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ON THE IMPLOSION OF A COMPRESSIBLE FLUID I: Smooth self-similar inviscid profiles

FRANK MERLE, PIERRE RAPHAËL, IGOR RODNIANSKI, AND JEREMIE SZEFTEL

Abstract. In this paper and its sequel, we construct a set of finite energy smooth initial data for which the corresponding solutions to the compressible three dimensional Navier-Stokes and Euler equations implode (with infinite density) at a later time at a point, and completely describe the associated formation of singularity. This paper is concerned with existence of smooth self-similar profiles for the barotropic Euler equations in dimension $d \geq 2$ with decaying density at spatial infinity. The phase portrait of the nonlinear ode governing the equation for spherically symmetric self-similar solutions has been introduced in the pioneering work of Guderley. It allows to construct global profiles of the self-similar problem, which however turn out to be generically non-smooth across the associated acoustic cone. In a suitable range of barotropic laws and for a sequence of *quantized speeds* accumulating to a critical value, we prove the existence of non-generic $\mathcal{C}^\infty$ self-similar solutions with suitable decay at infinity. The $\mathcal{C}^\infty$ regularity is used in a *fundamental way* in our companion paper (part II) in the analysis of the associated linearized operator and leads, in turn, to the construction of finite energy blow up solutions of the compressible Euler and Navier-Stokes equations in dimensions $d = 2, 3$.

Contents

1. Introduction

1.1. **Setting of the problem.** In this paper and its sequel [9], we construct a set (of finite co-dimension) of finite energy smooth initial data for which the corresponding solutions to the compressible three dimensional Navier-Stokes and Euler equations implode (with infinite density) at a point, and completely describe the associated formation of singularity. The first paper will deal exclusively with the isentropic compressible Euler equations in dimension $d \geq 2$, $y \in \mathbb{R}^d$,

$$\text{(Euler)} \quad \left| \begin{array}{l} \partial_t \rho + \nabla \cdot (\rho u) = 0 \\ \rho \partial_t u + \rho u \cdot \nabla u + \nabla p = 0 \\ p = \frac{\gamma-1}{\gamma} \rho^\gamma \\ \rho(t,y) > 0. \end{array} \right. \tag{1.1}$$

Our goal is the construction of a family of *smooth, global in space, self-similar* profiles corresponding to spherically symmetric solutions of (1.1) which arise from smooth initial data and blow up at a chosen point $(T, 0)$. We are specifically interested in solutions which do not exhibit *growth* at spatial infinity. In fact, our solutions will obey

$$\lim_{|y| \to +\infty} (\rho(t,y), u(t,y)) = 0. \tag{1.2}$$

Existence of self-similar solutions with spherical symmetry for the Euler equation with *a continuum* of admissible blow up speeds has been known since the pioneering work of Guderley [4] and Sedov [13]. However, the known solutions are either non-global or non-smooth (in self-similar variables). The former solutions are ubiquitous in the physics literature and describe physical phenomena of denotation, implosion, flame propagation etc. and by design contain a shock, discontinuously connecting a smooth solution to another state. We refer to them as non-global even if, more accurately, they should be called discontinuous, to differentiate them from the second class of solutions – non-smooth ones. The latter appear to be much less known but can be easily constructed from the phase portrait analysis introduced in [4]. The solutions are global in a sense that they connect the behavior (1.2) at infinity with the regular behavior at the center of symmetry. In the process, they have to cross the so called *sonic line* – a point on it representing a backward acoustic cone from the singular point in the original variables. It turns out that *generically* this crossing is non-smooth. The regularity of the associated solution depends on the values of various parameters (dimension, equation of state, scaling) but the standard Lyapunov analysis suggests that it is always finite (although it can be made arbitrarily high for a particular choice of parameters.) In principle, such solutions, together with the finite speed of propagation, immediately lead to the existence of *finite energy* well localized blow up solutions of the Euler equations. Note however that for the reason explained above these solutions do not arise from smooth initial data. But, in principle, such solutions would correspond to a formation of a *strong singularity*, in which the solution concentrates at a point and the density blows up. In the previous rigorous work on compressible Euler and Navier-Stokes equations such phenomenon was not known and the singularity formation was tied to either formation of shocks (in the Euler case) or to qualitative arguments based on

the virial type arguments (for both Euler and Navier-Stokes). For a more detailed discussion see our second paper in the series [9].

The aim of this paper is to show the existence of non-generic $\mathcal{C}^\infty$ self-similar solutions for *quantized values of the blow up speed in the vicinity of a certain critical value*. This is contrary to the Lyapunov analysis which would suggest that it might never be possible. In the companion paper [9], we will use these solutions as the leading order blow up profiles for the compressible Navier-Stokes equation (as well as its inviscid Euler limit) to produce (a finite co-dimension set of) blowing up solutions arising from smooth initial data. The $\mathcal{C}^\infty$ *regularity* of the profile is needed not only for the regularity of initial data but much more importantly, in fact, crucially, for the stability analysis. The stability analysis itself is needed not only to establish existence of a whole finite co-dimensional manifold (in the moduli space of initial data) of blowing up solutions but also for the existence result of even just one such solution. These profiles are merely *approximate* solutions for the Navier-Stokes problem and their stability and thus their $\mathcal{C}^\infty$ *regularity* are essential to ensuring that the approximation holds until the blow up time.

Remark 1.1. In [10], we will also use these $\mathcal{C}^\infty$ self-similar solutions as the leading order profiles to produce blowing up solutions arising from smooth initial data of finite energy for the energy super-critical defocusing nonlinear Schrödinger equations.

Self-similar motion has long been recognized as an important concept in hydrodynamics (see e.g. [13] and the references therein). It could be said that it originated in the dimensional analysis of Reynolds and eventually crystallized as a model for both the simplest and universal behaviors in fluid and gas dynamics. The simplicity stems from the fact that the assumption of self-similar motion (together with spherical or cylindrical symmetry in higher dimensions) reduces the Euler equations to a system of ode's. The universality is supported by the ubiquity of self-similar solutions as well as the belief that self-similar motions act as an attractor for many different phenomena in hydro/gas dynamics. In that respect, two types of self-similar motions have been discussed in the physics literature, [16]. In the first kind, all self-similar parameters are determined from the dimensional analysis, while in the second kind, an undetermined (free) parameter is fixed by the boundary conditions or some other physical requirements on the solution. Self-similar solutions have been extensively used and analyzed in the study of problems involving detonation and implosion waves, combustion, reflected shocks, etc. The current approach has been pioneered by Guderley [4] and has been given numerous treatments, see [3, 13]. In that approach, the study of self-similar (spherically symmetric) solutions of the Euler equations is reduced to the system

$$\left| \begin{array}{l} \frac{dw}{dx} = -\frac{\Delta_1}{\Delta} \\ \frac{d\sigma}{dx} = -\frac{\Delta_2}{\Delta} \end{array} \right. \tag{1.3}$$

where $\Delta, \Delta_{1,2}$ are polynomials in w, σ and the similarity variable $Z = e^x$ is related to the original (t, y) via

$$Z = \frac{|y|}{(T-t)^{\frac{1}{r}}}$$

The $1/r$ parameter is the similarity exponent (free parameter) and is the inverse of what in this paper we call *speed* r. The equations (1.3) are an autonomous system of ode's. Its phase portrait and, specifically, the set where Δ, Δ_1, Δ_2 vanish determine the qualitative properties of all solutions.

1.2. **The self-similar equation.** Let

$$\ell = \frac{2}{\gamma - 1} > 0, \quad r > 1, \tag{1.4}$$

then the self-similar renormalization

$$\left| \begin{array}{l} \rho(t,y) = \left(\frac{\lambda}{\nu}\right)^{\frac{2}{\gamma-1}} \hat{\rho}(\tau, Z) \\ u(t,y) = \frac{\lambda}{\nu}\hat{u}(\tau, Z) \\ Z = \frac{y}{\lambda}, \quad \frac{d\tau}{dt} = \frac{1}{\nu} \\ -\frac{\lambda_\tau}{\lambda} = 1, \quad -\frac{\nu_\tau}{\nu} = r \end{array} \right. \tag{1.5}$$

maps (1.1) on $[0, T)$ onto the global in time τ renormalized flow

$$\left| \begin{array}{l} \partial_\tau \hat{\rho} + \ell(r-1)\hat{\rho} + \Lambda\hat{\rho} + \nabla \cdot (\hat{\rho}\hat{u}) = 0 \\ \partial_\tau \hat{u} + (r-1)\hat{u} + \Lambda\hat{u} + \hat{u} \cdot \nabla\hat{u} + \nabla(\hat{\rho}^{\gamma-1}) = 0 \\ \Lambda = Z \cdot \nabla \end{array} \right. \tag{1.6}$$

A self-similar profile is a stationary solution to (1.6):

$$\left| \begin{array}{l} \ell(r-1)\hat{\rho} + \Lambda\hat{\rho} + \nabla \cdot (\hat{\rho}\hat{u}) = 0 \\ (r-1)\hat{u} + \Lambda\hat{u} + \hat{u} \cdot \nabla\hat{u} + \nabla(\hat{\rho}^{\gamma-1}) = 0 \end{array} \right. \tag{1.7}$$

which produces a blow up solution for (1.1) with the rate of concentration

$$\lambda(t) = \lambda_0 (T-t)^{\frac{1}{r}}, \quad \nu(t) = \nu_0 r (T-t).$$

1.3. **Emden transform and Guderley's phase portrait.** In the pioneering work [4], see also [13, 11], all solutions to (1.7) with spherical symmetry are mapped through the Emden transform

$$\left| \begin{array}{l} (\hat{\rho}(Z))^{\frac{\gamma-1}{2}} = \sqrt{\frac{\ell}{2}} Z \sigma(x) \\ \hat{u}(Z) = -Z w(x) \\ Z = e^x \end{array} \right. \tag{1.8}$$

with w the component along the radial outgoing normal vector, onto the *autonomous* system of nonlinear ode's:

$$\left| \begin{array}{l} (w-1)\frac{dw}{dx} + \ell\sigma\frac{d\sigma}{dx} + (w^2 - rw + \ell\sigma^2) = 0 \\ \frac{\sigma}{\ell}\frac{dw}{dx} + (w-1)\frac{d\sigma}{dx} + \sigma\left[w\left(\frac{d}{\ell}+1\right) - r\right] = 0 \end{array} \right. \Leftrightarrow \left| \begin{array}{l} \Delta\frac{dw}{dx} = -\Delta_1 \\ \Delta\frac{d\sigma}{dx} = -\Delta_2 \end{array} \right. \tag{1.9}$$

with the explicit

$$\left| \begin{array}{l} \Delta = (w-1)^2 - \sigma^2 \\ \Delta_1 = w(w-1)(w-r) - d(w-w_e)\sigma^2 \\ \Delta_2 = \frac{\sigma}{\ell}\left[(\ell+d-1)w^2 - w(\ell+d+\ell r - r) + \ell r - \ell\sigma^2\right] \end{array} \right. \tag{1.10}$$

and

$$w_e = \frac{\ell(r-1)}{d}. \tag{1.11}$$

The above system can be fully analyzed through *the phase portrait in the* (σ, w) *plane*. The shape of the phase portrait is highly dependent on the values of the parameters (r, ℓ, d). Let us introduce the following critical speeds which will play a fundamental role in the forthcoming analysis

$$\left| \begin{array}{l} r^*(d,\ell) = \frac{d+\ell}{\ell+\sqrt{d}}, \\ r_+(d,\ell) = 1 + \frac{d-1}{(1+\sqrt{\ell})^2}, \end{array} \right. \tag{1.12}$$

where an explicit computation[1] shows

$$1 < r^*(d,\ell) \leq r_+(d,\ell) \quad \text{with equality iff} \quad \ell = d. \tag{1.13}$$

Let us denote

$$r_\infty(d,\ell) = \left| \begin{array}{l} r^*(d,\ell) \quad \text{for} \quad 0 < \ell < d \\ r_+(d,\ell) \quad \text{for} \quad \ell > d. \end{array} \right. \tag{1.14}$$

Then, in the range of parameters

$$d \geq 2, \quad \ell > 0, \quad \left| \begin{array}{l} 1 < r < r^*(d,\ell) \quad \text{for} \quad \ell < d \\ r^*(d,\ell) < r < r_+(d,\ell) \quad \text{for} \quad \ell > d, \end{array} \right. \tag{1.15}$$

we will prove that the phase portrait contains five fundamental points $(P_i)_{1\leq i\leq 5}$, see figure 1, figure 2 and section 2, as well as two *sonic lines*.

Sonic lines $w - 1 = \pm\sigma$. This is exactly the set where $\Delta = 0$. In the original variables of the Euler equations, the sonic lines correspond to the equation

$$\frac{|y|}{r(T-t)} = -\left(u \pm c\right), \qquad c = \sqrt{\frac{\partial p}{\partial \rho}} \tag{1.16}$$

where c is the sound speed and the right hand side is a function of $\frac{|y|}{\lambda_0 (T-t)^{\frac{1}{r}}}$. On the other hand, the equation

$$\frac{d|y|}{dt} = -\left(u \pm c\right)$$

describes solutions $(t, |y|(t))$ – *acoustical cones* (radial characteristics) of the metric associated with a solution of the compressible Euler equations. It then follows that for any point Z^* on the sonic line, the set $(T-t, Z^*\lambda_0 (T-t)^{\frac{1}{r}})$ is the backward acoustical cone from the singular point $(0,0)$. On the phase diagram, the points to the right of the sonic line $w - 1 = \sigma$ will correspond to the (t, y) points in the interior of the acoustical (light) cone, while the points to the left of the sonic line lie in the exterior of the cone. Moreover, the absolute value of characteristic speed $u + c$ – particle velocity plus the sound speed – is smaller in the exterior and larger in the interior of the cone.

P_5 point. The point

$$P_5 = \left(\sigma_5 = \frac{r\sqrt{d}}{d+\ell}, w_5 = \frac{\ell r}{d+\ell} \right)$$

is an endpoint of the dynamical system (1.9), i.e.

$$\left| \begin{array}{l} \Delta_1(P_5) = \Delta_2(P_5) = 0 \\ \Delta(P_5) \neq 0 \end{array} \right.$$

and the relative position of P_5 with respect to the sonic line is given by

$$w_5 + \sigma_5 < 1 \Leftrightarrow r < r^*(d,\ell).$$

Points P_2, P_3. Trajectories can only cross the sonic line at the triple points, where

$$\Delta = \Delta_1 = \Delta_2,$$

which are $(0,0), (1,0), (r,0)$ and two other points on the sonic line $w + \sigma = 1$ which we refer to as P_2, P_3 and which exist thank to the constraint (1.15).

P_6 point. The point $P_6 = (w = w_e, \sigma = +\infty)$ is a saddle point at $+\infty$ and corresponds to $x = -\infty$, i.e., $Z = 0$.

[1] see (2.29)

P_4 point. The point $P_4 = (0, 0)$ attracts solutions which vanish near $x \to +\infty$, i.e., $Z \to +\infty$.

A classical analysis of the phase portrait, figure 1, figure 2, yields the following result (see Lemma 3.1 and 3.2 for a proof).

Lemma 1.2 (General structure of spherically symmetric self-similar solutions)**.** *Assume* (1.15)*. Then,*

(1) *Solutions near the origin: there exists a unique trajectory of* (1.9) *which connects P_6 to P_2. This trajectory corresponds to the unique (up to scaling*[2]*) (local) spherically symmetric solution to* (1.7) *which is $\mathcal{C}^\infty$ on the ball $|Z| \in [0, Z_2)$ with* 0 *corresponding to P_6 and Z_2 to P_2.*

(2) *Solutions near infinity: there exists a one parameter family (indexed by the scaling of* (1.7)*) of trajectories which connect P_2 to P_4, and one can fix the scaling so that P_2 is reached at the same $Z = Z_2$. The corresponding curve then corresponds to the spherically symmetric solution to* (1.7) *and is in $\mathcal{C}^\infty$ on $|Z| \in (Z_2, \infty)$ with ∞ corresponding to P_4.*

(3) *Connection at P_2: in both cases, P_2 is reached for a finite value of Z, i.e., $0 < Z_2 < \infty$, and the solutions constructed in (1) and (2) can be glued continuously to each other at $Z = Z_2$.*

In other words, the unique[3], spherically symmetric solution $(\hat\rho(Z), \hat u(Z))$ to (1.7), which is smooth at $Z = 0$, extends to the point Z_2 (which corresponds to P_2 on the phase diagram) where it can be *glued* to any of the one parameter family of solutions $(\hat\rho(Z), \hat u(Z))$ defined for $[Z_2, \infty)$ (which correspond to the trajectories connecting P_2 to P_4). This procedure yields spherically symmetric solutions to (1.7) which are $\mathcal{C}^\infty(\mathbb{R}^d \backslash \{Z = Z_2\})$ and vanish as $Z \to +\infty$.

1.4. **Regularity at P_2 and the reconnection problem.** It remains to understand the regularity at P_2. The above gluing procedure produces a solution with limited $\mathcal{C}^{k(r)}$ regularity[4] at the degenerate point P_2, see Remark 3.7. As it turns out, this limited regularity has a dramatic effect on the spectral structure of the linearized operator for (1.6) close to this self-similar profile, and we do not know how to use these non-$\mathcal{C}^\infty$ solutions to produce finite energy imploding solutions to the viscous perturbations of (1.1) studied in our second paper [9].

On the other hand, P_2 is a regular singular point of (1.9). As a result, there always[5] exists one trajectory which is $\mathcal{C}^\infty$ at P_2. Thus, the problem becomes: *can we find parameters (d, ℓ, r) for which the $P_6 - P_2$ solution is $\mathcal{C}^\infty$ at P_2* and *can be glued to a $P_2 - P_4$ solution which is also $\mathcal{C}^\infty$ at P_2 (see figure 1)?* Such a solution would produce a $\mathcal{C}^\infty$ self-similar profile with vanishing density and velocity as $Z \to +\infty$.

The Lyapunov type (linear) analysis at P_2 predicts that the regularity of solutions passing through is determined by the eigenvalues $\lambda_\pm$ of the Jacobian matrix. Both $\lambda_- < \lambda_+ < 0$, so that P_2 is a stable node. As a result, all curves through P_2 but one will have limited regularity $\mathcal{C}^{\frac{\lambda_-}{\lambda_+}}$. The exceptional solution is $\mathcal{C}^\infty$ but it does

[2]note that (1.7) is invariant under the scaling $(\hat\rho(Z), \hat u(Z)) \to (\lambda^{-\frac{\gamma-1}{2}} \hat\rho(\lambda Z), \lambda^{-1} \hat u(\lambda Z))$ for any $\lambda > 0$. This scaling invariance in Z for (1.7) corresponds to a translation invariance in x at the level of (1.9).

[3]up to scale invariance

[4]with $k(r) \sim \frac{c(d,\ell)}{r_\infty(d,\ell) - r}$ as $r \uparrow r_\infty(d, \ell)$.

[5]at least away from critical integer values, see Lemma 3.6.

not go to P_6 and is thus inadmissible as a global profile. The $\mathcal{C}^{\frac{\lambda_-}{\lambda_+}}$ regularity is *sharply* insufficient for our purposes (linear stability analysis of these solutions as solutions of the Euler equations misses exactly one derivative, see [10]). Instead, we consider the regime where $\lambda_+ \uparrow 0$ (and λ_- stays uniformly bounded away from 0 and $-\infty$) which corresponds to a limiting degeneracy of the phase portrait for the critical speed $r \uparrow r_{\text{👁}}(d,\ell)$ given by (1.14) where a vanishing "👁" structure appears to the left of P_2. We perform a careful *nonlinear* and *global* analysis near the eye configuration to show the existence of a discrete sequence of $\mathcal{C}^\infty$ solutions as $r = r_n \uparrow r_{\text{👁}}$.

1.5. **Statement of the main result.** Our main result in this paper is the existence of such $\mathcal{C}^\infty$ profiles for *quantized values of the speed* r near the critical speed *whose value depends on* ℓ given by (1.4). We introduce the set $\mathcal{O}_d \subset (0,+\infty) \setminus \{d\}$ defined as the union of the interval $(d,+\infty)$ and a subset O_d^* of $(0,d)$ defined by the condition that $\ell \in \mathcal{O}_d$ for $\ell < d$ if the value of the function $\nu_\infty(\ell,d)$ in (F.10) is > 0. In dimensions $d = 2, 3$, $\mathcal{O}_d = (0,+\infty) \setminus \{d\}$ and for $5 \le d \le 9$ the subset $\mathcal{O}_d^*$ is a finite collection of subintervals and it is non-empty, see (F.9).

Theorem 1.3 (Existence of $\mathcal{C}^\infty$ asymptotically vanishing self-similar profiles)**.** *Let $d \ge 2$. Let the critical speed $r_{\text{👁}}(d,\ell)$ be given by* (1.14)*. Then there exists a function*

$$S_\infty(d,\ell) : \mathbb{N}^* \backslash \{1\} \times \mathcal{O}_d \to \mathbb{R}$$

such that for any $\ell \in \mathcal{O}_d$ obeying the condition

$$S_\infty(d,\ell) \neq 0, \tag{1.17}$$

there exists a discrete sequence

$$\left| \begin{array}{l} r_n < r_{\text{👁}}(d,\ell), \quad |r_n - r_{\text{👁}}(d,\ell)| \ll 1 \\ \lim_{n\to+\infty} r_n = r_{\text{👁}}(d,\ell) \end{array} \right.$$

such that (1.7) *with $r = r_n$ admits a global $\mathcal{C}^\infty$ spherically symmetric solution $(\hat\rho(Z), \hat u(Z))$ which terminates at P_4 at spatial infinity (i.e. as $Z \to +\infty$), see figure 1.*

Corollary 1.4. *Under the assumptions of Theorem 1.3, the Euler equations* (1.1) *admit a family of spherically symmetric self-similar solutions, smooth away from the concentration point* $(T,0)$*:*

$$\left| \begin{array}{l} \rho(t,y) = \frac{1}{(T-t)^{\frac{2(r-1)}{r(\gamma-1)}}} \hat\rho(Z) \\ u(t,y) = \frac{1}{(T-t)^{\frac{(r-1)}{r}}} \hat u(Z) \\ Z = \frac{y}{\lambda_0 (T-t)^{\frac{1}{r}}} \end{array} \right.$$

for any $\lambda_0 > 0$ with the asymptotics

$$\rho(t,y) = \frac{\rho_*(1 + o_{|Z|\to+\infty}(1))}{|y|^{\frac{2(r-1)}{r(\gamma-1)}}}, \qquad u(t,y) = \frac{u_*(1 + o_{|Z|\to+\infty}(1))}{|y|^{\frac{(r-1)}{r}}},$$

for some $\rho^* > 0$. In particular, these solutions decay at infinity but do not have finite energy.

The function $S_\infty(d,\ell)$ appears in the asymptotic analysis of the flow near P_2. It can be explicitly expressed as a normally convergent series

$$S_\infty(d,\ell) = \sum_{n=0}^{+\infty} u_n(d,\ell), \quad |u_n(d,\ell)| \le \frac{c_{d,\ell}}{1+n^2}$$

where the series $u_n(d,\ell)$ satisfies an explicit though complicated non linear induction relation. The proof of convergence of the series yields the analyticity of the mapping $\ell \mapsto S_\infty(d,\ell)$ in a suitable open set of the complex plane.

More precisely, the function $S_\infty(d,\ell)$ is in fact composed of two different functions $S^*_\infty(d,\ell)$, defined for the critical value of the speed r_* on the subset $\mathcal{O}^*_d \subset (0,d)$, and $S^+_\infty(d,\ell)$, defined for the critical value of the speed r^+ for $\ell > d$. It turns out that the *explicit* definitions of these functions allow us to extend them holomorphically.

Lemma 1.5 (Holomorphic extension)**.** *The following hold*

(1) *For $d \geq 4$ the function $\ell \to S^*_\infty(d,\ell)$ extends holomorphically to a complex neighborhood of $\mathcal{O}^*_d$.*
(2) *For $d \geq 4$ the function $\ell \to S^+_\infty(d,\ell)$ extends holomorphically to a complex neighborhood of $(d,+\infty)$.*
(3) *For $d = 2,3$ the function $\ell \to S^*_\infty(d,\ell)$ extends holomorphically to a complex neighborhood of $(0,d)$.*
(4) *For $d = 2,3$, the function $\ell \to S^+_\infty(d,\ell)$ extends holomorphically to an open set Ω^+_d of the complex plane, with $\mathbb{R}^*_+\backslash\{d\} \subset \Omega^+_d$, and $(0,d)$ and $(d,+\infty)$ belong to the same connected component of Ω^+_d.*

We do not know how to compute analytically the zeroes of $S_\infty(d,\ell)$. However, for $d = 2,3$, Lemma 1.5 ensures that, unless the function vanishes identically, the possible zeroes are isolated with possible accumulation points $(0,d,+\infty)$ only. For $d \geq 4$ the same conclusions can be reached about each of the intervals in $\mathcal{O}^*_d$ and $(d+\infty)$.

Moreover, since $u_n(d,\ell)$ is given by an explicit induction relation on the coefficients, we can perform an elementary numerical computation of the series. We first give the results in dimension $d = 2$ and $d = 3$ which will be used for the study of the compressible Euler and Navier-Stokes equations in [9], see Appendix I. The assertion in (4) allows us to check the non-vanishing condition for small values of ℓ only.

Numerical claim [Numerical study of the zeroes of $S_\infty(d,\ell)$, case $d = 2,3$] *In the case $d = 2$ and $d = 3$, we have*

$$\left| \begin{array}{l} S^*_\infty(2,\ell) > 0 \quad \textit{for} \quad \ell = 0.1, \\ S^*_\infty(3,\ell) > 0 \quad \textit{for} \quad \ell = 0.1, \end{array} \right. \tag{1.18}$$

and

$$\left| \begin{array}{l} S^+_\infty(2,\ell) > 0 \quad \textit{for} \quad \ell = 0.1, \\ S^+_\infty(3,\ell) > 0 \quad \textit{for} \quad \ell = 0.1. \end{array} \right. \tag{1.19}$$

The case of higher dimensions will be relevant for the study of the energy super critical defocusing (NLS) equation in [10], for which the power nonlinearity involves the real number p given by

$$p = 1 + \frac{4}{\ell}. \tag{1.20}$$

We numerically check the non degeneracy $S_\infty(d,\ell) \neq 0$ for a range of dimensions and integer nonlinearities, see Appendix I.

Numerical claim [Numerical study of the zeroes of $S_\infty(d,\ell)$, case $d \geq 5$]
Let $p(\ell)$ be given by (1.20)*. Then the condition* (1.17) *holds for*

$$(d,p) \in \{(5,9),(6,5),(7,4),(8,3),(9,3)\}. \tag{1.21}$$

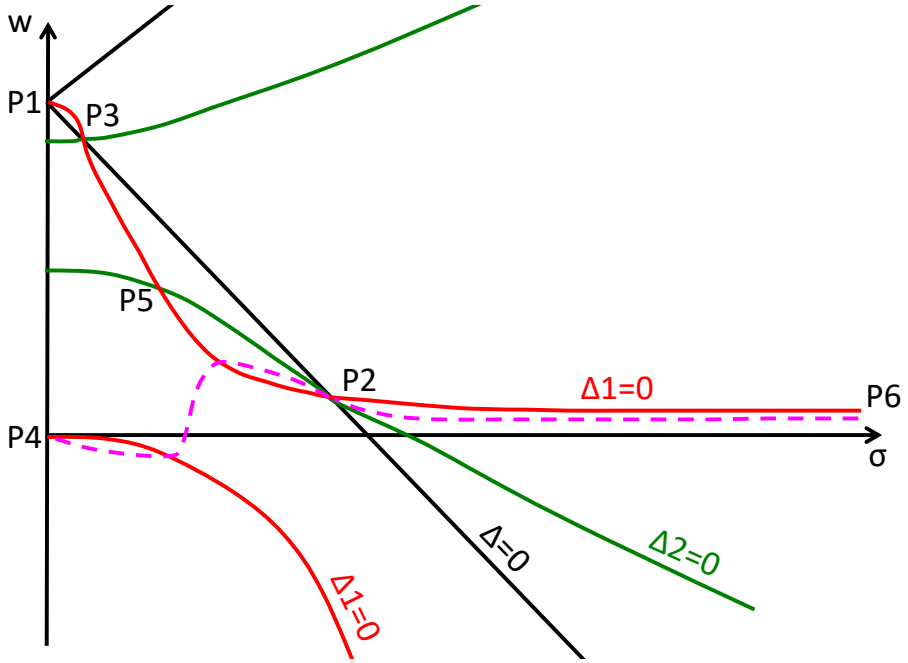

FIGURE 1. Phase portrait in the range $1 < r < r^*(d,\ell)$. Dashed curve is the trajectory of the solution constructed in Theorem 1.3.

Comments on the results.

1. The set $\mathcal{O}_d$. It is defined by the requirement $\nu_\infty(d,\ell) > 0$ with ν_∞ explicitly given by the formulas (F.10) for $\ell < d$ and (F.15) for $\ell > d$. It turns out that $\nu_\infty > 0$ for all $\ell \in \mathbb{R}_+^* \backslash \{d\}$ in dimension $d = 2, 3$. In higher dimensions, $\mathcal{O}_d$ also includes the interval $\ell > d$ and its intersection with $\ell < d$ consists of a finite union of open intervals. The latter intersection is non-empty for $5 \leq d \leq 9$. The condition $\nu_\infty(d,\ell) > 0$ is convenient for our analysis but is clearly not sharp and could be relaxed. Let us emphasize that (1.18), (1.21) merely provide explicit examples of admissible couples (d,ℓ).

2. The asymptotic analysis $r \uparrow r_\circledcirc(d,\ell)$. The proof of Theorem 1.3 involves a careful renormalization of the flow (1.9) for $|r - r_\circledcirc| \ll 1$, $r < r_\circledcirc$. The reason for this choice is the presence of an "eye" of the phase portrait at the left of P_2: as $r \uparrow r_\circledcirc$, the critical point at the left of P_2 which is P_5 for $\ell < d$ and P_3 for $\ell > d$, converges to P_2, and this convergence happens with a suitable ordering of the points P_2, P_3, P_5 which depends on whether $\ell > d$ or $\ell < d$, see Lemma 2.6. The choice of the critical values r_+ or r^* in (1.14) is dictated by this geometry of the phase portrait. The fact that two of the points P_2, P_3, P_5 collide at $r_\circledcirc$ induces a degeneracy of the flow at the critical value which is the starting point of a renomalization process in terms of a small parameter $b = o_{r\uparrow r_\circledcirc}(1)$. We would however like to stress the fact that, given the smallness parameter b, the understanding of the $\mathcal{C}^\infty$ regularity of the solution involves formally the expansion of the solution *to the order* $\frac{1}{b}$ which is too large for a WKB type analysis. Instead, we rely on a holomorphic expansion of the solution at P_2 to extract a formal limit involving the limiting series $S_\infty(d,\ell)$. The condition $S_\infty(d,\ell) \neq 0$ turns out to be sufficient to prove the existence of a $\mathcal{C}^\infty$ profile, but we

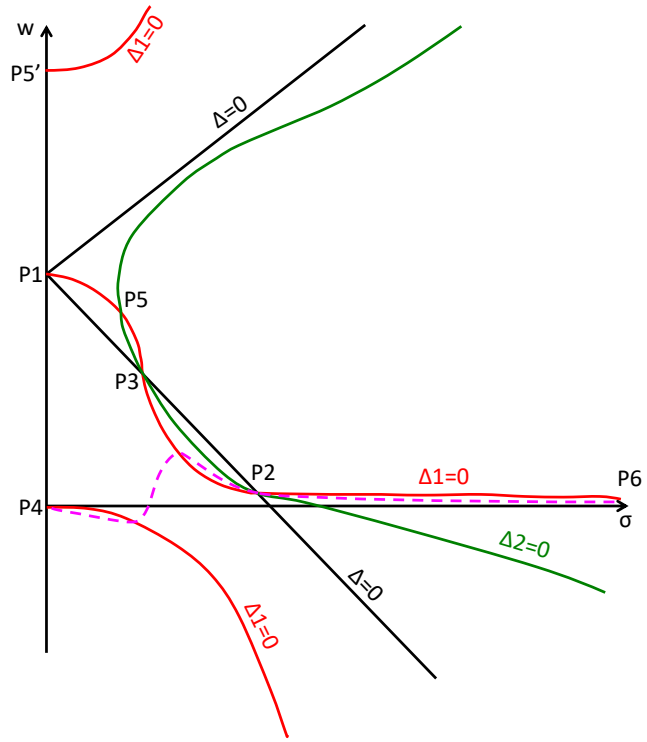

FIGURE 2. Phase portrait in the range $r^*(d,\ell) < r < r_+(d,\ell)$, $\ell > d$. Dashed curve is the trajectory of the solution constructed in Theorem 1.3.

do not know if it is also necessary, as this would require pushing the asymptotic expansions to the next order. We refer to section 4.1 for a detailed strategy of the proof of Theorem 1.3 and the description of the role played by the limiting series $S_\infty(d,\ell)$.

3. The sequence r_n. There is an abundant literature devoted to the existence of self-similar solutions in various semilinear problems. Let us for example consider the semilinear focusing wave equation

$$\partial_t^2 u = \Delta u + u^p.$$

Then, in dimension $d \geq 3$, for a suitable range of p, the equation admits a discrete family of exact selfsimilar solutions of the form

$$u_n(t,x) = \frac{1}{(T-t)^{\frac{2}{p-1}}} U_n(y), \qquad y = \frac{x}{T-t},$$

where $U_n(y)$ is $\mathcal{C}^\infty$ in y and decays like $U_n(y) \sim |y|^{-\frac{2}{p-1}}$ as $|y| \to +\infty$, see [1][6]. Let us emphasize that the blow up speed in this case is dictated by the scaling of the equation, and the profiles U_n are automatically $\mathcal{C}^\infty$ across the light cone. The quasilinear situation of Theorem 1.3 is however different: the Euler equations possess a 2-parameter family of scalings, and, as a result, the blow up speed r of the self-similar profile (1.5) is a free parameter. The choice of the sequence r_n in Theorem 1.3 is related to the requirement that the profile is $\mathcal{C}^\infty$ through the acoustic light cone which is no longer automatic. Theorem 1.3 thus provides a family of

[6]see also [5] for an extension to the defocusing case, and [2, 7, 15] for the case of the semilinear heat equation.

admissible[7] blow up profiles which *accumulate* at the critical speed $r_\infty(d,\ell)$. This is, to our knowledge, a completely new phenomenon.

