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Sergiu Klainerman, Jeremie Szeftel. Effective Results on Uniformization and Intrinsic GCM Spheres in Perturbations of Kerr. *Annals of PDE*, 2022, 8 (2), Art. 18, 89 pp. 10.1007/s40818-022-00132-7 . hal-03797541

HAL Id: hal-03797541

<https://hal.science/hal-03797541v1>

Submitted on 7 Oct 2022

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Effective results on uniformization and intrinsic GCM spheres in perturbations of Kerr

Sergiu Klainerman, Jérémie Szeftel
(with an appendix by Camillo De Lellis)

June 1, 2022

Abstract. *This is a follow-up of our paper [17] on the construction of general covariant modulated (GCM) spheres in perturbations of Kerr, which we expect to play a central role in establishing their nonlinear stability. We reformulate the main results of that paper using a canonical definition of $\ell = 1$ modes on a 2-sphere embedded in a $1 + 3$ vacuum manifold. This is based on a new, effective, version of the classical uniformization theorem which allows us to define such modes and prove their stability for spheres with comparable metrics. The reformulation allows us to prove a second, intrinsic, existence theorem for GCM spheres, expressed purely in terms of geometric quantities defined on it. A natural definition of angular momentum for such GCM spheres is also introduced, which we expect to play a key role in determining the final angular momentum for general perturbations of Kerr.*

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1 Introduction

This is a follow-up of our paper [17] on the construction of general covariant modulated (GCM) spheres in perturbations of Kerr. We reformulate the main results of that paper using a canonical definition of $\ell = 1$ modes on a 2-sphere embedded in a $1 + 3$ vacuum manifold. This is based on a new, effective, version of the classical uniformization theorem which allows us to define such modes and prove their stability for spheres with comparable metrics. The reformulation allows us to prove a second, intrinsic, existence theorem for GCM spheres, expressed purely in terms of geometric quantities defined on them. The result can be regarded as a codimension 2, spacetime version, of the well known Huisken-Yau result [12] concerning mean curvature foliations in asymptotically flat Riemannian manifolds. A natural definition of angular momentum for such GCM

spheres is also introduced, which we expect to play a key role in the definition of the final angular momentum for general perturbations of Kerr. Though our immediate motivation is to apply the main GCM results of this paper to the stability of Kerr conjecture, we expect that our construction, or some variation of it, will prove useful to other final states problems¹ in GR, in which a suitable choice of gauge, tailored to the final state, is of paramount importance.

We refer to the introduction in [17] for a short description of the role of GCM spheres in the proof of the nonlinear stability of Schwarzschild in [16] and the crucial role we expect them to play in extending that result to Kerr.

1.1 Review of the main results of [17]

1.1.1 Background space

As in [17] we consider spacetime regions \mathcal{R} foliated by a geodesic foliation $S(u, s)$ induced by an outgoing optical function² u with s a properly normalized affine parameter along the null geodesic generators of $L = -\mathbf{g}^{\alpha\beta}\partial_\beta u\partial_\alpha$. We denote by $r = r(u, s)$ the area radius of $S(u, s)$ and let (e_3, e_4, e_1, e_2) be an adapted null frame with e_4 proportional to L and e_1, e_2 tangent to spheres $S = S(u, s)$, see section 5.1. The main assumptions made in [17] were that the Ricci and curvature coefficients, relative to the adapted null frame, have the same asymptotics in powers of r as in Schwarzschild space. Note that these assumptions hold true in the far region of Kerr and is expected to hold true for the far region of realistic perturbations of Kerr. The actual size of the perturbation from Kerr is measured with respect to a small parameter $\epsilon > 0$, see sections 5.1.2 and 5.1.3 for precise definitions.

1.1.2 Definition of GCM spheres

These are topological spheres \mathbf{S} embedded in \mathcal{R} endowed with a null frame $e_3^{\mathbf{S}}, e_4^{\mathbf{S}}, e_1^{\mathbf{S}}, e_2^{\mathbf{S}}$ adapted to \mathbf{S} (i.e. $e_1^{\mathbf{S}}, e_2^{\mathbf{S}}$ tangent to \mathbf{S}), relative to which the null expansions $\kappa^{\mathbf{S}} = \text{tr}\chi^{\mathbf{S}}$, $\underline{\kappa}^{\mathbf{S}} = \text{tr}\underline{\chi}^{\mathbf{S}}$ and mass aspect function $\mu^{\mathbf{S}}$ take Schwarzschildian values, i.e.

$$\kappa^{\mathbf{S}} - \frac{2}{r^{\mathbf{S}}} = 0, \quad \underline{\kappa}^{\mathbf{S}} + \frac{2\Upsilon^{\mathbf{S}}}{r^{\mathbf{S}}} = 0, \quad \mu^{\mathbf{S}} - \frac{2m^{\mathbf{S}}}{(r^{\mathbf{S}})^3} = 0, \quad (1.1)$$

¹Such as formation of black holes, stability of multiple black holes, final state conjecture,...

²As explained in Remark 1.3 of [17], this is not strictly necessary. Any other foliation satisfying comparable asymptotic assumptions would also work.

where $r^{\mathbf{S}}, m^{\mathbf{S}}$ denote the area radius and Hawking mass of \mathbf{S} and $\Upsilon^{\mathbf{S}} = 1 - \frac{2m^{\mathbf{S}}}{r^{\mathbf{S}}}$, see section 5.1 for precise definitions.

As explained in the introduction to [17], these conditions have to be relaxed with respect to the $\ell = 0, 1$ modes on \mathbf{S} . To make sense of this statement requires us to fix a basis of $\ell = 1$ modes on \mathbf{S} , which generalize the $\ell = 1$ spherical harmonics of the standard sphere³. Assuming the existence of such a basis $J^{(p)}, p \in \{-, 0, +\}$, we define, for a scalar function h ,

$$(h)_{\ell=1}^{\mathbf{S}} := \left\{ \int_{\mathbf{S}} h J^{(p)}, \quad p = -, 0, + \right\}. \quad (1.2)$$

A scalar function h is said to be supported on $\ell \leq 1$ modes, i.e. $(f)_{\ell \geq 2}^{\mathbf{S}} = 0$, if there exist constants A_0, B_-, B_0, B_+ such that

$$h = A_0 + B_- J^{(-)} + B_0 J^{(0)} + B_+ J^{(+)}. \quad (1.3)$$

With this definition, (1.1) is replaced by

$$\kappa^{\mathbf{S}} - \frac{2}{r^{\mathbf{S}}} = 0, \quad \left(\underline{\kappa}^{\mathbf{S}} + \frac{2\Upsilon^{\mathbf{S}}}{r^{\mathbf{S}}} \right)_{\ell \geq 2} = 0, \quad \left(\mu^{\mathbf{S}} - \frac{2m^{\mathbf{S}}}{(r^{\mathbf{S}})^3} \right)_{\ell \geq 2} = 0. \quad (1.4)$$

1.1.3 Deformations of spheres and frame transformations

The construction of GCM spheres in [17] was obtained by deforming a given sphere $\overset{\circ}{S} = S(\overset{\circ}{u}, \overset{\circ}{s})$ of the background foliation of \mathcal{R} . An $O(\overset{\circ}{\delta})$ deformation of $\overset{\circ}{S}$ is defined by a map $\Psi : \overset{\circ}{S} \rightarrow \mathbf{S}$ of the form

$$\Psi(\overset{\circ}{u}, \overset{\circ}{s}, y^1, y^2) = \left(\overset{\circ}{u} + U(y^1, y^2), \overset{\circ}{s} + S(y^1, y^2), y^1, y^2 \right) \quad (1.5)$$

with (U, S) smooth functions on $\overset{\circ}{S}$, vanishing at a fixed point of $\overset{\circ}{S}$, of size proportional to the small constant $\overset{\circ}{\delta}$. Given such a deformation we identify, at any point on \mathbf{S} , two important null frames.

1. The null frame (e_3, e_4, e_1, e_2) of the background foliation of \mathcal{R} .
2. A null frame $(e_3^{\mathbf{S}}, e_4^{\mathbf{S}}, e_1^{\mathbf{S}}, e_2^{\mathbf{S}})$ adapted to the sphere \mathbf{S} .

³Recall that on the standard sphere \mathbb{S}^2 , in spherical coordinates (θ, φ) , these are $J^{(0, \mathbb{S}^2)} = \cos \theta$, $J^{(+, \mathbb{S}^2)} = \sin \theta \cos \varphi$, $J^{(-, \mathbb{S}^2)} = \sin \theta \sin \varphi$.

In general, two null frames (e_3, e_4, e_1, e_2) and (e'_3, e'_4, e'_1, e'_2) are related by a frame transformation of the form, see Lemma 5.6,

$$\begin{aligned} e'_4 &= \lambda \left(e_4 + f^b e_b + \frac{1}{4}|f|^2 e_3 \right), \\ e'_a &= \left(\delta_{ab} + \frac{1}{2} \underline{f}_a f_b \right) e_b + \frac{1}{2} \underline{f}_a e_4 + \left(\frac{1}{2} f_a + \frac{1}{8} |f|^2 \underline{f}_a \right) e_3, \\ e'_3 &= \lambda^{-1} \left(\left(1 + \frac{1}{2} f \cdot \underline{f} + \frac{1}{16} |f|^2 |\underline{f}|^2 \right) e_3 + \left(\underline{f}^b + \frac{1}{4} |\underline{f}|^2 f^b \right) e_b + \frac{1}{4} |\underline{f}|^2 e_4 \right), \end{aligned} \quad (1.6)$$

where the scalar λ and the 1-forms f and \underline{f} are called the transition coefficients of the transformation. As explained in the introduction to [17], one can relate all Ricci and curvature coefficients of the primed frame in terms of the Ricci and curvature coefficients of the un-primed one. In particular, the GCM conditions (1.4) can be expressed in terms of differential conditions for the transition coefficients $(f, \underline{f}, \lambda)$. The condition that the horizontal part of the frame $(e_1^{\mathbf{S}}, e_2^{\mathbf{S}})$ is tangent to \mathbf{S} also leads to a relation between the gradients of the scalar functions U, S , introduced in (1.5), and (f, \underline{f}) . Roughly, we thus expect to derive a coupled system⁴ between Laplace equations for U, S , on \mathring{S} and an elliptic Hodge system for $F = (f, \underline{f}, \lambda - 1)$ on \mathbf{S} .

1.1.4 GCM spheres with non canonical $\ell = 1$ modes in [17]

Here is a short version of our main result in [17].

Theorem 1.1 (Existence of GCM spheres in [17]). *Let \mathcal{R} be fixed spacetime region, endowed with an outgoing geodesic foliation $S(u, s)$, verifying specific asymptotic assumptions⁵ expressed in terms of two parameters $0 < \mathring{\delta} \leq \mathring{\epsilon}$. In particular we assume that the GCM quantities of the background spheres in \mathcal{R} , i.e.*

$$\kappa - \frac{2}{r}, \quad \left(\frac{\underline{\kappa}}{r} + \frac{2\underline{\Upsilon}}{r} \right)_{\ell \geq 2}, \quad \left(\mu - \frac{2m}{r^3} \right)_{\ell \geq 2}, \quad (1.7)$$

are small with respect to the parameter $\mathring{\delta}$. Let $\mathring{S} = S(\mathring{u}, \mathring{s})$ be a fixed sphere of the foliation with \mathring{r} and \mathring{m} denoting respectively its area radius and Hawking mass, with \mathring{r} sufficiently large. Then, for any fixed triplets $\Lambda, \underline{\Lambda} \in \mathbb{R}^3$ verifying

$$|\Lambda|, |\underline{\Lambda}| \lesssim \mathring{\delta}, \quad (1.8)$$

⁴See the introduction in [17] for further explanations.

⁵Compatible with small perturbations of Kerr.

there exists a unique GCM sphere $\mathbf{S} = \mathbf{S}(\Lambda, \underline{\Lambda})$, which is a deformation of \mathring{S} , such that⁶

$$\kappa^{\mathbf{S}} - \frac{2}{r^{\mathbf{S}}} = 0, \quad \left(\frac{\underline{\kappa}^{\mathbf{S}}}{r^{\mathbf{S}}} + \frac{2\Upsilon^{\mathbf{S}}}{r^{\mathbf{S}}} \right)_{\ell \geq 2} = 0, \quad \left(\mu^{\mathbf{S}} - \frac{2m^{\mathbf{S}}}{(r^{\mathbf{S}})^3} \right)_{\ell \geq 2} = 0, \quad (1.9)$$

and

$$(\operatorname{div}^{\mathbf{S}} f)_{\ell=1} = \Lambda, \quad (\operatorname{div}^{\mathbf{S}} \underline{f})_{\ell=1} = \underline{\Lambda}, \quad (1.10)$$

where $(f, \underline{f}, \lambda)$ denote the transition coefficients of the transformation (1.6) from the background frame of \mathcal{R} to the frame adapted to \mathbf{S} .

Remark 1.2. The conditions (1.7), (1.9) and (1.10) depend on the definition of $\ell = 1$ modes respectively on \mathring{S} and \mathbf{S} . In [17], once a choice of $\ell = 1$ modes on \mathring{S} is made, it is then extended to \mathbf{S} using the background foliation. As a consequence, the GCM spheres of Theorem 1.1 depend on the particular choice of $\ell = 1$ modes on \mathring{S} .

1.2 Main new results on uniformization of spheres

One of the main goals of this paper is to remove the arbitrariness in the definition of the $\ell = 1$ modes on the deformed GCM spheres \mathbf{S} . We do that by appealing to two new results concerning uniformization of nearly round spheres. The first result, based on works⁷ by Onofri [19] and Chang-Yang [3], [4], given in Theorem 3.1, provides an effective version of the classical uniformization theorem. The second result, given in Theorem 4.1, based on Theorem 3.1 and previous work⁸ of Frisicke James and Müller [10], allows us to formulate and prove a stability result for effective uniformization of nearby spheres. Together with a notion of calibration, which we introduce in section 4.4, this result allows us to define a canonical definition of $\ell = 1$ modes for deformed spheres \mathbf{S} .

1.2.1 Classical uniformization

According to the classical uniformization theorem, if S is a closed, oriented and connected surface of genus 0 and g^S is a Riemannian metric on S , then, there exists a smooth diffeomorphism $\Phi : \mathbb{S}^2 \rightarrow S$ and a smooth conformal factor u on \mathbb{S}^2 such that

$$\Phi^\#(g^S) = (r^S)^2 e^{2u} \gamma_0 \quad (1.11)$$

⁶Note that the GCM conditions (1.9) require a choice of $\ell = 1$ modes on \mathbf{S} , see Remark 1.2.

⁷We thank A. Chang for bringing this works to our attention. The precise formulation given in Theorem 3.1 does not seem to appear in the literature we are aware of.

⁸We thank C. De Lellis for bringing this paper to our attention.

where γ_0 is the canonical metric on the standard sphere \mathbb{S}^2 and r^S the area radius of S . Unfortunately the conformal factor u is not unique and it is thus difficult to control its size and, more importantly for our applications to GCM spheres, it does not allow one to compare the the conformal factors for two nearby spheres. We state below two results which overcome these difficulties. Both results are applicable to almost round spheres, i.e. closed 2-surfaces S with Gauss curvature K^S sufficiently close to that of a round sphere. More precisely, we have for some sufficiently small $\epsilon > 0$

$$\left| K^S - \frac{1}{(r^S)^2} \right| \leq \frac{\epsilon}{(r^S)^2}. \quad (1.12)$$

In what follows we state, in a simplified version, our main new results on effective uniformization.

1.2.2 Effective uniformization

Theorem 1.3 (Effective uniformization 1). *Given an almost round sphere (S, g^S) as above there exists, up to isometries⁹ of \mathbb{S}^2 , a unique diffeomorphism $\Phi : \mathbb{S}^2 \rightarrow S$ and a unique conformal factor u such that*

$$\begin{aligned} \Phi^\#(g^S) &= (r^S)^2 e^{2u} \gamma_{\mathbb{S}^2}, \\ \int_{\mathbb{S}^2} e^{2u} x^i &= 0, \quad i = 1, 2, 3. \end{aligned} \quad (1.13)$$

Moreover the size of the conformal factor u is small with respect to the parameter ϵ , i.e.

$$\|u\|_{L^\infty(\mathbb{S}^2)} \lesssim \epsilon.$$

The result is restated in Theorem 3.1, in the case of radius 1 spheres, and in general in Corollary 3.8.

We use this effective uniformization to define a basis of $\ell = 1$ modes on S as follows

$$J^{(p,S)} = J^{(p,\mathbb{S}^2)} \circ \Phi^{-1} \quad (1.14)$$

where $J^{(p,\mathbb{S}^2)}$ are the standard $\ell = 1$ spherical harmonics of the round sphere \mathbb{S}^2 . We note however that this definition is still ambiguous with respect to arbitrary rotations of \mathbb{S}^2 . We can remove this arbitrariness for deformations \mathbf{S} of \mathring{S} by calibrating the effective uniformization of \mathbf{S} with that of \mathring{S} , see the discussion in section 4.4. This requires however our stable uniformization result which we discuss below.

⁹i.e. all the solutions are of the form $(\Phi \circ O, u \circ O)$ for $O \in O(3)$.

1.2.3 Stability of the effective uniformization

Theorem 1.4 (Effective uniformization 2). *Consider two almost round spheres (S_1, g^{S_1}) and (S_2, g^{S_2}) and their respective canonical uniformizations (Φ_1, u_1) , (Φ_2, u_2) provided by the effective uniformization result mentioned above, and a smooth map $\Psi : S_1 \rightarrow S_2$. Assume also that the metrics g^{S_1} and $\Psi^\#(g^{S_2})$ are close to each other in S^1 ,*

$$(r^{S_1})^{-2} \|g^{S_1} - \Psi^\#(g^{S_2})\|_{L^\infty(S^1)} \leq \delta \quad (1.15)$$

relative to a small parameter $\delta > 0$. Then, there exists $O \in O(3)$ such that the map $\widehat{\Psi} := (\Phi_2)^{-1} \circ \Psi \circ \Phi_1$ verifies,

$$\|\widehat{\Psi} - O\|_{L^\infty(\mathbb{S}^2)} \lesssim \delta. \quad (1.16)$$

Moreover the conformal factors u_1, u_2 verify

$$\|u_1 - \widehat{\Psi}^\# u_2\|_{L^\infty(\mathbb{S}^2)} \lesssim \delta. \quad (1.17)$$

The result is restated in Theorem 4.1. Its proof is based on a result due to G. Friesecke, R. James and S. Müller [10] which improves on a classical result of F. John [13].

We also provide the higher regularity analogue of Theorem 1.4, stated in Proposition 4.7, based on a recent personal communication of Camillo De Lellis and Stefan Müller. The De Lellis-Müller result is stated in Theorem 4.6 and proved in Appendix A to this paper by De Lellis.

To avoid the indeterminacy with respect to $O \in O(3)$, we introduce a notion of calibration between (Φ_1, v_1) and (Φ_2, v_2) with respect to the map Ψ , see section 4.4. Under this notion of calibration we can replace (1.16) by,

$$\|\widehat{\Psi} - I\|_{L^\infty(\mathbb{S}^2)} \lesssim \delta. \quad (1.18)$$

1.2.4 Stability of the canonical $\ell = 1$ modes

Consider two almost round spheres (S_1, g^{S_1}) and (S_2, g^{S_2}) and their respective canonical uniformizations (Φ_1, u_1) , (Φ_2, u_2) provided by the effective uniformization result mentioned above, and a smooth map $\Psi : S_1 \rightarrow S_2$. Assume also that the metrics g^{S_1} and $\Psi^\#(g^{S_2})$ are close to each other in S^1 , i.e.,

$$\|g^{S_1} - \Psi^\#(g^{S_2})\|_{H^2(S^1)} \leq \delta. \quad (1.19)$$

Assume also that the uniformization maps of S_1, S_2 are calibrated, as in section 4.4 and let

$$J^i = J^{S_i} = J \circ \Phi_i^{-1}, \quad i = 1, 2,$$

be the $\ell = 1$ canonical modes of S_1, S_2 . Then

$$\left| J^1 - J^2 \circ \Psi \right| \lesssim \delta. \quad (1.20)$$

The result, stated in Proposition 4.14, follows from Corollary 4.11 which is an immediate consequence of Theorem 4.1 and calibration.

1.3 Construction of GCM spheres with canonical $\ell = 1$ modes

1.3.1 Canonical version of Theorem 1.1

Theorem 1.5 (GCM spheres with canonical $\ell = 1$ modes). *Under the same assumptions as in Theorem 1.1, there exists a unique¹⁰ GCM sphere $\mathbf{S} = \mathbf{S}(\Lambda, \underline{\Lambda})$, which is a deformation of $\overset{\circ}{S}$, such that*

$$\kappa^{\mathbf{S}} - \frac{2}{r^{\mathbf{S}}} = 0, \quad \left(\underline{\kappa}^{\mathbf{S}} + \frac{2\Upsilon^{\mathbf{S}}}{r^{\mathbf{S}}} \right)_{\ell \geq 2} = 0, \quad \left(\mu^{\mathbf{S}} - \frac{2m^{\mathbf{S}}}{(r^{\mathbf{S}})^3} \right)_{\ell \geq 2} = 0, \quad (1.21)$$

and

$$(\operatorname{div}^{\mathbf{S}} f)_{\ell=1} = \Lambda, \quad (\operatorname{div}^{\mathbf{S}} \underline{f})_{\ell=1} = \underline{\Lambda}, \quad (1.22)$$

with canonically defined $\ell = 1$ modes, i.e. $J^{(p, \mathbf{S})} = J^{(p, \mathbb{S}^2)} \circ \Phi^{-1}$, see (1.14).

The result is restated in Theorem 6.5.

1.3.2 Intrinsic GCM spheres

Using Theorem 1.5 we prove an intrinsic result, i.e. without reference to the background foliation, on the existence and uniqueness of GCM surfaces. These are such that the $\ell = 1$ vanishing conditions (1.22) on $\operatorname{div}^{\mathbf{S}}(f)$ and $\operatorname{div}^{\mathbf{S}}(\underline{f})$ are replaced by the vanishing of the $\ell = 1$ modes of $\operatorname{div}^{\mathbf{S}} \beta^{\mathbf{S}}$ and $\widetilde{\operatorname{tr}} \chi^{\mathbf{S}}$. The horizontal 1-form $\beta^{\mathbf{S}}$ is a curvature component of the Riemann curvature tensor with respect to the null frame adapted to \mathbf{S} , constructed in Theorem 1.1.

¹⁰Up to a rotation of \mathbb{S}^2 .

Theorem 1.6 (Intrinsic GCM spheres with canonical $\ell = 1$ modes). *Under slightly stronger assumptions on the background foliation of \mathcal{R} there exists a unique¹¹ GCM deformation of \mathring{S} verifying, in addition to (1.21),*

$$(\operatorname{div}^{\mathbf{S}} \beta^{\mathbf{S}})_{\ell=1} = 0, \quad \widetilde{\operatorname{tr} \chi}_{\ell=1}^{\mathbf{S}} = 0, \quad (1.23)$$

relative to the canonical $\ell = 1$ modes of \mathbf{S} .

The result, which is restated more precisely in Theorem 7.3, can be regarded as a codimension-2 spacetime version of the well known Huisken-Yau result [12] on the existence of stable, constant mean curvature spheres, in 3D asymptotically flat Riemannian manifolds. Though Theorem 1.5 is strictly motivated by our interest in the Kerr stability conjecture, we strongly believe that it is of independent interest and has the potential for other applications.

Remark 1.7. *The assumptions on the spacetime region \mathcal{R} in Theorem 1.6 are in particular satisfied in Kerr for r sufficiency large, see Lemma 7.9. We can thus apply Theorem 1.6 in that context, and obtain the existence of intrinsic GCM spheres \mathbf{S}_{Kerr} in Kerr for r sufficiency large, see Corollary 7.10. The intrinsic GCM spheres \mathbf{S} of Theorem 1.6 thus correspond to the analog of \mathbf{S}_{Kerr} in perturbations of Kerr for r sufficiency large.*

1.3.3 Definition of angular momentum

The result of Theorem 1.6 is unique up to a rotation of \mathbb{S}^2 in the definition of the canonical $\ell = 1$ modes on \mathbf{S} , see (1.14). We can remove this final ambiguity¹² by adjusting the choice of canonical modes $J^{(p,\mathbf{S})}$ on \mathbf{S} , performing a suitable rotation in \mathbb{S}^2 , such that

$$\int_{\mathbf{S}} \operatorname{curl}^{\mathbf{S}} \beta^{\mathbf{S}} J^{(\pm,\mathbf{S})} = 0. \quad (1.24)$$

This allows us to define on \mathbf{S} the angular parameter $a^{\mathbf{S}}$ by the formula¹³

$$a^{\mathbf{S}} := \frac{(r^{\mathbf{S}})^3}{8\pi m^{\mathbf{S}}} \int_{\mathbf{S}} \operatorname{curl}^{\mathbf{S}} \beta^{\mathbf{S}} J^{(0,\mathbf{S})}. \quad (1.25)$$

The result is stated more precisely in Corollary 7.7.

¹¹Up to a rotation of \mathbb{S}^2 .

¹²The only case where the ambiguity cannot be removed is when (1.24) holds for any one choice of canonical modes, i.e. $(\operatorname{curl} \beta^{\mathbf{S}})_{\ell=1} = 0$. In that case, we set $a^{\mathbf{S}} = 0$ which is consistent with (1.25).

¹³Note that in a Kerr space $\mathcal{K}(a, m)$, relative to a geodesic foliation normalized on \mathcal{I}^+ , we have $\int_{\mathbf{S}} \operatorname{curl}^{\mathbf{S}} \beta^{\mathbf{S}} J^{(\pm,\mathbf{S})} = 0$ and $\int_{\mathbf{S}} \operatorname{curl}^{\mathbf{S}} \beta^{\mathbf{S}} J^{(0,\mathbf{S})} = \frac{8\pi am}{(r^{\mathbf{S}})^3} + O\left(\frac{ma^2}{(r^{\mathbf{S}})^4}\right)$.

Remark 1.8. *Given its relation with GCM spheres, we expect that the definition given above will play a key role in determining the final angular momentum for general perturbations of Kerr. We also note that other definitions of angular momentum have been proposed in the literature, see [21] for a comprehensive review, and [20] and [6] for recent interesting proposals.*

1.4 Structure of the paper

The structure of the paper is as follows.

- In section 2, we review some of the classic results on the uniformization theorem for spheres.
- In section 3, we provide a proof of our first effective uniformization result stated in Theorem 1.3 and use it to define canonical basis of $\ell = 1$ modes.
- In Section 4, we prove our main stability result for effective uniformization stated in Theorem 1.4.
- In Section 5, we review the geometric set-up we need to state our GCM results.
- In Section 6, we review the main result of [17], stated in Theorem 1.1, and prove its new version, stated in Theorem 1.5, using the notion of canonical $\ell = 1$ basis introduced in section 3.
- In section 7, we prove our intrinsic GCM result stated in Theorem 1.6 and use it to define our notion of angular momentum. We also illustrate the result in Corollary 7.10 applied to a far region of a Kerr spacetime.
- In section 8, we derive a priori estimates for GCM spheres.

1.5 Acknowledgments

The authors are grateful to A. Chang and C. De Lellis for very helpful suggestions and references in connection to our results on effective uniformization in sections 3 and 4.

The first author is supported by the NSF grant DMS 180841 as well as by the Simons grant 10011738. He would like to thank the Laboratoire Jacques-Louis Lions of Sorbonne Université and IHES for their hospitality during his many visits. The second author is supported by ERC grant ERC-2016 CoG 725589 EPGR.

2 Review of uniformization results for the sphere

In this section, we review some well-known results concerning the uniformization theorem for the sphere that will be used in section 3.

2.1 Uniformization for metrics on \mathbb{S}^2

We start with the following well known calculation.

Lemma 2.1. *Let S a surface and let g^S be a Riemannian metric on S . For a scalar function u on S , the Gauss curvature of $e^{2u}g^S$ is connected to that of g^S by the formula*

$$K(e^{2u}g^S) = e^{-2u}(K^S - \Delta_S u) \quad (2.1)$$

where K^S denotes the Gauss curvature of S and Δ_S the Laplace-Beltrami on S .

According to the classical uniformization theorem, if S is a closed, oriented and connected surface of genus 0 and g^S is a Riemannian metric on S , then, there exists a smooth diffeomorphism $\Phi : \mathbb{S}^2 \rightarrow S$ and a smooth conformal factor u on \mathbb{S}^2 such that

$$\Phi^\#(g^S) = e^{2u}\gamma_0 \quad (2.2)$$

where γ_0 is the canonical metric on the standard sphere \mathbb{S}^2 . In view of Lemma 2.1 above, if we denote by g the metric on \mathbb{S}^2 , $g = \Phi^\#(g^S) = e^{2u}\gamma_0$, we derive

$$\Delta_0 u + K(g)e^{2u} = 1 \quad (2.3)$$

where we have used the notation

$$\Delta_0 := \Delta_{\gamma_0}.$$

We thus have the following corollary of Lemma 2.1.

Corollary 2.2. *If $\Phi^\#g^S = e^{2u}\gamma_0$, then u verifies the equation*

$$\Delta_0 u + (K^S \circ \Phi)e^{2u} = 1, \quad (2.4)$$

where K^S is the Gauss curvature of g^S on S .

Proof. The proof follows from (2.3) in view of the fact that $K(\Phi^\#g^S) = K(g^S) \circ \Phi$. \square

Definition 2.3. Let \mathbf{M} denote the group of conformal transformations of \mathbb{S}^2 , i.e. the set of diffeomorphisms Φ of \mathbb{S}^2 such that $\Phi^\# \gamma_0 = e^{2u} \gamma_0$ for some scalar function u on \mathbb{S}^2 .

Remark 2.4. Let $\Phi \in \mathbf{M}$ so that $\Phi^\# \gamma_0 = e^{2u} \gamma_0$. Then, u satisfies¹⁴

$$u = \frac{1}{2} \log |\det d\Phi|.$$

Also, in view of Corollary 2.2, we have

$$\Delta_0 u + e^{2u} = 1. \tag{2.5}$$

Lemma 2.5. All solutions of (2.5) are of the form $u = \frac{1}{2} \log |\det d\Phi|$ with $\Phi \in \mathbf{M}$.

Proof. We have already checked that for every $\Phi \in \mathbf{M}$, $u = \frac{1}{2} \log |\det d\Phi|$ is a solution of (2.5), see Remark 2.4. On the other hand, if u is a solution of that equation, then $K(e^{2u} \gamma_0) = 1$ by Lemma 2.1 which implies that there exist a diffeomorphism $\Phi : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ such that $\Phi^\# \gamma_0 = e^{2u} \gamma_0$, i.e. $\Phi \in \mathbf{M}$ and $u = \frac{1}{2} \log |\det d\Phi|$. \square

Corollary 2.6. If u is a solution of (2.3) with $K = K(g)$, and if $\Phi \in \mathbf{M}$, then

$$u_\Phi := u \circ \Phi + \frac{1}{2} \log |\det d\Phi|$$

is also a solution of (2.3) with K replaced by $K \circ \Phi$.

Proof. If $\Phi \in \mathbf{M}$ then, since $\Phi^\# \gamma_0 = |\det(d\Phi)| \gamma_0$ and $g = e^{2u} \gamma_0$,

$$\Phi^\# g = e^{2u \circ \Phi} \Phi^\# \gamma_0 = e^{2u \circ \Phi} |\det(d\Phi)| \gamma_0 = e^{2v} \gamma_0, \quad v = u \circ \Phi + \frac{1}{2} \log |\det d\Phi|.$$

Hence if u is a solution of (2.3), with $K = K(g)$, then so is $u_\Phi = u \circ \Phi + \frac{1}{2} \log |\det d\Phi|$ with K replaced by $K \circ \Phi$, as $K(\Phi^\# g) = K(g) \circ \Phi$. \square

2.2 Conformal isometries of \mathbb{S}^2 and the Möbius group

We represent the standard sphere \mathbb{S}^2 as $\{x \in \mathbb{R}^3, |x|^2 = 1\}$. Let $N = (0, 0, 1)$ denote the north pole of \mathbb{S}^2 . Through the stereographic projection from the North pole to the equatorial plane plane (x^1, x^2) we consider the complex coordinate

$$z = \frac{x^1 + ix^2}{1 - x^3} \tag{2.6}$$

¹⁴This follows immediately from writing $\Phi^\# \gamma_0 = e^{2u} \gamma_0$ in matrix form, by evaluating on an orthonormal frame, and then taking the absolute value of the determinant on both sides. Also, recall that $|\det d\Phi|$ is an intrinsic scalar on \mathbb{S}^2 , i.e. it does not depend on the particular choice of orthonormal frame.

with the inverse transformation

$$x^1 = \frac{2}{1 + |z|^2} \Re z, \quad x^2 = \frac{2}{1 + |z|^2} \Im z, \quad x^3 = \frac{|z|^2 - 1}{|z|^2 + 1} \quad (2.7)$$

and pull-back of the standard metric γ_0 on \mathbb{S}^2

$$4(1 + |z|^2)^{-2} |dz|^2. \quad (2.8)$$

The conformal isometry group \mathbf{M} of \mathbb{S}^2 consists in fact of Möbius transforms and conjugation of Möbius transforms, see for example Theorem 18.10.4 and section 18.10.2.4 in [2], and can thus be identified with $SL(2, \mathbb{C})$,

$$z \rightarrow \frac{az + b}{cz + d}, \quad z \rightarrow \frac{a\bar{z} + b}{c\bar{z} + d}, \quad ad - bc = 1, \quad a, b, c, d \in \mathbb{C}. \quad (2.9)$$

The particular case of Möbius transforms where $d = a^{-1} > 0$ and $b = c = 0$ will play an important role. Given $t > 0$ and a point $p \in \mathbb{S}^2$, we can choose coordinates such that p is at the north pole and obtain scale transformations defined by

$$\Phi_{p,t} z = tz. \quad (2.10)$$

2.3 Onofri functional and inequality

Following [19], as well as [3] [4], we introduce the functional

$$S[u] = \frac{1}{4\pi} \left(\int_{\mathbb{S}^2} |\nabla u|^2 + 2 \int_{\mathbb{S}^2} u \right). \quad (2.11)$$

Lemma 2.7. *The functional S is invariant with respect to the transformations $u \rightarrow u_\Phi$, i.e.*

$$S[u_\Phi] = S[u]. \quad (2.12)$$

Proof. See [19]. □

Definition 2.8. *We define the center of mass of e^{2u} to be*

$$CM[e^{2u}] = \frac{\int_{\mathbb{S}^2} x e^{2u}}{\int_{\mathbb{S}^2} e^{2u}} \quad (2.13)$$

where $x = (x^1, x^2, x^3)$ on the sphere \mathbb{S}^2 . Also, we define the spaces of functions¹⁵

$$\begin{aligned}\mathcal{S} &:= \left\{ u \in H^1(\mathbb{S}^2) \text{ such that } CM[e^{2u}] = 0 \right\}, \\ \mathcal{S}_0 &:= \left\{ u \in \mathcal{S} \text{ such that } \int_{\mathbb{S}^2} e^{2u} = 4\pi \right\}.\end{aligned}\tag{2.14}$$

Remark 2.9. If u belongs to \mathcal{S} (resp. \mathcal{S}_0), then $u \circ O$ also belongs to \mathcal{S} (resp. \mathcal{S}_0) for any $O \in O(3)$.

Lemma 2.10. Given u a smooth function on \mathbb{S}^2 there exists a conformal transformation Φ such that $u_\Phi \in \mathcal{S}$, i.e. $\int_{\mathbb{S}^2} e^{2u_\Phi} x = 0$.

Proof. See [19], as well as Proposition 2.2. in [3] and Lemma 2 in [5]. □

Remark 2.11. In fact, Φ in Lemma 2.10 is unique up to isometries of \mathbb{S}^2 , see Lemma 3.5.

Proposition 2.12 (Onofri). Given $u \in H^1(\mathbb{S}^2)$ we have,

$$\log \left(\int_{\mathbb{S}^2} e^{2u} \right) \leq S[u]\tag{2.15}$$

with equality iff $u = \frac{1}{2} \log |\det \Phi|$ for some conformal $\Phi \in \mathbf{M}$.

Proof. See [19]. □

2.4 Improved Onofri inequality and applications

The following is Proposition B in [3] which improves the Onofri inequality for $u \in \mathcal{S}$.

