6.1.61. *Find a primitive to each of the following function.*

1. $f(x) = 25\cos(5x+3)$, 2. $g(x) = \cos(x+3)$, 3. $h(x) = -\cos\left(\dfrac{5}{2}x+5\right)$.

To refresh memory to the readers, we give the following table.

$f(x)$	Domain of f	$F(x)$	Domain of F
$u'.u$	D_u	$\dfrac{u^2}{2}$	D_u
$u'.u^n$	D_u	$\dfrac{u^{n+1}}{n+1}$	D_u
$\dfrac{u'}{u^2}$	D_1	$-\dfrac{1}{u}$	D_1
$\dfrac{u'}{u^n},\ n \geq 2$	D_1	$\dfrac{-1}{(n-1)u^{n-1}},$	D_1
$\dfrac{u'}{\sqrt{u}}$	D_3	$2\sqrt{u(x)}$	D_2
$u'.\sin(u)$	D_u	$-\cos(u)$	D_u
$u'.\cos(u)$	D_u	$\sin(u)$	D_u
$u'.\exp(u)$	D_u	$\exp(u)$	D_u
$\dfrac{u'}{u}$	D_3	$\ln(u)$	D_3

Here we define $D_1 = \{x \in \mathbb{R} : u(x) \neq 0\}$, $D_2 = \{x \in \mathbb{R} : u(x) \geq 0\}$ and

$$D_3 = \{x \in \mathbb{R} : u(x) > 0\}.$$

One may ask, why we are trying to determine primitives of functions. Why we engage in such complicated work . Do we need to spend so much time in finding such primitives. To answer this question, we consider a function F which is differentiable in some interval I, such that $F'(x) = f(x)$. Let x_0 belong to I. For some real number $h > 0$, we define the quantity

$$\Delta F = \frac{F(x_0 + h) - F(x_0)}{x_0 + h - x_0} = \frac{F(x_0 + h) - F(x_0)}{h}.$$

For h small enough [h tends to 0 but remains different to 0], we have

$$\Delta F = \frac{F(x_0 + h) - F(x_0)}{h} \simeq f(x_0).$$

Multiplying by h we obtain $F(x_0+h)-F(x_0) \simeq h \cdot f(x_0)$. To simplify the presentation we assume $f(x_0) > 0$. Then the positive real $h \cdot f(x_0)$ is the area of the rectangle that has vertices the points $(x_0\,,\,0)$, $(x_0 + h\,,\,0)$, $(x_0\,,\,f(x_0)\,)$ and $(x_0 + h\,,\,f(x_0)\,)$. This area is an approximation of the function in the interval $[\,x_0\,,\,x_0 + h\,]$. In other words, when h goes to 0, we have

$$F(x_0 + h) - F(x_0) = \int_{x_0}^{x_0+h} f(t)\,dt \tag{6.1.3}$$

As we can see it identity (6.1.3) justifies why we went to so much trouble to define primitives. In the next section we show the close link that exists between *primitives* and *integrals*.

6.2 Intriduction to Integration

This section deals with integration. Indeed , here we define an *integral*. We also specify that *integrals* are just some particular primitives. We will define some properties of integrals. To end this section we introce integration by parts and integration by using substitution method.

6.2.1 Defitions and Notations

Let f be a continuous function in $[a\,,\,b]$, with $a < b$. We denote

$$\int_a^b f(x)dx, \tag{6.2.1}$$

to mean the integral from a to b of f. One may wonder how to find the value of (6.2.1). That is why we state the following definition.

Definition 6.2.1. *Let f be a continuous function in $[a\,,\,b]$. We consider a function F which is a primitive of f in $[a\,,\,b]$. We define the integral from a to b of f by*

$$\int_a^b f(x)dx = \Big[F(x)\Big]_a^b = F(x)\Big|_a^b = F(b) - F(a). \tag{6.2.2}$$

Remark 6.2.2. *We make the following remarks*

- *The name of the variable x in which we integrate does not matter. We can integrate in the variable y or u or t. Indeed, we have*

$$\int_a^b f(x)\,dx = \int_a^b f(y)\,dy = \int_a^b f(t)\,dt = \int_a^b f(u)\,du$$

- *To find the value of the integral of f from a to b, $\int_a^b f(x)\,dx$, we should follow at least three steps:*
  - *Step 1: to define the function F a primitive of the function f*
  - *Step 2: we compute $F(a)$ and $F(b)$.*
  - *Step 3: To finish we write*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

We make the precision that it is not necessary to observe these steps or to follow them by dot. They can be done in one step by writing directly the value of the integral.

Example 6.2.3. *To define the integral $\int_0^2 2x\,dx$ we set $f(x) = 2x$. Since f is an affine function, it is continuous in the interval $[0\,,\,2]$. Then it has primitives in $[\,0\,,\,2\,]$.*

- *Step 1: We know from what that a primitive of the function $2\,x$ can be defined by $F(x) = x^2$.*
- *Step 2: Now, we have to find $F(2)$ and $F(0)$, which are: $F(2) = 2^2 = 4$, $F(0) = 0^2 = 0$*
- *Step 3: In this step, we define the value of the integral:*

$$\int_0^2 2x\,dx = \Big[x^2\Big]_0^2 = x^2\Big|_0^2 = 2^2 - 0^2 = 4.$$

Example 6.2.4. *Compute the value of the following integral $\int_1^2 x^2 + 2x + 1\,dx$. We define the function $f(x) = x^2 + 2x + 1$. The function*

$$F(x) = \frac{x^3}{3} + x^2 + x$$

is a primitive of f.

On the other hand we have $F(2) = \frac{8}{3} + 4 + 2 = \frac{26}{3}$ and $F(1) = \frac{1}{3} + 1 + 1 = \frac{7}{3}$. Therefore we obtain

$$\int_1^2 (x^2 + x + 1)\,dx = F(2) - F(1) = \frac{26}{3} - \frac{7}{3} = \frac{19}{3}\cdot$$

Exercise 6.2.5. *Define the value of the following integrals*

$$\int_0^{3\pi} \cos(x)\,dx, \quad \int_2^{\mathrm{e}} \frac{1}{x}\,dx, \quad \int_3^5 \frac{1}{\sqrt{x+3}}\,dx, \quad \int_0^7 x\,\mathrm{e}^{x^2+3}\,dx, \quad \int_1^3 (3x+2)^2\,dx.$$

Before going further we specify the following notations.

Theorem 6.2.6. *Let f be continuous function in the interval $[a\,,\,b]$. For every real number x in $[a\,,\,b]$ we define the function*

$$F(x) = \int^{x} f(t)\,dt \tag{6.2.3}$$

Then, the function F is a primitive of f.

In the next subsection, we study properties of integrals

6.2.2 Some Properties of Integrals

In this section, we deal with some properties of integrals. We do not pretend to give the full list of such properties.

Let f be a continuous function in $[a\,,\,b]$. We consider the function F which is a primitive of f in $[a\,,\,b]$. Using the definition above one deduces

$$\int_a^a f(x)\,dx = F(a) - F(a) = 0.$$

Lemma 6.2.7. *Let f be a continuous function in the interval $[a\,,\,b]$. Then, we have*

$$\int_a^a f(x)\,dx = 0$$

Example 6.2.8. *We have $\int_2^2 x^2 + 2x\,dx = 0$.*

Lemma 6.2.9. *Let f be a continuous function in the interval $[a\,,\,b]$. Then, the following property holds*

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

Proof. Let f be a continuous function in $[a\,,\,b]$. Then, the function f has primitives. We take one among such primitives and we denote it F. Therefore, we have

$$\int_b^a f(x)\,dx = F(a) - F(b) = -\Big(F(b) - F(a)\Big) = -\int_a^b f(x)\,dx.$$

□

Example 6.2.10. *We have*

$$\int_2^1 \frac{1}{x}\,dx = -\int_1^2 \frac{1}{x}\,dx = -\ln(2).$$

Lemma 6.2.11. *Let f be a continuous function in $[a\,,\,b]$ and k a real number. Then,*

$$\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx.$$

Proof. Let f be a continuous function in $[a\,,\,b]$. Therefore f has primitives. We consider the function F as one of such primitives. Then, for any real number k, the function $k\,F$ is a primitive of $k\,f$. Consequently

$$\int_a^b k\,f(x)\,dx = k\,F(b) - k\,F(a) = k\,(F(b) - F(a)) = k\int_a^b f(x)\,dx.$$

□

Example 6.2.12. *Let $f(x) = 3\,\sin(x)$, we have*

$$I = \int_0^\pi 3\,\sin(x)\,dx = 3\int_0^\pi \sin(x)\,dx = 3\Big[-\cos(x)\Big]_0^\pi = 3(-\cos(\pi) + \cos(0)) = 6.$$

Exercise 6.2.13. *Determine the following integrals*

$$\int_0^2 2x\,dx, \qquad \int_2^3 \frac{5}{u}\,du, \qquad \int_5^7 \frac{3}{x^4}\,dx, \qquad \int_{\pi/6}^{\pi/2} \frac{1}{2}\cos(t)\,dt, \qquad \int_0^2 4\mathrm{e}^y\,dy.$$

Lemma 6.2.14. *Let f and g be two continuous functions in $[a\,,\,b]$. Then,*

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

Proof. We consider two functions F and G that are respective primitives of f and g. The function $F + G$ is a primitive of the function $f + g$. This implies that,

$$\begin{aligned}\int_a^b f(x) + g(x)\,dx &= F(b) + G(b) - F(a) + G(a) = F(b) - F(a) + G(b) - G(a)\\ &= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.\end{aligned}$$

□

Example 6.2.15. *We consider the following integral*

$$\int_0^1 x^2 + 3x + 1\,dx.$$

To compute this integral , we proceed as follows

$$\begin{aligned}\int_0^1 (x^2 + x + 1)\,dx &= \int_0^1 x^2\,dx + \int_0^1 3x\,dx + \int_0^1 dx = \left[\frac{x^3}{3}\right]_0^1 + \left[\frac{3x^2}{2}\right]_0^1 + \Big[x\Big]_0^1\\ &= \frac{1}{3} + \frac{3}{2} + 1 = \frac{17}{6}.\end{aligned}$$

Exercise 6.2.16. *Determine the values of the following integrals*

$$\int_0^\pi x^2 + \sin(x) + \mathrm{e}^x\,dx, \qquad \int_1^3 \frac{1}{x} + \cos(x) + \frac{1}{\sqrt{x}}\,dx, \qquad \int_5^1 \frac{1}{x^3} + \frac{1}{x^2}\,dx.$$

Exercise 6.2.17. *Compute the values of the following integrals*

$$\int_0^\pi x^2 + \frac{1}{x} + x + \mathrm{e}^x\,dx, \qquad \int_1^3 \frac{1}{\sqrt{x}} + x^3 + 3x + 1\,dx, \qquad \int_6^1 \mathrm{e}^x + x^2\,dx.$$

Theorem 6.2.18 (Chasles Relation)**.** *Let f be a continuous function in $[a\,,\,b]$. We consider some real number c in $(a\,,\,b)$. Then, we have*

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx.$$

Proof. Let F be a primitive of f in $[a\,,\,b]$. Therefore we have

$$\begin{aligned}\int_a^c f(x)\,dx + \int_c^b f(x)\,dx &= (F(c) - F(a)) + (F(b) - F(c)) = F(b) - F(a)\\ &= \int_a^b f(x)\,dx.\end{aligned}$$

□

Example 6.2.19. *As an example we take*

$$\begin{aligned}\int_{-\pi}^{\pi} -\sin(x)\,dx &= \int_{-\pi}^{0} -\sin(x)\,dx + \int_0^{\pi} -\sin(x)\,dx = \Big[\cos(x)\Big]_{-\pi}^{0} + \Big[\cos(x)\Big]_0^{\pi}\\ &= (\cos(0) - \cos(-\pi)) + (\cos(\pi) - \cos(0)) = \cos(\pi) - \cos(-\pi) = 0\end{aligned}$$

Below we give three important theorems without proofs.

Theorem 6.2.20. *Let f be continuous function in $[a\,,\,b]$. For every real number x in $[a\,,\,b]$ we define the function*

$$F(x) = \int_a^x f(x)\,dx.$$

Then, the function F is the primitive of f, that satisfies , $F(a) = 0$.

Remark 6.2.21. *From this theorem, we deduce, any continuous function has a primitive.*

Theorem 6.2.22 (Mean Value Theorem)**.** *Let f be a continuous function in $[a\,,\,b]$. Then, there exists at least a real number c in $]a\,,\,b[$, such that,*

$$\mu = f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Remark 6.2.23. *We point out that, here, the real number $f(c)$ is the mean value of the function f in the interval $[a\,,\,b]$. In other words, if it was possible to list all values taken by f in $[a\,,\,b]$ and to calculate the average of these values, we will find $f(c)$. But here, we know that it is not possible to get the full list of all values taken by f in $[a\,,\,b]$.*

Example 6.2.24. *Let $f(x) = 2x + 1$. The mean value of f in $[1\,,\,3]$ is*

$$\mu = \frac{1}{3-1}\int_1^3 2x + 1\,dx = \frac{1}{2}\Big[x^2 + x\Big]_1^3 = \frac{1}{2}(9 + 3 - 1 - 1) = \frac{10}{2} = 5.$$

Exercise 6.2.25. *Determine the mean value of the following functions in the given interval I.*

1. $f(x) = \mathrm{e}^x + x, \quad I = [-1\,,\,1]$ 2. $g(x) = \dfrac{1}{x} + x^2, \quad I = [2\,,\,5]$
3. $h(x) = \sin(x), \quad I = \left[-\dfrac{\pi}{2}\,,\,\pi\right].$

Theorem 6.2.26 (Fundamental Theorem of Analysis)**.** *Let f be a differentiable function in $[a\,,\,b]$. Moreover, we assume the function f' is continuous in $[a\,,\,b]$. Then, for all x and y in $[a\,,\,b]$,*

$$f(y) - f(x) = \int_x^y f'(t)\,dt. \tag{6.2.4}$$

Here, we define a table which allows us to give integrals of some elementary functions . We consider a continuous function in $[a\,,\,b]$

$\int_a^b f(u(x))\,dx$	$F(b) - F(a)$
$\int_a^b u'(x).u(x)\,dx$	$\dfrac{u^2(b) - u^2(a)}{2}$
$\int_a^b u'.u^n(x)\,dx$	$\dfrac{u^{n+1}(b) - u^{n+1}(a)}{n+1}$
$\int_a^b \frac{u'(x)}{u^2(x)}\,dx$	$-\dfrac{1}{u(b)} + \dfrac{1}{u(a)}$
$\int_a^b \frac{u'(x)}{u^n(x)}\,dx, \quad n \geq 2$	$-\dfrac{1}{(n-1)u^{n-1}(b)} + \dfrac{1}{(n-1)u^{n-1}(a)}$
$\int_a^b \frac{u'(x)}{\sqrt{u(x)}}\,dx$	$2\sqrt{u(b)} - 2\sqrt{u(a)}$
$\int_a^b u'.\sin(u(x))\,dx$	$-\cos(u(b)) + \cos(u(a))$
$\int_a^b u'(x).\cos(u(x))\,dx$	$\sin(u(b)) - \sin(u(a))$
$\int_a^b u'(x).\exp(u(x))\,dx$	$\exp(u(b)) - \exp(u(a))$
$\int_a^b \frac{u'(x)}{u(x)}\,dx$	$\ln(u(b)) - \ln(u(a))$

Example 6.2.27. *To Determine the value of the following integral*

$$\int_1^2 3(3x+1)\,dx,$$

we define, $u(x) = 3x + 1$, then, $u'(x) = 3$. Hence,

$$\int_1^2 3\,(3x+1)\,dx = \int_1^2 u'(x)\,u(x)\,dx = \left[\frac{(3x+1)^2}{2}\right]_1^2 = \frac{49}{2} - \frac{16}{2} = \frac{33}{2}.$$

Exercise 6.2.28. *Determine the values of the following integrals*

$$\int_0^1 2(2x+3)\,dx, \quad \int_1^5 (2x+3)(x^2+3x+1)\,dx, \quad \int_2^3 (7x+5)\,dx.$$

Example 6.2.29. *To find the value of the following integral*

$$\int_0^1 2(2x+1)^3\,dx,$$

we set $u(x) = 2x + 1$*, then,* $u'(x) = 2$*. Therefore,*

$$\int_0^1 2(2x+1)^3\,dx = \int_1^2 u'(x)\,u^3(x)\,dx = \left[\frac{(2x+1)^4}{4}\right]_0^1 = \frac{3^4}{4} - \frac{1}{4} = 20\cdot$$

Exercise 6.2.30. *Determine the exact value of the following integrals*

$$\int_0^1 (3x^2+2x)(x^3+x^2)^4\,dx,\quad \int_1^2 (2x+3)(x^2+3x)^5\,dx,\quad \int_2^3 \frac{1}{2\sqrt{x}}(\sqrt{x}+4)^2\,dx.$$

Example 6.2.31. *Now we want to determine the following integral*

$$\int_1^2 \frac{2x}{(x^2+1)^2}\,dx.$$

To this end we define the function $u(x) = x^2 + 1$*. Then,* $u'(x) = 2x$*. From this one deduces,*

$$\int_1^2 \frac{2x}{(x^2+1)^2}\,dx = \int_1^2 \frac{u'(x)}{u^2(x)}\,dx = \left[-\frac{1}{x^2+1}\right]_1^2 = -\frac{1}{5} + \frac{1}{2} = \frac{3}{10}\cdot$$

Exercise 6.2.32. *Find the values of the following integrals*

$$\int_5^6 \frac{7}{(7x+1)^2}\,dx,\qquad \int_0^1 \frac{1}{(4x+1)^2}\,dx,\qquad \int_3^5 \frac{x^3+2x}{(x^4+4x^2+7)^2}\,dx.$$

Example 6.2.33. *We consider the following integral*

$$\int_0^1 \frac{4}{(4x+1)^5}\,dx.$$

To determine the value of the integral above we set $u(x) = 4x+1$*. The derivative of* u *is* $u'(x) = 4$*. This implies ,*

$$\int_0^1 \frac{4}{(4x+1)^5}\,dx = \int_0^1 \frac{u'(x)}{u^5(x)}\,dx = \left[-\frac{1}{(4x+1)^4}\right]_0^1 = -\frac{1}{4}\left(\frac{1}{5^4} - 1\right)\cdot$$

Exercise 6.2.34. *Determine the following integrals*

$$\int_2^3 \frac{2x+1}{(x^2+x+7)^3}\,dx,\qquad \int_0^1 \frac{1}{(5x+2)^6}\,dx,\qquad \int_3^4 \frac{1}{(3x+5)^4}\,dx.$$

Example 6.2.35. *We consider this integral*

$$\int_0^1 \frac{1}{\sqrt{x+1}}\,dx.$$

To compute its value we define the function $u(x) = x + 1$*. Then,* $u'(x) = 1$*. Therefore one has*

$$\int_0^1 \frac{1}{\sqrt{x+1}}\,dx = \int_0^1 \frac{u'(x)}{\sqrt{u(x)}}\,dx = \left[2\sqrt{x+1}\right]_0^1 = 2\sqrt{2} - 2\sqrt{1} = 2\sqrt{2} - 2.$$

Exercise 6.2.36. *Determine the following integrals*

$$\int_1^2 \frac{2x+1}{\sqrt{x^2+x+3}}\,dx, \qquad \int_2^3 \frac{1}{\sqrt{5x+1}}\,dx, \qquad \int_4^7 \frac{1}{\sqrt{2x+3}}\,dx.$$

Example 6.2.37. *To determine the value of the integral*

$$\int_0^\pi 4\sin(4x+\pi)\,dx,$$

we set $u(x) = 4x+\pi$. *Then, we have* $u'(x) = 4$. *From, this we deduce,*

$$\begin{aligned}\int_0^\pi 4\sin(4x+\pi)\,dx &= \int_0^\pi u'(x)\,\sin(u(x))\,dx = [-\cos(4x+\pi)]_0^\pi \\ &= -\cos(5\pi)+\cos(\pi) = -\cos(\pi)+\cos(\pi) = 0.\end{aligned}$$