4. The case $\ell = d$. The case $d = \ell = 3$ is $\gamma = \frac{5}{3}$, i.e., the law for the monoatomic gas, and is always degenerate since for $d = \ell$:

$$r = r^*(d,d) = r_+(d,d) \Rightarrow P_2 = P_3 = P_5$$

and the phase portrait has a triple point degeneracy which requires a separate treatment.

5. Decay of self-similar solutions. As stated in Corollary 1.4, the constructed self-similar solutions $(\rho(t,y), u(t,y))$ decay as $y \to \infty$ but do not have finite energy. Nonetheless, this decay is sufficient for us to construct, in [9], solutions to the Euler and Navier-Stokes equations which arise from smooth, well localized initial data (in particular, of finite energy) and form a singularity in finite time via a profile given in this paper. We also are able to utilize these profiles to produce blowing up solutions to the super-critical defocusing nonlinear Schrödinger equations (in $d \geq 5$) which also arise from smooth, well localized (in particular, finite energy) initial data, [10].

1.6. **Further qualitative properties of the solution.** We emphasize again that the main motivation of Theorem 1.3 is the *dynamical study* of (1.6), central for the construction of well localized smooth blow up solutions in the second paper [9], as well as in [10]. The study of the linearized operator close to the solution given by Theorem 1.3 and performed in [10] requires, in an essential way, the $\mathcal{C}^\infty$ regularity at P_2 as well as suitable positivity properties which we now collect. These properties are responsible for the coercivity of the linearized operator, which is why we refer to them as *repulsivity.*

Positivity inside the light cone. We first claim the following positivity property inside the light cone $\sigma > \sigma(P_2)$ $(Z < Z_2)$ for the $P_6 - P_2$ trajectory.

Lemma 1.6 (Repulsivity inside the light cone)**.** *Let $d \geq 2$ then there exists $0 < \varepsilon \ll 1$ such that for all (ℓ, r) in the range*

$$\left|\begin{array}{l} d \geq 2 \\ 0 < \ell < d \\ r^*(d,\ell) - \varepsilon(d,\ell) < r < r^*(d,\ell) \end{array}\right. \quad or \quad \left|\begin{array}{l} d = 2,3 \\ \ell > d \\ r_+(d,\ell) - \varepsilon(d,\ell) < r <_+ (d,\ell) \end{array}\right. \tag{1.22}$$

there exists $c_r > 0$ such that the $P_6 - P_2$ trajectory $(\sigma(x), w(x))$ given by Lemma 1.2 satisfies the following bound. Let

$$F = \sigma + \frac{d\sigma}{dx}, \tag{1.23}$$

then

$$\forall \sigma \geq \sigma(P_2), \quad \left|\begin{array}{l} \left(1 - w - \frac{dw}{dx}\right)^2 - F^2 \geq c_r \\ 1 - w - \frac{dw}{dx} - \frac{(1-w)F}{\sigma} \geq c_r \\ 1 - w - \frac{dw}{dx} \geq c_r \end{array}\right. \tag{1.24}$$

Taking $r = r_n$ for n large enough ensures that (1.24) holds for the solution of Theorem 1.3. Property (1.24) is sharp: the constant $c_r \to 0$ as $r \uparrow r_\infty$. Note that the fact that the repulsivity property (1.24), which is at the heart of the stability

[7] selected by the fact that their $\mathcal{C}^\infty$ regularity and the decay at $+\infty$ guarantee that the linearized operator displays dynamical stability properties, [10].

of the linearized flow, can be proved in the full range $\ell > 0$ is surprising, and will require a substantial amount of algebra.

Positivity outside the light cone. We claim another positivity property in the exterior of the light cone ($Z > Z_2$) in a range of parameters which includes the solutions of Theorem 1.3.

Lemma 1.7 (Repulsivity outside the light cone)**.** *Let*

$$\left| \begin{array}{l} d = 3, \quad \ell > \sqrt{3} \quad \textit{satisfying} \quad (1.17), \\ \textit{or} \quad (d, \ell) \quad \textit{satisfies} \quad (1.21). \end{array} \right. \tag{1.25}$$

Let r_n be given by Theorem 1.3 and (w, σ) be the corresponding $\mathcal{C}^\infty$ integral curve. Then for all n large enough, there exists $c_{r_n} > 0$ such that

$$\forall 0 < \sigma \leq \sigma(P_2), \quad \left| \begin{array}{l} \left(1 - w - \frac{dw}{dx}\right)^2 - F^2 > c_{r_n}, \\ 1 - w - \frac{dw}{dx} > c_{r_n}. \end{array} \right. \tag{1.26}$$

1.7. **Organization of the paper.** In section 2, we establish the main geometric properties of the phase diagram 1, in particular related to the location of the roots of the polynomials $\Delta, \Delta_1, \Delta_2$ in the suitable range of parameters. In section 3, we recall the main dynamical properties of solutions to (1.9), and in particular discuss the existence of the $P_6 - P_2$ trajectory, and the regularity of integral curves passing through P_2. Both sections 2 and 3 are classical and we recall most details for the convenience of the reader. In section 4, we start the semi classical analysis $r \uparrow r_\circledcirc(d, \ell)$, provide an overview of the strategy of the proof of Theorem 1.3 in section 4.1 and complete the proof in sections 4, 5, 6, 7. The positivity properties of Lemma 1.6 and Lemma 1.7 are proved in section 8 and section 9, respectively. The holomorphic dependence on ℓ of $S_\infty(d, \ell)$ is proved in Appendix E, and numerical results for the computation of $S_\infty(d, \ell)$ are collected in Appendix I.

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2. The geometry of the phase portrait

The aim of this section is to start the analysis of the nonlinear ODE system (1.9)

$$\left| \begin{array}{l} (w-1)\frac{dw}{dx} + \ell\sigma\frac{d\sigma}{dx} + (w^2 - rw + \ell\sigma^2) = 0 \\ \frac{\sigma}{\ell}\frac{dw}{dx} + (w-1)\frac{d\sigma}{dx} + \sigma\left[w\left(\frac{d}{\ell} + 1\right) - r\right] = 0 \end{array} \right. \Leftrightarrow \left| \begin{array}{l} \Delta\frac{dw}{dx} = -\Delta_1 \\ \Delta\frac{d\sigma}{dx} = -\Delta_2 \end{array} \right. \tag{2.1}$$

by examining the roots of the polynomials $\Delta, \Delta_1, \Delta_2$ given by (1.10)

$$\left| \begin{array}{l} \Delta = (w-1)^2 - \sigma^2 \\ \Delta_1 = w(w-1)(w-r) - d(w - w_e)\sigma^2 \\ \Delta_2 = \frac{\sigma}{\ell}\left[(\ell + d - 1)w^2 - w(\ell + d + \ell r - r) + \ell r - \ell\sigma^2\right] \end{array} \right. \tag{2.2}$$

$$w_e = \frac{\ell(r-1)}{d}. \tag{2.3}$$

Their location is heavily dependent on the values of the parameters (d, ℓ, r). Let

$$w_\ell^- = w_e(r_+) = \frac{\ell(r_+ - 1)}{d} = \frac{\ell(d-1)}{d(1 + \sqrt{\ell})^2}.$$

We observe

$$w_\ell^- < 1$$

and from the beginning we restrict to the case

$$\left| \begin{array}{l} d \geq 2 \\ 0 < w_e < w_\ell^- \Leftrightarrow 1 < r < r_+(d,\ell) \end{array} \right. \tag{2.4}$$

where we recall (1.11), (1.12). Note also that $r^* \leq r_+$, see (1.13), so that the above condition is not a restriction in view of $r < r_{\text{ᔕ}}$.

2.1. **Roots of $\Delta, \Delta_1\Delta_2$.** Δ has been normalized to vanish on the sonic lines

$$\{\Delta = 0\} = \{w = 1+\sigma\} \cup \{w = 1-\sigma\}$$

which are independent of the parameters. We now study the roots of Δ_2.

Lemma 2.1 (Roots of Δ_2)**.** *Assume* (2.4)*. There exists $\sigma_2^{(0)}(d,\ell) \in [0,+\infty)$ such that the roots of Δ_2 in the range $\sigma \geq 0$ are given by*

$$\left| \begin{array}{l} w_2^\pm(\sigma) = \frac{1}{2(\ell+d-1)} \left[2\ell + d - 1 - \frac{dw_e(1-\ell)}{\ell} \pm \sqrt{I(\sigma)} \right] \\ \sigma \geq \sigma_2^{(0)} \end{array} \right. \tag{2.5}$$

where

$$\left| \begin{array}{rcl} J(w_e) & = & d^2 \left(\frac{1-\ell}{\ell} \right)^2 w_e^2 - \frac{2d(d-1)(\ell+1)}{\ell} w_e + (d-1)^2, \\ I(\sigma) & = & J(w_e) - 4dw_e + 4\ell(\ell+d-1)\sigma^2. \end{array} \right. \tag{2.6}$$

Moreover,

$$\forall \sigma > \sigma_2^{(0)}, \quad (w_2^-)'(\sigma) < 0, \quad (w_2^+)'(\sigma) > 0. \tag{2.7}$$

Remark 2.2. The value $\sigma_2^{(0)}$ is explicitly given by (2.8) if $J(w_e) - 4dw_e < 0$ and $\sigma_2^{(0)} = 0$ otherwise. In the range of parameters (2.4), both cases $\sigma_2^{(0)} = 0$ and $\sigma_2^{(0)} > 0$ are possible which means that the parabola defining the set of zeroes of Δ_2 may or may not touch the line $\sigma = 0$ in figure 1. This will play no role in our qualitative study of the flow.

Proof of Lemma 2.1. We solve $\Delta_2 = 0$ for $\sigma \neq 0$ which is

$$\begin{aligned} 0 &= (\ell+d-1)w^2 - (\ell+d+\ell r - r)w + \ell r - \ell\sigma^2 \\ &= (\ell+d-1)w^2 - \left(\ell + d + \ell\left(1 + \frac{dw_e}{\ell}\right) - 1 - \frac{dw_e}{\ell} \right) w + \ell\left(1+\frac{dw_e}{\ell}\right) - \ell\sigma^2 \\ &= (\ell+d-1)w^2 - \left(2\ell + d - 1 + dw_e\left(1 - \frac{1}{\ell}\right) \right) w + (\ell + dw_e) - \ell\sigma^2. \end{aligned}$$

The discriminant is given by

$$\begin{aligned} I(\sigma) &= \left(2\ell + d - 1 + dw_e\left(1-\frac{1}{\ell}\right) \right)^2 - 4\left[(\ell + dw_e) - \ell\sigma^2 \right](\ell+d-1) \\ &= 4\ell(\ell+d-1)\sigma^2 + \frac{d^2(1-\ell)^2}{\ell^2} w_e^2 - \frac{2d}{\ell}(\ell d + \ell + d - 1) w_e + (d-1)^2 \\ &= J(w_e) - 4dw_e + 4\ell(\ell+d-1)\sigma^2 \end{aligned}$$

which justifies the formula for $w_2^\pm(\sigma)$. We now study the sign of $J(w_e) - 4dw_e$. The equation $J(w_e) - 4dw_e = 0$ has real roots given by

$$w_{\ell,\pm}^{**} = \frac{\ell}{d(1-\ell)^2} \left[\ell d + \ell + d - 1 \pm 2\sqrt{\ell d(d+\ell-1)} \right].$$

Hence $J(w_e) - 4dw_e < 0$ for $w_{\ell,-}^{**} < w_e < w_{\ell,+}^{**}$ in which case $I(\sigma) \geq 0$ for

$$\sigma \geq \sigma_2^{(0)} = \sqrt{\frac{4dw_e - J(w_e)}{4\ell(\ell+d-1)}}. \tag{2.8}$$

Next $J(w_e) - 4dw_e \geq 0$ for $w_e \geq w_{\ell,+}^{**}$ or $w_e \leq w_{\ell,-}^{**}$ in which case $I(\sigma) \geq 0$ for all $\sigma \geq \sigma_2^{(0)} = 0$. The monotonicity (2.7) follows directly from the fact that we have $I'(\sigma) > 0$ for all $\sigma > 0$. □

Lemma 2.3 (Roots of Δ_1)**.** *Assume* (2.4). *For all* $\sigma \geq 0$, *the equation* $\Delta_1(w, \sigma) = 0$ *has exactly three distinct root branches* $w_1(\sigma) < w_2(\sigma) < w_3(\sigma)$ *which satisfy the following:*
relative positions: $\forall \sigma \geq 0$,

$$w_1(\sigma) \leq 0 < w_e < w_2(\sigma) \leq 1 < r \leq w_3(\sigma). \tag{2.9}$$

monotonicity: $\forall \sigma > 0$,

$$w_1'(\sigma) < 0, \quad w_2'(\sigma) < 0, \quad w_3'(\sigma) > 0. \tag{2.10}$$

asymptotics:

$$\left| \begin{array}{l} w_1(\sigma) = -\dfrac{dw_e}{r}\sigma^2 + O(\sigma^3), \\ w_2(\sigma) = 1 - \dfrac{d(1-w_e)\sigma^2}{r-1} + O(\sigma^3), \quad as \ \ \sigma \to 0 \\ w_3(\sigma) = r + \dfrac{d(r-w_e)}{r(r-1)}\sigma^2 + O(\sigma^3), \end{array} \right. \tag{2.11}$$

and

$$\left| \begin{array}{l} w_1(\sigma) = -\sqrt{d}\sigma + O(1), \\ w_2(\sigma) = w_e + O(\sigma^{-2}), \quad as \ \ \sigma \to +\infty. \\ w_3(\sigma) = \sqrt{d}\sigma + O(1) \end{array} \right. \tag{2.12}$$

Proof of Lemma 2.3. At $\sigma = 0$, we have the three obvious roots

$$w_1(0) = 0, \quad w_2(0) = 1, \quad w_3(0) = r$$

where we recall that (2.4), (1.11) imply

$$r > 1. \tag{2.13}$$

Also, for each fixed $\sigma > 0$, we have

$$\left| \begin{array}{l} \lim_{w\to+\infty} \Delta_1(w, \sigma) = +\infty, \quad \lim_{w\to-\infty} \Delta_1(w, \sigma) = -\infty \\ \Delta_1(0, \sigma) = dw_e\sigma^2 > 0, \quad \Delta_1(w_e, \sigma) = w_e(w_e - 1)(w_e - r) > 0 \\ \Delta_1(1, \sigma) = -d(1-w_e)\sigma^2 < 0, \quad \Delta_1(r, \sigma) = -d(r-w_e)\sigma^2 < 0, \end{array} \right.$$

where we used from (1.11), (2.13) that $w_e < 1 < r$. Thus, for all $\sigma \geq 0$, $\Delta_1(w, \sigma)$ has exactly three distinct simple roots which then satisfy (2.9) and

$$\partial_w \Delta_1(w_1(\sigma), \sigma) > 0, \quad \partial_w \Delta_1(w_2(\sigma), \sigma) < 0, \quad \partial_w \Delta_1(w_3(\sigma), \sigma) > 0. \tag{2.14}$$

We may now apply the implicit function theorem and conclude that the roots $w_1(\sigma)$, $w_2(\sigma)$, $w_3(\sigma)$ are smooth function of σ for $\sigma \geq 0$. Furthermore, we have the formula for $j = 1, 2, 3$

$$w_j'(\sigma) = -\frac{\partial_\sigma \Delta_1(w_j(\sigma), \sigma)}{\partial_w \Delta_1(w_j(\sigma), \sigma)} = \frac{2d\sigma(w_j(\sigma) - w_e)}{\partial_w \Delta_1(w_j(\sigma), \sigma)}, \quad \sigma \geq 0 \tag{2.15}$$

which together with (2.14) and the location of the roots (2.9) ensures the monotonicity (2.10).
We now compute the limiting asymptotics. Near $\sigma = 0$, we compute from (2.15):

$$w_j'(0) = 0, \quad w_j''(0) = \frac{2d(w_j(0) - w_e)}{\partial_w \Delta_1(w_j(0), 0)}, \quad j = 1, 2, 3,$$

which together with the above explicit values $w_j(0)$, $j = 1, 2, 3$, and the fact that

$$\partial_w \Delta_1(w, 0) = w(w-1) + w(w-r) + (w-1)(w-r)$$

yields the expansion (2.11) as $\sigma \to 0$. To compute the expansion of the three roots near $+\infty$, we notice that

$$\begin{aligned} \Delta_1(\pm\sqrt{d}\sigma + c_1, \sigma) &= d(2c_1 - r - 1 + w_e)\sigma^2 + O(\sigma), \\ \Delta_1(w_e + c_2\sigma^{-2}, \sigma) &= w_e(w_e - 1)(w_e - r) - dc_2 + O(\sigma^{-2}), \end{aligned}$$

and an appropriate choice of the constants c_1, c_2 yields (2.12) from the mean value theorem. □

2.2. **Double roots.** We now discuss the double roots $\Delta_1 = \Delta_2 = 0$ which play a *fundamental* role in the study of (1.9).

Lemma 2.4 (Double roots)**.** *Assume* (2.4)*. The solutions to* $\Delta_1 = \Delta_2 = 0$ *are:*

$$\left| \begin{array}{l} P_1 = (0, 1), \quad P_2 = (1 - w_-, w_-) \quad P_3 = (1 - w_+, w_+), \\ P_4 = (0, 0), \quad P_5 = (\sigma_5, w_5), \quad P_5' = (0, r), \end{array} \right. \tag{2.16}$$

where the points are defined as follows:
P_5 point.

$$P_5 = \left(w_5 = \frac{\ell r}{d + \ell}, \sigma_5 = \frac{r\sqrt{d}}{d + \ell} \right) \tag{2.17}$$

P_2, P_3 points. Let $J(w_e)$ given (2.6)*, then*

$$w_\pm = \frac{1}{2(d-1)} \left(dw_e + d - 1 - \frac{dw_e}{\ell} \pm \sqrt{J(w_e)} \right) \tag{2.18}$$

with

$$w_e < w_\pm < 1. \tag{2.19}$$

Location. P_2, P_3, P_5 are located on the curve of the middle root $(\sigma, w_2(\sigma))$ of Δ_1. Moreover, P_2, P_5 are on the curve of the lower root w_2^- of Δ_2.

Position of the middle root. Let $w_e < w_\ell^-$ and $w_2(\sigma)$ be the middle root of Δ_1, then the relative position of the middle root with respect to the sonic line is:

$$\sigma + w_2(\sigma) \left| \begin{array}{ll} > 1 & \text{for } \ 0 < \sigma < \sigma(P_3) \\ < 1 & \text{for } \ \sigma(P_3) < \sigma < \sigma(P_2) \\ > 1 & \text{for } \ \sigma > \sigma(P_2). \end{array} \right. \tag{2.20}$$

Proof of Lemma 2.4. It relies on the factorization

$$\left| \begin{array}{l} \Delta = a_1 b_2 - b_1 a_2 \\ \Delta_1 = -b_1 d_2 + b_2 d_1 \\ \Delta_2 = d_2 a_1 - d_1 a_2. \end{array} \right. \tag{2.21}$$

with

$$\left| \begin{array}{l} a_1 = w - 1, \quad b_1 = \ell\sigma, \quad d_1 = w^2 - rw + \ell\sigma^2, \\ a_2 = \frac{\sigma}{\ell}, \quad b_2 = w - 1, \quad d_2 = \sigma \left[\left(1 + \frac{d}{\ell}\right) w - r \right]. \end{array} \right. \tag{2.22}$$

step 1 Computation of the triple points. From (2.21):

$$(\Delta_1 = \Delta_2 = 0) \Leftrightarrow (b_1 d_2 = b_2 d_1 \ \text{ and } \ d_2 a_1 = d_1 a_2)\,. \tag{2.23}$$

Also, by definition, $a_1 = b_2$, and we argue below according to the cases $a_1 = b_2 = 0$ and $a_1 = b_2 \neq 0$.

Case $a_1 = b_2 = 0$, i.e. $w = 1$. If $\sigma = 0$, we have the point P_1. If $\sigma \neq 0$, $b_1 \neq 0$ and hence from (2.22) (2.23)

$$0 = d_2 = \sigma\left(1 + \frac{d}{\ell} - r\right) = \frac{\sigma d}{\ell}(1 - w_e) \neq 0,$$

a contradiction.

Case $a_1 = b_2 \neq 0$. If $\sigma = 0$, then $d_2 = b_1 = 0$ and hence $0 = d_1 = w(w - r)$, and hence the points $P_4 = (0, 0)$ and $P_5' = (0, r)$. If $\sigma \neq 0$, then $a_1, b_2, b_1, a_2 \neq 0$. If $d_2 = 0$, then $d_1 = 0$ and hence

$$\left|\begin{array}{l} w = w_5 = \frac{r}{1+\frac{d}{\ell}}, \\ \sigma_5^2 = \frac{w_5(r-w_5)}{\ell} = \frac{dr^2}{(d+\ell)^2}, \quad \sigma_5 = \frac{r\sqrt{d}}{d+\ell}. \end{array}\right.$$

We observe

$$w_5 < 1 \Leftrightarrow w_e < 1$$

and hence (2.4) and (2.9) implies that P_5 lies on the middle root $w_2(\sigma)$ of Δ_1.
If $d_2 \neq 0$ then

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{d_1}{d_2}, \quad \Delta_1 = \Delta_2 = \Delta = 0.$$

Hence $w = 1 + \sigma$ or $w + \sigma = 1$ which we consider separately.

Points on the lower sonic line, i.e., $w = 1 - \sigma$. First, note that it suffices to consider the solutions to $\Delta_2(\sigma, 1 - \sigma) = 0$. Indeed, since $a_1 = b_2 \neq 0$ and $d_2 \neq 0$, then, since we have $\Delta_2 = \Delta = 0$, we infer

$$\frac{b_1}{b_2} = \frac{a_1}{a_2} = \frac{d_1}{d_2}.$$

It then follows that $\Delta_1 = 0$. Thus, we now consider the solutions to $\Delta_2(\sigma, 1-\sigma) = 0$. Since $\sigma \neq 0$:

$$\begin{aligned} 0 = P(w) &= (\ell + d - 1)w^2 - (\ell + d + \ell r - r)w + \ell r - \ell(1 - w)^2 \\ &= (\ell + d - 1)w^2 - (\ell + d + \ell r - r)w + \ell r - \ell(w^2 - 2w + 1) \\ &= (d - 1)w^2 - (\ell(r - 1) + d - r)w + \ell(r - 1) \\ &= (d - 1)w^2 - \left(dw_e + d - 1 - \frac{dw_e}{\ell}\right)w + dw_e. \end{aligned} \tag{2.24}$$

The roots of P are real iff

$$\begin{aligned} J &= \left(dw_e + d - 1 - \frac{dw_e}{\ell}\right)^2 - 4d(d - 1)w_e \\ &= d^2\left(\frac{1 - \ell}{\ell}\right)^2 w_e^2 - \frac{2d(d - 1)(\ell + 1)}{\ell} w_e + (d - 1)^2 \geq 0. \end{aligned} \tag{2.25}$$

The discriminant of J, as a second order polynomial in w_e, is > 0, and the roots of J are given by

$$w_\ell^\pm = \frac{\ell^2}{2d^2(1-\ell)^2}\left[\frac{2d(d-1)(\ell+1)}{\ell} \pm \frac{4d(d-1)}{\sqrt{\ell}}\right] = \frac{\ell(d-1)}{d(1-\ell)^2}\left[\ell+1\pm 2\sqrt{\ell}\right].$$

Hence

$$J \geq 0 \iff (w_e \leq w_\ell^- \text{ or } w_e \geq w_\ell^+). \tag{2.26}$$

If $J \geq 0$, the roots of P are given by

$$w_\pm = \frac{1}{2(d-1)}\left(dw_e + d - 1 - \frac{dw_e}{\ell} \pm \sqrt{J}\right).$$

We compute from (2.24):

$$\left| \begin{array}{l} P(1) = r-1 > 0 \\ P'(1) = d-1-(\ell-1)(r-1) \end{array} \right.$$

and

$$\left| \begin{array}{l} P(0) = \ell(r-1) > 0 \\ P'(0) = -\left[d-1+(\ell-1)(r-1)\right] \end{array} \right.$$

Since P is a second order polynomial with non negative second order term, we conclude

$$0 < w_\pm < 1 \Leftrightarrow \left| \begin{array}{l} P'(0) < 0 \\ P'(1) > 0 \end{array} \right. \Leftrightarrow \left| \begin{array}{l} d-1+(\ell-1)(r-1) > 0 \\ d-1-(\ell-1)(r-1) > 0 \end{array} \right. \tag{2.27}$$

This holds for $\ell = 1$, and we now distinguish the following two cases:
<u>case $\ell < 1$</u>. We want

$$d-1-(1-\ell)(r-1) > 0 \Leftrightarrow r < 1 + \frac{d-1}{1-\ell}$$

and we observe from (2.4)

$$r < r_+ = 1 + \frac{d-1}{(1+\sqrt{\ell})^2} < 1 + \frac{d-1}{1-\ell}.$$

<u>case $\ell > 1$</u>. We want

$$d-1-(\ell-1)(r-1) > 0 \Leftrightarrow r < 1 + \frac{d-1}{\ell-1}$$

and we observe from (2.4)

$$r < r_+ = 1 + \frac{d-1}{(1+\sqrt{\ell})^2} < 1 + \frac{d-1}{\ell-1}.$$

This concludes the proof of the inequality

$$0 < w_\pm < 1.$$

This implies from (2.9) that P_2, P_3 lie on the middle root $w_2(\sigma)$ of Δ_1 which is a non increasing function of σ from (2.10), and hence necessarily

$$w_e < w_\pm.$$

<u>Points on the upper sonic line, i.e., $w = 1+\sigma$.</u> We consider the solutions to $\Delta_2(\sigma, 1+\sigma) = 0$ or, equivalently,

$$Q(\sigma) = (\ell+d-1)(1+\sigma)^2 - (\ell+d+\ell r-r)(1+\sigma) + \ell r - \ell\sigma^2 = 0.$$

We compute

$$Q(0) = d+r-1, \quad Q'(0) = d+r-\ell(r-1) > r$$

from (2.4), and hence the second order polynomial Q is > 0 for $\sigma > 0$ and there is no intersection point on the upper sonic line.

step 2 Location of P_2, P_3, P_5. We have established that P_2, P_3, P_5 lie on the middle root of Δ_1. Next, note that $w = 1 - \sigma$ is decreasing and $w_2^+(\sigma)$ in increasing by (2.7), so they can intersect at most once, and hence at least one point among P_2, P_3 must be on the root $(\sigma, w_2^-(\sigma))$ of Δ_2. Since $w_- < w_+$, we infer that P_2 is on the root $(\sigma, w_2^-(\sigma))$ of Δ_2.

step 3 Proof of (2.20). Assume now $w_e < w_\ell^-$, then $J(w_e) > 0$ ensures $w_- < w_+$ and, since P_2, P_3 lie on the sonic line $w + \sigma = 1$,

$$\sigma(P_3) < \sigma(P_2).$$

Hence $0 < \sigma(P_3) < \sigma(P_2)$ are three distinct roots of

$$R(\sigma) = \Delta_1(1 - \sigma, \sigma) = (1 - \sigma)(-\sigma)(1 - \sigma - r) - d(1 - \sigma - w_e)\sigma^2$$

which is an order three polynomial, and hence these are the only roots which are simple. Since $w_2(\sigma)$ lies above the sonic line $\sigma + w = 1$ near $\sigma = 0$ and near $\sigma = +\infty$ from (2.11), the ordering (2.20) follows □

2.3. **Relative positions of P_2, P_3, P_5.** We now discuss a very important property for our forthcoming analysis regarding the relative position of P_2 and P_5. We recall the definition (1.12), (1.14) of the critical speed exponents. Let us start with comparing these values.

Lemma 2.5 (Comparison of r_+ and r^*)**.** *Let $d \geq 2$, then*

$$\left| \begin{array}{l} w_\ell^- = w_e(r_+) < 1 \\ r^*(d, \ell) < r_+(d, \ell). \end{array} \right.$$

and

$$w_e < w_\ell^- \Leftrightarrow r \leq r_+(d, \ell). \tag{2.28}$$

Proof of Lemma 2.5. We compute from (2.19):

$$w_\ell^- = \frac{\ell(d-1)}{d(1-\ell)^2}\left[\ell + 1 \pm 2\sqrt{\ell}\right] = \frac{\ell(d-1)}{d(1+\sqrt{\ell})^2} < 1$$

and hence from (1.12):

$$w_e < w_\ell^- \Leftrightarrow \frac{\ell(r-1)}{d} < \frac{\ell(d-1)}{d(1+\sqrt{\ell})^2} \Leftrightarrow r \leq r_+(d, \ell).$$

Then

$$\begin{aligned} r_+(d, \ell) - r^*(d, \ell) &= 1 + \frac{d-1}{(1+\sqrt{\ell})^2} - \frac{d+\ell}{\ell + \sqrt{d}} \\ &= \frac{(d-1)(\ell + \sqrt{d}) - (d - \sqrt{d})(1 + 2\sqrt{\ell} + \ell)}{(\ell + \sqrt{d})(1 + \sqrt{\ell})^2} \\ &= (\sqrt{d} - 1)\frac{(\sqrt{\ell} - \sqrt{d})^2}{(\ell + \sqrt{d})(1 + \sqrt{\ell})^2} > 0. \end{aligned} \tag{2.29}$$

□

We now design an admissible portrait as follows.

Lemma 2.6 (Admissible phase portrait, see figure 3)**.** *Assume*

$$d \geq 2, \quad \ell > 0, \quad 1 < r \leq r_+(d,\ell). \tag{2.30}$$

then the conclusions of Lemma 2.4 hold with $0 < w_e \leq w_\ell^-$ *and* P_2, P_3, P_5 *are well defined. More precisely, denoting* $\sigma_2 = \sigma(P_2)$, $\sigma_3 = \sigma(P_3)$ *and* $\sigma_5 = \sigma(P_5)$*:*
1. below the r^* *speed: for* $1 < r < r^*(d,\ell)$, P_5 *lies strictly below the sonic line* $0 < w_5 + \sigma_5 < 1$ *and*

$$0 < \sigma_3 < \sigma_5 < \sigma_2. \tag{2.31}$$

2. r^* *speed: for* $r = r^*(d,\ell)$,

$$\left| \begin{array}{l} \ell < d \Leftrightarrow P_5 = P_2 \\ \ell > d \Leftrightarrow P_5 = P_3 \\ \ell = d \Leftrightarrow P_5 = P_2 = P_3. \end{array} \right. \tag{2.32}$$

3. below the r_+ *speed: for* $r^*(d,\ell) < r < r_+(d,\ell)$ *and* $\ell > d$,

$$\sigma_5 < \sigma_3 < \sigma_2 \tag{2.33}$$

and

$$P_3 \to P_2 \quad as \quad r \uparrow r_+.$$

Remark 2.7. We note here the fundamental role payed by the case $\ell = d$ which corresponds to $\gamma = 1 + \frac{2}{d}$ and is a degenerate triple point configuration

$$r = r^*(d,d) = r_+(d,d) \Leftrightarrow P_2 = P_3 = P_5.$$

Proof of Lemma 2.6. We observe

$$w_\ell^- < 1 \Leftrightarrow \frac{\ell(d-1)(\sqrt{\ell}-1)^2}{d(1-\ell)^2} < 1 \Leftrightarrow \frac{\ell(d-1)}{d(1+\sqrt{\ell})^2} < 1 \Leftrightarrow d(2\sqrt{\ell}+1) + \ell > 0$$

which holds, and this together with (2.28) ensures that the conclusions of Lemma 2.4 hold and P_2, P_3 are well defined, distinct and on the sonic line.
Subcritical speed. For $1 < r < r^*(d,\ell)$, we compute from (2.17):

$$w_5 + \sigma_5 = \frac{\ell r}{d+\ell} + \frac{r\sqrt{d}}{d+\ell} = \frac{r}{r^*(d,\ell)} < 1 \tag{2.34}$$

and hence P_5 lies strictly below the sonic line $w + \sigma = 1$. Since P_5 lies on the curve $(\sigma, w_2(\sigma))$ where $w_2(\sigma)$ is the middle root of Δ_1, the ordering (2.20) implies (2.31).