Proposition 2.13. There exists an $a < 1$ such that for all $u \in \mathcal{S}$

$$\frac{1}{4\pi} \int e^{2u} \leq \exp \left(a \frac{1}{4\pi} \int |\nabla u|^2 + 2 \frac{1}{4\pi} \int u \right).\tag{2.16}$$

Corollary 2.14. Let $a < 1$ as in Proposition 2.13. If $u \in \mathcal{S}_0$ then,

$$\frac{1}{4\pi} \int |\nabla u|^2 \leq (1 - a)^{-1} S[u].\tag{2.17}$$

¹⁵ To the best of our knowledge the condition $\int_{\mathbb{S}^2} x e^{2u} = 0$ appears first in [1].

Proof. Since $u \in \mathcal{S}_0$, we have in view of Proposition 2.13

$$1 = \frac{1}{4\pi} \int e^{2u} \leq \exp \left(a \frac{1}{4\pi} \int |\nabla u|^2 + 2 \frac{1}{4\pi} \int u \right) \quad (2.18)$$

from which we deduce

$$a \frac{1}{4\pi} \int |\nabla u|^2 + 2 \frac{1}{4\pi} \int u \geq 0.$$

By definition of $S[u]$ we infer

$$(1-a) \frac{1}{4\pi} \int |\nabla u|^2 = S[u] - \left(a \frac{1}{4\pi} \int |\nabla u|^2 + 2 \frac{1}{4\pi} \int u \right) \leq S[u]$$

as desired. \square

The following is Proposition 4.1 in [4].

Proposition 2.15. *Let $K > 0$ be the Gauss curvature of the metric $e^{2w} \gamma_0$ on \mathbb{S}^2 , so that w verifies (2.3), i.e. $\Delta w + K e^{2w} = 1$. Then $S[w] \leq C(K)$ for some constant C depending only on $\max_{\mathbb{S}^2} K$ and $\min_{\mathbb{S}^2} K$.*

Proof. In view of Lemma 2.10 and Lemma 2.7, we may assume that w is centered, i.e. $w \in \mathcal{S}$. The proof then follows by exploiting that w satisfies $\Delta w + K e^{2w} = 1$, and relying on an estimate satisfied by functions in \mathcal{S} . See [4] for the details. \square

The following is Corollary 4.3. in [4].

Corollary 2.16. *Given any $\delta > 0$, there exists $\epsilon > 0$ sufficiently small such that if $\|K - 1\|_\infty \leq \epsilon$ then $S[w] \leq \delta$ for all w, K which verify equation (2.3).*

Proof. The proof, based on a contradiction argument, relies on Lemma 2.7, Proposition 2.15 and the fact that any centered solution of $\Delta_0 w + e^{2w} = 1$ must vanish identically. See [4] for the details. \square

We restate the result in the following.

Corollary 2.17. *Let g a metric on \mathbb{S}^2 such that $g = e^{2u} \gamma_0$ with $u \in \mathcal{S}_0$. Then, given any $\delta > 0$, there exists $\epsilon > 0$ sufficiently small such that if $\|K(g) - 1\|_{L^\infty} \leq \epsilon$, then*

$$\|u\|_{H^1} \leq \delta. \quad (2.19)$$

Proof. According to Corollary 2.16, given $\delta > 0$ sufficiently small, we can find $\epsilon > 0$ such that if $\|K(g) - 1\|_{L^\infty} \leq \epsilon$, then $S[u] \leq \delta^2$. In view of Corollary 2.14, we deduce

$$\frac{1}{4\pi} \int |\nabla u|^2 \leq (1-a)^{-1} \delta^2.$$

Together with the bound $S[u] \leq \delta^2$, we infer

$$\|\nabla u\|_{L^2} + \left| \int u \right| \lesssim \delta.$$

Together with the Poincaré inequality for scalars on \mathbb{S}^2 , we infer

$$\|u\|_{H^1} \lesssim \delta$$

as desired. □

3 Effective uniformization for nearly round spheres

Let (S, g^S) be a fixed sphere of area radius 1, i.e. $|S| = 4\pi$. The goal of this section is to obtain the following improvement of the results reviewed in section 2.

Theorem 3.1 (Effective uniformization). *Let (S, g^S) be a fixed sphere with $|S| = 4\pi$. There exists, up to isometries¹⁶ of \mathbb{S}^2 , a unique diffeomorphism $\Phi : \mathbb{S}^2 \rightarrow S$ and a unique centered conformal factor u , i.e. $u \in \mathcal{S}$, such that $\Phi^\#(g^S) = e^{2u}\gamma_0$. Moreover, under the almost round condition*

$$\|K^S - 1\|_{L^\infty} \leq \epsilon \tag{3.1}$$

where $K^S = K(g^S)$, the following properties are verified for sufficiently small $\epsilon > 0$.

1. We have

$$\|u \circ \Phi^{-1}\|_{L^\infty(S)} \lesssim \epsilon. \tag{3.2}$$

2. If in addition $\|K^S - 1\|_{H^s(S)} \leq \epsilon$ for some $s \geq 0$, then

$$\|u \circ \Phi^{-1}\|_{H^{2+s}(S)} \lesssim \epsilon. \tag{3.3}$$

¹⁶i.e. all the solutions are of the form $(\Phi \circ O, u \circ O)$ for $O \in O(3)$. In particular, recall that if u is centered, then so is $u \circ O$ for $O \in O(3)$, see Remark 2.9.

Remark 3.2. *To the best of our knowledge the estimates (3.2), (3.3) have not been stated as such in the literature. The uniqueness statement also appears to be new.*

Remark 3.3. *One can easily adapt the statement of Theorem 3.1 to the case $|S| \neq 4\pi$, see Corollary 3.8.*

Theorem 3.1 will be proved in section 3.3. We first provide improvements of Lemma 2.10 and Corollary 2.17.

3.1 Uniqueness for Lemma 2.10

In this section, we provide a uniqueness statement for Lemma 2.10, see Lemma 3.5. We also strengthen the conclusions of Lemma 2.10 for nearly round spheres, see Lemma 3.6. To this end, we start with the following decomposition of conformal isometries of \mathbb{S}^2 .

Lemma 3.4. *Any $\Phi \in \mathbf{M}$ admits the following decomposition*

$$\Phi = O_1 \circ \Phi_{N,t} \circ O_2$$

where $O_1, O_2 \in O(3)$, $N = (0, 0, 1)$, $t > 0$, and the scale transformation $\Phi_{N,t}$ has been introduced in (2.10).

Proof. Upon multiplying Φ by a reflexion, which belongs to $O(3)$, we may assume that Φ corresponds, in view of section 2.2, to a Möbius transformation, and hence to a matrix in $SL(2, \mathbb{C})$. Then, using that, see for example [14],

1. composition of Möbius transformations corresponds to matrix multiplication in $SL(2, \mathbb{C})$,
2. $SO(3)$ corresponds to Möbius transformations with matrices in $SU(2)$,
3. scale transformations $\Phi_{N,t}$ with $t > 0$ correspond to the following matrix in $SL(2, \mathbb{C})$

$$\begin{pmatrix} \sqrt{t} & 0 \\ 0 & \frac{1}{\sqrt{t}} \end{pmatrix},$$

the lemma reduces to proving the following decomposition for a matrix $A \in SL(2, \mathbb{C})$

$$A = U_1 D U_2, \quad U_1, U_2 \in SU(2), \quad D = \begin{pmatrix} \sqrt{t} & 0 \\ 0 & \frac{1}{\sqrt{t}} \end{pmatrix}. \quad (3.4)$$

To prove the claim (3.4), we apply the polar decomposition to A

$$A = RU_3, \quad R = R^*, \quad R > 0, \quad \det(R) = 1, \quad U_3 \in SU(2),$$

where $\det(R) = 1$ and $U_3 \in SU(2)$ since $\det(A) = 1$. Also, R being hermitian semidefinite positive, we may diagonalize it as follows

$$R = U_1 \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} U_1^*, \quad U_1 \in SU(2), \quad \lambda > 0,$$

where we have used the fact that $\det(R) = 1$. Finally, we have

$$A = U_1 \begin{pmatrix} \sqrt{t} & 0 \\ 0 & \frac{1}{\sqrt{t}} \end{pmatrix} U_2, \quad \text{where } U_2 = U_1^* U_3, \quad t = \lambda^2 > 0,$$

which is (3.4) as desired. \square

Lemma 3.5. *Let u be a smooth function on \mathbb{S}^2 and let Φ be a conformal transformation. Assume that both u and u_Φ belong to \mathcal{S} . Then, we have¹⁷ $\Phi \in O(3)$.*

Proof. Decomposing Φ using Lemma 3.4, it suffices in fact to prove that if u and $u_{\Phi_{N,t}}$ are in \mathcal{S} , where $t > 0$, then $\Phi_{N,t} = I$, i.e. $t = 1$. Thus, from now on, we assume that u and $u_{\Phi_{N,t}}$ are in \mathcal{S} with $t > 0$.

Since u and $u_{\Phi_{N,t}}$ are in \mathcal{S} , we have

$$\begin{aligned} 0 &= \int_{\mathbb{S}^2} e^{2u_{\Phi_{N,t}}} x - \int_{\mathbb{S}^2} e^{2u} x = \int_{\mathbb{S}^2} e^{2u \circ \Phi_{N,t}} |\det(d\Phi_{N,t})| x - \int_{\mathbb{S}^2} e^{2u} x \\ &= \int_{\mathbb{S}^2} e^{2u} x \circ \Phi_{N,t}^{-1} - \int_{\mathbb{S}^2} e^{2u} x = \int_{\mathbb{S}^2} e^{2u} \left(x \circ \Phi_{N,t}^{-1} - x \right) \end{aligned} \quad (3.5)$$

where we used the definition of $u_{\Phi_{N,t}}$, the change of variable formula, and the fact that $\Phi_{N,t}^{-1} = \Phi_{N,t^{-1}}$. Now, in view of the formula for $\Phi_{N,t}$, we have, using the formula for stereographic coordinates, see (2.7),

$$x^1 \circ \Phi_{N,t^{-1}} = \frac{2t^{-1}}{1+t^{-2}|z|^2} \Re z, \quad x^2 \circ \Phi_{N,t^{-1}} = \frac{2t^{-1}}{1+t^{-2}|z|^2} \Im z, \quad x^3 \circ \Phi_{N,t^{-1}} = \frac{t^{-2}|z|^2 - 1}{t^{-2}|z|^2 + 1}.$$

Since

$$|z|^2 = \frac{1+x^3}{1-x^3},$$

¹⁷Recall from Remark 2.9 that if u belongs to \mathcal{S} , then $u \circ O$ also belongs to \mathcal{S} for any $O \in O(3)$.

we infer, using (2.7) again,

$$\begin{aligned}
x^1 \circ \Phi_{N,-t} &= \frac{2t^{-1}}{t^{-2}(1+x^3) + (1-x^3)} x^1, \\
x^2 \circ \Phi_{N,-t} &= \frac{2t^{-1}}{t^{-2}(1+x^3) + (1-x^3)} x^2, \\
x^3 \circ \Phi_{N,-t} &= \frac{t^{-2}(1+x^3) - (1-x^3)}{t^{-2}(1+x^3) + (1-x^3)}.
\end{aligned} \tag{3.6}$$

In particular, the last identity of (3.6) yields

$$x^3 \circ \Phi_{N,-t} - x^3 = (t^{-2} - 1) \frac{1 - (x^3)^2}{t^{-2}(1+x^3) + (1-x^3)}$$

and thus,

$$\int_{\mathbb{S}^2} e^{2u} \left(x^3 \circ \Phi_{N, \frac{1}{t}} - x^3 \right) = (t^{-2} - 1) \int_{\mathbb{S}^2} e^{2u} \frac{1 - (x^3)^2}{t^{-2}(1+x^3) + (1-x^3)}$$

so that we have, in view of (3.5),

$$(t^{-2} - 1) \int_{\mathbb{S}^2} e^{2u} \frac{1 - (x^3)^2}{t^{-2}(1+x^3) + (1-x^3)} = 0.$$

Since the integral is strictly positive and $t > 0$, we infer $t = 1$ as desired. \square

Lemma 3.6. *Let ϵ and δ two constants such that $0 < \delta \leq \epsilon$. Given u a smooth function on \mathbb{S}^2 such that*

$$\left| \int_{\mathbb{S}^2} e^{2u} x \right| \leq \delta, \quad \|u\|_{L^\infty} \leq \epsilon.$$

Then, for $\epsilon > 0$ sufficiently small, there exists a conformal transformation Φ such that $u_\Phi \in \mathcal{S}$ and satisfying in addition $|\Phi - I| \lesssim \delta$.

Proof. Let $p_1 = (1, 0, 0)$, $p_2 = (0, 1, 0)$ and $p_3 = (0, 0, 1)$, and let us consider the map

$$\Theta : \mathbb{R}^3 \rightarrow \mathbb{R}^3, \quad \Theta(t) = \int_{\mathbb{S}^2} e^{2u_{\Phi_{p_1, t_1} \circ \Phi_{p_2, t_2} \circ \Phi_{p_3, t_3}}} x.$$

To prove the lemma, it suffices to exhibit $t \in \mathbb{R}^3$ such that $\Theta(t) = 0$ and $|t - (1, 1, 1)| \lesssim \delta$.

Using the definition of $u_{\Phi_{p_i, t_i}}$, the change of variable formula, and the fact that $\Phi_{p_i, t_i}^{-1} = \Phi_{p_i, t_i^{-1}}$, we have

$$\Theta(t) = \int_{\mathbb{S}^2} e^{2u} x \circ \Phi_{p_3, t_3^{-1}} \circ \Phi_{p_2, t_2^{-1}} \circ \Phi_{p_1, t_1^{-1}}.$$

Since $|\int_{\mathbb{S}^2} e^{2u} x| \leq \delta$, and since $\Phi_{p_i,1}$ is the identity, we have

$$\Theta(1, 1, 1) = \int_{\mathbb{S}^2} e^{2u} x = O(\delta). \quad (3.7)$$

Next, we compute the differential of Θ at $t = (1, 1, 1)$. To this end, note that we have in view of (3.6)

$$\begin{aligned} \Theta(1, 1, t_3) &= \int_{\mathbb{S}^2} e^{2u} x \circ \Phi_{p_3, \frac{1}{t_3}} = \Theta(1, 1, 1) + \int_{\mathbb{S}^2} e^{2u} (x \circ \Phi_{N, \frac{1}{t_3}} - x) \\ &= \Theta(1, 1, 1) + \int_{\mathbb{S}^2} e^{2u} \begin{pmatrix} \frac{2t^{-1}}{t^{-2}(1+x^3)+(1-x^3)} x^1 \\ \frac{2t^{-1}}{t^{-2}(1+x^3)+(1-x^3)} x^2 \\ \frac{t^{-2}(1+x^3)-(1-x^3)}{t^{-2}(1+x^3)+(1-x^3)} \end{pmatrix}. \end{aligned}$$

We infer, using also $\|u\|_{L^\infty} \leq \epsilon$,

$$\partial_{t_3} \Theta(1, 1, 1) = \int_{\mathbb{S}^2} e^{2u} \begin{pmatrix} 2x^1 x^3 \\ 2x^2 x^3 \\ -(1 - (x^3)^2) \end{pmatrix} = O(\epsilon) + \int_{\mathbb{S}^2} \begin{pmatrix} 2x^1 x^3 \\ 2x^2 x^3 \\ -(1 - (x^3)^2) \end{pmatrix}$$

and hence

$$\partial_{t_3} \Theta(1, 1, 1) = -\frac{8\pi}{3} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + O(\epsilon).$$

One easily derives corresponding identities for $\partial_{t_1} \Theta(1, 1, 1)$ and $\partial_{t_2} \Theta(1, 1, 1)$ which yields

$$d\Theta|_{(1,1,1)} = -\frac{8\pi}{3} I + O(\epsilon). \quad (3.8)$$

Thus, finding t such that $\Theta(t) = 0$ amounts to solving the following fixed point

$$t - (1, 1, 1) = -(d\Theta|_{(1,1,1)})^{-1} \left[\Theta(1, 1, 1) + \left(\Theta(t) - \Theta(1, 1, 1) - d\Theta|_{(1,1,1)}(t - (1, 1, 1)) \right) \right]$$

whose existence, together with the desired estimate $|t - (1, 1, 1)| \lesssim \delta$, follows, in view of (3.7) and (3.8), from the Banach fixed point theorem. \square

3.2 Improvement of Corollary 2.17 for nearly round spheres

In the proposition below, we improve the results of Corollary 2.17 for nearly round spheres.

Proposition 3.7. *Assume that the metric $g = e^{2u}\gamma_0$ on \mathbb{S}^2 , with $u \in \mathcal{S}_0$, is such that $\|K(g) - 1\|_{L^\infty} \leq \epsilon$. Then, for $\epsilon > 0$ sufficiently small, we have*

$$\|u\|_{L^\infty(\mathbb{S}^2)} \lesssim \epsilon. \quad (3.9)$$

Proof. Let $\delta > 0$ a sufficiently small universal constant which will be chosen later. Then, according to Corollary 2.17, if ϵ , such that $\|K - 1\|_{L^\infty} \leq \epsilon$ holds, is sufficiently small, i.e. $0 < \epsilon \leq \epsilon(\delta) \ll 1$, we have

$$\|u\|_{H^1} \leq \delta.$$

Our goal is to improve this estimate by showing that the stronger bound (3.9) holds.

Since $g = e^{2u}\gamma_0$, the scalar function u verifies equation (2.3), i.e.

$$\Delta_0 u + Ke^{2u} = 1,$$

where $K = K(g)$, which we rewrite in the form,

$$\Delta_0 u + (e^{2u} - 1) = -(K - 1)e^{2u}$$

or,

$$\Delta_0 u + 2u = -(K - 1)e^{2u} - f(u), \quad \text{where} \quad f(u) := e^{2u} - 1 - 2u = \sum_{n \geq 2} \frac{2^n}{n!} u^n.$$

Also, since $u \in \mathcal{S}_0$, we have $\int e^{2u} x = 0$. Together with $\int x = 0$, we deduce,

$$0 = \int e^{2u} x = \int (1 + 2u + f(u))x = 2 \int ux + \int f(u)x$$

and hence

$$\left| \int ux \right| \lesssim \int |f(u)| \lesssim \|f(u)\|_{L^2}.$$

Note that $f(u) \leq 4u^2 e^{2u}$. By standard elliptic estimates we have¹⁸

$$\|u\|_{H^2} \lesssim \|(\Delta_0 + 2)u\|_{L^2} + \left| \int ux \right|$$

¹⁸Note that the kernel of $\Delta_0 + 2$ is given by the $\ell = 1$ spherical harmonics, and that $\int ux$ corresponds to the projection on these spherical harmonics.

and hence

$$\begin{aligned}
\|u\|_{H^2} &\lesssim \|(K-1)e^{2u}\|_{L^2} + \|f(u)\|_{L^2} + \left| \int u x \right| \\
&\lesssim \epsilon \|e^{2u}\|_{L^2} + \|f(u)\|_{L^2} \\
&\lesssim \epsilon \|e^{2u}\|_{L^2} + 4\|u\|_{L^\infty}^2 \|e^{2u}\|_{L^2}.
\end{aligned}$$

In view of Onofri inequality (2.15), we have

$$\log \int e^{4u} \leq S[2u] \lesssim \|u\|_{H^1}^2 + \|u\|_{H^1} \lesssim \delta \leq 1$$

where we have used $\|u\|_{H^1} \leq \delta$ and chosen $\delta > 0$ sufficiently small. We deduce,

$$\|u\|_{H^2} \lesssim \epsilon + \|u\|_{L^\infty}^2.$$

By a calculus inequality

$$\|u\|_{L^\infty} \lesssim \|u\|_{L^2}^{\frac{1}{2}} \|u\|_{H^2}^{\frac{1}{2}}.$$

Hence,

$$\|u\|_{H^2} \lesssim \epsilon + \|u\|_{L^2} \|u\|_{H^2}.$$

Thus, since $\|u\|_{H^1} \leq \delta$, we infer that, in fact, for $\delta > 0$ small

$$\|u\|_{H^2} \lesssim \epsilon.$$

In other words, if $\|K-1\|_{L^\infty} \leq \epsilon$, for $\epsilon > 0$ sufficiently small, then $\|u\|_{H^2} \lesssim \epsilon$. In particular, by Sobolev inequality, we have $\|u\|_{L^\infty} \lesssim \epsilon$ as stated. \square

3.3 Proof of Theorem 3.1

3.3.1 Existence part

According to the standard uniformization theorem, there exists a scalar function \tilde{u} and a map $\tilde{\Phi} : \mathbb{S}^2 \rightarrow S$ such that we have $\tilde{\Phi}^\#(g^S) = e^{2\tilde{u}}\gamma_0$ with $\int_{\mathbb{S}^2} e^{2\tilde{u}} = 4\pi$. In view of Lemma 2.10, we can find $\Psi \in \mathbf{M}$ such that $u = \tilde{u}_\Psi \in \mathcal{S}_0$, and then, $g = \Phi^\#(g^S) = e^{2u}\gamma_0$ with $\Phi = \tilde{\Phi} \circ \Psi$.

3.3.2 Uniqueness part

We address below the issue of uniqueness, up to isometries of \mathbb{S}^2 , of (Φ, u) . Assume that we have

$$(\Phi_i)^\# g^S = e^{2u_i} \gamma_0, \quad i = 1, 2, \quad (3.10)$$

with $\Phi_i : \mathbb{S}^2 \rightarrow S$ diffeomorphisms and u_i centered conformal factors. Let $\Psi = \Phi_2^{-1} \circ \Phi_1$ so that $\Psi : \mathbb{S}^2 \rightarrow \mathbb{S}^2$. In view of the above, we have

$$\Psi^\# \gamma_0 = \Phi_1^\# (\Phi_2^{-1})^\# \gamma_0 = \Phi_1^\# (e^{-2u_2 \circ \Phi_2^{-1}} g^S) = e^{-2u_2 \circ \Phi_2^{-1} \circ \Phi_1} e^{2u_1} \gamma_0$$

and hence

$$\Psi^\# \gamma_0 = e^{2(u_1 - u_2 \circ \Psi)} \gamma_0.$$

Therefore Ψ is a conformal isometry of \mathbb{S}^2 , and we deduce in view of Remark 2.4

$$u_1 - u_2 \circ \Psi = \frac{1}{2} \log |\det \Psi|,$$

i.e.,

$$u_1 = (u_2)_\Psi.$$

Since both u_1 and u_2 are centered we deduce, in view of Lemma 3.5, $\Psi = O$ with $O \in O(3)$ and therefore $\Phi_1 = \Phi_2 \circ O$ and $u_1 = u_2 \circ O$ as stated.

3.3.3 Estimates

From now on, we assume in addition the almost round condition (3.1). In view of (3.1), and the fact that $g = e^{2u} \gamma_0$ with $u \in \mathcal{S}_0$, we deduce by Proposition 3.7 that $\|u\|_{L^\infty(\mathbb{S}^2)} \lesssim \epsilon$. This implies $\|u \circ \Phi^{-1}\|_{L^\infty(S)} \lesssim \epsilon$ as stated in (3.2). To prove the higher derivative estimates we observe that

$$(\Phi^{-1})^\# \gamma_0 = e^{-2v} g^S, \quad v := u \circ \Phi^{-1}, \quad (3.11)$$

which together with Lemma 2.1 implies

$$1 = K((\Phi^{-1})^\# \gamma_0) = K(e^{-2v} g^S) = e^{2v} (K^S + \Delta_S v)$$

so that v verifies

$$\Delta_S v = (e^{-2v} - 1) - (K^S - 1), \quad \|v\|_{L^\infty(S)} \lesssim \epsilon.$$

In view of the almost round condition (3.1), we easily deduce that in fact $\|v\|_{H^2(S)} \lesssim \epsilon$. Moreover, if we have in addition $\|K^S - 1\|_{H^s} \leq \epsilon$, then, by standard elliptic regularity, we obtain $\|v\|_{H^{2+s}(S)} \lesssim \epsilon$ as stated.

3.4 Effective uniformization for nearly round spheres of arbitrary area

Let (S, g^S) be a fixed sphere, and let r^S denote its area radius, i.e. r^S satisfies

$$|S| = 4\pi(r^S)^2.$$

Given a positive integer s , we introduce the following norm on S

$$\|f\|_{\mathfrak{h}_s(S)} := \sum_{i=0}^s \|(r^S \nabla^S)^i f\|_{L^2(S)}. \quad (3.12)$$

The goal of the following corollary is to extend Theorem 3.1 to the case $r^S \neq 1$.

Corollary 3.8. *Let (S, g^S) be a fixed sphere. There exists, up to isometries¹⁹ of \mathbb{S}^2 , a unique diffeomorphism $\Phi : \mathbb{S}^2 \rightarrow S$ and a unique centered conformal factor u , i.e. $u \in \mathcal{S}$, such that*

$$\Phi^\#(g^S) = (r^S)^2 e^{2u} \gamma_0.$$

Moreover, under the almost round condition

$$\left\| K^S - \frac{1}{(r^S)^2} \right\|_{L^\infty(S)} \leq \frac{\epsilon}{(r^S)^2}, \quad (3.13)$$

the following properties are verified for sufficiently small $\epsilon > 0$.

1. We have

$$\|u \circ \Phi^{-1}\|_{L^\infty(S)} \lesssim \epsilon. \quad (3.14)$$

2. If in addition

$$\left\| K^S - \frac{1}{(r^S)^2} \right\|_{\mathfrak{h}_s(S)} \leq \frac{\epsilon}{r^S} \quad (3.15)$$

for some $s \geq 0$, then

$$\|u \circ \Phi^{-1}\|_{\mathfrak{h}_{s+2}(S)} \lesssim \epsilon r^S. \quad (3.16)$$

Proof. Consider the metric \tilde{g}^S on S given by

$$\tilde{g}^S := \frac{1}{(r^S)^2} g^S.$$

Then, (S, \tilde{g}^S) has area radius 1, and $K(\tilde{g}^S) = (r^S)^2 K(g^S)$ so that, in view of (3.13), the almost round condition (3.1) holds for \tilde{g}^S . Thus, we may apply Theorem 3.1 to (S, \tilde{g}^S) which then implies the first two conclusions of Corollary 3.8 for (S, g^S) . \square

¹⁹i.e. all the solutions are of the form $(\Phi \circ O, u \circ O)$ for $O \in O(3)$.

3.5 Canonical basis of $\ell = 1$ modes on S

Let S be an almost round sphere, i.e. verifying (3.13). The goal of this section is to define on S a canonical generalization of the $\ell = 1$ spherical harmonics.

Recall that the $\ell = 1$ spherical harmonics $J^{\mathbb{S}^2} = (J^{(-,\mathbb{S}^2)}, J^{(0,\mathbb{S}^2)}, J^{(+,\mathbb{S}^2)})$ are given by the restriction of x^1, x^2, x^3 to \mathbb{S}^2 . More precisely, in polar coordinates,

$$J^{(0,\mathbb{S}^2)} = x^3 = \cos \theta, \quad J^{(+,\mathbb{S}^2)} = x^1 = \sin \theta \cos \varphi, \quad J^{(-,\mathbb{S}^2)} = x^2 = \sin \theta \sin \varphi. \quad (3.17)$$

Lemma 3.9. *We have, for $p, q \in \{-, 0, +\}$,*

$$\begin{aligned} \Delta_0 J^{(p,\mathbb{S}^2)} &= -2J^{(p,\mathbb{S}^2)}, \\ \int_{\mathbb{S}^2} J^{(p,\mathbb{S}^2)} J^{(q,\mathbb{S}^2)} da_{\gamma_0} &= \frac{4\pi}{3} \delta_{pq}, \\ \int_{\mathbb{S}^2} J^{(p,\mathbb{S}^2)} da_{\gamma_0} &= 0. \end{aligned} \quad (3.18)$$

Proof. Straightforward verification. □

Definition 3.10 (Basis of canonical $\ell = 1$ modes on S). *Let (S, g^S) be an almost round sphere, i.e. verifying (3.13). Let (Φ, u) the unique, up to isometries of \mathbb{S}^2 , uniformization pair given by Corollary 3.8, i.e.,*

$$\Phi : \mathbb{S}^2 \longrightarrow S, \quad \Phi^\#(g^S) = (r^S)^2 e^{2u} \gamma_0, \quad u \in \mathcal{S}.$$

We define the basis of canonical $\ell = 1$ modes on S by

$$J^S := J^{\mathbb{S}^2} \circ \Phi^{-1}, \quad (3.19)$$

where $J^{\mathbb{S}^2}$ denotes the $\ell = 1$ spherical harmonics, see (3.17).

Remark 3.11. *Note that the canonical basis is unique up to a rotation on \mathbb{S}^2 .*

Lemma 3.12. *Consider (S, g^S) a sphere of area radius r^S verifying the almost round condition (3.13). Let (Φ, u) the unique, up to isometries of \mathbb{S}^2 , uniformization pair given by Corollary 3.8. Let J^S denote the basis of canonical $\ell = 1$ modes on S of Definition 3.10. Then, we have*

$$\begin{aligned} \Delta_S J^{(p,S)} &= -\frac{2}{(r^S)^2} J^{(p,S)} + \frac{2}{(r^S)^2} (1 - e^{-2v}) J^{(p,S)}, \\ \int_S J^{(p,S)} J^{(q,S)} da_g &= \frac{4\pi}{3} (r^S)^2 \delta_{pq} + \int_S J^{(p,S)} J^{(q,S)} (1 - e^{-2v}) da_{g^S}, \\ \int_S J^{(p,S)} da_g &= 0, \end{aligned} \quad (3.20)$$

with Δ^S the Laplace-Beltrami of the metric g^S and with $v := u \circ \Phi^{-1}$. Moreover we have,

$$\begin{aligned} \Delta_S J^{(p,S)} &= \left(-\frac{2}{(r^S)^2} + O\left(\frac{\epsilon}{(r^S)^2}\right) \right) J^{(p,S)}, \\ \int_S J^{(p,S)} J^{(q,S)} da_g &= \frac{4\pi}{3} (r^S)^2 \delta_{pq} + O(\epsilon (r^S)^2), \end{aligned} \quad (3.21)$$

where $\epsilon > 0$ is the smallness constant appearing in the almost round condition (3.13).

Proof. Since (Φ, u) is the unique, up to isometries of \mathbb{S}^2 , uniformization pair given by Corollary 3.8, we have

$$\Phi : \mathbb{S}^2 \longrightarrow S, \quad \Phi^\#(g^S) = (r^S)^2 e^{2u} \gamma_0, \quad u \in \mathcal{S}.$$

We rewrite this as follows

$$g^S = (r^S)^2 e^{2v} g_0, \quad g_0 := (\Phi^{-1})^\# \gamma_0, \quad v := u \circ \Phi^{-1}. \quad (3.22)$$

Since $\Delta_0 J^{\mathbb{S}^2} = -2J^{\mathbb{S}^2}$ and $g_0 = (\Phi^{-1})^\# \gamma_0$, we have, in view of the definition of J^S , i.e. $J^S = J^{\mathbb{S}^2} \circ \Phi^{-1} = (\Phi^{-1})^\# J^{\mathbb{S}^2}$,

$$\Delta_{g_0} J^S = -2J^S.$$

On the other hand, since $g^S = (r^S)^2 e^{2v} g_0$, we have in view of the conformal invariance of the Laplacian, $\Delta_S = (r^S)^{-2} e^{-2v} \Delta_{g_0}$. We deduce

$$\Delta_S J^S = (r^S)^{-2} e^{-2v} \Delta_{g_0} J^S = -2(r^S)^{-2} e^{-2v} J^S = -\frac{2}{(r^S)^2} J^S + \frac{2}{(r^S)^2} \left(-e^{-2v} + 1 \right) J^S.$$

Also,

$$\begin{aligned} \int_S J^{(p,S)} J^{(q,S)} da_{g^S} &= \int_S J^{(p,S)} J^{(q,S)} (r^S)^2 e^{2v} da_{g_0} \\ &= (r^S)^2 \int_S (\Phi^{-1})^\# J^{(p,\mathbb{S}^2)} (\Phi^{-1})^\# J^{(q,\mathbb{S}^2)} da_{(\Phi^{-1})^\# \gamma_0} \\ &\quad + (r^S)^2 \int_S J^{(p,S)} J^{(q,S)} (e^{2v} - 1) da_{g_0} \\ &= (r^S)^2 \int_{\mathbb{S}^2} J^{(p,\mathbb{S}^2)} J^{(q,\mathbb{S}^2)} da_{\gamma_0} + \int_S J^{(p,S)} J^{(q,S)} (e^{2v} - 1) e^{-2v} da_{g^S} \\ &= \frac{4\pi}{3} (r^S)^2 \delta_{pq} + \int_S J^{(p,S)} J^{(q,S)} (1 - e^{-2v}) da_{g^S}. \end{aligned}$$

Also, in view of the centeredness of u ,

$$\begin{aligned}\int_S J^S da_{g^S} &= \int_S J^{\mathbb{S}^2} \circ \Phi^{-1} (r^S)^2 e^{2v} da_{g_0} = (r^S)^2 \int_S (\Phi^{-1})^\# (e^{2u} J^{\mathbb{S}^2}) da_{(\Phi^{-1})^\# \gamma_0} \\ &= (r^S)^2 \int_{\mathbb{S}^2} e^{2u} x da_{\gamma_0} = 0\end{aligned}$$

as stated. The proof of (3.21) follows immediately in view of the fact that $v = u \circ \Phi$ satisfies $\|v\|_{L^\infty} \lesssim \epsilon$, see (3.14). \square

Corollary 3.13. *Let (S, g^S) verifying (3.13). Let (Φ, u) the unique, up to isometries of \mathbb{S}^2 , uniformization pair given by Corollary 3.8. Let J^S denote the basis of canonical $\ell = 1$ modes on S of Definition 3.10. Then, for sufficiently small $\epsilon > 0$, the following holds²⁰*

$$\int_S \left(K^S - \frac{1}{(r^S)^2} \right) J^{(p,S)} = O(\epsilon^2), \quad p = 0, +, -, \quad (3.23)$$

where K^S and r^S denote respectively the Gauss curvature and the area radius of S .

Proof. Since $\Phi^*(g^S) = (r^S)^2 e^{2u} \gamma_0$, we have $K((r^S)^{-2} e^{-2u} \Phi^*(g^S)) = 1$ and hence

$$\begin{aligned}1 &= K((r^S)^{-2} e^{-2u} \Phi^*(g^S)) = K((r^S)^{-2} \Phi^*(e^{-2v} g^S)) \\ &= K((r^S)^{-2} e^{-2v} g^S) \circ \Phi, \quad v := u \circ \Phi^{-1},\end{aligned}$$

which together with Lemma 2.1 yields

$$1 = K((r^S)^{-2} e^{-2v} g^S) = (r^S)^2 e^{2v} (K^S + \Delta_S v).$$

We infer

$$K^S - \frac{1}{(r^S)^2} = - \left(\Delta_S + \frac{2}{(r^S)^2} \right) v + \frac{f(-v)}{(r^S)^2}, \quad f(v) := e^{2v} - 1 - 2v. \quad (3.24)$$

Multiplying by $J^{(p,S)}$, integrating on S and using integration by parts, we deduce

$$\begin{aligned}\int_S \left(K^S - \frac{1}{(r^S)^2} \right) J^{(p,S)} &= - \int_S J^{(p,S)} \left(\Delta_S + \frac{2}{(r^S)^2} \right) v + \int_S \frac{f(-v)}{(r^S)^2} J^{(p,S)} \\ &= - \int_S v \left(\Delta_S + \frac{2}{(r^S)^2} \right) J^{(p,S)} + \int_S \frac{f(-v)}{(r^S)^2} J^{(p,S)}.\end{aligned}$$

Together with (3.21), the control $|v| \lesssim \epsilon$ provided by (3.14), and the definition of f , we infer

$$\left| \int_S \left(K^S - \frac{1}{(r^S)^2} \right) J^{(p,S)} \right| \lesssim \int_S \frac{\epsilon}{(r^S)^2} |v| + \int_S \frac{|v|^2}{(r^S)^2} \lesssim \epsilon^2$$

as desired. \square

²⁰Note that, a priori, one would expect the right-hand side of (3.23) to be $O(\epsilon)$. The fact that it is actually $O(\epsilon^2)$ is an application of Corollary 3.8 and (3.21), see the proof below.