Exercise 6.2.38. *Determine the value of the following integrals*

$$\int_{-\pi}^0 3\sin(3x))\,dx, \qquad \int_{\pi/6}^{\pi/2} \sin(2x)\,dx, \qquad \int_{\pi/4}^{\pi/2} \sin\left(\frac{1}{2}x+\frac{\pi}{4}\right)\,dx.$$

Example 6.2.39. *We consider the following integral*

$$\int_0^{\pi/2} 2\cos(2x)\,dx.$$

To find its value we define $u(x) = 2x$. *We have* $u'(x) = 2$. *Hence,*

$$\int_0^{\pi/2} 2\cos(2x)\,dx = \int_0^{\pi/2} u'(x)\cos(u(x))\,dx = [\sin(2x)\,]_0^{\pi/2} = 0.$$

Exercise 6.2.40. *Compute the following integrals*

$$\int_{-\pi}^{-\pi/2} 3\cos(3x+\pi)\,dx, \qquad \int_0^{\pi/6} \cos(4x)\,dx, \qquad \int_{\pi/4}^{3\pi/4} \cos(5x)\,dx.$$

Example 6.2.41. *To determine the following integral*

$$\int_2^3 2\mathrm{e}^{2x+3}\,dx,$$

we consider the function $u(x) = 2x+3$. *Then,* $u'(x) = 2$. *Hence, we have*

$$\int_2^3 2\mathrm{e}^{2x+3}\,dx = \int_2^3 u'(x)\,\mathrm{e}^{u(x)}\,dx = \left[\mathrm{e}^{2x+3}\right]_2^3 = \mathrm{e}^9 - \mathrm{e}^7.$$

Exercise 6.2.42. *Determine the following integrals*

$$\int_0^1 \mathrm{e}^{3x+5}\,dx, \qquad \int_2^3 \mathrm{e}^{x+2}\,dx, \qquad \int_2^3 \mathrm{e}^{7x-5}\,dx.$$

Example 6.2.43. *Now we consider the following integral*

$$\int_2^4 \frac{5}{5x+1}\,dx.$$

We define the function $u(x) = 5x+1$. *Therefore we obtain* $u'(x) = 5$. *Consequently,*

$$\int_2^4 \frac{5}{5x+1}\,dx = \int_2^4 \frac{u'(x)}{u(x)}\,dx = \Big[\ln(5x+1)\Big]_2^4 = \ln(21) - \ln(11) = \ln\left(\frac{21}{11}\right).$$

Exercise 6.2.44. *Find the value of the following integrals*

$$\int_2^3 \frac{2}{2x+5}\,dx, \qquad \int_0^1 \frac{x}{x^2+1}\,dx, \qquad \int_4^5 \frac{11}{11x+1}\,dx.$$

In the next subsection, we introduce one of the most famous technique to calculate integrals. In the theory of integration we call this method *integration by parts.*

6.3 Introduction to Integration by Parts

This section is devoted to inetgration by parts. Here we explain how runs this important method in the theory of integration.

Before defining integration by parts, we specify some notations.

Definition 6.3.1. *Let* f *be a real function defined in the interval* I. *We say the function* f *is of class* $\mathscr{C}^1$, *if it satisfies these properties*

- f *is differentiable in* I *and*
- *the derivative of* f, f' *is continuous in* I.

When the function f *is of class* $\mathscr{C}^1$, *we say* f *is a* $\mathscr{C}^1$ *function. We denote* $f \in \mathscr{C}^1(I)$.

Now, we consider two functions f and g which are of class $\mathscr{C}^1$ in the interval $[\,a\,,\,b\,]$ $\big[f \in \mathscr{C}\,([a\,;\,b])$ and $g \in \mathscr{C}^1\,([a\,;\,b])\,.\big]$ Since f and g are differentiable, the fnction $f \cdot g$ is differentiable in $[a\,,\,b]$. Let $x \in [a\,,\,b]$, then, we have

$$(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x).$$

From the identity above, we deduce the function $f \cdot g$ is a primitive of the function $h(x) = f'(x)\,g(x) + f(x)\,g'(x)$. Consequently, we obtain

$$\int_a^b h(x)\,dx = \Big[f(x)g(x)\Big]_a^b.$$

Replacing h by its expression, involves

$$\int_a^b (f'(x)g(x) + f(x)g'(x))\,dx = \Big[f(x)g(x)\Big]_a^b.$$

As we know ,

$$\int_a^b (f'(x)g(x) + f(x)g'(x))\,dx = \int_a^b f'(x)g(x)\,dx + \int_a^b f(x)g'(x)\,dx$$

Therefore,

$$\int_a^b f'(x)g(x)\,dx + \int_a^b f(x)g'(x)\,dx = \Big[f(x)g(x)\Big]_a^b.$$

This leads to the following identity

$$\int_a^b f'(x)g(x)\,dx = \Big[f(x)g(x)\Big]_a^b - \int_a^b f(x)g'(x)\,dx. \tag{6.3.1}$$

Theorem 6.3.2 (Integration by Parts)**.** *Let f and g be two functions of class $\mathscr{C}^1$ in $[a\,,\,b]$. Then, we have*

$$\int_a^b f'(x)g(x)\,dx = \Big[f(x)g(x)\Big]_a^b - \int_a^b f(x)g'(x)\,dx.$$

Example 6.3.3. *We consider the following integral*

$$I_0 = \int_1^2 x\ln(x)\,dx.$$

To find the value of this integral, we define $f'(x) = x$ and $g(x) = \ln(x)$. Therefore, we have $f(x) = \dfrac{x^2}{2}$ and $g'(x) = \dfrac{1}{x}$. Using (6.3.1)*, we obtain*

$$\begin{aligned}\int_1^2 x\ln(x)\,dx &= \left[\frac{x^2}{2}\ln(x)\right]_1^2 - \int_1^2 \frac{x^2}{2}\cdot\frac{1}{x}\,dx = \left(\frac{4}{2}\ln(2) - \frac{1}{2}\ln(1)\right) - \frac{1}{2}\int_1^2 x\,dx\\ &= 2\ln(2) - \frac{1}{2}\left[\frac{x^2}{2}\right]_1^2 = 2\ln(2) - \frac{1}{2}\left[\frac{4}{2} - \frac{1}{2}\right] = 2\ln(2) - \frac{3}{4}.\end{aligned}$$

Example 6.3.4. *Now we aim to find the value of the following integral*

$$\int_0^1 x\,\mathrm{e}^x\,dx.$$

To this end we set $f'(x) = \exp(x)$ and $g(x) = x$. Then, $f(x) = \exp(x)$ and $g'(x) = 1$ Applying theorem 6.3.2, we obtain

$$\int_0^1 x\mathrm{e}^x\,dx = \Big[x\mathrm{e}^x\Big]_0^1 - \int_0^1 \mathrm{e}^x\cdot 1\,dx = \Big(\mathrm{e} - 0\Big) - \int_0^1 \mathrm{e}^x\,dx = \mathrm{e} - \Big[\mathrm{e}^x\Big]_0^1 = 1.$$

Exercise 6.3.5. *Determine the value of the following integrals*

$$1.\ \int_0^\pi x\cos(x)\,dx, \qquad 2.\ \int_2^3 \frac{\ln(x)}{x^2}\,dx, \qquad 3.\ \int_0^5 x\,\sin(x)\,dx.$$

Now we consider the function $f(x) = \dfrac{x}{\sqrt{x^2+3}}$. Let us define the variable $u = x^2 + 3$. The variable u depends on x. Therefore the variable u is a function of x. From this we deduce $u'(x) = 2x$. This means that $x = \dfrac{u'}{2}$. If we substitute in the expression of f, we obtain

$$f(x) = \frac{u'}{2\sqrt{u}}.$$

Now it is easy to find a primitive to the function f. Indeed we know the function $F(x) = \sqrt{u} = \sqrt{x^2+3}$ is one of such primitives. As this example showed it is sometimes usefull to make a change in variable in which we integrate in order to be able to find easily a primitive. Here we made a substitution of the variable. In the next subsection we will set the theory of the *substitution method*.

6.4 Introduction to the Substitution Method

The objective of this subsection is to settle the theory of the substitution method. This technique is one of the most important in the elementary theory of integration. It can be summed up in the following theorem. We assume the set I is an interval.

Theorem 6.4.1. *Let $\phi : [a\,,\,b] \longrightarrow I$ be a $\mathscr{C}^1$- function. We consider the function $f : I \longrightarrow \mathbb{R}$ and for every x in $[a\,,\,b]$, we define $u = \phi(x)$. Then,*

$$\int_a^b f\big(\phi(x)\big)\,\phi'(x)\,dx = \int_{\phi(a)}^{\phi(b)} f(u)\,du. \tag{6.4.1}$$

Proof. Let $\phi : [a\,,\,b] \longrightarrow I$ be a function of class $\mathscr{C}^1$. We consider the function F a primitive of f. We define the function $F \circ \phi$ which is differentiable in $[a\,,\,b]$. For every real number $x \in [a\,,\,b]$,

$$\Big[F(\phi(x)\Big]' = F'(\phi(x)) \cdot \phi'(x) = f(\phi(x)) \cdot \phi'(x).$$

From this we deduce

$$\int_b^b \big[F(\phi(x)\big]'\,dx = \int_a^b f(\phi(x)) \cdot \phi'(x)\,dx = F(\phi(b)) - F(\phi(a)).$$

Now we introduce the following substitution. For every x in $[\,a\,,\,b\,]$, we define

$$u = \phi(x).$$

When $x = a$, the variable u takes the value $\phi(a)$. When $x = b$, we have $u = \phi(b)$. Formally speaking one has $\dfrac{du}{dx} = \dfrac{d\phi}{dx}$ Multiplying formally by dx, we obtain

$$du = \phi'(x)\,dx.$$

This leds to the identity

$$F(\phi(b)) - F(\phi(a)) = \int_b^a f(\phi(x)) \cdot \phi'(x)\,dx = \int_{\phi(a)}^{\phi(b)} f(u)\,du$$

This completes the proof. □

When we make a substitution, we should be careful. Because most of the students forget to change the bounds of the integral.

Example 6.4.2. *We consider the following integral*

$$I_0 = \int_0^1 \frac{x+1}{\sqrt{x^2+2x+1}}\, dx.$$

To calculate the value of this integral, we define $u = x^2 + 2x + 1$. *We see when* $x = 0$, *we have* $u = 1$. *If we take* $x = 1$, *the value of the variable* u *is* 4. *In this case* $du = 2\,(x+1)dx$. *This involves* $\dfrac{du}{2} = (x+1)dx$.

$$I_0 = \int_0^1 \frac{x+1}{\sqrt{x^2+2x+1}}\, dx = \int_1^4 \frac{du}{2\sqrt{u}}.$$

Therefore,

$$I_0 = \int_1^4 \frac{du}{2\sqrt{u}} = \Big[\sqrt{u}\Big]_1^4 = \sqrt{4} - \sqrt{1} = 2 - 1 = 1$$

Example 6.4.3. *We define the following integral*

$$I_1 = \int_1^2 \frac{\ln(t)}{t}\, dt.$$

To compute the value of I_1, *we set* $u = \ln(t)$. *In this case when* $t = 1$ $u = \ln(1) = 0$. *Taken* $t = 2$, *we have* $u = \ln(2)$. *The function* u *is differentiable and* $du = \dfrac{1}{t}\, dt$. *This implies*

$$I_1 = \int_1^2 \frac{\ln(t)}{t}\, dt = \int_0^{\ln(2)} u\, du = \left[\frac{u^2}{2}\right]_0^{\ln(2)} = \frac{\Big(\ln(2)\Big)^2}{2}.$$

Exercise 6.4.4. *Compute the following integrals*

1. $\displaystyle\int_0^2 x^2\, \mathrm{e}^{x^3+6}\, dx,$ 2. $\displaystyle\int_0^1 \frac{1}{\sqrt{2x+3}}\, dx,$ 3. $\displaystyle\int_{\pi/3}^{3\pi/4} \cos\big(3x - \pi\big)\, dx.$

CHAPTER 7

INTRODUCTION TO FIRST ORDER LINEAR ODES

Contents

After staying at home more than two months, because of Covid 19, Gunay, Tanya, Nubar, Allahverdi, Farida, Kanan, Suad, Arzu, Lala, and Khumar were bored. Thus they decided to organize a trip for a week-end, in order to enjoy their selves. They rent a car. After a day of travel, they decided to calculate the distance traveled.

- Gunay : Since the distance is a function depending on the time, we can denote the distance by $x(t)$.

- Tanya : Yes I agree with you. But the problem is how to calculate it?

-Kanan : I have seen somewhere this formula $x(t) = v \cdot t$. Here v is the speed of the car and t represent the time.

Automatically, students remember, they learned from their physical course, that $v(t) = \dfrac{dx(t)}{dt}$.

-Nubar : If we replace $v(t)$ by $\dfrac{dx}{dt}$ in the formula given by Kanan we will have

$$t \cdot \frac{dx}{dt} = x(t). \tag{7.0.1}$$

This equation is equivalent to

$$t \cdot \frac{dx}{dt} - x(t) = 0.$$

Thus, students get a relation between a function and its derivative. They, were a little bit confused, because they never face this types of equation.

After enjoying their week-end, students came back to school. A nice surprise awaited for them: Chemestry teachers planned to organize a *practical work*. They gave them a subject in which they can read: we consider the following chemical reaction

$$2\,C_4H_{10} + 13\,O_2 \rightarrow 8\,CO_2 + 10\,H_20$$

- we denote by $C(t)$ the concentration of the reagent C_4H_{10}
- the velocity of the reaction is defined by : $\dfrac{dC(t)}{dt}$
- The velocity is proportional to the concentration of the reagent

It was asked to students to find a relation between the concentration and the velocity of the reaction.

After three or five minutes thinking, Allahverdiyev put up his hand. Yes Allahverdiyev says the teacher.

- Allahverdiyev : Since the velocity is proportional to the concentration, we can find a constant real number k, such that,

$$k\frac{dC(t)}{dt} = C(t). \tag{7.0.2}$$

Oh my God, exclaimed one of the students, we got this type of equations before. They asked to the professor how do we call these types of equations?

- Chimistry Teacher: These type of equations are called Ordinary Differential Equations. To shorten it, they are called ODEs.

If you want to learn more about these equations, you need to have a discussion with maths teachers.

Once Chimistry teacher finished, students went to office 211, in order to meet Professor Ulviya. They want to know more about ordinary differential equations. Students knocked the door and enter the room.

-Students : Hi Teacher !

-Maths Teacher: How you are students? Can I do something for you?

-Students : Dear Professor, we were facing ordinary differential equations, but we did not know what were their applications and how to solve them.

Ordinary differential equations have several applications. They can be used in physics, biology, chemistry, geo-sciences. To study ordinary differential equations we will proceed step by step.

-Suad : What is a differential equation?

To answer Suad's question the teacher stated the following definition

Definition 7.0.1. *An ordinary differential equation in an interval I is a relation between a function and its derivatives.*

To simplify notations, we write f' instead of $\dfrac{df}{dt}(x)$ to mean the derivative of the function f.

Example 7.0.2. *We consider the following examples*

$$1.\ 2y'(x)+3y(x)=(x^2+2), \qquad 2.\ 5y''(x)+2y'(x)+3y(x)=0,$$
$$3.\ y'''(x)+3y''(x)+2x\,y'(x)+3x^2y(x)=(4x+5).$$

Here, you can see in the first equation,we have a relation between a function y and its first derivative. In the second equation, we have a relation between a function and its first and second derivatives. In the third equation, we have a relation between y, y', y'' and y'''.

Remark 7.0.3. *As we can observe it, in these equations the unknown is a function. This means, to solve these type of equations, we should find a function, which satisfies the relation.*

The first equation in the previous example is called a *first order ordinary differential equations.*

- Nubar : teacher what do you mean by first order differential equation ?

To make himself understood by students he stated the following definition

Definition 7.0.4. *We say that an ordinary differential equation is of the first order if we have a relation between a function and its firstderivative.*

Example 7.0.5. *The following differential equations are ordinary differential equations of first order*

$$1.\ y'+2x\,y=(2x^2+3), \qquad 2.\ 2y'+5y^2=0, \qquad 3.\ x\,y'+y^5=\ln(x).$$

In the same way we defined second order ordinary differential equations.

Definition 7.0.6. *A differential equation is said to be of second order, if we have a relation between a function and its first and second derivative.*

Example 7.0.7. *The following equations are second ordinary differential equations*

$$1.\ y''+2y'+2y=0, \qquad 2.\ x^2\,y''+y=x^2\mathrm{e}^x, \qquad 3.\ x\,y''+3x^2y'+y=\mathrm{e}^x.$$

In this chapter, we will study first order ordinary differential equations. To make the presentation clear, we organize it as follows. In the first section, we will define and study homogeneous first order linear differential equations. In the second section, we study nonhomogeneous first order linear differential equations. In the section 3 we study lsecond order linear differential equations.

We specify again in this course we will only deal with linear ordinary differential equations.

7.1 First Order linear Differential Equations

In this section, we define first order linear differential equations. We study some properties of the solutions. We show how to solve first order linear differential equations.

Before going further we need to specify the following definitions and notations.

Definition 7.1.1. *Let E be a set. We say the set E is a real vector space if it satisfies these conditions:*

1. *the null vector* 0 *is in* E
2. *for all* x *and* y *in* E,*the sum* $x+y$ *is in* E
3. *for all* $\lambda \in \mathbb{R}$ *and* $x \in E$, $\lambda \cdot x \in E$.

We reexplain this definition by saying a real vector space E is a set which contains the vector 0 and which is stable by addition of vector and multiplication by a real number.

To clarify its definition the teacher introduce the following example

Example 7.1.2. *Let* $E = \mathbb{R}$ *denote the set of all real numbers. Then,* E *is a vector space. Indeed we have*

- $0_{\mathbb{R}}$ *belongs to* $\mathbb{R}$
- *if* x *and* y *are real numbers then,* $x+y$ *is a real number.*
- *if* λ *and* x *are real number,* $\lambda \cdot x$ *is a real number.*

The set of real numbers satisfies the conditions of the definition. Therefore the set $\mathbb{R}$ *is a real vector space.*

Students were surprise to discover how it was easy to find vector spaces. To make the presentation undurstandable, the teacher added another example.

Example 7.1.3. *Let* $E = \mathbb{C}$ *denote the set of all complex numbers . Then,* $\mathbb{C}$ *is a real vector space.*

To prove this, we check,

- *the complex number* 0 *belongs to* $\mathbb{C}$
- *if* z_1 *and* z_2 *are in* $\mathbb{C}$, *therefore* z_1+z_2 *is in* $\mathbb{C}$ *because* z_1+z_2 *is a complex number.*
- *if we take* $k \in \mathbb{R}$ *and* $z \in \mathbb{C}$, *then* $k \cdot z$ *is a complex number.*

Hence we conclude the set $\mathbb{C}$ *is a real vector space.*

Definition 7.1.4. *Let* E *be a real vector space. We say* $F \subset E$ *is a real vector subspace of* E, *if* F *satisfies these two conditions:*

- $0_E \in F$
- *for any real numbers* α *and* β, *and vectors* x *and* y *in* F, *the vector* $\alpha\, x + \beta\, y$ *belongs to* F

Let E and F be two vector spaces, we study how to define correspondance between two vector spaces. We start by defining linear applications

Definition 7.1.5. *Let* E *and* F *be two vector spaces. We consider an application* $L : E \longrightarrow F$. *We say* L *is linear if it satisfies the following conditions:*

- *for every* x *and* y *in* E, *we have* $L(x+y) = L(x) + L(y)$
- *for all* $k \in \mathbb{R}$ *and* $x \in E$, $L(k\,x) = k\,L(x)$.