Critical speed. Let now $r = r^*(d,\ell)$. Then P_5 is on the sonic line and since $(0,1), P_2, P_3$ are the only intersections of $w_2(\sigma)$ with the sonic line, P_5 coincides necessarily with P_2 or P_3. For $d = \ell$,

$$r^*(d,\ell) = r_+(d,\ell) \Rightarrow w_e = w_\ell^- \Rightarrow P_2 = P_3$$

and hence $P_2 = P_3 = P_5$. For $d \neq \ell$, let $J = J(w_e) \geq 0$ be given by (2.6) and, from (2.18), let

$$\begin{aligned} \sigma_2 \equiv \sigma(P_2) &= 1 - w_- = 1 - \frac{1}{2(d-1)} \left(dw_e + d - 1 - \frac{dw_e}{\ell} - \sqrt{J} \right) \\ &= \frac{1}{2(d-1)} \left[d - 1 + d \left(\frac{1}{\ell} - 1 \right) w_e + \sqrt{J} \right] \end{aligned}$$

and

$$\begin{aligned}\sigma_3 &\equiv \sigma(P_3) = 1 - w_+ = 1 - \frac{1}{2(d-1)}\left(dw_e + d - 1 - \frac{dw_e}{\ell} + \sqrt{J}\right)\\ &= \frac{1}{2(d-1)}\left[d - 1 + d\left(\frac{1}{\ell} - 1\right)w_e - \sqrt{J}\right]\end{aligned}$$

so that from (2.17):

$$\left|\begin{array}{l}\sigma_2 - \sigma_5 = \frac{\sqrt{J}-A}{2(d-1)}\\ \sigma_3 - \sigma_5 = \frac{-\sqrt{J}-A}{2(d-1)}\end{array}\right. \tag{2.35}$$

with

$$\left|\begin{array}{l}A = 2(d-1)\sqrt{d}\frac{r}{d+\ell} - d + 1 - (1-\ell)(r-1) = \left(\frac{2(d-1)\sqrt{d}}{d+\ell} - 1 + \ell\right)(r - r_0(\ell))\\ r_0(\ell) = \frac{d+\ell-2}{\frac{2(d-1)\sqrt{d}}{d+\ell} - 1 + \ell} > 0,\end{array}\right.$$

where we used that

$$\forall \ell > 0, \quad \forall d \geq 2, \quad \frac{2(d-1)\sqrt{d}}{d+\ell} - 1 + \ell > 0.$$

We compute

$$\begin{aligned}r^*(d,\ell) - r_0 &= \frac{d+\ell}{\sqrt{d}+\ell} - \frac{d+\ell-2}{\frac{2(d-1)\sqrt{d}}{d+\ell} + \ell - 1}\\ &= \frac{d+\ell}{(\sqrt{d}+\ell)(2(d-1)\sqrt{d} + (\ell-1)(d+\ell))}\left[2(d-1)\sqrt{d} + (\ell-1)(d+\ell) - (\sqrt{d}+\ell)(d+\ell-2)\right]\\ &= \frac{(\sqrt{d}-1)(d+\ell)}{(\sqrt{d}+\ell)(2(d-1)\sqrt{d} + (\ell-1)(d+\ell))}(d-\ell)\end{aligned}$$

Therefore,

$$A(r^*)\left|\begin{array}{ll}> 0 & \text{for } \ell < d\\ < 0 & \text{for } \ell > d.\end{array}\right. \tag{2.36}$$

Since P_5 coincides with P_2 or P_3, (2.35) implies

$$J(r^*) = A^2(r^*) \tag{2.37}$$

and hence the sign of $A(r^*)$ is given by (2.36) and (2.35) yield (2.32).

step 3 We now turn to the case $r^* < r < r_+$ for $\ell > d$. We claim

$$r_+ - r_0 < 0 \tag{2.38}$$

and

$$J - A^2 = -c_1(d,\ell)(r + c_2(d,\ell))(r - r^*), \quad c_1, c_2 > 0. \tag{2.39}$$

Assume (2.38), (2.39), then for $r^* < r < r_+$, $A < 0$ and $J - A^2 < 0$ implies from (2.35) that $\sigma_3 - \sigma_5 > 0$, and since $\sigma_2 > \sigma_3$ by definition, (2.33) is proved.
Proof of (2.38). We compute

$$\begin{aligned}r^*(d,\ell) - r_0 &= \frac{d+\ell}{\sqrt{d}+\ell} - \frac{d+\ell-2}{\frac{2(d-1)\sqrt{d}}{d+\ell} + \ell - 1}\\ &= \frac{d+\ell}{(\sqrt{d}+\ell)(2(d-1)\sqrt{d} + (\ell-1)(d+\ell))}\left[2(d-1)\sqrt{d} + (\ell-1)(d+\ell) - (\sqrt{d}+\ell)(d+\ell-2)\right]\\ &= \frac{(\sqrt{d}-1)(d+\ell)}{(\sqrt{d}+\ell)(2(d-1)\sqrt{d} + (\ell-1)(d+\ell))}(d-\ell)\end{aligned}$$

and hence recalling (2.29):

$$\begin{aligned} & r_+ - r_0 = r_+ - r^* + r^* - r_0 \\ = \ & \frac{(\sqrt{d}-1)(d+\ell)(d-\ell)}{(\sqrt{d}+\ell)(2(d-1)\sqrt{d}+(\ell-1)(d+\ell))} + (\sqrt{d}-1)\frac{(\sqrt{\ell}-\sqrt{d})^2}{(\ell+\sqrt{d})(1+\sqrt{\ell})^2} \\ = \ & \frac{(\sqrt{d}-1))(\sqrt{d}-\sqrt{\ell})}{\ell+\sqrt{d}}\left[\frac{(d+\ell)(\sqrt{\ell}+\sqrt{d})}{2(d-1)\sqrt{d}+(\ell-1)(d+\ell))} - \frac{\sqrt{\ell}-\sqrt{d}}{(1+\sqrt{\ell})^2}\right] \end{aligned}$$

and hence the sign is dictated by

$$\begin{aligned} & P(d,\ell) = (d+\ell)(\sqrt{\ell}+\sqrt{d})(1+\sqrt{\ell})^2 - (\sqrt{\ell}-\sqrt{d})\left[2(d-1)\sqrt{d}+(\ell-1)(d+\ell))\right] \\ = \ & (d+\ell)\left[(\sqrt{\ell}+\sqrt{d})(1+2\sqrt{\ell}+\ell) - (\sqrt{\ell}-\sqrt{d})(\ell-1)\right] - 2(d-1)\sqrt{d}(\sqrt{\ell}-\sqrt{d}) \\ = \ & (d+\ell)\left[\sqrt{d}+\sqrt{\ell}+2\sqrt{\ell}\sqrt{d}+2\ell+\ell\sqrt{d}+\ell\sqrt{\ell}-(\ell\sqrt{\ell}-\sqrt{\ell}-\ell\sqrt{d}+\sqrt{d})\right] \\ - \ & 2(d-1)\sqrt{d}(\sqrt{\ell}-\sqrt{d}) \\ = \ & (d+\ell)[(2\sqrt{d}+2)\sqrt{\ell}+(\sqrt{d}+3)\ell] - 2(d-1)\sqrt{d}(\sqrt{\ell}-\sqrt{d}) \\ = \ & (d+\ell)(\sqrt{d}+3)\ell + 2\sqrt{d}(d-1) + \sqrt{\ell}[(d+\ell)(2\sqrt{d}+2) - 2(d-1)\sqrt{d}] > 0 \end{aligned}$$

and (2.38) is proved.
Proof of (2.39). We compute

$$\begin{aligned} J \ &= \ d^2\left(\frac{1-\ell}{\ell}\right)^2 w_e^2 - \frac{2d(d-1)(\ell+1)}{\ell}w_e + (d-1)^2 \\ &= \ (1-\ell)^2(r-1)^2 - 2(d-1)(\ell+1)(r-1) + (d-1)^2 \\ &= \ (1-\ell)^2 r^2 - 2r\left[(1-\ell)^2 + (\ell+1)(d-1)\right] + (d-1)^2 + (1-\ell)^2 + 2(d-1)(\ell+1). \end{aligned}$$

and hence injecting the value of A:

$$\begin{aligned} J - A^2 \ &= \ -\left\{ r^2\left[\left(\frac{2(d-1)\sqrt{d}}{d+\ell} - 1 + \ell\right)^2 - (1-\ell)^2\right]\right. \\ &+ \ 2r\left[(1-\ell)^2 + (\ell+1)(d-1) - \left(\frac{2(d-1)\sqrt{d}}{d+\ell} - 1 + \ell\right)^2 r_0\right] r \\ &- \ \left.(d-1)^2 - (1-\ell)^2 - 2(d-1)(\ell+1) + \left(\frac{2(d-1)\sqrt{d}}{d+\ell} - 1 + \ell\right)^2 r_0^2\right\} \\ &= \ -(ar^2 + 2br - c) \end{aligned}$$

with using $\ell > d > 1$:

$$a = \left(\frac{2(d-1)\sqrt{d}}{d+\ell} + \ell - 1\right)^2 - (\ell-1)^2 = \frac{4(d-1)\sqrt{d}}{d+\ell}\left[\frac{(d-1)\sqrt{d}}{d+\ell} + \ell - 1\right] > 0$$

$$
\begin{aligned}
b &= (1-\ell)^2 + (\ell+1)(d-1) - \left(\frac{2(d-1)\sqrt{d}}{d+\ell} - 1 + \ell\right)^2 r_0 \\
&= (1-\ell)^2 + (\ell+1)(d-1) - (d+\ell-2)\left(\frac{2(d-1)\sqrt{d}}{d+\ell} - 1 + \ell\right) \\
&= 1 - 2\ell + \ell^2 + \ell d - \ell + d - 1 - \frac{2(d-1)(d+\ell-2)\sqrt{d}}{d+\ell} - (\ell d + \ell^2 - 2\ell - d - \ell + 2) \\
&= 2(d-1)\left[1 - \frac{(d+\ell-2)\sqrt{d}}{d+\ell}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
c &= (d-1)^2 + (1-\ell)^2 + 2(d-1)(\ell+1) - \left(\frac{2(d-1)\sqrt{d}}{d+\ell} - 1 + \ell\right)^2 r_0^2 \\
&= (d-1)^2 + (1-\ell)^2 + 2(d-1)(\ell+1) - (d+\ell-2)^2 = 4(d-1).
\end{aligned}
$$

Since $a, c > 0$, the roots are given by

$$
r^*\pm = \frac{1}{a}(-b \pm \sqrt{b^2 + ac}), \quad r_-^* < 0.
$$

We now observe that (2.37) implies

$$
r_+^* = r^* \tag{2.40}
$$

which can also be checked directly, and (2.39) is proved. □

2.4. **Slopes of $w_2(\sigma)$ and $w_2^-(\sigma)$ at P_2.** The point P_2 will play a fundamental role in the proof of Theorem 1.3. In this section we collect the main geometric properties of the phase portrait near P_2 in the regime (2.30). We note

$$
\sigma_2 = \sigma(P_2), \quad w_2 = 1 - \sigma_2.
$$

Definition of the slopes We compute the slopes of root curves $w_2(\sigma)$ and $w_2^-(\sigma)$ at P_2 by defining the following coefficients

$$
\left|\begin{array}{l}
c_1 = \partial_w \Delta_1(P_2) = 3w_2^2 - 2(r+1)w_2 + r - d\sigma_2^2 \\
c_2 = \partial_w \Delta_2(P_2) = \frac{\sigma_2}{\ell}[2w_2(\ell+d-1) - (\ell+d+\ell r - r)] \\
c_3 = \partial_\sigma \Delta_1(P_2) = -2d\sigma_2 w_2 + 2\ell(r-1)\sigma_2 \\
c_4 = \partial_\sigma \Delta_2(P_2) = -2\sigma_2^2,
\end{array}\right. \tag{2.41}
$$

so that the corresponding slopes are $-\frac{c_3}{c_1}$ and $-\frac{c_4}{c_2}$. We now claim:

Lemma 2.8 (Sign of the slopes)**.** *Assume $d \geq 2$ and*

$$
\left|\begin{array}{l}
1 < r < r^*(d,\ell) \quad \textit{for} \quad \ell < d \\
r^*(d,\ell) < r < r_+(d,\ell) \quad \textit{for} \quad \ell > d
\end{array}\right. \tag{2.42}
$$

then

$$
\left|\begin{array}{l}
c_i < 0, \quad 1 \leq i \leq 4 \\
c_2 c_3 - c_1 c_4 < 0.
\end{array}\right. \tag{2.43}
$$

Proof of Lemma 2.8. The argument relies solely on the consideration of the relative positions of the red and green curves locally near P_2 which is the same in the range (2.42). Indeed, in view of the discussion of the roots of Δ_1, and since we have established that P_2 is on the middle root of Δ_1, we have

$$
c_1 = \partial_\omega \Delta_1(P_2) < 0.
$$

Since P_2 corresponds to the smallest root w_2^- of Δ_2 and Δ_2 is a second order polynomial in w with a strictly positive coefficient in front of the w^2 term, we have

$$c_2 = \partial_\omega \Delta_2(P_2) < 0.$$

Also, since $\sigma > 0$ at P_2, and since $w_e < w_- < 1$, we have

$$\begin{aligned} c_3 &= \partial_\sigma \Delta_1(P_2) = -2d(w_- - w_e)(1 - w_-) < 0, \\ c_4 &= \partial_\sigma \Delta_2(P_2) = -2(1 - w_-)^2 < 0. \end{aligned}$$

Finally, we compute

$$\begin{aligned} c_2c_3 - c_1c_4 &= \partial_\omega \Delta_2(P_2)\partial_\sigma \Delta_1(P_2) - \partial_\omega \Delta_1(P_2)\partial_\sigma \Delta_2(P_2) \\ &= \partial_\omega \Delta_2(P_2)\partial_\omega \Delta_1(P_2)\left(\frac{\partial_\sigma \Delta_1(P_2)}{\partial_\omega \Delta_1(P_2)} - \frac{\partial_\sigma \Delta_2(P_2)}{\partial_\omega \Delta_2(P_2)}\right) \\ &= c_1c_2\Big((w_2^-)'(\sigma) - w_1'(\sigma)\Big)_{|_{\sigma=1-w_-}} \end{aligned}$$

where we used the fact that $\sigma = 1 - w_-$ and $w = w_-$ at P_2, the fact that w_2 is both the middle root of Δ_1, i.e., w_1, and the smallest root of Δ_2, i.e., w_2^-, with the formula for ${w_2^-}'(\sigma)$ and $w_1'(\sigma)$ following from the implicit function theorem. Now, at P_2, the slope of Δ_2 is strictly more negative than the slope of Δ_1 since P_2 is the last intersection in σ and Δ_1 asymptotes to w_e while Δ_2 goes to $-\infty$. Thus, we have $(w_2^-)'(1 - w_-) - w_2'(1 - w_-) < 0$ and hence

$$c_2c_3 - c_1c_4 < 0$$

as desired. □

Slopes and eigenvalues. In additions to the slopes of the root curves $w_2(\sigma)$ and $w_2^-(\sigma)$ we will also compute the slopes of any integral curve passing through P_2. it turns out that there are only two possible values:

$$c_\pm = \frac{c_4 - c_1 \pm \sqrt{(c_1 - c_4)^2 + 4c_2c_3}}{2|c_2|} \tag{2.44}$$

This follows since $c_\pm$ are the solutions of the equation

$$c_\pm = \frac{c_1c_\pm + c_3}{c_2c_\pm + c_4}. \tag{2.45}$$

The characteristic matrix

$$\mathcal{A}(P_2) = \begin{pmatrix} c_1 & c_3 \\ c_2 & c_4 \end{pmatrix},$$

coming from the linearization of (1.9) at P_2 (see (3.16) (3.17)), possesses the following eigenvalues:

$$\lambda_\pm = \frac{c_1 + c_4 \pm \sqrt{(c_1 - c_4)^2 + 4c_2c_3}}{2} \tag{2.46}$$

It may be diagonalized as follows:

$$P^{-1}\left[\mathcal{A}(P_2)\right]P = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}$$

with

$$P = \begin{pmatrix} c_- & c_+ \\ 1 & 1 \end{pmatrix}, \quad P^{-1} = \frac{1}{c_+ - c_-}\begin{pmatrix} -1 & c_+ \\ 1 & -c_- \end{pmatrix}. \tag{2.47}$$

Lemma 2.9 (Estimates on the slopes)**.** *Assume* (2.42) *and let*

$$A = \frac{\lambda_-}{\lambda_+} = \frac{c_1c_4 - c_2c_3}{(c_4 + c_2c_-)^2}, \tag{2.48}$$

Then,

$$\left| \begin{array}{l} c_- < 0 < c_+ \\ c_4 + c_2c_- < 0 \\ A > 1 \\ -1 < -\frac{c_4}{c_2} < c_- < -\frac{c_3}{c_1} < 0 \\ \lambda_- < \lambda_+ < 0 \end{array} \right. \tag{2.49}$$

Proof of Lemma 2.9. From $c_2c_3 > 0$:

$$\frac{c_4 - c_1 - \sqrt{(c_1 - c_4)^2 + 4c_2c_3}}{2|c_2|} < 0 < \frac{c_4 - c_1 + \sqrt{(c_1 - c_4)^2 + 4c_2c_3}}{2|c_2|}$$

and hence

$$c_- < 0 < c_+.$$

Next, we compute

$$c_4 + c_2c_- = \frac{c_4 + c_1 + \sqrt{(c_1 - c_4)^2 + 4c_2c_3}}{2} = \frac{2(c_2c_3 - c_1c_4)}{-c_4 - c_1 + \sqrt{(c_1 - c_4)^2 + 4c_2c_3}} < 0.$$

We now observe

$$\left| \begin{array}{l} c_2c_- + c_4 = c_4 - \frac{c_4 - c_1 - \sqrt{\Delta}}{2} = \frac{c_4 + c_1 + \sqrt{\Delta}}{2} = \lambda_+ \\ c_2c_+ + c_4 = c_4 - \frac{c_4 - c_1 + \sqrt{\Delta}}{2} = \frac{c_4 + c_1 - \sqrt{\Delta}}{2} = \lambda_-. \end{array} \right. \tag{2.50}$$

and hence $\lambda_- < \lambda_+ < 0$. We now estimate A:

$$\begin{aligned} & c_4c_1 - c_3c_2 - (c_4 + c_2c_-)^2 \\ = \; & c_4c_1 - c_3c_2 - \frac{4(c_2c_3 - c_1c_4)^2}{(-c_4 - c_1 + \sqrt{(c_1 - c_4)^2 + 4c_2c_3})^2} \\ = \; & -(c_4 + c_2c_-)\frac{\Big[(-c_4 - c_1 + \sqrt{(c_1 - c_4)^2 + 4c_2c_3})^2 - 4(c_4c_1 - c_2c_3)\Big]}{2(-c_4 - c_1 + \sqrt{(c_1 - c_4)^2 + 4c_2c_3})} \\ = \; & -(c_4 + c_2c_-)\frac{\Big[(c_4 + c_1)^2 + 2(-c_4 - c_1)\sqrt{(c_1 - c_4)^2 + 4c_2c_3} + (c_1 - c_4)^2 + 4c_2c_3 - 4(c_4c_1 - c_2c_3)\Big]}{2(-c_4 - c_1 + \sqrt{(c_1 - c_4)^2 + 4c_2c_3})} \\ = \; & -(c_4 + c_2c_-)\frac{\Big[(-c_4 - c_1)\sqrt{(c_1 - c_4)^2 + 4c_2c_3} + (c_1 - c_4)^2 + 4c_2c_3\Big]}{(-c_4 - c_1 + \sqrt{(c_1 - c_4)^2 + 4c_2c_3})^2} \end{aligned}$$

and hence

$$\begin{aligned} A - 1 &= \frac{c_1c_4 - c_2c_3}{(c_4 + c_2c_-)^2} - 1 \\ &= -\frac{\Big((-c_4 - c_1)\sqrt{(c_1 - c_4)^2 + 4c_2c_3} + (c_1 - c_4)^2 + 4c_2c_3\Big)}{(c_4 + c_2c_-)(-c_4 - c_1 + \sqrt{(c_1 - c_4)^2 + 4c_2c_3})}. \end{aligned}$$

Using in particular that $c_1, c_2, c_3, c_4 < 0$, this yields

$$A > 1.$$

Next, note that we have

$$-\frac{c_3}{c_1} < 0$$

since $c_3 < 0$ and $c_1 < 0$. Also, since $\{\Delta_2 = 0\}$ intersects $w = 1 - \sigma$ at P_2 and P_3, $(\sigma, w_2^-(\sigma))$ is above $w = 1 - \sigma$ for $\sigma > 1 - w_-$ and below for $\sigma < 1 - w_-$. Since $-c_4/c_2$ is the slope of $w = w_2^-(\sigma)$ at P_2, we infer

$$-\frac{c_4}{c_2} > -1.$$

It remains to compare c_- to $-c_3/c_1$. We compute

$$\begin{aligned} c_- + \frac{c_3}{c_1} &= c_- + \frac{|c_3|}{|c_1|} \\ &= \frac{|c_1|(c_4 - c_1 - \sqrt{(c_1 - c_4)^2 + 4c_2c_3}) + 2c_2c_3}{2|c_1||c_2|} \\ &= \frac{c_1(c_1 - c_4) + 2c_2c_3 - |c_1|\sqrt{(c_1 - c_4)^2 + 4c_2c_3}}{2|c_1||c_2|}. \end{aligned}$$

Now, we have

$$\begin{aligned} & \Big(c_1(c_1 - c_4) + 2c_2c_3\Big)^2 - \Big(|c_1|\sqrt{(c_1 - c_4)^2 + 4c_2c_3}\Big)^2 \\ = \;& c_1^2(c_1 - c_4)^2 + 4c_2c_3c_1(c_1 - c_4) + 4c_2^2c_3^2 - c_1^2(c_1 - c_4)^2 - 4c_1^2c_2c_3 \\ = \;& 4c_2c_3(c_2c_3 - c_1c_4) < 0 \end{aligned}$$

and hence

$$c_- + \frac{c_3}{c_1} < 0.$$

This concludes the proof of (2.49). □

3. General properties of the dynamical system (1.9)

In this section we establish the general properties of Lemma 1.2 for the dynamical system (1.9). We assume

$$d \geq 2, \quad \left| \begin{array}{l} 1 < r < r^*(\ell, d) \quad \text{for} \quad \ell < d \\ r^*(d, \ell) < r < r_+(d, \ell) \quad \text{for} \quad \ell > d \end{array} \right. \tag{3.1}$$

so that the shape of the phase portrait is given by respectively figure 1 or figure 2. We recall that $w_2(\sigma)$ is the middle root of Δ_1 and $w_2^-(\sigma)$ is the smallest root of Δ_2 given by (2.5). The arguments in this section are classical and are given for the reader's convenience.
We note

$$P_\infty = \left| \begin{array}{l} P_5 \quad \text{for} \quad \ell < d \\ P_3 \quad \text{for} \quad \ell > d. \end{array} \right.$$

3.1. **The spherically symmetric solution emerging from the origin.** We first claim the existence and uniqueness (up to the scaling symmetry) of a spherically symmetric solution to (1.7) which exists on the interval $[0, Z_2]$ and, in the variables of Emden transform, corresponds to the integral curve $P_6 - P_2$.

Lemma 3.1 (The solution emerging from P_6)**.** *Assume* (3.1).

(1) *Existence: there is $\sigma_0 > 0$ large enough and a unique curve $w(\sigma)$ solution to* (1.9) *on $(\sigma_0, +\infty)$ with*

$$\lim_{\sigma \to +\infty} w(\sigma) = w_e. \tag{3.2}$$

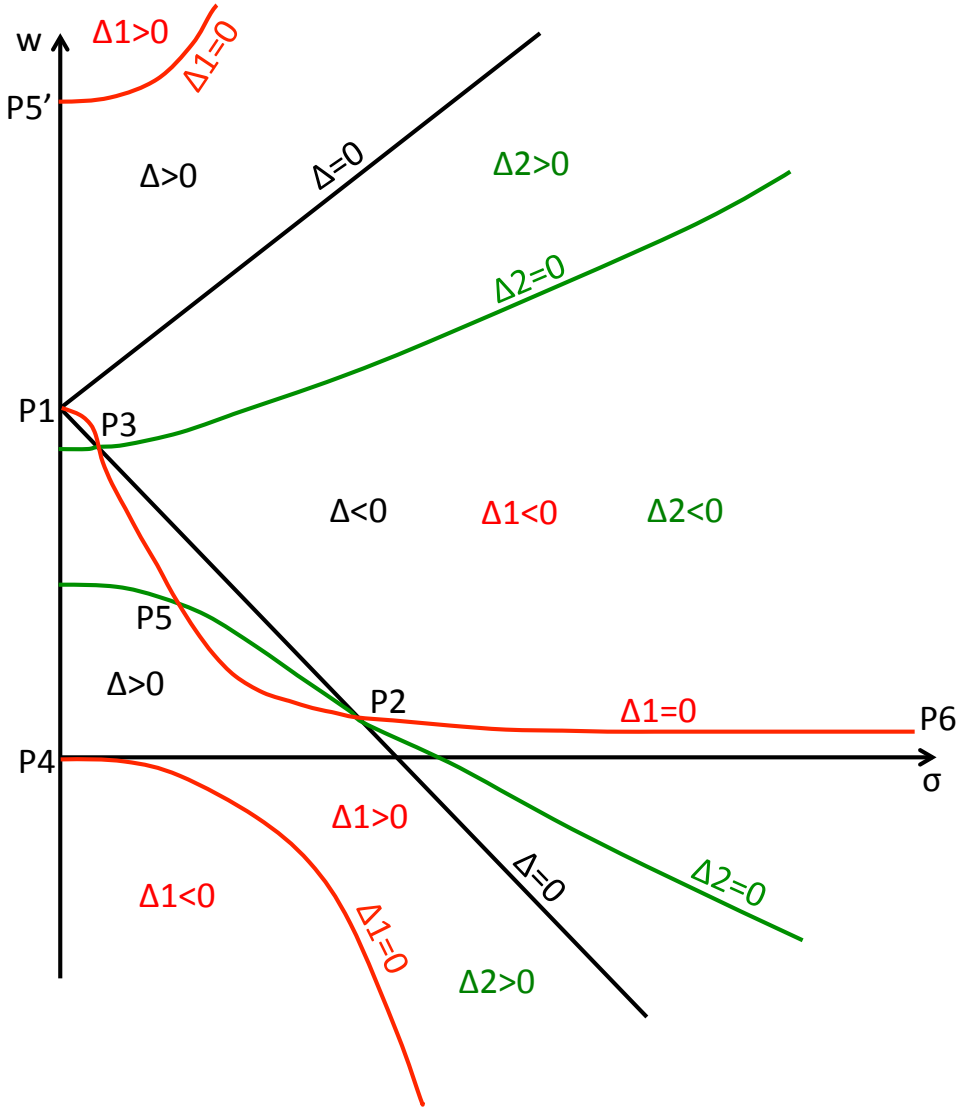

FIGURE 3. Shape of the phase portrait for $1 < r < r^*(\ell, d)$

It admits the asymptotic expansion as $\sigma \to +\infty$*:*

$$w(\sigma) = w_e + \frac{w_e(w_e-1)(w_e-r)}{d+2}\frac{1}{\sigma^2} + O\left(\frac{1}{\sigma^4}\right). \tag{3.3}$$

(2) *Original variables: The curve corresponds to the unique (up to scaling invariance) spherically symmetric solution of* (1.7) *defined on the interval* $|Z| \in [0, Z_0]$*. This solution belongs to* $\mathcal{C}^\infty(|Z| \le Z_0)$*.*

(3) *Reaching* P_2*:* w *exists in fact on* $(\sigma_2, +\infty)$ *and we have* $w \in \mathcal{C}^\infty(\sigma_2, +\infty)$ *with*

$$\left| \begin{array}{l} \forall \sigma_2 < \sigma < +\infty, \quad w_2^-(\sigma) < w(\sigma) < w_2(\sigma) \\ \lim_{\sigma \downarrow \sigma_2} w(\sigma) = w_2, \end{array} \right. \tag{3.4}$$

see figure 1.

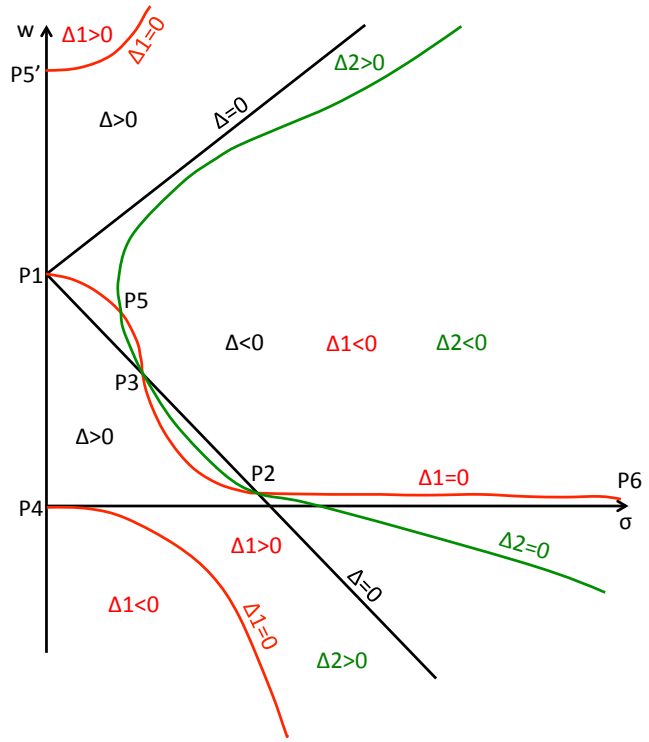

FIGURE 4. Shape of the phase portrait for $r^*(\ell,d) < r < r_+(d,\ell)$, $\ell > d$

Proof of Lemma 3.1. This follows from the asymptotic behavior of the polynomials Δ_1, Δ_2.

step 1 Flow near P_6. We decompose

$$w = w_e + \widehat{w}, \quad \lim_{\sigma\to+\infty} \widehat{w}(\sigma) = 0.$$

Then, $\widehat{w}$ satisfies

$$\begin{aligned}
\frac{d\widehat{w}}{d\sigma} - \frac{d}{\sigma}\widehat{w} &= \frac{w(w-1)(w-r) - d\widehat{w}\sigma^2}{\frac{\sigma}{\ell}\Big[(\ell+d-1)w^2 - w(\ell+d+\ell r-r) + \ell r - \ell\sigma^2\Big]} - \frac{d}{\sigma}\widehat{w} \\
&= \left(\frac{\widehat{w} - \frac{w(w-1)(w-r)}{d\sigma^2}}{1 + \frac{-(\ell+d-1)w^2 + w(\ell+d+\ell r-r) - \ell r}{\ell\sigma^2}} - \widehat{w}\right)\frac{d}{\sigma} \\
&= \left(\frac{-\ell w(w-1)(w-r) + d\Big((\ell+d-1)w^2 - w(\ell+d+\ell r-r) + \ell r\Big)\widehat{w}}{1 + \frac{-(\ell+d-1)w^2 + w(\ell+d+\ell r-r) - \ell r}{\ell\sigma^2}}\right)\frac{1}{\ell\sigma^3}
\end{aligned}$$

and hence

$$\begin{aligned}
\frac{d}{d\sigma}\left(\frac{\widehat{w}}{\sigma^d}\right) &= \frac{1}{\sigma^d}\left(\frac{d\widehat{w}}{d\sigma} - \frac{d}{\sigma}\widehat{w}\right) \qquad (3.5) \\
&= \left(\frac{-\ell w(w-1)(w-r) + d\Big((\ell+d-1)w^2 - w(\ell+d+\ell r-r) + \ell r\Big)\widehat{w}}{1 + \frac{-(\ell+d-1)w^2 + w(\ell+d+\ell r-r) - \ell r}{\ell\sigma^2}}\right)\frac{1}{\ell\sigma^{d+3}}.
\end{aligned}$$

Existence. We solve

$$\begin{aligned} & \widehat{w} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.6) \\ = \quad & -\sigma^d \int_\sigma^{+\infty} \left(\frac{-\ell w(w-1)(w-r) + d\Big((\ell+d-1)w^2 - w(\ell+d+\ell r-r) + \ell r\Big)\widehat{w}}{1 + \frac{-(\ell+d-1)w^2 + w(\ell+d+\ell r-r) - \ell r}{\ell \sigma'^2}} \right) \frac{d\sigma'}{\ell \sigma^{d+3}}. \end{aligned}$$

using an elementary fixed point argument which yields the existence and uniqueness on $\sigma_0 \leq \sigma \leq +\infty$ of $\widehat{w}$ such that

$$|\widehat{w}| \lesssim \frac{1}{\sigma^2} \text{ for } \sigma \geq \sigma_0 \tag{3.7}$$

provided σ_0 has been chosen large enough.

Uniqueness. Let now $w(\sigma)$ be a solution to (3.5) on $[\sigma_0, +\infty)$ with $\lim_{\sigma \to +\infty} w(\sigma) = w_e$. We integrate (3.5). The integral of the RHS of (3.5) converges at $+\infty$ from the a priori bound $|w| \lesssim 1$, and hence $\widehat{w}$ satisfies (3.6) and the a priori bound (3.7). The uniqueness claim follows.

Asymptotics as $\sigma \to +\infty$. The integral equation (3.6) for $\widehat{w}$ and the fact that

$$\begin{aligned} & \frac{-\ell w(w-1)(w-r) + d\Big((\ell+d-1)w^2 - w(\ell+d+\ell r-r) + \ell r\Big)\widehat{w}}{1 + \frac{-(\ell+d-1)w^2 + w(\ell+d+\ell r-r) - \ell r}{\ell \sigma'^2}} \\ = \quad & -\ell w_e(w_e-1)(w_e-r) + O\left(\frac{1}{\sigma^2}\right) \end{aligned}$$

and

$$\begin{aligned} & -\sigma^d \int_\sigma^{+\infty} \left(-\ell w_e(w_e-1)(w_e-r) + O\left(\frac{1}{\sigma'^2}\right)\right) \frac{d\sigma'}{\ell \sigma'^{d+3}} \\ = \quad & \frac{w_e(w_e-1)(w_e-r)}{d+2} \frac{1}{\sigma^2} + O\left(\frac{1}{\sigma^4}\right). \end{aligned}$$

yield (3.3).