4 Stability of uniformization for nearby spheres

Consider two almost round spheres (S_1, g^{S_1}) and (S_2, g^{S_2}) , i.e. verifying (3.13), and their respective uniformization pairs (Φ_1, u_1) , (Φ_2, u_2) , i.e.

$$\begin{aligned}\Phi_1 : \mathbb{S}^2 &\longrightarrow S_1, & g_1 &:= \Phi_1^\#(g^{S_1}) = (r^{S_1})^2 e^{2u_1} \gamma_0, \\ \Phi_2 : \mathbb{S}^2 &\longrightarrow S_2, & g_2 &:= \Phi_2^\#(g^{S_2}) = (r^{S_2})^2 e^{2u_2} \gamma_0,\end{aligned}\tag{4.1}$$

and u_1, u_2 defined on \mathbb{S}^2 verifying the conclusions of Corollary 3.8. We assume in addition given a smooth diffeomorphism $\Psi : S_1 \rightarrow S_2$ such that the metrics g^{S_1} and $\Psi^\#(g^{S_2})$ are close to each other in S^1 with respect to the coordinate chart provided by Φ_1 , i.e. for some $0 < \delta \leq \epsilon$,

$$\|g^{S_1} - \Psi^\#(g^{S_2})\|_{L^\infty(S_1)} + \frac{1}{(r^{S_1})} \|g^{S_1} - \Psi^\#(g^{S_2})\|_{\mathfrak{h}_4(S_1)} \leq (r^{S_1})^2 \delta.\tag{4.2}$$

The goal of this section is to show the existence of a canonical diffeomorphism $\widehat{\Psi} : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ which relates the two uniformization maps. More precisely we prove the following.

Theorem 4.1. *Under the assumptions above, let $\widehat{\Psi} : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ be the unique smooth diffeomorphism such that $\Psi \circ \Phi_1 = \Phi_2 \circ \widehat{\Psi}$. Then, the following holds true.*

1. *The diffeomorphism $\widehat{\Psi}$ is smooth and there exists $O \in O(3)$ such that*

$$\|\widehat{\Psi} - O\|_{L^\infty(\mathbb{S}^2)} + \|\widehat{\Psi} - O\|_{H^1(\mathbb{S}^2)} \lesssim \delta.\tag{4.3}$$

2. *The conformal factors u_1, u_2 verify*

$$\|u_1 - \widehat{\Psi}^\# u_2\|_{L^\infty(\mathbb{S}^2)} \lesssim \delta.\tag{4.4}$$

Remark 4.2. *Let us note the following concerning assumption (4.2).*

- *It is clearly not sharp in terms of regularity. Sharper results could be obtained by working in Hölder spaces. On the other hand, in view of our applications, $\mathfrak{h}_s(S)$ are the natural spaces.*
- *It is coordinate dependent. Though it is sufficient for our applications, it would be nice find a coordinate independent condition sufficient to recover the conclusions of Theorem 4.1.*

The proof of Theorem 4.1 is postponed to section 4.2. It will rely in particular on the control of almost isometries of \mathbb{S}^2 discussed in the next section.

4.1 Almost isometries of \mathbb{S}^2

Proposition 4.3. *Let $\Theta : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ be a C^2 diffeomorphism such that,*

$$\|\Theta^\# \gamma_0 - \gamma_0\|_{L^\infty(\mathbb{S}^2)} \leq \delta.$$

Then, there exists $O \in O(3)$ such that

$$\|\Theta - O\|_{L^\infty(\mathbb{S}^2)} + \|\Theta - O\|_{H^1(\mathbb{S}^2)} \lesssim \delta. \quad (4.5)$$

Proof. We first extend the map Θ to a map $\tilde{\Theta} : \mathbb{R}^3 \setminus \{0\} \rightarrow \mathbb{R}^3 \setminus \{0\}$ such that for every $x = r\omega, \omega \in \mathbb{S}^2, r > 0$,

$$\tilde{\Theta}(r\omega) = r\Theta(\omega). \quad (4.6)$$

It is easy to check that, denoting by e the euclidian metric,

$$\|\tilde{\Theta}^\# e - e\|_{L^\infty(D)} \lesssim \delta, \quad D := \left\{ x \in \mathbb{R}^3 \mid \frac{1}{2} \leq |x| \leq 2 \right\}. \quad (4.7)$$

We deduce

$$\|\nabla \tilde{\Theta} (\nabla \tilde{\Theta})^t - I\|_{L^\infty(D)} \lesssim \delta,$$

i.e.

$$\|AA^t - I\|_{L^\infty(D)} \lesssim \delta, \quad A := \nabla \tilde{\Theta},$$

where A is a function on D taking values in 3 by 3 matrices.

By the polar decomposition of A , there exists, at every point $p \in D$, a unique orthogonal matrix $O \in O(3)$ such that

$$A = (AA^t)^{\frac{1}{2}} O.$$

We deduce, at every $p \in D$,

$$|A - O| = |(AA^t)^{\frac{1}{2}} - I| \lesssim \delta.$$

Therefore,

$$\sup_D \min_{O \in O(3)} |\nabla \tilde{\Theta} - O| \lesssim \delta$$

i.e.,

$$\|\text{dist}(\nabla\tilde{\Theta}, O(3))\|_{L^\infty(D)} \lesssim \delta.$$

In particular, the determinant of $\nabla\tilde{\Theta}$ cannot vanish on D , and hence, since D is connected, is either strictly positive or strictly negative everywhere on D . From now on, we assume for simplicity that the determinant of $\nabla\tilde{\Theta}$ is strictly positive everywhere on D , the other case being similar. We infer

$$\|\text{dist}(\nabla\tilde{\Theta}, SO(3))\|_{L^\infty(D)} \lesssim \delta. \quad (4.8)$$

We rely on the following result²¹ of G. Friesecke, R. James and S. Müller in [10] pointed to us by Camillo De Lellis.

Theorem 4.4. *Let $D \subset \mathbb{R}^n$ a bounded Lipschitz domain, $n \geq 2$ and $1 < p < +\infty$. There exists a constant $C > 0$ depending only on D and p with the property:*

For every $\psi \in W^{1,p}(D, \mathbb{R}^n)$ there exists a rotation $O_0 \in SO(n)$ such that,

$$\|\nabla\psi - O_0\|_{L^p(D)} \lesssim C \|\text{dist}(\nabla\psi, SO(n))\|_{L^p(D)}. \quad (4.9)$$

Remark 4.5. *The estimate of Theorem 4.4 fails in L^∞ but still holds in BMO from a well known estimate of F. John [13].*

We infer from (4.8) and Theorem 4.4 the existence of a rotation $O_0 \in SO(3)$ such that

$$\|\nabla\tilde{\Theta} - O_0\|_{L^3(D)} + \|\nabla\tilde{\Theta} - O_0\|_{L^2(D)} \lesssim \delta.$$

Restricting to the sphere, we infer,

$$\|\nabla\Theta - O_0\|_{L^3(\mathbb{S}^2)} + \|\nabla\Theta - O_0\|_{L^2(\mathbb{S}^2)} \lesssim \delta.$$

Together with Poincaré inequality and the Sobolev embedding, this yields

$$\|\Theta - O_0 - x_0\|_{L^\infty(\mathbb{S}^2)} + \|\nabla(\Theta - O_0)\|_{L^2(\mathbb{S}^2)} \lesssim \delta,$$

where x_0 is given by

$$x_0 = \frac{1}{|\mathbb{S}^2|} \int_{\mathbb{S}^2} (\Theta - O_0) = \frac{1}{|\mathbb{S}^2|} \int_{\mathbb{S}^2} \Theta, \quad x_0 \in \mathbb{R}^3, \quad |x_0| \leq 1.$$

²¹The result in [10] actually covers only $p = 2$. The case $1 < p < +\infty$ is in section 2.4 of S. Conti and B. Schweizer [9].

Now, we have

$$\begin{aligned} |\mathbb{S}^2| x_0 &= \int_{\mathbb{S}^2} \Theta(x) da_{\gamma_0} = \int_{\mathbb{S}^2} x da_{(\Theta^{-1})^\# \gamma_0} \\ &= \int_{\mathbb{S}^2} x d\gamma_0 + O(\delta) = O(\delta) \end{aligned}$$

and hence $x_0 = O(\delta)$. Plugging in the above, we infer

$$\|\Theta - O_0\|_{L^\infty(\mathbb{S}^2)} + \|\nabla(\Theta - O_0)\|_{L^2(\mathbb{S}^2)} \lesssim \delta$$

as stated. This concludes the proof of Proposition 4.3. \square

4.2 Proof of Theorem 4.1

We note that the following is an immediate consequence of (4.2)

$$\left| \frac{r^{S_1}}{r^{S_2}} - 1 \right| \lesssim \delta. \quad (4.10)$$

Denote

$$g_1 = \Phi_1^\#(g^{S_1}), \quad g_2 = (\Psi \circ \Phi_1)^\#(g^{S_2}).$$

In view of our assumptions, we have

$$g_1 = (r^{S_1})^2 e^{2u_1} \gamma_0, \quad \int_{\mathbb{S}^2} e^{2u_1} x = 0, \quad \|u_1\|_{H^4(\mathbb{S}^2)} \leq \epsilon, \quad (4.11)$$

and

$$\|g_1 - g_2\|_{L^\infty(\mathbb{S}^2)} + \|g_1 - g_2\|_{H^4(\mathbb{S}^2)} \lesssim (r^{S_1})^2 \delta. \quad (4.12)$$

We deduce,

$$\left\| e^{2u_1} \gamma_0 - \frac{1}{(r^{S_1})^2} g_2 \right\|_{L^\infty(\mathbb{S}^2)} + \left\| e^{2u_1} \gamma_0 - \frac{1}{(r^{S_1})^2} g_2 \right\|_{H^4(\mathbb{S}^2)} \lesssim \delta,$$

or, in view of the control of u_1 as well as (4.10)

$$\|\gamma_0 - e^{-2u_1} \tilde{g}_2\|_{L^\infty(\mathbb{S}^2)} + \|\gamma_0 - e^{-2u_1} \tilde{g}_2\|_{H^4(\mathbb{S}^2)} \lesssim \delta, \quad \tilde{g}_2 := \frac{1}{(r^{S_2})^2} g_2. \quad (4.13)$$

Consequently

$$\|K(e^{-2u_1}\tilde{g}_2) - 1\|_{H^2(\mathbb{S}^2)} \lesssim \delta.$$

We can thus apply Corollary 3.8 to the metric $e^{-2u_1}\tilde{g}_2$ and deduce that there exists, up to an isometry of \mathbb{S}^2 , a unique centered conformal v and a unique smooth diffeomorphism $\Theta : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ such that

$$\Theta^\#(e^{-2u_1}\tilde{g}_2) = (r_{1,2})^2 e^{2v} \gamma_0, \quad \|v\|_{H^4(\mathbb{S}^2)} \lesssim \delta. \quad (4.14)$$

Hence,

$$e^{-2u_1}\tilde{g}_2 = (r_{1,2})^2 e^{2v \circ \Theta^{-1}} (\Theta^{-1})^\# \gamma_0.$$

Since $\|\gamma_0 - e^{-2u_1}\tilde{g}_2\|_{H^4} \lesssim \delta$ it follows that,

$$\|\gamma_0 - (r_{1,2})^2 e^{2v \circ \Theta^{-1}} (\Theta^{-1})^\# \gamma_0\|_{H^4(\mathbb{S}^2)} \lesssim \delta, \quad |r_{1,2} - 1| \lesssim \delta,$$

where we also used $\|v\|_{L^\infty(\mathbb{S}^2)} \lesssim \delta$ for the second inequality. Since $\|v\|_{H^4(\mathbb{S}^2)} \lesssim \delta$ and Θ is smooth we deduce,

$$\|\gamma_0 - \Theta^\# \gamma_0\|_{H^4(\mathbb{S}^2)} \lesssim \delta. \quad (4.15)$$

We now appeal to Proposition 4.3 which yields the existence of $O \in O(3)$ such that

$$\|\Theta - O\|_{L^\infty(\mathbb{S}^2)} + \|\Theta - O\|_{H^1(\mathbb{S}^2)} \lesssim \delta. \quad (4.16)$$

We rewrite (4.14) in the form

$$\Theta^\# \tilde{g}_2 = (r_{1,2})^2 e^{2v+2u_1 \circ \Theta} \gamma_0 = e^{2\tilde{u}_2} \gamma_0, \quad \tilde{u}_2 := v + u_1 \circ \Theta + \log(r_{1,2}).$$

Note that \tilde{u}_2 is a priori not centered. On the other hand, we have

$$\|\tilde{u}_2\|_{L^\infty} \leq \|v\|_{L^\infty} + \|u_1\|_{L^\infty} + |\log(r_{1,2})| \lesssim \epsilon. \quad (4.17)$$

Also, since $|r_{1,2} - 1| \lesssim \delta$, $\|v\|_{H^4} \lesssim \delta$ and $\|\Theta - O\|_{L^\infty} \lesssim \delta$,

$$\begin{aligned} \int_{\mathbb{S}^2} e^{\tilde{u}_2} x da_{\gamma_0} &= \int_{\mathbb{S}^2} e^{2v+2u_1 \circ \Theta + 2\log(r_{1,2})} x da_{\gamma_0} \\ &= \int_{\mathbb{S}^2} e^{2u_1 \circ \Theta} x da_{\gamma_0} + \int_{\mathbb{S}^2} e^{2u_1 \circ \Theta} ((r_{1,2})^2 e^{2v} - 1) x da_{\gamma_0} \\ &= \int_{\mathbb{S}^2} e^{2u_1} (x \circ \Theta^{-1}) \frac{1}{|\det d\Theta|} da_{\gamma_0} + O(\delta) \\ &= \int_{\mathbb{S}^2} e^{2u_1} x \circ O^{-1} da_{\gamma_0} + O(\delta). \end{aligned}$$

We deduce, remembering that O is a matrix in $O(3)$ and $x \in \mathbb{R}^3$,

$$\begin{aligned} \int_{\mathbb{S}^2} e^{\tilde{u}_2} x da_{\gamma_0} &= \int_{\mathbb{S}^2} e^{2u_1} O^{-1} x da_{\gamma_0} + O(\delta) \\ &= O^{-1} \left(\int_{\mathbb{S}^2} e^{2u_1} x da_{\gamma_0} \right) + O(\delta). \end{aligned}$$

Since u_1 is centered, we infer

$$\int_{\mathbb{S}^2} e^{2\tilde{u}_2} x = O(\delta). \quad (4.18)$$

In view of (4.17) and (4.18), Lemma 3.6 applies so that there exists $\psi \in \mathbf{M}$ such that²²

$$u'_2 := (\tilde{u}_2)_\psi = \tilde{u}_2 \circ \psi + \frac{1}{2} \log |\det \psi| \in \mathcal{S}, \quad \|\psi - O^{-1}\|_{L^\infty(\mathbb{S}^2)} \lesssim \delta. \quad (4.19)$$

We deduce

$$\psi^\# \Theta^\# \tilde{g}_2 = e^{\tilde{u}_2 \circ \psi} \psi^\# \gamma_0 = e^{2\tilde{u}_2 \circ \psi} |\det d\psi| \gamma_0 = e^{2\tilde{u}_2 \circ \psi + \log |\det d\psi|} \gamma_0 = e^{2u'_2} \gamma_0.$$

Also, in view of the control on Θ and ψ , we have

$$\|\Theta \circ \psi - I\|_{L^\infty(\mathbb{S}^2)} + \|\Theta \circ \psi - I\|_{H^1(\mathbb{S}^2)} \lesssim \delta.$$

Therefore we have found a diffeomorphism

$$\widehat{\Psi}' := (\Theta \circ \psi)^{-1}, \quad \widehat{\Psi}' : \mathbb{S}^2 \longrightarrow \mathbb{S}^2, \quad (4.20)$$

with the properties

$$(\widehat{\Psi}'^{-1})^\# \tilde{g}_2 = e^{2u'_2} \gamma_0, \quad u'_2 \in \mathcal{S}, \quad (4.21)$$

and

$$\|\widehat{\Psi}' - I\|_{L^\infty(\mathbb{S}^2)} + \|\widehat{\Psi}' - I\|_{H^1(\mathbb{S}^2)} \lesssim \delta. \quad (4.22)$$

Also, recall that

$$\begin{aligned} u'_2 &= \tilde{u}_2 \circ \psi + \frac{1}{2} \log |\det \psi| = (v + u_1 \circ \Theta + 2 \log(r_{1,2})) \circ \psi + \frac{1}{2} \log |\det \psi| \\ &= u_1 \circ \widehat{\Psi}'^{-1} + v_\psi + 2 \log(r_{1,2}). \end{aligned}$$

²²The statement of Lemma 3.6 actually yields $|\psi - I| \lesssim \delta$. Here we use the fact that \mathcal{S} is invariant by $O(3)$, see Remark 2.9, so that we may indeed assume, by composing with $O^{-1} \in O(3)$, that $|\psi - O^{-1}| \lesssim \delta$.

We infer

$$\widehat{\Psi}'\#u_2' - u_1 = \widehat{\Psi}'\#(v_\psi) + 2\log(r_{1,2}),$$

and thus, since $|r_{1,2} - 1| \lesssim \delta$, $\|v\|_{H^4} \leq \delta$, $\|\widehat{\Psi}' - I\|_{L^\infty} \lesssim \delta$ and $\|\psi - O^{-1}\|_{L^\infty} \lesssim \delta$, we deduce,

$$\|u_1 - \widehat{\Psi}'\#(u_2')\|_{L^\infty(\mathbb{S}^2)} \lesssim \delta. \quad (4.23)$$

Next, let

$$\Phi_2' := \Psi \circ \Phi_1 \circ \widehat{\Psi}'^{-1}.$$

Then, by the definition of g_2 and \tilde{g}_2 ,

$$\tilde{g}_2 = \frac{1}{(r^{S_2})^2} g_2 = \frac{1}{(r^{S_2})^2} (\Psi \circ \Phi_1)\#g^{S_2},$$

and the construction of $\widehat{\Psi}'$ we have

$$(\Phi_2')\#g^{S_2} = (\widehat{\Psi}'^{-1})\#(\Psi \circ \Phi_1)\#g^{S_2} = (r^{S_2})^2 (\widehat{\Psi}'^{-1})\#\tilde{g}_2 = (r^{S_2})^2 e^{2u_2'} \gamma_0$$

and by the definition of Φ_2, u_2 we also have,

$$(\Phi_2)\#g^{S_2} = (r^{S_2})^2 e^{2u_2} \gamma_0.$$

Therefore, since both u_2 and u_2' are centered, from the uniqueness statement of Corollary 3.8, there exists $O \in O(3)$ such that,

$$u_2' = u_2 \circ O, \quad \Psi \circ \Phi_1 = \Phi_2 \circ O \circ \widehat{\Psi}', \quad (4.24)$$

where the last identity corresponds to $\Phi_2' = \Phi_2 \circ O$. We now define

$$\widehat{\Psi} := O \circ \widehat{\Psi}'.$$

Then, together with (4.22), (4.23) and (4.24), we infer

$$\Psi \circ \Phi_1 = \Phi_2 \circ \widehat{\Psi}, \quad \|\widehat{\Psi} - O\|_{L^\infty} + \|\widehat{\Psi} - O\|_{H^1} \lesssim \delta, \quad \|u_1 - \widehat{\Psi}\#(u_2)\|_{L^\infty(\mathbb{S}^2)} \lesssim \delta,$$

as stated. This concludes the proof of Theorem 4.1.

4.3 Higher regularity estimates for Theorem 4.1

We provide in this section higher regularity estimates corresponding to (4.3) (4.4). This relies on a recent personal communication of Camillo De Lellis and Stefan Müller.

Theorem 4.6 (C. De Lellis, S. Müller [18]). *Let $n \geq 2$. Consider the balls $B_1 \subset B_2 \subset \mathbb{R}^n$ and e the euclidian metric on \mathbb{R}^n . Consider a map $u \in H^1(B_2, \mathbb{R}^n)$ verifying $\det \nabla u > 0$ and,*

$$\|u^\#e - e\|_{L^\infty(B_1)} \leq \frac{1}{2}. \quad (4.25)$$

Then, there exists a rotation $O \in SO(n)$ such that, relative to the Hölder norms C^α , $0 < \alpha < 1$,

$$\|\nabla u - O\|_{C^\alpha(B_1)} \lesssim \|u^\#e - e\|_{C^\alpha(B_2)}.$$

Proof. We thank C. De Lellis for agreeing to put his proof in appendix A. □

The goal of this section is to prove the following proposition that extends the estimates (4.3), (4.4) of Theorem 4.1 to corresponding higher regularity estimates.

Proposition 4.7. *Assume, in addition to the assumptions of Theorem 4.1, that*

$$\|g^{S_1} - \Psi^\#(g^{S_2})\|_{\mathfrak{h}_{4+s}(S^1)} \leq (r^{S_1})^3 \delta \quad (4.26)$$

for some $s \geq 0$. Then, the following higher regularity analogs of (4.3), (4.4) hold true.

1. *The diffeomorphism $\widehat{\Psi}$ is smooth and there exists $O \in O(3)$ such that*

$$\|\widehat{\Psi} - O\|_{H^{5+s}(S^2)} \lesssim \delta. \quad (4.27)$$

2. *The conformal factors u_1, u_2 verify*

$$\|u_1 - \widehat{\Psi}^\#u_2\|_{H^{4+s}(S^2)} \lesssim \delta. \quad (4.28)$$

The only part of the proof of Theorem 4.1 which does not immediately extend to higher regularity is when applying Proposition 4.3 to infer (4.16) from (4.15). Thus, to complete the proof of Proposition 4.7, it suffices to prove the following extension of Proposition 4.3 to higher regularity.

Proposition 4.8. *Let $\Theta : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ be a C^2 diffeomorphism such that,*

$$\|\Theta^\# \gamma_0 - \gamma_0\|_{H^{s+2}(\mathbb{S}^2)} \leq \delta \quad (4.29)$$

for some $s \geq 0$. Then, there exists $O \in O(3)$ such that

$$\|\Theta - O\|_{H^{s+3}(\mathbb{S}^2)} \lesssim \delta. \quad (4.30)$$

Proof. Recall that the proof of Proposition 4.3 proceeds by extending the map Θ to a map $\tilde{\Theta} : \mathbb{R}^3 \setminus \{0\} \rightarrow \mathbb{R}^3 \setminus \{0\}$ such that for every $x = r\omega, \omega \in \mathbb{S}^2, r > 0$,

$$\tilde{\Theta}(r\omega) = r\Theta(\omega),$$

see (4.6). Under the assumption (4.29), we obtain the following higher regularity analog of the estimate (4.7)

$$\|\tilde{\Theta}^\# e - e\|_{H^{s+2}(D)} \lesssim \delta, \quad D = \left\{ x \in \mathbb{R}^3 \mid \frac{1}{2} \leq |x| \leq 2 \right\}. \quad (4.31)$$

In view of Theorem 4.6, covering \mathbb{S}^2 by finitely many euclidean balls²³, and using that $H^2(D)$ embeds in $C^{1/2}(D)$ by Sobolev, we infer the existence of $O \in O(3)$ such that

$$\|\nabla \tilde{\Theta} - O\|_{C^{\frac{1}{2}}(D_1)} \lesssim \delta, \quad D_1 := \left\{ x \in \mathbb{R}^3 \mid \frac{3}{4} \leq |x| \leq \frac{5}{4} \right\}.$$

We deduce the following decomposition for $\tilde{\Theta}$

$$\tilde{\Theta} = (I + \tilde{\Theta}')O, \quad \|\tilde{\Theta}'\|_{L^\infty(D_1)} + \|\nabla \tilde{\Theta}'\|_{L^\infty(D_1)} \lesssim \delta. \quad (4.32)$$

Next, we rewrite (4.31) in matrix form

$$\nabla \tilde{\Theta} (\nabla \tilde{\Theta})^t = I + E, \quad \|E\|_{H^{s+2}(D)} \lesssim \delta. \quad (4.33)$$

Linearizing this equation based on the decomposition of $\tilde{\Theta}$ provided by (4.32), we infer on D_1

$$S(\tilde{\Theta}') = E - \nabla \tilde{\Theta}' (\nabla \tilde{\Theta}')^t, \quad S(\tilde{\Theta}') := \nabla \tilde{\Theta}' + (\nabla \tilde{\Theta}')^t. \quad (4.34)$$

Note that S satisfies the well-known Korn inequality, see for example [8],

$$\|f\|_{H^1(D_1)} \leq C_{D_1} \left(\|f\|_{L^2(D_1)} + \|S(f)\|_{L^2(D_1)} \right), \quad f : D_1 \rightarrow \mathbb{R}^3. \quad (4.35)$$

²³More precisely, B_1 and B_2 of Theorem 4.6 are chosen to be euclidean balls centered on points of \mathbb{S}^2 of radius given respectively by $1/4$ and $1/2$ so that both B_1 and B_2 are included in D and the union of the B_1 's includes D_1 .

where $S(f)$ denotes the symmetric part of the gradient of f . Differentiating (4.34), relying on the estimate (4.33) for E and the a priori bound (4.32) for $\tilde{\Theta}'$ on D_1 , and making use of Korn inequality (4.35), we infer

$$\|\tilde{\Theta}'\|_{H^{s+3}(D_1)} \lesssim \delta$$

and hence

$$\|\tilde{\Theta} - O\|_{H^{s+3}(D_1)} \lesssim \delta.$$

Restricting to the sphere, we obtain

$$\|\Theta - O\|_{H^{s+3}(\mathbb{S}^2)} \lesssim \delta$$

as desired. □

4.4 Calibration of uniformization maps between spheres

In order to eliminate the arbitrariness with respect to isometries of \mathbb{S}^2 in Theorem 4.1 and in Proposition 4.7, see (4.3) and (4.27), we calibrate the effective uniformization maps²⁴ $\Phi_1 : \mathbb{S}^2 \rightarrow S_1$, $\Phi_2 : \mathbb{S}^2 \rightarrow S_2$, for given diffeomorphism $\Psi : S_1 \rightarrow S_2$, as follows.

Definition 4.9. *On \mathbb{S}^2 we fix²⁵ a point N and a unit vector v in the tangent space $T_N\mathbb{S}^2$. Given $\Psi : S_1 \rightarrow S_2$, we say that the effective uniformization maps $\Phi_1 : \mathbb{S}^2 \rightarrow S_1$, $\Phi_2 : \mathbb{S}^2 \rightarrow S_2$ are calibrated relative to Ψ if the map $\widehat{\Psi} := (\Phi_2)^{-1} \circ \Psi \circ \Phi_1 : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ is such that*

1. *The map $\widehat{\Psi}$ fixes the point N , i.e. $\widehat{\Psi}(N) = N$.*
2. *The tangent map $\widehat{\Psi}_\#$ fixes the direction of v , i.e. $\widehat{\Psi}_\#(v) = a_{1,2}v$ where $a_{1,2} > 0$.*
3. *The tangent map $\widehat{\Psi}_\#$ preserves the orientation of $T_N\mathbb{S}^2$.*

Lemma 4.10. *Given $\Psi : S_1 \rightarrow S_2$ and a fixed effective uniformization map $\Phi_1 : \mathbb{S}^2 \rightarrow S_1$ for S_1 . Then, there exists a unique, effective, uniformization for S_2 calibrated with that of S_1 relative to Ψ .*

²⁴Given by Corollary 3.8.

²⁵In particular, one can choose $N = (0, 0, 1)$ and $v = (1, 0, 0)$.

Proof. Let Φ'_2 be an effective uniformization map for S_2 and let $\widehat{\Psi}' := (\Phi'_2)^{-1} \circ \Psi \circ \Phi_1$ so that $\widehat{\Psi}' : \mathbb{S}^2 \rightarrow \mathbb{S}^2$. Recall that for any $O \in O(3)$, Φ_2 given by $\Phi_2 = \Phi'_2 \circ O^{-1}$ is also an effective uniformization map for S_2 . We have in this case $\widehat{\Psi} = O\widehat{\Psi}'$, and Φ_1 and Φ_2 are calibrated relative to Ψ if and only if O satisfies the following.

1. $O(\widehat{\Psi}'(N)) = N$.
2. $\widehat{\Psi}'_{\#}(Ov) = av$ where $a > 0$.
3. $\widehat{\Psi}'_{\#} \circ O$ preserves the orientation of $T_N\mathbb{S}^2$.

Since $\widehat{\Psi}'$ is $O(\delta)$ close to an element of $O(3)$, see (4.3), there exists a unique $O \in O(3)$ satisfying the conditions above. \square

We now state the following corollaries to Theorem 4.1 and Proposition 4.7.

Corollary 4.11. *In addition to the assumptions of Theorem 4.1, assume that the maps Φ_1, Φ_2 are calibrated relative to Ψ according to Definition 4.9. Then $\widehat{\Psi}$ verifies*

$$\|\widehat{\Psi} - I\|_{L^\infty(\mathbb{S}^2)} + \|\widehat{\Psi} - I\|_{H^1(\mathbb{S}^2)} \lesssim \delta. \quad (4.36)$$

The conformal factors u_1, u_2 verify

$$\|u_1 - \widehat{\Psi}^{\#}u_2\|_{L^\infty(\mathbb{S}^2)} \lesssim \delta. \quad (4.37)$$

Proof. In view of Theorem 4.1, there exists $O \in O(3)$ such that $\widehat{\Psi}$ satisfies

$$\|\widehat{\Psi} - O\|_{L^\infty(\mathbb{S}^2)} + \|\widehat{\Psi} - O\|_{H^1(\mathbb{S}^2)} \lesssim \delta \quad (4.38)$$

and (4.37) holds.

It remains to check (4.36). In view of (4.38) and the fact that Φ_1, Φ_2 are calibrated relative to Ψ according to Definition 4.9, we infer that

1. $O(N) = N + O(\delta)$,
2. $O(v) = av + O(\delta)$ with $a > 0$.
3. O preserves the orientation of $T_N\mathbb{S}^2$.

These conditions imply immediately that $|O - I| \lesssim \delta$ which together with (4.38) yields (4.36) as desired. \square

Corollary 4.12. *In addition to the assumptions of Proposition 4.7, assume that the maps Φ_1, Φ_2 are calibrated relative to Ψ according to Definition 4.9. Then $\widehat{\Psi}$ verifies*

$$\|\widehat{\Psi} - I\|_{H^{5+s}(\mathbb{S}^2)} \lesssim \delta. \quad (4.39)$$

The conformal factors u_1, u_2 verify

$$\|u_1 - \widehat{\Psi}^\# u_2\|_{H^{4+s}(\mathbb{S}^2)} \lesssim \delta. \quad (4.40)$$

Proof. In view of Proposition 4.7, there exists $O \in O(3)$ such that $\widehat{\Psi}$ satisfies

$$\|\widehat{\Psi} - O\|_{L^\infty(\mathbb{S}^2)} + \|\widehat{\Psi} - O\|_{H^{5+s}(\mathbb{S}^2)} \lesssim \delta$$

and (4.40) holds. It remains to check (4.39) which follows immediately from the bound $|O - I| \lesssim \delta$ established in the proof of Corollary 4.11. \square

Lemma 4.13 (Transitivity of calibrations). *Let $\Phi_1 : \mathbb{S}^2 \rightarrow S_1$, $\Phi_2 : \mathbb{S}^2 \rightarrow S_2$ and $\Phi_3 : \mathbb{S}^2 \rightarrow S_3$ three effective uniformization maps. Let $\Psi_{12} : S_1 \rightarrow S_2$ and $\Psi_{13} : S_1 \rightarrow S_3$ satisfying (4.2) and assume that Φ_1, Φ_2 are calibrated relative to Ψ_{12} , while Φ_1, Φ_3 are calibrated relative to Ψ_{13} . Then, Φ_2, Φ_3 are calibrated relative to $\Psi_{23} := \Psi_{13} \circ \Psi_{12}^{-1}$.*

Proof. Since Φ_1, Φ_2 are calibrated relative to Ψ_{12} , and Φ_1, Φ_3 are calibrated relative to Ψ_{13} , $\widehat{\Psi}_{12} := (\Phi_2)^{-1} \circ \Psi_{12} \circ \Phi_1$ and $\widehat{\Psi}_{13} := (\Phi_3)^{-1} \circ \Psi_{13} \circ \Phi_1$ satisfy the three properties of Definition 4.9. Then, introducing

$$\Psi_{23} = \Psi_{13} \circ \Psi_{12}^{-1}, \quad \widehat{\Psi}_{23} := (\Phi_3)^{-1} \circ \Psi_{23} \circ \Phi_2,$$

we have $\widehat{\Psi}_{23} = \widehat{\Psi}_{13} \circ \widehat{\Psi}_{12}^{-1}$ so that $\widehat{\Psi}_{23}$ also satisfy the three properties of Definition 4.9. Hence, Φ_2, Φ_3 are calibrated relative to Ψ_{23} as desired. \square

4.5 Comparison of $\ell = 1$ modes between two spheres

Consider, as in Theorem 4.1, two almost round spheres (S_1, g^{S_1}) and (S_2, g^{S_2}) and a smooth map $\Psi : S_1 \rightarrow S_2$ such that, as in (4.2), that the metrics g^{S_1} and $\Psi^\#(g^{S_2})$ are close to each other in S^1 . Assume that (Φ_1, u_1) , (Φ_2, u_2) are effective uniformization maps of S_1 and S_2 , calibrated as in definition 4.9. We define

$$J^i = J^{S_i} = J^{\mathbb{S}^2} \circ \Phi_i^{-1}, \quad i = 1, 2,$$

to be the $\ell = 1$ canonical modes of S_1, S_2 according to Definition 3.10. We want to compare J^1 with $J^2 \circ \Psi$. We prove the following result.

Proposition 4.14. *Under the assumptions of Corollary 4.11, we have*

$$\sup_{S_1} \left| J^1 - J^2 \circ \Psi \right| \lesssim \delta. \quad (4.41)$$

Also, under the assumptions of Corollary 4.12, we have

$$\left\| J^1 - J^2 \circ \Psi \right\|_{\mathfrak{h}_{5+s}(S_1)} \lesssim r^{S_1} \delta. \quad (4.42)$$

Proof. Indeed, using that $\Psi \circ \Phi_1 = \Phi_2 \circ \widehat{\Psi}$,

$$J^2 \circ \Psi = J^{\mathbb{S}^2} \circ \Phi_2^{-1} \circ \Psi = J^{\mathbb{S}^2} \circ \widehat{\Psi} \circ \Phi_1^{-1}.$$

Hence,

$$J^1 - J^2 \circ \Psi = J^{\mathbb{S}^2} \circ \Phi_1^{-1} - J^{\mathbb{S}^2} \circ \widehat{\Psi} \circ \Phi_1^{-1}.$$

This implies, together with (4.36),

$$\left| J^1 - J^2 \circ \Psi \right| \lesssim \sup_{\mathbb{S}^2} \left| I - \widehat{\Psi} \right| \lesssim \delta$$

and, together with (4.39),

$$\left\| J^1 - J^2 \circ \Psi \right\|_{\mathfrak{h}_{5+s}(S_1)} \lesssim r^{S_1} \delta$$

as stated. \square

We conclude this section with a comparison result between a non canonical basis of $\ell = 1$ modes and a canonical one.