Remark 7.1.6. *From the definition above, we deduce when x belongs to the vector space E, $L(x) = L(x+0) = L(x) + L(0)$. Hence $L(0) = 0$.*

Example 7.1.7. *The application $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = 2x$ is linear.*

Example 7.1.8. *Generally speaking if E is a vector space and a a real number, the application $g : E \longrightarrow E$, defined by $g(x) = a\,x$ is linear.*

Definition 7.1.9. *Let I be an interval. A first order ordinary differential equations is called a first order linear differential equations if it is in the form*

$$a(x)\,y'(x) + b(x)\,y(x) = f(x), \tag{7.1.1}$$

where a, b, f and y are functions defined in I.

Students , did not understand why the teacher claimed equation (7.1.1) is linear. To explain this, the teacher defined the application

$$L(y) = a(x)y' + b(x)y. \tag{7.1.2}$$

He took y_1 and y_2 two functions satisfying equation (7.1.1). He showed that

$$\begin{aligned} L(y_1 + y_2) &= a(x)(y_1' + y_2') + b(x)(y_1 + y_2) \\ &= \big[a(x)y_1' + b(x)y_1\big] + \big[a(x)y_2' + b(x)y_2\big] = L(y_1) + L(y_2). \end{aligned}$$

On the other hand, if y is a function satisfying (7.1.1) and k a real number, then we have

$$L(k\,y) = a(x)(k\,y') + b(x)(k\,y) = ka(x)y' + kb(x)y = k\,\big[a(x)y' + b(x)y\big] = kL(y).$$

Therefore, the application L is linear. From this, we deduce that the equation (7.1.1) can be rewritten as follows:

$$L(y) = f(x). \tag{7.1.3}$$

Since the left hand side part is linear, we say that (7.1.1) is linear.

In other words a differential equation $L(y) = f(x)$ is linear if the left hand side part $L(y)$ is linear.

Let us consider the following differential equation $a(x)y' + b(x)y = f(x)$ defined in the interval I. According to the value of f, we say the linear differential equation is homogeneous or nonhomogeneous. We make the following definitions:

Definition 7.1.10. *We say a first order linear differential equation is homogeneous if it is in the form*

$$a(x)y' + b(x)y = 0. \tag{7.1.4}$$

To clarify the definition and illustration its meaning the teacher gave the following examples

Example 7.1.11. *These differential equations are homogeneous:*

$$1.\ g' + 3g = 0, \qquad 2.\ 2x\,y' + \frac{1}{x}\,y = 0, \qquad 3.\ (x^2 + 2x + 1)\,y' + \mathrm{e}^x\,y = 0.$$

Once the teacher finished to write its examples on the board, Gunay asked to the teacher, what happed when $f \not\equiv 0$. We recall that $f \not\equiv 0$ means f is a nonzero function. When the second member f is a nonzero function we have the following definition:

Definition 7.1.12. *Let f be a nonzero function. The first order linear differential equations*

$$a(x)y' + b(x)y = f(x),$$

is called nonhomogeneous differential equations.

Example 7.1.13. *The following differential equations are nonhomogeneous*

1. $y' + 5xy = (x^3 + 2x + 2)$, 2. $x^2y' + \ln(x)\, y = (x^2 + x + 3)\ln(x)$,
3. $(2x + 7)y' + \mathrm{e}^x y = \mathrm{e}^{2x+1}$.

Let a, b and f be three functions defined in the interval I. We consider the differential equation

$$(*) : \ a(x)y' + b(x)y = c(x).$$

We call equation $(*)$ a generalized differential equation.

This terminology is justified by definition 7.1.14 which will be stated below. To make the definition easy to understand, we specify the following notations. We define the set I^* as follows: $I^* = \{x \in I\,;\, a(x) \neq 0\}$. We can observe that for every $x \in I^*$,we have $a(x) \neq 0$. Therefore , if we restrict our selves in the set I^*, we can divide by $a(x)$. This means that in I^* the equation $(*)$ is equivalent to the following equation

$$y' + \frac{b(x)}{a(x)}y = \frac{c(x)}{a(x)}\cdot \tag{7.1.5}$$

The first order linear differential equation (7.1.5) is called a *normalized* first order linear differential equation. In this case we set the following definition.

Definition 7.1.14. *We say a first order linear differential equation defined in I is normalized , if it is in the form:*

$$y' + a(x)\, y = f(x), \tag{7.1.6}$$

where a, y and f are functions defined in the interval I.

Example 7.1.15. *The following differential equations are normalized differential equations:*

$$1.\ y' + 2y = 2x + 1, \quad 2.\ y' + (x^2 + x + 1)y = 0, \quad 3.\ y' + \frac{1}{x+1}y = (x^3 + 2).$$

After the definitions generalized and normalized, Nubar was a little bit confused. She wanted to know the technique which allows to pass from a generilazed to a normalized equation. To answer her question the teacher explain the following general rule: Let b, c and d be functions defined in the interval I. We consider the equation

$$(**) : \ b(x)y' + c(x)y = d(x).$$

We define $I_1 \subset I$, such that, for all $x \in I_1$, $b(x) \neq 0$. Therefore, in the interval I_1, $(**)$ is equivalent to

$$y' + \frac{c(x)}{b(x)}\, y = \frac{d(x)}{b(x)}\cdot$$

Setting $a(x) := \dfrac{c(x)}{b(x)}$ and $f(x) := \dfrac{d(x)}{b(x)}$, we obtain the following equation

$$y' + a(x)y = f(x).$$

This is the rule to obtain normalized equations from general equations.

Example 7.1.16. *Take the general differential equation* $2y' + 3y = x + 1$. *To obtain a normalized form, we divide both sides by 2. Then, we have*

$$y' + \frac{3}{2}\, y = \frac{x+1}{2}\cdot$$

Example 7.1.17. *We consider the general differential equation* $(x+1)y' + 2xy = \mathrm{e}^x$.

Let $x \neq -1$, *then* $x + 1 \neq 0$. *In this case we can divide by* $x + 1$ *and we obtain, for all* $x \in \mathbb{R}\backslash\{-1\}$,

$$y' + \frac{2x}{x+1}\, y = \frac{\mathrm{e}^x}{x+1}\cdot$$

Example 7.1.18. *Now we consider the equation* $x\, y' + \ln(x)y = 0$ *defined in* $(0\,, +\infty)$. *For all* $x > 0$, *we divide by* x *and obtain the following normalized equation*

$$y' + \frac{\ln(x)}{x} y = 0\cdot$$

Suad did not understand why we were defining normalized differential equations. To convince him to the usefulness of this form the teacher made the following parallel parallel with exponential function.

Let $f(x) = \mathrm{e}^x$. The function f is differentiable in $\mathbb{R}$ and $f'(x) = \mathrm{e}^x$. That is , $f(x) = \mathrm{e}^x$ is the solution to the following differential equation

$$y' - y = 0,$$

which satisfies $y(0) = 1$. Generaly speaking if we define the function

$$g(x) = \exp(A(x)),$$

where the function A is a primitive of the function a, we can easily check that

$$g'(x) = a(x)\exp(A(x)).$$

Then, the function g is a solution to the differential equation $y' - a(x)\, y = 0$. We see from these examples that the normalized form, can help to simplify calculations.

In the next subsection, we study homogeneous linear differential equations.

7.1.1 Homogeneous Linear Differential Equations

In this subsection we deal with homogeneous differential equation. Here we start by defining some properties shared by solution of homogeneous linear equations.

Now, we consider the following homogeneous equation: find a function y in $\mathscr{C}^1(I)$, which satisfies, for all $x \in I$,

$$y' + a(x)y = 0. \tag{7.1.7}$$

Having a look on this equation, Gunay pointed out that the function $y(x) = 0$, for all $x \in I$ is a solution to (7.1.7). This leads to the following lemma.

Lemma 7.1.19. *The function $y \equiv 0$ is a solution to any homogeneous linear differential equation.*

The function defined by $y(x) = 0$, for every $x \in I$ is called the evident solution of a linear homogeneous differential equation.

Remark 7.1.20. *We remark that the evident solution is not so interesting. That is why, solving a homogeous differential equation means to find a nonzero function. That is, here we are looking for a function $y \not\equiv 0$, which satisfies* (7.1.7).

Let us consider the set of all solutions of (7.1.7) defined in the interval I. We denote it $\mathscr{E}(I)$. We take two functions y_1 and y_2 solutions to (7.1.7) which are elements of $\mathscr{E}(I)$. Since y_1 and y_2 are solutions of (7.1.7), we have

$$y_1' + a(x)\, y_1 = 0 \quad \text{and} \quad y_2' + a(x)\, y_2 = 0.$$

Therefore, we have

$$y_1' + a(x)\, y_1' + y_2' + a(x)\, y_2 = (y_1 + y_2') + a(x)\,(y_1 + y_2) = 0.$$

Defining the function $z = y_1 + y_2$, we obtain the equation $z' + a(x)z = 0$.

This means that , the function $y_1 + y_2$ is a solution of (7.1.7). In other words the function $y_1 + y_2$ belongs to $\mathscr{E}(I)$.

Let k be a real number and y a solution of (7.1.7). The function $y \in \mathscr{E}(I)$. On the other hand, we know $(k\,y)' = k\,y'$. This implies

$$(k\,y)' + a(x)(k\,y) = k\,y' + ka(x)\,y = k(y' + a(x)\,y) = 0.$$

Hence, the function $z_1 = k\,y$ satisfies the differential equation $z_1' + a(x)\,z_1 = 0$. The function $k\,y$ is a solution of (7.1.7). That is, $k\,y \in \mathscr{E}(I)$. Since the function $y \equiv 0 \in \mathscr{E}(I)$, this involves the following lemma

Lemma 7.1.21. *Let $\mathscr{E}(I)$ be the set of all solutions of the differential equation $y' + a(x)\,y = 0$ defined in I. Then, $\mathscr{E}(I)$ is a vector space.*

Now we aim to establish the set $\mathscr{E}(I)$ is not reduce to the null solution. The chalenge is now to find conditions under which equation (7.1.7) has a solution $y \not\equiv 0$ in I. To this end we consider a function a which is continuous on I. We look at the differential equation: find a differentiable function, y on I solution to

$$y'(x) + a(x)\,y(x) = 0.$$

This equation is equivalent to $y'(x) = -a(x)\,y(x)$. Since the function a is continuous and y is differentiable, then y' is continuous. We know that $y \not\equiv 0$. This means that there exists $x_1 \in I$, such that $y(x_1) \neq 0$. We can find a subinterval $I_1 \subset I$, such that for all $x \in I_1$, $y(x) \neq 0$. In I_1 the equation $y' + a(x)\,y = 0$ is equivalent to

$$\frac{y'(x)}{y(x)} = -a(x).$$

The function $\dfrac{y'}{y}$ is continuous in I_1, then it has a primitive. To make the presentation clear and simplest, we assum $y(x) > 0$ for every $x \in I_1$. Hence we have

$$\int^x \frac{y'(t)}{y(t)}\,dt = \ln(y) = -\int^x a(t)\,dt + C.$$

Using the exponential function we obtain, for all $x \in I_1$,

$$y(x) = \exp\left(-\int^x a(t)\,dt + C\right) = K\exp\left(-\int^x a(t)\,dt\right),$$

where $K = \exp(C)$. Therefore, there exists $y \in \mathscr{C}^1(I_1)$ solution to the homogeneous linear differential equation $y' + a(x)\,y = 0$.

Theorem 7.1.22. *Let a be a continuous function on I. Then there exists a function $y \in \mathscr{C}^1(I)$ solution to the homogeneous linear differential equation* (7.1.7)

This theorem is a corollary of one of the most important theorem in analysis: the Cauchy-Lipschitz theorem. We will recall this theorem in the next subsection.

Tanya pointed out that since we stated in the previous theorem that equation (7.1.7) has solutions, they wanted to know what these solutions look like. In other words they wounder if there were an explicite formula for these solutions. To answer this question the teacher enunciate the following theorem.

Theorem 7.1.23. *Let a be a continuous function on I and A be a primitive of the function a on I. Then, a general solution to differential equation* (7.1.7) *are defined by the function*

$$y(x) = K \cdot \exp(-A(x)), \tag{7.1.8}$$

for every $x \in I$ and $K \in \mathbb{R}$.

Proof. We will do the proof in two steps
Step 1: Let a be a continuous function on I. We consider the function A a primitive of a on I. For every $x \in I$, we define the function $y(x) = K\exp(-A(x))$. Thus the function y is differentiable in I and we have

$$y'(x) = -A'(x)K\,\exp(-A(x)) = -a(x)K\,\exp(-A(x)).$$

Since the function a is continuous, then, y' is continuous on I. Therefore, $y \in \mathscr{C}^1(I)$ and $y'(x) + a(x)K\mathrm{e}^{-A(x)} = 0$. As the function $K\,\exp(-A(x))$ is nothing but y, we get the following equation: for all $x \in I$, $y' + a(x)y = 0$. This means that (7.1.8) is a solution to (7.1.7)
Step 2: Now we consider the equation

$$(*):\ y' + a(x)y = 0,$$

where a is a continuous function on I. We take the function A as a primitive of a in I. We multiply $(*)$ by $\exp(A(x))$. The following equation holds in I

$$\exp(A(x))\, y' + a(x)\exp(A(x))\, y = 0.$$

Setting $z(x) = \exp(A(x))$ for every $x \in I$, one has $z'(x) = a(x)\exp(A(x))$. Substituting in the equation above, we obtain $z\, y' + y\, z' = 0$. We know that

$$y'z + z'y = (zy)' = 0.$$

This implies $(z \cdot y)' = 0$. In this case one can find a constant real number K, such that, $z \cdot y = K$. This is equivalent to

$$\exp(A(x))\, y(x) = K,$$

for every $x \in I$. Since $\exp(A(x)) \neq 0$, for all $x \in I$, we can divide by $\exp(A(x))$. However, dividing by $\exp(A(x))$ is equivalent to multiply by $\exp(-A(x))$. Consequently, we have $y(x) = K\exp(-A(x))$. □

Example 7.1.24. *We consider the following differential equation $y' + 2x\, y = 0$ defined in $\mathbb{R}$*
Solution: In this example we define the function $a(x) = 2x$. This function is continuous in $\mathbb{R}$. A primitive of a is defined by the function $A(x) = x^2$. Applying theorem 7.1.23, we obtain $y(x) = K\exp(-x^2)$, for all $x \in \mathbb{R}$ and $K \in \mathbb{R}$.

Example 7.1.25. *Find solutions of the equation $y' + \frac{1}{x}\, y = 0$ in the interval $(\,0\,,\,+\infty\,)$.*

Solution : Let $x \in (\,0\,,\,+\infty\,)$. We define the function $a(x) = \dfrac{1}{x}\cdot$ Then, the function a is continuous on $(\,0\,,\,+\infty\,)$. The function $A(x) = \ln(x)$ is a primitive of a in $(\,0\,,\,+\infty\,)$. Applying theorem 7.1.23, we obtain

$$y(x) = K\exp(-\ln(x)) = \frac{K}{\exp(\ln(x))} = \frac{K}{x},$$

for all $x \in (\,0\,,\,+\infty\,)$ and $K \in \mathbb{R}$.

Example 7.1.26. *Let $x \in (-1\,,\,+\infty)$. Find the solution of the following homogeneous linear differential equation $y' - \dfrac{1}{x+1}\, y = 0$.*

Solution : Here we define the function $a(x) = -\dfrac{1}{x+1}\cdot$ The function $A(x) = -\ln(x+1)$ is a primitive of a. Therefore, the function $y(x) = K\exp(\ln(x+1)) = K\,(x+1)$, for all $x \in (\,0\,,\,+\infty\,)$ and $K \in \mathbb{R}$.

Exercise 7.1.27. *Find solutions to the following differential equations in the indicated interval I*

$$1.\ y' + 2\, y = 0, \quad I = \mathbb{R}, \quad 2.\ y' + (x^2 + 2x + 6)\, y = 0, \quad I = \mathbb{R},$$
$$3.\ y' + (x^3 + 3x + 1)\, y = 0, \quad I = \mathbb{R}.$$

Exercise 7.1.28. *Find solutions to the following homogeneous differential equations in the indicated interval I*

$$1.\ y' + \frac{2x}{x^2+1}\, y = 0, \quad I = \mathbb{R}, \quad 2.\ y' + \frac{1}{(x^2+1)}\, y = 0, \quad I = \mathbb{R}$$
$$3.\ y' + \frac{2}{(2x+4)}\, y = 0, \quad I = (-2\,,\,+\infty\,).$$

Exercise 7.1.29. *Find solutions to the following differential equations on the indicated interval I*

1. $y' + \cos(x)\, y = 0, \quad I = \mathbb{R}$, 2. $y' + 2\sin(2x+3)\, y = 0, \quad I = \mathbb{R}$
3. $y' + \tan(x)\, y = 0, \quad I = (-\pi/2\,,\, \pi/2)$.

Exercise 7.1.30. *Find solutions to the following differential equations in the indicated interval I*

1. $y' + \dfrac{\ln(x+2)}{x+2}\, y = 0, \quad I = (-2\ +\infty\,)$, 2. $y' + \mathrm{e}^x\, y = 0, \quad I = \mathbb{R}$
3. $y' + (2x+1)(x^2+x+1)\, y = 0, \quad I = \mathbb{R}$, 4. $y' + 3(3x+1)^3\, y = 0, \quad I = \mathbb{R}$
5. $y' + \dfrac{1}{(x+2)^2}\, y = 0, \quad I = \mathbb{R}\backslash\{-2\}$.

Exercise 7.1.31. *Find the normalized form and the solutions of each of the following equations*

1. $x^3\, y' + x\, y = 0$, 2. $(x^3 + 2x + 5)\, y' + (3x^2+2)\, y = 0$,
3. $(2x+3)\, y' + 2\, y = 0$, 4. $\mathrm{e}^{x^3+2}\, y' - 3x^2\, y = 0$, 5. $\sin(x)\, y' + \cos(x)\, y = 0$.

After a little break Sanan make the following observations. Teacher we could see that each time we change the constant K in the expression (7.1.8) we get a new solution . We wanted to know how to obtain a unique solution. The teacher explained that to find a unique solution we join to the differential equation an initial condition. Then, if we wanted to have a unique solution we have to study what we call a *Cauchy problems.*

7.1.2 Homogeneous Linear Cauchy Problem

Formally speaking, a *Cauchy problem* is a differential equation to which we join an initial condition. An initial condition is just a starting point. Mathematically speaking we define a *Cauchy Problem* by :

Definition 7.1.32. *Let a be a continuous function on I. We consider $x_0 \in I$ and $y_0 \in \mathbb{R}$. A homogeneous first order linear Cauchy problem is a system in the following form*

$$\begin{cases} y' + a(x)\, y = 0, \quad x \in I \\ \\ y(x_0) = y_0, \quad x = x_0. \end{cases}$$

The differential equation $y' + a(x)\, y = 0$ is called the dynamic. Because, it describes the evolution of phenomenon studied. The expression $y(x_0) = y_0$ is called the initial condition. It is the starting point. This means that we start studying the system at the time x_0 or when $x = x_0$ and we consider this system after $x = x_0$.

Example 7.1.33. *Below,we can see two examples of Cauchy problems.*

$$\begin{cases} y' + (3x+1)\, y = 0,\ x \in \mathbb{R} \\ \\ y(0) = 3, \quad x = 0, \end{cases} \quad , \quad \begin{cases} y' + \dfrac{1}{x^2}\, y = 0,\ x \in \mathbb{R}\backslash\{0\} \\ \\ y(1) = 5, \quad x = 1. \end{cases}$$

The examples shown, Farida asked how to find a solution to a Cauchy problem. The teacher replayed: before solving Cauchy problem, we would prove the existence and uniqueness of a solution to the Cauchy problem in a first time. In a second time we would give the expression of the solution to a Cauchy problem.