Spherical symmetry and regularity at the origin. From (3.3), on the solution, as $\sigma \to +\infty$:

$$\left| \begin{array}{l} \Delta = -\sigma^2 \left[1 + O\left(\frac{1}{\sigma^2}\right)\right] \\ \Delta_2 = -\sigma^3 \left[1 + O\left(\frac{1}{\sigma^2}\right)\right] \end{array} \right.$$

As a result,

$$\frac{dx}{d\sigma} = -\frac{\Delta}{\Delta_2} = -\frac{1}{\sigma}\left[1 + O\left(\frac{1}{\sigma^2}\right)\right].$$

Up to a constant

$$x = -\log\sigma + O\left(\frac{1}{\sigma^2}\right) \Rightarrow \sigma = e^{-x}\left[1 + O(e^{2x})\right] \quad \text{as} \quad x \to -\infty.$$

Recalling the Emden transform formula (1.8), we obtain from (3.3) the asymptotics as $Z = e^x \to 0$:

$$\left| \begin{array}{l} (\hat{\rho}(Z))^{\frac{\gamma-1}{2}} = \sqrt{\frac{\ell}{2}} Z\sigma(x) = \sqrt{\frac{\ell}{2}}(1 + O(Z^2)) \\ \hat{u}(Z) = -Zw(x) = -Z\left[w_e + O(Z^2)\right] \end{array} \right. \tag{3.8}$$

We also observe that the system (2.1) can be rewritten with respect to the variables

$$s := Z^2, \qquad f := w - w_e, \qquad g = Z\sigma$$

in the form

$$\left| \begin{array}{l} \frac{df}{ds} = -\frac{d}{2s} f + \mathcal{F}(s,f,g) \\ \frac{dg}{ds} = \mathcal{G}(s,f,g) \end{array} \right.$$

for some functions $\mathcal{F}, \mathcal{G}$ with the property that they are smooth functions of all the variables in the neighborhood of the point $(0,0,\sqrt{\frac{\ell}{2}})$. We may rewrite the above in the form

$$\left| \begin{array}{l} f(s) = s^{-\frac{d}{2}} \int_0^s (\tilde{s})^{\frac{d}{2}} \mathcal{F}(\tilde{s}, f, g) d\tilde{s} \\ g(s) = \sqrt{\frac{\ell}{2}} + \int_0^s \mathcal{G}(\tilde{s}, f, g) d\tilde{s} \end{array} \right.$$

This produces a unique fixed point solution which is $\mathcal{C}^1$ in s and, obviously, coincides with the solution constructed from (3.6). Rewriting the above again as

$$\left| \begin{array}{l} f(s) = s \int_0^1 (\kappa)^{\frac{d}{2}} \mathcal{F}(s\kappa, f(s\kappa), g(s\kappa)) d\kappa \\ g(s) = \sqrt{\frac{\ell}{2}} + s \int_0^1 \mathcal{G}(\kappa, f(s\kappa), g(s\kappa)) d\kappa \end{array} \right.$$

implies that

$$\begin{aligned} f'(s) &= \int_0^1 (\kappa)^{\frac{d}{2}} \mathcal{F}(s\kappa, f(s\kappa), g(s\kappa)) d\kappa + s \int_0^1 (\kappa)^{\frac{d}{2}} \frac{d}{ds} \left(\mathcal{F}(s\kappa, f(s\kappa), g(s\kappa)) \right) d\kappa \\ &= \int_0^1 (\kappa)^{\frac{d}{2}} \mathcal{F}(s\kappa, f(s\kappa), g(s\kappa)) d\kappa + \int_0^1 (\kappa)^{\frac{d}{2}+1} \frac{d}{d\kappa} \left(\mathcal{F}(s\kappa, f(s\kappa), g(s\kappa)) \right) d\kappa \\ &= \int_0^1 (\kappa)^{\frac{d}{2}} \mathcal{F}(s\kappa, f(s\kappa), g(s\kappa)) d\kappa + \mathcal{F}(s, f(s), g(s)) \\ &\quad - \frac{d+2}{2} \int_0^1 (\kappa)^{\frac{d}{2}} \mathcal{F}(s\kappa, f(s\kappa), g(s\kappa)) d\kappa \end{aligned}$$

Similarly, for g. This expresses $f'(s)$ (and $g'(s)$) in terms of regular kernels involving f and g and immediately implies (by step by step differentiation) that both f and g are $\mathcal{C}^\infty$ with respect to s including at $s = 0$. Recalling that $s = Z^2$ and the Emden transform relation (3.8) to the original variables $\hat{u}, \hat{\rho}$, yields the desired local (defined in a neighborhood of the origin) $\mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R}^2)$ spherically symmetric solution.

step 2 Reaching P_2. Let $w_r(\sigma)$ be the unique curve entering P_6 constructed in step 1. From (3.3),

$$\begin{aligned} \Delta_1(\sigma, w_r(\sigma)) &= w_r(\sigma)(w_r(\sigma)-1)(w_r(\sigma)-r) - d(w_r(\sigma)-w_e)\sigma^2 \\ &= w_e(w_e-1)(w_e-r)\left(1 - \frac{d}{d+2}\right) + O\left(\frac{1}{\sigma^2}\right) \\ &= w_e(w_e-1)(w_e-r)\frac{2}{d+2} + O\left(\frac{1}{\sigma^2}\right) > 0 \end{aligned}$$

near $+\infty$. By Lemma 2.3, $\Delta_1 < 0$ for $w < w_1(\sigma)$ and for $w \in (w_2(\sigma), w_3(\sigma))$, and $\Delta_1 > 0$ for $w > w_3(\sigma)$ and for $w \in (w_1(\sigma), w_2(\sigma))$. Similarly, by Lemma 2.1, $\Delta_2 > 0$ for $w \in (w_2^-(\sigma), w_2^+(\sigma))$ and $\Delta_2 < 0$ for $w < w_2^-(\sigma)$ and for $w > w_2^+(\sigma)$. From the asymptotic expansion of $w_r(\sigma)$ we can conclude that $w_r'(\sigma) < 0$ for $\sigma > \sigma_0$. Given that $w_2^\pm(\sigma) \to \pm\infty$ as $\sigma \to \infty$, it follows that $w_r(\sigma) \in (w_2^-(\sigma), w_2^+(\sigma))$ for $\sigma > \sigma_0$ and thus $\Delta_2(\sigma, w_r(\sigma)) > 0$. Since,

$$0 > w_r'(\sigma) = \frac{\Delta_1}{\Delta_2},$$

it follows that $\Delta_1(\sigma, w_r(\sigma)) < 0$. Combining this with the fact that $w_r(\sigma) \to w_e = \lim_{\sigma\to\infty} w_2(\sigma)$ and examining the regions of constant signs of Δ_1 described above, imply that $w_r(\sigma) < w_2(\sigma)$ for all $\sigma > \sigma_0$. Thus, we have obtained

$$w_2^-(\sigma) < w_r(\sigma) < w_2(\sigma) \text{ for } \sigma \geq \sigma_0.$$

In view of the phase portrait of figure 3 *and the strict monotonicity* $w_2' < 0$ given by (2.10), this implies that the curve w_r reaches P_2 with

$$\forall \sigma > \sigma_2, \quad w_2^-(\sigma) < w_r(\sigma) < w_2(\sigma). \tag{3.9}$$

Let us give a quick proof. From (3.9), we consider the region below the middle root of Δ_1, above the smallest root of Δ_2 and to the right of P_2.

$$\mathcal{R}_r = \{(\sigma, w),\, w_2^-(\sigma) \leq w \leq w_2(\sigma), \sigma \geq 1 - w_-\}$$

The point P_2 is the utmost right joint root of Δ_1 and Δ_2 . The curves $(\sigma, w_2(\sigma))$ and $(\sigma, w_2^-(\sigma))$ intersect there and $\lim_{\sigma\to\infty} w_2(\sigma) = w_e$, $\lim_{\sigma\to\infty} w_2^-(\sigma) = -\infty$. Therefore, $w_2^-(\sigma) < w_2(\sigma)$ to the right of P_2. Moreover, in that region $w_1(\sigma) < w_2^-(\sigma)$ and $w_2(\sigma) < w_2^+(\sigma)$. As a consequence of the above, in the region $\mathcal{R}_r$

$$\Delta_1 \leq 0, \quad \Delta_2 \geq 0.$$

Now, assume by contradiction that there exists $\sigma_1 > 1 - w_-$ such that $(\sigma, w_r(\sigma))$ is in $\mathcal{R}_r$ for $\sigma > \sigma_1$ and $(\sigma_1, w_r(\sigma_1))$ is the last point in that region. Then either $(\sigma_1, w_r(\sigma_1))$ is on the middle root of Δ_1 to the right of P_2, in which case we have

$$w_r'(\sigma_1) = 0$$

But in view of (2.10), $w_2'(\sigma_1) < 0$, and we have a contradiction since $w_r(\sigma) < w_2(\sigma)$ for all $\sigma > \sigma_1$. Or $(\sigma_1, w_r(\sigma_1))$ is on the smallest root of Δ_2 to the right of P_2, in which case we have

$$w_r'(\sigma_1) = -\infty$$

This is again a contradiction since $w_r(\sigma) > w_2^-(\sigma)$ all $\sigma > \sigma_1$. Therefore, the curve can only exit the region $\mathcal{R}_r$ at $\sigma = 1 - w_-$, i.e., through P_2. Since there are no other accumulation points in this zone, the curve must approach P_2 as $\sigma \downarrow \sigma_2$. Therefore, the curve $w_r(\sigma)$ is defined on $(\sigma_2, +\infty)$, is $\mathcal{C}^\infty$ on this interval and satisfies (3.4). □

3.2. **Solutions crossing red between P_2 and P_∞.** We now analyze trajectories that cross the middle root $w_2(\sigma)$ of Δ_1 between[8] P_2 and P_∞.

Lemma 3.2 (Solutions crossing red between P_2 and P_∞)**.** *Assume* (3.1). *Let* $\sigma_5 < \sigma^* < \sigma_2$ *and* $w(\sigma)$ *be the solution to* (1.9) *with the data* $w(\sigma^*) = w_2(\sigma^*)$. *Then:*

1. backward flow*:* $w \in \mathcal{C}^\infty(0, \sigma^*]$ *and*

$$\lim_{\sigma\downarrow 0} w(\sigma) = 0.$$

Moreover, $\sigma \to 0$ *corresponds to* $x \to +\infty$ *and there exist* $(w_\infty, \sigma_\infty) \in \mathbb{R} \times \mathbb{R}_+^*$ *such that for* $x \to +\infty$

$$\left| \begin{array}{l} \sigma(x) = \sigma_\infty e^{-rx}\left(1 + O(e^{-rx})\right), \\ w(x) = w_\infty e^{-rx}\left(1 + O(e^{-rx})\right). \end{array} \right. \tag{3.10}$$

[8]Since the roots of Δ_1 are depicted in red on the various figures of this paper, we will sometimes say that such solutions cross red between P_2 and P_∞.

2. forward flow*:* $w \in \mathcal{C}^\infty[\sigma^*, \sigma_2)$ *and*

$$\left| \begin{array}{l} \forall \sigma^* < \sigma < \sigma_2, \quad w_2(\sigma) < w(\sigma) < w_2^-(\sigma) \\ \lim_{\sigma \uparrow \sigma_2} w(\sigma) = w_2, \end{array} \right. \tag{3.11}$$

see figure 1.

Remark 3.3. The above lemma shows that all such solutions provide admissible $P_2 - P_4$ connections.

Proof of Lemma 3.2. The fact that the solution generates a $P_4 - P_2$ connection with forward flow trapped in the region (3.11) follows again directly from the phase portrait of figure 3 and the monotonicity (2.10). We leave that to the reader, while we focus on the proof of the asymptotic expansion (3.10) near $\sigma = 0$.

As $\sigma \to 0$, we have

$$\begin{aligned} \frac{dx}{d\sigma} &= -\frac{\Delta(\sigma, w(\sigma))}{\Delta_2(\sigma, w(\sigma))} = -\frac{(w(\sigma)-1)^2 - \sigma^2}{\frac{\sigma}{\ell}\Big[(\ell+d-1)w(\sigma)^2 - (\ell+d+\ell r - r)w(\sigma) + \ell r - \ell\sigma^2\Big]} \\ &= -\frac{1+o(1)}{r\sigma} \end{aligned}$$

so that $x \to +\infty$ as $\sigma \to 0$. It thus suffices to consider the flow for x large enough.

Next, note that for $(w, \sigma) \to 0$, we have

$$\left|\frac{\Delta_1}{\Delta} - rw\right| + \left|\frac{\Delta_2}{\Delta} - r\sigma\right| \lesssim w^2 + \sigma^2.$$

Decomposing w and σ under the form

$$w(x) = w_\infty e^{-rx} + \tilde{w}, \qquad \sigma(x) = \sigma_\infty e^{-rx} + \tilde{\sigma},$$

we have

$$\frac{d\tilde{w}}{dx} + r\tilde{w} = -\left(\frac{\Delta_1}{\Delta} - rw\right), \qquad \frac{d\tilde{\sigma}}{dx} + r\tilde{\sigma} = -\left(\frac{\Delta_2}{\Delta} - r\sigma\right).$$

Assuming

$$\tilde{w}(x),\, \tilde{\sigma}(x) = o(e^{-rx}), \qquad x \to +\infty,$$

we infer

$$\begin{aligned} \tilde{w}(x) &= e^{-rx}\int_x^{+\infty} e^{ry}\left(\frac{\Delta_1}{\Delta} - rw\right)dy, \\ \tilde{\sigma}(x) &= e^{-rx}\int_x^{+\infty} e^{ry}\left(\frac{\Delta_2}{\Delta} - r\sigma\right)dy. \end{aligned}$$

An elementary fixed point argument then allows to obtain for any $(w_\infty, \sigma_\infty)$ with $|w_\infty| + |\sigma_\infty| = 1$ a unique corresponding solution $(w(x), \sigma(x))$ satisfying (3.10), which is defined for $e^{-rx} \geq \delta(\ell, r, d)$ with $\delta(\ell, r, d) > 0$ small enough, and hence for $|w| + |\sigma| \leq \delta(\ell, r, d)$. We have thus described the behavior of all solution curves (w, σ) converging to P_4 as $\sigma \to 0$, and shown that they all satisfy (3.10). □

The results of the previous two sections provide the proof of all the statements of Lemma 1.2. We summarize them as follows. We have constructed the unique spherically symmetric smooth solution of (1.7) on the interval $[0, Z_2)$ and a one parameter of spherically symmetric smooth solutions of (1.7) on the interval (Z_2, ∞). These solutions agree at Z_2 and thus can be glued to each other continuously. Our goal however is to construct a *global* $\mathcal{C}^\infty$ solution. At this point it is already clear that the crux of the matter is the point P_2.

3.3. **Diagonalized system at P_2.** The point P_2 will play an essential role in the proof of Theorem 1.3. The dynamical properties of this point in the regime (3.1) can only be seen after passing to the diagonalized variables (2.47). We recall the values of the slopes (2.41), (2.44), (2.46), the diagonalization matrices (2.47) and the non degeneracy properties of Lemma 2.8 and Lemma 2.9 in the range (3.1). We rewrite the system in coordinates which diagonalize its linear part. We will also introduce a time variable and recast the system as a dynamical flow approaching the point P_2 (from either side) as $t \to \infty$.

Lemma 3.4 (Equations in the diagonal form)**.** *Assume* (3.1)*. Let*

$$\left| \begin{array}{l} w = w_2 + W \\ \sigma = \sigma_2 + \Sigma \\ \frac{dt}{dx} = -\frac{1}{\Delta} \end{array} \right. , \quad X = \left| \begin{array}{l} W \\ \Sigma \end{array} \right. , \quad Y = P^{-1}X = \left| \begin{array}{l} \tilde{W} \\ \tilde{\Sigma} \end{array} \right. \tag{3.12}$$

then (1.9) *becomes:*

$$\frac{dY}{dt} = \frac{1}{c_+ - c_-} \left| \begin{array}{l} \mathcal{G}_1 \\ \mathcal{G}_2 \end{array} \right. \tag{3.13}$$

with

$$\begin{aligned} \mathcal{G}_1 &= (c_+ - c_-)\lambda_+ \tilde{W} + \tilde{d}_{20}\tilde{W}^2 + \tilde{d}_{11}\tilde{W}\tilde{\Sigma} + \tilde{d}_{02}\tilde{\Sigma}^2 + \tilde{d}_{30}\tilde{W}^3 + \tilde{d}_{21}\tilde{W}^2\tilde{\Sigma} + \tilde{d}_{12}\tilde{W}\tilde{\Sigma}^2 + \tilde{d}_{03}\Sigma^3 \\ &= -\Delta_1 + c_+\Delta_2, \end{aligned} \tag{3.14}$$

$$\begin{aligned} \mathcal{G}_2 &= (c_+ - c_-)\lambda_- \tilde{\Sigma} + \tilde{e}_{20}\tilde{W}^2 + \tilde{e}_{11}\tilde{W}\tilde{\Sigma} + \tilde{e}_{02}\tilde{\Sigma}^2 + \tilde{e}_{30}\tilde{W}^3 + \tilde{e}_{21}\tilde{W}^2\tilde{\Sigma} + \tilde{e}_{12}\tilde{W}\tilde{\Sigma}^2 + \tilde{e}_{03}\Sigma^3 \\ &= \Delta_1 - c_-\Delta_2 \end{aligned} \tag{3.15}$$

and where the values of the coefficients are collected in (F.1)*,* (F.2)*.*

Proof of Lemma 3.4. This is a direct computation.

step 1 Reexpressing $\Delta, \Delta_1, \Delta_2$. Let $w = w_2 + W$, $\sigma = \sigma_2 + \Sigma$, we compute the nonlinear terms

$$\begin{aligned} & \Delta_1 = w^3 - (r+1)w^2 + rw - dw\sigma^2 + \ell(r-1)\sigma^2 \\ = \; & w_2^3 + 3w_2^2 W + 3w_2 W^2 + W^3 - (r+1)(w_2^2 + 2w_2 W + W^2) + r(w_2 + W) \\ - \; & d(w_2 + W)(\sigma_2^2 + 2\sigma_2\Sigma + \Sigma^2) + \ell(r-1)(\sigma_2^2 + 2\sigma_2\Sigma + \Sigma^2) \\ = \; & W(3w_2^2 - 2(r+1)w_2 + r - d\sigma_2^2) + \Sigma(-2d\sigma_2 w_2 + 2\ell(r-1)\sigma_2) \\ + \; & W^2(3w_2 - (r+1)) + \Sigma^2(\ell(r-1) - dw_2) + \Sigma W(-2d\sigma_2) \\ + \; & W^3 - dW\Sigma^2 \\ = \; & c_1 W + c_3\Sigma + d_{20}W^2 + d_{11}W\Sigma + d_{02}\Sigma^2 + W^3 - dW\Sigma^2 \end{aligned} \tag{3.16}$$

and

$$\begin{aligned}
&\Delta_2 = \frac{\sigma}{\ell}\Big[(\ell+d-1)w^2 - w(\ell+d+\ell r-r)+\ell r-\ell\sigma^2\Big]\\
=\;& \frac{\sigma_2+\Sigma}{\ell}\left[(\ell+d-1)(w_2^2+2w_2W+W^2)-(\ell+d+\ell r-r)(w_2+W)+\ell r-\ell(\sigma_2^2+2\sigma_2\Sigma+\Sigma^2)\right]\\
=\;& \frac{\sigma_2+\Sigma}{\ell}\left[(\ell+d-1)(2w_2W+W^2)-(\ell+d+\ell r-r)W-\ell(2\sigma_2\Sigma+\Sigma^2)\right]\\
=\;& W\frac{\sigma_2}{\ell}[2w_2(\ell+d-1)-(\ell+d+\ell r-r)]-2\sigma_2^2\Sigma\\
+\;& W^2\frac{\sigma_2(\ell+d-1)}{\ell}+\Sigma^2[-3\sigma_2]+W\Sigma\left[\frac{2w_2(\ell+d-1)-(\ell+d+\ell r-r)}{\ell}\right]\\
+\;& W^2\Sigma\left[\frac{\ell+d-1}{\ell}\right]+\Sigma^3[-1]\\
=\;& c_2W+c_4\Sigma+e_{20}W^2+e_{11}W\Sigma+e_{02}\Sigma^2+e_{21}W^2\Sigma+\Sigma^3[-1] \qquad (3.17)
\end{aligned}$$

where the parameters are given by (F.1).

step 2 Y variable. We now pass to the Y variable:

$$\begin{aligned}
&\Delta_1(X) = c_1W+c_3\Sigma+d_{20}W^2+d_{11}W\Sigma+d_{02}\Sigma^2+W^3-dW\Sigma^2\\
=\;& c_1(c_-\tilde W+c_+\tilde\Sigma)+c_3(\tilde W+\tilde\Sigma)+d_{20}(c_-\tilde W+c_+\tilde\Sigma)^2+d_{11}(c_-\tilde W+c_+\tilde\Sigma)(\tilde W+\tilde\Sigma)\\
+\;& d_{02}(\tilde W+\tilde\Sigma)^2+(c_-\tilde W+c_+\tilde\Sigma)^3-d(c_-\tilde W+c_+\tilde\Sigma)(\tilde W+\tilde\Sigma)^2\\
=\;& (c_1c_-+c_3)\tilde W+(c_1c_++c_3)\tilde\Sigma+d_{20}(c_-^2\tilde W^2+2c_-c_+\tilde W\tilde\Sigma+c_+^2\tilde\Sigma^2)\\
+\;& d_{11}(c_-\tilde W^2+(c_-+c_+)\tilde W\tilde\Sigma+c_+\tilde\Sigma^2)+d_{02}(\tilde W^2+2\tilde W\tilde\Sigma+\tilde\Sigma^2)\\
+\;& (c_-^3\tilde W^3+3c_-^2\tilde W^2c_+\tilde\Sigma+3c_-\tilde Wc_+^2\tilde\Sigma^2+c_+^3\tilde\Sigma^3)-d(c_-\tilde W+c_+\tilde\Sigma)(\tilde W^2+2\tilde W\tilde\Sigma+\tilde\Sigma^2)\\
=\;& (c_1c_-+c_3)\tilde W+(c_1c_++c_3)\tilde\Sigma\\
+\;& (d_{20}c_-^2+d_{11}c_-+d_{02})\tilde W^2+(2c_-c_+d_{20}+(c_-+c_+)d_{11}+2d_{02})\tilde W\tilde\Sigma+(d_{20}c_+^2+d_{11}c_++d_{02})\tilde\Sigma^2\\
+\;& (c_-^3-dc_-)\tilde W^3+(3c_-^2c_+-2dc_--dc_+)\tilde W^2\tilde\Sigma+(3c_-c_+^2-dc_--2dc_+)\tilde W\tilde\Sigma^2+(c_+^3-dc_+)\Sigma^3
\end{aligned}$$

and

$$\begin{aligned}
&\Delta_2(X) = c_2W + c_4\Sigma + e_{20}W^2 + e_{11}W\Sigma + e_{02}\Sigma^2 + \left(\frac{\ell+d-1}{\ell}\right)W^2\Sigma - \Sigma^3\\
=\;& c_2(c_-\tilde{W} + c_+\tilde{\Sigma}) + c_4(\tilde{W}+\tilde{\Sigma})\\
+\;& e_{20}(c_-\tilde{W} + c_+\tilde{\Sigma})^2 + e_{11}(c_-\tilde{W} + c_+\tilde{\Sigma})(\tilde{W}+\tilde{\Sigma}) + e_{02}(\tilde{W}+\tilde{\Sigma})^2\\
+\;& \left(\frac{\ell+d-1}{\ell}\right)(c_-\tilde{W} + c_+\tilde{\Sigma})^2(\tilde{W}+\tilde{\Sigma}) - (\tilde{W}+\tilde{\Sigma})^3\\
=\;& (c_2c_- + c_4)\tilde{W} + (c_2c_+ + c_4)\tilde{\Sigma} + e_{20}(c_-^2\tilde{W}^2 + 2c_-c_+\tilde{W}\tilde{\Sigma} + c_+^2\tilde{\Sigma}^2)\\
+\;& e_{11}(c_-\tilde{W}^2 + (c_- + c_+)\tilde{W}\tilde{\Sigma} + c_+\tilde{\Sigma}^2) + e_{02}(\tilde{W}^2 + 2\tilde{W}\tilde{\Sigma} + \tilde{\Sigma}^2)\\
+\;& \frac{\ell+d-1}{\ell}\left[c_-^2\tilde{W}^3 + (c_-^2 + 2c_+c_-)\tilde{W}^2\tilde{\Sigma} + (2c_-c_+ + c_+^2)\tilde{W}\tilde{\Sigma}^2 + c_+^2\tilde{\Sigma}^3\right]\\
-\;& (\tilde{W}^3 + 3\tilde{W}^2\tilde{\Sigma} + 3\tilde{W}\tilde{\Sigma}^2 + \tilde{\Sigma}^3)\\
=\;& (c_2c_- + c_4)\tilde{W} + (c_2c_+ + c_4)\tilde{\Sigma}\\
+\;& (e_{20}c_-^2 + e_{11}c_- + e_{02})\tilde{W}^2 + (2e_{20}c_-c_+ + e_{11}(c_- + c_+) + 2e_{02})\tilde{W}\tilde{\Sigma} + (e_{20}c_+^2 + e_{11}c_+ + e_{02})\tilde{\Sigma}^2\\
+\;& \left[\frac{\ell+d-1}{\ell}c_-^2 - 1\right]\tilde{W}^3 + \left[\frac{\ell+d-1}{\ell}(c_-^2 + 2c_-c_+) - 3\right]\tilde{W}^2\tilde{\Sigma}\\
+\;& \left[\frac{\ell+d-1}{\ell}(2c_-c_+ + c_+^2) - 3\right]\tilde{W}\tilde{\Sigma}^2 + \left[\frac{\ell+d-1}{\ell}c_+^2 - 1\right]\tilde{\Sigma}^3.
\end{aligned}$$

Linear terms yield

$$\begin{aligned}
&\frac{1}{c_+ - c_-}\left|\begin{array}{l} -((c_1c_- + c_3)\tilde{W} + (c_1c_+ + c_3)\tilde{\Sigma}) + c_+[(c_2c_- + c_4)\tilde{W} + (c_2c_+ + c_4)\tilde{\Sigma}] \\ (c_1c_- + c_3)\tilde{W} + (c_1c_+ + c_3)\tilde{\Sigma} - c_-[(c_2c_- + c_4)\tilde{W} + (c_2c_+ + c_4)\tilde{\Sigma}]\end{array}\right.\\
=\;& \frac{1}{c_+ - c_-}\left|\begin{array}{l} [-(c_1c_- + c_3) + c_+(c_2c_- + c_4)]\tilde{W} + [-(c_1c_+ + c_3) + c_+(c_2c_+ + c_4)]\tilde{\Sigma} \\ ((c_1c_- + c_3) - c_-(c_2c_- + c_4))\tilde{W} + [(c_1c_+ + c_3) - c_-(c_2c_+ + c_4)]\tilde{\Sigma}\end{array}\right.
\end{aligned}$$

We then recall the $c_\pm$ equation (2.44) which kills the off diagonal term

$$\left|\begin{array}{l} -(c_1c_+ + c_3) + c_+(c_2c_+ + c_4) = 0 \\ (c_1c_- + c_3) - c_-(c_2c_- + c_4) = 0\end{array}\right.$$

and compute using (2.50):

$$\left|\begin{array}{l} \frac{-(c_1c_- + c_3) + c_+(c_2c_- + c_4)}{c_+ - c_-} = \frac{-c_-(c_2c_- + c_4) + + c_+(c_2c_- + c_4)}{c_+ - c_-} = c_2c_- + c_4 = \lambda_+ \\ \frac{(c_1c_+ + c_3) - c_-(c_2c_+ + c_4)}{c_+ - c_-} = \frac{c_+(c_2c_+ + c_4) - c_-(c_2c_+ + c_4)}{c_+ - c_-} = c_2c_+ + c_4 = \lambda_-\end{array}\right.$$

For the quadratic terms, we compute for the first coordinate

$$\begin{aligned}
\tilde{d}_{20} &= -(d_{20}c_-^2 + d_{11}c_- + d_{02}) + c_+(e_{20}c_-^2 + e_{11}c_- + e_{02})\\
&= (c_+e_{20} - d_{20})c_-^2 + (c_+e_{11} - d_{11})c_- + c_+e_{02} - d_{02}, \qquad (3.18)
\end{aligned}$$

and

$$\begin{aligned}
\tilde{d}_{11} &= -(2c_-c_+d_{20} + (c_- + c_+)d_{11} + 2d_{02}) + c_+(2e_{20}c_-c_+ + e_{11}(c_- + c_+) + 2e_{02})\\
&= 2c_-c_+(c_+e_{20} - d_{20}) + (c_- + c_+)(c_+e_{11} - d_{11}) + 2(c_+e_{02} - d_{02})
\end{aligned}$$

and

$$\begin{aligned}
\tilde{d}_{02} &= -(d_{20}c_+^2 + d_{11}c_+ + d_{02}) + c_+(e_{20}c_+^2 + e_{11}c_+ + e_{02})\\
&= (c_+e_{20} - d_{20})c_+^2 + (c_+e_{11} - d_{11})c_+ + c_+e_{02} - d_{02}
\end{aligned}$$

and

$$\tilde{d}_{30} = -(c_-^3 - dc_-) + c_+ \left(\frac{\ell + d - 1}{\ell} c_-^2 - 1 \right)$$

and

$$\tilde{d}_{21} = -(3c_-^2 c_+ - 2dc_- - dc_+) + c_+ \left(\frac{\ell + d - 1}{\ell} (c_-^2 + 2c_- c_+) - 3 \right)$$

and

$$\tilde{d}_{12} = -(3c_- c_+^2 - dc_- - 2dc_+) + c_+ \left(\frac{\ell + d - 1}{\ell} (2c_- c_+ + c_+^2) - 3 \right).$$

Similarly for the second coordinate:

$$\begin{aligned} \tilde{e}_{20} &= (d_{20} c_-^2 + d_{11} c_- + d_{02}) - c_-(e_{20} c_-^2 + e_{11} c_- + e_{02}) \\ &= (d_{20} - c_- e_{20}) c_-^2 + (d_{11} - c_- e_{11}) c_- + d_{02} - c_- e_{02} \end{aligned} \tag{3.19}$$

and

$$\begin{aligned} \tilde{e}_{11} &= 2c_- c_+ d_{20} + (c_- + c_+) d_{11} + 2d_{02} - c_-(2e_{20} c_- c_+ + e_{11}(c_- + c_+) + 2e_{02}) \\ &= 2c_- c_+ (d_{20} - c_- e_{20}) + (c_- + c_+)(d_{11} - c_- e_{11}) + 2(d_{02} - c_- e_{02}) \end{aligned}$$

and

$$\begin{aligned} \tilde{e}_{02} &= d_{20} c_+^2 + d_{11} c_+ + d_{02} - c_-(e_{20} c_+^2 + e_{11} c_+ + e_{02}) \\ &= (d_{20} - c_- e_{20}) c_+^2 + (d_{11} - c_- e_{11}) c_+ + d_{02} - c_- e_{02} \end{aligned}$$

and

$$\tilde{e}_{21} = (3c_-^2 c_+ - 2dc_- - dc_+) - c_- \left(\frac{\ell + d - 1}{\ell} (c_-^2 + 2c_- c_+) - 3 \right),$$

this is (F.2).
Introducing the time variable t (we note that $t \to +\infty$ corresponds to both $Z \uparrow\downarrow Z_2$):

$$\frac{dt}{dx} = -\frac{1}{\Delta}$$

yields

$$\frac{dY}{dt} = \left| \begin{array}{l} \frac{(c_+ - c_-)\lambda_+ \tilde{W} + \tilde{d}_{20} \tilde{W}^2 + \tilde{d}_{11} \tilde{W}\tilde{\Sigma} + \tilde{d}_{02} \tilde{\Sigma}^2 + \tilde{d}_{30} \tilde{W}^3 + \tilde{d}_{21} \tilde{W}^2 \tilde{\Sigma} + \tilde{d}_{12} \tilde{W}\tilde{\Sigma}^2 + \tilde{d}_{03} \Sigma^3}{c_+ - c_-} \\ \frac{(c_+ - c_-)\lambda_- \tilde{\Sigma} + \tilde{e}_{20} \tilde{W}^2 + \tilde{e}_{11} \tilde{W}\tilde{\Sigma} + \tilde{e}_{02} \tilde{\Sigma}^2 + \tilde{e}_{30} \tilde{W}^3 + \tilde{e}_{21} \tilde{W}^2 \tilde{\Sigma} + \tilde{e}_{12} \tilde{W}\tilde{\Sigma}^2 + \tilde{e}_{03} \Sigma^3}{c_+ - c_-}. \end{array} \right.$$

This is (3.14), (3.15). □

3.4. **Integral curves passing through P_2.**

Lemma 3.5 (Slope of the curves converging to P_2)**.** *Assume* (3.1). *Let c_+, c_-, A be given by* (2.44), (2.48). *Then,*

1. *all integral curves of* (1.9) *converging to P_2 have slope given either by c_+ or c_-, where the slope refers to the limit of $\frac{dw}{d\sigma}$ as $\sigma \to \sigma_2$,*
2. *there are only two curves with slope c_+, with one converging to P_2 from the region $\Delta_1 > 0$ and the other converging to P_2 from the region $\Delta_1 < 0$, while all the other curves have slope c_-,*
3. *the unique curve converging to P_2 from the region $\Delta_1 > 0$ with the slope c_+ exists on $0 < \sigma < \sigma_2$ and converges to P_4 as $\sigma \to 0$.*

Proof. The Jacobian matrix of the autonomous system (3.13) at the equilibrium $(\tilde{W}, \tilde{\Sigma}) = (0,0)$ is diagonal with two negative eigenvalues $\lambda_- < \lambda_+ < 0$. Thus, standard results imply that $(0,0)$ is an asymptotically stable node, and the following holds for the trajectories converging to $(0,0)$ as $t \to +\infty$

(1) there exists exactly two trajectories tangent to the eigenvector of the Jacobian matrix corresponding to the smallest eigenvalue λ_-, i.e. these trajectories satisfy

$$\lim_{t\to+\infty} \frac{\tilde{W}}{\tilde{\Sigma}} = 0,$$

(2) all the other trajectories are tangent to the eigenvector of the Jacobian matrix corresponding to the largest eigenvalue λ_+, i.e. these trajectories satisfy

$$\lim_{t\to+\infty} \frac{\tilde{\Sigma}}{\tilde{W}} = 0.$$

Coming back to (W, Σ), we infer that the slope of any curve converging to P_2 is either c_+ or c_-, and there are exactly two with slope c_+, while all the others have slope c_-. Also, since $c_+ > 0$ while the slope $w_2'(\sigma_2)$ of the middle root of Δ_1 is negative, see Lemma 2.9, among the two curves converging to P_2 with c_+ slope, one converges to P_2 from the region $\Delta_1 > 0$ and the other converges to P_2 from the region $\Delta_1 < 0$.