Proposition 4.15. *Let (S, g^S) an almost round sphere, i.e. verifying (3.13), let (Φ, u) the unique, up to isometries of \mathbb{S}^2 , uniformization pair given by Corollary 3.8, and let J^S denote the basis of canonical $\ell = 1$ modes on S of Definition 3.10. Also, for δ such that $0 < \delta \leq \epsilon$, and $s \geq 2$, let a diffeomorphism $\widetilde{\Phi} : \mathbb{S}^2 \rightarrow S$ and a smooth scalar function \widetilde{u} on \mathbb{S}^2 such that we have on coordinates charts of \mathbb{S}^2*

$$\left\| \widetilde{\Phi}^\#(g^S) - (r^S)^2 e^{2\widetilde{u}} \gamma_0 \right\|_{H^s(\mathbb{S}^2)} \leq (r^S)^2 \delta, \quad (4.43)$$

and

$$\|\widetilde{u}\|_{H^2(\mathbb{S}^2)} \leq \epsilon. \quad (4.44)$$

Finally, assume that the scalar functions $\tilde{J}^{(p)} := J^{(p, \mathbb{S}^2)} \circ \tilde{\Phi}^{-1}$, $p = 0, +, -$, satisfy

$$\left| \int_S \tilde{J}^{(p)} \right| \leq (r^S)^2 \delta. \quad (4.45)$$

Then, recalling that (Φ, u) is unique modulo isometries of \mathbb{S}^2 , we may choose (Φ, u) such that

$$(r^S)^{-1} \|\tilde{u} \circ (\tilde{\Phi})^{-1} - u \circ \Phi^{-1}\|_{\mathfrak{h}_s(S)} + \max_{p=0,+,-} (r^S)^{-1} \|\tilde{J}^{(p)} - J^{(p,S)}\|_{\mathfrak{h}_{s+1}(S)} \lesssim \delta. \quad (4.46)$$

Proof. Let the map $\Psi : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ and the scalar function v on \mathbb{S}^2 given by

$$\Psi := (\tilde{\Phi})^{-1} \circ \Phi, \quad v := \tilde{u} \circ \Psi - u.$$

By definition of (Φ, u) , we have $\Phi^\#(g^S) = (r^S)^2 e^{2u} \gamma_0$ and hence

$$\tilde{\Phi}^\#(g^S) = (\Phi \circ \Psi^{-1})^\#(g^S) = (\Psi^{-1})^\#((r^S)^2 e^{2u} \gamma_0) = (r^S)^2 e^{2u \circ \Psi^{-1}} (\Psi^{-1})^\# \gamma_0.$$

Plugging in (4.43), we infer

$$\|\Psi^\# \gamma_0 - \gamma_0\|_{H^s(\mathbb{S}^2)} \lesssim \delta + \|v\|_{H^s(\mathbb{S}^2)}.$$

Since $s \geq 2$, we may apply Proposition 4.8 and infer the existence of $O \in O(3)$ such that

$$\|\Psi - O\|_{H^{s+1}(\mathbb{S}^2)} \lesssim \delta + \|v\|_{H^s(\mathbb{S}^2)}.$$

Now, recall that (Φ, u) is unique, up to isometries of \mathbb{S}^2 , i.e. the uniformization pair given by Corollary 3.8 are all of the form $(\Phi \circ O, u \circ O)$ for $O \in O(3)$. Thus, choosing the pair $(\Phi \circ O^{-1}, u \circ O^{-1})$ instead of (Φ, u) , we may assume $O = I$ and hence

$$\|\Psi - I\|_{H^{s+1}(\mathbb{S}^2)} \lesssim \delta + \|v\|_{H^s(\mathbb{S}^2)}. \quad (4.47)$$

Next, we estimate v . By definition of (Φ, u) , we have $\Phi^\#(g^S) = (r^S)^2 e^{2u} \gamma_0$ and hence, using Lemma 2.1, we infer

$$K(\Phi^\#(g^S)) = K((r^S)^2 e^{2u} \gamma_0) = (r^S)^{-2} e^{-2u} (1 - \Delta_0 u).$$

On the other hand, in view of (4.43), we have, recalling also that $s \geq 2$,

$$\left\| K(\tilde{\Phi}^\#(g^S)) - K((r^S)^2 e^{2\tilde{u}} \gamma_0) \right\|_{H^{s-2}(\mathbb{S}^2)} \lesssim (r^S)^{-2} \delta,$$

and hence, using Lemma 2.1 and $K(\tilde{\Phi}^\#(g^S)) = K((\Phi \circ \Psi^{-1})^\#(g^S)) = K(\Phi^\#(g^S)) \circ \Psi^{-1}$, we infer

$$\left\| K(\Phi^\#(g^S)) \circ \Psi^{-1} - (r^S)^{-2} e^{-2\tilde{u}} (1 - \Delta_0 \tilde{u}) \right\|_{H^{s-2}(\mathbb{S}^2)} \lesssim \delta.$$

Plugging the above, we deduce

$$\left\| e^{2v} (1 - \Delta_0 u) - (1 - \Delta_0((u+v) \circ \Psi^{-1})) \circ \Psi \right\|_{H^{s-2}(\mathbb{S}^2)} \lesssim \delta.$$

Now, in view of (4.47), and the fact that $\|u\|_{H^s(\mathbb{S}^2)} \lesssim \epsilon$ in view of Corollary 3.8, we have

$$\left\| \Delta_0((u+v) \circ \Psi^{-1}) \circ \Psi - \Delta_0(u+v) \right\|_{H^{s-2}(\mathbb{S}^2)} \lesssim \epsilon \delta + \epsilon \|v\|_{H^s(\mathbb{S}^2)}$$

and hence

$$\left\| e^{2v} (1 - \Delta_0 u) - 1 + \Delta_0(u+v) \right\|_{H^{s-2}(\mathbb{S}^2)} \lesssim \delta + \epsilon \|v\|_{H^s(\mathbb{S}^2)},$$

or, using again $\|u\|_{H^s(\mathbb{S}^2)} \lesssim \epsilon$,

$$(\Delta_0 + 2)v = h, \quad \|h\|_{H^{s-2}(\mathbb{S}^2)} \lesssim \delta + \epsilon \|v\|_{H^s(\mathbb{S}^2)}.$$

By standard elliptic estimates, recalling also that $s \geq 2$, we have²⁶

$$\|v\|_{H^s(\mathbb{S}^2)} \lesssim \|(\Delta_0 + 2)v\|_{H^{s-2}(\mathbb{S}^2)} + \left| \int_{\mathbb{S}^2} vx \right|$$

and thus, for $\epsilon > 0$ small enough

$$\|v\|_{H^s(\mathbb{S}^2)} \lesssim \delta + \left| \int_{\mathbb{S}^2} vx \right|.$$

Together with (4.47), this yields

$$\|\Psi - I\|_{H^{s+1}(\mathbb{S}^2)} + \|v\|_{H^s(\mathbb{S}^2)} \lesssim \delta + \left| \int_{\mathbb{S}^2} vx \right|. \quad (4.48)$$

To complete the estimate of Ψ and v , it remains to control the integral on the RHS of (4.48). Now, recall that $\tilde{J}^{(p)} := J^{(p, \mathbb{S}^2)} \circ \tilde{\Phi}^{-1}$ which implies

$$\begin{aligned} \int_S \tilde{J} da_{g^S} &= \int_S J^{\mathbb{S}^2} \circ \tilde{\Phi}^{-1} da_{g^S} = \int_{\mathbb{S}^2} J^{\mathbb{S}^2} da_{\tilde{\Phi}^\#(g^S)} = \int_{\mathbb{S}^2} x da_{\tilde{\Phi}^\#(g^S)} \\ &= \int_{\mathbb{S}^2} (r^S)^2 e^{2\tilde{u}} x da_{\gamma_0} + \left(\int_{\mathbb{S}^2} x \left(da_{\tilde{\Phi}^\#(g^S)} - da_{(r^S)^2 e^{2\tilde{u}} \gamma_0} \right) \right). \end{aligned}$$

²⁶Note that the kernel of $\Delta_0 + 2$ is given by the $\ell = 1$ spherical harmonics, and that $\int vx$ corresponds to the projection on these spherical harmonics.

Together with the assumptions (4.43) and (4.45), we infer

$$\left| \int_{\mathbb{S}^2} e^{2\tilde{u}} x da_{\gamma_0} \right| \lesssim \delta$$

and thus

$$\left| \int_{\mathbb{S}^2} e^{2(u+v) \circ \Psi^{-1}} x \right| \lesssim \delta.$$

Together with (4.48), we obtain

$$\left| \int_{\mathbb{S}^2} e^{2(u+v)} x \right| \lesssim \delta + \epsilon \left| \int_{\mathbb{S}^2} vx \right|.$$

Since u is balanced, this yields

$$\left| \int_{\mathbb{S}^2} e^{2u} (e^{2v} - 1) x \right| \lesssim \delta + \epsilon \left| \int_{\mathbb{S}^2} vx \right|$$

and hence

$$\left| \int_{\mathbb{S}^2} vx \right| \lesssim \delta + \epsilon \left| \int_{\mathbb{S}^2} vx \right|.$$

For $\epsilon > 0$ small enough, we finally deduce, together with (4.48),

$$\|\Psi - I\|_{H^{s+1}(\mathbb{S}^2)} + \|v\|_{H^s(\mathbb{S}^2)} \lesssim \delta. \quad (4.49)$$

We are now ready to conclude. Recall that we have by definition

$$\tilde{J}^{(p)} = J^{(p, \mathbb{S}^2)} \circ \tilde{\Phi}^{-1}, \quad J^{(p, S)} = J^{(p, \mathbb{S}^2)} \circ \Phi^{-1}, \quad p = 0, +, -,$$

so that

$$\begin{aligned} \tilde{J}^{(p)} - J^{(p, S)} &= J^{(p, \mathbb{S}^2)} \circ \tilde{\Phi}^{-1} - J^{(p, \mathbb{S}^2)} \circ \Phi^{-1} \\ &= \left(J^{(p, \mathbb{S}^2)} \circ \Psi - J^{(p, \mathbb{S}^2)} \right) \circ \Phi^{-1}. \end{aligned}$$

Together with (4.49), we deduce

$$\max_{p=0,+, -} (r^S)^{-1} \|\tilde{J}^{(p)} - J^{(p, S)}\|_{\mathfrak{h}_{s+1}(S)} \lesssim \|\Psi - I\|_{H^{s+1}(\mathbb{S}^2)} \lesssim \delta$$

and

$$(r^S)^{-1} \|\tilde{u} \circ (\tilde{\Phi})^{-1} - u \circ \Phi^{-1}\|_{\mathfrak{h}_s(S)} \lesssim \|v\|_{H^s(\mathbb{S}^2)} \lesssim \delta$$

as stated. □

5 Review of the geometric set-up in [17]

5.1 Background spacetime

As in [17], we consider given a vacuum spacetime \mathcal{R} with metric \mathbf{g} endowed with an outgoing geodesic foliation by spheres $S(u, s)$ of fixed (u, s) , where u is an outgoing optical function²⁷ with $L = -\mathbf{g}^{\alpha\beta}\partial_\beta u\partial_\alpha$ its null geodesic generator and s chosen such that

$$L(s) = \frac{1}{\varsigma}, \quad L(\varsigma) = 0.$$

Let $e_4 = \varsigma L$ and e_3 the unique null vectorfield orthogonal to $S(u, s)$ and such that $\mathbf{g}(e_3, e_4) = -2$. We then let (e_1, e_2) an orthogonal basis of the tangent space of $S(u, s)$. The corresponding connection coefficients relative to the null frame (e_3, e_4, e_1, e_2) are denoted by $\chi, \underline{\chi}, \xi, \underline{\xi}, \omega, \underline{\omega}, \eta, \underline{\eta}, \zeta$ and the null components of the curvature tensor by $\alpha, \underline{\alpha}, \beta, \underline{\beta}, \rho, \star\rho$. For the convenience of the reader we recall their definition below.

$$\begin{aligned} \underline{\chi}_{ab} &= g(\mathbf{D}_a e_3, e_b), & \chi_{ab} &= g(\mathbf{D}_a e_4, e_b), \\ \underline{\xi}_a &= \frac{1}{2}\mathbf{g}(\mathbf{D}_3 e_3, e_a), & \xi_a &= \frac{1}{2}g(\mathbf{D}_4 e_4, e_a), \\ \underline{\omega} &= \frac{1}{4}g(\mathbf{D}_3 e_3, e_4), & \omega &= \frac{1}{4}g(\mathbf{D}_4 e_4, e_3), \\ \underline{\eta}_a &= \frac{1}{2}(\mathbf{D}_4 e_3, e_a), & \eta_a &= \frac{1}{2}g(\mathbf{D}_3 e_4, e_a), \\ \zeta_a &= \frac{1}{2}g(\mathbf{D}_{e_a} e_4, e_3), \end{aligned}$$

and

$$\begin{aligned} \alpha_{ab} &= \mathbf{R}(e_a, e_4, e_b, e_4), & \underline{\alpha}_{ab} &= \mathbf{R}(e_a, e_3, e_b, e_3), \\ \beta_a &= \frac{1}{2}\mathbf{R}(e_a, e_4, e_3, e_4), & \underline{\beta}_a &= \frac{1}{2}\mathbf{R}(e_a, e_3, e_3, e_4), \\ \rho &= \frac{1}{4}\mathbf{R}(e_3, e_4, e_3, e_4), & \star\rho &= \star R(e_3, e_4, e_3, e_4). \end{aligned}$$

The null second fundamental forms $\chi, \underline{\chi}$ are further decomposed in their traces $\kappa = \text{tr}\chi$ and $\underline{\kappa} = \text{tr}\underline{\chi}$, and traceless parts $\hat{\chi}$ and $\hat{\underline{\chi}}$. We recall, see Lemma 2.4 in [17], that the geodesic nature of the foliation implies

$$\omega = \xi = 0, \quad \underline{\eta} = -\zeta, \quad \varsigma = \frac{2}{e_3(u)}.$$

²⁷That is $\mathbf{g}^{\alpha\beta}\partial_\alpha u\partial_\beta u = 0$.

The volume radius $r = r(u, s)$ is defined such that the volume of S is given by $4\pi r^2$. The Hawking mass $m = m(u, s)$ of $S = S(u, s)$ is given by the formula

$$\frac{2m}{r} = 1 + \frac{1}{16\pi} \int_S \text{tr}\chi \text{tr}\underline{\chi}. \quad (5.1)$$

The Gauss curvature of S is denoted by K . It verifies the Gauss equation

$$K = -\rho - \frac{1}{4} \text{tr}\chi \text{tr}\underline{\chi} + \frac{1}{2} \widehat{\chi} \cdot \widehat{\underline{\chi}}. \quad (5.2)$$

The mass aspect function μ is defined by

$$\mu := -\text{div}\zeta - \rho + \frac{1}{2} \widehat{\chi} \cdot \widehat{\underline{\chi}}. \quad (5.3)$$

As in [17], we define the renormalized quantities

$$\begin{aligned} \widetilde{\text{tr}\chi} &:= \text{tr}\chi - \frac{2}{r}, & \widetilde{\text{tr}\underline{\chi}} &:= \text{tr}\underline{\chi} + \frac{2\Upsilon}{r}, & \widetilde{\omega} &:= \omega - \frac{m}{r^2}, \\ \check{K} &:= K - \frac{1}{r^2}, & \check{\rho} &:= \rho + \frac{2m}{r^3}, & \check{\mu} &:= \mu - \frac{2m}{r^3}, \\ \check{\underline{\Omega}} &:= \underline{\Omega} + \Upsilon, & \check{\zeta} &:= \zeta - 1, \end{aligned}$$

where

$$\Upsilon := 1 - \frac{2m}{r},$$

and the sets

$$\begin{aligned} \Gamma_g &:= \left\{ \widetilde{\text{tr}\chi}, \widehat{\chi}, \zeta, \widetilde{\text{tr}\underline{\chi}}, r\check{\mu}, r\check{\rho}, r^*\rho, r\beta, r\alpha, r\check{K}, r^{-1}(e_4(r) - 1), r^{-1}e_4(m) \right\}, \\ \Gamma_b &:= \left\{ \eta, \widehat{\underline{\chi}}, \widetilde{\omega}, \underline{\xi}, r\check{\beta}, \underline{\alpha}, r^{-1}\check{\underline{\Omega}}, r^{-1}\check{\zeta}, r^{-1}(e_3(r) + \Upsilon), r^{-1}e_3(m) \right\}. \end{aligned} \quad (5.4)$$

5.1.1 Adapted coordinates

Recall, see [17], that a coordinate system (u, s, y^1, y^2) is said to be adapted to an outgoing geodesic foliation as above if $e_4(y^1) = e_4(y^2) = 0$. In that case the spacetime metric can be written in the form

$$\mathbf{g} = -2\zeta dud s + \zeta^2 \underline{\Omega} du^2 + g_{ab} (dy^a - \zeta \underline{B}^a du) (dy^b - \zeta \underline{B}^b du), \quad (5.5)$$

where

$$\underline{\Omega} = e_3(s), \quad \underline{B}^a = \frac{1}{2}e_3(y^a), \quad g_{ab} = \mathbf{g}(\partial_{y^a}, \partial_{y^b}). \quad (5.6)$$

Relative to these coordinates

$$e_4 = \partial_s, \quad \partial_u = \varsigma \left(\frac{1}{2}e_3 - \frac{1}{2}\underline{\Omega}e_4 - \underline{B}^a\partial_{y^a} \right), \quad \partial_{y^a} = \sum_{c=1,2} Y_{(a)}^c e_c, \quad a = 1, 2, \quad (5.7)$$

with coefficients $Y_{(a)}^b$ verifying

$$g_{ab} = \sum_{c=1,2} Y_{(a)}^c Y_{(b)}^c.$$

As in [17] we assume that \mathcal{R} is covered by two coordinate systems , i.e. $\mathcal{R} = \mathcal{R}_N \cup \mathcal{R}_S$, such that

1. The North coordinate chart \mathcal{R}_N is given by the coordinates (u, s, y_N^1, y_N^2) with $(y_N^1)^2 + (y_N^2)^2 < 2$.
2. The South coordinate chart \mathcal{R}_S is given by the coordinates (u, s, y_S^1, y_S^2) with $(y_S^1)^2 + (y_S^2)^2 < 2$.
3. The two coordinate charts intersect in the open equatorial region $\mathcal{R}_{Eq} := \mathcal{R}_N \cap \mathcal{R}_S$ in which both coordinate systems are defined.
4. In \mathcal{R}_{Eq} the transition functions between the two coordinate systems are given by the smooth functions φ_{SN} and $\varphi_{NS} = \varphi_{SN}^{-1}$.

The metric coefficients for the two coordinate systems are given by

$$\begin{aligned} \mathbf{g} &= -2\varsigma dud s + \varsigma^2 \underline{\Omega} du^2 + g_{ab}^N (dy_N^a - \varsigma \underline{B}_N^a du) (dy_N^b - \varsigma \underline{B}_N^b du), \\ \mathbf{g} &= -2\varsigma dud s + \varsigma^2 \underline{\Omega} du^2 + g_{ab}^S (dy_S^a - \varsigma \underline{B}_S^a du) (dy_S^b - \varsigma \underline{B}_S^b du), \end{aligned}$$

where

$$\underline{\Omega} = e_3(s), \quad \underline{B}_N^a = \frac{1}{2}e_3(y_N^a), \quad \underline{B}_S^a = \frac{1}{2}e_3(y_S^a).$$

On a 2-sphere $S(u, s)$ and $f \in \mathcal{S}_p(S)$, $p = 0, 1, 2$, we consider the following norms,

$$\begin{aligned} \|f\|_\infty &:= \|f\|_{L^\infty(S)}, & \|f\|_2 &:= \|f\|_{L^2(S)}, \\ \|f\|_{\infty, k} &= \sum_{i=0}^k \|\mathfrak{d}^i f\|_\infty, & \|f\|_{2, k} &= \sum_{i=0}^k \|\mathfrak{d}^i f\|_2, \end{aligned} \quad (5.8)$$

where \mathfrak{d}^i stands for any combination of length i of operators of the form $e_3, r e_4, r \nabla$.

5.1.2 Reduced region \mathcal{R}

In what follows we restrict our attention to smaller region of the original \mathcal{R} , still denoted \mathcal{R} defined as follows.

Definition 5.1. Let $m_0 > 0$ a constant. Let $\overset{\circ}{\epsilon} > 0$ two sufficiently small constants, and let $(\overset{\circ}{u}, \overset{\circ}{s}, \overset{\circ}{r})$ three real numbers with $\overset{\circ}{r}$ sufficiently large so that

$$\overset{\circ}{\epsilon} \ll m_0, \quad \overset{\circ}{r} \gg m_0. \quad (5.9)$$

We define $\mathcal{R} = \mathcal{R}$ to be the region

$$\mathcal{R} := \left\{ |u - \overset{\circ}{u}| \leq \overset{\circ}{\epsilon}, \quad |s - \overset{\circ}{s}| \leq \overset{\circ}{\epsilon} \right\}, \quad (5.10)$$

such that assumptions **A1-A4** below with constant $\overset{\circ}{\epsilon}$ on the background foliation of \mathcal{R} , are verified.

5.1.3 Main assumptions for \mathcal{R}

Given an integer $s_{max} \geq 3$, we assume²⁸ the following.

A1. For $k \leq s_{max}$

$$\begin{aligned} \|\Gamma_g\|_{k,\infty} &\leq \overset{\circ}{\epsilon} r^{-2}, \\ \|\Gamma_b\|_{k,\infty} &\leq \overset{\circ}{\epsilon} r^{-1}. \end{aligned} \quad (5.11)$$

A2. The Hawking mass $m = m(u, s)$ of $S(u, s)$ verifies

$$\sup_{\mathcal{R}} \left| \frac{m}{m_0} - 1 \right| \leq \overset{\circ}{\epsilon}. \quad (5.12)$$

A3. In the region of their respective validity²⁹ we have

$$\underline{B}_N^a, \underline{B}_S^a \in r^{-1}\Gamma_b, \quad Z_N^a, Z_S^a \in \Gamma_b, \quad r^{-2}\check{g}_{ab}^N, r^{-2}\check{g}_{ab}^S \in r\Gamma_g \quad (5.13)$$

where

$$\begin{aligned} \check{g}_{ab}^N &= g_{ab}^N - \frac{4r^2}{1 + (y_N^1)^2 + (y_N^2)^2} \delta_{ab}, \\ \check{g}_{ab}^S &= g_{ab}^S - \frac{4r^2}{(1 + (y_S^1)^2 + (y_S^2)^2)} \delta_{ab}. \end{aligned}$$

²⁸In view of (5.11), we will often replace Γ_g by $r^{-1}\Gamma_b$.

²⁹That is the quantities on the left verify the same estimates as those for Γ_b , respectively Γ_g .

A4. We assume the existence of a smooth family of scalar functions³⁰ $J^{(p)} : \mathcal{R} \rightarrow \mathbb{R}$, for $p = 0, +, -$, verifying the following properties

(a) On the sphere \mathring{S} of the background foliation, there holds

$$\begin{aligned} \left((\mathring{r})^2 \mathring{\Delta} + 2 \right) J^{(p)} &= O(\mathring{\epsilon}), \quad p = 0, +, -, \\ \frac{1}{|\mathring{S}|} \int_{\mathring{S}} J^{(p)} J^{(q)} &= \frac{1}{3} \delta_{pq} + O(\mathring{\epsilon}), \quad p, q = 0, +, -, \\ \frac{1}{|\mathring{S}|} \int_{\mathring{S}} J^{(p)} &= O(\mathring{\epsilon}), \quad p = 0, +, -, \end{aligned} \quad (5.14)$$

(b) We extend $J^{(p)}$ from \mathring{S} to \mathcal{R} by $\partial_s J^{(p)} = \partial_u J^{(p)} = 0$, i.e.

$$J^{(p)}(u, s, y^1, y^2) = J^{(p)}(\mathring{u}, \mathring{s}, y^1, y^2). \quad (5.15)$$

Remark 5.2. We note that the assumptions **A1**, **A2**, **A3**, **A4**, are expected to be valid in the far regions, i.e. r large, of a perturbed Kerr.

Given a scalar function f defined on a sphere S we define the $\ell = 1$ modes of f by the triplet

$$(f)_{\ell=1} := \left\{ \int_S f J^{(p)}, \quad p = 0, +, - \right\}.$$

5.1.4 $O(\mathring{\epsilon})$ -spheres

Definition 5.3. Given a compact 2-surface $\mathbf{S} \subset \mathcal{R}$, not necessarily a leaf $S(u, s)$ of the background geodesic foliation of \mathcal{R} , we denote

- by $\chi^{\mathbf{S}}, \underline{\chi}^{\mathbf{S}}, \zeta^{\mathbf{S}}, \dots$, the corresponding Ricci coefficients,

³⁰The properties (5.14) of the scalar functions $J^{(p)}$ are motivated by the fact that the $\ell = 1$ spherical harmonics on the standard sphere \mathbb{S}^2 , given by $J^{(0, \mathbb{S}^2)} = \cos \theta$, $J^{(+, \mathbb{S}^2)} = \sin \theta \cos \varphi$, $J^{(-, \mathbb{S}^2)} = \sin \theta \sin \varphi$, satisfy (5.14) with $\mathring{\epsilon} = 0$. Note also that on \mathbb{S}^2 ,

$$\int_{\mathbb{S}^2} (\cos \theta)^2 = \int_{\mathbb{S}^2} (\sin \theta \cos \varphi)^2 = \int_{\mathbb{S}^2} (\sin \theta \sin \varphi)^2 = \frac{4\pi}{3}, \quad |\mathbb{S}^2| = 4\pi.$$

- by $\alpha^{\mathbf{S}}, \beta^{\mathbf{S}}, \rho^{\mathbf{S}}, \dots$, the corresponding curvature coefficients,
- by $r^{\mathbf{S}}, m^{\mathbf{S}}, K^{\mathbf{S}}$ and $\mu^{\mathbf{S}}$ respectively the corresponding area radius, Hawking mass, Gauss curvature and mass aspect function,
- by $\nabla^{\mathbf{S}}$ the corresponding covariant derivative.

Definition 5.4. We will work with the following weighted Sobolev norms on \mathbf{S}

$$\|f\|_{\mathfrak{h}_s(\mathbf{S})} := \sum_{i=0}^s \|(\mathfrak{d}^{\mathbf{S}})^i f\|_{L^2(\mathbf{S})}, \quad \mathfrak{d}^{\mathbf{S}} = r^{\mathbf{S}} \nabla^{\mathbf{S}}. \quad (5.16)$$

The goal of this paper is to construct new spheres $\mathbf{S} \subset \mathcal{R}$ which verify special properties we call GCM conditions. In particular these spheres are close to being a round sphere in the sense of the definition below.

Definition 5.5 ($O(\overset{\circ}{\epsilon})$ -sphere). A compact surface $\mathbf{S} \subset \mathcal{R}$ of area radius $r^{\mathbf{S}}$ is called a $O(\overset{\circ}{\epsilon})$ -sphere provided that the Gauss curvature $K^{\mathbf{S}}$ of \mathbf{S} verifies

$$K^{\mathbf{S}} = \frac{1 + O(\overset{\circ}{\epsilon})}{(r^{\mathbf{S}})^2} \quad (5.17)$$

as well as

$$\left\| K^{\mathbf{S}} - \frac{1}{(r^{\mathbf{S}})^2} \right\|_{\mathfrak{h}_{s_{max}-1}(\mathbf{S})} \lesssim (r^{\mathbf{S}})^{-1} \overset{\circ}{\epsilon}, \quad (5.18)$$

and the area radius $r^{\mathbf{S}}$ verifies

$$\sup_{\mathbf{S}} |r^{\mathbf{S}} - r| \lesssim r^{\mathbf{S}} \overset{\circ}{\epsilon}. \quad (5.19)$$

5.1.5 General frame transformations

Lemma 5.6. Given a null frame (e_3, e_4, e_1, e_2) , a general null transformation from the null frame (e_3, e_4, e_1, e_2) to another null frame (e'_3, e'_4, e'_1, e'_2) can be written in the form,

$$\begin{aligned} e'_4 &= \lambda \left(e_4 + f^b e_b + \frac{1}{4} |f|^2 e_3 \right), \\ e'_a &= \left(\delta_{ab} + \frac{1}{2} \underline{f}_a f_b \right) e_b + \frac{1}{2} \underline{f}_a e_4 + \left(\frac{1}{2} f_a + \frac{1}{8} |f|^2 \underline{f}_a \right) e_3, \quad a = 1, 2, \\ e'_3 &= \lambda^{-1} \left(\left(1 + \frac{1}{2} \underline{f} \cdot \underline{f} + \frac{1}{16} |f|^2 |\underline{f}|^2 \right) e_3 + \left(\underline{f}^b + \frac{1}{4} |\underline{f}|^2 f^b \right) e_b + \frac{1}{4} |\underline{f}|^2 e_4 \right), \end{aligned} \quad (5.20)$$

where λ is a scalar, f and \underline{f} are horizontal 1-forms. The dot product and magnitude $|\cdot|$ are taken with respect to the standard euclidian norm of \mathbb{R}^2 . We call $(f, \underline{f}, \lambda)$ the transition coefficients of the change of frame. We denote $F := (f, \underline{f}, \overset{\circ}{\lambda})$ where $\overset{\circ}{\lambda} = \lambda - 1$.

Relative to the primed frame the connection coefficients Γ' can be related to the connection coefficients Γ of the background frame by specific formulas involving the transition coefficients and their derivatives. Similarly the curvature coefficients R' of the primed frame are connected to R by formulas involving the transition coefficients. The precise formulas are given in Proposition 3.3 in [17]. In this paper we only make use of the transformation formulas for $\kappa = \text{tr}\chi$, $\underline{\kappa} = \text{tr}\underline{\chi}$ and β . The transformation formulas for the first two have the form, see Proposition 3.4 in [17],

$$\begin{aligned}\kappa' &= \kappa + \kappa \overset{\circ}{\lambda} + \text{div}' f + F \cdot \Gamma_b + F \cdot \nabla' F + r^{-1} F^2, \\ \underline{\kappa}' &= \underline{\kappa} - \underline{\kappa} \overset{\circ}{\lambda} + \text{div}' \underline{f} + F \cdot \Gamma_b + F \cdot \nabla' F + r^{-1} F^2.\end{aligned}\tag{5.21}$$

The transformation formula for β has the form, see Proposition 3.3. in [17],

$$\beta' = \beta + \frac{3}{2}(f\rho + {}^*f{}^*\rho) + r^{-1}\Gamma_g \cdot F.\tag{5.22}$$

5.2 Deformation of surfaces in \mathcal{R}

Definition 5.7. We say that \mathbf{S} is a deformation of $\overset{\circ}{S}$ if there exist smooth scalar functions U, S defined on $\overset{\circ}{S}$ and a map $\Psi : \overset{\circ}{S} \rightarrow \mathbf{S}$ verifying, on any coordinate chart (y^1, y^2) of $\overset{\circ}{S}$,

$$\Psi(\overset{\circ}{u}, \overset{\circ}{s}, y^1, y^2) = \left(\overset{\circ}{u} + U(y^1, y^2), \overset{\circ}{s} + S(y^1, y^2), y^1, y^2 \right).\tag{5.23}$$

Definition 5.8. Given a deformation $\Psi : \overset{\circ}{S} \rightarrow \mathbf{S}$ we say that a new frame (e'_3, e'_4, e'_1, e'_2) on \mathbf{S} , obtained from the standard frame (e_3, e_4, e_1, e_2) via the transformation (5.20), is \mathbf{S} -adapted if the horizontal vectorfields e'_1, e'_2 are tangent to \mathbf{S} or, equivalently e'_3, e'_4 are perpendicular to S .

We will need the following lemmas.

Lemma 5.9. Let $\Psi : \overset{\circ}{S} \rightarrow \mathbf{S}$ be a deformation in \mathcal{R} as in Definition 5.7, F a scalar function on \mathcal{R} and $F^\# = F \circ \Psi$ its pull back to $\overset{\circ}{S}$ by Ψ . We have

$$\|F^\# - F\|_{L^\infty(\overset{\circ}{S})} \lesssim \|U\|_{L^\infty(\overset{\circ}{S})} \sup_{\mathcal{R}} |e_3 F| + \|(U, S)\|_{L^\infty(\overset{\circ}{S})} \sup_{\mathcal{R}} (|e_4 F| + |\nabla F|).\tag{5.24}$$

Proof. See Lemma 5.7 in [17]. □

Lemma 5.10. *Let $\mathring{S} \subset \mathcal{R}$. Let $\Psi : \mathring{S} \rightarrow \mathbf{S}$ be a deformation generated by the functions (U, S) as in Definition 5.7 and denote by $g^{\mathbf{S}, \#}$ the pull back of the metric $g^{\mathbf{S}}$ to \mathring{S} . Assume the bound*

$$\|(U, S)\|_{L^\infty(\mathring{S})} + r^{-1} \|(U, S)\|_{\mathfrak{h}_{s_{max}+1}(\mathring{S})} \lesssim \mathring{\delta}. \quad (5.25)$$

Then

1. We have

$$\|g^{\mathbf{S}, \#} - \mathring{g}\|_{L^\infty} + r^{-1} \|g^{\mathbf{S}, \#} - \mathring{g}\|_{\mathfrak{h}_{s_{max}}(\mathring{S})} \lesssim \mathring{\delta} r \quad (5.26)$$

where $g^{\mathbf{S}, \#}$ denotes the pull-back by Ψ of the metric $g^{\mathbf{S}}$ on \mathbf{S} .

2. For any tensor h on \mathcal{R}

$$\|h\|_{\mathfrak{h}_s(\mathbf{S})} \lesssim r \sup_{\mathcal{R}} \left(|\mathring{\partial}^{\leq s} h| + \mathring{\delta} |\mathring{\partial}^{\leq s} h| \right), \quad 0 \leq s \leq s_{max}. \quad (5.27)$$

3. If $V \in \mathfrak{h}_s(\mathbf{S})$ and $V^\#$ is its pull-back by Ψ , we have for all $0 \leq s \leq s_{max}$,

$$\|V\|_{\mathfrak{h}_s(\mathbf{S})} = \|V^\#\|_{\mathfrak{h}_s(\mathring{S}, g^{\mathbf{S}, \#})} = \|V^\#\|_{\mathfrak{h}_s(\mathring{S}, \mathring{g})} (1 + O(r^{-1} \mathring{\delta})). \quad (5.28)$$

4. As a corollary of (5.28) (choosing $V = 1$ and $s = 0$), we deduce

$$\frac{r^{\mathbf{S}}}{\mathring{r}} = 1 + O(r^{-1} \mathring{\delta}) \quad (5.29)$$

where $r^{\mathbf{S}}$ is the area radius of \mathbf{S} and \mathring{r} that of \mathring{S} .

Proof. See Lemma 5.8 and Proposition 5.10 in [17]. □

As a corollary, we deduce the following.

Lemma 5.11. *Under the same assumptions as in Lemma 5.10, the following estimate holds for a scalar function F defined on \mathcal{R} ,*

$$\left| \int_{\mathbf{S}} F - \int_{\mathring{S}} F \right| \lesssim \mathring{\delta} r \left(\sup_{\mathcal{R}} |F| + r \sup_{\mathcal{R}} (|\partial_u F| + |\partial_s F|) \right).$$

Proof. See Corollary 5.9 in [17]. □

Corollary 5.12. *Let $\mathring{S} \subset \mathcal{R}$. Let $\Psi : \mathring{S} \rightarrow \mathbf{S}$ be a deformation generated by the functions (U, S) as in Definition 5.7. Assume that (U, S) satisfy the bound*

$$\|(U, S)\|_{\mathfrak{h}_{s_{max}+1}(\mathring{S})} \lesssim r\mathring{\delta}.$$

Then, for all $j, k \geq 0$ with $0 \leq j + k \leq s_{max}$, we have

$$\begin{aligned} \|\mathfrak{d}^{\leq j} \Gamma_g\|_{\mathfrak{h}_k(\mathbf{S})} &\lesssim \mathring{\epsilon} r^{-1}, \\ \|\mathfrak{d}^{\leq j} \Gamma_b\|_{\mathfrak{h}_k(\mathbf{S})} &\lesssim \mathring{\epsilon}, \end{aligned} \tag{5.30}$$

and

$$r \|\mathfrak{d}^{\leq j} \underline{B}\|_{\mathfrak{h}_k(\mathbf{S})} + \|\mathfrak{d}^{\leq j} Z\|_{\mathfrak{h}_k(\mathbf{S})} + r^{-1} \|\mathfrak{d}^{\leq j} \check{Y}\|_{\mathfrak{h}_k(\mathbf{S})} \lesssim \mathring{\epsilon}. \tag{5.31}$$

Proof. See Corollary 5.11 in [17]. □

Corollary 5.13. *Let $\mathring{S} \subset \mathcal{R}$. Let $\Psi : \mathring{S} \rightarrow \mathbf{S}$ be a deformation generated by the functions (U, S) as in Definition 5.7. Assume the bound*

$$\|(U, S)\|_{\mathfrak{h}_{s_{max}+1}(\mathring{S})} \lesssim r\mathring{\delta}$$

with $s_{max} \geq 3$. Then

$$K^{\mathbf{S}} = \frac{1 + O(\mathring{\epsilon})}{(r^{\mathbf{S}})^2}, \quad \left\| K^{\mathbf{S}} - \frac{1}{(r^{\mathbf{S}})^2} \right\|_{\mathfrak{h}_{s_{max}-1}(\mathbf{S})} \lesssim (r^{\mathbf{S}})^{-1} \mathring{\epsilon}.$$

Proof. See Corollary 5.18 in [17]. □

Remark 5.14. *In view of Corollary 5.13 and Lemma 5.10, a deformed sphere \mathbf{S} with (U, S) satisfying (5.25) is an $O(\mathring{\epsilon})$ -sphere in the sense of Definition 5.5.*

Corollary 5.15. *Let $\mathring{S} \subset \mathcal{R}$. Let $\Psi : \mathring{S} \rightarrow \mathbf{S}$ be a deformation generated by the functions (U, S) as in Definition 5.7. Assume the bound*

$$\|(U, S)\|_{\mathfrak{h}_{s_{max}+1}(\mathring{S})} \lesssim r\mathring{\delta}$$

with $s_{max} \geq 3$. Then, we have

$$\sup_{\mathbf{S}} |r - \mathring{r}| + \sup_{\mathbf{S}} |r - r^{\mathbf{S}}| \lesssim \mathring{\delta}, \quad |m^{\mathbf{S}} - \mathring{m}| + \sup_{\mathbf{S}} |m - m^{\mathbf{S}}| \lesssim \mathring{\delta}, \quad \sup_{\mathbf{S}} |m - \mathring{m}| \lesssim \mathring{\epsilon} \mathring{\delta}.$$

Proof. See Corollary 5.12 and Corollary 5.17 in [17]. □

5.3 Non canonical basis of $\ell = 1$ modes on a deformed sphere

Consider a deformation $\Psi : \overset{\circ}{S} \rightarrow \mathbf{S}$ and recall the existence of the family of scalar functions $J^{(p)}$, $p \in \{0, +, -\}$, on \mathcal{R} satisfying **A4**.