We know from theorem 7.1.22, the homogeneous linear differential equation

$$(*) : y' + a(x)y = 0$$

has solutions which belong to $\mathscr{C}^1(I)$. Let A be a primitive of a. Solution of $(*)$ are given by $y(x) = K\exp(-A(x))$, for some real number K. Since $y(x_0) = K\exp(-A(x_0)) = y_0$, we have $K = y_0 \cdot \exp(A(x_0)$. From this we deduce a unique value of K which gives the unique solution

$$y(x) = y_0 \exp\Big(- \big[A(x) - A(x_0)\big]\Big).$$

This proves the following theorem we prove.

Theorem 7.1.34. *Let a be a continuous function on some interval I. We consider a real number $x_0 \in I$. For some real number $y_0 \in \mathbb{R}$, we define the following Cauchy Problem :*

$$\begin{cases} y' + a(x)\, y = 0, \quad x \in I \\ \\ y(x_0) = y_0, \quad x = x_0. \end{cases} \tag{7.1.9}$$

Then, there exists a unique function $y \in \mathscr{C}^1(I)$ solution to (7.1.9).

To find the solution of a Cauchy problem we have to follow these two steps.
Step 1: Find solutions to the differential equation

$$y' + a(x)\, y = 0.$$

We know from previous section, if A is a primitive of $a(x)$ on I. Then, for all $x \in I$, the function $y(x) = K\exp(-A(x))$, where $K \in \mathbb{R}$ is a solution to the linear differential equation above.
Step 2: We use the initial condition to determine the value of the constant real number K. Indeed the initial condition allows to establish $y(x_0) = K\exp(-A(x_0)) = y_0$. This implies that $K = y_0\exp(A(x_0))$. The solution of the Cauchy problem is defined by $y(x) = y_0 \exp\Big(- \big[A(x) - A(x_0)\big]\Big)$. This leads to the following theorem.

Theorem 7.1.35. *Let a be a continuous function on I. We consider $x_0 \in I$ and $y_0 \in \mathbb{R}$. Then, the expression of the unique solution to the Cauchy problem*

$$\begin{cases} y' + a(x)\, y = 0, \quad x \in I \\ \\ y(x_0) = y_0, \quad x = x_0 \end{cases}$$

is given by

$$y(x) = y_0 \cdot \exp\Big(- \big[A(x) - A(x_0)\big]\Big). \tag{7.1.10}$$

To illustrate this theorem, we consider the following examples

Example 7.1.36. *Let us define the following example: for $x \in \mathbb{R}$, find the function y, such that,*

$$\begin{cases} y' + 3x\, y = 0, \\ \\ y(0) = 2. \end{cases}$$

Solution : We follow steps decribe above
Step 1: We define the homogeneous linear differential equation $y' + 3xy = 0$. Here we take $a(x) = 3x$. Therefore, $A(x) = \dfrac{3}{2}\, x^2$ is a primitive of the function a. Its solution is defined by

$$y(x) = K \exp\Big(- 3x^2/2\Big).$$

Step 2: Using the initial condition $y(0) = 2$, one obtains $y(0) = K \exp(0) = 2$. That is, $K = 2$. In this case the unique solution to the Homogeneous Cauchy problem is

$$y(x) = 2 \exp\Big(- 3x^2/2\Big).$$

Example 7.1.37. *Find the solution to the following Cauchy problem*

$$\begin{cases} y' + 2y = 0, \\ \\ y(1) = 3. \end{cases}$$

Solution
Step1: We consider the linear homogeneous differential equation $y' + 2y = 0$. To find the solution to this differential equation we set $a(x) = 2$. The function $A(x) = 2x$ is a primitive of the function a. Therefore,

$$y(x) = K \exp\big(- 2x\big)$$

is a solution of the previous differential equation.
Step 2: Now we use the initial condition $y(1) = 3$, to determine the value of the constant real number K. We know that $y(1) = K \exp(-2) = 3$. This implies, the constant real number $K = 3 \exp(2)$. Consequently the unique solution of the Homogeneous Cauchy problem is

$$y(x) = 3 \exp\Big(- 2x + 2\Big).$$

Example 7.1.38. *Find the function y defined in $x \in \ (0\,, +\infty\,)$, solution to the following differential equation*

$$\begin{cases} y' + \dfrac{1}{x}\, y = 0, \\ \\ y(3) = 4. \end{cases}$$

Solution
Step 1: Let x belongs to $(0\,, +\infty\,)$. We define the differential equation $y' + \dfrac{1}{x}\, y = 0$. We define the function $a(x) = \dfrac{1}{x}\cdot$ Therefore, the function $A(x) = \ln(x)$ is a primitive of the function a. Therefore,

$$y(x) = K \exp\Big(- \ln(x)\Big) = \frac{K}{x}\cdot$$

Step 2: From the initial condition we deduce $y(3) = \dfrac{K}{3} = 4$. *This involves* $K = 12$. *The unique solution is of the Cauchy problem is*

$$y(x) = \frac{12}{x}.$$

Now we consider y as a variable and we separate function depending on y to functions depending on x. This method is called the "Method of separation of variables".

7.1.3 Method of Separation of Variables

The main objective of this method consist of separating functions depending on x to function depending on the variable y. In this section and only in this section, we consider y as a variable.

Let us take a first order homogeneous linear differential equation

$$y' + a(x)\, y = 0$$

To study the previous linear equation we remark when a is continuous in I the problem admits a nonzero solution. We mean by nonzero function a function $f \not\equiv 0$. Since $y \not\equiv 0$, there exists an interval $I_1 \subset I$, such that, for all $x \in I_1$, $y(x) \neq 0$. In this sub interval I_1, the equation $y' + a(x)\, y = 0$ is equivalent to

$$\frac{y'}{y} = -a(x).$$

We gather functions depending on y between them. We did the same with functions depending x in this way

Step 1 : Integration. We put functions depending on y on the left hand side and functions depending on x on the right hand side:

$$\frac{y'}{y} = -a(x).$$

Since the functions y'/y and a are continuous on I_1, we can find their integrals in this way :

$$\int^x \frac{y'}{y}\, dt = -\int^x a(t)\, dt.$$

The integral on the left hand side is equal to

$$\int^x \frac{y'}{y}\, dt = \ln\Big(|y(x)|\Big) + C \Longrightarrow \ln\Big(|y(x)|\Big) = -\int^x a(t)\, dt + C', \quad \text{with, } C' = -C.$$

Step 2: Composition with exp *Function.* Using exponential function we obtain

$$|y(x)| = \exp\Big(C'\Big) \cdot \exp\left(-\int^x a(t)\, dt\right).$$

Setting $K = \exp(C')$ and $A(x) = \int^x a(t)\, dt$, one has $|y(x)| = K \cdot \exp\big(-A(x)\big)$.

Step 3: Expression of the solution y. According to the signe of the function y we distinguish two cases.

Case 1 : $y \geq 0$**.** When, for every $x \in I_1$, we have $y(x) \geq 0$, then, the absolute value of $y(x)$ is equal to $y(x)$. That is $|y(x)| = y(x)$. Hence, the following identity holds

$$y(x) = \exp(C') \cdot \exp\left(- \int^x a(t)\, dt \right).$$

Case 2: $y < 0$**.** If for every $x \in I_1$, the following inequality $y(x) < 0$ holds, one has $|y(x)| = -y(x)$. Therefore, the general solution of homogeneous linear differential equation is

$$y(x) = -\exp(C') \cdot \exp\left(- \int^x a(x)\, dx \right).$$

Remark 7.1.39. *We emphasize this method can be used if and only if we can separate functions depending on y to functions depending x. Otherwise it cannot be applied. One more advantage is the fact that it works also for nonlinear differential equations.*

As applications we consider the following examples.

Example 7.1.40. *Let $x \in \mathbb{R}$. Find solutions of the following differential equation:*

$$y' + 2x\, y = 0.$$

Solution : Let $x \in \mathbb{R}$, such that, $y(x) \neq 0$. Then we have, $\dfrac{y'}{y} = -2x$. Integrating in x, we obtain the following identity

$$\ln(|y(x)|) = - \int^x 2t\, dt = -x^2 + C$$

This implies that, $|y(x)| = \exp(C) \cdot \exp\big(- x^2\big)$. Now we define the sets I_1 and I_2, such that, $\mathbb{R} = I_1 \cup I_2$ We assume that for all $x \in I_1$, $y(x) < 0$ and for all $x \in I_2$, $y(x) \geq 0$. Then, we obtain the following expression of the solution y:

$$y(x) = \begin{cases} -K\mathrm{e}^{-x^2}, & \text{if,} \quad x \in I_1 \\ \\ K\mathrm{e}^{-x^2}, & \text{if} \quad x \in I_2, \end{cases}$$

where $K = \exp(C)$.

Example 7.1.41. *Let x belongs to $(-\infty\,, 0\,)$. Find solutions to the following differential equation:*

$$y' + \frac{1}{x}\, y = 0.$$

Solution: Let $x \in (-\infty\,, 0\,)$. We assume $y(x) \neq 0$. for every $x \in (-\infty\,, 0)$. The equation above is equivalent to

$$\frac{y'}{y} = -\frac{1}{x}.$$

This yields $\ln(|y|) = -\ln(|x|) + C$. From this, we deduce

$$|y(x)| = \exp(C) \cdot \exp\big(- \ln(|x|)\big) = \frac{\exp(C)}{|x|}.$$

Since $x \in (-\infty\,,\,0)$*, we obtain*

$$|y(x)| = -\frac{\exp(C)}{x}.$$

Now, we define the set J_1 *and* J_2 *as follows:*

$$J_1 = \Big\{x \in (-\infty\,,\,0)\;;\;\; y(x) < 0\Big\} \quad \textit{and} \quad J_2 = \Big\{x \in (-\infty\,,\,0)\;\;;\;\; y(x) \geq 0\Big\}.$$

Therefore, the expression of y *is*

$$y(x) = \begin{cases} \dfrac{K}{x}, & \textit{if,} \quad x \in J_1 \\ \\ -\dfrac{K}{x}, & \textit{if} \quad x \in J_2, \end{cases}$$

here, we set $K = \exp(C)$.

Remark 7.1.42. *We remark again we can use the method for nonlinear equation and the principle is the same.*

After these examples the students were woundering, what happens if the second member f is different to 0. To remove any ambiguity the teacher introduce the nonhomogeneous linear differential equations.

7.2 Nonhomogeneous Linear Equation

In this section we study the first order differential equations with second member $f \not\equiv 0$. Here we will explain a general method to solve these type of differential equations.

Arzu asked if there was possible to recall the definition of nonhomogeneous linear differential equation. The teacher accessed to Arzu's request. He stated this definition

Definition 7.2.1. *We call nonhomogeneous first order linear differential equation a linear differential equation which is in the following form*

$$y'(x) + a(x)\, y(x) = f(x). \tag{7.2.1}$$

We take $f \not\equiv 0$, and we consider equation (7.2.1). We aim to establish an existence theorem for nonhomogeneous equation. To this end we assume the functions a and f are two continuous functions on I. Let y be a differentiable function in I, such that,

$$y' + a(x)\, y = f(x).$$

We consider a function A which is a primitive of a. The solution of the homogeneous equation $y'_H + a(x)\, y_H = 0$ is $y_H(x) = K \exp\big(-A(x)\big)$.

Let k be a differentiable function on I. We define the function

$$y(x) = k(x) \cdot \exp(-A(x))$$

and we assume that y is a solution to the nonhomogeneous differential equation (7.2.1). The function y is differentiable in I and we have

$$y'(x) = \exp\big(-A(x)\big)\big[k'(x) - k(x)\,a(x)\big].$$

Substituting in (7.2.1) we obtain

$$\exp(-A(x))\big[k'(x) - k(x)\,a(x) + k(x)\,a(x)\big] = k'(x)\exp(-A(x)) = f(x).$$

This implies $k'(x) = f(x)\exp(A(x))$. From this, we deduce that

$$k(x) = \int^x f(t)\cdot \mathrm{e}^{A(t)}\,dt.$$

This leads to the following formula: for every $x \in I$

$$y(x) = \mathrm{e}^{-A(x)}\int^x \mathrm{e}^{A(t)} f(t)\,dt = \int^x \mathrm{e}^{-(A(x)-A(t))}\cdot f(t)\,dt.$$

This means that if a and f are continuous functions on I, there exist a solution to the nonhomogeneous equation (7.2.1).

Theorem 7.2.2. *Let a and f be two continuous functions on I. Then, there exists a function $y \in \mathscr{C}^1(I)$ solution to* (7.2.1).

In this case we say equation (7.2.1) has at least a particular solution.

Students pointed out that refering to the previous section they know how to find $y_H(x) = K\exp(-A(x))$, they woundered if there was efficient methods to determine particular solutions y_p.

7.2.1 Methods to Define a Particular solution

This subsection devoted to methods which will allow to find a particular solution. We define a particular solution as

Definition 7.2.3. *We say that y_p is a particular solution of the nonhomgeneous equation if y_p satisfies equation* (7.2.1).

To determine y_p, there are several methods but here we will only study two of them:

1. *Method of Variations of constant*
2. *Method of undetermined coefficients*

Method of Variations of Constant

We consider the nonhomogeneous linear differential equation $y' + a(x)\,y = f(x)$ We know the homogeneous equation $y'_H + a(x)\,y_H = 0$ has solution

$$y_H(x) = K\ \exp(-A(x)),$$

where K is a real number and the function A is a primitive of a. To find a particular solution y_p, we replace the constant K in the expression of y_H by a function $\lambda(x)$ depending on x. Hence, we define

$$y_p(x) = \lambda(x)\ \exp(-A(x)).$$

The function y_p is differentiable and

$$y_p'(x) = \big(\lambda'(x) - a(x)\lambda(x)\big)\, \exp(-A(x)).$$

Now we must find the expression of the function λ. Since the function y_p is a particular solution of the equation $y' + a(x)\, y = f(x)$, then, we have

$$y_p' + a(x)\, y_p = \Big(\lambda'(x) - a(x)\lambda(x)\Big)\, \exp(-A(x)) + a(x)\lambda(x)\, \exp(-A(x)) = f(x).$$

This implies $\lambda'(x)\, \exp(-A(x)) = f(x)$. Multiplying by $\exp(A(x))$, one has

$$\lambda'(x) = f(x)\, \exp(A(x)).$$

This means the function λ is a primitive of the function $f(x)\, \exp(A(x))$. Consequently

$$\lambda(x) = \int^x \exp(A(t))\, f(t)\, dt.$$

Example 7.2.4. *Find a particular solution to the equation $y' + 2y = x^2 \exp(-2x)$.*

A general solution of the homogeneous equation $y_H' + 2y_H = 0$ is

$$y_H(x) = K\, \exp(-2x).$$

To determine a particular solution to the equation, we apply the method of variations of constant. We define the function $y_p(x) = \lambda(x) \exp(-2x)$. Then, we have

$$y_p'(x) = \big(\lambda'(x) - 2\,\lambda(x)\big)\, \exp(-2x).$$

The function y_p satisfies

$$\big(\lambda'(x) - 2\,\lambda(x)\big)\, \exp(-2x) + 2\lambda(x)\, \exp(-2x) = x^2\, \exp(-2x).$$

This implies the identity $\lambda'(x) = x^2$. From this we deduce $\lambda(x) = \frac{x^3}{3}\cdot$ This means that

$$y_p(x) = \frac{x^3}{3}\, \exp\big(-2x\big).$$

Example 7.2.5. *Determine a particular solution to the nonhomogeneous linear differential equation $y' + 2x\, y = x\, \exp\big(-x^2\big)$.*

We start by emphasizing that the homogeneous equation $y_H' + 2\,x\, y_H = 0$ has solution

$$y_H(x) = K\, \exp\Big(-x^2\Big).$$

Now we define the function $y_p(x) = \lambda(x)\, \exp\big(-x^2\big)$ as a particular solution to the equation above. It then, follows

$$y_p'(x) = \Big(\lambda'(x) - 2x \cdot \lambda(x)\Big) \exp\Big(-x^2\Big).$$

Since y_p is a particular solution of $y' + 2xy = x \exp(-2x)$, we have

$$\Big(\lambda'(x) - 2x \cdot \lambda(x)\Big) \exp\Big(-x^2\Big) + 2x \cdot \lambda(x) \exp\Big(-x^2\Big) = x \cdot \exp\Big(-x^2\Big).$$

This is equivalent to $\lambda'(x) = x$. *This involves*

$$\lambda(x) = \frac{x^2}{2}$$

From this we deduce

$$y_p(x) = \frac{x^2}{2}\,\exp\Big(-x^2\Big).$$

Example 7.2.6. *Define a particular solution to the following differential equation* $xy' - 2y = x^2$. *For any real number* $x \neq 0$, *the equation above is equivalent to*

$$y' - \frac{2}{x}y = x.$$

A general homogeneous solution of this latter is defined by

$$y_H(x) = K\mathrm{e}^{\ln(x^2)} = K\,x^2,$$

where K *is a real number.*

We define the function $y_p(x) = \lambda(x)\cdot x^2$ *as a particular solution to the equation above . Hence we have* $y_p'(x) = x^2\lambda'(x) + 2x\lambda(x)$. *The function* y_p *is a particular solution to the nonhomogeneous then, we have*

$$\lambda'(x)x^2 + 2x\lambda(x) - 2x\lambda(x) = x.$$

This equation is equivalent to $\lambda'(x) = \dfrac{1}{x}$, *for every* $x \neq 0$. *Consequently, we get* $\lambda(x) = \ln(|x|)$ *One deduces*

$$y_p(x) = x^2\,\ln(|x|).$$

Example 7.2.7. *Define a particular solution to the following nonhomogeneous linear differential equation*

$$(2x+1)\,y' = 2x + 2\,y$$

This equation is equivalent to $(2x+1)\,y' - 2y = 2\,x$. *Let* x *be a real number such that* $x \neq -\dfrac{1}{2}$. *Dividing the equation above by* $2x+1$, *we obtain the following equation*

$$y' - \frac{2}{2x+1}\,y = \frac{2x}{2x+1}\cdot$$

The associated homogeneous equation is $y_H' - \frac{2}{2x+1}\,y_H = 0$. *For every* $x \neq -\dfrac{1}{2}$ *the solution of this latter is defined by*

$$y_H(x) = K\exp\Big(\ln(|2x+1|)\Big) = K\,|2x+1|.$$

Hence

$$y_H(x) = \begin{cases} K(2x+1), & \text{if} \quad x > -\dfrac{1}{2} \\ \\ -K(2x+1), & \text{if} \quad x < -\dfrac{1}{2} \end{cases}$$

where K *is a real number.*

Now we consider $x > -\frac{1}{2}$ *and we define the function*

$$y_p(x) = \lambda(x)\,(2x+1).$$

We have $y_p'(x) = \lambda(x)'(2x+1) + 2\,\lambda(x)$ *Replacing in the nonhomogeneous equation one has* $(2x+1)^2\,\lambda'(x) + 2\,(2x+1)\,\lambda(x) - 2\,(2x+1)\,\lambda(x) = 2x$. *This implies*

$$\lambda'(x) = \frac{2x}{(2x+1)^2} = \frac{2x+1-1}{(2x+1)^2} = \frac{1}{2x+1} - \frac{1}{(2x+1)^2}.$$

Therefore,

$$\lambda(x) = \frac{1}{2}\Big(\ln(2x+1) + \frac{1}{2x+1}\Big).$$

For every $x > -\dfrac{1}{2}$, *we have*

$$y_p(x) = \frac{1}{2}(2x+1)\ln(2x+1) + \frac{1}{2}.$$

Here we suppose $x < -\frac{1}{2}$, *and we take the function* $y_p(x) = -\lambda(x)(2x+1)$ *Then, we obtain* $y_p'(x) = -\lambda'(x)(2x+1) - 2\lambda(x)$ *Substuting in the nonhomogeneous equation one gets*

$$-\lambda'(x)(2x+1)^2 - 2(2x+1)\lambda(x) + 2(2x+1)\lambda(x) = 2x.$$

That is, $\lambda'(x) = -\dfrac{2x}{(2x+1)^2}$. *This means for every* $x < -\dfrac{1}{2}$

$$\lambda(x) = -\frac{1}{2}\left(\ln(-2x-1) + \frac{1}{2x+1}\right).$$

In this case we have

$$y_p(x) = \frac{1}{2}\left((2x+1)\ln(-2x-1) + 1\right),$$

for every $x < -\frac{1}{2}$.