The fact that the unique curve converging to P_2 from the region $\Delta_1 > 0$ with slope c_+ is defined all the way to $\sigma = 0$ and attracted to P_4 is a straightforward consequence of the phase portrait in Figure 3. □

Lemma 3.6. *Assume* (3.1). *Let* c_-, A *be given by* (2.44), (2.48) *and assume that*

$$A = K + \alpha, \quad K \in \mathbb{N}\backslash\{0,1\}, \quad 0 < \alpha < 1.$$

Then, there exists a unique solution curve (w, σ) *to* (1.9) *which is a* $\mathcal{C}^\infty$ *function of* x *at* P_2 *with slope* c_-. *This solution also satisfies that* $w(\sigma)$ *is a* $\mathcal{C}^\infty$ *function of* σ *at* σ_2.

Remark 3.7. One can show that all the other curves converging to P_2 with slope c_-, see Lemma 3.5, are $\mathcal{C}^{K+\alpha}$ both on the right and on the left of P_2.

Proof. Recall from Lemma 3.5 that the curves converging to P_2 with slope c_- satisfy $\tilde{W} = o(\tilde{\Sigma})$ near P_2, and hence, we have at P_2, i.e., at $\tilde{W} = 0$,

$$\tilde{\Psi} := \frac{\tilde{\Sigma}}{\tilde{W}}, \qquad \tilde{\Psi}(0) = 0. \tag{3.20}$$

Now

$$\begin{aligned}
&\tilde{W}\frac{d\tilde{\Psi}}{d\tilde{W}} + \tilde{\Psi} = \frac{d\tilde{\Sigma}}{d\tilde{W}} \\
=\ & \frac{(c_+ - c_-)\lambda_-\tilde{\Sigma} + \tilde{e}_{20}\tilde{W}^2 + \tilde{e}_{11}\tilde{W}\tilde{\Sigma} + \tilde{e}_{02}\tilde{\Sigma}^2 + \tilde{e}_{30}\tilde{W}^3 + \tilde{e}_{21}\tilde{W}^2\tilde{\Sigma} + \tilde{e}_{12}\tilde{W}\tilde{\Sigma}^2 + \tilde{e}_{03}\Sigma^3}{(c_+ - c_-)\lambda_+\tilde{W} + \tilde{d}_{20}\tilde{W}^2 + \tilde{d}_{11}\tilde{W}\tilde{\Sigma} + \tilde{d}_{02}\tilde{\Sigma}^2 + \tilde{d}_{30}\tilde{W}^3 + \tilde{d}_{21}\tilde{W}^2\tilde{\Sigma} + \tilde{d}_{12}\tilde{W}\tilde{\Sigma}^2 + \tilde{d}_{03}\Sigma^3} \\
=\ & \frac{(c_+ - c_-)\lambda_-\tilde{\Psi} + \tilde{e}_{20}\tilde{W} + \tilde{e}_{11}\tilde{W}\tilde{\Psi} + \tilde{e}_{02}\tilde{\Psi}^2\tilde{W} + \tilde{e}_{30}\tilde{W}^2 + \tilde{e}_{21}\tilde{W}^2\tilde{\Psi} + \tilde{e}_{12}\tilde{W}^2\tilde{\Psi}^2 + \tilde{e}_{03}\tilde{\Psi}^3\tilde{W}^2}{(c_+ - c_-)\lambda_+ + \tilde{d}_{20}\tilde{W} + \tilde{d}_{11}\tilde{W}\tilde{\Psi} + \tilde{d}_{02}\tilde{\Psi}^2\tilde{W} + \tilde{d}_{30}\tilde{W}^2 + \tilde{d}_{21}\tilde{W}^2\tilde{\Psi} + \tilde{d}_{12}\tilde{W}^2\Psi^2 + \tilde{d}_{03}\tilde{\Psi}^3\tilde{W}^2}
\end{aligned}$$

and hence

$$\tilde{W}\frac{d\tilde{\Psi}}{d\tilde{W}} - \left[\frac{\lambda_-}{\lambda_+} - 1\right]\tilde{\Psi} = \frac{\tilde{W}G_1(\tilde{W}, \tilde{\Psi})}{G_2(\tilde{W}, \tilde{\Psi})}$$

with $G_2(0,0) \neq 0$. Since $A = \lambda_-/\lambda_+$, we infer

$$\tilde{W}\frac{d\tilde{\Psi}}{d\tilde{W}} - [A-1]\,\tilde{\Psi} = \frac{\tilde{W}G_1(\tilde{W},\tilde{\Psi})}{G_2(\tilde{W},\tilde{\Psi})}.$$

Let $k \geq 1$. Assuming that $\tilde{W}$ is $\mathcal{C}^\infty$, we differentiate k times, and evaluate at $\tilde{W} = 0$. We infer

$$[k+1-A]\,\tilde{\Psi}^{(k)}(0) = k\left[\frac{G_1(\tilde{W},\tilde{\Psi})}{G_2(\tilde{W},\tilde{\Psi})}\right]^{(k-1)}_{|_{(\tilde{W},\tilde{\Psi})=(0,0)}}, \quad k \geq 1, \qquad \tilde{\Psi}(0) = 0. \quad (3.21)$$

Since $A \notin \mathbb{N}$, (3.21) yields the uniqueness of the Taylor expansion of $\tilde{\Psi}$ at any order.

Let $k \in \mathbb{N}^*$ with $k > A-2$. We denote by $P_k(\tilde{W})$ the unique Taylor polynomial of degree k provided by solving the iteration (3.21) for $j = 1, \cdots, k$. Then, $P_k(\tilde{W})$ satisfies

$$\tilde{W}\frac{dP_k}{d\tilde{W}} - [A-1]\,P_k = \frac{\tilde{W}G_1(\tilde{W},P_k(\tilde{W}))}{G_2(\tilde{W},P_k(\tilde{W}))} + \tilde{W}^{k+1}Q_k,$$

where Q_k is a polynomial uniquely prescribed by k, A, G_1 and G_2. We decompose

$$\tilde{\Psi} \;=\; P_k + \tilde{\varepsilon}_k, \qquad \tilde{\varepsilon}_k = O(\tilde{W}^{k+1}).$$

Plugging this decomposition into the equation for $\tilde{\Psi}$, we infer

$$\begin{aligned}\tilde{W}\frac{d\tilde{\varepsilon}_k}{d\tilde{W}} - [A-1]\,\tilde{\varepsilon}_k &= \frac{\tilde{W}G_1(\tilde{W},P_k(\tilde{W})+\tilde{\varepsilon}_k)}{G_2(\tilde{W},P_k(\tilde{W})+\tilde{\varepsilon}_k)} - \left(\tilde{W}\frac{dP_k}{d\tilde{W}} - [A-1]\,P_k\right)\\ &= -\tilde{W}^{k+1}Q_k + \tilde{W}\tilde{\varepsilon}_k H_k(\tilde{W},\tilde{\varepsilon}_k)\end{aligned}$$

where H_k is a smooth function near $(0,0)$ uniquely determined by k, A, G_1 and G_2. We deduce

$$\frac{d}{d\tilde{W}}\Big(\tilde{W}^{1-A}\tilde{\varepsilon}_k\Big) \;=\; -\tilde{W}^{k+1-A}Q_k + \frac{\tilde{\varepsilon}_k}{\tilde{W}^{A-1}}H_k(\tilde{W},\tilde{\varepsilon}_k).$$

Now, since $\tilde{\varepsilon}_k = O(\tilde{W}^{k+1})$ and $k > A-2$, we deduce that if a solution curve is $\mathcal{C}^\infty$ at P_2 with slope c_-, then it must satisfy

$$\tilde{\varepsilon}_k \;=\; \tilde{W}^{A-1}\int_0^{\tilde{W}}\left(-\tilde{w}^{k+1-A}Q_k + \frac{\tilde{\varepsilon}_k}{\tilde{w}^{A-1}}H_k(\tilde{w},\tilde{\varepsilon}_k)\right)d\tilde{w}. \quad (3.22)$$

Using a fixed point argument, there exists a unique solution to (3.22) defined in a neighborhood of $\tilde{W} = 0$. Uniqueness follows.

We now focus on the existence of a solution curve $\mathcal{C}^\infty$ at P_2 with slope c_-. Let

$$\tilde{\Psi} \;:=\; P_{\lceil A\rceil-2} + \tilde{\varepsilon}_{\lceil A\rceil-2},$$

where $\tilde{\varepsilon}_{\lceil A\rceil-2}$ is the unique solution to the fixed point argument fixed (3.22) for $k = \lceil A\rceil - 2 > A-2$. With this definition, $\tilde{\Psi}$ yields a solution curve with slope c_- at P_2, and it remains to prove that it is $\mathcal{C}^\infty$ at P_2. In fact, it suffices to prove that

$$P_k + \tilde{\varepsilon}_k \;=\; P_{\lceil A\rceil-2} + \tilde{\varepsilon}_{\lceil A\rceil-2} \text{ for all } k \geq \lceil A\rceil - 2, \quad (3.23)$$

as it will then imply that $\tilde{\Psi}$ has a Taylor expansion at P_2 at any order and is hence $\mathcal{C}^\infty$.

Next, we focus on proving (3.23). For $k > A-2$, we define $\delta\tilde{\Psi}_{k,A}$ by

$$\delta\tilde{\Psi}_{k,A} \;:=\; P_k - P_{\lceil A\rceil-2} + \tilde{\varepsilon}_k - \tilde{\varepsilon}_{\lceil A\rceil-2}$$

which satisfies in view of the properties of $P_{\lceil A\rceil-2}$, P_k, $\tilde{\varepsilon}_{\lceil A\rceil-2}$ and $\tilde{\varepsilon}_k$

$$|\delta\tilde{\Psi}_{k,A}| \;=\; O(\tilde{W}^{\lceil A\rceil-1})$$

and

$$\tilde{W}\frac{d\delta\tilde{\Psi}_{k,A}}{d\tilde{W}} - [A-1]\,\delta\tilde{\Psi}_{k,A} \;=\; \tilde{W}\delta\tilde{\Psi}_{k,A}J_{k,A}(\tilde{W},P_k,P_{\lceil A\rceil-2},\tilde{\varepsilon}_k,\tilde{\varepsilon}_{\lceil A\rceil-2})$$

where $J_{k,A}$ is a smooth function near $(0,0)$ uniquely prescribed by k, A, G_1 and G_2. In view of the above control for $\delta\tilde{\Psi}_{k,A}$, we deduce

$$|\delta\tilde{\Psi}_{k,A}| \;\lesssim\; \tilde{W}^{A-1}\int_0^{\tilde{W}}\frac{|\delta\tilde{\Psi}_{k,A}|}{\tilde{w}^{A-1}}d\tilde{w}.$$

Hence $\delta\tilde{\Psi}_{k,A}$ vanishes identically, so that (3.23) holds as desired.

We have thus obtained a unique solution $(\tilde{\Sigma},\tilde{W})$ such that $\tilde{\Psi}$ is $\mathcal{C}^\infty$ at $\tilde{W}=0$. Now, proceeding as in the proof of Lemma 3.4, we easily obtain from (2.47) (3.12)

$$\begin{aligned}
\Delta &= \tilde{W}\Big[c_\Delta + \tilde{W}F_0(\tilde{W},\tilde{\Psi}) + \tilde{\Psi}G_0(\tilde{W},\tilde{\Psi})\Big],\\
\Delta_1 &= \tilde{W}\Big[c_{\Delta_1} + \tilde{W}F_1(\tilde{W},\tilde{\Psi}) + \tilde{\Psi}G_1(\tilde{W},\tilde{\Psi})\Big],\\
\Delta_2 &= \tilde{W}\Big[c_{\Delta_2} + \tilde{W}F_2(\tilde{W},\tilde{\Psi}) + \tilde{\Psi}G_2(\tilde{W},\tilde{\Psi})\Big],
\end{aligned}$$

where F_j, G_j, $j=0,1,2$ are polynomials in $(\tilde{W},\tilde{\Psi})$ and c_{Δ_j}, $j=0,1,2$ are nonzero constants. Together with (3.14), this implies that

$$\frac{\mathcal{G}_1}{\Delta},\qquad \frac{\Delta_1}{\Delta_2},$$

are $\mathcal{C}^\infty$ and nonzero at $\tilde{W}=0$. In view of (3.13), we have

$$\frac{d\tilde{W}}{dx} \;=\; \frac{1}{c_+-c_-}\mathcal{G}_1\frac{dt}{dx} = -\frac{1}{c_+-c_-}\frac{\mathcal{G}_1}{\Delta}$$

so that $\tilde{W}$ is a $\mathcal{C}^\infty$ function of x at P_2. Then $\tilde{\Psi}$ and hence $\tilde{\Sigma}$ are also $\mathcal{C}^\infty$ function of x at P_2, and hence, (w,σ) is a solution to (1.9) which is a $\mathcal{C}^\infty$ function of x at P_2. Finally, since $\frac{\Delta_1}{\Delta_2}$ is $\mathcal{C}^\infty$ and nonzero at $\tilde{W}=0$, $w(\sigma)$ is a $\mathcal{C}^\infty$ function of σ at σ_2. □

4. **Renormalization of the flow near P_2**

Our aim in this section is to start the proof of Theorem 1.3 with the renormalization of the flow (1.9) for $0 < r_\backsim - r \ll 1$ near P_2 with $r_\backsim$ given by (1.14). We first detail the strategy of the proof in section 1.3, and then proceed to the expected renormalization. The proof of Theorem 1.3 will then be completed in sections 5, 6, 7. For the rest of this paper, we assume

$$\ell\in\mathbb{R}^*\backslash\{d\},\quad 0<r_\backsim(d,\ell)-r\ll 1.$$

We will also denote

$$P_\backsim = \left|\begin{array}{ll} P_5 & \text{for}\quad \ell<d\\ P_3 & \text{for}\quad \ell<d.\end{array}\right.$$

4.1. **Strategy of the proof of Theorem 1.3.** We describe the main steps of the proof of Theorem 1.3.

step 1 Renormalization. We introduce a suitable semi classical parameter, see (4.11),

$$0 < b = o_{r \uparrow r_{\circledcirc}}(1)$$

and use the geometry of the "eye" $P_{\circledcirc}, P_2$ to produce a suitable renormalization. Here we use a fundamental degeneracy of the phase portrait, see Lemma 2.6, which gives the eye property

$$\lim_{r \uparrow r_{\circledcirc}} |P_{\circledcirc}(r) - P_2(r)| = 0, \tag{4.1}$$

and the slopes of $w_2(\sigma)$ and $w_2^-(\sigma)$ converge to the same value, see figure 3, 4. Passing to the diagonalized variables (2.47) and after explicit suitable reductions, this degeneracy leads to a quadratic cancellation, (4.13). Solving for $w(\sigma)$, we are left with the study of a problem of the form[9]:

$$u(1-u)\frac{d\Theta}{du} - [\gamma - 2 + (\nu_b + 3)u]\,\Theta = -\frac{\mathcal{G}}{b}. \tag{4.2}$$

Here we introduced the parameters which appear in the renormalization process

$$\left| \begin{array}{l} \gamma = \frac{c_\infty(d,\ell)(1+o_{b\to 0}(1))}{b}, \quad c_\infty > 0 \\ \nu_b = \nu_\infty(d,\ell) + o_{b\to 0}(1), \quad \nu_\infty \neq 0 \end{array} \right. \tag{4.3}$$

The trajectory is $\Theta(u)$ with $u = 0$ at σ_2, $u = 1$ at $\sigma_{\circledcirc}$. $\mathcal{G}$ is an explicit nonlinear term. An analogous reduction can be performed to the right of P_2.

step 2 Main part of the solution. The nonlinear ode (4.2) has a regular singular point at the origin $u = 0$. Therefore, it admits a $\mathcal{C}^\infty$ solution with an holomorphic expansion at the origin

$$\Theta(u) = \sum_{k=0}^{+\infty} \theta_k(b,d,\ell) u^k$$

where $\theta_k(b,d,\ell)$ is given by an explicit b dependent induction relation. We let

$$\gamma - 1 = K + \alpha_\gamma, \quad 0 < \alpha_\gamma < 1 \tag{4.4}$$

and truncate the holomorphic expansion at the critical frequency[10]:

$$\Theta(u) = \sum_{k=0}^{K-2} \theta_k u^k + \theta_{K-1} u^{K-1} + (-1)^{K-1} S_{K-1} \Theta_{\rm main}(u) - \mathcal{T}(r_{\mathcal{G}}) \tag{4.5}$$

where $\Theta_{\rm main} = O(u^K)$ is an explicit integral, and $\mathcal{T}(r_{\mathcal{G}})$ is a remainder which is of higher order. Our first fundamental observation is that there exists a strong limit

$$\lim_{b\to 0} S_{K-1} = S_\infty(d,\ell). \tag{4.6}$$

The proof relies on bounding the formal series solution to a limiting problem with $b = 0$ first. This is done in Proposition 5.1, which belongs to the realm of nonlinear Maillet theorems, [8, 14]. The original problem (4.2) can be thought of as a b-deformation of the limiting problem. The challenge however is that we need uniform estimates for all frequencies up to the critical value K *which itself is of size* $\sim \frac{1}{b}$

[9] observe that the notation u for the variable appearing in (4.2) should not be confused with the one for the solution to the Euler equation (1.1).

[10] which corresponds to the limit of regularity of a generic solution, see Lemma 3.6.

step 3 Non vanishing of $S_\infty(d,\ell)$. The proof of finiteness of $S_\infty(d,\ell)$ implies the analycity of the mapping $\ell \to S_\infty(d,\ell)$ on suitable domains of the complex plane (see Lemma 1.5). This number can be reexpressed as an explicit normally convergent series, but we do not know how to prove analytically that it is non zero. We therefore perform a numerical study of this convergent series which allows us to provide windows of parameters (d,ℓ) for which

$$S_\infty(d,\ell) \neq 0. \tag{4.7}$$

step 4 No oscillation at the left of P_2. In the variable Θ, it is easily seen that the $P_2 - P_\infty$ separatrix satisfies $|\Theta| \lesssim_{\ell,d} 1$, Lemma 7.11. Hence we pick a large enough (in absolute value) constant $\Theta^* \gg 1$ and aim at reaching the value

$$\Theta(u^*) = \Theta^*, \quad 0 < u^* < 1. \tag{4.8}$$

The second fundamental observation is that the function $\Theta_{\rm main}$ can be analyzed explicitly near $u = 0$

$$\begin{aligned} \Theta_{\rm main} &= \Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)K^{\nu_b+3-\alpha_\gamma}u^{K-1} \\ &\times \left\{[1+o_{b\to 0}(1)]\left[\frac{1}{\Gamma(1-\alpha_\gamma)} + \frac{K+\nu_b+2}{\Gamma(2-\alpha_\gamma)}u\right] + \text{lot}\right\}. \end{aligned} \tag{4.9}$$

In a boundary layer close to the integer values

$$\alpha_\gamma \in (\varepsilon_1, 2\varepsilon_1) \cup (1-2\varepsilon_2, 1-\varepsilon_2), \quad \varepsilon_i = o_{b\to 0}(1), \tag{4.10}$$

we can ensure that (4.8) happens for a small $0 < u^*(\alpha_\gamma) \ll 1$. For the interval $2\varepsilon_1 < \alpha_\gamma < 1 - 2\varepsilon_2$, we need to understand $\Theta_{\rm main}$ away from $u = 0$ where the truncated Taylor expansion no longer dominates, and here we use the *explicit integral representation of* $\Theta_{\rm main}$ to show that (4.8) happens for $u^*(\alpha_\gamma) < \frac{1}{2}$. The conclusion is that for every $\alpha_\gamma \in (0,1)$ except maybe a very small b dependent boundary layer around the integer values, the solution to (4.5) reaches (4.8) in time $0 < u^* < \frac{1}{2}$. This means that we are leaving a large neighborhood around the $P_2 - P_5$ separatrix[11] with a prescribed sign $\Theta^* \gg 1$. A further use of monotonicity properties of the flow (1.9) allows us to conclude that the integral curve will intersect either the root branch $w_2(\sigma)$ or $w_2^-(\sigma)$ for some $\sigma_5 < \sigma^* < \sigma_2$. The $(-1)^K$ prefactor in (4.5) dictates that the former happens when K is even while the latter holds when K is odd (if $S_\infty > 0$ and the other way around if $S_\infty < 0$.) Once the trajectory reaches $w_2(\sigma)$, by Lemma 3.2, it then continues on to P_4, as desired.

step 5 Oscillations at the right of P_2. The analysis of the flow to the right of P_2 produces the same decomposition (4.9) but with $u < 0$. We then observe since $\lim_{\alpha_\gamma \uparrow 1} \Gamma(1-\alpha_\gamma) = +\infty$ that by choosing α_γ in a boundary layer close to respectively 0 or 1, the sign $u < 0$ allows us to reach

$$\Theta(u^*) = \left| \begin{array}{l} \Theta^* \ \text{ for } \ \varepsilon_1 < \alpha_\gamma < 2\varepsilon_1 \\ -\Theta^* \ \text{ for } \ \varepsilon_2 < 1-\alpha_\gamma < 2\varepsilon_2 \end{array} \right. , \quad u^* < 0, \quad |u^*| \ll 1.$$

Then, if K is even, the curve will exit through $w_2(\sigma)$ in the first case, in which case we say it exits red, and through the branch $w_2^-(\sigma)$ in the second case, in which case we say it exits green. For odd K the situation is reversed (Again, this holds for $S_\infty > 0$. For $S_\infty < 0$ the picture is reversed.)

step 6 Conclusion by continuity. Given K with the suitable parity (depending on the sign of $S_\infty(d,\ell) \neq 0$), we vary the parameter $\alpha_\gamma \in (\varepsilon_1, 1-\varepsilon_2)$ continuously

[11] i.e. the only solution curve connecting P_2 to P_5.

and conclude that the $\mathcal{C}^\infty$ curve through P_2 crosses the red at the left of P_2 for all $\alpha_\gamma \in (e_1, 1-\varepsilon_2)$, while it also crosses the red at the right of P_2 at the beginning of the α_γ interval, green at the end. Hence an elementary continuity argument implies the existence of at least one value $\alpha_\gamma^* \in (\varepsilon_1, 1-\varepsilon_2)$ such that the solution curve intersects the $P_2 - P_6$ trajectory given by Lemma 3.1. The two curves intersect each other away from singular points and thus, by uniqueness, must coincide. It follows easily that this constructed solution satisfies the conclusions of Theorem 1.3. In other words, as long as the non degeneracy condition (4.7) is satisfied, the integer interval $\gamma \in [K+1, K+2]$, with K given by (4.4) large enough and of suitable parity, contains at least one $\mathcal{C}^\infty$ $P_6 - P_4$ solution. Since γ is related to b through (4.3) and $b = o_{r\uparrow r_\infty}(1)$, the above construction produces an infinite family of global $\mathcal{C}^\infty$ solutions parametrized by the speeds $r_k \uparrow r_\infty(d,\ell)$ for each (d,ℓ) such that $S_\infty(d,\ell) \neq 0$.

4.2. **Degeneracy of the geometry at $r_\infty(d,\ell)$.** We use Lemma 3.4 and the Y variable (3.12) to map (1.9) onto (3.13). The starting point of the renormalization procedure is the following fundamental degeneracy property as $r \uparrow r_\infty(d,\ell)$.

Remark 4.1 (Notation for the parameters)**.** From now on and for the rest of this paper we adopt the following notation: all slopes, characteristic eigenvalues and geometrical parameters involved in the renormalization of the flow, Lemma 3.4, depend of r, and will be noted with an $^\infty$ subscript when evaluated at $r = r_\infty$. The non degeneracy and signs of some of these limiting values will be crucial in the forthcoming analysis, and all relevant values are collected in Appendix F.

Lemma 4.2 (Degeneracy in the diagonalized system)**.** *Let*

$$b = \left| \begin{array}{l} r^* - r \quad \textit{for} \quad \ell < d \\ \sqrt{r_+ - r} \quad \textit{for} \quad \ell > d \end{array} \right. \tag{4.11}$$

and define

$$\mu_+ = \frac{\lambda_+}{b} \tag{4.12}$$

then, denoting by $\tilde{W}_\infty$ and $\tilde{\Sigma}_\infty$ the value at P_∞ of $\tilde{W}$ and $\tilde{\Sigma}$ defined in (3.12)*, we have*

$$\left| \begin{array}{l} \tilde{W}_\infty = -b\frac{(c_+ - c_-)\mu_+}{\tilde{d}_{20}} + O(b^2) \\ \tilde{\Sigma}_\infty = -\frac{\tilde{e}_{20}\tilde{W}_\infty^2}{(c_+ - c_-)\lambda_-} + O(b^3) \\ \mu_+ = \mu_+^\infty + O(b) \end{array} \right. \tag{4.13}$$

where the non degenerate limiting values are computed in Appendix F.

Proof. The value of $\sigma_2(r)$ is computed from (2.18) and hence $\sigma_2 \in \mathcal{C}^\infty(1, r_+)$. Moreover, $J(r)$ is from (2.25) a second order polynomial with roots $r_+ = 1 + \frac{d-1}{(1+\sqrt{\ell})^2} < r_- = 1 + \frac{d-1}{(1-\sqrt{\ell})^2}$ and hence the root r_+ is simple. Since $r^* < r_+$, we conclude that with the definition (4.11), $\sigma_2(r)$ and the slopes coefficients $c_i(r)$ given by (2.41) are smooth functions of b on $[0, b^*]$, $0 < b^*(d,\ell) \ll 1$ universal small enough. We now explicitly check that the determinant $(c_1 - c_4)^2 + 4c_2c_3$ which appears in the definition of the slopes and eigenfunctions (2.44), (2.46) is non degenerate at r_∞, see limiting values in Appendix F and the non degeneracy of λ_-, which ensures that $c_\pm, \lambda_\pm$ are smooth functions of b all the way to $b = 0$. The eye property (4.1) thus implies

$$|W_\infty| + |\Sigma_\infty| = |w_\infty - w_2| + |\sigma_{\rm eye} - \sigma_2| \lesssim Cb.$$

Since the coefficients of the matrix P^{-1} given by (2.47) do not degenerate at r_∞ from direct check, we conclude

$$|\tilde{W}_\infty| + |\tilde{\Sigma}_\infty| \leq Cb.$$

The coefficients $\tilde{d}_{ij}, \tilde{e}_{ij}$ of the polynomials of the RHS of (3.13) are computed from (F.2) and are $O(1)$ at r_∞. Moreover $\mathcal{G}_1, \mathcal{G}_2$ vanish at P_∞ from (3.14), (3.15), and this forces (4.13). □

4.3. **Renormalization.** We now proceed to the renormalization of (3.13) for $0 < b \ll 1$.

Lemma 4.3 (Renormalization and quasilinear formulation)**.** *Let*

$$\left| \begin{array}{l} \tilde{W} = -b\tilde{w} \\ \tilde{\Sigma} = b^2\tilde{\sigma} \end{array} \right. , \quad \left| \begin{array}{l} \tilde{\psi} = \frac{\tilde{\sigma}}{\tilde{w}} = \tilde{\psi}_\infty \phi \\ \tilde{w} = \tilde{w}_\infty u \end{array} \right. , \quad \phi(u) = u + (1-u)\Psi(u) \tag{4.14}$$

where $\tilde{\psi}_\infty$ and $\tilde{w}_\infty$ denote respectively the value at P_∞ of $\tilde{\psi}$ and $\tilde{w}$. Then (3.13) *is mapped to the quasilinear problem*

$$\begin{aligned} & \left[1 + H_2 + G_2\Psi + \widetilde{NL}_2\right] u(1-u)\frac{d\Psi}{du} \\ + & \left[(1-2u)(1 + H_2 + G_2\Psi + \widetilde{NL}_2) - \gamma(1+G_1) + 2uG_2\right]\Psi \\ = & \; u\left[\gamma b H_1 - 2(1+H_2) + \frac{\gamma b \widetilde{NL}_1}{bu} - 2\widetilde{NL}_2\right] \end{aligned} \tag{4.15}$$

where H_1, H_2, G_1, G_2 are explicit polynomials in (b,u) given by (G.2), (G.4)*, and the nonlinear terms $(\widetilde{NL}_i)_{i=1,2}$ are given by* (G.5)*. Moreover,*

$$\left| \begin{array}{l} \tilde{w}_\infty = \frac{(c_+^\infty - c_-^\infty)\mu_+^\infty}{\tilde{d}_{20}^\infty} + O(b) \\ \tilde{\psi}_\infty = -\frac{\tilde{e}_{20}^\infty \mu_+^\infty}{\tilde{d}_{20}^\infty \lambda_-^\infty} + O(b). \end{array} \right. \tag{4.16}$$

Remark 4.4. Unfortunately, we need to keep track of *all* terms in (4.15) since they will create the limiting problem which, in turn, will give rise to the $S_\infty(d,\ell)$ function, evaluated numerically.

Remark 4.5. In the quasilinear formulation (4.14), (4.15), $u = 0$ is P_2 and $u = 1$ is P_∞.