Proposition 5.16. *Assume that the parameters (U, S) of the deformation verifies the bounds*

$$\|(U, S)\|_{\mathfrak{h}_{s_{max}+1}(\overset{\circ}{S})} \lesssim r\overset{\circ}{\delta}. \quad (5.32)$$

Then, $J^{(p)}$, $p \in \{0, +, -\}$, satisfies on \mathbf{S} the following

$$\begin{aligned} \left((r^{\mathbf{S}})^2 \Delta^{\mathbf{S}} + 2 \right) J^{(p)} &= O(\overset{\circ}{\epsilon}), \\ \frac{1}{|\mathbf{S}|} \int_{\mathbf{S}} J^{(p)} J^{(q)} &= \frac{1}{3} \delta_{pq} + O(\overset{\circ}{\epsilon}), \\ \frac{1}{|\mathbf{S}|} \int_{\mathbf{S}} J^{(p)} &= O(\overset{\circ}{\epsilon}). \end{aligned} \quad (5.33)$$

Proof. See Proposition 5.21 in [17]. □

Remark 5.17. While $J^{(p)}$, $p \in \{0, +, -\}$, is, in general, not the canonical basis of $\ell = 1$ modes of \mathbf{S} introduced in Definition 3.10, the properties (5.33) are similar to the ones of Lemma 3.12 satisfies by canonical basis of $\ell = 1$ modes. Thus, from now on, we will call a family of scalar functions $J^{(p)}$, $p \in \{0, +, -\}$, on \mathcal{R} satisfying **A4** a non canonical basis of $\ell = 1$ modes.

Definition 5.18. Let a deformation $\Psi : \overset{\circ}{S} \rightarrow \mathbf{S}$ with (U, S) satisfying (5.25). According to Corollary 5.13, \mathbf{S} satisfies the almost round condition (3.13). Thus there exists $J^{\mathbf{S}}$ a basis of canonical $\ell = 1$ modes on \mathbf{S} as in Definition 3.10. We define $J^{(p)}[\mathbf{S}] : \mathcal{R} \rightarrow \mathbb{R}$, for $p = 0, +, -$, as follows

1. On the sphere $\overset{\circ}{S}$ of the background foliation, $J^{(p)}[\mathbf{S}]$ is defined by

$$J^{(p)}[\mathbf{S}] = J^{(p, \mathbf{S})} \circ \Psi. \quad (5.34)$$

2. We extend $J^{(p)}[\mathbf{S}]$ from $\overset{\circ}{S}$ to \mathcal{R} by $\partial_s J^{(p)}[\mathbf{S}] = \partial_u J^{(p)}[\mathbf{S}] = 0$, i.e.

$$J^{(p)}[\mathbf{S}](u, s, y^1, y^2) = J^{(p, \mathbf{S})}(\Psi(\overset{\circ}{u}, \overset{\circ}{s}, y^1, y^2)). \quad (5.35)$$

Remark 5.19. Note in particular that $J^{(p)}[\mathbf{S}]$ satisfies

$$J^{(p)}[\mathbf{S}] = J^{(p,\mathbf{S})} \quad \text{on } \mathbf{S}. \quad (5.36)$$

Indeed, a point on \mathbf{S} is of the form $(u, s, y^1, y^2) = \Psi(\overset{\circ}{u}, \overset{\circ}{s}, y^1, y^2)$ so that (5.36) follows immediately from (5.35).

Proposition 5.20. Let a deformation $\Psi : \overset{\circ}{S} \rightarrow \mathbf{S}$ with (U, S) satisfying (5.25) and let $J^{(p)}[\mathbf{S}] : \mathcal{R} \rightarrow \mathbb{R}$, for $p = 0, +, -$, given by Definition 5.18. Then, $J^{(p)}[\mathbf{S}]$ satisfies **A4**.

Proof. In view of its definition, to prove that $J^{(p)}[\mathbf{S}]$ satisfies **A4**, it suffices to show that $J^{(p)}[\mathbf{S}]$ verifies (5.14). In view of Lemma 3.12, the basis of canonical $\ell = 1$ modes $J^{(p,\mathbf{S})}$ satisfies

$$\begin{aligned} \Delta_{\mathbf{S}} J^{(p,\mathbf{S})} &= \left(-\frac{2}{(r^{\mathbf{S}})^2} + O\left(\frac{\overset{\circ}{\epsilon}}{(r^{\mathbf{S}})^2}\right) \right) J^{(p,\mathbf{S})}, \\ \int_{\mathbf{S}} J^{(p,\mathbf{S})} J^{(q,\mathbf{S})} &= \frac{4\pi}{3} (r^{\mathbf{S}})^2 \delta_{pq} + O(\overset{\circ}{\epsilon} (r^{\mathbf{S}})^2), \\ \int_{\mathbf{S}} J^{(p,\mathbf{S})} &= 0. \end{aligned}$$

Pulling back to $\overset{\circ}{S}$ by the map Ψ , and since $(J^{(p,\mathbf{S})})^{\#} = J^{(p)}[\mathbf{S}]$, we infer

$$\begin{aligned} \Delta^{\mathbf{S},\#} (J^{(p)}[\mathbf{S}]) &= \left(-\frac{2}{(r^{\mathbf{S}})^2} + O\left(\frac{\overset{\circ}{\epsilon}}{(r^{\mathbf{S}})^2}\right) \right) J^{(p)}[\mathbf{S}], \\ \int_{\overset{\circ}{S}} J^{(p)}[\mathbf{S}] J^{(q)}[\mathbf{S}] da_{g^{\mathbf{S},\#}} &= \frac{4\pi}{3} (r^{\mathbf{S}})^2 \delta_{pq} + O(\overset{\circ}{\epsilon} (r^{\mathbf{S}})^2), \\ \int_{\overset{\circ}{S}} J^{(p)}[\mathbf{S}] da_{g^{\mathbf{S},\#}} &= 0. \end{aligned}$$

Together with the control of $r^{\mathbf{S}} - \overset{\circ}{r}$ provided by Lemma 5.10, we infer

$$\begin{aligned} \left((\overset{\circ}{r})^2 \Delta^{\mathbf{S},\#} + 2 \right) J^{(p)}[\mathbf{S}] &= O(\overset{\circ}{\epsilon}), \\ \frac{1}{|\overset{\circ}{S}|} \int_{\overset{\circ}{S}} J^{(p)}[\mathbf{S}] J^{(q)}[\mathbf{S}] da_{g^{\mathbf{S},\#}} &= \frac{1}{3} \delta_{pq} + O(\overset{\circ}{\epsilon}), \\ \frac{1}{|\overset{\circ}{S}|} \int_{\overset{\circ}{S}} J^{(p)}[\mathbf{S}] da_{g^{\mathbf{S},\#}} &= 0. \end{aligned}$$

Using the control of $g^{\mathbf{S},\#} - \overset{\circ}{g}$ provided by Lemma 5.10, we obtain the last two estimates of (5.14)

$$\begin{aligned} \frac{1}{|\overset{\circ}{S}|} \int_{\overset{\circ}{S}} J^{(p)}[\mathbf{S}] J^{(q)}[\mathbf{S}] da_{\overset{\circ}{g}} &= \frac{1}{3} \delta_{pq} + O(\overset{\circ}{\epsilon}), \\ \frac{1}{|\overset{\circ}{S}|} \int_{\overset{\circ}{S}} J^{(p)}[\mathbf{S}] da_{\overset{\circ}{g}} &= O(\overset{\circ}{\epsilon}). \end{aligned}$$

Next, we use

$$\begin{aligned} (\Delta^{\mathbf{S},\#} - \overset{\circ}{\Delta}) J^{(p)}[\mathbf{S}] &= \left((g^{\mathbf{S},\#})^{ab} - (\overset{\circ}{g})^{ab} \right) \left(\partial_{y^a} \partial_{y^b} J^{(p)}[\mathbf{S}] + (\Gamma^{\mathbf{S},\#})_{ab}^c \partial_{y^c} J^{(p)}[\mathbf{S}] \right) \\ &\quad + (\overset{\circ}{g})^{ab} \left((\Gamma^{\mathbf{S},\#})_{ab}^c - (\overset{\circ}{\Gamma})_{ab}^c \right) \partial_{y^c} J^{(p)}[\mathbf{S}]. \end{aligned}$$

In view of the estimates of Lemma 5.10, we deduce

$$\left| (\Delta^{\mathbf{S},\#} - \overset{\circ}{\Delta}) J^{(p)}[\mathbf{S}] \right| \lesssim r^{-2} \overset{\circ}{\delta} \left(\sum_{a=1,2} \left| \partial_{y^a} J^{(p)}[\mathbf{S}] \right| + \sum_{a,b=1,2} \left| \partial_{y^a} \partial_{y^b} J^{(p)}[\mathbf{S}] \right| \right) \lesssim r^{-2} \overset{\circ}{\delta}.$$

Therefore

$$\left((\overset{\circ}{r})^2 \overset{\circ}{\Delta} + 2 \right) J^{(p)}[\mathbf{S}] = \left((\overset{\circ}{r})^2 \Delta^{\mathbf{S},\#} + 2 \right) J^{(p)}[\mathbf{S}] + O(\overset{\circ}{\epsilon}) = O(\overset{\circ}{\epsilon})$$

which concludes the proof that $J^{(p)}[\mathbf{S}]$ verifies (5.14), and hence **A4**. \square

6 GCM spheres with canonical $\ell = 1$ modes

6.1 GCM spheres with non-canonical $\ell = 1$ modes in [17]

We review below the result on existence and uniqueness of GCM spheres proved in [17].

The result was proved in the context of an arbitrary choice of a $\ell = 1$ modes on $\overset{\circ}{S}$, denoted $J^{(p)}$, which verify the assumptions **A4**.

Theorem 6.1 (GCM spheres with non-canonical $\ell = 1$ modes [17]). *Let $m_0 > 0$ a constant. Let $0 < \overset{\circ}{\delta} \leq \overset{\circ}{\epsilon}$ two sufficiently small constants, and let $(\overset{\circ}{u}, \overset{\circ}{s}, \overset{\circ}{r})$ three real numbers with $\overset{\circ}{r}$ sufficiently large so that*

$$\overset{\circ}{\epsilon} \ll m_0, \quad \overset{\circ}{r} \gg m_0.$$

Let a fixed spacetime region \mathcal{R} , as in Definition 5.1, together with a (u, s) outgoing geodesic foliation verifying the assumptions **A1-A4**. Let $\mathring{S} = S(\mathring{u}, \mathring{s})$ be a fixed sphere from this foliation with \mathring{r} and \mathring{m} denoting its area radius and Hawking mass. Assume that the GCM quantities $\kappa, \underline{\kappa}, \mu$ of the background foliation verify the following:

$$\begin{aligned}\kappa &= \frac{2}{r} + \dot{\kappa}, \\ \underline{\kappa} &= -\frac{2\Upsilon}{r} + \underline{C}_0 + \sum_p \underline{C}^{(p)} J^{(p)} + \underline{\dot{\kappa}}, \\ \mu &= \frac{2m}{r^3} + M_0 + \sum_p M^{(p)} J^{(p)} + \dot{\mu},\end{aligned}\tag{6.1}$$

where the scalar functions $\underline{C}_0 = \underline{C}_0(u, s)$, $\underline{C}^{(p)} = \underline{C}^{(p)}(u, s)$, $M_0 = M_0(u, s)$ and $M^{(p)} = M^{(p)}(u, s)$, defined on the spacetime region \mathcal{R} , depend only on the coordinates (u, s) , and where $\dot{\kappa}$, $\underline{\dot{\kappa}}$ and $\dot{\mu}$ satisfy the following estimates

$$\sup_{\mathcal{R}} |\mathfrak{P}^{\leq s_{max}}(\dot{\kappa}, \underline{\dot{\kappa}})| \lesssim r^{-2} \mathring{\delta}, \quad \sup_{\mathcal{R}} |\mathfrak{P}^{\leq s_{max}} \dot{\mu}| \lesssim r^{-3} \mathring{\delta}.\tag{6.2}$$

Then for any fixed triplets $\Lambda, \underline{\Lambda} \in \mathbb{R}^3$ verifying

$$|\Lambda|, |\underline{\Lambda}| \lesssim \mathring{\delta},\tag{6.3}$$

there exists a unique GCM sphere $\mathbf{S} = \mathbf{S}(\Lambda, \underline{\Lambda})$, which is a deformation of \mathring{S} , such that there exist constants $\underline{C}_0^{\mathbf{S}}, \underline{C}^{(\mathbf{S}, p)}, M_0^{\mathbf{S}}, M^{(\mathbf{S}, p)}$, $p \in \{-, 0, +\}$ for which the following GCM conditions are verified

$$\begin{aligned}\kappa^{\mathbf{S}} &= \frac{2}{r^{\mathbf{S}}}, \\ \underline{\kappa}^{\mathbf{S}} &= -\frac{2}{r^{\mathbf{S}}} \Upsilon^{\mathbf{S}} + \underline{C}_0^{\mathbf{S}} + \sum_p \underline{C}^{(\mathbf{S}, p)} J^{(p)}, \\ \mu^{\mathbf{S}} &= \frac{2m^{\mathbf{S}}}{(r^{\mathbf{S}})^3} + M_0^{\mathbf{S}} + \sum_p M^{(\mathbf{S}, p)} J^{(p)},\end{aligned}\tag{6.4}$$

where we recall from Remark 5.17 that we choose $J^{(p)}$ as our non canonical basis of $\ell = 1$ modes on \mathbf{S} . Relative to these modes we also have

$$(\operatorname{div}^{\mathbf{S}} f)_{\ell=1} = \Lambda, \quad (\operatorname{div}^{\mathbf{S}} \underline{f})_{\ell=1} = \underline{\Lambda}.\tag{6.5}$$

The resulting deformation has the following additional properties:

1. The triplet $(f, \underline{f}, \overset{\circ}{\lambda})$ verifies

$$\|(f, \underline{f}, \overset{\circ}{\lambda})\|_{\mathfrak{h}_{s_{max}+1}(\mathbf{S})} \lesssim \overset{\circ}{\delta}. \quad (6.6)$$

2. The GCM constants $\underline{C}_0^{\mathbf{S}}, \underline{C}^{(\mathbf{S},p)}, M_0^{\mathbf{S}}, M^{(\mathbf{S},p)}, p \in \{-, 0, +\}$ verify

$$\begin{aligned} |\underline{C}_0^{\mathbf{S}} - \overline{C}_0^{\mathbf{S}}| + |\underline{C}^{(\mathbf{S},p)} - \overline{C}^{(p)\mathbf{S}}| &\lesssim r^{-2}\overset{\circ}{\delta}, \\ |M_0^{\mathbf{S}} - \overline{M}_0^{\mathbf{S}}| + |M^{(\mathbf{S},p)} - \overline{M}^{(p)\mathbf{S}}| &\lesssim r^{-3}\overset{\circ}{\delta}. \end{aligned} \quad (6.7)$$

3. The volume radius $r^{\mathbf{S}}$ verifies

$$\left| \frac{r^{\mathbf{S}}}{\overset{\circ}{r}} - 1 \right| \lesssim r^{-1}\overset{\circ}{\delta}. \quad (6.8)$$

4. The parameter functions U, S of the deformation verify

$$\|(U, S)\|_{\mathfrak{h}_{s_{max}+1}(\overset{\circ}{S})} \lesssim r\overset{\circ}{\delta}. \quad (6.9)$$

5. The Hawking mass $m^{\mathbf{S}}$ of \mathbf{S} verifies the estimate

$$|m^{\mathbf{S}} - \overset{\circ}{m}| \lesssim \overset{\circ}{\delta}. \quad (6.10)$$

6. The well defined³¹ Ricci and curvature coefficients of \mathbf{S} verify,

$$\begin{aligned} \|\Gamma_g^{\mathbf{S}}\|_{\mathfrak{h}_{s_{max}}(\mathbf{S})} &\lesssim \overset{\circ}{\epsilon}r^{-1}, \\ \|\Gamma_b^{\mathbf{S}}\|_{\mathfrak{h}_{s_{max}}(\mathbf{S})} &\lesssim \overset{\circ}{\epsilon}. \end{aligned} \quad (6.11)$$

7. The transition parameters $f, \underline{f}, \overset{\circ}{\lambda}$ are continuously differentiable with respect to $\Lambda, \underline{\Lambda}$ and

$$\begin{aligned} \frac{\partial f}{\partial \Lambda} &= O(r^{-1}), & \frac{\partial f}{\partial \underline{\Lambda}} &= O(\overset{\circ}{\delta}r^{-1}), & \frac{\partial \underline{f}}{\partial \Lambda} &= O(\overset{\circ}{\delta}r^{-1}), & \frac{\partial \underline{f}}{\partial \underline{\Lambda}} &= O(r^{-1}), \\ \frac{\partial \overset{\circ}{\lambda}}{\partial \Lambda} &= O(\overset{\circ}{\delta}r^{-1}), & \frac{\partial \overset{\circ}{\lambda}}{\partial \underline{\Lambda}} &= O(\overset{\circ}{\delta}r^{-1}). \end{aligned} \quad (6.12)$$

³¹Note that while the Ricci coefficients $\kappa^{\mathbf{S}}, \underline{\kappa}^{\mathbf{S}}, \widehat{\chi}^{\mathbf{S}}, \widehat{\underline{\chi}}^{\mathbf{S}}, \zeta^{\mathbf{S}}$ as well as all curvature components and mass aspect function $\mu^{\mathbf{S}}$ are well defined on \mathbf{S} , this is not the case of $\eta^{\mathbf{S}}, \underline{\eta}^{\mathbf{S}}, \xi^{\mathbf{S}}, \underline{\xi}^{\mathbf{S}}, \omega^{\mathbf{S}}, \underline{\omega}^{\mathbf{S}}$ which require the derivatives of the frame in the $e_3^{\mathbf{S}}$ and $e_4^{\mathbf{S}}$ directions.

8. The parameter functions (U, S) of the deformation are continuously differentiable with respect to $\Lambda, \underline{\Lambda}$ and

$$\frac{\partial U}{\partial \Lambda} = O(1), \quad \frac{\partial U}{\partial \underline{\Lambda}} = O(1), \quad \frac{\partial S}{\partial \Lambda} = O(1), \quad \frac{\partial S}{\partial \underline{\Lambda}} = O(1). \quad (6.13)$$

9. Relative to the coordinate system induced by Ψ the metric $g^{\Lambda, \underline{\Lambda}}$ of $\mathbf{S} = \mathbf{S}(\Lambda, \underline{\Lambda})$ is continuous with respect to the parameters $\Lambda, \underline{\Lambda}$ and verifies

$$\|\partial_{\Lambda} g^{\mathbf{S}}, \partial_{\underline{\Lambda}} g^{\mathbf{S}}\|_{L^{\infty}(\mathbf{S})} \lesssim O(r^2). \quad (6.14)$$

Remark 6.2. The conclusions of Theorem 6.1 still hold if we replace (6.2) with the weaker condition³²

$$\|(\dot{\kappa}, \dot{\underline{\kappa}})\|_{\mathfrak{h}_{s_{max}}(\mathbf{S})} \lesssim r^{-1} \mathring{\delta}, \quad \|\dot{\mu}\|_{\mathfrak{h}_{s_{max}}(\mathbf{S})} \lesssim r^{-2} \mathring{\delta}, \quad (6.15)$$

for any deformed sphere \mathbf{S} with (U, S) satisfying (6.9), where (6.15) is uniform w.r.t such spheres. See Remark 6.2 in [17].

Note that the basis of $\ell = 1$ modes $J^{(p)}$ appearing in (6.4) are not canonical on \mathbf{S} , i.e. they do not correspond to the modes defined using the effective uniformization map for \mathbf{S} . In the next section we show that we can modify the result of Theorem 6.1 so that the same statements hold when we replace $J^{(p)}$ by canonical modes.

6.2 Construction of GCM spheres with canonical $\ell = 1$ modes

We state and prove a new version of Theorem 6.1 stated relative to a canonical basis of $\ell = 1$ modes of \mathbf{S} as in Definition 3.10. In other words we replace the non canonical basis of $\ell = 1$ modes $J^{(p)}$ by the canonical basis of $\ell = 1$ modes $J^{(p, \mathbf{S})}$ on \mathbf{S} . The following definition prescribes the calibration of the basis $J^{(p, \mathbf{S})}$ of canonical $\ell = 1$ modes for deformed surfaces \mathbf{S} .

Definition 6.3. Given a deformation map $\Psi : \overset{\circ}{S} \rightarrow \mathbf{S}$ and a fixed effective uniformization map $(\overset{\circ}{\Phi}, \overset{\circ}{\phi})$ for $\overset{\circ}{S}$ we let (Φ, ϕ) be the unique effective uniformization map of \mathbf{S} calibrated with $(\overset{\circ}{\Phi}, \overset{\circ}{\phi})$ relative to the map Ψ , in the sense of Definition 4.9. With this choice, we define the canonical $\ell = 1$ modes of \mathbf{S} by the formula

$$J^{(p, \mathbf{S})} = J^{(p, \mathbb{S}^2)} \circ \Phi^{-1} \quad (6.16)$$

with $J^{(p, \mathbb{S}^2)}$ denoting the $\ell = 1$ spherical harmonics, see (3.17).

³²Note that (6.2) implies (6.15) in view of (5.27).

Remark 6.4. Assume given two deformations $\Psi_i : \mathring{S} \rightarrow \mathbf{S}_i$, $i = 1, 2$ with canonical effective uniformization maps (Φ_i, u_i) , $\Phi_i : \mathbb{S}^2 \rightarrow \mathbf{S}_i$, $i = 1, 2$, both calibrated through the respective maps with the fixed effective uniformization map $(\mathring{\Phi}, \mathring{\phi})$ for \mathring{S} . Then, in view of Lemma 4.13, (Φ_1, u_1) , (Φ_2, u_2) are also calibrated with respect to the map $\Psi_{12} : \mathbf{S}_1 \rightarrow \mathbf{S}_2$, with $\Psi_{12} = \Psi_2 \circ \Psi_1^{-1}$.

Theorem 6.5 (GCM spheres with canonical $\ell = 1$ modes). *Under the same assumptions as in Theorem 6.1, and assuming moreover that the choice of $\ell = 1$ modes $J^{(p)}$ on \mathring{S} , in addition to **A4**, verifies³³*

$$\max_{p=0,-,+} \|J^{(p)} - J^{(p,\mathring{S})}\|_{\mathfrak{h}_{\max+1}(\mathring{S})} \lesssim r\mathring{\delta}, \quad (6.17)$$

for any fixed triplets $\Lambda, \underline{\Lambda} \in \mathbb{R}^3$ verifying

$$|\Lambda|, |\underline{\Lambda}| \lesssim \mathring{\delta}, \quad (6.18)$$

there exists a unique GCM sphere $\mathbf{S} = \mathbf{S}(\Lambda, \underline{\Lambda})$, which is a deformation of \mathring{S} , such that there exist constants $\underline{C}^{(\mathbf{S},p)}$, $M^{(\mathbf{S},p)}$, $p \in \{-, 0, +\}$ for which the following GCM conditions are verified

$$\begin{aligned} \kappa^{\mathbf{S}} &= \frac{2}{r^{\mathbf{S}}}, \\ \underline{\kappa}^{\mathbf{S}} &= -\frac{2}{r^{\mathbf{S}}} \Upsilon^{\mathbf{S}} + \sum_p \underline{C}^{(\mathbf{S},p)} J^{(p,\mathbf{S})}, \\ \mu^{\mathbf{S}} &= \frac{2m^{\mathbf{S}}}{(r^{\mathbf{S}})^3} + \sum_p M^{(\mathbf{S},p)} J^{(p,\mathbf{S})}. \end{aligned} \quad (6.19)$$

Moreover,

$$(\operatorname{div}^{\mathbf{S}} f)_{\ell=1} = \Lambda, \quad (\operatorname{div}^{\mathbf{S}} \underline{f})_{\ell=1} = \underline{\Lambda}, \quad (6.20)$$

where $J^{(p,\mathbf{S})}$ form the basis of canonical $\ell = 1$ modes of \mathbf{S} in the sense of Definition 6.3 and $(\operatorname{div}^{\mathbf{S}} f)_{\ell=1}$, $(\operatorname{div}^{\mathbf{S}} \underline{f})_{\ell=1}$ are defined with respect to them. Finally, the resulting deformation verifies all the properties 1-9 as in Theorem 6.1.

³³ $J^{(p,\mathring{S})}$ denotes the canonical basis of $\ell = 1$ modes on \mathring{S} corresponding to the effective uniformization map $(\mathring{\Phi}, \mathring{\phi})$ for \mathring{S} appearing in Definition 6.3.

Remark 6.6. Note that the canonical $\ell = 1$ modes $J^{(p, \mathbf{S})}$ verify,

$$\begin{aligned} \Delta_{\mathbf{S}} J^{(p, \mathbf{S})} &= \left(-\frac{2}{(r^{\mathbf{S}})^2} + O\left(\frac{\overset{\circ}{\epsilon}}{(r^{\mathbf{S}})^3}\right) \right) J^{(p, \mathbf{S})}, \\ \int_{\mathbf{S}} J^{(p, \mathbf{S})} J^{(q, \mathbf{S})} da_g &= \frac{4\pi}{3} (r^{\mathbf{S}})^2 \delta_{pq} + O(\overset{\circ}{\epsilon} r^{\mathbf{S}}), \\ \int_{\mathbf{S}} J^{(p, \mathbf{S})} da_g &= 0. \end{aligned} \tag{6.21}$$

Indeed, (6.21) follows from the fact that $K^{\mathbf{S}} - \frac{1}{(r^{\mathbf{S}})^2} \in r^{-1}\Gamma_g$, i.e.

$$\left| K^{\mathbf{S}} - \frac{1}{(r^{\mathbf{S}})^2} \right| \lesssim \frac{\overset{\circ}{\epsilon}}{(r^{\mathbf{S}})^3}$$

and the result of Lemma 3.12 applied to the case when $\epsilon = r^{-1}\overset{\circ}{\epsilon}$.

6.3 Proof of Theorem 6.5

We proceed by an iteration argument in which we will make a sequence of choices for $J^{(p)}$ which will converge in the limit to a canonical basis of $\ell = 1$ modes for the limiting GCM sphere. To this end, we will rely on the following corollary of Theorem 6.1 which investigates the dependance of the GCM spheres of Theorem 6.1 on the particular choice of $J^{(p)}$.

Corollary 6.7. Under the same assumptions than Theorem 6.1, assume in addition to $J^{(p)}$, the existence of a second family of scalar function $\tilde{J}^{(p)}$ satisfying **A4**, and such that

$$\sum_p \left\| J^{(p)} - \tilde{J}^{(p)} \right\|_{\mathfrak{h}_s(\overset{\circ}{\mathbf{S}})} \lesssim r\overset{\circ}{\epsilon} \tag{6.22}$$

for some $2 \leq s \leq s_{max} - 1$. We consider

- $\mathbf{S} = \mathbf{S}(\Lambda, \underline{\Lambda})$ the GCM sphere obtained by applying Theorem 6.1 using $J^{(p)}$, (U, S) the corresponding deformation parameters, $(f, \underline{f}, \lambda)$ the corresponding change of frame coefficients, and $(\underline{C}_0, \underline{C}^{(p)}, M_0, M^{(p)})$ the corresponding GCM constants,
- $\tilde{\mathbf{S}} = \tilde{\mathbf{S}}(\tilde{\Lambda}, \tilde{\underline{\Lambda}})$ the GCM sphere obtained by applying Theorem 6.1 using $\tilde{J}^{(p)}$, (\tilde{U}, \tilde{S}) , the corresponding deformation parameters, $(\tilde{f}, \tilde{\underline{f}}, \tilde{\lambda})$ the corresponding change of frame coefficients, and $(\tilde{\underline{C}}_0, \tilde{\underline{C}}^{(p)}, \tilde{M}_0, \tilde{M}^{(p)})$ the corresponding GCM constants.

Then, the following comparison estimate holds true

$$\begin{aligned}
& r^{-1} \left\| (\tilde{U}, \tilde{S}) - (U, S) \right\|_{\mathfrak{h}_{s+1}(\mathring{S})} + \left\| (\tilde{f}, \tilde{f}, \tilde{\lambda})^{\#_{\tilde{\Psi}}} - (f, \underline{f}, \lambda)^{\#_{\Psi}} \right\|_{\mathfrak{h}_{s+1}(\mathring{S})} + r^2 |\tilde{\underline{C}}_0 - \underline{C}_0| \\
& + r^3 |\tilde{M}_0 - M_0| + \sum_p \left(r^2 |\tilde{\underline{C}}^{(p)} - \underline{C}^{(p)}| + r^3 |\tilde{M}^{(p)} - M^{(p)}| \right) \\
& \lesssim \overset{\circ}{\epsilon} \sum_p r^{-1} \|J^{(p)} - \tilde{J}^{(p)}\|_{\mathfrak{h}_s(\mathring{S})}. \tag{6.23}
\end{aligned}$$

Proof. See Corollary 6.11 in [17]. □

To control the right-hand side of (6.23), we will use the following lemma.

Lemma 6.8. *Assume given two deformations $\Psi_1 : \mathring{S} \rightarrow \mathbf{S}_1$ and $\Psi_2 : \mathring{S} \rightarrow \mathbf{S}_2$ with the corresponding deformation parameters (U_1, S_1) and (U_2, S_2) satisfying*

$$\|(U_1, S_1)\|_{\mathfrak{h}_{s_{max}+1}(\mathring{S})} + \|(U_2, S_2)\|_{\mathfrak{h}_{s_{max}+1}(\mathring{S})} \lesssim \overset{\circ}{\delta}.$$

Let $J^{\mathbf{S}_1}, J^{\mathbf{S}_2}$ be the corresponding canonical basis of $\ell = 1$ modes of \mathbf{S}_1 and \mathbf{S}_2 calibrated according to Definition 6.3. To $J^{\mathbf{S}_1}$ and $J^{\mathbf{S}_2}$, we associate the maps $J^{(p)}[\mathbf{S}(j)] : \mathcal{R} \rightarrow \mathbb{R}$, $j = 1, 2$, as in Definition 5.18. Then, the following estimate holds true, for all $5 \leq s \leq s_{max} + 1$,

$$\max_{p=0,-,+} \|J^{(p)}[\mathbf{S}(1)] - J^{(p)}[\mathbf{S}(2)]\|_{\mathfrak{h}_s(\mathring{S})} \lesssim r^{-1} \|(U_1 - U_2, S_1 - S_2)\|_{\mathfrak{h}_s(\mathring{S})}. \tag{6.24}$$

Proof. Let (Φ_1, ϕ_1) and (Φ_2, ϕ_2) be the unique effective uniformization map of \mathbf{S} calibrated with $(\overset{\circ}{\Phi}, \overset{\circ}{\phi})$ relative respectively to the map Ψ_1 and Ψ_2 , in the sense of Definition 4.9. Recall from (6.16) that

$$J^{(p, \mathbf{S}_1)} = J^{(p, \mathbf{S}^2)} \circ \Phi_1^{-1}, \quad J^{(p, \mathbf{S}_2)} = J^{(p, \mathbf{S}^2)} \circ \Phi_2^{-1}.$$

Next, note that Lemma 5.10 implies

$$\|g^{\mathbf{S}_1, \#1} - g^{\mathbf{S}_2, \#2}\|_{\mathfrak{h}_{s_{max}}(\mathring{S})} \lesssim \|g^{\mathbf{S}_1, \#1} - \overset{\circ}{g}\|_{\mathfrak{h}_{s_{max}}(\mathring{S})} + \|\overset{\circ}{g} - g^{\mathbf{S}_2, \#2}\|_{\mathfrak{h}_{s_{max}}(\mathring{S})} \lesssim r^2 \overset{\circ}{\delta}.$$

Hence, using also (5.28),

$$\begin{aligned}
\|g^{\mathbf{S}_1} - (\Psi_2 \circ \Psi_1^{-1})^{\#} g^{\mathbf{S}_2}\|_{\mathfrak{h}_{s_{max}}(\mathbf{S}_1)} &= \|g^{\mathbf{S}_1, \#1} - g^{\mathbf{S}_2, \#2}\|_{\mathfrak{h}_{s_{max}}(\mathring{S}, g^{\mathbf{S}_1, \#1})} \\
&\lesssim \|g^{\mathbf{S}_1, \#1} - g^{\mathbf{S}_2, \#2}\|_{\mathfrak{h}_{s_{max}}(\mathring{S})} \lesssim r^2 \overset{\circ}{\delta}
\end{aligned}$$

so that (4.26) holds with $\Psi = \Psi_2 \circ \Psi_1^{-1}$ and $\delta = r^{-1}\mathring{\delta}$. We may thus apply Proposition 4.14 which yields, for $5 \leq s \leq s_{max} + 1$,

$$\left\| J^{\mathbf{S}_1} - J^{\mathbf{S}_2} \circ \Psi \right\|_{\mathfrak{h}_s(\mathbf{S}_1)} \lesssim r^{-2} \|g^{\mathbf{S}_1} - (\Psi_2 \circ \Psi_1^{-1})^\# g^{\mathbf{S}_2}\|_{\mathfrak{h}_{s-1}(\mathbf{S}_1)}.$$

Using (5.28) on both sides, we deduce

$$\left\| J^{\mathbf{S}_1} \circ \Psi_1 - J^{\mathbf{S}_2} \circ \Psi_2 \right\|_{\mathfrak{h}_s(\mathring{S})} \lesssim r^{-2} \|g^{\mathbf{S}_1, \#1} - g^{\mathbf{S}_2, \#2}\|_{\mathfrak{h}_{s-1}(\mathring{S})}.$$

Finally, a simple adaptation of the proof of (5.26) implies

$$r^{-2} \|g^{\mathbf{S}_1, \#1} - g^{\mathbf{S}_2, \#2}\|_{\mathfrak{h}_{s-1}(\mathring{S})} \lesssim r^{-1} \|(U_1 - U_2, S_1 - S_2)\|_{\mathfrak{h}_s(\mathring{S})}$$

and hence

$$\left\| J^{\mathbf{S}_1} \circ \Psi_1 - J^{\mathbf{S}_2} \circ \Psi_2 \right\|_{\mathfrak{h}_s(\mathring{S})} \lesssim r^{-1} \|(U_1 - U_2, S_1 - S_2)\|_{\mathfrak{h}_s(\mathring{S})}$$

as desired. \square

Corollary 6.9. *Let $J^{(p)}$ satisfying **A4**, and let $J^{(p, \mathring{S})}$ denotes the canonical basis of $\ell = 1$ modes on \mathring{S} corresponding to the effective uniformization map $(\mathring{\Phi}, \mathring{\phi})$ for \mathring{S} appearing in Definition 6.3. Let a deformation $\Psi : \mathring{S} \rightarrow \mathbf{S}$ with the corresponding deformation parameters (U, S) satisfying (6.9). Let $J^{\mathbf{S}}$ be the corresponding canonical basis of $\ell = 1$ modes of \mathbf{S} calibrated according to Definition 6.3. To $J^{\mathbf{S}}$, we associate the map $J^{(p)}[\mathbf{S}] : \mathcal{R} \rightarrow \mathbb{R}$ as in Definition 5.18. Then, the following estimate holds true*

$$\max_{p=0, -, +} \|J^{(p)} - J^{(p)}[\mathbf{S}]\|_{\mathfrak{h}_{s_{max}}(\mathring{S})} \lesssim \max_{p=0, -, +} \|J^{(p)} - J^{(p, \mathring{S})}\|_{\mathfrak{h}_{s_{max}}(\mathring{S})} + r^{-1} \|(U, S)\|_{\mathfrak{h}_{s_{max}}(\mathring{S})} \quad (6.25)$$

Proof. We have

$$\begin{aligned} & \max_{p=0, -, +} \|J^{(p)} - J^{(p)}[\mathbf{S}]\|_{\mathfrak{h}_{s_{max}}(\mathring{S})} \\ & \lesssim \max_{p=0, -, +} \|J^{(p)} - J^{(p, \mathring{S})}\|_{\mathfrak{h}_{s_{max}}(\mathring{S})} + \max_{p=0, -, +} \|J^{(p, \mathring{S})} - J^{(p)}[\mathbf{S}]\|_{\mathfrak{h}_{s_{max}}(\mathring{S})}. \end{aligned}$$

It thus remains to control the term $J^{(p, \mathring{S})} - J^{(p)}[\mathbf{S}]$. Note that \mathring{S} corresponds to a deformed sphere with trivial deformation map $\mathring{\Psi} = I$, trivial deformation parameters $(\mathring{U}, \mathring{S}) = (0, 0)$. In particular, we have $J^{\mathring{S}} = J[\mathring{S}]$. Thus, together with Lemma 6.8, we infer

$$\begin{aligned} \max_{p=0, -, +} \|J^{(p, \mathring{S})} - J^{(p)}[\mathbf{S}]\|_{\mathfrak{h}_{s_{max}}(\mathring{S})} &= \max_{p=0, -, +} \|J^{(p)}[\mathring{S}] - J^{(p)}[\mathbf{S}]\|_{\mathfrak{h}_{s_{max}}(\mathring{S})} \\ &\lesssim r^{-1} \|(U - \mathring{U}, S - \mathring{S})\|_{\mathfrak{h}_{s_{max}}(\mathring{S})} \end{aligned}$$

and hence, since $(\overset{\circ}{U}, \overset{\circ}{S}) = (0, 0)$, we deduce

$$\max_{p=0,-,+} \|J^{(p)} - J^{(p)}[\mathbf{S}]\|_{\mathfrak{h}_{s_{max}}(\overset{\circ}{S})} \lesssim \max_{p=0,-,+} \|J^{(p)} - J^{(p,\overset{\circ}{S})}\|_{\mathfrak{h}_{s_{max}}(\overset{\circ}{S})} + r^{-1} \|(U, S)\|_{\mathfrak{h}_{s_{max}}(\overset{\circ}{S})}.$$

as desired. \square

We will also rely on the following simple lemma.