Finally we obtain the following particular solution

$$y(x) = \begin{cases} \dfrac{1}{2}(2x+1)\ln(2x+1) + \dfrac{1}{2}, & \text{if} \quad x > -\dfrac{1}{2} \\ \\ \dfrac{1}{2}(2x+1)\ln(-2x-1) + \dfrac{1}{2}, & \text{if} \quad x < -\dfrac{1}{2} \end{cases}$$

Example 7.2.8. *Find a particular solution to the following nonhomogeneous differential equation*

$$\sin(x)y' + \cos(x)\,y = x.$$

Let $x \in (-\pi\,,\,0\,) \cup (\,0\,,\,\pi\,)$. *The homogeneous equation associated to the previous equation is*

$$\sin(x)y_H' + \cos(x)y_H = 0.$$

Dividing by $\sin(x)$ *for* $x \in (-\pi\,,\,0\,) \cup (\,0\,,\,\pi\,)$ *one obtains the following equivalent equation*

$$y_H' + \frac{\cos(x)}{\sin(x)}\,y_H = 0.$$

For every $x \in (-\pi\,,\,0\,) \cup (\,0\,,\,\pi\,)$, *we define* $a(x) = \dfrac{\cos(x)}{\sin(x)} = \dfrac{u'}{u}\cdot$ *A primitive of a is defined by* $A(x) = \ln(|\sin(x)|)$. *Then, a general homogeneous solution is defined by* $y_H(x) = K\,\exp(-\ln(|\sin(x)|)) = \frac{K}{|\sin(x)|}$, *where* K *is a real number. Hence we have*

$$y_H(x) = \begin{cases} \dfrac{-K}{\sin(x)}, & \text{if} \quad x \in (-\pi\,,\,0\,) \\ \\ \dfrac{K}{\sin(x)}, & \text{if} \quad x \in (\,0\,,\,\pi\,) \end{cases}$$

Let $x \in (\,0\,,\,\pi\,)$, *we define the function* $y_p(x) = \dfrac{\lambda(x)}{\sin(x)}\cdot$ *Then we have*

$$\frac{\lambda'(x)}{\sin(x)} = \frac{x}{\sin(x)}\cdot$$

One deduces $\lambda'(x) = x$. *That is,* $\lambda(x) = \dfrac{x^2}{2}$ *and for every* $x \in (\,0\,,\,\pi\,)$ *the particular solution is defined by* $y_p(x) = \dfrac{x^2}{2\sin(x)}\cdot$

Now we consider $x \in (-\pi\,,\,0\,)$ *and we define the function* $y_p(x) = \dfrac{-\lambda(x)}{\sin(x)}\cdot$ *This leads to the following identity* $\dfrac{-\lambda'(x)}{\sin(x)} = \dfrac{x}{\sin(x)}\cdot$ *Then, we have* $\lambda(x) = \dfrac{-x^2}{2}$ *and for every* $x \in (-\pi\,,\,0\,)$,

$$y_p(x) = -\frac{x^2}{2\sin(x)}\cdot$$

Therefore, we obtain the following solution

$$y(x) = \begin{cases} -\dfrac{x^2}{2\,\sin(x)}, & \text{if} \quad x \in (-\pi\,,\,0\,) \\ \\ \dfrac{x^2}{2\,\sin(x)}, & \text{if} \quad x \in (\,0\,,\,\pi\,). \end{cases}$$

When the second member has particular form we can simplify calculations by applying the method of undetermined coefficients. We specify that this method works only for some particular forms of the second mmber f. Indeed it can be applied when the second member f has one of the following forms

Form 1: Polynomial Form. when the function $f(x) = P(x)$, where P is a polynomial function.

Form 2 : Polynomial function multiplied by exponential. if the function $f(x) = P(x)\,\mathrm{e}^{\lambda x}$, where P is a polynomial function and λ is real number.

Form 3: Polynonial function multiplied by cosine or sine. if the funcion $f(x) = \alpha\,\cos(kx)$ or $f(x) = \beta\,\sin(kx)$, the function P is a polynomial function.

It is important to remark that when the expression of f is different to the form 1, 2, or 3, applying this method is useless. This justify the fact that this method is adopted to some particular situations. That is why, we say it is particular method. The second method is more general, but seems to be more difficult to be applied. As an example to which we cannot apply the Method of undetermined coefficients we take $f(x) = \ln(x)$. To illustrate this method we proceed by examples.

7.2.2 Undetermined Coefficients Methods

The objective of this subsection is to define the method of *undetermined coefficients Methods*. For the sake of clarity and simplicity we proceed by examples to explain how the method runs.

We start with the case $f(x) = P(x)$, where P is a polynomial function.

The Second Member Polynomial Function

When the second member f is a polynomial function, we define the function

$$y_p(x) = \sum_{k=0}^{n} a_k\, x^k = a_n\, x^n + \cdots + a_1\, x + a_0.$$

To determine the function y_p we need just to find the coefficients $a_0, \cdots, a_n$.

Example 7.2.9. *Using the method of undetermined coefficients, find a particular solution for the following nonhogeneous linear differential equation*

$$y' + 2\,y = x^2 + 2x + 1.$$

Solution . Here we observe $f(x) = x^2 + 2x + 1$ and $a = 2$. Then, we can define the function

$$y_p = bx^2 + cx + d.$$

To determine the function y_p we should find a, c and d. One can directly apply the formulas above and find b, c and d. Indeed we have

$$b = \frac{1}{2}, \quad c = \frac{2}{2} - \frac{2\times 1}{4} = \frac{1}{2}, \quad d = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

Therefore

$$y_p(x) = \frac{1}{2}\big(x^2 + x + \frac{1}{2}\big).$$

To explain in details the previous result, we define $y_p = bx^2+cx+d$. The function y_p is differentiable in $\mathbb{R}$ and $y_p'(x) = 2bx + c$ Replacing in the previous equation one obtains

$$2bx + c + 2(bx^2 + cx + d) = x^2 + 2x + 1.$$

This is equivalent to

$$2bx^2 + 2(b + c)x + (c + 2d) = x^2 + 2x + 1.$$

We identify the two polynomial functions to get $2b = 1$, $2(b+c) = 2$ and $c+2d = 1$. This means that $b = \frac{1}{2}$, $c = \frac{1}{2}$, and $d = \frac{1}{4}$. That is,

$$y_p(x) = \frac{1}{2}x^2 + \frac{1}{2}x + \frac{1}{4}.$$

Example 7.2.10. *Determine a particular solution to the following nonhomogeneous equation*

$$y' + \frac{1}{2}y = 2.$$

Since $f(x) = 2$ we define as a polynomial function. That is we define $y_p = bx + c$, where b and c are real numbers. In this case $y_p'(x) = a$. Substituting in the equation we obtain

$$y_p'(x) + \frac{1}{2}y_p(x) = 2 \iff a + \frac{1}{2}(ax + b) = 2 \iff \frac{1}{2}\,a\,x + a + \frac{1}{2}\,b = 2$$

Identifying the ploynomes one gets $\frac{1}{2}a = 0$ and $a + \frac{1}{2}b = 2$. This involves $a = 0$ and $b = 4$. Therefore, $y_p(x) = 4$.

Example 7.2.11. *Find a particular solution to the following equation*

$$y' + 3y = x^2.$$

Here we observe the second member $f(x) = x^2$ is a polynomial function. Thus we define the particular solution as a polynomial function. That is why we consider $y_p(x) = ax^2 + bx + c$. It then follows $y_p'(x) = 2ax + b$. As y_p is a particular solution of $y' + 3y = x^2$, we have $2ax + b + 3(ax^2 + bx + c) = x^2$. This is equivalent to $3\,a\,x^2 + (2a + 3b)x + b + 3c = x^2$. This leads to the following equations $3a = 1$, $2a + 3b = 0$ and $b + 3c = 0$. From the previous equations we deduce that $a = \dfrac{1}{3}$, $b = -\dfrac{2}{9}$ and $c = \dfrac{2}{27}\cdot$ Therefore for any real number x,

$$y_p(x) = \frac{1}{3}x^2 - \frac{2}{9}x + \frac{2}{27}\cdot$$

Example 7.2.12. *Determine a particular solution to the following differential equation*

$$y' + \frac{1}{x}y = x.$$

In this example we have $f(x) = x$. That is why we can define

$$y_p(x) = ax^2 + bx + c.$$

This involves , $y_p'(x) = 2ax + b$. Substituting in the previous equation we obtain

$$3ax + 2b + \frac{c}{x} = x.$$

This identity implies $3a = 1$, $2b = 0$ and $c = 0$. Therefore, $a = \dfrac{1}{3}$, and $b = c = 0$. The particular solution is

$$y_p(x) = \frac{1}{3}x^2.$$

Example 7.2.13. *Determine a particular solution of $y' + y = x$.*

To find a particular solution y_p, we point out that f is a polynomial solution. Hence, we define $y_p(x) = ax + b$. The derivative of y_p is $y_p' = a$. Since y_p is a particular solution, it satisfies $a + ax + b = x$. This implies $a = 1$ and $a + b = 0$. This means that $a = 1$ and $b = -1$. The particular solution is defined by

$$y_p(x) = x - 1.$$

Polynomial Function Multplied by Exponential

When the second member $f = P(x) \cdot \exp\big(\lambda\, x\big)$, where P is a polynomial function and λ a real number different to zero, we define the particular solution y_p as a polynomial function multiplied by an exponential. Indeed, if here we assume the function f is in the form polynomial function multiplied by exponential:

$$f(x) = \left(\sum_{i=0}^{n} b_i\, x^i\right) \cdot \exp\Big(\lambda\, x\Big).$$

In this case we define the particular solution

$$y_p = \left(\sum_{i=0}^{n} a_i\, x^i\right) \cdot \exp\Big(\lambda\, x\Big).$$

There exist efficient methods to choose efficiently the order of the polynome P, but here we will only show how the method works by using examples.

Example 7.2.14. *Determine a particular solution to the equation* $y' + 3y = x\mathrm{e}^x$.

The second member f *is in the form* $f(x) = x\mathrm{e}^x$. *We should determine* y_p *in the form polynomial multiplied by exponential. That is, we define* $y_p = (bx + c)\,\mathrm{e}^x$. *We have* $y_p'(x) = b\mathrm{e}^x + (bx + c)\mathrm{e}^x$. *Then, we obtain*

$$y_p'(x) + 3y_p(x) = x\mathrm{e}^x \iff \mathrm{e}^x(bx + b + c) + 3(bx + c)\mathrm{e}^x = x\mathrm{e}^x.$$

This is equivalent to $4bx + b + 4c = x$. *This means that,* $4b = 1$ *and* $b + 4c = 0$. *Therefore* $b = \dfrac{1}{4}$ *and* $c = -\dfrac{1}{16}$. *The particular solution is*

$$y_p(x) = \left(\frac{1}{4}x - \frac{1}{16}\right)\mathrm{e}^x.$$

Example 7.2.15. *Define a particular solution to differential equation*

$$y' - 4y = (2x + 5)\mathrm{e}^x.$$

Since $f(x) = (2x + 5)\,\mathrm{e}^x$, *we define* $y_p(x) = (ax + b)\,\mathrm{e}^x$. *In this case, we have* $y_p'(x) = (ax + a + b)\mathrm{e}^x$. *This is equivalent to*

$$(-3ax + a - 3b)\mathrm{e}^x = (2x + 5)\mathrm{e}^x.$$

This equation leads to the following identity We deduce $a = -\dfrac{2}{3}$ *and* $b = -\dfrac{17}{9}$. *This gives*

$$y_p(x) = \left(-\frac{2}{3}x - \frac{17}{9}\right)\mathrm{e}^x.$$

Second Member in the Form $\alpha\cos(ax)$ **or** $\beta\sin(ax)$

If the function f is equal to $\alpha\cos(ax)$ or $\beta\sin(ax)$, where a, α and β are real numbers, we define y_p as a combination of $\cos(ax)$ and $\sin(ax)$. In other words, we set

$$y_p(x) = A\cos(ax) + B\sin(ax),$$

where A and B are real numbers. To determine y_p, we just need to find the values of A and B. We show how to find A and B by using examples.

Example 7.2.16. *Determine a particular solution solutions to*

$$y' + y = \cos(x).$$

Here the second member $f(x) = \cos(x)$. *We should define* y_p *as a combination of* $\cos(x)$ *and* $\sin(x)$. *We set*

$$y_p(x) = A\cos(x) + B\sin(x).$$

Then, we have

$$y_p'(x) = -A\sin(x) + B\cos(x).$$

Substituting in the differential equation, we obtain

$$(B - A)\sin(x) + (A + B)\cos(x) = \cos(x).$$

This is equivalent to $B - A = 0$ *and* $A + B = 1$. *From this we deduce* $A = \dfrac{1}{2}$ *and* $B = \dfrac{1}{2}$ *Therefore,*

$$y_p(x) = \frac{1}{2}\cos(x) + \frac{1}{2}\sin(x).$$

As asserted in the previous theorem the existence of a solution to (7.2.1), Gunay would like to know if the solutions had a known expression. To answer this question we introduce the following subsection

7.2.3 Solution to the Nonhomogeneous Equation

The teacher confirmed there were one such formulas. But he needed to precise some propertie shared by thes solutions before establishing their expressions.

He took two functions y_H and y_p, which are defined in the real interval I and satisfied these differential equations

$$y_H' + a(x)\, y_H = 0 \quad \text{and} \quad y_p' + a(x)\, y_p = f(x).$$

The function y_H is a solution to the homogeneous linear equation. We called it a homogeneous solution. The function y_p is called a particular solution to the nonhomogeneous linear differential equation. Now, let $x \in I$ and define the function $z(x) = y_H(x) + y_p(x)$. The functions y_H and y_p were assumed to be differentiable. Then the function z is differentiable and the following holds $z'(x) = y_H'(x) + y_p'(x)$. This yields

$$z'(x) + a(x)\, z = y_H' + a(x)\, y_p' + a(x)\,(y_H + y_p) = (y_H' + a(x)\, y_H) + (y_p' + a(x)\, y_p).$$

The functions y_H and y_p which are in $\mathscr{C}^1(I)$ satisfy equations $y_H' + a(x)\, y_H = 0$ and $y_p' + a(x)\, y_p = f(x)$. Hence $z' + a(x)\, z = f(x)$. We conclude the function z is a solution to (7.2.1).

Let $y \in \mathscr{C}^1$ be a solution to (7.2.1). We assume there are $x_0 \in I$, and $y_0 \in \mathbb{R}$, such that $y(x_0) = y_0$. Then, the function y satisfies the nonhomogeous linear differential equation $y' + a(x)\, y = f(x)$. We supposed a to be a continuous function on I. We define A as the primitive of a on I, which satisfies $A(x_0) = 0$. Multiplying (7.2.1) by $\exp(A(x))$, we get

$$y' \exp(A(x)) + a(x) \exp(A(x))\, y = \Big(y \exp(A(x))\Big)' = \exp(A(x))\, f(x).$$

Integrating from x_0 to $x \in I$, one has

$$\int_{x_0}^{x} \Big(y \exp(A(t))\Big)' \, dt = \exp(A(x))y(x) - y_0 = \int_{x_0}^{x} \exp(A(t))\, f(t)\, dt.$$

This leads to the following identity

$$y(x) = y_0 \cdot \exp(-A(x)) + \exp(-A(x)) \int_{x_0}^{x} \exp(A(t))\, f(t)\, dt.$$

We can observe that $y_H = y_0 \exp(-A(x))$ is a solution to the homogeneous linear differential equation $y_H' + a(x)\, y_H = 0$.

Students asked the signification of the second part of the right hand side of the formula. To show that the second part corresponds to a particular solution of (7.2.1), he recalled how to derivate under the sign integral.

Proposition 7.2.17. *Let g be a continuous function on the interval $[\,a\,,\,b\,]$. For every real number x in $[\,a\,,\,b\,]$, we define the function*

$$F(x) = \int^{x} g(t)\, dt.$$

Then, the function F is differential in $(\,a\,,\,b\,)$ and for every $x \in (a\,,\,b)$,

$$F'(x) = g(x). \tag{7.2.2}$$

Now, we take a real number x in I and we consider the function

$$y_p(x) = \exp(-A(x)) \int_{x_0}^{x} \exp(A(t))\, f(t)\, dt.$$

The function y_p is a multiplication of two differentiable functions. Then, the funcion y_p is differentiable. Let $x \in I$, we have

$$y_p'(x) = -A'(x) \exp(-A(x)) \int_{x_0}^{x} \exp(A(t))\, f(t) dt + \ exp(-A(x)) \exp(A(x))\, f(x).$$

The Proposition above involves

$$\left(\int_{x_0}^{x} \exp(A(t))\, f(t)\, dt \right)' = \exp(A(x))\, f(x).$$

As we defined A, such that, $A'(x) = a(x)$. One has

$$y_p'(x) = -a(x) \exp(-A(x)) \int_{x_0}^{x} \exp(A(t))\, f(t)\, dt + f(x) = -a(x) y_p + f(x).$$

This leads to the identity: $y_p = -a(x)\, y_p + f(x)$. Therefore $y_p' + a(x)\, y_p = f(x)$. Consequently the function y_p is a particular solution to (7.2.1). This is equivalent to say $y = y_H + y_p$.

Theorem 7.2.18. *Let y be a solution to (7.2.1) in some interval I. Then there exist two functions y_H and y_p in $\mathscr{C}^1(I)$, such that, for every $x \in I$*

$$y(x) = y_H(x) + y_p(x). \tag{7.2.3}$$

The function y_H satisfies $y_H' + a(x)\, y_H = 0$ and the function y_p is a particular solution of (7.2.1).

Example 7.2.19. *Find a solution to* $y' + 2\,x\,y = x\,\mathrm{e}^{-x^2}$

We remind that the solution y *is defined by*

$$y(x) = y_H(x) + y_p(x).$$

where y_H *satisfies the homogeneous equation* $y'_H + 2\,x\,y_H = 0$ *and* y_p *is a particular solution.*

We know from the previous section that a general homogeneous solution is defined by

$$y_H(x) = K\mathrm{e}^{-x^2}.$$

One can easily check that the function $y_p(x) = \dfrac{x^2}{2}\,\exp\big(-x^2\big)$ *is a particular solution. Therefore a general solution to the nonhomogeneous equation is defined by*

$$y(x) = K\mathrm{e}^{-x^2} + \frac{x^2}{2}\mathrm{e}^{-x^2} = \left(\frac{x^2}{2} + K\right)\exp\left(-x^2\right),$$

where K *is some constant real number.*

Example 7.2.20. *Find a solution to* $\sin(x)y' + \cos(x)y = x$.