Proof of Lemma 4.3. This is a brute force computation.

step 1 b renormalization. We renormalize (3.13):

$$\left| \begin{array}{l} \tilde{W} = -b\tilde{w} \\ \tilde{\Sigma} = b^2\tilde{\sigma} \\ \lambda_+ = b\mu_+ \\ \frac{d\tau}{dt} = b \end{array} \right.$$

and define the variable

$$\tilde{\psi} = \frac{\tilde{\sigma}}{\tilde{w}}. \tag{4.17}$$

We compute from (4.13), (F.4), (F.12) the expansion as $b \to 0$:

$$\left| \begin{array}{l} \mu_+(r) = \frac{\lambda_+(r)}{b} = \mu_+^\infty + O(b) < 0 \\ \tilde{w}_\infty(r) = -\frac{\tilde{W}_\infty(r)}{b} = \frac{(c_+^\infty - c_-^\infty)\mu_+^\infty}{\tilde{d}_{20}^\infty} + O(b) \\ \tilde{\psi}_\infty = \frac{\tilde{\sigma}_\infty}{\tilde{w}_\infty} = -\frac{\tilde{\Sigma}_\infty}{b^2\frac{\tilde{W}_\infty}{b}} = \frac{\tilde{e}_{20}\tilde{W}_\infty}{b(c_+ - c_-)\lambda_-} = -\frac{\tilde{e}_{20}^\infty \mu_+^\infty}{\tilde{d}_{20}^\infty \lambda_-^\infty} + O(b) \end{array} \right.$$

and (4.16) is proved. We now compute the flow from (3.13).
First equation. We compute

$$\begin{aligned}
&\frac{\mathcal{G}_1}{c_+-c_-} = -b^2\frac{d\tilde{w}}{d\tau}\\
= \;& \frac{-b^2(c_+-c_-)\mu_+\tilde{w} + b^2\tilde{d}_{20}\tilde{w}^2 - b^3\tilde{d}_{11}\tilde{w}\tilde{\sigma} + b^4\tilde{d}_{02}\tilde{\sigma}^2 - b^3\tilde{d}_{30}\tilde{w}^3 + b^4\tilde{d}_{21}\tilde{w}^2\tilde{\sigma} - b^5\tilde{d}_{12}\tilde{w}\tilde{\sigma}^2 + b^6\tilde{d}_{03}\tilde{\sigma}^3}{c_+-c_-}
\end{aligned}$$

i.e.

$$\frac{d\tilde{w}}{d\tau} = \frac{-(c_+-c_-)|\mu_+|\tilde{w} + |\tilde{d}_{20}|\tilde{w}^2 + b(\tilde{d}_{11}\tilde{w}\tilde{\sigma} + \tilde{d}_{30}\tilde{w}^3) - b^2(\tilde{d}_{02}\tilde{\sigma}^2 + \tilde{d}_{21}\tilde{w}^2\tilde{\sigma}) + b^3\tilde{d}_{12}\tilde{w}\tilde{\sigma}^2 - b^4\tilde{d}_{03}\tilde{\sigma}^3}{c_+-c_-}.$$

We insert (4.17) and compute:

$$\begin{aligned}
&\mathcal{F}_2 = -\frac{\mathcal{G}_1}{b^2} \qquad (4.18)\\
= \;& -(c_+-c_-)|\mu_+|\tilde{w} + |\tilde{d}_{20}|\tilde{w}^2 + b(\tilde{d}_{11}\tilde{w}\tilde{\sigma} + \tilde{d}_{30}\tilde{w}^3) - b^2(\tilde{d}_{02}\tilde{\sigma}^2 + \tilde{d}_{21}\tilde{w}^2\tilde{\sigma}) + b^3\tilde{d}_{12}\tilde{w}\tilde{\sigma}^2 - b^4\tilde{d}_{03}\tilde{\sigma}^3\\
= \;& \tilde{w}\Big[-(c_+-c_-)|\mu_+| + |\tilde{d}_{20}|\tilde{w} + b(\tilde{d}_{11}\tilde{w}\tilde{\psi} + \tilde{d}_{30}\tilde{w}^2) - b^2(\tilde{d}_{02}\tilde{\psi}^2\tilde{w} + \tilde{d}_{21}\tilde{w}^2\tilde{\psi})\\
+ \;& b^3\tilde{d}_{12}\tilde{w}^2\tilde{\psi}^2 - b^4\tilde{d}_{03}\tilde{w}^2\tilde{\psi}^3\Big]\\
= \;& \tilde{w}\Big\{-(c_+-c_-)|\mu_+| + \Big[|\tilde{d}_{20}| + b\tilde{d}_{11}\tilde{\psi} - b^2\tilde{d}_{02}\tilde{\psi}^2\Big]\tilde{w} + \Big[b\tilde{d}_{30} - b^2\tilde{d}_{21}\tilde{\psi} + b^3\tilde{d}_{12}\tilde{\psi}^2 - b^4\tilde{d}_{03}\tilde{\psi}^3\Big]\tilde{w}^2\Big\}
\end{aligned}$$

Second equation. We compute

$$\begin{aligned}
&\frac{\mathcal{G}_2}{c_+-c_-} = b^3\frac{d\tilde{\sigma}}{d\tau}\\
= \;& \frac{b^2(c_+-c_-)\lambda_-\tilde{\sigma} + b^2\tilde{e}_{20}\tilde{w}^2 - b^3\tilde{e}_{11}\tilde{w}\tilde{\sigma} + b^4\tilde{e}_{02}\tilde{\sigma}^2 - b^3\tilde{e}_{30}\tilde{w}^3 + b^4\tilde{e}_{21}\tilde{w}^2\tilde{\sigma} - b^5\tilde{e}_{12}\tilde{w}\tilde{\sigma}^2 + b^6\tilde{e}_{03}\sigma^3}{c_+-c_-}.
\end{aligned}$$

and hence

$$\begin{aligned}
&b\frac{d\tilde{\sigma}}{d\tau} = \frac{1}{c_+-c_-}\\
\times \;& \big[-(c_+-c_-)|\lambda_-|\tilde{\sigma} + \tilde{e}_{20}\tilde{w}^2 - b(\tilde{e}_{11}\tilde{w}\tilde{\sigma} + \tilde{e}_{30}\tilde{w}^3) + b^2(\tilde{e}_{02}\tilde{\sigma}^2 + \tilde{e}_{21}\tilde{w}^2\tilde{\sigma}) - b^3\tilde{e}_{12}\tilde{w}\tilde{\sigma}^2 + b^4\tilde{e}_{03}\tilde{\sigma}^3\big]
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{F}_1 = \frac{\mathcal{G}_2}{b^2} \qquad (4.19)\\
= \;& -(c_+-c_-)|\lambda_-|\tilde{\sigma} + \tilde{e}_{20}\tilde{w}^2 - b(\tilde{e}_{11}\tilde{w}\tilde{\sigma} + \tilde{e}_{30}\tilde{w}^3) + b^2(\tilde{e}_{02}\tilde{\sigma}^2 + \tilde{e}_{21}\tilde{w}^2\tilde{\sigma}) - b^3\tilde{e}_{12}\tilde{w}\tilde{\sigma}^2 + b^4\tilde{e}_{03}\tilde{\sigma}^3\\
= \;& \tilde{w}\Big\{-(c_+-c_-)|\lambda_-|\tilde{\psi} + \tilde{e}_{20}\tilde{w} - b(\tilde{e}_{11}\tilde{w}\tilde{\psi} + \tilde{e}_{30}\tilde{w}^2) + b^2(\tilde{e}_{02}\tilde{w}\tilde{\psi}^2 + \tilde{e}_{21}\tilde{w}^2\tilde{\psi}) - b^3\tilde{e}_{12}\tilde{w}^2\tilde{\psi}^2\\
+ \;& b^4\tilde{e}_{03}\tilde{w}^2\tilde{\psi}^3)\Big\}\\
= \;& \tilde{w}\Big\{-(c_+-c_-)|\lambda_-|\tilde{\psi} + \Big[\tilde{e}_{20} - b\tilde{e}_{11}\tilde{\psi} + b^2\tilde{e}_{02}\tilde{\psi}^2\Big]\tilde{w} + \Big[-b\tilde{e}_{30} + b^2\tilde{e}_{21}\tilde{\psi} - b^3\tilde{e}_{12}\tilde{\psi}^2 + b^4\tilde{e}_{03}\tilde{\psi}^3\Big]\tilde{w}^2\Big\}.
\end{aligned}$$

Conclusion. We have obtained the system

$$\begin{aligned}
&\frac{d\tilde{w}}{d\tau} = \frac{\mathcal{F}_2}{c_+-c_-} \qquad (4.20)\\
= \;& \frac{\tilde{w}\Big\{-(c_+-c_-)|\mu_+| + \Big[|\tilde{d}_{20}| + b\tilde{d}_{11}\tilde{\psi} - b^2\tilde{d}_{02}\tilde{\psi}^2\Big]\tilde{w} + \Big[b\tilde{d}_{30} - b^2\tilde{d}_{21}\tilde{\psi} + b^3\tilde{d}_{12}\tilde{\psi}^2 - b^4\tilde{d}_{03}\tilde{\psi}^3\Big]\tilde{w}^2\Big\}}{c_+-c_-}
\end{aligned}$$

and

$$b\frac{d\tilde{\sigma}}{d\tau} = \frac{\mathcal{F}_1}{c_+ - c_-} \tag{4.21}$$

$$= \frac{\tilde{w}\left\{-(c_+ - c_-)|\lambda_-|\tilde{\psi} + \left[\tilde{e}_{20} - b\tilde{e}_{11}\tilde{\psi} + b^2\tilde{e}_{02}\tilde{\psi}^2\right]\tilde{w} + \left[-b\tilde{e}_{30} + b^2\tilde{e}_{21}\tilde{\psi} - b^3\tilde{e}_{12}\tilde{\psi}^2 + b^4\tilde{e}_{03}\tilde{\psi}^3\right]\tilde{w}^2\right\}}{c_+ - c_-}.$$

Remark 4.6. Note that from (4.13)

$$\tilde{\psi}_{\backsim} = -\frac{\tilde{\Sigma}_{\backsim}}{b\tilde{W}_{\backsim}} = O_{b\to 0}(1)$$

(4.21) then forces the relations for $b = 0$:

$$\left|\begin{array}{l} \tilde{\psi}^{\infty}_{\backsim} = \frac{\tilde{e}^{\infty}_{20}}{(c^{\infty}_{+} - c^{\infty}_{-})|\lambda^{\infty}_{-}|}\tilde{w}^{\infty}_{\backsim} \\ |\mu^{\infty}_{+}| = \frac{|\tilde{d}^{\infty}_{20}|}{c^{\infty}_{+} - c^{\infty}_{-}}\tilde{w}^{\infty}_{\backsim}. \end{array}\right. \tag{4.22}$$

Note also that we have the signs, valid for all $d \geq 2$ and $\ell \neq d$:

$$\tilde{d}^{\infty}_{20} < 0, \quad \tilde{e}^{\infty}_{20} > 0,$$

see (F.8).

step 2 Normalization of $\tilde{w}$. We further renormalize the flow to obtain the leading order size 1 constants leading the nonlinear dynamics as $b \to 0$. Let

$$\tilde{w} = \tilde{w}_{\backsim} u, \quad \tilde{\psi} = \tilde{\psi}_{\backsim}\phi \tag{4.23}$$

and define

$$\gamma = \frac{|\lambda_-|}{|\lambda_+|} = \frac{|\lambda_-|}{b|\mu_+|} \tag{4.24}$$

then recalling (4.20)

$$\frac{du}{d\tau} = \frac{1}{\tilde{w}_{\backsim}}\frac{d\tilde{w}}{d\tau} = \frac{\mathcal{F}_2}{\tilde{w}_{\backsim}(c_+ - c_-)} = |\mu_+|u \tag{4.25}$$

$$\times \left[-1 + \frac{|\tilde{d}_{20}| + b\tilde{d}_{11}\tilde{\psi}_{\backsim}\phi - b^2\tilde{d}_{02}\tilde{\psi}^2_{\backsim}\phi^2}{|\mu_+|(c_+ - c_-)}\tilde{w}_{\backsim}u\right.$$

$$\left. + \frac{b\tilde{d}_{30} - b^2\tilde{d}_{21}\tilde{\psi}_{\backsim}\phi + b^3\tilde{d}_{12}\tilde{\psi}^2_{\backsim}\phi^2 - b^4\tilde{d}_{03}\tilde{\psi}^3_{\backsim}\phi^3}{|\mu_+|(c_+ - c_-)}\tilde{w}^2_{\backsim}u^2\right] \tag{4.26}$$

$$= |\mu_+|u$$

$$\times \left[-1 + \left[|\tilde{D}_{20}| + b\tilde{D}_{11}\phi - b^2\tilde{D}_{02}\phi^2\right]u + b\left[\tilde{D}_{30} - b\tilde{D}_{21}\phi + b^2\tilde{D}_{12}\phi^2 - b^3\tilde{D}_{03}\phi^3\right]u^2\right]$$

with $\tilde{D}_{ij}$ given by (G.1). Observe that by definition of $P_{\backsim}$:

$$-1 + |\tilde{D}_{20}| + b\tilde{D}_{11} - b^2\tilde{D}_{02} + b\tilde{D}_{30} - b^2\tilde{D}_{21} + b^3\tilde{D}_{12} - b^4\tilde{D}_{03} = 0.$$

We then compute

$$\frac{d\tilde{\psi}}{d\tau} = \frac{1}{\tilde{w}}\frac{d\tilde{\sigma}}{d\tau} - \frac{\tilde{\sigma}}{\tilde{w}^2}\frac{d\tilde{w}}{d\tau} = \frac{1}{\tilde{w}}\frac{d\tilde{\sigma}}{d\tau} - \tilde{\psi}\frac{1}{\tilde{w}}\frac{d\tilde{w}}{d\tau}$$

$$\Leftrightarrow \quad \frac{d\phi}{d\tau} + \frac{1}{u}\frac{du}{d\tau}\phi = \frac{1}{\tilde{\psi}_{\backsim}\tilde{w}}\frac{d\tilde{\sigma}}{d\tau}$$

and recalling (4.21):

$$\frac{b}{\tilde{w}}\frac{d\tilde{\sigma}}{d\tau} = \frac{\mathcal{F}_1}{(c_+ - c_-)\tilde{w}} = |\lambda_-|\tilde{\psi}_{\backsim} \tag{4.27}$$

$$\begin{aligned} \times \quad & \left[-\phi + \frac{\tilde{e}_{20} - b\tilde{e}_{11}\tilde{\psi}_{\backsim}\phi + b^2\tilde{e}_{02}\tilde{\psi}_{\backsim}^2\phi^2}{(c_+ - c_-)|\lambda_-|\tilde{\psi}_{\backsim}}\tilde{w}_{\backsim}u\right. \\ + \quad & \left.\frac{-b\tilde{e}_{30} + b^2\tilde{e}_{21}\tilde{\psi}_{\backsim}\phi - b^3\tilde{e}_{12}\tilde{\psi}_{\backsim}^2\phi^2 + b^4\tilde{e}_{03}\tilde{\psi}_{\backsim}^3\phi^3}{(c_+ - c_-)|\lambda_-|\tilde{\psi}_{\backsim}}\tilde{w}_{\backsim}^2u^2\right] \end{aligned} \tag{4.28}$$

$$\begin{aligned} \Leftrightarrow \quad & \frac{1}{|\mu_+|\tilde{\psi}_{\backsim}\gamma}\frac{1}{\tilde{w}}\frac{d\tilde{\sigma}}{d\tau} \\ = \quad & \left[-\phi + \left(\tilde{E}_{20} - b\tilde{E}_{11}\phi + b^2\tilde{E}_{02})\phi^2\right)u + \left(-b\tilde{E}_{30} + b^2\tilde{E}_{21}\phi - b^3\tilde{E}_{12}\phi^2 + b^4\tilde{E}_{03}\phi^3\right)u^2\right] \end{aligned}$$

with $\tilde{E}_{ij}$ given by (G.1), and again by definition of $P_{\backsim}$:

$$-1 + \tilde{E}_{20} - b\tilde{E}_{11} + b^2\tilde{E}_{02} - b\tilde{E}_{30} + b^2\tilde{E}_{21} - b^3\tilde{E}_{12} + b^4\tilde{E}_{03} = 0.$$

This yields the renormalized ϕ equation

$$\begin{aligned} & \frac{d\phi}{d\tau} + \frac{1}{u}\frac{du}{d\tau}\phi \\ = \quad & |\mu_+|\gamma\left[-\phi + \left(\tilde{E}_{20} - b\tilde{E}_{11}\phi + b^2\tilde{E}_{02}\phi^2\right)u + b\left(-\tilde{E}_{30} + b\tilde{E}_{21}\phi - b^2\tilde{E}_{12}\phi^2 + b^3\tilde{E}_{03}\phi^3\right)u^2\right]. \end{aligned}$$

step 3 Quasilinear formulation. Let

$$\Lambda = Z\frac{d}{dZ} = -\frac{1}{|\mu_+|}\frac{d}{d\tau},$$

then equivalently:

$$\left|\begin{array}{l} \Lambda u = u\left[1 - \left(|\tilde{D}_{20}| + b\tilde{D}_{11}\phi - b^2\tilde{D}_{02}\phi^2\right)u - b\left(\tilde{D}_{30} - b\tilde{D}_{21}\phi + b^2\tilde{D}_{12}\phi^2 - b^3\tilde{D}_{03}\phi^3\right)u^2\right] \\ \Lambda\phi + \frac{\Lambda u}{u}\phi = \gamma\left[\phi - \left(\tilde{E}_{20} - b\tilde{E}_{11}\phi + b^2\tilde{E}_{02}\phi^2\right)u - b\left(-\tilde{E}_{30} + b\tilde{E}_{21}\phi - b^2\tilde{E}_{12}\phi^2 + b^3\tilde{E}_{03}\phi^3\right)u^2\right]. \end{array}\right.$$

with the relation on the parameters:

$$\left|\begin{array}{l} -1 + |\tilde{D}_{20}| + b\tilde{D}_{11} - b^2\tilde{D}_{02} + b\tilde{D}_{30} - b^2\tilde{D}_{21} + b^3\tilde{D}_{12} - b^4\tilde{D}_{03} = 0 \\ -1 + \tilde{E}_{20} - b\tilde{E}_{11} + b^2\tilde{E}_{02} - b\tilde{E}_{30} + b^2\tilde{E}_{21} - b^3\tilde{E}_{12} + b^4\tilde{E}_{03} = 0. \end{array}\right. \tag{4.29}$$

Let

$$\left|\begin{array}{l} F_1(u,\phi) = \phi - \left(\tilde{E}_{20} - b\tilde{E}_{11}\phi + b^2\tilde{E}_{02}\phi^2\right)u - b\left(-\tilde{E}_{30} + b\tilde{E}_{21}\phi - b^2\tilde{E}_{12}\phi^2 + b^3\tilde{E}_{03}\phi^3\right)u^2 \\ F_2(u,\phi) = 1 - \left(|\tilde{D}_{20}| + b\tilde{D}_{11}\phi - b^2\tilde{D}_{02}\phi^2\right)u - b\left(\tilde{D}_{30} - b\tilde{D}_{21}\phi + b^2\tilde{D}_{12}\phi^2 - b^3\tilde{D}_{03}\phi^3\right)u^2 \end{array}\right.$$

then this is

$$\left|\begin{array}{l} \Lambda u = uF_2(u,\phi) \\ \Lambda\phi + \frac{\Lambda u}{u}\phi = \gamma F_1(u,\phi). \end{array}\right. \tag{4.30}$$

We have from (4.27), (4.25):

$$\left|\begin{array}{l} \mathcal{F}_1 = |\lambda_-|\tilde{\psi}_{\backsim}\tilde{w}_{\backsim}(c_+ - c_-)u(-F_1) \\ \mathcal{F}_2 = \tilde{w}_{\backsim}(c_+ - c_-)|\mu_+|u(-F_2) \end{array}\right. \tag{4.31}$$

Then

$$\Lambda\phi = Z\frac{d\phi}{dZ} = Z\frac{d\phi}{du}\frac{du}{dZ} = \Lambda u\frac{d\phi}{du}$$

and hence the $\phi(u)$ renormalized quasi linear formulation

$$\left| \begin{array}{l} \frac{d\phi}{du} + \frac{\phi}{u} = \frac{\gamma F_1(\phi,u)}{uF_2(\phi,u)} \\ \lim_{u\to 0} \phi = 0, \quad \lim_{u\to 1} \phi = 1. \end{array} \right. \tag{4.32}$$

Reexpression of the nonlinear terms. From (4.29):

$$\tilde{E}_{20} = 1 + b(\tilde{E}_{11} + \tilde{E}_{30}) - b^2(\tilde{E}_{02} + \tilde{E}_{21}) + b^3\tilde{E}_{12} - b^4\tilde{E}_{03}$$

Then,

$$\begin{aligned} F_1(u,\phi) &= \phi - \left(1 + b(\tilde{E}_{11} + \tilde{E}_{30}) - b^2(\tilde{E}_{02} + \tilde{E}_{21}) + b^3\tilde{E}_{12} - b^4\tilde{E}_{03} - b\tilde{E}_{11}\phi + b^2\tilde{E}_{02}\phi^2\right) u \\ &- b\left(-\tilde{E}_{30} + b\tilde{E}_{21}\phi - b^2\tilde{E}_{12}\phi^2 + b^3\tilde{E}_{03}\phi^3\right) u^2 \\ &= \phi - u + b\left[-(\tilde{E}_{11} + \tilde{E}_{30})u + \tilde{E}_{11}\phi u + \tilde{E}_{30}u^2\right] + b^2\left[(\tilde{E}_{02} + \tilde{E}_{21})u - \tilde{E}_{02}\phi^2 u - \tilde{E}_{21}\phi u^2\right] \\ &+ b^3\left[-\tilde{E}_{12}u + \tilde{E}_{12}\phi^2u^2\right] + b^4\left[\tilde{E}_{03}u - \tilde{E}_{03}\phi^3u^2\right]. \end{aligned}$$

Similarly,

$$|\tilde{D}_{20}| = 1 - b(\tilde{D}_{11} + \tilde{D}_{30}) + b^2(\tilde{D}_{02} + \tilde{D}_{21}) - b^3\tilde{D}_{12} + b^4\tilde{D}_{03} = 0$$

and

$$\begin{aligned} F_2(u,\phi) &= 1 - \left(1 - b(\tilde{D}_{11} + \tilde{D}_{30}) + b^2(\tilde{D}_{02} + \tilde{D}_{21}) - b^3\tilde{D}_{12} + b^4\tilde{D}_{03} + b\tilde{D}_{11}\phi - b^2\tilde{D}_{02}\phi^2\right) u \\ &- b\left(\tilde{D}_{30} - b\tilde{D}_{21}\phi + b^2\tilde{D}_{12}\phi^2 - b^3\tilde{D}_{03}\phi^3\right) u^2 \\ &= 1 - u + b\left[(\tilde{D}_{11} + \tilde{D}_{30})u - \tilde{D}_{11}\phi u - \tilde{D}_{30}u^2\right] + b^2\left[-(\tilde{D}_{02} + \tilde{D}_{21})u + \tilde{D}_{02}\phi^2 u + \tilde{D}_{21}\phi u^2\right] \\ &+ b^3\left[\tilde{D}_{12}u - \tilde{D}_{12}\phi^2u^2\right] + b^4\left[-\tilde{D}_{03}u + \tilde{D}_{03}\phi^3u^2\right]. \end{aligned}$$

step 5 Changing variables. We change variables to make the critical points of the ode appear explicitly. Let

$$\widetilde{\Phi} = \phi - u. \tag{4.33}$$

Reexpressing F_1. Recall

$$\begin{aligned} F_1(u,\phi) &= \phi - u + b\left[-(\tilde{E}_{11} + \tilde{E}_{30})u + \tilde{E}_{11}\phi u + \tilde{E}_{30}u^2\right] + b^2\left[(\tilde{E}_{02} + \tilde{E}_{21})u - \tilde{E}_{02}\phi^2 u - \tilde{E}_{21}\phi u^2\right] \\ &+ b^3\left[-\tilde{E}_{12}u + \tilde{E}_{12}\phi^2u^2\right] + b^4\left[\tilde{E}_{03}u - \tilde{E}_{03}\phi^3u^2\right] \end{aligned}$$

then

$$-(\tilde{E}_{11} + \tilde{E}_{30})u + \tilde{E}_{11}\phi u + \tilde{E}_{30}u^2 = -(\tilde{E}_{11} + \tilde{E}_{30})u(1-u) + \tilde{E}_{11}u\widetilde{\Phi}$$

and

$$\begin{aligned} &(\tilde{E}_{02} + \tilde{E}_{21})u - \tilde{E}_{02}\phi^2 u - \tilde{E}_{21}\phi u^2 = (\tilde{E}_{02} + \tilde{E}_{21})u - \tilde{E}_{02}(u + \widetilde{\Phi})^2 u - \tilde{E}_{21}u^2(u + \widetilde{\Phi}) \\ = \;& (\tilde{E}_{02} + \tilde{E}_{21})u(1-u)(1+u) - u^2(2\tilde{E}_{02} + \tilde{E}_{21})\widetilde{\Phi} - \tilde{E}_{02}u\widetilde{\Phi}^2 \end{aligned}$$

and

$$\begin{aligned} -\tilde{E}_{12}u + \tilde{E}_{12}\phi^2u^2 &= -\tilde{E}_{12}u + \tilde{E}_{12}(u + \widetilde{\Phi})^2u^2 \\ &= -\tilde{E}_{12}u(1-u)(1+u+u^2) + 2\tilde{E}_{12}u^3\widetilde{\Phi} + \tilde{E}_{12}u^2\widetilde{\Phi}^2 \end{aligned}$$

and

$$\begin{aligned} \tilde{E}_{03}u - \tilde{E}_{03}\phi^3u^2 &= \tilde{E}_{03}u - \tilde{E}_{03}(u + \widetilde{\Phi})^3u^2 \\ &= \tilde{E}_{03}u(1-u)(1+u+u^2+u^3) - \tilde{E}_{03}(3u^4\widetilde{\Phi} + 3u^3\widetilde{\Phi}^2 + u^2\widetilde{\Phi}^3) \end{aligned}$$

Thus,

$$\begin{aligned}
& F_1(u,\phi) = \widetilde{\Phi} + b\left[-(\tilde{E}_{11}+\tilde{E}_{30})u(1-u) + \tilde{E}_{11}u\widetilde{\Phi}\right] \\
+ \;& b^2\left[(\tilde{E}_{02}+\tilde{E}_{21})u(1-u)(1+u) - u^2(2\tilde{E}_{02}+\tilde{E}_{21})\widetilde{\Phi} - \tilde{E}_{02}u\widetilde{\Phi}^2\right] \\
+ \;& b^3\left[-\tilde{E}_{12}u(1-u)(1+u+u^2) + 2\tilde{E}_{12}u^3\widetilde{\Phi} + \tilde{E}_{12}u^2\widetilde{\Phi}^2\right] \\
+ \;& b^4\left[\tilde{E}_{03}u(1-u)(1+u+u^2+u^3) - \tilde{E}_{03}(3u^4\widetilde{\Phi} + 3u^3\widetilde{\Phi}^2 + u^2\widetilde{\Phi}^3)\right] \\
= \;& u(1-u)\left[-b(\tilde{E}_{11}+\tilde{E}_{30}) + b^2(\tilde{E}_{02}+\tilde{E}_{21})(1+u) - b^3\tilde{E}_{12}(1+u+u^2) + b^4\tilde{E}_{03}(1+u+u^2+u^3)\right] \\
+ \;& \widetilde{\Phi}\left[1 + \tilde{E}_{11}bu - (2\tilde{E}_{02}+\tilde{E}_{21})b^2u^2 + 2\tilde{E}_{12}b^3u^3 - 3\tilde{E}_{03}b^4u^4\right] \\
+ \;& b\widetilde{\Phi}^2\left[-\tilde{E}_{02}bu + \tilde{E}_{12}b^2u^2 - 3\tilde{E}_{03}b^3u^3\right] - b^2\widetilde{\Phi}^3\left[\tilde{E}_{03}b^2u^2\right] \\
= \;& u(1-u)bH_1(b,u) + (1+G_1(bu))\widetilde{\Phi} + \mathrm{NL}_1(u,\widetilde{\Phi}) \qquad (4.34)
\end{aligned}$$

with

$$\left|\begin{array}{l}
H_1(b,u) = -(\tilde{E}_{11}+\tilde{E}_{30}) + b(\tilde{E}_{02}+\tilde{E}_{21})(1+u) - b^2\tilde{E}_{12}(1+u+u^2) + b^3\tilde{E}_{03}(1+u+u^2+u^3) \\
G_1(x) = \tilde{E}_{11}x - (2\tilde{E}_{02}+\tilde{E}_{21})x^2 + 2\tilde{E}_{12}x^3 - 3\tilde{E}_{03}x^4 \\
\mathrm{NL}_1(u,\widetilde{\Phi}) = b\widetilde{\Phi}^2\left[-\tilde{E}_{02}bu + \tilde{E}_{12}b^2u^2 - 3\tilde{E}_{03}b^3u^3\right] - b^2\widetilde{\Phi}^3\left[\tilde{E}_{03}b^2u^2\right]
\end{array}\right. \qquad (4.35)$$

Reexpressing F_2. Recall

$$\begin{aligned}
F_2(u,\phi) \;=\;& 1 - u + b\left[(\tilde{D}_{11}+\tilde{D}_{30})u - \tilde{D}_{11}\phi u - \tilde{D}_{30}u^2\right] + b^2\left[-(\tilde{D}_{02}+\tilde{D}_{21})u + \tilde{D}_{02}\phi^2u + \tilde{D}_{21}\phi u^2\right] \\
+\;& b^3\left[\tilde{D}_{12}u - \tilde{D}_{12}\phi^2u^2\right] + b^4\left[-\tilde{D}_{03}u + \tilde{D}_{03}\phi^3u^2\right]
\end{aligned}$$

then

$$(\tilde{D}_{11}+\tilde{D}_{30})u - \tilde{D}_{11}\phi u - \tilde{D}_{30}u^2 = (\tilde{D}_{11}+\tilde{D}_{30})u(1-u) - \tilde{D}_{11}u\widetilde{\Phi}$$

and

$$\begin{aligned}
& -(\tilde{D}_{02}+\tilde{D}_{21})u + \tilde{D}_{02}\phi^2u + \tilde{D}_{21}\phi u^2 = -(\tilde{D}_{02}+\tilde{D}_{21})u + \tilde{D}_{02}(u+\widetilde{\Phi})^2u + \tilde{D}_{21}(u+\widetilde{\Phi})u^2 \\
=\;& -(\tilde{D}_{02}+\tilde{D}_{21})u(1-u)(1+u) + u^2(2\tilde{D}_{02}+\tilde{D}_{21})\widetilde{\Phi} + \tilde{D}_{02}u\widetilde{\Phi}^2
\end{aligned}$$

and

$$\begin{aligned}
& \tilde{D}_{12}u - \tilde{D}_{12}\phi^2u^2 = \tilde{D}_{12}u - \tilde{D}_{12}(u+\widetilde{\Phi})^2u^2 \\
=\;& \tilde{D}_{12}u(1-u)(1+u+u^2) - 2\tilde{D}_{12}u^3\widetilde{\Phi} - \tilde{D}_{12}u^2\widetilde{\Phi}^2
\end{aligned}$$

and

$$\begin{aligned}
& -\tilde{D}_{03}u + \tilde{D}_{03}\phi^3u^2 = -\tilde{D}_{03}u + \tilde{D}_{03}(u+\widetilde{\Phi})^3u^2 \\
=\;& -\tilde{D}_{03}u(1-u)(1+u+u^2+u^3) + \tilde{D}_{03}(3u^4\widetilde{\Phi} + 3u^3\widetilde{\Phi}^2 + u^2\widetilde{\Phi}^3).
\end{aligned}$$

Then,

$$\begin{aligned}
& F_2(u,\phi) = 1-u+b\left[(\tilde{D}_{11}+\tilde{D}_{30})u(1-u)-\tilde{D}_{11}u\widetilde{\Phi}\right] \\
+ \;& b^2\left[-(\tilde{D}_{02}+\tilde{D}_{21})u(1-u)(1+u)+u^2(2\tilde{D}_{02}+\tilde{D}_{21})\widetilde{\Phi}+\tilde{D}_{02}u\widetilde{\Phi}^2\right] \\
+ \;& b^3\left[\tilde{D}_{12}u(1-u)(1+u+u^2)-2\tilde{D}_{12}u^3\widetilde{\Phi}-\tilde{D}_{12}u^2\widetilde{\Phi}^2\right] \\
+ \;& b^4\left[-\tilde{D}_{03}u(1-u)(1+u+u^2+u^3)+\tilde{D}_{03}(3u^4\widetilde{\Phi}+3u^3\widetilde{\Phi}^2+u^2\widetilde{\Phi}^3)\right] \\
= \;& (1-u)\left[1+b(\tilde{D}_{11}+\tilde{D}_{30})u-b^2(\tilde{D}_{02}+\tilde{D}_{21})u(1+u)+b^3\tilde{D}_{12}u(1+u+u^2)\right. \\
- \;& \left.b^4\tilde{D}_{03}u(1+u+u^2+u^3)\right] \\
+ \;& \widetilde{\Phi}\left[-\tilde{D}_{11}bu+(2\tilde{D}_{02}+\tilde{D}_{21})b^2u^2-2\tilde{D}_{12}b^3u^3+3\tilde{D}_{03}b^4u^4\right] \\
+ \;& b\widetilde{\Phi}^2\left[\tilde{D}_{02}bu-\tilde{D}_{12}b^2u^2+3\tilde{D}_{03}b^3u^3\right]+b^2\widetilde{\Phi}^3\left[\tilde{D}_{03}b^2u^2\right] \\
= \;& (1-u)\left[1+H_2(b,u)\right]+G_2(bu)\widetilde{\Phi}+\mathrm{NL}_2(u,\widetilde{\Phi}) \qquad (4.36)
\end{aligned}$$

with

$$\begin{aligned}
H_2(b,u) \;=\; & b(\tilde{D}_{11}+\tilde{D}_{30})u-b^2(\tilde{D}_{02}+\tilde{D}_{21})u(1+u)+b^3\tilde{D}_{12}u(1+u+u^2) \\
- \;& b^4\tilde{D}_{03}u(1+u+u^2+u^3) \qquad (4.37)
\end{aligned}$$

and

$$\left|\begin{array}{l}
G_2(x) = -\tilde{D}_{11}x+(2\tilde{D}_{02}+\tilde{D}_{21})x^2-2\tilde{D}_{12}x^3+3\tilde{D}_{03}x^4 \\
\mathrm{NL}_2(u,\widetilde{\Phi}) = b\widetilde{\Phi}^2\left[\tilde{D}_{02}bu-\tilde{D}_{12}b^2u^2+3\tilde{D}_{03}b^3u^3\right]+b^2\widetilde{\Phi}^3\left[\tilde{D}_{03}b^2u^2\right]
\end{array}\right. \qquad (4.38)$$

step 6 Final change of variables. We now reexpress (4.32) as

$$\begin{aligned}
& F_2(\phi,u)\left(u\frac{d\phi}{du}+\phi\right)=\gamma F_1 \\
\Leftrightarrow \;& \left[(1-u)(1+H_2)+G_2\widetilde{\Phi}+\mathrm{NL}_2\right]\left(u\frac{d\widetilde{\Phi}}{du}+\widetilde{\Phi}+2u\right) \\
= \;& \gamma\left[u(1-u)bH_1+(1+G_1)\widetilde{\Phi}+\mathrm{NL}_1\right]
\end{aligned}$$

We change variables

$$\widetilde{\Phi}=(1-u)\Psi \qquad (4.39)$$

and define

$$x=bu.$$

Nonlinear terms. We rewrite from (4.35)

$$\left|\begin{array}{l}
\mathrm{NL}_1 = bM_{11}(x)\widetilde{\Phi}^2+b^2M_{12}(x)\widetilde{\Phi}^3 \\
M_{11} = -\tilde{E}_{02}x+\tilde{E}_{12}x^2-3\tilde{E}_{03}x^3 \\
M_{12} = \tilde{E}_{03}x^2
\end{array}\right.$$

and from (4.38):

$$\left|\begin{array}{l}
\mathrm{NL}_2 = bM_{21}(x)\widetilde{\Phi}^2+b^2M_{22}(x)\widetilde{\Phi}^3 \\
M_{21} = \tilde{D}_{02}x-\tilde{D}_{12}x^2+3\tilde{D}_{03}x^3 \\
M_{22} = \tilde{D}_{03}x^2
\end{array}\right. \qquad (4.40)$$

Then,

$$\begin{aligned} \mathrm{NL}_1 &= bM_{21}(1-u)^2\Psi^2 + b^2M_{12}(1-u)^3\Psi^3 \\ &= (1-u)\left[(b-x)M_{11}(x)\Psi^2 + (b^2-2bx+x^2)M_{12}\Psi^3\right] \\ &\equiv (1-u)\widetilde{\mathrm{NL}}_1 \end{aligned} \tag{4.41}$$

and

$$\begin{aligned} \mathrm{NL}_2 &= bM_{21}(1-u)^2\Psi^2 + b^2M_{22}(1-u)^3\Psi^3 \\ &= (1-u)\left[(b-x)M_{21}(x)\Psi^2 + (b^2-2bx+x^2)M_{22}\Psi^3\right] \\ &\equiv (1-u)\widetilde{\mathrm{NL}}_2 \end{aligned} \tag{4.42}$$

Ψ equation. We compute

$$u\frac{d\widetilde{\Phi}}{du} + \widetilde{\Phi} + 2u = u\left[(1-u)\frac{d\Psi}{du} - \Psi\right] + (1-u)\Psi + 2u = u(1-u)\frac{d\Psi}{du} + (1-2u)\Psi + 2u.$$

Then,

$$\begin{aligned} &(1-u)(1+H_2) + G_2\widetilde{\Phi} + \mathrm{NL}_2 = (1-u)\left[1 + H_2 + G_2\Psi + \widetilde{\mathrm{NL}}_2\right] \\ &u(1-u)bH_1 + (1+G_1)\widetilde{\Phi} + \mathrm{NL}_1 = (1-u)\left[buH_1 + (1+G_1)\Psi + \widetilde{\mathrm{NL}}_1\right]. \end{aligned}$$

This gives the Ψ equation:

$$\begin{aligned} &\left[1 + H_2 + G_2\Psi + \widetilde{\mathrm{NL}}_2\right]\left[u(1-u)\frac{d\Psi}{du} + (1-2u)\Psi + 2u\right] = \gamma\left[xH_1 + (1+G_1)\Psi + \widetilde{\mathrm{NL}}_1\right] \\ = \;& u\left[\gamma bH_1 + \left(\frac{\gamma}{u} + \frac{\gamma bG_1}{x}\right)\Psi + \frac{\gamma b\widetilde{\mathrm{NL}}_1}{x}\right]. \end{aligned}$$

Equivalently:

$$\begin{aligned} &\left[1 + H_2 + G_2\Psi + \widetilde{\mathrm{NL}}_2\right]\left[u(1-u)\frac{d\Psi}{du} + (1-2u)\Psi\right] \\ = \;& u\left[\gamma bH_1 + \left(\frac{\gamma}{u} + \frac{\gamma bG_1}{x}\right)\Psi + \frac{\gamma b\widetilde{\mathrm{NL}}_1}{x} - 2\left(1 + H_2 + G_2\Psi + \widetilde{\mathrm{NL}}_2\right)\right] \end{aligned}$$

i.e.,

$$\begin{aligned} &\left[1 + H_2 + G_2\Psi + \widetilde{\mathrm{NL}}_2\right]u(1-u)\frac{d\Psi}{du} \\ + \;& \left[(1-2u)(1 + H_2 + G_2\Psi + \widetilde{\mathrm{NL}}_2) - \gamma(1+G_1) + 2uG_2\right]\Psi \\ = \;& u\left[\gamma bH_1 - 2(1+H_2) + \frac{\gamma b\widetilde{\mathrm{NL}}_1}{x} - 2\widetilde{\mathrm{NL}}_2\right]. \end{aligned}$$

Reordering terms. We split H_2:

$$\begin{aligned} &H_2(b,u) = b(\tilde{D}_{11} + \tilde{D}_{30})u - b^2(\tilde{D}_{02} + \tilde{D}_{21})u(1+u) + b^3\tilde{D}_{12}u(1+u+u^2) \\ - \;& b^4\tilde{D}_{03}u(1+u+u^2+u^3) = \sum_{j=0}^{3} b^j H_{2,j}(x), \end{aligned}$$

and similarly

$$\begin{aligned} H_1(b,u) &= -(\tilde{E}_{11}+\tilde{E}_{30}) + b(\tilde{E}_{02}+\tilde{E}_{21})(1+u) - b^2\tilde{E}_{12}(1+u+u^2) + b^3\tilde{E}_{03}(1+u+u^2+u^3) \\ &= \sum_{j=0}^{3} b^j H_{1,j}(bu) \end{aligned}$$

with $H_{i,j}$ given by (G.3). We reorder the nonlinear terms using the same rule:

$$\left| \begin{array}{l} \widetilde{\mathrm{NL}}_1 = (b-x)M_{11}(x)\Psi^2 + (b^2-2bx+x^2)M_{12}\Psi^3 = \sum_{j=0}^{2} b^j \widetilde{\mathrm{NL}}_{1j} \\ \widetilde{\mathrm{NL}}_2 = (b-x)M_{21}(x)\Psi^2 + (b^2-2bx+x^2)M_{22}\Psi^3 = \sum_{j=0}^{2} b^j \widetilde{\mathrm{NL}}_{2j} \end{array} \right. \tag{4.43}$$

with (G.5). □

5. **Bounding the Taylor series of the formal limit problem**

We now start the analysis of the non linear ode (4.15) for $0 < u < 1$. It has a regular singular point at the origin and our first task is to estimate the growth of the Taylor coefficients of solutions' expansions at $u = 0$. This will be done in two steps. First, in this section we estimate the growth of the coefficients for a formal $b = 0$ limiting system, Proposition 5.1. This will make appear the function $S_\infty(d,\ell)$. Then, in section 6 we will obtain uniform bounds in b for the Taylor coefficients associated to (4.15) *for frequencies* $k \lesssim \frac{1}{b}$.