Lemma 6.10. *Let $J^{(p)}$ and $\tilde{J}^{(p)}$ two families of scalar functions satisfying **A4**. Let also scalar functions $\underline{C}^{(p)}$, $M^{(p)} = M^{(p)}$, defined on the spacetime region \mathcal{R} , and satisfying*

$$\sum_{p=0,-,+} \sup_{\mathcal{R}} \left(r^2 |\mathfrak{d}^{\leq s_{max}} \underline{C}^{(p)}| + r^3 |\mathfrak{d}^{\leq s_{max}} M^{(p)}| \right) \lesssim \overset{\circ}{\epsilon}.$$

Then, we have, for $s_{max} \geq 2$,

$$\begin{aligned} & \sum_p \left(r^2 \|\underline{C}^{(p)}(\tilde{J}^{(p)} - J^{(p)})\|_{\mathfrak{h}_{s_{max}}(\mathbf{S})} + r^3 \|M^{(p)}(\tilde{J}^{(p)} - J^{(p)})\|_{\mathfrak{h}_{s_{max}}(\mathbf{S})} \right) \\ & \lesssim \overset{\circ}{\epsilon} \max_{p=0,-,+} \|\tilde{J}^{(p)} - J^{(p)}\|_{\mathfrak{h}_{s_{max}}(\overset{\circ}{S})} \end{aligned} \quad (6.26)$$

for any deformed sphere \mathbf{S} with (U, S) satisfying (5.25), where (6.26) is uniform w.r.t. such spheres.

Proof. Using the assumptions for $\underline{C}^{(p)}$ and $M^{(p)}$ and (5.27), we have

$$\sum_p \left(r^2 \|\underline{C}^{(p)}\|_{\mathfrak{h}_{s_{max}}(\mathbf{S})} + r^3 \|M^{(p)}\|_{\mathfrak{h}_{s_{max}}(\mathbf{S})} \right) \lesssim r \overset{\circ}{\epsilon}.$$

Together with Sobolev and the fact that $s_{max} \geq 2$, we deduce

$$\begin{aligned} & \sum_p \left(r^2 \|\underline{C}^{(p)}(\tilde{J}^{(p)} - J^{(p)})\|_{\mathfrak{h}_{s_{max}}(\mathbf{S})} + r^3 \|M^{(p)}(\tilde{J}^{(p)} - J^{(p)})\|_{\mathfrak{h}_{s_{max}}(\mathbf{S})} \right) \\ & \lesssim \overset{\circ}{\epsilon} \max_{p=0,-,+} \|\tilde{J}^{(p)} - J^{(p)}\|_{\mathfrak{h}_{s_{max}}(\mathbf{S})}. \end{aligned}$$

To conclude, it suffices to prove

$$\max_{p=0,-,+} \|\tilde{J}^{(p)} - J^{(p)}\|_{\mathfrak{h}_{s_{max}}(\mathbf{S})} \lesssim \max_{p=0,-,+} \|\tilde{J}^{(p)} - J^{(p)}\|_{\mathfrak{h}_{s_{max}}(\overset{\circ}{S})}.$$

To this end, we rely on (5.28) which yields

$$\max_{p=0,-,+} \|\tilde{J}^{(p)} - J^{(p)}\|_{\mathfrak{h}_{s_{max}}(\mathbf{S})} \lesssim \max_{p=0,-,+} \|(\tilde{J}^{(p)} - J^{(p)})^\# \|_{\mathfrak{h}_{s_{max}}(\overset{\circ}{S})}$$

Also, $J^{(p)}$ and $\tilde{J}^{(p)}$ satisfy (5.15)

$$J^{(p)}(u, s, y^1, y^2) = J^{(p)}(\overset{\circ}{u}, \overset{\circ}{s}, y^1, y^2), \quad \tilde{J}^{(p)}(u, s, y^1, y^2) = \tilde{J}^{(p)}(\overset{\circ}{u}, \overset{\circ}{s}, y^1, y^2),$$

and since $\Psi(\overset{\circ}{u}, \overset{\circ}{s}, y^1, y^2) = (\overset{\circ}{u} + U(y^1, y^2), \overset{\circ}{s} + S(y^1, y^2), y^1, y^2)$, we deduce on $\overset{\circ}{S}$

$$(\tilde{J}^{(p)} - J^{(p)})^\# = \tilde{J}^{(p)} - J^{(p)}$$

and hence

$$\max_{p=0,-,+} \|\tilde{J}^{(p)} - J^{(p)}\|_{\mathfrak{h}_{smax}(\mathbf{S})} \lesssim \max_{p=0,-,+} \|\tilde{J}^{(p)} - J^{(p)}\|_{\mathfrak{h}_{smax}(\overset{\circ}{S})}$$

as desired. □

6.3.1 Iteration procedure

We define iteratively a sequence of maps $\Psi(n) : \overset{\circ}{S} \rightarrow \mathbf{S}(n)$, for fixed $\Lambda, \underline{\Lambda}$, as follows.

1. Step $n = 0$ of the iteration.

- (a) The basis of $\ell = 1$ modes $J^{(p)}$ satisfies **A4** by the assumptions of the theorem, so we may apply Theorem 6.1 which yields the existence of a deformation map $\Psi(1) : \overset{\circ}{S} \rightarrow \mathbf{S}(1)$ such that $\mathbf{S}(1)$ satisfies the conclusions of Theorem 6.1, with $J^{(p)}$ as basis of $\ell = 1$ modes in (6.4) (6.5).
- (b) The corresponding deformation parameters $(U(1), S(1))$ verify (6.9). Hence, we may then associate to $\mathbf{S}(1)$ its canonical basis of $\ell = 1$ modes $J^{\mathbf{S}(1)}$ calibrated as in Definition 6.3. Also, we may associate to $J^{\mathbf{S}(1)}$ the map $J^{(p)}[\mathbf{S}(1)] : \mathcal{R} \rightarrow \mathbb{R}$ as in Definition 5.18.

2. Step $n = 1$ of the iteration.

- (a) The basis of $\ell = 1$ modes $J^{(p)}[\mathbf{S}(1)]$ satisfies **A4** in view of Proposition 5.20. Also, (6.17), Corollary 6.9 and Lemma 6.10 imply that (6.1) (6.15) hold with $J^{(p)}$ being replaced by $J^{(p)}[\mathbf{S}(1)]$. Thus the assumptions of Theorem 6.1 hold, so we may apply Theorem 6.1 which yields the existence of a deformation map $\Psi(2) : \overset{\circ}{S} \rightarrow \mathbf{S}(2)$ such that $\mathbf{S}(2)$ satisfies the conclusions of Theorem 6.1, with $J[\mathbf{S}(1)]$ as basis of $\ell = 1$ modes in (6.4) (6.5).

- (b) The corresponding deformation parameters $(U(2), S(2))$ verify (6.9). Hence, we may then associate to $\mathbf{S}(2)$ its canonical basis of $\ell = 1$ modes $J^{\mathbf{S}(2)}$ calibrated as in Definition 6.3. Also, we may associate to $J^{\mathbf{S}(2)}$ the map $J^{(p)}[\mathbf{S}(1)] : \mathcal{R} \rightarrow \mathbb{R}$ as in Definition 5.18.
3. Let $n \geq 2$. Assume by iteration that the sequence of deformation maps $\Psi(n) : \overset{\circ}{S} \rightarrow \mathbf{S}(n)$ has already been constructed by Theorem 6.1 with $J[\mathbf{S}(n-1)]$ as basis of $\ell = 1$ modes in (6.4) (6.5). Then
- (a) The corresponding deformation parameters $(U(n), S(n))$ verify (6.9). Hence, we may then associate to $\mathbf{S}(n)$ its canonical basis of $\ell = 1$ modes $J^{\mathbf{S}(n)}$ calibrated as in Definition 6.3. Also, we may associate to $J^{\mathbf{S}(n)}$ the map $J^{(p)}[\mathbf{S}(1)] : \mathcal{R} \rightarrow \mathbb{R}$ as in Definition 5.18.
- (b) The basis of $\ell = 1$ modes $J^{(p)}[\mathbf{S}(n)]$ satisfies **A4** in view of Proposition 5.20. Also, (6.17), Corollary 6.9 and Lemma 6.10 imply that (6.1) (6.15) hold with $J^{(p)}$ being replaced by $J^{(p)}[\mathbf{S}(n)]$. Thus the assumptions of Theorem 6.1 hold, so we may apply Theorem 6.1 which yields the existence of a deformation map $\Psi(n+1) : \overset{\circ}{S} \rightarrow \mathbf{S}(n+1)$ such that $\mathbf{S}(n+1)$ satisfies the conclusions of Theorem 6.1, with $J[\mathbf{S}(n+1)]$ as basis of $\ell = 1$ modes in (6.4) (6.5).

6.3.2 Boundedness and contraction

To the deformed sphere $\mathbf{S}(n)$ of the iterative procedure, we associate the following ninetets

$$\mathcal{Q}^{(n)} := \left(U^{(n)}, S^{(n)}, f^{(n)}, \underline{f}^{(n)}, \overset{\circ}{\lambda}^{(n)}; \underline{C}_0^{(n)}, M_0^{(n)}, \underline{C}^{(n),p}, M^{(n),p} \right),$$

where $(U^{(n)}, S^{(n)})$ are the defining functions of the map $\Psi(n)$, $(f^{(n)}, \underline{f}^{(n)}, \overset{\circ}{\lambda}^{(n)})$ the corresponding transition functions, and $\underline{C}_0^{(n)}, M_0^{(n)}, \underline{C}^{(n),p}, M^{(n),p}$ the corresponding constants appearing in Theorem 6.1.

We define the norms,

$$\begin{aligned} \|\mathcal{Q}^{(n)}\|_k &:= r^{-1} \left\| (U^{(n)}, S^{(n)}) \right\|_{\mathfrak{h}_k(\overset{\circ}{S})} + \left\| (f^{(n)}, \underline{f}^{(n)}, \overset{\circ}{\lambda}^{(n)}) \right\|_{\mathfrak{h}_k(\mathbf{S})} \\ &+ r^2 \left(|\underline{C}_0^{(n)}| + \sum_p |\underline{C}^{(n),p}| + r |M_0^{(n)}| + r \sum_p |M^{(n),p}| \right) \end{aligned} \quad (6.27)$$

where,

$$\begin{aligned}\dot{\underline{C}}_0^{(n)} &= \underline{C}_0^{(n)} - \overline{\underline{C}}_0^{\mathbf{S}(n)}, & \dot{\underline{C}}^{(n),p} &= \underline{C}^{(n),p} - \overline{\underline{C}^{(p)}}^{\mathbf{S}(n)}, \\ \dot{M}_0^{(n)} &= M_0^{(n)} - \overline{M}_0^{\mathbf{S}(n)}, & \dot{M}^{(n),p} &= M^{(n),p} - \overline{M^{(p)}}^{\mathbf{S}(n)}.\end{aligned}$$

Recall that $\mathbf{S}(n)$ is obtained by an application of Theorem 6.1. In particular, the ninetets $\mathcal{Q}^{(n)}$ are uniformly bounded in the norm (6.27)

$$\|\mathcal{Q}^{(n)}\|_{s_{max}+1} \lesssim \overset{\circ}{\delta}, \quad (6.28)$$

where (6.28) is an immediate consequence of the estimates (6.6), (6.7) and (6.9) in Theorem 6.1.

To insure convergence we also need to establish a contraction estimate. We cannot compare directly the ninetets $\mathcal{Q}^{(n)}$, so we compare instead the modified ninetets, well defined on $\overset{\circ}{S}$,

$$\mathcal{P}^{(n)} := \left(U^{(n)}, S^{(n)}, \overset{\circ}{\lambda}^{n,\#}, f^{n,\#}, \underline{f}^{n,\#}, \underline{C}_0^{(n)}, M_0^{(n)}, \underline{C}^{(n),p}, M^{(n),p} \right) \quad (6.29)$$

where $\overset{\circ}{\lambda}^{n,\#}, f^{n,\#}, \underline{f}^{n,\#}$ are the pull-backs by $\#_{n-1}$ of the triplet $\overset{\circ}{\lambda}^{(n)}, f^{(n)}, \underline{f}^{(n)}$ defined on the sphere $\mathbf{S}(n-1)$. We also introduce the modified norms

$$\begin{aligned}\|\mathcal{P}^{(n)}\|_{k,\overset{\circ}{S}} &:= r^{-1} \left\| (U^{(n)}, S^{(n)}) \right\|_{\mathfrak{h}_k(\overset{\circ}{S})} + \left\| (f^{n,\#}, \underline{f}^{n,\#}, \overset{\circ}{\lambda}^{n,\#}) \right\|_{\mathfrak{h}_k(\overset{\circ}{S})} \\ &+ r^2 \left(|\dot{\underline{C}}_0^{(n)}| + \sum_p |\dot{\underline{C}}^{(n),p}| + r |\dot{M}_0^{(n)}| + r \sum_p |\dot{M}^{(n),p}| \right).\end{aligned} \quad (6.30)$$

In view of the Sobolev norm comparison of (5.28), we deduce from (6.27)

$$\|\mathcal{P}^{(n)}\|_{s_{max},\overset{\circ}{S}} \lesssim \overset{\circ}{\delta}. \quad (6.31)$$

Contraction in this modified norms is established in the following.

Proposition 6.11. *The following estimate holds true for $n \geq 2$*

$$\|\mathcal{P}^{n+1} - \mathcal{P}^n\|_{6,\overset{\circ}{S}} \lesssim r^{-1} \overset{\circ}{\epsilon} \|\mathcal{P}^n - \mathcal{P}^{n-1}\|_{5,\overset{\circ}{S}}. \quad (6.32)$$

Proof. In view of the definition of $\mathbf{S}(n)$ and $\mathbf{S}(n+1)$, Corollary 6.7 with $s = 6$ yields immediately, for $n \geq 2$,

$$\|\mathcal{P}^{n+1} - \mathcal{P}^n\|_{6,\overset{\circ}{S}} \lesssim \overset{\circ}{\epsilon} \sum_p r^{-1} \|J^{(p)}[\mathbf{S}(n)] - J^{(p)}[\mathbf{S}(n-1)]\|_{\mathfrak{h}_5(\overset{\circ}{S})}.$$

Using Lemma 6.8 to estimate the right-hand side, we infer for $n \geq 2$

$$\|\mathcal{P}^{n+1} - \mathcal{P}^n\|_{6, \mathring{S}} \lesssim r^{-2\circ} \epsilon \|(U^{(n)}, S^{(n)}) - (U^{(n-1)}, S^{(n-1)})\|_{\mathfrak{b}_5(\mathring{S})},$$

and hence

$$\|\mathcal{P}^{n+1} - \mathcal{P}^n\|_{6, \mathring{S}} \lesssim r^{-1\circ} \epsilon \|\mathcal{P}^n - \mathcal{P}^{n-1}\|_{5, \mathring{S}}$$

as desired. \square

6.3.3 Convergence

In view of (6.28), (6.31), and Proposition 6.11, it follows that

- the sequence $\Psi(n)$ converges to a deformation map $\Psi : \mathring{S} \rightarrow \mathbf{S}$, $\mathbf{S}(n) \rightarrow \mathbf{S}$,
- $J^{\mathbf{S}(n)} \rightarrow J^{\mathbf{S}}$ and $J[\mathbf{S}(n)] \rightarrow J[\mathbf{S}]$, where $J^{\mathbf{S}}$ is the canonical basis of $\ell = 1$ modes on \mathbf{S} calibrated as in Definition 6.3, and $J^{(p)}[\mathbf{S}]$ is associated to $J^{\mathbf{S}}$ as in Definition 5.18.
- $(\underline{C}_0^{(n)}, M_0^{(n)}, \underline{C}^{(n),p}, M^{(n),p})$ converges to constants $(\underline{C}_0^{\mathbf{S}}, \underline{C}^{(\mathbf{S},p)}, M_0^{\mathbf{S}}, M^{(\mathbf{S},p)})$.

Moreover, \mathbf{S} satisfies the conclusions of Theorem 6.1, with $J[\mathbf{S}]$ as basis of $\ell = 1$ modes in (6.4) (6.5). Now, recall from (5.36) that we have $J[\mathbf{S}] = J^{\mathbf{S}}$ on \mathbf{S} . Thus, \mathbf{S} satisfies the conclusions of Theorem 6.1, with the canonical basis $J^{\mathbf{S}}$ of $\ell = 1$ modes in (6.4) (6.5). In particular, we have

$$\begin{aligned} \kappa^{\mathbf{S}} &= \frac{2}{r^{\mathbf{S}}}, \\ \underline{\kappa}^{\mathbf{S}} &= -\frac{2}{r^{\mathbf{S}}} \Upsilon^{\mathbf{S}} + \underline{C}_0^{\mathbf{S}} + \sum_p \underline{C}^{(\mathbf{S},p)} J^{(p,\mathbf{S})}, \\ \mu^{\mathbf{S}} &= \frac{2m^{\mathbf{S}}}{(r^{\mathbf{S}})^3} + M_0^{\mathbf{S}} + \sum_p M^{(\mathbf{S},p)} J^{(p,\mathbf{S})}. \end{aligned} \tag{6.33}$$

To infer (6.19) from (6.33), it remains to prove that $\underline{C}_0^{\mathbf{S}} = M_0^{\mathbf{S}} = 0$. We start with $M^{\mathbf{S}}$. Using the Gauss equations (5.2), the definition (5.3) of the mass aspect function, the

definition (5.1) of the Hawking mass, we infer

$$\begin{aligned}
4\pi &= \int_{\mathbf{S}} K^{\mathbf{S}} = \int_{\mathbf{S}} \left(-\rho^{\mathbf{S}} - \frac{1}{4} \kappa^{\mathbf{S}} \underline{\kappa}^{\mathbf{S}} + \frac{1}{2} \widehat{\chi}^{\mathbf{S}} \cdot \widehat{\chi}^{\mathbf{S}} \right) \\
&= \int_{\mathbf{S}} \left(\mu^{\mathbf{S}} + \operatorname{div}^{\mathbf{S}} \zeta^{\mathbf{S}} - \frac{1}{4} \kappa^{\mathbf{S}} \underline{\kappa}^{\mathbf{S}} \right) = \int_{\mathbf{S}} \mu^{\mathbf{S}} - \frac{1}{4} \int_{\mathbf{S}} \kappa^{\mathbf{S}} \underline{\kappa}^{\mathbf{S}} \\
&= \int_{\mathbf{S}} \mu^{\mathbf{S}} + 4\pi - \frac{8\pi m^{\mathbf{S}}}{r^{\mathbf{S}}}
\end{aligned}$$

where we have integrated $\operatorname{div}^{\mathbf{S}} \zeta^{\mathbf{S}}$ by parts. Plugging the identity for $\mu^{\mathbf{S}}$ in (6.33), and using the fact that $\int_{\mathbf{S}} J^{(p,\mathbf{S})} = 0$, we deduce

$$\begin{aligned}
4\pi &= \int_{\mathbf{S}} \mu^{\mathbf{S}} + 4\pi - \frac{8\pi m^{\mathbf{S}}}{r^{\mathbf{S}}} = \int_{\mathbf{S}} \left(\frac{2m^{\mathbf{S}}}{(r^{\mathbf{S}})^3} + M_0^{\mathbf{S}} + \sum_p M^{(\mathbf{S},p)} J^{(p,\mathbf{S})} \right) + 4\pi - \frac{8\pi m^{\mathbf{S}}}{r^{\mathbf{S}}} \\
&= \frac{8\pi m^{\mathbf{S}}}{r^{\mathbf{S}}} + 4\pi (r^{\mathbf{S}})^2 M_0^{\mathbf{S}} + 4\pi - \frac{8\pi m^{\mathbf{S}}}{r^{\mathbf{S}}}
\end{aligned}$$

and hence

$$M_0^{\mathbf{S}} = 0.$$

Therefore,

$$\mu^{\mathbf{S}} = \frac{2m^{\mathbf{S}}}{(r^{\mathbf{S}})^3} + \sum_p M^{(\mathbf{S},p)} J^{(p,\mathbf{S})}$$

as desired.

Next, using the identities for $\kappa^{\mathbf{S}}$ and $\underline{\kappa}^{\mathbf{S}}$ in (6.33), the definition (5.1) of the Hawking mass, and using the fact that $\int_{\mathbf{S}} J^{(p,\mathbf{S})} = 0$, we infer

$$\begin{aligned}
\Upsilon^{\mathbf{S}} &= 1 - \frac{2m^{\mathbf{S}}}{r^{\mathbf{S}}} = -\frac{1}{16\pi} \int_{\mathbf{S}} \kappa^{\mathbf{S}} \underline{\kappa}^{\mathbf{S}} = -\frac{1}{8\pi r^{\mathbf{S}}} \int_{\mathbf{S}} \left(-\frac{2}{r^{\mathbf{S}}} \Upsilon^{\mathbf{S}} + \underline{C}_0^{\mathbf{S}} + \sum_p \underline{C}^{(\mathbf{S},p)} J^{(p,\mathbf{S})} \right) \\
&= \Upsilon^{\mathbf{S}} - \frac{r^{\mathbf{S}}}{2} \underline{C}_0^{\mathbf{S}}
\end{aligned}$$

and hence

$$\underline{C}_0^{\mathbf{S}} = 0.$$

Therefore, we deduce

$$\begin{aligned}\kappa^{\mathbf{S}} &= \frac{2}{r^{\mathbf{S}}}, \\ \underline{\kappa}^{\mathbf{S}} &= -\frac{2}{r^{\mathbf{S}}}\Upsilon^{\mathbf{S}} + \sum_p \underline{C}^{(\mathbf{S},p)} J^{(p,\mathbf{S})}, \\ \mu^{\mathbf{S}} &= \frac{2m^{\mathbf{S}}}{(r^{\mathbf{S}})^3} + \sum_p M^{(\mathbf{S},p)} J^{(p,\mathbf{S})}.\end{aligned}$$

Thus, \mathbf{S} satisfies the conclusions of Theorem 6.1, with the canonical basis $J^{\mathbf{S}}$ of $\ell = 1$ modes in (6.19) (6.20). This concludes the proof of Theorem 6.5.

7 Construction of intrinsic GCM spheres

In what follows we prove an important corollary of Theorem 6.5 which makes use of the arbitrariness of $\Lambda, \underline{\Lambda}$ to ensure the vanishing of the $\ell = 1$ modes of β and $\widetilde{\text{tr}}\chi$. The result requires stronger assumptions than those made in **A1**, see (5.11), namely we assume that Γ_b has the same behavior as Γ_g , i.e.

A1-Strong. For $k \leq s_{max}$, and for a small enough constant $\delta_1 > 0$, with $\delta_1 \geq \mathring{\epsilon}$,

$$\left\| (\Gamma_g, \Gamma_b) \right\|_{k,\infty} \lesssim \delta_1 r^{-2}, \quad \left\| \nabla_3 \Gamma_g \right\|_{k,\infty} \lesssim \delta_1 r^{-3}. \quad (7.1)$$

Remark 7.1. *In applications to our envisioned proof of the nonlinear stability of Kerr, we plan to apply the result below in spacetime regions where we have $r \sim u$. Such regions appear naturally when we extend a given GCM admissible spacetime \mathcal{M} , see Theorem M7 in [16], in a neighborhood of its top GCM sphere. The improvement in r for Γ_b is a consequence of the fact that we can exchange, in such regions, the decay factors in u for decay in r , while the improvement for $\nabla_3 \Gamma_g$ is due to the fact that such terms behave like $r^{-1} \Gamma_b$ as a consequence of the structure equations.*

We also need to make stronger assumptions concerning the basis of $\ell = 1$ modes for the background foliation. These assumptions are also consistent with the what we expect in applications to the nonlinear stability of Kerr.

A4-Strong. We assume the existence of a smooth family of scalar functions $J^{(p)} : \mathcal{R} \rightarrow \mathbb{R}$, for $p = 0, +, -$, verifying the following properties

1. On the sphere \mathring{S} of the background foliation, there holds the following stronger version of (5.14)

$$\begin{aligned} \left((\mathring{r})^2 \mathring{\Delta} + 2 \right) J^{(p)} &= O(\mathring{\epsilon} r^{-1}), \quad p = 0, +, -, \\ \frac{1}{|\mathring{S}|} \int_{\mathring{S}} J^{(p)} J^{(q)} &= \frac{1}{3} \delta_{pq} + O(\mathring{\epsilon} r^{-1}), \quad p, q = 0, +, -, \\ \frac{1}{|\mathring{S}|} \int_{\mathring{S}} J^{(p)} &= O(\mathring{\epsilon} r^{-1}), \quad p = 0, +, -. \end{aligned} \quad (7.2)$$

2. There holds also on \mathring{S} the following stronger version of (6.17)

$$\max_{p=0,-,+} \|J^{(p)} - J^{(p,\mathring{S})}\|_{\mathfrak{h}_{s_{max}+1}(\mathring{S})} \lesssim \mathring{\delta}, \quad (7.3)$$

where $J^{(p,\mathring{S})}$ denotes the canonical basis of $\ell = 1$ modes on \mathring{S} corresponding to the effective uniformization map $(\mathring{\Phi}, \mathring{\phi})$ for \mathring{S} appearing in Definition 6.3.

3. We extend $J^{(p)}$ from \mathring{S} to \mathcal{R} as in (5.15), i.e. by $\partial_s J^{(p)} = \partial_u J^{(p)} = 0$.

We will rely on the following corollary of Corollary 6.9.

Corollary 7.2. *Let $J^{(p)}$ satisfying **A4-Strong**, and let $J^{(p,\mathring{S})}$ denotes the canonical basis of $\ell = 1$ modes on \mathring{S} corresponding to the effective uniformization map $(\mathring{\Phi}, \mathring{\phi})$ for \mathring{S} appearing in Definition 6.3. Let a deformation $\Psi : \mathring{S} \rightarrow \mathbf{S}$ with the corresponding deformation parameters (U, S) satisfying (6.9) with $s_{max} \geq 2$. Let $J^{\mathbf{S}}$ be the corresponding canonical basis of $\ell = 1$ modes of \mathbf{S} calibrated according to Definition 6.3. Then, the following estimate holds true*

$$\max_{p=0,-,+} \sup_{\mathbf{S}} |J^{(p)} - J^{(p,\mathbf{S})}| \lesssim r^{-1} \mathring{\delta}. \quad (7.4)$$

Proof. From Corollary 6.9, we have

$$\max_{p=0,-,+} \|J^{(p)} - J^{(p)}[\mathbf{S}]\|_{\mathfrak{h}_{s_{max}}(\mathring{S})} \lesssim \max_{p=0,-,+} \|J^{(p)} - J^{(p,\mathring{S})}\|_{\mathfrak{h}_{s_{max}}(\mathring{S})} + r^{-1} \|(U, S)\|_{\mathfrak{h}_{s_{max}}(\mathring{S})},$$

where $J^{(p)}[\mathbf{S}] : \mathcal{R} \rightarrow \mathbb{R}$ is associated to $J^{\mathbf{S}}$ as in Definition 5.18. Together with (7.3) and (6.9), we obtain

$$\max_{p=0,-,+} \|J^{(p)} - J^{(p)}[\mathbf{S}]\|_{\mathfrak{h}_{s_{max}}(\mathring{S})} \lesssim \mathring{\delta}.$$

Since $s_{max} \geq 2$, we infer from the Sobolev embedding

$$\max_{p=0,-,+} \sup_{\overset{\circ}{S}} |J^{(p)} - J^{(p)}[\mathbf{S}]| \lesssim r^{-1} \overset{\circ}{\delta}.$$

Finally, since, by definition, we have $J^{(p)}[\mathbf{S}] = J^{(p,\mathbf{S})} \circ \Psi$ on $\overset{\circ}{S}$, and since $J^{(p)} \circ \Psi = J^{(p)}$ in view of (5.15), we infer

$$\max_{p=0,-,+} \sup_{\mathbf{S}} |J^{(p)} - J^{(p,\mathbf{S})}| \lesssim r^{-1} \overset{\circ}{\delta}$$

as desired. □

We are ready to state and prove our theorem on the construction of intrinsic GCM spheres.

Theorem 7.3 (Existence of intrinsic GCM spheres). *Assume that the spacetime region \mathcal{R} verifies the assumptions **A1-Strong**, **A2**, **A3**, **A4-Strong**. We further assume that, relative to the $\ell = 1$ modes of the background foliation,*

$$(\operatorname{div} \beta)_{\ell=1} = O(\overset{\circ}{\delta} r^{-3}), \quad (\widetilde{\operatorname{tr}} \chi)_{\ell=1} = O(\overset{\circ}{\delta} r^{-1}), \quad (\widetilde{\operatorname{tr}} \underline{\chi})_{\ell=1} = O(\overset{\circ}{\delta} r^{-1}). \quad (7.5)$$

Then, there exist unique constants $M^{(\mathbf{S},p)}$, $p \in \{-, 0, +\}$, such that

$$\begin{aligned} \kappa^{\mathbf{S}} &= \frac{2}{r^{\mathbf{S}}}, \\ \underline{\kappa}^{\mathbf{S}} &= -\frac{2}{r^{\mathbf{S}}} \Upsilon^{\mathbf{S}}, \\ \mu^{\mathbf{S}} &= \frac{2m^{\mathbf{S}}}{(r^{\mathbf{S}})^3} + \sum_p M^{(\mathbf{S},p)} J^{(p,\mathbf{S})}, \end{aligned} \quad (7.6)$$

and

$$\int_{\mathbf{S}} \operatorname{div}^{\mathbf{S}} \beta^{\mathbf{S}} J^{(p,\mathbf{S})} = 0, \quad (7.7)$$

where $J^{(p,\mathbf{S})}$ is a canonical $\ell = 1$ basis for \mathbf{S} calibrated, relative by Ψ , with the canonical $\ell = 1$ basis of $\overset{\circ}{S}$. Moreover the deformation verifies the properties (6.8), (6.9), (6.10), (6.11) stated in Theorem 6.1.

Remark 7.4. *The assumptions (7.5) for $(\operatorname{div} \beta)_{\ell=1}$ and $(\widetilde{\operatorname{tr}} \chi)_{\ell=1}$ holds true in general under the assumptions **A1-A3**. The corresponding assumption for $(\widetilde{\operatorname{tr}} \underline{\chi})_{\ell=1}$ holds true in regions, discussed in Remark 7.1, where $r \sim u$, see the proof of Theorem M4 in [16].*

The proof of Theorem 7.3 relies on the following lemma.

Lemma 7.5. *Assume that the spacetime region \mathcal{R} verifies the assumptions **A1-Strong**, **A2**, **A3**, **A4-Strong**. Let $\mathbf{S} = \mathbf{S}(\Lambda, \underline{\Lambda})$ be a deformation of $\overset{\circ}{S}$ as constructed by Theorem 6.5 with*

$$\int_{\mathbf{S}} \operatorname{div}^{\mathbf{S}} f J^{(p, \mathbf{S})} = \Lambda, \quad \int_{\mathbf{S}} \operatorname{div}^{\mathbf{S}} \underline{f} J^{(p, \mathbf{S})} = \underline{\Lambda}.$$

The following identities hold true.

$$\begin{aligned} \Lambda &= \frac{(r^{\mathbf{S}})^3}{3m^{\mathbf{S}}} \left[-(\operatorname{div}^{\mathbf{S}} \beta^{\mathbf{S}})_{\ell=1} + (\operatorname{div} \beta)_{\ell=1} \right] + F_1(\Lambda, \underline{\Lambda}), \\ \underline{\Lambda} &= \Upsilon^{\mathbf{S}} \Lambda + \frac{r^{\mathbf{S}}}{3m^{\mathbf{S}}} \left[(\check{\kappa}^{\mathbf{S}})_{\ell=1} + \Upsilon^{\mathbf{S}}(\check{\kappa})_{\ell=1} - (\check{\kappa})_{\ell=1} \right] + F_2(\Lambda, \underline{\Lambda}), \end{aligned} \tag{7.8}$$

where F_1, F_2 are continuously differentiable functions of $\Lambda, \underline{\Lambda}$ verifying³⁴

$$|F_1, F_2| \lesssim \delta_1 \overset{\circ}{\delta}, \quad |\partial_{\Lambda}(F_1, F_2), \partial_{\underline{\Lambda}}(F_1, F_2)| \lesssim \delta_1. \tag{7.9}$$

We postpone the proof of Lemma 7.5 to section 7.1.

Proof of Theorem 7.3. We note that under the assumptions of the theorem the system

$$\begin{aligned} \Lambda &= \frac{(r^{\mathbf{S}})^3}{3m^{\mathbf{S}}} (\operatorname{div} \beta)_{\ell=1} + F_1(\Lambda, \underline{\Lambda}), \\ \underline{\Lambda} &= \Upsilon^{\mathbf{S}} \Lambda + \frac{r^{\mathbf{S}}}{3m^{\mathbf{S}}} \left(\Upsilon^{\mathbf{S}}(\check{\kappa})_{\ell=1} - (\check{\kappa})_{\ell=1} \right) + F_2(\Lambda, \underline{\Lambda}), \end{aligned}$$

has a unique solution $\Lambda_0, \underline{\Lambda}_0$ verifying the estimate

$$|\Lambda_0| + |\underline{\Lambda}_0| \lesssim \overset{\circ}{\delta}.$$

Therefore, taking $\Lambda = \Lambda_0, \underline{\Lambda} = \underline{\Lambda}_0$ in (7.8), we deduce

$$(\operatorname{div}^{\mathbf{S}} \beta^{\mathbf{S}})_{\ell=1} = 0, \quad (\check{\kappa}^{\mathbf{S}})_{\ell=1} = 0.$$

³⁴Recall that δ_1 is the smallness constant introduced in **A1-Strong**, see (7.1).