For every $x \in (-\pi\,,\,0\,) \cup (\,0\,,\,\pi\,)$, *the equation* $\sin(x)y'_H + \cos(x)y_H = 0$ *is equivalent to*

$$y'_H + \frac{\cos(x)}{\sin(x)}y_H = 0$$

Let $x \in (-\pi\,,\,0\,) \cup (\,0\,,\,\pi\,)$. *A general homogeneous solution is defined by*

$$y_H(x) = K\,\mathrm{e}^{-\,\ln(|\,\sin(x)|)} = \frac{K}{|\sin(x)|}$$

where K *is a real number. Hence we have*

$$y_H(x) = \begin{cases} \dfrac{-K}{\sin(x)} & \text{if} \quad x \in (-\pi\,,\,0\,) \\[2ex] \dfrac{K}{\sin(x)} & \text{if} \quad x \in (\,0\,,\,\pi\,). \end{cases}$$

The function

$$y_p(x) = \begin{cases} -\dfrac{x^2}{2\,\sin(x)}\,, & \text{if} \quad x \in (-\pi\,,\,0\,) \\[2ex] +\dfrac{x^2}{2\,\sin(x)}\,, & \text{if} \quad x \in (\,0\,,\,\pi\,) \end{cases}$$

is a particular solution. Then, we obtain the following solution

$$y(x) = \begin{cases} \dfrac{-K}{\sin(x)} - \dfrac{x^2}{2\,\sin(x)}\,, & \text{if} \quad x \in (-\pi\,,\,0\,) \\[2ex] \dfrac{K}{\sin(x)} + \dfrac{x^2}{2\,\sin(x)}\,, & \text{if} \quad x \in (\,0\,,\,\pi\,) \end{cases}$$

Example 7.2.21. *Determine a solutions to* $y' + 2y = x^2 + 2x + 1$.

A general Homogeneous solution is in the form $y_H(x) = K\exp(-2x)$. *A particular solution is given by*

$$y_p(x) = \frac{1}{2}\left(x^2 + x + \frac{1}{2}\right)$$

Therefore we have this general solution

$$y(x) = K\mathrm{e}^{-2x} + \frac{1}{2}\left(x^2 + x + \frac{1}{2}\right).$$

Let a, f_1 and f_2 be three continuous functions in I. We consider the following linear differential equations: Find y_1 and y_2 in $\mathscr{C}^1(I)$ solution to the linear differential equations

$$y_1' + a(x)\, y_1 = f_1(x) \tag{7.2.4}$$
$$y_2' + a(x)\, y_2 = f_2(x). \tag{7.2.5}$$

For any real numbers λ and μ , we define the function

$$y(x) = \lambda\, y_1(x) + \mu\, y_2(x).$$

The function y satisfies $y'(x) = \lambda\, y_1'(x) + \mu\, y_2'(x)$. This involves

$$\begin{aligned} y'(x) + a(x)y &= \lambda\, y_1' + \mu\, y_2' + a(x)[\lambda\, y_1 + \mu\, y_2] = \lambda\big[y_1' + a(x)\, y_1\big] + \mu\big[y_2' + a(x)\, y_2\big] \\ &= \lambda\, f_1(x) + \mu\, f_2(x). \end{aligned}$$

This means the function y is solution to the equation

$$y'(x) + a(x)\, y = \lambda\, f_1(x) + \mu\, f_2(x).$$

This is called the "principle of superposition". Let the functions f_i be continuous in I for $1 \leq i \leq n$. We define the functions y_i as solutions to the equations $y_i' + a(x)\, y_i = f_i$ and consider the function

$$y = \sum_{i=1}^{n} \alpha_i\, y_i \quad \text{and} \quad f(x) = \sum_{i=1}^{n} \alpha_i\, f_i.$$

Then y is a solution of the equation $y' + a(x)\, y = f(x)$.

Theorem 7.2.22 (Principle of Superposition)**.** *Let* a, *and* f_i *for* $1 \leq i \leq n$ *be* $n+1$ *continuous functions in* I. *We consider* n *function* $y_i \in \mathscr{C}^1(I)$ *for* $1 \leq i \leq n$ *solution to the equation*

$$y_i' + a(x)\, y_i = f_i(x). \tag{7.2.6}$$

Then, the function $y(x) = \alpha_1\, y_1(x) + \cdots + \alpha_n\, y_n(x)$ *is solution to the linear differential equation*

$$y' + a(x)\, y = \sum_{i=1}^{n} \alpha_i\, f_i. \tag{7.2.7}$$

Example 7.2.23. *Find a solution to* $y' + y = x - \mathrm{e}^x + cos(x)$

Here we use the principle of superposition by defing the function $y = y_1 + y_1 + y_3$, *where the functions* y_1, y_2 *and* y_3 *are solutions of the following equations*

$$\begin{aligned} y_1' + y_1 &= x \\ y_2' + y_2 &= -\mathrm{e}^x \\ y_3' + y_3 &= \cos(x). \end{aligned}$$

We point out that the linear differential equation $y_1' + y_1 = x$ *has solution*

$$y_1(x) = K_1\,\mathrm{e}^{-x} + x - 1.$$

A solution of $y_2' + y_2 = -\mathrm{e}^x$ *can be given by* $y_2(x) = K_2\,\exp(-x) - \frac{1}{2}\exp(x)$. *For the differential* $y_3' + y_3 = \cos(x)$, *one has* $y_3(x) = K_3\,\mathrm{e}^{-x} + \frac{1}{2}\left(\cos(x) + \sin(x)\right)$. *Applying theorem 7.2.22 we obtain*

$$y(x) = K'\,\mathrm{e}^{-x} - \frac{1}{2}\,\mathrm{e}^x + x - 1 + \frac{1}{2}\Big(\cos(x) + \sin(x)\Big),$$

where K' *is a constant real number.*

To obtain a unique solution for a nonhomogeneous equation, we add an initial condition. In this case we obtain a nonhomogeneous Cauchy problem

7.2.4 Nonhomogeneous Cauchy Problem

This subsection is devoted to the nonhomogeneous linear Cauchy problem. Indeed, for $x_0 \in I$ and $y_0 \in \mathbb{R}$, we are looking for a unique function y in $\mathscr{C}^1(I)$ solution to the following problem

$$\begin{cases} y' + a(x)\,y = f(x) \\ \\ y(x_0) = y_0 \end{cases} \tag{7.2.8}$$

We know that a solution to the differential equation $y' + a(x)\,y = f(x)$ is in the form

$$y(x) = K\,\exp\left(-\int^x a(t)\,dt\right) + y_p(x).$$

To get uniqueness we need to determine a fix value of K. This value is obtained by using the initial condition. We have

$$y_0 = y(x_0) = K\,\exp\left(-\int^{x_0} a(t)\,dt\right) + y_p(x_0)$$

This implies

$$K = \Big(y_0 - y(x_0)\Big)\cdot\exp\left(\int^{x_0} a(t)\,dt\right).$$

in this case we obtain a unique solution.

Theorem 7.2.24. *Let* a *and* f *be two continuous functions in an interval* I. *We consider* $x_0 \in I$ *and* $y_0 \in \mathbb{R}$. *Then, the Cauchy problem* (7.2.8) *has a unique solution.*

Example 7.2.25. *Determine in $I = (0\,;+\infty)$ the unique solution to the following Cauchy problem*

$$\begin{cases} y' + \dfrac{1}{x}y = x \\ \\ y(1) = 3. \end{cases}$$

We consider the differential equation $y' + \dfrac{1}{x}y = x$. The homogeneous solution is defined by

$$y_H(x) = K\exp\big(-\ln(x)\big) = \frac{K}{x},$$

where K is a constant real number. Since $f(x) = x$, we define $y_p(x) = ax^2 + bx + c$. Then, $y_p'(x) = 2ax + b$. We obtain $3ax + 2b + \dfrac{c}{x} = x$. this means $3a = 1$, $2b = 0$ and $c = 0$. Therefore, $a = \dfrac{1}{3}$, and $b = c = 0$. The particular solution is $y_p(x) = \dfrac{1}{3}x^2$. The solution of the general equation is

$$y(x) = \frac{K}{x} + \frac{1}{3}x^2.$$

Now we have to use the initial condition to find the constant K. We know that $y(1) = 3$. Hence , one has $K + \frac{1}{3} = 3$. We obtain $K = 3 - \dfrac{1}{3} = \dfrac{8}{3}$. This yields the unique solution to the Cauchy problem is defined by

$$y(x) = \frac{8}{3x} + \frac{1}{3}x^2.$$

Example 7.2.26. *Determine the unique solution to the Cauchy problem*

$$\begin{cases} y' + \cos(x)\, y = \cos(x) \\ \\ y(0) = 2. \end{cases}$$

We remark that the function $y_p(x) = 1$ is a particular solution to the equation

$$y' + \cos(x)\, y = \cos(x).$$

On the other hand we have $y_H(x) = K\,\exp(-\sin(x))$, where K is a real number Then ,

$$y(x) = K\,\exp(-\sin(x)) + 1.$$

Since $y(0) = 2$, we have $K + 1 = 2$. This means that $K = 1$. The unique solution of the initial value problem is defined by

$$y(x) = \mathrm{e}^{-\sin(x)} + 1.$$

Example 7.2.27. *Find the unique solution to the following Cauchy problem*

$$\begin{cases} y' + y = x + \mathrm{e}^x \\ \\ y(0) = 0 \end{cases}$$

The homogeneous equation has solution $y_H(x) = K\exp(-x)$, *where* K *is a constant real number. We can easily check that the function*

$$y_p(x) = \frac{1}{2}\exp(x) + x - 1.$$

Hence

$$y(x) = K\mathrm{e}^{-x} + \frac{\mathrm{e}^x}{2} + x - 1.$$

Now we use the initial condition $y(0) = 0$. *Then, we obtain*

$$y(0) = K + \frac{1}{2} - 1 = 0$$

This implies $K = \dfrac{1}{2}$. *Therefore,*

$$y(x) = \frac{1}{2}\mathrm{e}^{-x} + \frac{\mathrm{e}^x}{2} + x - 1 = \mathrm{ch}(x) + x - 1$$

CHAPTER 8

SECOND ORDER LINEAR ODES

Contents

This chapter is devoted to second order linear ordinary differential equations. We specify here we will only deal with second order linear differential equation with constant coefficients: find a function y of class $\mathscr{C}^2$ satisfying

$$ay'' + by' + cy = f(x). \tag{8.0.1}$$

Here, a, b and c are constant real numbers. Moreover we assume $a \neq 0$. Otherwise we will have $b\,y' + c\,y = f(x)$, which is a first order linear differential equation. An example of such equation is given by

$$2y'' + y' + 3y = x^3 + 3x + 1 \quad \text{or} \quad 3y'' + 5y' + 4y = \mathrm{e}^{x^2+2}.$$

These equations occured in the formulation of Newton's second law

$$\mathbf{m}\ddot{\mathrm{x}} = \sum \mathbf{F}_{\mathrm{ext}}. \tag{8.0.2}$$

This equation describes the motion of an object of mass $\mathbf{m}$ subjected to the forces $\mathbf{F}_{\mathrm{ext}}$. In (8.0.2) we use physicist's notations $\ddot{\mathrm{x}}$. We mean by $\ddot{\mathrm{x}}$ the second derivative of the trajectory x. Here $\ddot{\mathrm{x}}$ representate the acceleration of the mouvement of an object subjected to the external forces. In other words Newton second Law's claims that the acceleration is proportional to the sum of the external forces. Equation (8.0.2) is one of the most important equation in Physics. Several other physical

phenomenons can be described by these equations. As we did it for first order linear differential equations, we are looking for necessary conditions such that there will be a solution for (8.0.1).

In the first time we will deal with the homogeneous second order linear differential equation $ay'' + by' + cy = 0$. In the second section we will be concerned by the Cauchy problems.

8.1 Homogeneous Equation

The main purpose of this section is to define a solution to the equation

$$ay'' + by' + cy = 0,$$

where a, b and c are constant real numbers such that, $a \neq 0$. As an example of homogeneous equation we consider the equation $y'' + 4y' + 9y = 0$. In a first time we will establish an exsitence theorem. In the second part of this section we define the expression of the solutions to $ay'' + by' + cy = 0$, we remind that the coefficients a, b and c are constant real numbers such that $a \neq 0$.

Let a, b and c be three real numbers such that $a \neq 0$. We are looking for a function y which is twice differentiable in some interval of $\mathbb{R}$ and satisfies the following differential equation

$$ay'' + by' + cy = 0. \tag{8.1.1}$$

We introduce the following function $z(x) := y'(x)$. This involves $z'(x) = y''(x)$. Substituting in the equation above we obtain

$$a\,z' + b\,z = -cy.$$

Since $a \neq 0$ we can divide by a in both sides in the equation including the function z. Therefore, one has

$$z' + \frac{b}{a}\,z = -\frac{c}{a}\,y. \tag{8.1.2}$$

As we assume y differentiable in $\mathbb{R}$, then, the function y is continuous. Consequently the function $f(x) = -\dfrac{c}{a}\,y$ is continuous in $\mathbb{R}$. One deduces equation (8.1.2) admits a solution in $\mathbb{R}$. Since (8.1.1) and (8.1.2) are equivalent , we conclude (8.1.1) has a solution in $\mathbb{R}$. This proves this theorem

Theorem 8.1.1. *Let a, b and c be three real numbers such that, $a \neq 0$. Then, the second order linear differential equation $ay'' + by' + cy = 0$ admits a solution.*

As we know there exists a solutions, our next step will consist of studying some properties of these solutions.

8.1.1 Properties of the Homogeneous Solutions

In this subsection we study the properties of the homogeneous solution.

Let y_1 and y_2 be two different solutions of (8.1.1). We consider two real numbers α and β and we define the function $y = \alpha\,y_1 + \beta\,y_2$. The function y is twice

differentiable, because the functions y_1 and y_2 are twice differentiable. We have $y' = \alpha\, y_1' + \beta\, y_2'$ and $y'' = \alpha\, y_1'' + \beta\, y_2''$. This involves

$$\begin{aligned} a\,y'' + b\,y' + c\,y &= a\big(\alpha\, y_1'' + \beta\, y_2''\big) + b\big(\alpha\, y_1' + \beta\, y_2'\big) + c\big(\alpha\, y_1 + \beta\, y_2\big) \\ &= \alpha\big(a\, y_1'' + b\, y_1' + c\, y_1\big) + \beta\big(a\, y_2'' + b\, y_2' + c\, y_2\big) \end{aligned}$$

As y_1 and y_2 are solutions of (8.1.1), therefore $a\, y_1'' + b\, y_1' + c\, y_1 = 0$ and $a\, y_2'' + b\, y_2' + c\, y_2 = 0$. Hence $a\, y'' + b\, y' + c\, y = 0$. This means the function y which is a combination of solution is also a solution. Since the null function is a solution to the differential equation, we have the following theorem.

Theorem 8.1.2. *The set of all solutions of* (8.1.1) *is a vector space.*

We can make the following comments on theorem 8.1.2. We can prove that the set $\mathcal{S}_0$ of all solution of equation (8.1.1) is a vector space of dimension 2. Taking advantage of the fact that $\mathcal{S}_0$ is a vector space of dimension 2, we can define a basis of two function f_1 and f_2 on this set. This allows to establish

Theorem 8.1.3. *Let $\mathcal{S}_0$ be the st of all solution of equation* (8.1.1) *and $\{f_1\,;\, f_2\}$ a basis of $\mathcal{S}_0$. Then, for any solution y of* (8.1.1) *there exist two real numbers α and β such that*

$$y = \alpha\, f_1 + \beta\, f_2.$$

This theorem means that any solution should be a combination of two elements of a basis of $\mathcal{S}_0$. That is, determining $\mathcal{S}_0$ consists of finding a basis of this set.

8.1.2 Definition of a Basis to the Set of Solutions

The goal of this subsection is to define a basis of $\mathcal{S}_0$. A basis of $\mathcal{S}_0$ should be a set of two functions which are linearly independant.

Definition 8.1.4. *Let f_1 and f_2 be two elements of a set E. We say f_1 and f_2 are linearly independant if for any real numbers α and β, $\alpha\, f_1 + \beta\, f_2 = 0$ implies $\alpha = \beta = 0$.*

Example 8.1.5. *We consider the vectors $u_1 = (2\,,\,5)$ and $u_2 = (3\,,\,0)$. We consider two real numbers α and β such that, $\alpha\, u_1 + \beta\, u_2 = 0$. This is equivalent to the following linear system*

$$\begin{cases} 2\alpha + 3\beta = 0 \\ 5\alpha \qquad = 0 \end{cases}$$

This implies $\alpha = 0$ and $\beta = 0$. We conclude u_1 and u_1 are linearly independant.

Example 8.1.6. *Let $v_1 = (1\,,\,2\,,\,5)$ and $v_2 = (2\,,\,3\,,\,1)$ be two in $\mathbb{R}^3$, which satisfy $\alpha\, v_1 + \beta\, v_2 = 0$ for some real numbers α and β. One has the following linear system*

$$\begin{cases} \alpha + 2\,\beta = 0 \\ 2\alpha + 3\,\beta = 0 \\ 5\alpha + \beta = 0 \end{cases}$$

This leads to the identities $\alpha = -2\beta$ and $\alpha = -\beta/5$. Therefore $\alpha = \beta = 0$. The vectors v_1 and v_2 are linearly independant.

Example 8.1.7. *We consider the vectors* $w_1 = (2\,,\,3)$ *and* $w_2 = (12\,,\,18)$ *in* $\mathbb{R}^2$. *We observe* $w_2 = 6\,w_1$. *Then, the vectors* w_1 *and* w_2 *are linearly dependant.*

To simplify we say f_1 and f_2 are linearly independant if they are not collinear. If elements are not linearly independant, we say they are linearly dependant. Two linear dependant elements are collinear.

Since we will deal with functions it is important to get a criterion which will allows us to check whether two functions are linearly independant.

Theorem 8.1.8. *Let* f_1 *and* f_2 *be two* $\mathscr{C}^1$ *functions defined in some interval* I. *Then,* f_1 *and* f_2 *are lineraly independant if and only if for all real number* $x \in I$,

$$f_1(x) \cdot f_2'(x) - f_1'(x) \cdot f_2(x) \neq 0.$$

In truth we can estabblish a more simplest result which is equivalent to the previous theorem. This theorem is more easy to use in applications.

Theorem 8.1.9. *Let* f_1 *and* f_2 *be two* $\mathscr{C}^1$ *functions defined in some interval* I. *Then,* f_1 *and* f_2 *are lineraly independant if and only if there exist* $x_0 \in I$,

$$f_1(x_0) \cdot f_2'(x_0) - f_1'(x_0) \cdot f_2(x_0) \neq 0.$$

Example 8.1.10. *We consider the function* $f_1(x) = x + 2$ *and* $f_2(x) = 2x^2 + 2$. *Since* f_1 *and* f_2 *are differentiable, one has* $f_1'(x) = 1$ *and* $f_2'(x) = 4x$. *We obtain* $f_1(0) = 2$, $f_1'(0) = 1$, $f_2(0) = 2$ *and* $f_2'(0) = 0$. *This implies*

$$f_1(0) \cdot f_2'(0) - f_1'(0) \cdot f_2(0) = 2 \times 0 - 1 \times 1 = -1 \neq 0.$$

The functions f_1 *and* f_2 *are linearly independant.*

Example 8.1.11. *Let* a *and* b *be real numbers such that* $a \neq b$. *We define the functions* $u_1(x) = \exp(a\,x)$ *and* $u_2(x) = \exp(b\,x)$. *We have* $u_1'(x) = a\,\exp(ax)$ *and* $u_2'(x) = b\,\exp(b\,x)$. *Taking* $x_0 = 0$, *we obtain* $u_1(0) = 1$, $u_2(0) = 1$, $u_1'(0) = a$ *and* $u_2'(0) = b$. *Therefore*

$$u_1(0) \cdot u_2'(0) - u_1'(0) \cdot u_2(0) = b - a \neq 0.$$

The functions u_1 *and* u_2 *are linearly independant.*

Here, we choose 0 in order to simplify notations, but one can choose another real number.

Exercise 8.1.12. *Check whether the following functions are linearly independant*

1. $f_1(x) = \cos(x)$ *and* $f_2(x) = \sin(x)$, 2. $f_1(x) = \exp(ax)$ *and* $f_2(x) = \exp(-ax)$,
3. $f_1(x) = x^4 \quad$ *and* $\quad f_2(x) = x^7$.

There are several methods to define a basis of a set of function. But here we favor the less abstract method. In comparasion with first differential equation, here also we are looking for solution in the form $\exp(A(x))$.