5.1. **Formal limit** $b = 0$**.** Recall (4.15) and let[12]

$$\Psi(u) = \tilde{\Psi}(x), \quad x = bu$$

then

$$\begin{aligned} & \left[1 + H_2 + G_2\Psi + \widetilde{\mathrm{NL}}_2\right] \frac{x}{b}\left(1-\frac{x}{b}\right) b\frac{d\tilde{\Psi}}{dx} \\ + & \left[\left(1-\frac{2x}{b}\right)(1+H_2+G_2\Psi+\widetilde{\mathrm{NL}}_2) - \gamma(1+G_1) + 2\frac{x}{b}G_2\right]\tilde{\Psi} \\ = & \frac{x}{b}\left[\gamma b H_1 - 2(1+H_2) + \frac{\gamma b\widetilde{\mathrm{NL}}_1}{x} - 2\widetilde{\mathrm{NL}}_2\right] \\ \Leftrightarrow & \left[1 + H_2 + G_2\tilde{\Psi} + \widetilde{\mathrm{NL}}_2\right] x(b-x)\frac{d\tilde{\Psi}}{dx} \\ + & \left[(b-2x)(1+H_2+G_2\tilde{\Psi}+\widetilde{\mathrm{NL}}_2) - b\gamma(1+G_1) + 2xG_2\right]\tilde{\Psi} \\ = & x\left[\gamma b H_1 - 2(1+H_2) + \frac{\gamma b\widetilde{\mathrm{NL}}_1}{x} - 2\widetilde{\mathrm{NL}}_2\right]. \end{aligned} \tag{5.1}$$

We introduce the parameters

$$\left| \begin{array}{l} a = \gamma b = \frac{|\lambda_-|}{|\mu_+|} > 0 \\ \nu = -\gamma b(\tilde{D}_{11} + \tilde{D}_{30} - \tilde{E}_{11}), \end{array} \right. \tag{5.2}$$

which have a well defined limit as $b \to 0$ noted a_∞, ν_∞, and assume

$$\nu_\infty(d,\ell) > 0 \tag{5.3}$$

[12]Observe that the new variable x introduced here should not be confused with the variable $x = \log Z$ of the Emden transform (1.8).

which is checked in sections F.3 and F.5. In view of the explicit formulas of Appendix G and remark G.1, the *formal* limit $b \to 0$ is:

$$\begin{aligned} & \left[1 + H_{20}^{\infty} + G_2^{\infty}\tilde{\Psi} + \widetilde{\mathrm{NL}}_{20}^{\infty}\right](-x^2)\frac{d\tilde{\Psi}}{dx} \\ + & \left[-2x(1 + H_{20}^{\infty} + G_2^{\infty}\tilde{\Psi} + \widetilde{\mathrm{NL}}_{20}^{\infty}) - a_{\infty}(1 + G_1^{\infty}) + 2xG_2^{\infty}\right]\tilde{\Psi} \\ = & \; x\left[a_{\infty}H_{10}^{\infty} - 2(1 + H_{20}^{\infty}) + \frac{a_{\infty}\widetilde{\mathrm{NL}}_{10}^{\infty}}{x} - 2\widetilde{\mathrm{NL}}_{20}^{\infty}\right] \end{aligned} \tag{5.4}$$

where we recall[13] that the subscript $^{\infty}$ means that we compute all parameters $(\tilde{D}_{ij}, \tilde{E}_{ij})$ given by (G.1) in their well defined limit $b = 0$. Let us stress the fact that this is *not a dynamical limit*, since the change variables $x = bu$ maps the original flow on the set $x = 0$. Our claim is that for fixed order k, the Taylor coefficients associated to (5.4) are the $b = 0$ limit of the Taylor coefficients associated to (4.15) up to a suitable renormalization, see (6.16).

5.2. **Boundedness of the limiting series.** The condition (5.3) holds, by a direct examination, at r^+ and $\ell > d$ from (F.15), and at r_* on a collection of subintervals of $(0, d)$ from (F.10). In fact, (F.9) implies that in the latter set is non-empty and for $d = 2, 3$ coincides with $(0, d)$. We also note the fact that the condition (5.3) is not necessary for the following arguments. We also make the following note that in the case of r^+, in principle, all the arguments below can be extended to the values of $\ell \in (0, d)$. In particular, from (F.15) the function ν_{∞} is still positive there. This extension will be important for the analyticity argument in Appendix E.

Our aim in this section is to prove the following bound.

Proposition 5.1 (Boundedness for (5.4))**.** *Assume* (5.3)*. Then there exists* $c_{\nu_{\infty}, a_{\infty}} > 0$ *such that the following holds. Let* $\tilde{\Psi}$ *be the unique* $\mathcal{C}^{\infty}$ *local solution to* (5.4) *on* $[0, x_0]$*, then the sequence*

$$\tilde{\psi}_k = \frac{\tilde{\Psi}^{(k)}(0)}{k!}$$

satisfies

$$|\tilde{\psi}_k| \leq c_{\nu,a}\frac{\Gamma(k + \nu_{\infty} + 2)}{a_{\infty}^k}. \tag{5.5}$$

Remark 5.2. The bound (5.5) falls within the range of nonlinear Maillet type theorems, see [8, 12, 14]. We shall give a self contained proof which will allow us to obtain quantitative bounds. The latter is crucial for future uniform b independent bounds for all frequencies $k \lesssim \frac{1}{b}$ for the full problem, see (6.34). Note also that the toy model problem $x^2\Theta' + \Theta = 1 + x$ with $\tilde{\Psi} = x\Theta$ leads to the bound $\frac{|\Theta^{(k)}(0)|}{k!} = \Gamma(k)$ suggesting that (5.5) is essentially optimal.

5.3. **Conjugation formula.** We start by conjugating (5.4) to an explicitly solvable (at the linear level) problem.

Lemma 5.3 (Conjugation)**.** *There exist functions* $\xi(x), (\mu_j(x), \nu_j(x))_{1 \leq j \leq 4}$*, holomorphic in a neighborhood of* $x = 0$*, dependent on* (d, ℓ)*, such that the change of variables*

$$\left|\begin{array}{l} \tilde{\Psi}(x) = M(x)\Phi(x), \quad M(x) = e^{-\int_0^x \xi(y)dy} \\ \Theta(x) = \frac{\Phi(x)}{x} \end{array}\right. \tag{5.6}$$

[13] Remark 4.1

maps (5.4) *to*

$$x^2\frac{d\Theta}{dx}+[a_\infty+(\nu_\infty+3)x]\Theta=\mu_0+x\left[x\sum_{j=1}^{4}\mu_j\Theta^j+x^2\left(x\frac{d\Theta}{dx}\right)\sum_{j=1}^{4}\nu_j\Theta^j\right]. \quad (5.7)$$

Proof of Lemma 5.3. This is an explicit computation.

step 1 Linear conjugation. We solve the linear problem

$$[x+xH_{20}^\infty](-x)\frac{d\tilde\Psi}{dx}+[-2x(1+H_{20}^\infty)-a_\infty(1+G_1^\infty)+2xG_2^\infty]\,\tilde\Psi=xF. \quad (5.8)$$

by conjugating it back to an explicitly solvable problem:

$$\begin{aligned}(5.8)\quad&\Leftrightarrow\quad \frac{d\tilde\Psi}{dx}+\frac{2x(1+H_{20}^\infty)+a_\infty(1+G_1^\infty)-2xG_2^\infty}{x^2(1+H_{20})}\tilde\Psi=-\frac{xF}{x^2(1+H_{20})}\\ &\Leftrightarrow\quad \frac{d\tilde\Psi}{dx}+\left[\frac{2}{x}+\frac{a_\infty(1+\tilde E_{11}^\infty x)}{x^2(1+H_{20}^\infty)}+\frac{a_\infty(G_1^\infty-\tilde E_{11}^\infty x)-2xG_2^\infty}{x^2(1+H_{20}^\infty)}\right]\tilde\Psi=-\frac{F}{x(1+H_{20}^\infty)}.\end{aligned}$$

We have

$$\begin{aligned}&\frac{a_\infty(1+\tilde E_{11}^\infty x)}{x^2(1+H_{20}^\infty)}=\frac{a_\infty}{x^2}(1+\tilde E_{11}^\infty x)\left[1-(\tilde D_{11}^\infty+\tilde D_{30}^\infty)x+\frac{1}{1+H_{20}^\infty}-1+(\tilde D_{11}^\infty+\tilde D_{30}^\infty)x\right]\\ =\;&\frac{a_\infty}{x^2}(1+\tilde E_{11}^\infty x)\left[\frac{1}{1+H_{20}^\infty}-1+(\tilde D_{11}^\infty+\tilde D_{30}^\infty)x\right]-a\tilde E_{11}^\infty(\tilde D_{11}^\infty+\tilde D_{30}^\infty)\\ +\;&\frac{a_\infty}{x^2}+\frac{a_\infty[\tilde E_{11}^\infty-(\tilde D_{11}^\infty+\tilde D_{30}^\infty)]}{x}\end{aligned}$$

Let

$$\begin{aligned}\xi(x)\;=\;&\frac{a_\infty}{x^2}(1+\tilde E_{11}^\infty x)\left[\frac{1}{1+H_{20}^\infty}-1+(\tilde D_{11}^\infty+\tilde D_{30}^\infty)x\right]-a_\infty\tilde E_{11}^\infty(\tilde D_{11}^\infty+\tilde D_{30}^\infty)\\ +\;&\frac{a_\infty(G_1^\infty-\tilde E_{11}^\infty x)-2xG_2^\infty}{x^2(1+H_{20}^\infty)}\end{aligned}$$

Observe that $H_{20}^\infty-(\tilde D_{11}^\infty+\tilde D_{30}^\infty)x$ and $G_1^\infty-\tilde E_{11}^\infty x$ are polynomials in x starting with x^2, while G_2^∞ is a polynomial beginning with x. The function $\xi(x)$ is then holomorphic in a neighborhood of $x=0$. We have

$$(5.8)\Leftrightarrow\frac{d\tilde\Psi}{dx}+\left[\frac{a_\infty}{x^2}+\frac{\nu_\infty+2}{x}+\xi(x)\right]\tilde\Psi=-\frac{F}{x(1+H_{20}^\infty)}.$$

Define

$$M(x)=e^{-\int_0^x\xi(y)dy} \quad (5.9)$$

a holomorphic function in a neighborhood of $x=0$, and introduce the change of variables

$$\tilde\Psi(x)=M(x)\Phi(x),\quad G(x)=-\frac{F(x)}{M(x)(1+H_{20}^\infty(x))}.$$

We have obtained the conjugation formula:

$$(5.8)\Leftrightarrow\frac{d\Phi}{dx}+\left[\frac{a_\infty}{x^2}+\frac{\nu_\infty+2}{x}\right]\Phi=\frac{G}{x}. \quad (5.10)$$

step 2 Conjugation for the nonlinear problem. The equation (5.4) is in the form (5.8)

$$[x+xH_{20}^\infty](-x)\frac{d\tilde\Psi}{dx}+[-2x(1+H_{20}^\infty)-a_\infty(1+G_1^\infty)+2xG_2^\infty]\,\tilde\Psi=x(F_0+F)$$

for the source term

$$F_0 = a_\infty H_{10}^\infty - 2(1 + H_{20}^\infty)$$

and the nonlinear term

$$F = \frac{a_\infty \widetilde{\mathrm{NL}}_{10}^\infty}{x} - 2\widetilde{\mathrm{NL}}_{20}^\infty + \left[G_2^\infty \tilde\Psi + \widetilde{\mathrm{NL}}_{20}^\infty\right]\left(2\tilde\Psi + x\frac{d\tilde\Psi}{dx}\right)$$

We conjugate this using (5.10):

$$\left|\begin{array}{l} \tilde\Psi(x) = M(x)\Phi(x), \quad G(x) = -\frac{F(x)+F_0}{M(x)(1+H_{20}^\infty(x))} \\ x^2\frac{d\Phi}{dx} + [a_\infty + (\nu_\infty + 2)x]\,\Phi = xG \end{array}\right.$$

We express the nonlinearity in terms of Φ and obtain a representation:

$$G = \xi_0 + \sum_{j=2}^4 \xi_j \Phi^j + (x\Phi')\sum_{j=1}^3 (\tilde\xi_j \Phi^j)$$

where $(\xi_j, \tilde\xi_j)$ are explicit holomorphic functions in a neighborhood of the origin. We have obtained the equivalent nonlinear problem:

$$x^2\frac{d\Phi}{dx} + [a_\infty + (\nu_\infty + 2)x]\,\Phi = x\left[\xi_0 + \sum_{j=2}^4 \xi_j\Phi^j + \left(x\frac{d\Phi}{dx}\right)\sum_{j=1}^3(\tilde\xi_j\Phi^j)\right]. \tag{5.11}$$

A direct computation shows

$$\Phi(0) = 0$$

and we let

$$\Theta = \frac{\Phi}{x}, \tag{5.12}$$

so that:

$$\begin{aligned} &(5.11) \Leftrightarrow x^2\left[x\frac{d\Theta}{dx} + \Theta\right] + [a_\infty + (\nu_\infty + 2)x]\,x\Theta \\ = \;& x\left[\xi_0 + \sum_{j=2}^4 \xi_j x^j\Theta^j + x(x\Theta' + \Theta)\sum_{j=1}^3(\tilde\xi_j x^j\Theta^j)\right] \\ \Leftrightarrow \;& x^2\frac{d\Theta}{dx} + [a_\infty + (\nu_\infty + 3)x]\Theta = \xi_0 + \sum_{j=2}^4 \xi_j x^j\Theta^j + x\left(x\frac{d\Theta}{dx}\right)\sum_{j=1}^3(\tilde\xi_j x^j\Theta^j) + \sum_{j=1}^3 \tilde\xi_j(x\Theta)^{j+1} \\ \Leftrightarrow \;& x^2\frac{d\Theta}{dx} + [a_\infty + (\nu_\infty + 3)x]\Theta = \mu_0 + x\left[x\sum_{j=1}^4 \mu_j\Theta^j + x^2\left(x\frac{d\Theta}{dx}\right)\sum_{j=1}^4 \nu_j\Theta^j\right] \end{aligned}$$

where μ_j, ν_j are holomorphic functions of x in a neighborhood of 0, and (5.7) is proved. □

5.4. **The nonlinear induction relation.** The uniqueness of a local $\mathcal{C}^\infty$ solution near $x = 0$ to (5.7), and thus (5.4), follows from an elementary fixed point argument which is left to the reader. We let

$$\mu_{jk} = \frac{\mu_j^{(k)}(0)}{k!}, \quad \nu_{jk} = \frac{\nu_j^{(k)}(0)}{k!}, \quad \theta_k = \frac{\Theta^{(k)}(0)}{k!}, \quad \phi_k = \frac{\Phi^{(k)}(0)}{k!}$$

and claim the following fundamental nonlinear bound.

Lemma 5.4 (Bound on the limiting sequence)**.** *For some large enough universal constant $c(\nu_\infty, a_\infty) > 0$:*

$$|\phi_k| \leq c(\nu_\infty, a_\infty) \frac{\Gamma(k + \nu_\infty + 2)}{a_\infty^k}, \quad \forall k \geq 1. \tag{5.13}$$

Also, we have

$$\sum_{k=0}^{+\infty} \left| \frac{a_\infty^k g_k}{\Gamma(\nu_\infty + k + 3)} \right| \leq c(\nu_\infty, a_\infty) < +\infty \tag{5.14}$$

where g_k is defined below, see (5.15)*.*

Proof of Lemma 5.4. We compute the nonlinear induction relation and estimate the sequence using convolution estimates for the Γ function.

step 1 The induction relation. We formally expand[14]

$$\left| \begin{array}{l} \Theta = \sum_{k=0}^{+\infty} \theta_k x^k \\ G = \mu_0 + x \left[x \sum_{j=1}^4 \mu_j \Theta^j + x^2 (x\Theta') \sum_{j=1}^4 \nu_j \Theta^j \right] = \sum_{k=0}^{+\infty} g_k x^k \end{array} \right. \tag{5.15}$$

and obtain, from (5.7), the induction relation:

$$\begin{aligned}
& x^2 \frac{d\Theta}{dx} + (a_\infty + (\nu_\infty + 3)x)\Theta = G \\
\Leftrightarrow \quad & \sum_{k=1}^{+\infty} k\theta_k x^{k+1} + \sum_{k=0}^{+\infty} a_\infty \theta_k x^k + \sum_{k=0}^{+\infty} (\nu_\infty + 3)\theta_k x^{k+1} = \sum_{k=0}^{+\infty} g_k x^k \\
\Leftrightarrow \quad & \sum_{k=2}^{+\infty} (k-1)\theta_{k-1} x^k + \sum_{k=0}^{+\infty} a_\infty \theta_k x^k + \sum_{k=1}^{+\infty} (\nu_\infty + 3) x^k \theta_{k-1} = \sum_{k=0}^{+\infty} g_k x^k \\
\Leftrightarrow \quad & \left| \begin{array}{l} a_\infty \theta_0 = g_0 \\ (\nu_\infty + 3)\theta_0 + a_\infty \theta_1 = g_1 \\ (k + \nu_\infty + 2)\theta_{k-1} + a_\infty \theta_k = g_k, \quad k \geq 2 \end{array} \right. \\
\Leftrightarrow \quad & \left| \begin{array}{l} a_\infty \theta_0 = g_0 \\ (k + \nu_\infty + 3)\theta_k + a_\infty \theta_{k+1} = g_{k+1}, \quad k \geq 0 \end{array} \right.
\end{aligned} \tag{5.16}$$

We now compute g_k from (5.15). By Leibniz

$$(fg)_k = \frac{1}{k!} \frac{d^k}{dx^k}(fg)(0) = \frac{1}{k!} \sum_{k_1 + k_2 = k} \frac{k!}{k_1! k_2!} f^{(k_1)}(0) g^{(k_2)}(0) = \sum_{k_1 + k_2 = k} f_{k_1} g_{k_2}$$

Therefore,

$$(x^2 \mu_j \Theta^j)_{k+1} = (\mu_j \Theta^j)_{k-1} = \sum_{k_1 + \ldots k_{j+1} = k-1} \mu_{jk_1} \theta_{k_2} \ldots \theta_{k_{j+1}}.$$

Then

$$\left(x \frac{d\Theta}{dx} \right)_k = k\theta_k$$

yields

$$\left(x^3 \left(x \frac{d\Theta}{dx} \right) \nu_j \Theta^j \right)_{k+1} = \left(\left(x \frac{d\Theta}{dx} \right) \nu_j \Theta^j \right)_{k-2} = \sum_{k_1 + \ldots k_{j+2} = k-2} \nu_{jk_1} \theta_{k_2} \ldots \theta_{k_{j+1}} (k_{j+2} \theta_{k_{j+2}})$$

[14] Θ being smooth but not analytic, the expansions are only formal and the equalities below correspond to equalities between Taylor expansions at any order.

and (5.16) yields the induction relation for $k \geq 5$:

$$(k+\nu_\infty+3)\theta_k + a_\infty\theta_{k+1} = (\mu_0)_{k+1} + \sum_{j=1}^{4} \sum_{k_1+\ldots k_{j+1}=k-1} \mu_{jk_1}\theta_{k_2}\ldots\theta_{k_{j+1}}$$

$$+ \sum_{j=1}^{4} \sum_{k_1+\ldots k_{j+2}=k-2} \nu_{jk_1}\theta_{k_2}\ldots\theta_{k_{j+1}}(k_{j+2}\theta_{k_{j+2}}) \tag{5.17}$$

step 2 Renormalization of the sequence. We claim the bound:

$$|\theta_k| \leq c(\nu_\infty, a_\infty)\frac{\Gamma(k+\nu_\infty+3)}{a_\infty^k}, \quad \forall k \geq 1 \tag{5.18}$$

and

$$|(k+\nu_\infty+2)\theta_{k-1} + a_\infty\theta_k| \leq c(\nu, a)\frac{\Gamma(k+\nu_\infty+3)}{(1+k^2)a_\infty^k}, \quad \forall k \geq 1. \tag{5.19}$$

These imply, from (5.12) and (5.16)

$$\left| \begin{array}{l} |\phi_k| = |(x\Theta)_k| = |\theta_{k-1}| \leq c(\nu_\infty, a_\infty)\frac{\Gamma(k-1+\nu_\infty+3)}{a_\infty^k}, \\ \left|\frac{a_\infty^k g_k}{\Gamma(\nu_\infty+k+3)}\right| \leq \frac{c(\nu_\infty,a_\infty)}{1+k^2}, \end{array} \right.$$

and (5.13) and (5.14) follow. We therefore focus on the proof of (5.18), (5.19). We start by renormalizing the sequence. Let

$$\left| \begin{array}{l} w_k = \frac{a_\infty^k}{\Gamma(\nu_\infty+k+3)}\theta_k \\ (h_0)_k = \frac{a_\infty^k}{\Gamma(\nu_\infty+k+3)}(\mu_0)_{k+1} \\ h_k = \frac{a_\infty^k}{\Gamma(\nu_\infty+k+3)}\mu_k \\ \tilde{h}_k = \frac{a_\infty^k}{\Gamma(\nu_\infty+k+3)}\nu_k. \end{array} \right. \tag{5.20}$$

then (5.17) becomes

$$\frac{\Gamma(\nu_\infty+k+4)a_\infty}{a_\infty^{k+1}}w_{k+1} + (k+\nu_\infty+3)\frac{\Gamma(\nu_\infty+k+3)}{a_\infty^k}w_k = \frac{\Gamma(\nu_\infty+k+3)}{a_\infty^k}(h_0)_k$$

$$+ \quad \frac{1}{a_\infty^{k-1}}\sum_{j=1}^{4}\sum_{k_1+\ldots k_{j+1}=k-1} h_{jk_1}w_{k_2}\ldots w_{k_{j+1}}\Pi_{i=1}^{j+1}\Gamma(\nu_\infty+k_i+3)$$

$$+ \quad \frac{1}{a_\infty^{k-2}}\sum_{j=1}^{4}\sum_{k_1+\ldots k_{j+2}=k-2} \tilde{h}_{jk_1}w_{k_2}\ldots w_{k_{j+1}}(k_{j+2}w_{k_{j+2}})\Pi_{i=1}^{j+2}\Gamma(\nu_\infty+k_i+3)$$

$$\Leftrightarrow \quad w_{k+1} + w_k = \frac{1}{\nu_\infty+k+3}(h_0)_k$$

$$+ \quad \frac{a_\infty}{(k+\nu_\infty+3)(k+\nu_\infty+2)}\sum_{j=1}^{4}\sum_{k_1+\ldots k_{j+1}=k-1} h_{jk_1}w_{k_2}\ldots w_{k_{j+1}}\frac{\Pi_{i=1}^{j+1}\Gamma(\nu_\infty+k_i+3)}{\Gamma(k-1+\nu_\infty+3)}$$

$$+ \quad \frac{a_\infty^2}{\Pi_{j=1}^{3}(k+\nu_\infty+j)}\sum_{j=1}^{4}\sum_{k_1+\ldots k_{j+2}=k-2} \tilde{h}_{jk_1}w_{k_2}\ldots w_{k_{j+1}}(k_{j+2}w_{k_{j+2}})\frac{\Pi_{i=1}^{j+2}\Gamma(\nu_\infty+k_i+3)}{\Gamma(k-2+\nu_\infty+3)}$$

We obtain the upper bound

$$|w_{k+1}| < |w_k| + \frac{c_{\nu_\infty,a_\infty}}{k+1}(h_0)_k \tag{5.21}$$
$$+ \quad \frac{c_{\nu_\infty,a_\infty}}{1+k^2}\sum_{j=1}^{4}\sum_{k_1+\ldots k_{j+1}=k-1}|h_{jk_1}w_{k_2}\ldots w_{k_{j+1}}|\frac{\Pi_{i=1}^{j+1}\Gamma(\nu_\infty+k_i+3)}{\Gamma(k-1+\nu_\infty+3)}$$
$$+ \quad \frac{c_{\nu_\infty,a_\infty}}{1+k^2}\sum_{j=1}^{4}\sum_{k_1+\ldots k_{j+2}=k-2}|\tilde{h}_{jk_1}w_{k_2}\ldots w_{k_{j+1}}w_{k_{j+2}}|\frac{\Pi_{i=1}^{j+2}\Gamma(\nu_\infty+k_i+3)}{\Gamma(k-2+\nu_\infty+3)}.$$

We now proceed via a bootstrap argument and iteratively improve the control of the sequence w_k from a large exponential bound to boundedness, the key being the use of suitable *uniform* convolution estimates. The latter are proved in Appendix B.

step 3 Large exponential bound. Since the functions $h_0, h, \tilde{h}$ are holomorphic and (5.20), there exists $A = A(\nu_\infty, a_\infty) > 0$ such that

$$\forall k \geq 1, \quad |(h_0)_k| + |h_k| + |\tilde{h}_k| \leq \frac{A^k}{\Gamma(k+1)}. \tag{5.22}$$

Pick $C_0 \geq C_0(\nu_\infty, a_\infty) \gg 1$ large enough, we claim

$$|w_k| \leq C_0^k. \tag{5.23}$$

The bound holds for $k \leq 5$ by choosing C_0 large enough. We assume the bound for $j \leq k$ and prove it for $k+1$. We have from (5.22) the rough bound

$$|(h_0)_k| + |h_k| + |\tilde{h}_k| < C_0^k$$

provided C_0 has been chosen large enough. Then, from (5.21), (B.5):

$$|w_{k+1}| < C_0^k + \frac{c_{\nu_\infty,a_\infty}C_0^k}{1+k} + \frac{c_{\nu_\infty,a_\infty}}{1+k^2}\sum_{j=1}^{4}C_\nu^{j+2}C_0^{k-1}$$

and

$$\frac{|w_{k+1}|}{C_0^{k+1}} < \frac{1}{C_0} + \frac{c_{a_\infty,\nu_\infty}}{C_0(1+k)} < 1$$

for $k \geq 1$ and $C_0 > C_0(\nu_\infty, a_\infty)$ large enough. (5.23) is proved.

step 4 Improvement of the exponential bound. Let $C_0(\nu, a)$ be a large enough constant such that (5.23) holds. Pick a small enough universal constant $\delta(\nu_\infty, a_\infty) > 0$ such that $\delta(\nu_\infty, a_\infty) \ll (C_0(\nu_\infty, a_\infty))^{-1}$. We claim the following: assume that there exist

$$\left| \begin{array}{l} K_n > 1 \\ e^\delta \leq C_n \leq C_0 \end{array} \right. \tag{5.24}$$

such that

$$\forall k \geq 1, \quad |w_k| < K_n C_n^k, \tag{5.25}$$

then there exists $K_{n+1} = K_{n+1}(K_n, C_n) > 1$ such that

$$\forall k \geq 1, \quad |w_k| < K_{n+1}\left(C_n e^{-\delta}\right)^k. \tag{5.26}$$

Pick $k^*(K_n, \nu_\infty, a_\infty)$ large enough, then the bound (5.25) implies (5.26) for $k \leq k^*$ provided we choose $K_{n+1}(K_n, \nu_\infty, a_\infty) \geq K_n e^{k^*\delta}$. We now bootstrap the bound (5.26) by induction for $k \geq k^*$, assuming it for $j \leq k$ and proving it for $k+1$. Assume

$$k_1 \leq k_2 \cdots \leq k_{j+1},$$

and let $m = k_1 + \cdots + k_j$ and use the rough upper bound from (5.22) to estimate

$$\begin{aligned} & |h_{jk_1}||w_{k_2}|\dots|w_{k_{j+1}}|\frac{\Pi_{i=1}^{j+1}\Gamma(\nu_\infty+k_i+3)}{\Gamma(k-1+\nu_\infty+3)} \\ \lesssim\ & (K_nC_n^{k_1})|w_{k_2}|\dots|w_{k_{j+1}}|\frac{\Pi_{i=1}^{j+1}\Gamma(\nu_\infty+k_i+3)}{\Gamma(k-1+\nu_\infty+3)} \\ \lesssim\ & (K_nC_n^{k_1})|w_{k_2}|\dots|w_{k_{j+1}}|\frac{\Pi_{i=1}^{j}\Gamma(\nu_\infty+k_i+3)}{\Gamma(\nu_\infty+m+3)}\frac{\Gamma(\nu_\infty+m+3)\Gamma(\nu_\infty+k_{j+1}+3)}{\Gamma(\nu_\infty+k-1+3)}. \end{aligned}$$

<u>case $m \le k_{j+1}$</u> In this case, we let

$$m = x(k-1), \quad k_{j+1} = (1-x)(k-1), \quad x \le \frac{1}{2}.$$

For $\delta < x \le \frac{1}{2}$, from (B.2) and the monotonicity of $\Phi_{k-1}^{(2)}$:

$$\Phi_{k-1}^{(2)}(x) \ge \Phi_{k-1}^{(2)}(\delta) \ge \frac{1}{2}\delta|\log\delta|$$

Then, using (5.25), (B.1), (B.5):

$$\begin{aligned} & \frac{c_{\nu_\infty,a_\infty}}{1+k^2}\sum_{k_1+\cdots+k_j=m}\sum_{\delta<x\le\frac12}(K_nC_n^{k_1})|w_{k_2}|\dots|w_{k_{j+1}}| \\ \times\ & \frac{\Pi_{i=1}^{j}\Gamma(\nu_\infty+k_i+3)}{\Gamma(\nu_\infty+m+3)}\frac{\Gamma(\nu_\infty+m+3)(\Gamma(\nu_\infty+k_{j+1}+3)}{\Gamma(\nu_\infty+k-1+3)} \\ \lesssim\ & \frac{c_{\nu_\infty,a_\infty}}{1+k^2}\sum_{k_1+\cdots+k_j=m}K_n^jC_n^m\frac{\Pi_{i=1}^{j}\Gamma(\nu_\infty+k_i+3)}{\Gamma(\nu_\infty+m+3)}\sum_{\delta<x\le\frac12}K_nC_n^{k_{j+1}}e^{-\frac{\delta|\log\delta|}{4}k} \\ \le\ & \frac{c_{\nu_\infty,a_\infty}}{1+k^2}K_n^5C_n^{k-1}e^{-\frac{\delta|\log\delta|}{8}k}\sum_{k_1+\cdots+k_j=m}\frac{\Pi_{i=1}^{j}\Gamma(\nu_\infty+k_i+3)}{\Gamma(\nu_\infty+m+3)} \le \frac{c_{\nu_\infty,a_\infty}}{1+k^2}K_n^5\left(C_ne^{-\frac{\delta|\log\delta|}{8}}\right)^k \end{aligned}$$