It remains to check (7.6). According to Theorem 6.5, there exist constants $\underline{C}^{(\mathbf{S},p)}$, $M^{(\mathbf{S},p)}$, $p \in \{-, 0, +\}$ such that

$$\begin{aligned}\kappa^{\mathbf{S}} &= \frac{2}{r^{\mathbf{S}}}, \\ \underline{\kappa}^{\mathbf{S}} &= -\frac{2}{r^{\mathbf{S}}}\Upsilon^{\mathbf{S}} + \sum_p \underline{C}^{(\mathbf{S},p)} J^{(p,\mathbf{S})}, \\ \mu^{\mathbf{S}} &= \frac{2m^{\mathbf{S}}}{(r^{\mathbf{S}})^3} + \sum_p M^{(\mathbf{S},p)} J^{(p,\mathbf{S})}.\end{aligned}$$

Thus, it suffices to prove that $\underline{C}^{(\mathbf{S},p)} = 0$ for $p \in \{-, 0, +\}$. To this end, we rewrite the identity for $\underline{\kappa}^{\mathbf{S}}$ as follows

$$\underline{\kappa}^{\mathbf{S}} = \sum_p \underline{C}^{(\mathbf{S},p)} J^{(p,\mathbf{S})}.$$

Projecting on the canonical $\ell = 1$ basis and using the condition $(\underline{\kappa}^{\mathbf{S}})_{\ell=1} = 0$ and the property $\int_S J^{(p,\mathbf{S})} J^{(q,\mathbf{S})} da_g = \frac{4\pi}{3} (r^{\mathbf{S}})^2 \delta_{pq} + O(\epsilon r^{\mathbf{S}})$ in (6.21), we then conclude that $\underline{C}^{(\mathbf{S},p)} = 0$. Hence, we have finally obtained

$$\begin{aligned}\kappa^{\mathbf{S}} &= \frac{2}{r^{\mathbf{S}}}, \\ \underline{\kappa}^{\mathbf{S}} &= -\frac{2}{r^{\mathbf{S}}}\Upsilon^{\mathbf{S}}, \\ \mu^{\mathbf{S}} &= \frac{2m^{\mathbf{S}}}{(r^{\mathbf{S}})^3} + \sum_p M^{(\mathbf{S},p)} J^{(p,\mathbf{S})},\end{aligned}$$

and

$$\int_{\mathbf{S}} \operatorname{div}^{\mathbf{S}} \beta^{\mathbf{S}} J^{(p,\mathbf{S})} = 0$$

as desired. □

7.1 Proof of Lemma 7.5

Let $(f, \underline{f}, \lambda)$ denote the transition coefficients between the background frame of \mathcal{R} and the frame $(e_3^{\mathbf{S}}, e_4^{\mathbf{S}}, e_1^{\mathbf{S}}, e_2^{\mathbf{S}})$ adapted to the deformation \mathbf{S} .

Step 1. Consider a GCM sphere $\mathbf{S}(\Lambda, \underline{\Lambda})$ as in Theorem 6.5. Since ${}^* \rho \in r^{-1} \Gamma_g$ we rewrite the transformation (5.22) for β in the form

$$\beta^{\mathbf{S}} = \beta + \frac{3}{2} f \rho + r^{-1} \Gamma_g \cdot F.$$

Remark 7.6. Note that all quantities defined on \mathbf{S} , such as $r^{\mathbf{S}}$, $m^{\mathbf{S}}$, $(f, \underline{f}, \overset{\circ}{\lambda})$ and $J^{(p, \mathbf{S})}$ depend in fact continuously on the parameters $\Lambda, \underline{\Lambda}$ even though this dependence is not made explicit.

Taking the divergence, we infer

$$\begin{aligned} \operatorname{div}^{\mathbf{S}} \beta^{\mathbf{S}} &= \operatorname{div}^{\mathbf{S}} \beta + \frac{3}{2} \rho \operatorname{div}^{\mathbf{S}} f + \frac{3}{2} f \operatorname{div}^{\mathbf{S}} \rho + \operatorname{div}^{\mathbf{S}}(r^{-1} \Gamma_g \cdot F) \\ &= \operatorname{div}^{\mathbf{S}} \beta + \frac{3}{2} \rho \operatorname{div}^{\mathbf{S}} f + r^{-2} (\partial \Gamma_g) \cdot F + \operatorname{div}^{\mathbf{S}}(r^{-1} \Gamma_g \cdot F). \end{aligned}$$

Since $|m^{\mathbf{S}} - m| \lesssim \overset{\circ}{\delta}$ and $|r^{\mathbf{S}} - r| \lesssim \overset{\circ}{\delta}$ in view of Corollary 5.15, since $\check{\rho} \in r^{-1} \Gamma_g$, and in view of the assumptions for Γ_g , we have

$$\begin{aligned} \rho &= -\frac{2m}{r^3} + \check{\rho} = -\frac{2m^{\mathbf{S}}}{(r^{\mathbf{S}})^3} + \check{\rho} + \left(\frac{2m^{\mathbf{S}}}{(r^{\mathbf{S}})^3} - \frac{2m}{r^3} \right) \\ &= -\frac{2m^{\mathbf{S}}}{(r^{\mathbf{S}})^3} + r^{-1} \Gamma_g + O(r^{-3} \overset{\circ}{\delta}) = -\frac{2m^{\mathbf{S}}}{(r^{\mathbf{S}})^3} + O(r^{-3} \overset{\circ}{\epsilon}), \end{aligned}$$

where we used also $\overset{\circ}{\delta} \leq \overset{\circ}{\epsilon}$. We deduce

$$\begin{aligned} \operatorname{div}^{\mathbf{S}} \beta^{\mathbf{S}} + \frac{3m^{\mathbf{S}}}{(r^{\mathbf{S}})^3} \operatorname{div}^{\mathbf{S}} f &= \operatorname{div}^{\mathbf{S}} \beta + r^{-2} (\partial \Gamma_g) \cdot F + O(r^{-3} \overset{\circ}{\epsilon}) F + \operatorname{div}^{\mathbf{S}}(r^{-1} \Gamma_g \cdot F) \\ &= \operatorname{div} \beta + r^{-2} (\partial \Gamma_g) \cdot F + O(r^{-3} \overset{\circ}{\epsilon}) F + \operatorname{div}^{\mathbf{S}}(r^{-1} \Gamma_g \cdot F) \\ &= \operatorname{div} \beta + E_1(\Lambda, \underline{\Lambda}), \end{aligned}$$

with error term of the form

$$E_1(\Lambda, \underline{\Lambda}) := r^{-2} (\partial \Gamma_g) \cdot F + \operatorname{div}^{\mathbf{S}}(r^{-1} \Gamma_g \cdot F) + O(r^{-3} \overset{\circ}{\epsilon}) F. \quad (7.10)$$

Integrating on \mathbf{S} , we deduce,

$$\frac{3m^{\mathbf{S}}}{(r^{\mathbf{S}})^3} \int_{\mathbf{S}} (\operatorname{div}^{\mathbf{S}} f) J^{(p, \mathbf{S})} = - \int_{\mathbf{S}} (\operatorname{div}^{\mathbf{S}} \beta^{\mathbf{S}}) J^{(p, \mathbf{S})} + \int_{\mathbf{S}} (\operatorname{div} \beta) J^{(p, \mathbf{S})} + \int_{\mathbf{S}} E_1 J^{(p, \mathbf{S})}. \quad (7.11)$$

In view of (7.4), we have $|J^{(p, \mathbf{S})} - J^{(p)}| \lesssim \overset{\circ}{\delta} r^{-1}$. Therefore, since $\beta \in r^{-1} \Gamma_g$,

$$\begin{aligned} \int_{\mathbf{S}} (\operatorname{div} \beta) J^{(p, \mathbf{S})} &= \int_{\mathbf{S}} (\operatorname{div} \beta) J^{(p)} + \int_{\mathbf{S}} (\operatorname{div} \beta) (J^{(p, \mathbf{S})} - J^{(p)}) \\ &= \int_{\mathbf{S}} (\operatorname{div} \beta) J^{(p)} + O(r^{-3} \overset{\circ}{\epsilon} \overset{\circ}{\delta}) \\ &= \int_{\overset{\circ}{\mathbf{S}}} (\operatorname{div} \beta) J^{(p)} + \left(\int_{\mathbf{S}} (\operatorname{div} \beta) J^{(p)} - \int_{\overset{\circ}{\mathbf{S}}} (\operatorname{div} \beta) J^{(p)} \right) + O(r^{-3} \overset{\circ}{\epsilon} \overset{\circ}{\delta}). \end{aligned}$$

To estimate the term $\int_{\mathbf{S}}(\operatorname{div} \beta)J^{(p)} - \int_{\mathring{\mathbf{S}}}(\operatorname{div} \beta)J^{(p)}$ we appeal to Lemma 5.11 applied to $F = (\operatorname{div} \beta)J^{(p)}$. Thus

$$\left| \int_{\mathbf{S}} F - \int_{\mathring{\mathbf{S}}} F \right| \lesssim \mathring{\delta} r \left(\sup_{\mathcal{R}} |F| + r \sup_{\mathcal{R}} (|\partial_u F| + |\partial_s F|) \right).$$

Since $\partial_u J^{(p)} = \partial_s J^{(p)} = 0$ and $J^{(p)}$ is bounded, we infer

$$\left| \int_{\mathbf{S}} F - \int_{\mathring{\mathbf{S}}} F \right| \lesssim \mathring{\delta} r \left(\sup_{\mathcal{R}} |\operatorname{div} \beta| + r \sup_{\mathcal{R}} (|\partial_u(\operatorname{div} \beta)| + |\partial_s(\operatorname{div} \beta)|) \right).$$

Together with (5.7) and **A3**, we deduce

$$\left| \int_{\mathbf{S}} F - \int_{\mathring{\mathbf{S}}} F \right| \lesssim \mathring{\delta} r \left(\sup_{\mathcal{R}} |\operatorname{div} \beta| + r \sup_{\mathcal{R}} (|e_3(\operatorname{div} \beta)| + |e_4(\operatorname{div} \beta)| + |\nabla \operatorname{div} \beta|) \right).$$

In view of the control of β provided by **A1**, we infer that

$$\left| \int_{\mathbf{S}} F - \int_{\mathring{\mathbf{S}}} F \right| \lesssim \mathring{\epsilon} \mathring{\delta} r^{-3}.$$

We deduce

$$\int_{\mathbf{S}} (\operatorname{div} \beta) J^{(p, \mathbf{S})} = \int_{\mathring{\mathbf{S}}} (\operatorname{div} \beta) J^{(p)} + O(\mathring{\epsilon} \mathring{\delta} r^{-3}) = (\operatorname{div} \beta)_{\ell=1} + O(\mathring{\epsilon} \mathring{\delta} r^{-3}).$$

Since $\int_{\mathbf{S}} (\operatorname{div}^{\mathbf{S}} f) J^{(p, \mathbf{S})} = \Lambda$, after gathering all error terms in $F_1 = F_1(\Lambda, \underline{\Lambda})$, we deduce from 7.11

$$\Lambda = \frac{(r^{\mathbf{S}})^3}{3m^{\mathbf{S}}} \left[-(\operatorname{div}^{\mathbf{S}} \beta^{\mathbf{S}})_{\ell=1} + (\operatorname{div} \beta)_{\ell=1} \right] + F_1(\Lambda, \underline{\Lambda}) \quad (7.12)$$

where

$$F_1(\Lambda, \underline{\Lambda}) = \frac{(r^{\mathbf{S}})^3}{3m^{\mathbf{S}}} \int_{\mathbf{S}(\Lambda, \underline{\Lambda})} E_1(\Lambda, \underline{\Lambda}) J^{(p, \mathbf{S})} + \frac{(r^{\mathbf{S}})^3}{3m^{\mathbf{S}}} \left(\int_{\mathbf{S}} (\operatorname{div} \beta) J^{(p, \mathbf{S})} - (\operatorname{div} \beta)_{\ell=1} \right).$$

In view of the above, we easily check that

$$\left| F_1(\Lambda, \underline{\Lambda}) \right| \lesssim \mathring{\epsilon} \mathring{\delta}$$

as stated.

Step 2. We next consider the equations (5.21)

$$\begin{aligned} \kappa^{\mathbf{S}} &= \kappa + \kappa \mathring{\lambda} + \operatorname{div}^{\mathbf{S}} f + F \cdot \Gamma_b + F \cdot \nabla^{\mathbf{S}} F + r^{-1} F^2, \\ \underline{\kappa}^{\mathbf{S}} &= \underline{\kappa} - \underline{\kappa} \mathring{\lambda} + \operatorname{div}^{\mathbf{S}} \underline{f} + F \cdot \Gamma_b + F \cdot \nabla^{\mathbf{S}} F + r^{-1} F^2. \end{aligned}$$

Since $|F| \lesssim r^{-1}\overset{\circ}{\delta}$, and since we have $|\Gamma_b| \lesssim r^{-2}\delta_1$ in view of **A1-strong**, and using the GCM condition $\kappa^{\mathbf{S}} = 2/r^{\mathbf{S}}$, we deduce, using also $\overset{\circ}{\delta} \leq \overset{\circ}{\epsilon} \leq \delta_1$,

$$\begin{aligned}\operatorname{div}^{\mathbf{S}} f + \kappa \overset{\circ}{\lambda} &= \frac{2}{r^{\mathbf{S}}} - \kappa + O(\delta_1 \overset{\circ}{\delta} r^{-3}), \\ \operatorname{div}^{\mathbf{S}} \underline{f} - \underline{\kappa} \overset{\circ}{\lambda} &= \underline{\kappa}^{\mathbf{S}} - \underline{\kappa} + O(\delta_1 \overset{\circ}{\delta} r^{-3}).\end{aligned}$$

Differentiating w.r.t. $\Delta^{\mathbf{S}}$, and using **A1-strong** for κ and $\underline{\kappa}$, we infer

$$\begin{aligned}\Delta^{\mathbf{S}} \operatorname{div}^{\mathbf{S}} f + \kappa \Delta^{\mathbf{S}} \overset{\circ}{\lambda} &= -\Delta^{\mathbf{S}} \kappa + O(\delta_1 \overset{\circ}{\delta} r^{-5}), \\ \Delta^{\mathbf{S}} \operatorname{div}^{\mathbf{S}} \underline{f} - \underline{\kappa} \Delta^{\mathbf{S}} \overset{\circ}{\lambda} &= \Delta^{\mathbf{S}} \underline{\kappa}^{\mathbf{S}} - \Delta^{\mathbf{S}} \underline{\kappa} + O(\delta_1 \overset{\circ}{\delta} r^{-5}).\end{aligned}$$

This yields

$$\underline{\kappa} \Delta^{\mathbf{S}} \operatorname{div}^{\mathbf{S}} f + \kappa \Delta^{\mathbf{S}} \operatorname{div}^{\mathbf{S}} \underline{f} = \kappa \Delta^{\mathbf{S}} \underline{\kappa}^{\mathbf{S}} - \underline{\kappa} \Delta^{\mathbf{S}} \kappa - \kappa \Delta^{\mathbf{S}} \underline{\kappa} + O(\delta_1 \overset{\circ}{\delta} r^{-6}). \quad (7.13)$$

Hence, using the control of κ and $\underline{\kappa}$ provided by **A1-Strong**, and since $|m - m^{\mathbf{S}}| \lesssim \overset{\circ}{\delta}$ and $|r - r^{\mathbf{S}}| \lesssim \overset{\circ}{\delta}$ in view of Corollary 5.15, we deduce

$$-\frac{2\Upsilon^{\mathbf{S}}}{r^{\mathbf{S}}} \Delta^{\mathbf{S}} \operatorname{div}^{\mathbf{S}} f + \frac{2}{r^{\mathbf{S}}} \Delta^{\mathbf{S}} \operatorname{div}^{\mathbf{S}} \underline{f} = \frac{2}{r^{\mathbf{S}}} \Delta^{\mathbf{S}} \underline{\kappa}^{\mathbf{S}} + \frac{2\Upsilon^{\mathbf{S}}}{r^{\mathbf{S}}} \Delta^{\mathbf{S}} \kappa - \frac{2}{r^{\mathbf{S}}} \Delta^{\mathbf{S}} \underline{\kappa} + O(\delta_1 \overset{\circ}{\delta} r^{-6}),$$

or,

$$\Delta^{\mathbf{S}} (\operatorname{div}^{\mathbf{S}} \underline{f} - \Upsilon^{\mathbf{S}} \operatorname{div}^{\mathbf{S}} f) = \Delta^{\mathbf{S}} \underline{\kappa}^{\mathbf{S}} + \Upsilon^{\mathbf{S}} \Delta^{\mathbf{S}} \kappa - \Delta^{\mathbf{S}} \underline{\kappa} + O(\delta_1 \overset{\circ}{\delta} r^{-5}). \quad (7.14)$$

Next, we focus on $\Delta^{\mathbf{S}} \kappa$ and $\Delta^{\mathbf{S}} \underline{\kappa}$. Recall from (5.20) that

$$e_a^{\mathbf{S}} = \left(\delta_{ab} + \frac{1}{2} \underline{f}_a \underline{f}_b \right) e_b + \frac{1}{2} \underline{f}_a e_4 + \left(\frac{1}{2} \underline{f}_a + \frac{1}{8} |f|^2 \underline{f}_a \right) e_3, \quad a = 1, 2.$$

Recalling the definition of Γ_g and Γ_b in (5.4), we have $\kappa - \frac{2}{r} = \check{\kappa} \in \Gamma_g$, $\underline{\kappa} + \frac{2\underline{r}}{r} = \check{\underline{\kappa}} \in \Gamma_g$ and

$$e_4(r) - 1 \in r\Gamma_g, \quad e_3(r) + \Upsilon \in r\Gamma_b, \quad e_4(m) \in r\Gamma_g, \quad e_3(m) \in r\Gamma_b.$$

Thus using assumption **A1-Strong** and $|F| \lesssim r^{-1}\overset{\circ}{\delta}$, we infer that

$$\begin{aligned}\nabla^{\mathbf{S}} \kappa &= \nabla \kappa + \frac{\Upsilon}{r^2} f - \frac{1}{r^2} \underline{f} + O(r^{-4} \delta_1 \overset{\circ}{\delta}), \\ \nabla^{\mathbf{S}} \underline{\kappa} &= \nabla \underline{\kappa} - \frac{\Upsilon (1 - \frac{4m}{r})}{r^2} f + \frac{1 - \frac{4m}{r}}{r^2} \underline{f} + O(r^{-4} \delta_1 \overset{\circ}{\delta}).\end{aligned}$$

Taking the divergence, using **A1-strong** for κ and $\underline{\kappa}$, and since $|m - m^{\mathbf{S}}| \lesssim \overset{\circ}{\delta}$ and $|r - r^{\mathbf{S}}| \lesssim \overset{\circ}{\delta}$ in view of Corollary 5.15, we deduce

$$\begin{aligned}\Delta^{\mathbf{S}}\kappa &= \Delta\kappa + \frac{\Upsilon^{\mathbf{S}}}{(r^{\mathbf{S}})^2} \operatorname{div}^{\mathbf{S}} f - \frac{1}{(r^{\mathbf{S}})^2} \operatorname{div}^{\mathbf{S}} \underline{f} + O(r^{-5} \delta_1 \overset{\circ}{\delta}), \\ \Delta^{\mathbf{S}}\underline{\kappa} &= \Delta\underline{\kappa} - \frac{\Upsilon^{\mathbf{S}} \left(1 - \frac{4m^{\mathbf{S}}}{r^{\mathbf{S}}}\right)}{(r^{\mathbf{S}})^2} \operatorname{div}^{\mathbf{S}} f + \frac{1 - \frac{4m^{\mathbf{S}}}{r^{\mathbf{S}}}}{(r^{\mathbf{S}})^2} \operatorname{div}^{\mathbf{S}} \underline{f} + O(r^{-5} \delta_1 \overset{\circ}{\delta}).\end{aligned}$$

We introduce the notations

$$h := \underline{f} - \Upsilon^{\mathbf{S}} f.$$

and rewrite the above equations in the form

$$\Delta^{\mathbf{S}}(\operatorname{div}^{\mathbf{S}} h) = \Delta^{\mathbf{S}}\underline{\kappa}^{\mathbf{S}} + \Upsilon^{\mathbf{S}}\Delta^{\mathbf{S}}\kappa - \Delta^{\mathbf{S}}\underline{\kappa} + O(\delta_1 \overset{\circ}{\delta} r^{-5}),$$

and

$$\begin{aligned}\Delta^{\mathbf{S}}\kappa &= \Delta\kappa - \frac{1}{(r^{\mathbf{S}})^2} \operatorname{div}^{\mathbf{S}} h + O(r^{-5} \delta_1 \overset{\circ}{\delta}), \\ \Delta^{\mathbf{S}}\underline{\kappa} &= \Delta\underline{\kappa} + \frac{1 - \frac{4m^{\mathbf{S}}}{r^{\mathbf{S}}}}{(r^{\mathbf{S}})^2} \operatorname{div}^{\mathbf{S}} h + O(r^{-5} \delta_1 \overset{\circ}{\delta}).\end{aligned}$$

We infer that

$$\left(\Delta^{\mathbf{S}} + \frac{2}{(r^{\mathbf{S}})^2}\right) \operatorname{div}^{\mathbf{S}} h - \frac{6m^{\mathbf{S}}}{(r^{\mathbf{S}})^3} \operatorname{div}^{\mathbf{S}} h = \Delta^{\mathbf{S}}\underline{\kappa}^{\mathbf{S}} + \Upsilon^{\mathbf{S}}\Delta\kappa - \Delta\underline{\kappa} + O(\delta_1 \overset{\circ}{\delta} r^{-5}).$$

Projecting over the basis of canonical $\ell = 1$ modes $J^{(p, \mathbf{S})}$, integrating by parts and using, see (6.21),

$$\left(\Delta^{\mathbf{S}} + \frac{2}{(r^{\mathbf{S}})^2}\right) J^{(p, \mathbf{S})} = O\left(\frac{\overset{\circ}{\epsilon}}{(r^{\mathbf{S}})^3}\right) J^{(p, \mathbf{S})},$$

we deduce,

$$-\frac{6m^{\mathbf{S}}}{(r^{\mathbf{S}})^3} \int_{\mathbf{S}} \operatorname{div}^{\mathbf{S}} h J^{(p, \mathbf{S})} + \frac{O(\overset{\circ}{\epsilon})}{(r^{\mathbf{S}})^3} \int_{\mathbf{S}} \operatorname{div}^{\mathbf{S}} h J^{(p, \mathbf{S})} = \int_{\mathbf{S}} \left(\Delta^{\mathbf{S}}\underline{\kappa}^{\mathbf{S}} + \Upsilon^{\mathbf{S}}\Delta\kappa - \Delta\underline{\kappa}\right) J^{(p, \mathbf{S})} + O(\delta_1 \overset{\circ}{\delta} r^{-3})$$

i.e.,

$$\frac{6m^{\mathbf{S}}}{(r^{\mathbf{S}})^3} \int_{\mathbf{S}} \operatorname{div}^{\mathbf{S}} h J^{(p, \mathbf{S})} = - \int_{\mathbf{S}} \left(\Delta^{\mathbf{S}}\underline{\kappa}^{\mathbf{S}} + \Upsilon^{\mathbf{S}}\Delta\kappa - \Delta\underline{\kappa}\right) J^{(p, \mathbf{S})} + O(\delta_1 \overset{\circ}{\delta} r^{-3})$$

or,

$$\int_{\mathbf{S}} \operatorname{div}^{\mathbf{S}} h_{J^{(p,\mathbf{S})}} = -\frac{(r^{\mathbf{S}})^3}{6m^{\mathbf{S}}} \int_{\mathbf{S}} \left(\Delta^{\mathbf{S}} \underline{\kappa}^{\mathbf{S}} + \Upsilon^{\mathbf{S}} \Delta \kappa - \Delta \underline{\kappa} \right) J^{(p,\mathbf{S})} + O(\delta_1 \overset{\circ}{\delta}). \quad (7.15)$$

Next, we focus on the RHS of (7.15). We have

$$\begin{aligned} \int_{\mathbf{S}} \Delta \kappa J^{(p,\mathbf{S})} &= \int_{\mathbf{S}} \Delta \kappa J^{(p)} + \int_{\mathbf{S}} \Delta \kappa (J^{(p,\mathbf{S})} - J^{(p)}) \\ &= \int_{\overset{\circ}{\mathbf{S}}} \Delta \kappa J^{(p)} + \left(\int_{\mathbf{S}} \Delta \kappa J^{(p)} - \int_{\overset{\circ}{\mathbf{S}}} \Delta \kappa J^{(p)} \right) + \int_{\mathbf{S}} \Delta \kappa (J^{(p,\mathbf{S})} - J^{(p)}). \end{aligned}$$

Thus, using (7.4), **A1-Strong** for κ , and Lemma 5.11 applied to $F = (\Delta \kappa) J^{(p)}$, we infer, proceeding as in Step 1,

$$\begin{aligned} \int_{\mathbf{S}} \Delta \kappa J^{(p,\mathbf{S})} &= \int_{\overset{\circ}{\mathbf{S}}} \Delta \kappa J^{(p)} + r \overset{\circ}{\delta} \left(\sup_{\mathcal{R}} |\Delta \kappa| + r \sup_{\mathcal{R}} (|e_3(\Delta \kappa)| + |e_4(\Delta \kappa)| + |\nabla(\Delta \kappa)|) \right) \\ &\quad + O(r^{-3} \overset{\circ}{\epsilon} \overset{\circ}{\delta}) \\ &= \int_{\overset{\circ}{\mathbf{S}}} \Delta \kappa J^{(p)} + O(r^{-3} \delta_1 \overset{\circ}{\delta}) = -\frac{2}{r^2} (\check{\kappa})_{\ell=1} + O(r^{-3} \delta_1 \overset{\circ}{\delta}), \end{aligned}$$

where we also used (7.2) in the last equality. Similarly, we obtain

$$\int_{\mathbf{S}} \Delta \underline{\kappa} J^{(p,\mathbf{S})} = -\frac{2}{r^2} (\check{\underline{\kappa}})_{\ell=1} + O(r^{-3} \delta_1 \overset{\circ}{\delta}).$$

This yields

$$\begin{aligned} \int_{\mathbf{S}} \left(\Delta^{\mathbf{S}} \underline{\kappa}^{\mathbf{S}} + \Upsilon^{\mathbf{S}} \Delta \kappa - \Delta \underline{\kappa} \right) J^{(p,\mathbf{S})} &= \int_{\mathbf{S}} \Delta^{\mathbf{S}} \underline{\kappa}^{\mathbf{S}} J^{(p,\mathbf{S})} + \Upsilon^{\mathbf{S}} \int_{\mathbf{S}} \Delta \kappa J^{(p,\mathbf{S})} - \int_{\mathbf{S}} \Delta \underline{\kappa} J^{(p,\mathbf{S})} \\ &= -\frac{2}{(r^{\mathbf{S}})^2} \int_{\mathbf{S}} \check{\underline{\kappa}}^{\mathbf{S}} J^{(p,\mathbf{S})} - \frac{2\Upsilon^{\mathbf{S}}}{r^2} (\check{\kappa})_{\ell=1} + \frac{2}{r^2} (\check{\underline{\kappa}})_{\ell=1} + O(r^{-3} \delta_1 \overset{\circ}{\delta}). \end{aligned}$$

Since $|r - r^{\mathbf{S}}| \lesssim \overset{\circ}{\delta}$, we infer

$$\int_{\mathbf{S}} \left(\Delta^{\mathbf{S}} \underline{\kappa}^{\mathbf{S}} + \Upsilon^{\mathbf{S}} \Delta \kappa - \Delta \underline{\kappa} \right) J^{(p,\mathbf{S})} = -\frac{2}{(r^{\mathbf{S}})^2} (\check{\underline{\kappa}}^{\mathbf{S}})_{\ell=1} - \frac{2\Upsilon^{\mathbf{S}}}{(r^{\mathbf{S}})^2} (\check{\kappa})_{\ell=1} + \frac{2}{(r^{\mathbf{S}})^2} (\check{\underline{\kappa}})_{\ell=1} + O(r^{-3} \delta_1 \overset{\circ}{\delta}).$$

Together with (7.15), we deduce

$$\int_{\mathbf{S}} \operatorname{div}^{\mathbf{S}} h_{J^{(p,\mathbf{S})}} = \frac{r^{\mathbf{S}}}{3m^{\mathbf{S}}} \left((\check{\underline{\kappa}}^{\mathbf{S}})_{\ell=1} + \Upsilon^{\mathbf{S}} (\check{\kappa})_{\ell=1} - (\check{\underline{\kappa}})_{\ell=1} \right) + O(\delta_1 \overset{\circ}{\delta}).$$

Recalling $h = \underline{f} - \Upsilon^{\mathbf{S}} f$ and the definition of $\Lambda, \underline{\Lambda}$, we deduce,

$$\underline{\Lambda} - \Upsilon^{\mathbf{S}} \Lambda = \frac{r^{\mathbf{S}}}{3m^{\mathbf{S}}} \left((\check{\kappa}^{\mathbf{S}})_{\ell=1} + \Upsilon^{\mathbf{S}}(\check{\kappa})_{\ell=1} - (\check{\kappa})_{\ell=1} \right) + F_2(\Lambda, \underline{\Lambda})$$

with

$$|F_2(\Lambda, \underline{\Lambda})| \lesssim \delta_1 \overset{\circ}{\delta}$$

as stated. This concludes the proof of Lemma 7.5.

7.2 Definition of angular momentum

The result of Theorem 7.3 is unique up to a rotation of \mathbb{S}^2 in the definition of the canonical basis of $\ell = 1$ modes on \mathbf{S} , see Definition 6.3. We can choose that rotation such that the $p = \pm$ component of $(\text{curl}^{\mathbf{S}} \beta^{\mathbf{S}})_{\ell=1}$ vanish. We state this result in the following corollary of Theorem 7.3.

Corollary 7.7. *Under the same assumptions as in Theorem 7.3 we have, in addition to (7.6) and (7.7),*

- either, for any choice of a canonical $\ell = 1$ basis of \mathbf{S} ,

$$(\text{curl}^{\mathbf{S}} \beta^{\mathbf{S}})_{\ell=1} = 0,$$

- or there exists a canonical basis of $\ell = 1$ modes of \mathbf{S} such that

$$\int_{\mathbf{S}} \text{curl}^{\mathbf{S}} \beta^{\mathbf{S}} J^{(\pm, \mathbf{S})} = 0, \quad \int_{\mathbf{S}} \text{curl}^{\mathbf{S}} \beta^{\mathbf{S}} J^{(0, \mathbf{S})} \neq 0. \quad (7.16)$$

We then define the angular parameter $a^{\mathbf{S}}$ on \mathbf{S} by the formula³⁵

$$a^{\mathbf{S}} := \frac{(r^{\mathbf{S}})^3}{8\pi m^{\mathbf{S}}} \int_{\mathbf{S}} \text{curl}^{\mathbf{S}} \beta^{\mathbf{S}} J^{(0, \mathbf{S})}. \quad (7.17)$$

With this definition, we have $a^{\mathbf{S}} = 0$ in the first case, while $a^{\mathbf{S}} \neq 0$ in the second case.

³⁵Note that in a Kerr space $\mathcal{K}(a, m)$, relative to a geodesic foliation normalized on \mathcal{I}^+ , we have $\int_{\mathbf{S}} \text{curl}^{\mathbf{S}} \beta^{\mathbf{S}} J^{\pm, S} = 0$ and $\int_{\mathbf{S}} \text{curl}^{\mathbf{S}} \beta^{\mathbf{S}} J^{0, S} = \frac{8\pi am}{(r^{\mathbf{S}})^3} + O\left(\frac{ma^2}{(r^{\mathbf{S}})^4}\right)$.

Proof. Recall that the basis of canonical $\ell = 1$ modes on \mathbf{S} are given by the formula

$$J^{(p,\mathbf{S})} = J^{(p,\mathbb{S}^2)} \circ \Phi^{-1}$$

with $J^{(p,\mathbb{S}^2)}$ the $\ell = 1$ basis on \mathbb{S}^2 given by Definition 3.10, and (Φ, ϕ) an effective uniformization map of \mathbf{S} . In particular, we have by change of variable

$$\int_{\mathbf{S}} \text{curl}^{\mathbf{S}} \beta^{\mathbf{S}} J^{(p,\mathbf{S})} = \int_{\mathbb{S}^2} e^{2\phi} \text{curl}^{\mathbf{S}} \beta^{\mathbf{S}} \circ \Phi J^{(p,\mathbb{S}^2)}$$

and hence

$$(\text{curl}^{\mathbf{S}} \beta^{\mathbf{S}})_{\ell=1} = \int_{\mathbb{S}^2} e^{2\phi} \text{curl}^{\mathbf{S}} \beta^{\mathbf{S}} \circ \Phi x$$

where x is the position vector on \mathbb{S}^2 . Changing the definition of $\ell = 1$ modes by a rotation of \mathbb{S}^2 consist in changing (Φ, ϕ) to $(\Phi \circ O, \phi \circ O)$ with $O \in O(3)$ which yields

$$\begin{aligned} \int_{\mathbb{S}^2} e^{2\phi \circ O} \text{curl}^{\mathbf{S}} \beta^{\mathbf{S}} \circ \Phi \circ O x &= \int_{\mathbb{S}^2} e^{2\phi} \text{curl}^{\mathbf{S}} \beta^{\mathbf{S}} \circ \Phi O^{-1} x \\ &= O^{-1} \left(\int_{\mathbb{S}^2} e^{2\phi} \text{curl}^{\mathbf{S}} \beta^{\mathbf{S}} \circ \Phi x \right) \\ &= O^{-1} (\text{curl}^{\mathbf{S}} \beta^{\mathbf{S}})_{\ell=1}. \end{aligned}$$

In particular, $(\text{curl}^{\mathbf{S}} \beta^{\mathbf{S}})_{\ell=1}$ is identified with a vector v in \mathbb{R}^3 , and changing the definition of $\ell = 1$ modes by a rotation of \mathbb{S}^2 amounts to apply a rotation to v . The proof follows then from the fact that, given a vector v in \mathbb{R}^3 with $v \neq 0$, there exists a rotation of the (x^1, x^2, x^3) coordinates of \mathbb{R}^3 such that v points in the direction of the x^3 axis. \square

Remark 7.8. In the case $a^{\mathbf{S}} \neq 0$, with $a^{\mathbf{S}}$ given by (7.17), let a canonical basis $J^{(p,\mathbf{S})}$ satisfying the condition (7.16). Then, the set of all canonical basis satisfying the condition (7.16) forms a 1-parameter family of basis $J_{\varphi_0}^{(p,\mathbf{S})}$ given by

$$\begin{aligned} J_{\varphi_0}^{(0,\mathbf{S})} &= J^{(0,\mathbf{S})}, \\ J_{\varphi_0}^{(+,\mathbf{S})} &= \cos(\varphi_0) J^{(+,\mathbf{S})} - \sin(\varphi_0) J^{(-,\mathbf{S})}, \\ J_{\varphi_0}^{(-,\mathbf{S})} &= \sin(\varphi_0) J^{(+,\mathbf{S})} + \cos(\varphi_0) J^{(-,\mathbf{S})}, \end{aligned}$$

where $\varphi_0 \in [0, 2\pi)$. In particular, $J^{(0,\mathbf{S})}$ is unique so that, on an intrinsic GCM sphere \mathbf{S} ,

- $a^{\mathbf{S}}$ given by (7.17) is a well defined notion of angular momentum,
- the condition (7.16) is a canonical way to define a notion of axis³⁶ when $a^{\mathbf{S}} \neq 0$.

³⁶Note that in Kerr, the axis corresponds to the zeros of $J^{(0,\mathbf{S})}$.