Let r be a real or a complex number. For every real number x we define the function $y(x) = \exp(rx)$. We assume the fuction y is a solution to the linear homogeneous equation $ay'' + by' + cy = 0$. Now our objective is to determine the

values of the real nmber r. The function y is at least twice differentiable and for every x in $\mathbb{R}$, we have $y'(x) = r\,\exp(rx)$ and $y''(x) = r^2\,\exp(rx)$. Using our assumption, one obtains

$$a\,r^2\exp(rx) + b\,r\exp(rx) + c\exp(rx) = \big(a\,r^2 + b\,r + c\big)\exp(rx) = 0.$$

Multiplying the identity above by $\exp(-rx)$, we obtain

$$a\,r^2 + b\,r + c = 0. \tag{8.1.3}$$

Equation (8.1.3) is called the caracteristic equation of (8.1.1). To determine r we need to solve equation (8.1.3). There are three possibilities according to the sign of

$$\Delta = b^2 - 4ac.$$

Case 1:$\Delta > 0$. When $\Delta > 0$, equation (8.1.3) has two real solution $r_1 = \dfrac{-b - \sqrt{\Delta}}{2a}$ or $r_2 = \dfrac{-b + \sqrt{\Delta}}{2a}\cdot$ In this case we have

$$y(x) = \exp\left(r_1\,x\right) \qquad \text{or} \qquad y(x) = \exp\left(r_2\,x\right).$$

The functions $f_1(x) = \exp(r_1\,x)$ and $f_2(x) = \exp(r_2\,x)$ are indefinitely differentiable and for every real number x we have

$$\big(\exp(r_1\,x)\big)' = r_1\exp(r_1\,x) \qquad \text{and} \qquad \big(\exp(r_2\,x)\big)' = r_2\,\exp(r_2\,x).$$

Taking $x_0 = 0$, we obtain

$$f_1(0)\cdot f_2'(0) - f_1'(0)\cdot f_2(0) = r_2 - r_1 = \frac{\Delta}{a} \neq 0.$$

We have established this lemme

Lemma 8.1.13. *The functions* $\exp(r_1\,x)$ *and* $\exp(r_2\,x)$ *are linearly independant functions which belong to* $\mathcal{S}_0$.

We know from linear algebra if E is a vector space of dimension n any linearly independant familly of n elements is a basis. Since $\dim\big(\mathcal{S}_0\big) = 2$, any linearly independant set of two vectors is a basis. Then we have

Lemma 8.1.14. *The set of functions* $\big\{\exp(r_1\,x)\,;\,\exp(r_2\,x)\big\}$ *is a basis of* $\mathcal{S}_0$.

On the other hand we know that a combination of solutions is also a solution. As we have a basis of $\mathcal{S}_0$, we can establish

Theorem 8.1.15. *Let us consider equation* (8.1.1)*, such that,* $\Delta = b^2 - 4\,a\,c > 0$. *We define the real numbers* r_1 *and* r_2 *as the real solutions of the equation*

$$a\,r^2 + b\,r + c = 0.$$

Then, y_H *is a solution of* (8.1.1) *if there exist real numbers* C_1 *and* C_2 *such that,*

$$y_H(x) = C_1\,\exp(r_1\,x) + C_2\,\exp(r_2\,x). \tag{8.1.4}$$

Proof. The functions $\exp(r_1\,x)$ and $\exp(r_2\,x)$ are solutions of (8.1.1). Then, the combination $C_1\exp(r_1\,x)+C_2\exp(r_2\,x)$, where C_1 and C_2 are in $\mathbb{R}$, is a solution of(8.1.1).

On the other hand the set $\left\{\exp(r_1\,x)\,,\,\exp(r_2\,x)\right\}$ is a basis, then, any function in $\mathcal{S}_0$ is a combination of $\exp(r_1\,x)$ and $\exp(r_2)$. Therefore one can find real number C_1 and C_2 such that, $y_H(x)=C_1\,\exp(r_1\,x)+C_2\,\exp(r_2\,x)$. □

Example 8.1.16. *To find solutions of the equation $y''+4y'+3y=0$ we define its caracteristic equation*

$$(*)\quad r^2+4\,r+3=0.$$

The discriminant $\Delta=16-12=4>0$. The real solutions of $(*)$ are

$$r_1=\frac{-4-2}{2}=-3\quad or\quad r_2=\frac{-4+2}{2}=-1.$$

Then, a general solution of the second order linear differential equation $y''+4y'+3y=0$ is defined by

$$y_H(x)=C_1\,\mathrm{e}^{-3\,x}+C_2\,\mathrm{e}^{-x},$$

where C_1 and C_2 are real numbers.

Example 8.1.17. *Find solutions to the following second order linear differential equation*

$$y''+3y'+2y=0.$$

To solve this equation we define the caracteristical equation

$$(\star)\quad r^2+3\,r+2=0.$$

Since $\Delta=9-8=1>0$, the second order equation $(\star)$ has solution

$$r_1=\frac{-3-1}{2}=-2\quad or\quad r_2=\frac{-3+1}{2}=-1.$$

Hence we can find C_1 and C_2 such that

$$y_H(x)=C_1\,\mathrm{e}^{-2\,x}+C_2\,\mathrm{e}^{-x}.$$

Case 2: $\Delta=0$. When $\Delta=0$, equation (8.1.3) has only one real solution $r_0=-\dfrac{b}{2a}$. In this case we obtain

$$y(x)=\exp(r_0\,x).$$

To construct a basis of $\mathcal{S}_0$ we need to find a second solution y_2 of (8.1.1) which is linearly independant to $\exp(r_0\,x)$.

Now we define the function $y_2(x)=x\,\exp(r_0\,x)$. The function y_2 is indefinitely differentiable in $\mathbb{R}$ and for any real number x

$$y_2'(x)=(r_0\,x+1)\exp(r_0\,x)\quad\text{and}\quad y_2''(x)=\left(r_0^2\,x+2\,r_0\right)\exp(r_0\,x).$$

Substituting in (8.1.1) the following holds

$$\left(a\,r_0^2+b\,r_0+c\right)x+(2a\,r_0+b)=0,\quad\forall\quad x\in\mathbb{R}.$$

Since $a\,r^2+b\,r+c=0$ and $r_0=-\dfrac{b}{2a}$ we conclude that

Lemma 8.1.18. *The function* $y_2(x) = x \cdot \exp(r_0\, x)$ *is a solution to* (8.1.1).

Lemma 8.1.19. *The functions* $\exp(r_0\, x)$ *and* $x \cdot \exp(r_0\, x)$ *are linearly independant.*

Proof. Let us define $f_1(x) = \exp(r_0\, x)$ and $f_2(x) = x \cdot \exp(r_0\, x)$. From this one deduces $f_1'(x) = r_0 \exp(r_0\, x)$ and $f_2'(x) = (r_0\, x + 1) \exp(r_0\, x)$. Choosing $x_0 = 0$ one has

$$f_1(0) \cdot f_2'(0) - f_1'(0) \cdot f_2(0) = 1 \neq 0.$$

We conclude f_1 and f_2 are linearly independant. □

From this lemma we deduce the set $\left\{ \exp(r_0\, x)\,,\, x \cdot \exp(r_0\, x) \right\}$ is a basis of the two dimensional vector space $\mathcal{S}_0$.

Theorem 8.1.20. *We consider the second order linear differential equation* (8.1.1) *such that,* $\Delta = b^2 - 4a\, c = 0$. *We define the real number* $r_0 = -b/2a$. *Then,* y *is a real solution of* (8.1.1) *if there exist real numbers* C_1 *and* C_2 *such that, for every* $x \in \mathbb{R}$,

$$y_H(x) = (C_1\, x + C_2) \cdot \exp(r_0\, x). \tag{8.1.5}$$

Proof. The functions $\exp(r_0\, x)$ and $x \cdot \exp(r_0\, x)$ are solutions of (8.1.1). Then, any combination of these two function is a solution. Hence y_H is a solution of (8.1.1).

Now we take y_H a solution of (8.1.1). The function y_H belongs to $\mathcal{S}_0$. Since $\left\{ \exp(r_0\, x)\,,\, x \cdot \exp(r_0\, x) \right\}$ is a basis of $\mathcal{S}_0$, then, there exist two real numbers C_1 and C_2, such that, $y_H(x) = \big(C_1\, x + C_2\big) \cdot \exp(r_0\, x)$. □

Example 8.1.21. *As an example, we consider the second order linear differential equation*

$$y'' + 2y' + y = 0.$$

To find a solution to this equation, we define the equation $r^2 + 2\, r + 1 = 0$. *Here* $\Delta = 0$. *Therefore* $r_0 = -\dfrac{b}{2a} = -1$ *is the unique solution to the equation* $r^2 + 2\, r + 1 = 0$. *We deduce, for every* x *in* $\mathbb{R}$, *the solution of the differential equation is*

$$y(x) = \big(C_1\, x + C_2\big) \cdot \exp(-x),$$

where C_1 *and* C_2 *are real numbers.*

Example 8.1.22. *Find the solution of the following second order linear differential equation*

$$y'' - 6\, y' + 9\, y = 0.$$

The associated caracteristical equation is $r^2 - 6\, r + 9 = 0$. *Then, we have* $\Delta = 36 - 4 \times 9 = 0$. *The solution of the caracteristical equation is* $r_0 = \dfrac{6}{2} = 3$. *Consequently we can find two real numbers* α *and* β *such that,*

$$y(x) = (\alpha\, x + \beta) \cdot \exp(3x).$$

Case 3:$\Delta < 0$.

Now we assume $\Delta < 0$. Then, the associated carateriscal equation has two complex solution r_1 and r_2. If we define

$$\alpha = -\frac{b}{2\, a} \qquad \text{and} \qquad \omega = \frac{\sqrt{-\Delta}}{2\, a}. \tag{8.1.6}$$

We set $r_1 := \alpha - \mathrm{i}\,\omega$ and $r_2 := \alpha + \mathrm{i}\,\omega$. In this case we have

$$y_1(x) = \exp\big((\alpha - \mathrm{i}\,\omega)\,x\big) \quad or \quad y_2(x) = \exp\big((\alpha + \mathrm{i}\,\omega)\,x\big).$$

As we can see it these function are complex functions. From this we deduce any complex combination of $\exp\big((\alpha - \mathrm{i}\,\omega)\,x\big)$ and $\exp\big((\alpha + \mathrm{i}\,\omega)\,x\big)$ is a complex solution of (8.1.1).

Theorem 8.1.23. *We consider the second order linear differential equation* (8.1.1) *such that* $\Delta = b^2 - 4\,a\,c < 0$. *Let* α *and* ω *be defined by* (8.1.6). *Then,* y *is a complex solution of* (8.1.1), *if there exist two comlex numbers* C_1 *and* C_2 *such that, for every real number* x

$$y(x) = \exp\big(\alpha\,x\big)\Big(C_1\,\exp\big(\mathrm{i}\,\omega\,x\big) + C_2\,\exp\big(-\mathrm{i}\,\omega\,x\big)\Big). \tag{8.1.7}$$

We specify that (8.1.7) gives us complex solutions. One may wounder if there are real solutions. To answer this question we consider a real solution y. The function y satisfies $\overline{y(x)} = y(x)$. In other words the function

$$C_1\,\exp\big((\alpha + \mathrm{i}\,\omega)\,x\big) + C_2\,\exp\big((\alpha - \mathrm{i}\,\omega)\,x\big)$$

is a real solution if we have the following identity

$$C_1\,\exp\big(\mathrm{i}\,\omega\,x\big) + C_2\,\exp\big(-\mathrm{i}\,\omega\,x\big) = \overline{C}_1\exp\big(-\mathrm{i}\,\omega\,x\big) + \overline{C}_2\exp\big((\mathrm{i}\,\omega)\,x\big).$$

This is equivalent to the following identity

$$\Big(C_1 - \overline{C}_2\Big)\,\exp\big(2\,\mathrm{i}\,\omega\,x\big) + \Big(C_2 - \overline{C}_1\Big) = 0. \tag{8.1.8}$$

Taking $x = 0$ in (8.1.8), we get $C_1 - \overline{C}_1 = -(C_2 - \overline{C}_2)$. This implies

$$\mathrm{Im}(C_1) = -\mathrm{Im}(C_2). \tag{8.1.9}$$

Fixing $x = \dfrac{\pi}{2\,\omega}$ on the other hand, we obtain $-(C_1 + \overline{C}_1) + (C_2 + \overline{C}_2) = 0$. This involves

$$\mathrm{Re}(C_1) = \mathrm{Re}(C_2). \tag{8.1.10}$$

Combining (8.1.10) and (8.1.9) we conclude that y is a real solution if and only if $C_2 = \overline{C}_1$. In this case if we define $C_1 = \beta + \mathrm{i}\,\gamma$, one obtains $C_2 = \beta - \mathrm{i}\,\gamma$. Therefore

$$y(x) = \exp(\alpha\,x)\Big[(\beta + \mathrm{i}\,\gamma)\,\exp\big(\mathrm{i}\,\omega\,x\big) + (\beta - \mathrm{i}\,\gamma)\,\exp\big(-\mathrm{i}\,\omega\,x\big)\Big].$$

Using Euler's Formula $\exp(\mathrm{i}\,\omega\,x) = \cos(\omega\,x) + \mathrm{i}\,\sin(\omega x)$, we have

$$\begin{aligned} y(x) &= \exp(\alpha\,x)\Big[(\beta + \mathrm{i}\,\gamma)\,\big(\cos(\omega\,x) + \mathrm{i}\,\sin(\omega x)\big) + (\beta - \mathrm{i}\,\gamma)\,\big(\cos(\omega x) - \mathrm{i}\,\sin(\omega\,x)\big)\Big] \\ &= \exp(\alpha\,x)\Big[2\,\beta\,\cos(\omega\,x) - 2\gamma\,\sin(\omega\,x)\Big] \end{aligned}$$

Theorem 8.1.24. *Let* α *and* ω *be defined by* (8.1.6). *We consider the second order linear differential equation* (8.1.1), *such that,* $\Delta = b^2 - 4\,a\,c < 0$. *Then,* y *is a real solution of* (8.1.1) *if there exist two real numbers* C_1 *and* C_2, *such that,*

$$y(x) = \exp(\alpha\,x)\Big[C_1\,\cos(\omega\,x) + C_2\,\sin(\omega\,x)\Big] \tag{8.1.11}$$

We learn from this theorem that , if α and ω are defined by (8.1.6), then, the familly $\big\{\cos(\omega\,x)\,,\,\sin(\omega\,x)\big\}$, is a basis of the vector space $\mathcal{S}_0$.

Example 8.1.25. *Find the solution of the following second order linear differential equation*

$$y'' + 2\,y' + 3\,y = 0.$$

To solve this equation we define the associated carateriscal equation $r^2 + 2\,r + 3 = 0$. *We have* $\Delta = 4 - 12 = -8 < 0$. *In this case we obtain*

$$\alpha = -\frac{2}{2} = -1 \qquad \textit{and} \qquad \omega = \frac{\sqrt{8}}{2} = \frac{2\sqrt{2}}{2} = \sqrt{2}.$$

The complex solution of the second order linear differential equation is

$$y(x) = C_1\,\exp\Big(\Big(-1 + \mathrm{i}\sqrt{2}\Big)\,x\Big) + C_2\exp\Big(\Big(-1 - \mathrm{i}\sqrt{2}\Big)\,x\Big)\,,$$

where C_1 *and* C_2 *are complex numbers. The real solution of the second order differential equation is*

$$y(x) = \exp(-x)\Big[C_3\cos\Big(\sqrt{2}\,x\Big) + C_4\sin\Big(\sqrt{2}\,x\Big)\Big],$$

where the constant C_3 *and* C_4 *are real numbers.*

Example 8.1.26. *We consider the second order linear differential equation*

$$y'' + y' + y = 0.$$

We can establish that $\Delta = b^2 - 4a\,c = 1 - 4 = -3$. *To find a solution solution we define* $\alpha = -1/2$ *and* $\omega = \sqrt{3}/2$. *Therefore there are two complex numbers* C_1 *and* C_2, *such that*

$$y(x) = C_1\exp\left(\left(\frac{-1 + \mathrm{i}\sqrt{3}}{2}\,x\right)\right) + C_2\exp\left(\left(\frac{-1 - \mathrm{i}\sqrt{3}}{2}\,x\right)\right).$$

This solution is a complex function.

The real solution is define by

$$y(x) = \exp\left(\frac{-x}{2}\right)\left[C_3\,\cos\left(\frac{\sqrt{3}}{2}\,x\right) + C_4\,\sin\left(\frac{\sqrt{3}}{2}\,x\right)\right],$$

with real numbers C_3 *and* C_4.

Example 8.1.27. *Find a solution to the following second order linear differential equation*

$$y'' - y' + 3\,y = 0.$$

In this case we have $\Delta = b^2 - 4a\,c = 1 - 12 = -11$. *We set* $\alpha = \dfrac{1}{2}$ *and* $\omega = \dfrac{\sqrt{11}}{2}$. *Hence the complex solution of the second order linear differential equation is*

$$y(x) = C_1\exp\left(\left(\frac{1 + \mathrm{i}\sqrt{11}}{2}\,x\right)\right) + C_2\exp\left(\left(\frac{1 - \mathrm{i}\sqrt{11}}{2}\,x\right)\right),$$

where C_1 and C_2 are complex numbers.

For real numbers C_3 and C_4, we define a real solution of the second order linear differential equation as follows

$$y(x) = \exp\left(\frac{x}{2}\right)\left[C_3\,\cos\left(\frac{\sqrt{11}}{2}\,x\right) + C_4\,\sin\left(\frac{\sqrt{11}}{2}\,x\right)\right].$$

Exercise 8.1.28. *Find real solution to the following second order linear differential equations*

1. $y'' + 2\,y' + 3\,y = 0$, 2. $y'' + 3\,y' + y = 0$, 3. $2y'' + 5\,y' - y = 0$, 4. $y'' + 5y = 0$,
5. $y'' + 4\,y' + 2\,y = 0$, 6. $3\,y'' + 5\,y' + 2\,y = 0$, 7. $y'' + 3\,y = 0$, 8. $y'' + 3\,y' = 0$.

As we can see it, in the expressions of solutions we have two constant numbers. To determine the values of these constants we need at least a system of two equations with two unknown. This means that we have to fix two initial value conditions or boundaries conditions. This means that we should define initial values problems or boundary condition values problems.

8.2 Homogeneous Cauchy Problem

In the previous section we established the second order linear differential equation $a\,y'' + b\,y' + c\,y = 0$ has multiple solutions. Indeed each time we change the constants C_1 and C_2 in the expression of the solution (8.1.4) we obtain a new solution.

The main goal of this section is to define a unique solution of (8.1.1). This can be only done if and only if we determine fixed values of C_1 and C_2 in (8.1.4). To this end we need to fix two conditions in order to get a linear system of two equations with two unknowns. According to the nature of these conditions, we will say we have an initial values problem or a boundary conditions problem.

We start this section by defining initial values and boundary conditions problems. After these definitions we establish the uniqueness of their solutions.

Definition 8.2.1. *Let y_1 and y_2 be two real numbers and I an interval that contains x_0. We consider a real function $y \in \mathscr{C}^2(I)$ and we define the homogeneous initial values problem satisfied by y in I as follows*

$$\begin{cases} a\,y'' + b\,y' + c\,y = 0, \;\; in \;\; I \\ \\ y(x_0) = y_1, \;\; y'(x_0) = y_2. \end{cases} \tag{8.2.1}$$

Example 8.2.2. *As an example we consider the following initial values problem $y'' + 3\,y' + y = 0$ in $\mathbb{R}$, $y(0) = 1$ and $y'(0) = 0$.*

Example 8.2.3. *Another example can be given by the following initial values problem $2\,y'' - y' + 7y = 0$ in $\mathbb{R}$, $y(1) = 2$ and $y'(1) = 4$.*

Taking the values of y at the bounds of an interval one can define the following boundary conditions problem.

Definition 8.2.4. *Let y_1 and y_2 be two real numbers and $[a\,,\,b]$ an interval of $\mathbb{R}$. The boundary condition problem satisfied by $y \in \mathscr{C}^2\Big([a\,,\,b]\Big)$ is defined by*

$$\begin{cases} a\,y'' + b\,y' + c\,y = 0, \ \ in\ \ [a\,,\,b], \\ \\ y(a) = y_1, \ \ y(b) = y_2. \end{cases} \tag{8.2.2}$$

Example 8.2.5. *As an illustration we consider the following boundary conditions problem in the interval $[0\,,\,1]$, $y'' + 3\,y' + y = 0$ in $[0\,,\,1]$, $y(0) = 0$ and $y(1) = 2$.*

Example 8.2.6. *We can also consider the following boundary condition problem in the interval $[0\,,\,\pi/3]$, $2\,y'' - y' + 7y = 0$ in $[0\,,\,\pi/3]$, $y(0) = 0$ and $y(\pi/3) = 1$.*

8.2.1 Homogeneous Initial Value Problem

Let y_1 and y_2 be two real numbers. We consider an interval I and some point x_0 in I. We define the initial value problem: find $y \in \mathscr{C}^2(\mathbb{R})$ solution to

$$(*) \quad \begin{cases} a\,y'' + b\,y' + c\,y = 0, \ \ \text{in}\ \ \mathbb{R}, \\ \\ y(x_0) = y_1, \ \ y'(x_0) = y_2. \end{cases}$$

Definition 8.2.7. *We say a function $y \in \mathscr{C}^2(I)$ is a solution to the initial value problem $(*)$ if for every $x \in I$, $a\,y''(x) + b\,y'(x) + c\,y(x) = 0$ and the function y satisfies $y(x_0) = y_1$ and $y'(x_0) = y_2$.*

Example 8.2.8. *The real function $y(x) = \exp(-x) - \exp(-2\,x)$ defined in $\mathbb{R}$ is the solution to the initial value problem*

$$\begin{cases} y'' + 3\,y' + 2\,y = 0, \ \ in\ \ \mathbb{R}, \\ \\ y(0) = 0, \ \ y'(0) = 1. \end{cases}$$

Indeed, we have $y'(x) = -\exp(-x) + 2\exp(-2\,x)$ and $y''(x) = \exp(-x) - 4\exp(-2\,x)$ Then, one deduces $y(0) = 0$, $y'(0) = -1 + 2 = 1$ and

$$\begin{aligned} y'' + 3\,y' + 2\,y &= \mathrm{e}^{-x} - 4\mathrm{e}^{-2x} + 3\Big(-\mathrm{e}^{-x} + 2\mathrm{e}^{-2x}\Big) + 2\Big(\mathrm{e}^{-x} - \mathrm{e}^{-2x}\Big) \\ &= 3\mathrm{e}^{-x} - 3\mathrm{e}^{-x} - 6\mathrm{e}^{-2x} + 6\mathrm{e}^{-2x} = 0. \end{aligned}$$

Let us consider the general initial values problem (8.2.1). We know from the previous section, when $\Delta \neq 0$ the solution of the second linear differential equation $a\,y'' + b\,y' + c\,y = 0$ has solutions

$$y(x) = C_1\,\mathrm{e}^{r_1\,x} + C_2\,\mathrm{e}^{r_2\,x},$$

where r_1 and r_2 are solutions of (8.1.3) and C_1 and C_2 are constant real numbers.

Using initial conditions one has the following linear system:

$$\begin{cases} \exp(r_1\,x_0)\,C_1 \ \ + \ \ \exp(r_2\,x_0)\,C_2 = y_1 \\ \\ r_1\exp(r_1\,x_0)\,C_1 + r_2\exp(r_2\,x_0)\,C_2 = y_2. \end{cases}$$

In this linear system the unknowns are C_1 and C_2.

This linear system is equivalent to the following matricial equation

$$\begin{bmatrix} \exp(r_1\,x_0) & \exp(r_2\,x_0) \\ r_1\exp(r_1\,x_0) & r_2\exp(r_2\,x_0) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{8.2.3}$$

We know that (8.2.3) has a unique solution if and only if

$$r_2\exp((r_1+r_2)\,x_0) - r_1\exp((r_1+r_2)\,x_0) \neq 0.$$

Since $\exp((r_1+r_2)\,x_0) \neq 0$, this is equivalent to $r_2 - r_1 \neq 0$. As we know that $\Delta \neq 0$, then $r_1 \neq r_2$. Therefore, there exists a unique vector $(C_1\,,\,C_2)$ solution to (8.2.3). From we deduce, when $\Delta \neq 0$ the initial value problem (8.2.3) has a unique solution.

Now we are interested on the case $\Delta = 0$. We established in section 8.1 when $\Delta = 0$, the expression of solution of (8.1.1) is $y(x) = (C_1\,x + C_2)\exp(r\,x)$, where r is the unique solution of (8.1.3), C_1 and C_2 are constant real numbers. This means $y'(x) = \big((r\,x+1)C_1 + r\,C_2\big)\exp(r\,x)$. Applying the initial conditions it holds

$$\begin{cases} C_1\,x_0 \quad + \quad C_2 \;\; = y_1\,\exp(-r\,x_0) \\ \\ C_1(r\,x_0+1) + C_2\,r = y_2\,\exp(-r\,x_0)\,. \end{cases}$$

Using matrices one has

$$\begin{bmatrix} x_0 & 1 \\ r\,x_0+1 & r \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} y_1\,\exp(-r\,x_0) \\ y_2\,\exp(-r\,x_0) \end{bmatrix} \tag{8.2.4}$$

The linear system (8.2.4) has a unique solution because $r\,x_0 - r\,x_0 - 1 = -1 \neq 0$. Hence there exists a unique vector $(C_1\,,\,C_2)$ solution to (8.2.4). This leads to the conclusion that when $\Delta = 0$, the initial value problem (8.2.1) has a unique solution. We can summarize this study in this

Theorem 8.2.9. *We consider the initial values problem* (8.2.1)*. Then, there exists a unique function $y \in \mathscr{C}^2(I)$, solution to* (8.2.1)*.*

In other words the initial value problem (8.2.1) admits a unique solution.

Example 8.2.10. *Find the unique solution to the following initial value problem*

$$\begin{cases} y'' + 3\,y' + 2y = 0 \\ \\ y(0) = 0, \;\; y'(0) = 1\,. \end{cases}$$

In the firt step we consider the equation $y''+3y'+2y=0$. Here $\Delta = 9-8 = 1 > 0$. Defining $r_1 = \dfrac{-3-1}{2} = -2$ and $r_2 = \dfrac{-3+1}{2} = -1$, we obtain

$$y(x) = C_1\exp(-2\,x) + C_2\exp(-x),$$

with constant real numbers C_1 and C_2. The function y is differentiable in $\mathbb{R}$ and for every real number x,

$$y'(x) = -2\,C_1\exp(-2x) - C_2\exp(-x).$$

Now we use the initial conditions to obtain the following linear system

$$\begin{cases} y(0) = C_1 + C_2 = 0 \\ \\ y'(0) = -2\,C_1 - C_2 = 1\,. \end{cases}$$

This involves $C_2 = -C_1$. *Therefore* $C_1 = -1$ *and* $C_2 = 1$. *The unique solution to the initial value problem is*

$$y(x) = \exp(-x) - \exp(-2\,x).$$

Example 8.2.11. *Determine the unique solution to the following initial value problem*

$$\begin{cases} y'' + 6\,y' + 9y = 0 \\ \\ y(0) = 1, \;\; y'(0) = 0\,. \end{cases}$$

We define $\Delta = 36 - 36 = 0$. *Therefore* $r = -3$. *This implies*

$$y(x) = (C_1\,x + C_2)\,\exp(-3\,x),$$

where C_1 *and* C_2 *are real numbers. From this we deduce*

$$y'(x) = (-3\,x + C_1 - 3\,C_2)\,\exp(-3x).$$

Since $y(0) = 1$ *and* $y'(0) = 0$, *we get*

$$C_2 = 1 \quad \textit{and} \quad C_1 - 3\,C_2 = 0.$$

Consequently we have $C_1 = 3\,C_2 = 3$. *We conclude the unique solution of the initial values problem is*

$$y(x) = (3x + 1)\exp(-3\,x), \quad \forall\, x \in \mathbb{R}.$$

Example 8.2.12. *Find the unique complex solution of the following homogeneous initial value problem*

$$\begin{cases} y'' + y' + y = 0 \\ \\ y(0) = 0, \;\; y'(0) = 1\,. \end{cases}$$

In this case we have $\Delta = 1 - 4 = -3 < 0$. *Then we can define*

$$r_1 = \frac{-1 - \mathrm{i}\sqrt{3}}{2} \quad \textit{and} \quad r_2 = \frac{-1 = \mathrm{i}\sqrt{3}}{2}.$$

This means that $y(x) = C_1 \exp(r_1\,x) + C_2 \exp(r_2\,x)$, *for two complex numbers* C_1 *and* C_2. *The derivative of* y *is* $y'(x) = r_1\,C_1 \exp(r_1\,x) + r_2\,C_2 \exp(r_2\,x)$. *The initial value conditions implie* $y(0) = C_1 + C_2 = 0$ *and* $y'(0) = r_1\,C_1 + r_2\,C_2 = 1$. *Therefore* $C_2 = -C_1$ *and* $C_1(r_1 - r_2) = 1$. *Considering the expressions of* r_1 *and* r_2, *we hve*

$$r_1 - r_2 = \frac{-1 - \mathrm{i}\sqrt{3}}{2} - \frac{-1 + \mathrm{i}\sqrt{3}}{2} = \frac{2\mathrm{i}\sqrt{3}}{2} = -\mathrm{i}\sqrt{3}.$$

Hence we obtain $-\mathrm{i}\sqrt{3}\,C_1 = 1$. *This means that,* $C_1 = \frac{1}{-\mathrm{i}\sqrt{3}} = \frac{\mathrm{i}\sqrt{3}}{3}$ *and* $C_2 = -\frac{\mathrm{i}\sqrt{3}}{3}$. *The unique complex solution is defined by*

$$y(x) = \mathrm{i}\frac{\sqrt{3}}{3}\left(\exp\left(\frac{-1 - \mathrm{i}\sqrt{3}}{2}\,x\right) - \exp\left(\frac{-1 + \mathrm{i}\sqrt{3}}{2}\,x\right)\right).$$

Example 8.2.13. *Define the expression of the unique real solution to the following initial values problem*

$$\begin{cases} y'' + 4y' + 5y = 0 \\ \\ y(0) = 1, \;\; y'(0) = 0\,. \end{cases}$$

In this example we have $\Delta = 16 - 20 = -4$. *To find the real solution we define* $\alpha = \dfrac{-4}{2} = -2$ *and* $\omega = \dfrac{2}{2} = 1$. *The real solution of the equation* $y'' + 4\,y' + 5y = 0$ *is*

$$y(x) = \exp(-2x)\Big(C_1\cos(x) + C_2\sin(x)\Big),$$

with real numbers C_1 *and* C_2. *The function* y *is differentiable and for every real number* x,

$$y'(x) = \Big[\big(C_2 - 2C_1\big)\cos(x) - \big(2\,C_1 + C_2\big)\sin(x)\Big]\exp(-2\,x).$$

The initial values conditions involves $y(0) = C_1 = 1$ *and* $C_2 - 2\,C_1 = 0$. *Therefore we have* $C_2 = 2C_1 = 2$. *This leads to the following solution*

$$y(x) = \exp(-2\,x)\,\Big[\cos(x) + 2\,\sin(x)\Big].$$

Exercise 8.2.14. *For each of the following initial value problems, determine the unique solution*

1. $\begin{cases} y'' + 3y' + y = 0 \\ \\ y(0) = 1,\, y'(0) = 2, \end{cases}$ 2. $\begin{cases} 3\,y'' + 5y' + 2y = 0 \\ \\ y(0) = 2,\; y'(0) = 1, \end{cases}$ 3. $\begin{cases} y'' + 2y' + y = 0 \\ \\ y(2) = 1,\, y'(2) = 0 \end{cases}$

4. $\begin{cases} -2y'' + 5y' - 4y = 0 \\ \\ y(1) = 0,\, y'(1) = 1, \end{cases}$ 5. $\begin{cases} y'' + 2y' + 3y = 0 \\ \\ y(0) = y'(0) = 2, \end{cases}$ 6. $\begin{cases} y'' + 4y' + 6y = 0 \\ \\ y(0) = y'(0) = 1. \end{cases}$

8.2.2 Homogeneous Boundary Conditions Problem

We consider two real numbers y_1 and y_2. On the interval $[a\,,\,b] \subset \mathbb{R}$ we define the following problem: Find a function $y \in \mathscr{C}^2\big([a\,,\,b]\big)$ which satisfies

$$\begin{cases} a\,y'' + b\,y' + c\,y = 0 \\ \\ y(a) = y_1,\; y(b) = y_2. \end{cases}$$

In section (8.2) we studied equation $a\,y'' + b\,y' + c\,y = 0$ and we established its solution has expression $y(x) = K_1\exp(r_1\,x) + K_2\exp(r_2\,x)$ when $\Delta = b^2 - 4ac \neq 0$ we have $y(x) = \Big(K_1\,x + k_2\Big)\exp(r\,x)$. We remind that r, r_1, and r_2 are the solutions of the equation (8.1.3), k_1 and K_2 are constants.

It is obvious that equation $ay'' + by' + cy = 0$ has infinite number of solutions. Because each time we change the values of the constants K_1 and K_1 we get a new solution. To establish uniqueness of solutions we need to find fixed values for K_1 and K_1. This can be done only by asking the solutions to satisfy some more particular conditions. This means that among the solutions we are looking for such solutions

which only satisfy these conditions. Here we impose the conditions $y(a) = y_1$ and $y(b) = y_2$. We are looking for solutions which has these two properties. Using these assumptions, we will be able to select among all solutions of $ay'' + by' + cy = 0$, the unique one which satisfies the boundary conditions.

Now we have to apply the boudary condition when $\Delta \neq 0$ to establish there is a unique couple $(K_1\,,\,K_2)$. The first condition $y(a) = y_1$ leads to the equation $K_1 \exp(r_1\,a) + K_2 \exp(r_2\,a) = y_1$. The second condition $y(b) = y_2$ gives the equation $K_1 \exp(r_1\,b) + K_2 \exp(r_2\,b) = y_2$. Gathering these equations one obtains the following linear system

$$\begin{cases} K_1\, \exp(r_1\,a) + K_2 \exp(r_2\,b) = y_1 \\ \\ K_1 \exp(r_1\,b) + K_2 \exp(r_2\,b) = y_2. \end{cases}$$

This system is equivalent to the following matricial equation

$$\begin{bmatrix} \exp(r_1\,a) & \exp(r_2\,a) \\ \exp(r_1\,b) & \exp(r_2\,b) \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{8.2.5}$$

We remind that K_1 and K_2 are the unknowns here. Since $r_1\,a + r_2\,b \neq r_2\,a + r_1\,b$, then, $\exp(r_1\,a + r_2\,b) - \exp(r_1\,b + r_2\,a) \neq 0$. Therefore the system (8.2.5) has a unique solution $(K_1\,,\,K_2)$. We deduce there exists a unique $y \in \mathscr{C}^2\big([a\,,\,b]\big)$ solution to (8.2.2).

We assume $\Delta = b^2 - 4a\,c = 0$. Applying boundary conditions, one obtain the following linear system

$$\begin{cases} K_1\,a + K_2 = \exp(-r\,a)\,y_1 \\ \\ K_1\,b + K_2 = \exp(-r\,b)\,y_2. \end{cases}$$

We rewrite this system by using matrices. Then,

$$\begin{bmatrix} a & 1 \\ b & 1 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = \begin{bmatrix} \exp(-r\,a)\,y_1 \\ \exp(-r\,b)\,y_2 \end{bmatrix}$$

As we have $a \neq b$, there exists a unique vector $(K_1\,,\,K_2)$ solution to the linear system above. Therefore we have a unique function $y \in \mathscr{C}^2\big([a\,,\,b]\big)$ solution to (8.2.2). We have proved the following theorem.

Theorem 8.2.15. *We consider the problem* (8.2.2). *Then, there exists a unique function* $y \in \mathscr{C}^2\big([\,a\,,\,b\,]\big)$ *solution to* (8.2.2).

Example 8.2.16. *Determine the expression of the unique solution to the following boundary problem defined in* $[\,0\,,\,1\,]$

$$\begin{cases} y'' + 3\,y' + 2\,y = 0 \\ \\ y(0) = 1,\, y(1) = 0. \end{cases}$$

We consider the equation $y'' + 3\,y' + 2\,y = 0$. *One has* $\Delta = 9 - 8 = 1 > 0$. *Therefore,* $r_1 = \dfrac{-3-1}{2} = -2$ *and* $r_2 = \dfrac{-3+1}{2} = -1$. *This leads to the following general solution*

$$K_1 \exp(-2\,x) + K_2 \exp(-x),$$

with constant real numbers K_1 and K_2.

To determine the values of K_1 and K_2, we use the boundary conditions. This holds the following linear system

$$\begin{cases} K_1 + K_2 = 1 \\ \\ K_1\,\mathrm{e}^{-2} + K_2\,\mathrm{e}^{-1} = 0. \end{cases}$$

From this one deduces $K_1 = -K_2\,\mathrm{e}$ and $K_2\,(1-\mathrm{e}) = 1$. Thus, we obtain

$$K_2 = \frac{1}{1-\mathrm{e}} \qquad \text{and} \qquad K_1 = -\frac{\mathrm{e}}{1-\mathrm{e}}\cdot$$

Hence the function

$$y(x) = \frac{1}{1-\mathrm{e}}\,(\exp(-x) - \exp(-2x+1))$$

is the unique solution to the problem above.

Example 8.2.17. *Find the expression of the unique solution of the problem defined on $[\,0\,,\,1\,]$*

$$\begin{cases} y'' + 6\,y' + 9\,y = 0 \\ \\ y(0) = 0,\ y(1) = 2. \end{cases}$$

We take the equation $y'' + 6\,y' + 9\,y = 0$. In this case we obtain $\Delta = 36 - 36 = 0$. Then, $r = \dfrac{-6}{2} = -3$. The general solution of $y'' + 6\,y' + 9\,y = 0$ is

$$y(x) = \Big(K_1\,x + K_2\Big)\exp(-3\,x),$$

with constant real numbers K_1 and K_2. Now we have to determine fixed values for K_1 and K_2. The first condition implies $y(0) = K_2 = 0$. Using the second one we obtain $y(1) = K_1\exp(-3) = 2$. Hence $K_2 = 2\exp(3)$. This involves the following unique solution

$$y(x) = 2\,x\,\exp(-3x+3).$$

Example 8.2.18. *Give the expression of the unique solution of the following problem*

$$\begin{cases} y'' + 3\,y' + \dfrac{25}{4}\,y = 0,\ x \in\ (\,0\,,\,\pi/4\,) \\ \\ y(0) = 1,\ y(\pi/4) = -1. \end{cases}$$

First we study the equation $y'' + 3\,y' + \dfrac{25}{4}\,y = 0$, without fixing any conditions. Here we have $\Delta = 9 - 25 = -16$. We define $\alpha = -3/2$ and $\omega = 2$. The real solution of $y'' + 3\,y' + \dfrac{25}{4}\,y = 0$ is

$$y(x) = \exp\left(-\frac{3}{2}\,x\right)\Big[K_1\cos(2\,x) + K_2\sin(2\,x)\Big].$$

The initial value conditions imply $K_1 = 1$ *and* $K_2 \exp\left(-\frac{3\pi}{8}\right) = -1$. *Therefore* $K_2 = -\exp\left(\frac{3\pi}{8}\right)$. *Hence the unique solution is*

$$y(x) = \exp\left(-\frac{3x}{2}\right)\left[\cos(2\,x) - \exp\left(\frac{3\pi}{8}\right)\sin(2\,x)\right].$$

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