7.3 Intrinsic GCM spheres in Kerr

Consider $\mathcal{K}(a_0, m_0)$, $|a_0| < m_0$, a sub-extremal Kerr spacetime endowed with an outgoing optical function u normalized at \mathcal{I}^+ . We denote by $S(u, s)$ the spheres of the induced geodesic foliation, with s the affine parameter, and r the area radius, normalized on \mathcal{I}^+ such that $\lim_{r \rightarrow \infty} \frac{s}{r} = 1$. Let (e_4, e_3, e_2, e_1) be the associated null frame with $e_4 = -\mathbf{g}^{\alpha\beta} \partial_\beta u \partial_\alpha$. Define also the corresponding angular coordinates (θ, φ) , properly normalized at infinity, with $e_4(\theta) = e_4(\varphi) = 0$, and the corresponding $J^{(p)}$ defined by them, i.e.

$$J^{(0)} = \cos \theta, \quad J^{(+)} = \sin \theta \cos \varphi, \quad J^{(-)} = \sin \theta \sin \varphi. \quad (7.18)$$

Finally we consider the spacetime region

$$\mathcal{R}(r_0) = \left\{ r \geq r_0 \right\} \subset \mathcal{K}(a_0, m_0).$$

Lemma 7.9. *If $r_0 = r_0(m_0)$ is sufficiently large, the region $\mathcal{R}(r_0)$, endowed with the geodesic foliation described above, verifies the assumptions **A1-Strong**, **A2**, **A3** and **A4-Strong**, as well as (7.5), with the smallness constants*

$$\overset{\circ}{\epsilon} = \overset{\circ}{\delta} = \frac{a_0 m_0}{r_0}.$$

Proof. **A1-Strong**, **A2**, **A3** and **A4-Strong** follow immediately from Lemma 2.10 in [17]. It then remains to prove (7.5). Lemma 2.10 in [17] yields $\check{\kappa} = O(a_0^2 r^{-3})$ and $\check{\underline{\kappa}} = O(a_0^2 r^{-3})$. One can easily push the asymptotic to the next order to obtain

$$\check{\kappa} = \frac{a_0^2}{r^3} \left(c_1 + c_2 (\cos \theta)^2 \right) + O\left(\frac{m_0 a_0^2}{r^4}\right), \quad \check{\underline{\kappa}} = \frac{a_0^2}{r^3} \left(\underline{c}_1 + \underline{c}_2 (\cos \theta)^2 \right) + O\left(\frac{m_0 a_0^2}{r^4}\right),$$

for some universal constants $c_1, c_2, \underline{c}_1$ and \underline{c}_2 . Since

$$\left(c_1 + c_2 (\cos \theta)^2 \right)_{\ell=1} = 0, \quad \left(\underline{c}_1 + \underline{c}_2 (\cos \theta)^2 \right)_{\ell=1} = 0,$$

we infer

$$(\check{\kappa})_{\ell=1} = O\left(\frac{m_0 a_0^2}{r^2}\right), \quad (\check{\underline{\kappa}})_{\ell=1} = O\left(\frac{m_0 a_0^2}{r^2}\right),$$

as stated in (7.5). It remains to derive the estimate for $(\operatorname{div} \beta)_{\ell=1}$ in (7.5). Lemma 2.10 in [17] yields $\operatorname{div} \beta = O(a_0 m_0 r^{-5})$, and one can easily push the asymptotic to the next order to obtain $\operatorname{div} \beta = O(a_0^2 m_0 r^{-6})$ which immediately implies the estimate for $(\operatorname{div} \beta)_{\ell=1}$ in (7.5). This concludes the proof of (7.5), and hence the one of Lemma 7.9. \square

Corollary 7.10 (Existence of intrinsic GCM spheres in Kerr). *If $r_0 \gg m_0$ is sufficiently large, then any sphere $\mathring{S} \subset \mathcal{R}(r_0)$ admits a unique deformation $\Psi : \mathring{S} \rightarrow \mathbf{S}$ verifying (7.6) and (7.7). Moreover, the deformation verifies the properties (6.8), (6.9), (6.10) and (6.11) stated in Theorem 6.1.*

Proof. In view of Lemma 7.9, the assumptions **A1-Strong**, **A2**, **A3** and **A4-Strong**, as well as (7.5), are satisfied by the spacetime region $\mathcal{R}(r_0)$ provided that $r_0 = r_0(m_0)$ is sufficiently large, with the smallness constants $\overset{\circ}{\epsilon}$ and $\overset{\circ}{\delta}$ given by

$$\overset{\circ}{\epsilon} = \overset{\circ}{\delta} = \frac{a_0 m_0}{r_0}.$$

Thus, Theorem 7.3 applies which concludes the proof of the corollary. \square

Remark 7.11. *We may define the angular momentum $a^{\mathbf{S}}$ on \mathbf{S} as in (7.17), where \mathbf{S} denotes any of the intrinsic GCM spheres in Kerr constructed in Corollary 7.10. One can then easily prove³⁷ the estimate $|a^{\mathbf{S}} - a_0| \lesssim \overset{\circ}{\epsilon}$.*

8 Control of approximate intrinsic GCM spheres

The goal of this section is to prove the following proposition providing a priori estimates for the triplet $(f, \bar{f}, \bar{\lambda})$ corresponding to a given sphere $\mathbf{S} \subset \mathcal{R}$ satisfying an approximate version of the GCM conditions (7.6) (7.7).

Proposition 8.1. *Let a fixed spacetime region \mathcal{R} verifying assumptions **A1** – **A4** and (6.1) (6.2), as well as, for any background sphere S of \mathcal{R} ,*

$$|(div \beta)_{\ell=1}| \lesssim r^{-2} \overset{\circ}{\delta}, \quad |(\tilde{\kappa})_{\ell=1}| \lesssim \overset{\circ}{\delta}. \quad (8.1)$$

³⁷This relies in particular on the following analog of the estimate for $div \beta$ in the proof of Lemma 7.9

$$\text{curl } \beta = \frac{6a_0 m_0 \cos \theta}{r^5} + O(a_0^2 m_0 r^{-6}).$$

Assume that \mathbf{S} is a deformed sphere in \mathcal{R} which verifies the GCM conditions

$$\begin{aligned}\kappa^{\mathbf{S}} &= \frac{2}{r^{\mathbf{S}}}, \\ \underline{\kappa}^{\mathbf{S}} &= -\frac{2}{r^{\mathbf{S}}}\Upsilon^{\mathbf{S}} + \underline{C}_0^{\mathbf{S}} + \sum_p \underline{C}^{(\mathbf{S},p)} \tilde{J}^{(p)}, \\ \mu^{\mathbf{S}} &= \frac{2m^{\mathbf{S}}}{(r^{\mathbf{S}})^3} + M_0^{\mathbf{S}} + \sum_p M^{(\mathbf{S},p)} \tilde{J}^{(p)},\end{aligned}\tag{8.2}$$

for some basis³⁸ of $\ell = 1$ modes $\tilde{J}^{(p)}$ on \mathbf{S} , such that for a small enough constant $\delta_1 > 0$, the transition coefficients $(f, \underline{f}, \lambda)$ from the background frame of \mathcal{R} to that of \mathbf{S} verifies, for some $4 \leq s \leq s_{max}$, the bound

$$\|f\|_{\mathfrak{h}_s(\mathbf{S})} + (r^{\mathbf{S}})^{-1} \|(\underline{f}, \overset{\circ}{\lambda})\|_{\mathfrak{h}_s(\mathbf{S})} \leq \delta_1,\tag{8.3}$$

and the difference between the basis of $\ell = 1$ modes $\tilde{J}^{(p)}$ on \mathbf{S} and the basis of $\ell = 1$ modes of the background foliation $J^{(p)}$ verifies

$$r^{-1} \|\tilde{J}^{(p)} - J^{(p)}\|_{\mathfrak{h}_s(\mathbf{S})} \leq \delta_1.\tag{8.4}$$

Assume in addition that we have

$$|(\operatorname{div}^{\mathbf{S}} \beta^{\mathbf{S}})_{\ell=1}| \lesssim r^{-2} \overset{\circ}{\delta}, \quad |(\underline{\kappa}^{\mathbf{S}})_{\ell=1}| \lesssim \overset{\circ}{\delta}\tag{8.5}$$

with respect to the basis of $\ell = 1$ modes $\tilde{J}^{(p)}$ on \mathbf{S} . Then $(f, \underline{f}, \lambda)$ verify the estimates

$$\|(f, \underline{f}, \overset{\circ}{\lambda}^{\mathbf{S}})\|_{\mathfrak{h}_{s+1}(\mathbf{S})} \lesssim r \overset{\circ}{\delta} + r(\overset{\circ}{\epsilon})^2 + r\delta_1 \left(\frac{1}{r} + \overset{\circ}{\epsilon} + \delta_1 \right)$$

and

$$r |\overset{\circ}{\lambda}^{\mathbf{S}}| \lesssim r \overset{\circ}{\delta} + r(\overset{\circ}{\epsilon})^2 + r\delta_1 \left(\frac{1}{r} + \overset{\circ}{\epsilon} + \delta_1 \right) + \sup_{\mathbf{S}} |r - r^{\mathbf{S}}|.$$

Remark 8.2. The anomalous behavior for (f, λ) in the assumption (8.3) is motivated by the proof of the nonlinear stability of Schwarzschild in [16], where this behavior is due a substantial shift of the center of mass frame of the final black hole compared to initial one, known in the physics literature as a gravitational wave recoil, see Remark 4.5 in [16] for a more detailed heuristic explanation for this behavior of (f, λ) . We refer to Corollary 9.51 in [16] for an analogous statement to Proposition 8.1. We expect that Proposition 8.1 would play a similar role in a future proof of stability of Kerr.

³⁸ $\tilde{J}^{(p)}$ is not assumed to be a canonical basis of $\ell = 1$ modes on \mathbf{S} .

8.1 Deformed sphere with weaker assumptions on $(f, \underline{f}, \lambda)$

The proof of Proposition 8.1 requires the following extension of the results of section 5.2 to the case where $(f, \underline{f}, \lambda)$ satisfy the assumptions (8.3).

Lemma 8.3. *There exists a small enough constant δ_1 such that for given f, \underline{f} on \mathcal{R} satisfying*

$$\|f\|_{\mathfrak{h}_{smax}(\mathbf{S})} + (r^{\mathbf{S}})^{-1} \|\underline{f}\|_{\mathfrak{h}_{smax}(\mathbf{S})} \leq \delta_1,$$

the following holds

1. We have

$$\left| \frac{r^{\mathbf{S}}}{\mathring{r}} - 1 \right| + \sup_{\mathbf{S}} \left| \frac{r^{\mathbf{S}}}{r} - 1 \right| \lesssim \delta_1.$$

2. The following estimate holds true for an arbitrary scalar function h on \mathcal{R} ,

$$\left| \int_{\mathbf{S}} h - \int_{\mathring{S}} h \right| \lesssim \delta_1 (\mathring{r})^2 \left(\sup_{\mathcal{R}} (|f| + |\partial_u f|) + \mathring{r} \sup_{\mathcal{R}} |\partial_s f| \right).$$

Proof. See Lemma 7.3 in [17]. □

Lemma 8.4. *Assume that (f, \underline{f}) given on \mathcal{R} satisfy for a small enough constant δ_1*

$$\|f\|_{\mathfrak{h}_3(\mathbf{S})} + (r^{\mathbf{S}})^{-1} \|\underline{f}\|_{\mathfrak{h}_3(\mathbf{S})} \leq \delta_1.$$

Then, we have

$$|m^{\mathbf{S}} - \mathring{m}| \lesssim \delta_1 + (\mathring{\epsilon})^2.$$

Proof. See Corollary 7.7 in [17]. □

We will also need the following extension of Lemma 7.5.

Lemma 8.5. *Assume that the spacetime region \mathcal{R} verifies the assumptions **A1**, **A2**, **A3**, **A4**. Let $\mathbf{S} = \mathbf{S}^{(\tilde{\Lambda}, \tilde{\Delta})}$ be a deformation of \mathring{S} with*

$$\int_{\mathbf{S}} \operatorname{div}^{\mathbf{S}} f \tilde{J}^{(p)} = \tilde{\Lambda}, \quad \int_{\mathbf{S}} \operatorname{div}^{\mathbf{S}} \underline{f} \tilde{J}^{(p)} = \tilde{\Delta},$$

and with $(f, \underline{f}, \overset{\circ}{\lambda})$ satisfying (8.3). The following identities hold true.

$$\begin{aligned}\tilde{\Lambda} &= \frac{(r^{\mathbf{S}})^3}{3m^{\mathbf{S}}} \left[-(\operatorname{div}^{\mathbf{S}} \beta^{\mathbf{S}})_{\ell=1} + (\operatorname{div} \beta)_{\ell=1} \right] + F_1(\tilde{\Lambda}, \underline{\tilde{\Lambda}}), \\ \underline{\tilde{\Lambda}} &= \Upsilon^{\mathbf{S}} \tilde{\Lambda} + \frac{r^{\mathbf{S}}}{3m^{\mathbf{S}}} \left[(\check{\kappa}^{\mathbf{S}})_{\ell=1} + \Upsilon^{\mathbf{S}}(\check{\kappa})_{\ell=1} - (\check{\kappa})_{\ell=1} \right] + F_2(\tilde{\Lambda}, \underline{\tilde{\Lambda}}),\end{aligned}\tag{8.6}$$

where F_1, F_2 verify

$$|F_1, F_2| \lesssim r \delta_1 (\overset{\circ}{\epsilon} + \delta_1).\tag{8.7}$$

Proof. The proof is a simple adaptation³⁹ of the one of Lemma 7.5 where

1. the control (6.6) for $(f, \underline{f}, \overset{\circ}{\lambda})$ is replaced by (8.3),
2. **A1-strong** and **A4-strong** are replaced by the assumptions **A1** and **A4**,
3. the estimates for $r^{\mathbf{S}} - r$ and $m^{\mathbf{S}} - m$ of Corollary 5.15 are replaced respectively by the first property of Lemma 8.3 and Lemma 8.4,
4. Lemma 5.11 is replaced by the second property of Lemma 8.3,
5. the pair of triplets $(\Lambda, \underline{\Lambda})$ in Lemma 7.5, defined w.r.t. a canonical basis of $\ell = 1$ modes $J^{(p, \mathbf{S})}$ on \mathbf{S} , are replaced with the pair of triplets $(\tilde{\Lambda}, \underline{\tilde{\Lambda}})$ defined with respect to the basis of $\ell = 1$ modes $\tilde{J}^{(p)}$ on \mathbf{S} ,
6. the estimate (7.4) for $J^{(p, \mathbf{S})} - J^{(p)}$ is replaced by the assumption (8.4) on $\tilde{J}^{(p)} - J^{(p)}$.

This concludes the proof of the lemma. □

Finally, we also rely on the following proposition.

Proposition 8.6. *Let a fixed spacetime region \mathcal{R} verifying assumptions **A1** – **A4** and (6.1) (6.2). Assume that \mathbf{S} is a deformed sphere in \mathcal{R} which verifies the approximate*

³⁹The only step which requires a slight modification of the argument is when passing from (7.13) to (7.14) as $(\kappa - \kappa^{\mathbf{S}}) \Delta^{\mathbf{S}} \operatorname{div}^{\mathbf{S}} \underline{f}$ induces the term $(\frac{2}{r} - \frac{2}{r^{\mathbf{S}}}) \Delta^{\mathbf{S}} \operatorname{div}^{\mathbf{S}} \underline{f}$ which would loose one more power of r than allowed. Instead, one should first divide (7.13) by κ and the proof then proceeds along the same lines.

GCM conditions

$$\begin{aligned}
\kappa^{\mathbf{S}} &= \frac{2}{r^{\mathbf{S}}}, \\
\underline{\kappa}^{\mathbf{S}} &= -\frac{2}{r^{\mathbf{S}}}\Upsilon^{\mathbf{S}} + \underline{C}_0^{\mathbf{S}} + \sum_p \underline{C}^{(\mathbf{S},p)} J^{(p)} + Err[\underline{\kappa}^{\mathbf{S}}], \\
\mu^{\mathbf{S}} &= \frac{2m^{\mathbf{S}}}{(r^{\mathbf{S}})^3} + M_0^{\mathbf{S}} + \sum_p M^{(\mathbf{S},p)} J^{(p)} + Err[\mu^{\mathbf{S}}],
\end{aligned} \tag{8.8}$$

where $Err[\underline{\kappa}^{\mathbf{S}}]$ and $Err[\mu^{\mathbf{S}}]$ are two scalar functions on \mathbf{S} , and such that for a small enough constant $\delta_1 > 0$, the transition coefficients $(f, \underline{f}, \lambda)$ from the background frame of \mathcal{R} to that of \mathbf{S} verifies, for some $4 \leq s \leq s_{max}$, the bound

$$\|f\|_{\mathfrak{h}_s(\mathbf{S})} + (r^{\mathbf{S}})^{-1} \|(f, \underline{\lambda})\|_{\mathfrak{h}_s(\mathbf{S})} \leq \delta_1. \tag{8.9}$$

Then $(f, \underline{f}, \lambda)$ verify the estimates

$$\begin{aligned}
\|(f, \underline{f}, \overset{\circ}{\lambda})\|_{\mathfrak{h}_{s+1}(\mathbf{S})} + \sum_p \left(r^2 |\dot{\underline{C}}^{(p)}| + r^3 |\dot{M}^{(p)}| \right) &\lesssim \overset{\circ}{\delta} + (\overset{\circ}{\epsilon})^2 + r\delta_1 \left(\frac{1}{\overset{\circ}{r}} + \overset{\circ}{\epsilon} + \delta_1 \right) + |\Lambda| + |\underline{\Lambda}| \\
&\quad + r \|Err[\underline{\kappa}^{\mathbf{S}}]\|_{\mathfrak{h}_s(\mathbf{S})} + r^2 \|Err[\mu^{\mathbf{S}}]\|_{\mathfrak{h}_{s-1}(\mathbf{S})}
\end{aligned}$$

and

$$\begin{aligned}
r^2 |\dot{\underline{C}}_0| + r^3 |\dot{M}_0| + r \left| \overset{\circ}{\lambda} \right| &\lesssim \overset{\circ}{\delta} + (\overset{\circ}{\epsilon})^2 + r\delta_1 \left(\frac{1}{\overset{\circ}{r}} + \overset{\circ}{\epsilon} + \delta_1 \right) + |\Lambda| + |\underline{\Lambda}| + \sup_{\mathbf{S}} |r - r^{\mathbf{S}}| \\
&\quad + r \|Err[\underline{\kappa}^{\mathbf{S}}]\|_{L^2(\mathbf{S})} + r^2 \|Err[\mu^{\mathbf{S}}]\|_{L^2(\mathbf{S})},
\end{aligned}$$

where the pair of triplets $\Lambda, \underline{\Lambda} \in \mathbb{R}^3$ is defined by

$$\Lambda := (div^{\mathbf{S}} f)_{\ell=1}, \quad \underline{\Lambda} := (div^{\mathbf{S}} \underline{f})_{\ell=1},$$

with respect to the basis of $\ell = 1$ modes $J^{(p)}$ of the background foliation, and where we have introduced the notation

$$\begin{aligned}
\dot{\underline{C}}_0 &:= \underline{C}_0^{\mathbf{S}} - \overline{\underline{C}}_0^{\mathbf{S}}, & \dot{\underline{C}}^{(p)} &:= \underline{C}^{(\mathbf{S},p)} - \overline{\underline{C}}^{(p)\mathbf{S}}, \\
\dot{M}_0 &:= M_0^{\mathbf{S}} - \overline{M}_0^{\mathbf{S}}, & \dot{M}^{(p)} &:= M^{(\mathbf{S},p)} - \overline{M}^{(p)\mathbf{S}}.
\end{aligned} \tag{8.10}$$

Proof. See Corollary 7.1 in [17]. □

8.2 Proof of Proposition 8.1

First, we rewrite the GCM conditions (8.2) as follows

$$\begin{aligned}
\kappa^{\mathbf{S}} &= \frac{2}{r^{\mathbf{S}}}, \\
\underline{\kappa}^{\mathbf{S}} &= -\frac{2}{r^{\mathbf{S}}} \Upsilon^{\mathbf{S}} + \underline{C}_0^{\mathbf{S}} + \sum_p \underline{C}^{(\mathbf{S},p)} J^{(p)} + \text{Err}[\underline{\kappa}^{\mathbf{S}}], \\
\mu^{\mathbf{S}} &= \frac{2m^{\mathbf{S}}}{(r^{\mathbf{S}})^3} + M_0^{\mathbf{S}} + \sum_p M^{(\mathbf{S},p)} J^{(p)} + \text{Err}[\mu^{\mathbf{S}}],
\end{aligned} \tag{8.11}$$

where $\text{Err}[\underline{\kappa}^{\mathbf{S}}]$ and $\text{Err}[\mu^{\mathbf{S}}]$ are the scalar functions on \mathbf{S} given by

$$\text{Err}[\underline{\kappa}^{\mathbf{S}}] := \sum_p \underline{C}^{(\mathbf{S},p)} \left(\tilde{J}^{(p)} - J^{(p)} \right), \quad \text{Err}[\mu^{\mathbf{S}}] := \sum_p M^{(\mathbf{S},p)} \left(\tilde{J}^{(p)} - J^{(p)} \right),$$

and thus satisfy, in view of the assumption (8.4) on the control of $\tilde{J}^{(p)} - J^{(p)}$,

$$r \|\text{Err}[\underline{\kappa}^{\mathbf{S}}]\|_{b_s(\mathbf{S})} + r^2 \|\text{Err}[\mu^{\mathbf{S}}]\|_{b_s(\mathbf{S})} \lesssim \delta_1 \sum_p \left(r^2 |\underline{C}^{(\mathbf{S},p)}| + r^3 |M^{(\mathbf{S},p)}| \right). \tag{8.12}$$

Next, we apply Proposition 8.6 to the system (8.11), and rely on the estimate (8.12). We obtain

$$\begin{aligned}
\|(f, \underline{f}, \overset{\circ}{\lambda}^{\mathbf{S}})\|_{b_{s+1}(\mathbf{S})} + \sum_p \left(r^2 |\dot{\underline{C}}^{(p)}| + r^3 |\dot{M}^{(p)}| \right) &\lesssim \overset{\circ}{\delta} + (\overset{\circ}{\epsilon})^2 + r\delta_1 \left(\frac{1}{\overset{\circ}{r}} + \overset{\circ}{\epsilon} + \delta_1 \right) + |\Lambda| + |\underline{\Lambda}| \\
&\quad + \delta_1 \sum_p \left(r^2 |\underline{C}^{(\mathbf{S},p)}| + r^3 |M^{(\mathbf{S},p)}| \right)
\end{aligned}$$

and

$$\begin{aligned}
r^2 |\dot{\underline{C}}_0| + r^3 |\dot{M}_0| + r \left| \overset{\circ}{\lambda}^{\overline{\mathbf{S}}} \right| &\lesssim \overset{\circ}{\delta} + (\overset{\circ}{\epsilon})^2 + r\delta_1 \left(\frac{1}{\overset{\circ}{r}} + \overset{\circ}{\epsilon} + \delta_1 \right) + |\Lambda| + |\underline{\Lambda}| + \sup_{\mathbf{S}} |r - r^{\mathbf{S}}| \\
&\quad + \delta_1 \sum_p \left(r^2 |\underline{C}^{(\mathbf{S},p)}| + r^3 |M^{(\mathbf{S},p)}| \right).
\end{aligned}$$

Now, in view of (8.10) and the assumptions on the background foliation, we have

$$|\underline{C}^{(\mathbf{S},p)}| \leq |\overline{\underline{C}^{(p)}}^{\mathbf{S}}| + |\dot{\underline{C}}^{(p)}| \lesssim r^{-2\overset{\circ}{\epsilon}} + |\dot{\underline{C}}^{(p)}|, \quad |M^{(\mathbf{S},p)}| \leq |\overline{M^{(p)}}^{\mathbf{S}}| + |\dot{M}^{(p)}| \lesssim r^{-3\overset{\circ}{\epsilon}} + |\dot{M}^{(p)}|.$$

Plugging in the above, and using the smallness of δ_1 to absorb the terms $\underline{\dot{C}}^{(p)}$ and $\dot{M}^{(p)}$ on the RHS, we infer

$$\|(f, \underline{f}, \overset{\circ}{\lambda}^{\mathbf{S}})\|_{\mathfrak{h}_{s+1}(\mathbf{S})} \lesssim \overset{\circ}{\delta} + (\overset{\circ}{\epsilon})^2 + r\delta_1 \left(\frac{1}{\overset{\circ}{r}} + \overset{\circ}{\epsilon} + \delta_1 \right) + |\Lambda| + |\underline{\Lambda}| \quad (8.13)$$

and

$$r \left| \overset{\circ}{\lambda}^{\overline{\mathbf{S}}} \right| \lesssim \overset{\circ}{\delta} + (\overset{\circ}{\epsilon})^2 + r\delta_1 \left(\frac{1}{\overset{\circ}{r}} + \overset{\circ}{\epsilon} + \delta_1 \right) + |\Lambda| + |\underline{\Lambda}| + \sup_{\mathbf{S}} |r - r^{\mathbf{S}}|. \quad (8.14)$$

Also, we have from Lemma 8.5 the following identities

$$\begin{aligned} \tilde{\Lambda} &= \frac{(r^{\mathbf{S}})^3}{3m^{\mathbf{S}}} \left[-(\operatorname{div}^{\mathbf{S}} \beta^{\mathbf{S}})_{\ell=1} + (\operatorname{div} \beta)_{\ell=1} \right] + F_1(\tilde{\Lambda}, \tilde{\underline{\Lambda}}), \\ \tilde{\underline{\Lambda}} &= \Upsilon^{\mathbf{S}} \tilde{\Lambda} + \frac{r^{\mathbf{S}}}{3m^{\mathbf{S}}} \left[(\tilde{\kappa}^{\mathbf{S}})_{\ell=1} + \Upsilon^{\mathbf{S}}(\tilde{\kappa})_{\ell=1} - (\tilde{\kappa})_{\ell=1} \right] + F_2(\tilde{\Lambda}, \tilde{\underline{\Lambda}}), \end{aligned}$$

where F_1, F_2 verify

$$|F_1, F_2| \lesssim r\delta_1(\overset{\circ}{\epsilon} + \delta_1).$$

In view of (8.1) and (8.5), we infer

$$|\tilde{\Lambda}| + |\tilde{\underline{\Lambda}}| \lesssim r\overset{\circ}{\delta} + r\delta_1(\overset{\circ}{\epsilon} + \delta_1).$$

Since we have

$$|\Lambda| + |\underline{\Lambda}| \lesssim |\tilde{\Lambda}| + |\tilde{\underline{\Lambda}}| + \max_{p=0,+,-} \int_{\mathbf{S}} \left(|\operatorname{div}^{\mathbf{S}} f| + |\operatorname{div}^{\mathbf{S}} \underline{f}| \right) |\tilde{J}^{(p)} - J^{(p)}|,$$

we obtain, together with (8.3) and (8.4)

$$|\Lambda| + |\underline{\Lambda}| \lesssim r\overset{\circ}{\delta} + r\delta_1(\overset{\circ}{\epsilon} + \delta_1).$$

Together with (8.13) and (8.14) we deduce

$$\|(f, \underline{f}, \overset{\circ}{\lambda}^{\mathbf{S}})\|_{\mathfrak{h}_{s+1}(\mathbf{S})} \lesssim r\overset{\circ}{\delta} + r(\overset{\circ}{\epsilon})^2 + r\delta_1 \left(\frac{1}{\overset{\circ}{r}} + \overset{\circ}{\epsilon} + \delta_1 \right)$$

and

$$r \left| \overset{\circ}{\lambda}^{\overline{\mathbf{S}}} \right| \lesssim r\overset{\circ}{\delta} + r(\overset{\circ}{\epsilon})^2 + r\delta_1 \left(\frac{1}{\overset{\circ}{r}} + \overset{\circ}{\epsilon} + \delta_1 \right) + \sup_{\mathbf{S}} |r - r^{\mathbf{S}}|.$$

This concludes the proof of Proposition 8.1.

A Appendix by Camillo De Lellis

We denote by e the Euclidean metric on \mathbb{R}^n .

Theorem A.1. *For every $n \geq 2$ and every $\alpha \in (0, 1]$ there is a constant $C = C(n, \alpha)$ with the following property. Let $u \in H^1(B_2, \mathbb{R}^n)$ be such that*

$$\|u^\sharp e - e\|_{L^\infty(B_1)} \leq \frac{1}{2} \quad (\text{A.1})$$

and

$$\det Du \geq 0 \quad \text{a.e.} \quad (\text{A.2})$$

Then there is $A \in SO(n)$ such that

$$\|Du - A\|_{C^\alpha(B_1)} \leq C \|u^\sharp e - e\|_{C^\alpha(B_2)}. \quad (\text{A.3})$$

Remark A.2. *From now on, $|A|$ will denote the Hilbert-Schmidt norm of the matrix $A \in \mathbb{R}^{n \times n}$, namely $|A| = \sqrt{\text{tr } A^\top A}$ induced by the Hilbert-Schmidt scalar product $\langle A, B \rangle := \text{tr } A^\top B$. In particular, using standard coordinates in B_2 and identifying $u^\sharp e$ with the corresponding $n \times n$ matrix, we have $|Du|^2 = \text{tr } u^\sharp e$. Under the assumption (A.1), we thus conclude immediately that $\|Du\|_{L^\infty(B_2)} \leq C(n)$, namely that the map u is Lipschitz.*

Remark A.3. *Note that an assumption like (A.2) is needed because it is easy to give examples of Lipschitz maps whose derivative belong to $O(n)$ almost everywhere but which are not affine. In fact such maps are “abundant” in an appropriate sense: in particular they form a residual set in the space $X := \{u \in \text{Lip}(B_2, \mathbb{R}^n) : u^\sharp e \leq e\}$ endowed with the L^∞ distance, which makes X a compact metric space (cf. [15] for the latter and more subtle results).*

The two authors of the paper asked me while preparing their manuscript whether I could provide a reference or a proof for Theorem A.1. While I felt that this should be a well-known “classical fact”, I was unable to find a reference for it. I therefore suggested a simple argument which reduces (A.3) to an important work of [10] which essentially handles a corresponding “ L^2 -estimate”. The reduction is given in this appendix. It uses some elementary facts from Linear Algebra (which are well known and I just include for the reader’s convenience) and a Morrey-type decay. In what follows we denote by Id the identity matrix in $\mathbb{R}^{n \times n}$.

Lemma A.4. *We have*

$$\text{dist}(A, SO(n)) = \text{dist}(A, O(n)) \quad \text{for all } A \in \mathbb{R}^{n \times n} \text{ with } \det A \geq 0 \quad (\text{A.4})$$

and

$$\text{dist}(A, O(n)) \leq |A^\top A - Id| \quad \forall A \in \mathbb{R}^{n \times n}. \quad (\text{A.5})$$

Proof. In order to show (A.4) fix first an arbitrary matrix A with $\det A \geq 0$. Recalling the polar decomposition of matrices there is a symmetric S and a $O_1 \in SO(n)$ such that $A = O_1 S$. Next, recalling that every symmetric matrix is diagonalizable, there is $O_2 \in SO(n)$ such that $A = O_1 O_2^\top D O_2$ for some diagonal matrix D . Next recall that if O is a diagonal matrix with an even number of entries equal to -1 and the remaining equal to 1 , then $O \in SO(n)$. If one of the diagonal entries of D is zero, we can then assume without loss of generality that all entries of A are nonnegative. Otherwise, if no diagonal entry is 0 , we can assume without loss of generality that they are all positive but at most 1 . Since $\det A > 0$, we can exclude that one diagonal entry of D is negative and the others are all positive. Summarizing the arguments in the two cases, we can assume that all diagonal entries of D are nonnegative. Since $\text{dist}(A, O(n)) = \text{dist}(OA, O(n)) = \text{dist}(AO, O(n))$ and $\text{dist}(A, SO(n)) = \text{dist}(OA, SO(n)) = \text{dist}(AO, SO(n))$ for every $O \in SO(n)$, we conclude that it suffices to prove (A.4) for a diagonal matrix A which has all nonnegative entries. Denote them by $\lambda_1, \dots, \lambda_n$. For any $O \in O(n)$ we can then compute explicitly

$$|A - O|^2 = \sum_i \lambda_i^2 + n - 2 \sum_i \lambda_i O_{ii}.$$

Observe that $-1 \leq O_{ii} \leq 1$ because O is orthogonal. Since $\lambda_i \geq 0$ for every i we then conclude $|A - O|^2 \geq \sum_i \lambda_i^2 + n - 2 \sum_i \lambda_i = |A - Id|^2$. This however shows that $\text{dist}(A, O(n))^2 = |A - Id|^2 = \text{dist}(A, SO(n))^2$.

As for (A.5) fix $A \in \mathbb{R}^{n \times n}$ and let $O \in O(n)$ be such that $\text{dist}(A, O(n)) = |A - O|$. Since both sides of the inequality take the same value for A and $O^{-1}A$, we can assume that $O = Id$. By the minimality condition of Id we must have that $A - Id$ is orthogonal (in the Hilbert-Schmidt scalar product) to the tangent to $O(n)$ at Id , which is the space of skew-symmetric matrices. We therefore conclude that A is symmetric and, again applying the $O(n)$ invariance of the inequality, we can assume w.l.o.g. that it is diagonal. Let $\lambda_i = A_{ii}$ be the diagonal entries and observe that none of them can be negative: if $\lambda_k < 0$ then the matrix B which has $B_{ij} = 0$ for $i \neq j$, $B_{ii} = 1$ for $i \neq k$ and $B_{kk} = -1$ satisfies $B \in O(n)$ and $|A - B| < |A - Id|$. (A.5) is thus reduced to proving

$$\sum_i (\lambda_i - 1)^2 \leq \sum_i (\lambda_i^2 - 1)^2$$

under the assumption that $\lambda_i \geq 0$ for every i . This is equivalent to prove $(x - 1)^2 \leq (x^2 - 1)^2 = (x - 1)^2(x + 1)^2$ for $x \geq 0$, which is obvious. \square

Proof of Theorem A.1. Fix $x \in B_1$ and let $S(x)$ be the unique positive definite symmetric matrix such that $S(x)^2 = u^\sharp e(x)$. Observe that, by (A.1), we have

$$|S(x)| + |S(x)^{-1}| \leq C.$$

Let v be the map $v(y) := u(y)(S(x))^{-1}$ and observe further that

$$\begin{aligned} \text{dist}(Dv(y), SO(n)) &\leq \text{dist}(Dv(y), O(n)) \leq |Dv^\top(y)Dv(y) - Id| \\ &\leq |S(x)^{-1}Du^\top(y)Du(y)S(x)^{-1} - Id| \\ &= |S(x)^{-1}(Du^\top(y)Du(y) - S(y)^2)S(x)^{-1}| \\ &= |S(x)^{-1}(u^\sharp e(y) - u^\sharp e(x))S(x)^{-1}| \\ &\leq 4[u^\sharp e]_{\alpha, B_2}|x - y|^\alpha = 4[u^\sharp e - e]_{\alpha, B_2}|x - y|^\alpha \end{aligned}$$

(where we use the standard notation $[f]_{\alpha, \Omega} := \sup\{\frac{|f(x)-f(y)|}{|x-y|^\alpha} : x, y \in \Omega\}$). In particular, for every $r \leq 1$ we can apply the Friesecke-James-Müller inequality, namely [10, Theorem 3.1], to find a matrix $A(x, r) \in SO(n)$ with the property that

$$\int_{B_r(x)} |Dv - A(x, r)|^2 \leq C \int_{B_r(x)} \text{dist}(Dv, SO(n))^2 \leq C[u^\sharp e - e]_{\alpha, B_2}^2 r^{2\alpha} \quad (\text{A.6})$$

(note that [10, Theorem 3.1] is stated for a general open set U in place of $B_r(x)$, with a constant depending on U ; however an obvious scaling argument shows that the constant is the same for balls of arbitrary radii). Recalling that

$$\min_c \int |f - c|^2 = \int \left| f - \int f \right|^2,$$

we conclude

$$\begin{aligned} \int_{B_r(x)} \left| Du - \int_{B_r(x)} Du \right|^2 &\leq \int_{B_r(x)} |Du - A(x, r)S(x)|^2 \\ &= \int_{B_r(x)} |(Dv - A(x, r))S(x)|^2 \\ &\leq 4 \int_{B_r(x)} |Dv - A(x, r)|^2 \leq C[u^\sharp e - e]_{\alpha, B_2}^2 r^{2\alpha}. \end{aligned}$$

Morrey's estimate then implies that $Du \in C^\alpha(B_1)$ and

$$[Du]_{\alpha, B_1} \leq C[u^\sharp e - e]_{\alpha, B_2}. \quad (\text{A.7})$$

On the other hand, again by the Friesecke-James-Müller estimate, there is $A \in SO(n)$ such that

$$\int_{B_1} |Du - A|^2 \leq C \int_{B_1} \text{dist}(Du, SO(n))^2 \leq C \|u^\sharp e - e\|_{C^0(B_1)}^2. \quad (\text{A.8})$$

(A.7) and (A.8) immediately imply (A.3). \square

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