For $\frac{R_\nu}{k} < x < \delta$, with $R_\nu = R_{\nu_\infty}$ introduced in Lemma B.1, we have in view of (B.2),

$$\Phi_k^{(2)}(x) \ge \frac{x|\log x|}{2} \ge \frac{x|\log\delta|}{2}$$

and thus, since $0 < \delta \ll C_0^{-1}$,

$$\sqrt{1+m}C_n^me^{-k\Phi_k^{(2)}(x)} \le \sqrt{1+m}e^{m\log C_n-\frac{m|\log\delta|}{4}} \le \sqrt{1+m}e^{m\log C_0-\frac{m|\log\delta|}{4}} \le e^{-\frac{m|\log\delta|}{8}}$$

Using the bootstrap bounds (5.25) and (5.26) for $k_{j+1} \leq k-1$:

$$\begin{aligned}
& \frac{c_{\nu_\infty,a_\infty}}{1+k^2} \sum_{k_1+\cdots+k_j=m} \sum_{\frac{R_\nu}{k}<x\leq\delta} (K_n C_n^{k_1})|w_{k_2}|\dots|w_{k_{j+1}}| \\
\times \quad & \frac{\Pi_{i=1}^j \Gamma(\nu_\infty+k_i+3)}{\Gamma(\nu_\infty+m+3)} \frac{\Gamma(\nu_\infty+m+3)(\Gamma(\nu_\infty+k_{j+1}+3)}{\Gamma(\nu_\infty+k-1+3)} \\
\leq \quad & \frac{c_{\nu_\infty,a_\infty}}{1+k^2} \sum_{k_1+\cdots+k_j=m} K_n^j C_n^m \frac{\Pi_{i=1}^j \Gamma(\nu_\infty+k_i+3)}{\Gamma(\nu_\infty+m+3)} \sum_{\frac{R_\nu}{k}<x\leq\delta} K_{n+1} \left(C_n e^{-\delta}\right)^{k_{j+1}} \frac{e^{-\frac{m|\log\delta|}{8}}}{C_n^m} \\
\leq \quad & \frac{c_{\nu_\infty,a_\infty} K_n^5 K_{n+1}}{1+k^2} \left(C_n e^{-\delta}\right)^k \sum_{k_1+\cdots+k_j=m} \frac{\Pi_{i=1}^j \Gamma(\nu_\infty+k_i+3)}{\Gamma(\nu_\infty+m+3)} \\
\leq \quad & \frac{c_{\nu_\infty,a_\infty} K_n^5 K_{n+1}}{1+k^2} \left(C_n e^{-\delta}\right)^k
\end{aligned}$$

where we used (B.5) in the last step. For $x \leq \frac{R_\nu}{k}$, there are $m \leq xk \leq R_\nu$ terms in the sum and $k_j \leq m \leq kx \leq R_\nu$ fixed. We therefore use $\Phi_k^{(2)}(x) \geq 0$ and the bootstrap bounds (5.23) and (5.26) for $k_{j+1} \leq k-1$ to estimate:

$$\begin{aligned}
& \frac{c_{\nu,a}}{1+k^2} \sum_{k_1+\cdots+k_j=m} \sum_{x\leq\frac{R_\nu}{k}} (K_n C_n^{k_1})|w_{k_2}|\dots|w_{k_{j+1}}| \\
\times \quad & \frac{\Pi_{i=1}^j \Gamma(\nu_\infty+k_i+2)}{\Gamma(\nu_\infty+m+3)} \frac{\Gamma(\nu_\infty+m+3)(\Gamma(\nu_\infty+k_{j+1}+3)}{\Gamma(\nu_\infty+k-1+3)} \\
\leq \quad & \frac{c_{\nu_\infty,a_\infty} K_{n+1}}{1+k^2} \left(C_n e^{-\delta}\right)^{k_{j+1}} \sum_{k_1+\cdots+k_j=m} \frac{\Pi_{i=1}^j \Gamma(\nu_\infty+k_i+3)}{\Gamma(\nu+m+3)} \\
\leq \quad & \frac{c_{\nu,a} K_{n+1}}{1+k^2} \left(C_n e^{-\delta}\right)^k .
\end{aligned}$$

<u>case $k_{j+1} \leq m$</u> In this case, we let $\delta > 0$ as in Lemma B.1, and let

$$k_{j+1} = x(k-1), \quad m = (1-x)(k-1), \quad x \leq \frac{1}{2}.$$

Since $k_1 \leq k_2 \leq \dots k_{j+1}$, we have

$$k-1 = m + k_{j+1} \leq (j+1)k_{j+1}$$

and we are always in the range $\delta < \frac{1}{j+2} \leq x \leq \frac{1}{2}$. We then use verbatim the same chain of estimates as above: from (B.2) and the monotonicity of $\Phi_{k-1}^{(2)}$:

$$\Phi_{k-1}^{(2)}(x) \geq \Phi_{k-1}^{(2)}(\delta) \geq \frac{1}{2}\delta|\log\delta|$$

Therefore, using (5.25), (B.5):

$$\begin{aligned}
&\frac{c_{\nu_\infty,a_\infty}}{1+k^2}\sum_{k_1+\cdots+k_j=m}\sum_{\delta<x\leq\frac12}(K_nC_n^{k_1})|w_{k_2}|\dots|w_{k_{j+1}}||w_{k_{j+1}}|\\
\times\quad&\frac{\Pi_{i=1}^j\Gamma(\nu_\infty+k_i+3)}{\Gamma(\nu_\infty+m+3)}\frac{\Gamma(\nu_\infty+m+3)(\Gamma(\nu_\infty+k_{j+2}+3)}{\Gamma(\nu_\infty+k-1+3)}\\
\leq\quad&\frac{c_{\nu_\infty,a_\infty}}{1+k^2}\sum_{k_1+\cdots+k_j=m}K_n^{j+1}C_n^{k-1}\frac{\Pi_{i=1}^j\Gamma(\nu_\infty+k_i+3)}{\Gamma(\nu_\infty+m+3)}\sum_{\delta<x\leq\frac12}e^{-\frac{\delta|\log\delta|}{4}k}\\
\leq\quad&\frac{c_{\nu_\infty,a_\infty}}{1+k^2}K_n^5C_n^{k-1}e^{-\frac{\delta|\log\delta|}{8}k}\sum_{k_1+\cdots+k_j=m}\frac{\Pi_{i=1}^j\Gamma(\nu_\infty+k_i+2)}{\Gamma(\nu_\infty+m+2)}\leq\frac{c_{\nu_\infty,a_\infty}}{1+k^2}K_n^5\left(C_ne^{-\frac{\delta|\log\delta|}{8}}\right)^k
\end{aligned}$$

Conclusion The collection of above bounds inserted into (5.21) ensures:

$$|w_{k+1}|<|w_k|+\frac{c_{\nu_\infty,a_\infty}A^k}{\Gamma(k+1)}+\frac{c_{\nu_\infty,a_\infty}K_n^5K_{n+1}}{1+k^2}\left(C_ne^{-\delta}\right)^k$$

Then, provided that $k>k^*(K_n,\nu,a)$ has been chosen large enough:

$$\begin{aligned}
\frac{w_{k+1}}{K_{n+1}(C_ne^{-\delta})^{k+1}}&<\frac{K_n}{K_{n+1}C_ne^{-\delta}}+\frac{c_{\nu,a}A^k}{\Gamma(k+1)K_{n+1}(C_ne^{-\delta})^{k+1}}+\frac{c_{\nu,a}K_n^5}{(1+k^2)(C_ne^{-\delta})}\\
&<\frac12+\frac{c_{\nu_\infty,a_\infty}K_n^5}{1+k^2}<1
\end{aligned}$$

and (5.26) is proved.

step 5 Boundedness. The initial bound (5.23) allows us to apply (5.25), (5.26) iteratively a finite number of times[15] to obtain the bound for some large enough constant $C>1$ depending on ν_∞ and a_∞:

$$\forall k\geq1,\quad|w_k|<C$$

and (5.18) is proved. As a byproduct, we obtain

$$\begin{aligned}
&\left|\frac{a_\infty^kg_{k+1}}{\Gamma(\nu_\infty+k+1+3)}\right|=|w_k+w_{k+1}|<\frac{c_{\nu_\infty,a_\infty}}{k+1}(h_0)_k\\
+\quad&\frac{c_{\nu_\infty,a_\infty}}{1+k^2}\sum_{j=1}^4\sum_{k_1+\dots k_{j+1}=k-1}|h_{jk_1}w_{k_2}\dots w_{k_{j+1}}|\frac{\Pi_{i=1}^{j+1}\Gamma(\nu_\infty+k_i+3)}{\Gamma(k-1+\nu_\infty+3)}\\
+\quad&\frac{c_{\nu_\infty,a_\infty}}{1+k^2}\sum_{j=1}^4\sum_{k_1+\dots k_{j+2}=k-2}|\tilde h_{jk_1}w_{k_2}\dots w_{k_{j+1}}w_{k_{j+2}}|\frac{\Pi_{i=1}^{j+2}\Gamma(\nu_\infty+k_i+3)}{\Gamma(k-2+\nu_\infty+3)}\\
<\quad&\frac{c_{\nu_\infty,a_\infty}A^k}{\Gamma(k+1)}+\frac{c_{\nu_\infty,a_\infty}}{1+k^2}
\end{aligned}$$

and (5.16) follows as well. □

[15]Recall that $C_{n+1}=e^{-\delta}C_n$ so that $C_n=e^{-n\delta}C_0$. Thus, we need $C_n=C_0e^{-n\delta}=1$, i.e. it suffices to choose the number n of iterations as

$$n(\nu_\infty,a_\infty)=\frac{1}{\delta(\nu_\infty,a_\infty)}\log\big(C_0(\nu_\infty,a_\infty)\big)$$

where have adjusted the choice of δ to ensure that the formula provides an integer value for n.

5.5. **Proof of Proposition 5.1.** In view of (5.13) and the conjugation formula (5.6), the bound (5.5) directly follows from the following continuity lemma.

Lemma 5.5 (Continuity)**.** *Let $M(x)$ be holomorphic in a neighborhood of 0, then there exists $c_M > 0$ such that for all functions Φ which are $\mathcal{C}^\infty$ at the origin,*

$$\sup_{k\geq 0} \frac{a_\infty^k |(M\Phi)_k|}{\Gamma(k+\nu_\infty+2)} \leq c_M \sup_{k\geq 0} \frac{a_\infty^k |\phi_k|}{\Gamma(k+\nu_\infty+2)}. \tag{5.27}$$

Proof of Lemma 5.5. Since M is holomorphic in a neighborhood of the origin

$$|m_k| = \frac{1}{k!}\left|\frac{d^k M}{dx^k}(0)\right| \leq C^k$$

and then

$$\sup_{k\geq 0} \frac{a_\infty^k |m_k|}{\Gamma(k+\nu_\infty+2)} \leq c_M.$$

We estimate using (B.5):

$$\begin{aligned}
&\frac{a_\infty^k |(M\Phi)_k|}{\Gamma(k+\nu+2)} = a_\infty^k \sum_{k_1+k_2=k} \frac{|m_{k_1}\phi_{k_2}|}{\Gamma(k+\nu_\infty+2)} \\
\leq\ & \left(\sup_{k\geq 0} \frac{a_\infty^k |\phi_k|}{\Gamma(k+\nu_\infty+2)}\right)\left(\sup_{k\geq 0} \frac{a_\infty^k |m_k|}{\Gamma(k+\nu_\infty+2)}\right) \sum_{k_1+k_2=k} \frac{\Gamma(k_1+\nu_\infty+2)\Gamma(k_2+\nu_\infty+2)}{\Gamma(k+\nu_\infty+2)} \\
\leq\ & c_M \sup_{k\geq 0} \frac{a_\infty^k |\phi_k|}{\Gamma(k+\nu_\infty+2)}
\end{aligned}$$

and (5.27) is proved. □

6. **Bounding the Taylor series for** (4.15)

We now start the study of the full problem (4.15). We first aim at obtaining uniform in b bounds on the coefficients of the Taylor series as well as the convergence to the limiting problem as $b \to 0$. We recall the notation

$$\gamma - 1 = K + \alpha_\gamma, \quad K \in \mathbb{N}^*, \quad 0 < \alpha_\gamma < 1.$$

6.1. b **dependent conjugation.** We conjugate the b dependent problem (4.15) to an explicitly solvable (at the linear level) b-dependent equation.

Lemma 6.1 (Linear conjugation)**.** *There exist a function $\tilde{\xi}_b(x)$ which is holomorphic in a b-independent neighborhood of $x = 0$ such that the conjugation*

$$\Psi(u) = M_b(x)\Phi(u), \quad G = \frac{F}{M_b(1+H_{20})} \tag{6.1}$$

with

$$M_b(x) = e^{\tilde{\xi}_b(x)} \tag{6.2}$$

and

$$\left|\begin{array}{l} \nu_b = \nu + \xi(b) \\ \nu = -\gamma b(\tilde{D}_{11} + \tilde{D}_{30} - \tilde{E}_{11}) \end{array}\right. \tag{6.3}$$

maps[16]

$$[1+H_{20}]\, u(1-u)\frac{d\Psi}{du} + [(1-2u)(1+H_{20}) - \gamma(1+G_1) + 2uG_2]\,\Psi = -uF \tag{6.4}$$

[16]The link between (4.15) and (6.4) will be given later, see (6.20) (6.21). Also, see (G.3) for the definition of H_{20}.

to

$$\frac{d\Phi}{du} - \left[\frac{\gamma-1}{u} + \frac{\gamma+\nu_b+1}{1-u}\right]\Phi = -\frac{G}{1-u}. \tag{6.5}$$

Remark 6.2. We see immediately the fundamental difference between (6.5) and (5.10): for the b-dependent problem, the point $u = 0$ is a regular singular point, while it is a singular singular point for $b = 0$. As a result, we will see a change in the behavior of the Taylor series for large frequencies, which will be reflected in the nature of the weight $w_{\gamma,\nu}$, see Lemma 6.3.

Proof of Lemma 6.1. This is a direct computation.

step 1 Linear conjugation. We rewrite (6.4)

$$\frac{d\Psi}{du} - \frac{\frac{d\zeta_b}{du}}{\zeta_b}\Psi = -\frac{F}{(1-u)(1+H_{2,0})} \tag{6.6}$$

with

$$\begin{aligned}\frac{\frac{d\zeta_b}{du}}{\zeta_b} &= \frac{\gamma(1+G_1)}{u(1-u)(1+H_{2,0})} - \frac{2G_2}{(1-u)(1+H_{2,0})} - \frac{1-2u}{u(1-u)} \\ &= \frac{\gamma(1+G_1)}{u(1-u)(1+H_{2,0})} - \frac{2G_2}{(1-u)(1+H_{2,0})} - \frac{1}{u} + \frac{1}{1-u}\end{aligned}$$

From (4.35):

$$\left|\begin{array}{l} G_1(x) = \tilde{E}_{11}x + x\tilde{G}_1 \\ \tilde{G}_1 = -(2\tilde{E}_{02} + \tilde{E}_{21})x + 2\tilde{E}_{12}x^2 - 3\tilde{E}_{03}x^3 \end{array}\right.$$

and

$$\frac{\gamma(1+G_1)}{u(1-u)} = \frac{\gamma(1+\tilde{E}_{11}bu + bu\tilde{G}_1)}{u(1-u)} = \frac{\gamma}{u(1-u)} + \frac{\tilde{E}_{11}b\gamma}{1-u} + \frac{\gamma b\tilde{G}_1}{1-u}$$

yields

$$\frac{\frac{d\zeta_b}{du}}{\zeta_b} = \left[\frac{\gamma}{u(1-u)} + \frac{\tilde{E}_{11}b\gamma}{1-u}\right]\frac{1}{1+H_{2,0}} + \frac{\gamma b\tilde{G}_1 - 2G_2}{(1-u)(1+H_{2,0})} - \frac{1}{u} + \frac{1}{1-u}.$$

We recall that $x = bu$ and rewrite

$$\begin{aligned}\frac{\gamma}{u(1-u)(1+H_{2,0})} &= \frac{\gamma}{u(1-u)}\left[1 - (\tilde{D}_{11}+\tilde{D}_{30})bu + \left(\frac{1}{1+H_{2,0}} - 1 + (\tilde{D}_{11}+\tilde{D}_{30})x\right)\right] \\ &= \gamma\left(\frac{1}{u} + \frac{1}{1-u}\right) - \frac{\gamma b(\tilde{D}_{11}+\tilde{D}_{30})}{1-u} \\ &+ \frac{\gamma b}{1-u}\left[\frac{1}{x}\left(\frac{1}{1+H_{2,0}} - 1 + (\tilde{D}_{11}+\tilde{D}_{30})x\right)\right]\end{aligned}$$

and

$$\frac{\tilde{E}_{11}b\gamma}{(1-u)(1+H_{2,0})} = \frac{\tilde{E}_{11}b\gamma}{1-u} + \frac{\tilde{E}_{11}b\gamma}{1-u}\left[\frac{1}{1+H_{2,0}} - 1\right].$$

We have therefore obtained the formula:

$$\frac{\frac{d\zeta_b}{du}}{\zeta_b} = \gamma\left(\frac{1}{u} + \frac{1}{1-u}\right) - \frac{1}{u} + \frac{1}{1-u} - \frac{\gamma b(\tilde{D}_{11}+\tilde{D}_{30})}{1-u} + \frac{\tilde{E}_{11}b\gamma}{1-u} + \frac{\xi(x)}{1-u}$$

with

$$\xi(x) = \frac{\gamma b\tilde{G}_1 - 2G_2}{1+H_{2,0}} + \frac{\gamma b}{x}\left(\frac{1}{1+H_{2,0}} - 1 + (\tilde{D}_{11}+\tilde{D}_{30})x\right) + \gamma b\tilde{E}_{11}\left(\frac{1}{1+H_{2,0}} - 1\right).$$

We recall from (5.2)

$$\nu = -\gamma b(\tilde{D}_{11} + \tilde{D}_{30} - \tilde{E}_{11}),$$

and rewrite:

$$\frac{\frac{d\zeta_b}{du}}{\zeta_b} = \frac{\gamma - 1}{u} + \frac{\gamma + \nu + 1}{1 - u} + \frac{\xi(x)}{1 - u}. \tag{6.7}$$

step 2 Computation of the kernel. We now use the analyticity of ξ in $|x| < \frac{1}{C}$ and $\xi(0) = 0$ to compute:

$$\frac{\xi(x)}{1-u} = \frac{1}{1-u}\sum_{k=1}^{+\infty} \xi_k x^k = \frac{1}{1-u}\sum_{k=1}^{+\infty} \xi_k b^k u^k = \frac{\sum_{k=1}^{+\infty} \xi_k b^k}{1-u} - \sum_{k=1}^{+\infty} \xi_k b^k \frac{1-u^k}{1-u}$$

$$= \frac{\xi(b)}{1-u} - \sum_{k=1}^{+\infty} \xi_k b^k \sum_{j=0}^{k-1} u^j.$$

Let ν_b be given by (6.3), we have obtained:

$$\frac{\frac{d\zeta_b}{du}}{\zeta_b} = \frac{\gamma - 1}{u} + \frac{\gamma + \nu_b + 1}{1 - u} - \sum_{k=1}^{+\infty} \xi_k b^k \sum_{j=0}^{k-1} u^j.$$

We compute a primitive of the remaining term:

$$\sum_{k=1}^{+\infty} \xi_k b^k \sum_{j=0}^{k-1} \frac{u^{j+1}}{j+1} = \sum_{j=0}^{+\infty} \frac{u^{j+1}}{j+1} \sum_{k=j+1}^{+\infty} \xi_k b^k = \sum_{j=1}^{+\infty} \frac{u^j}{j} \sum_{k=j}^{+\infty} \xi_k b^k = \sum_{j=1}^{+\infty} \tilde{\xi}_{b,j} b^j u^j = \sum_{j=1}^{+\infty} \tilde{\xi}_{b,j} x^j$$

with

$$\tilde{\xi}_{b,j} = \frac{1}{j}\sum_{k=j}^{+\infty} \xi_k b^{k-j} = \frac{1}{j}\sum_{k=0}^{+\infty} \xi_{k+j} b^k.$$

The holomorphic bound $|\xi_j| \leq C^j$ ensures

$$|\tilde{\xi}_{b,j}| \leq \frac{1}{j}\sum_{k=0}^{+\infty} b^k C^{k+j} \leq \frac{C^j}{j}\frac{1}{1-bC} \leq \frac{C^j}{j} \tag{6.8}$$

for $0 < b < b^*$ universal small enough. We have therefore obtained the formula

$$\frac{\frac{d\zeta_b}{du}}{\zeta_b} = \frac{\gamma - 1}{u} + \frac{\gamma + \nu_b + 1}{1 - u} + \frac{d}{du}\tilde{\xi}_b(x) \tag{6.9}$$

where

$$\tilde{\xi}_b(x) = -\sum_{j=1}^{+\infty} \tilde{\xi}_{b,j} x^j, \quad |\tilde{\xi}_{b,j}| \leq C^j \tag{6.10}$$

is holomorphic in a neighborhood of $x = 0$ independent of b.

step 3 Conclusion. From (6.6), (6.9):

$$\frac{d\Psi}{du} - \left[\frac{\gamma - 1}{u} + \frac{\gamma + \nu_b + 1}{1 - u} + \frac{d}{du}\tilde{\xi}_b(x)\right]\Psi = -\frac{F}{(1-u)(1+H_{20})}$$

and (6.5) follows. □

6.2. **The b-dependent discrete weight.** We study the discrete weight associated to (6.5).

Lemma 6.3 (Properties of the weight)**.** *Let*

$$w_{\gamma,\nu}(k) = \frac{\Gamma(\gamma-1-k)\Gamma(\nu+k+2)}{\Gamma(\gamma-1)}, \quad k \in \mathbb{N}. \tag{6.11}$$

then:

(1) *value for $k \geq K$: $\forall j \geq 0$,*

$$w_{\gamma,\nu_b}(K+j) = (-1)^j \Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)\frac{\Gamma(K+j+\nu_b+2)}{\Gamma(K+\alpha_\gamma)\Gamma(j+1-\alpha_\gamma)}. \tag{6.12}$$

(2) *ν dependence:*

$$\forall k \geq 0, \quad \frac{\gamma w_{\gamma,\nu_b}(k+1)}{w_{\gamma-1,\nu_b+1}(k)} = \frac{\gamma}{\gamma-2} \tag{6.13}$$

(3) *induction property:*

$$\forall k \geq 1, \quad (\gamma-k-2)w_{\gamma-1,\nu_b+1}(k) - (k+\nu_b+2)w_{\gamma-1,\nu_b+1}(k-1) = 0. \tag{6.14}$$

Proof. (6.12) directly follows from (6.11), (A.10).
We then compute for $k \geq 0$:

$$\frac{\gamma w_{\gamma,\nu_b}(k+1)}{w_{\gamma-1,\nu_b+1}(k)} = \gamma\frac{\Gamma(\gamma-1-(k+1))\Gamma(k+1+\nu_b+2)}{\Gamma(\gamma-1)\frac{\Gamma(\gamma-2-k)\Gamma(k+\nu_b+3)}{\Gamma(\gamma-2)}} = \frac{\gamma\Gamma(\gamma-2)}{\Gamma(\gamma-1)} = \frac{\gamma}{\gamma-2}$$

and (6.13) is proved. We now turn to the induction formula: for $k \geq 1$,

$$\begin{aligned}
& (\gamma-k-2)w_{\gamma-1,\nu_b+1}(k) - (k+\nu_b+2)w_{\gamma-1,\nu_b+1}(k-1) \\
= \;& (\gamma-k-2)\frac{\Gamma(\gamma-2-k)\Gamma(k+\nu_b+1+2)}{\Gamma(\gamma-2)} \\
- \;& (k+\nu_b+2)\frac{\Gamma(\gamma-2-(k-1))\Gamma(k-1+\nu_b+1+2)}{\Gamma(\gamma-2)} \\
= \;& \frac{\Gamma(\gamma-k-1)\Gamma(k+\nu_b+3)}{\Gamma(\gamma-2)} - \frac{\Gamma(\gamma-k-1)\Gamma(k+\nu_b+3)}{\Gamma(\gamma-2)} = 0.
\end{aligned}$$

□

6.3. **Boundedness of the sequence.** We claim the following b-dependent nonlinear bound which, in a certain sense, is a deformation of (5.5).

Proposition 6.4 (b-dependent boundedness)**.** *There exist universal constants $c_{\nu,a} > 0$ and $0 < b^* \ll 1$ such that the following holds for all $0 < b < b^*$. Let Ψ be a solutions of (4.15) and define*

$$\psi_k = \frac{1}{k!}\frac{d^k\Psi}{du^k}(0),$$

then

$$\forall\, 0 \leq k \leq K, \quad |\psi_k| \leq c_{\nu,a} w_{\gamma,\nu_b}(k). \tag{6.15}$$

Moreover, let $\tilde{\Psi}^\infty$ be the unique local $\mathcal{C}^\infty$ solution to the limiting problem (5.4) and

$$\tilde{\psi}_k^\infty = \frac{1}{k!}\frac{d^k\tilde{\Psi}^\infty}{dx^k}(0),$$

then

$$\forall k \geq 0, \quad \lim_{b\to 0}\frac{\psi_k}{b^k} = \tilde{\psi}_k^\infty. \tag{6.16}$$

The rest of this section is devoted to the proof of Proposition 6.4.

6.4. **Proof of** (6.15) **for frequencies** $k \ll \gamma$**.** We claim the following small frequency bound.

Lemma 6.5 (Uniform small frequency bound)**.** *The limit* (6.16) *holds. Moreover, there exists* $c_{\nu,a} > 0$ *such that for all* $k^* \geq 0$*, there exists* $0 < b^*(k^*) \ll 1$ *such that*

$$\forall 0 < b < b^*(k^*), \quad \forall 0 \leq k \leq k^*, \quad |\psi_k| \leq c_{\nu,a} w_{\gamma,\nu_b}(k). \tag{6.17}$$

Proof of Lemma 6.5. Recall

$$\Psi(u) = \tilde{\Psi}(x), \quad x = bu$$

so that

$$\frac{\psi_k}{b^k} = \frac{1}{k! b^k} \frac{d^k \Psi}{du^k}(0) = \frac{1}{k!} \frac{d^k \tilde{\Psi}}{dx^k}(0) = \tilde{\psi}_k$$

and (6.16) is equivalent to

$$\lim_{b \to 0} \tilde{\psi}_k = \tilde{\psi}_k^{\infty}.$$

This follows immediately by passing to the $b \to 0$ limit for fixed k in the induction relation for the $\tilde{\psi}_k$ from (5.1). The details are straightforward and left to the reader. Similarly, note that we also have

$$\lim_{b \to 0} \frac{g_k}{b^k} = g_k^{\infty}. \tag{6.18}$$

We conclude from (5.5) that there exists $c_{\nu,a}$ such that

$$\forall k \geq 0, \quad |\tilde{\psi}_k^{\infty}| \leq c_{\nu,a} \frac{\Gamma(\nu + k + 2)}{a^k}. \tag{6.19}$$

Pick now an arbitrary $k^* \geq 0$, then from (6.16), (6.19):

$$\forall 0 < b < b^*(k^*), \quad \forall 0 \leq k \leq k^*, \quad |\tilde{\psi}_k| \leq 2 c_{\nu,a} \frac{\Gamma(\nu + k + 2)}{a^k}.$$

We now estimate using (C.1):

$$\begin{aligned} \frac{|\psi_k|}{w_{\gamma,\nu_b}(k)} &\leq c_1 \frac{|\psi_k| \gamma^k}{\Gamma(\nu_b + k + 2)} = c_1 \frac{|\tilde{\psi}_k| b^k \gamma^k}{\Gamma(\nu_b + k + 2)} = c_1 \frac{|\tilde{\psi}_k| a^k}{\Gamma(\nu_b + k + 2)} \\ &\leq 2 c_1 c_{\nu,a} \frac{\Gamma(\nu + k + 2)}{\Gamma(\nu_b + k + 2)}. \end{aligned}$$

Since $\nu_b \to \nu$ as $b \to 0$, we may choose $b^*(k^*)$ small enough so that

$$\forall 0 < b < b^*(k^*), \quad \forall 0 \leq k \leq k^*, \quad \frac{\Gamma(\nu + k + 2)}{\Gamma(\nu_b + k + 2)} \leq 2.$$

Then,

$$\frac{|\psi_k|}{w_{\gamma,\nu_b}(k)} \leq 4 c_1 c_{\nu,a}$$

and (6.17) is proved. □

6.5. **Nonlinear conjugation.** The proof of (6.15) now requires a careful track of the b-dependencies in the full problem (4.15). The first step is to use Lemma 6.1 and analyze the nonlinear conjugated problem.

step 1 Nonlinear conjugation. Recall (4.15):

$$\begin{aligned} & \left[1+H_2+G_2\Psi+\widetilde{\mathrm{NL}_2}\right] u(1-u)\frac{d\Psi}{du} \\ + & \left[(1-2u)(1+H_2+G_2\Psi+\widetilde{\mathrm{NL}_2})-\gamma(1+G_1)+2uG_2\right]\Psi \\ = & \ u\left[\gamma b H_1-2(1+H_2)+\frac{\gamma b\widetilde{\mathrm{NL}_1}}{x}-2\widetilde{\mathrm{NL}_2}\right] \\ \Leftrightarrow & \ [1+H_{20}]\,u(1-u)\frac{d\Psi}{du}+[(1-2u)(1+H_{20})-\gamma(1+G_1)+2uG_2]\,\Psi \\ = & \ u\mathcal{F} \end{aligned} \tag{6.20}$$

with recalling $\frac{1}{u}=\frac{b}{x}$:

$$\begin{aligned} & \mathcal{F}=\gamma b H_1-2(1+H_2) \\ + & \ \frac{\gamma b\widetilde{\mathrm{NL}_1}}{x}-2\widetilde{\mathrm{NL}_2}-(1-u)\left[\sum_{j=1}^{3}b^jH_{2j}(x)+G_2\Psi+\widetilde{\mathrm{NL}_2}\right]\frac{d\Psi}{du} \\ + & \left\{-\frac{b}{x}(G_2\Psi+\widetilde{\mathrm{NL}_2})+2(G_2\Psi+\widetilde{\mathrm{NL}_2})-\frac{b(1-2u)}{x}\sum_{j=1}^{3}b^jH_{2j}(x)\right\}\Psi \end{aligned} \tag{6.21}$$

From (6.1), (6.5) we now obtain the nonlinear conjugated problem:

$$\frac{d\Phi}{du}-\left[\frac{\gamma-1}{u}+\frac{\gamma+\nu_b+1}{1-u}\right]\Phi=-\frac{\mathcal{G}}{1-u} \tag{6.22}$$

step 2 Computation of $\mathcal{G}$. We plug (6.1) into (6.21) and decompose

$$\mathcal{G}=\mathcal{G}_0+\mathcal{L}(\Phi)+\mathrm{NL}(\Phi) \tag{6.23}$$

as follows.

Source term. We have

$$\mathcal{G}_0(b,x)=\frac{\gamma b H_1-2(1+H_2)}{M_b(1+H_{20})}$$

which, from (6.10), admits a holomorphic expansion in a neighborhood (independent of b) of $x=0$ i.e.,

$$\mathcal{G}_0(x)=\sum_{k=0}^{+\infty}g_{0k}x^k \tag{6.24}$$

for some b-dependent coefficients g_{0k} satisfying

$$|g_{0k}|\le C^k \tag{6.25}$$

for some $C>0$ independent of b.

Small linear term. It is given explicitly by

$$\begin{aligned}\mathcal{L}(\Phi) &= \frac{1}{M_b(1+H_{20})}\left\{-(1-u)\left[\sum_{j=1}^{3} b^j H_{2j}(x)\right](M_b(x)\Phi)' - \frac{b(1-2u)}{x}\sum_{j=1}^{3} b^j H_{2j}(x) M_b\Phi\right\}\\ &= \frac{1}{M_b(1+H_{20})}\left\{\left[-bM_b'(x)(1-u)\sum_{j=1}^{3} b^j H_{2j}(x) - \sum_{j=1}^{3}\left(\frac{b^{j+1}H_{2j}(x)}{x} - 2b^j H_{2j}\right)\right]\Phi\right.\\ &- \left.(1-u)M_b(x)\left[\sum_{j=1}^{3} b^j H_{2j}(x)\right]\Phi'\right\} \qquad (6.26)\end{aligned}$$

We rewrite

$$(1-u)\Phi' = \frac{1-u}{u}u\Phi' = \left(\frac{b}{x}-1\right)u\Phi'$$

and obtain

$$\begin{aligned}\mathcal{L}(\Phi) &= \frac{1}{M_b(1+H_{20})}\left\{\left[-M_b'(x)(b-x)\sum_{j=1}^{3} b^j H_{2j}(x) - \sum_{j=1}^{3}\left(\frac{b^{j+1}H_{2j}(x)}{x} - 2b^j H_{2j}\right)\right]\Phi\right.\\ &- \left.M_b(x)\sum_{j=1}^{3}\left(b^{j+1}\frac{H_{2j}(x)}{x} - b^j H_{2j}(x)\right)u\Phi'\right\}\end{aligned}$$

Therefore,

$$\mathcal{L}(\Phi) = b\left[(bh_1(x)+xh_2(x))\Phi + (bh_3(x)+xh_4(x))u\Phi'\right]$$

where

$$h_j(x) = \sum_{k=0}^{+\infty} h_{jk}x^k, \quad |h_{jk}| \le C^k$$

for some $C > 0$ independent of b.
Nonlinear term. We have

$$\begin{aligned}&\mathrm{NL}(\Phi) \qquad (6.27)\\ &= \frac{1}{M_b(1+H_{20})}\left\{\frac{\gamma b\widetilde{\mathrm{NL}}_1}{x} - 2\widetilde{\mathrm{NL}}_2 - (1-u)\left[G_2\Psi + \widetilde{\mathrm{NL}}_2\right]\Psi' + (G_2\Psi + \widetilde{\mathrm{NL}}_2)\left(2-\frac{b}{x}\right)\Psi\right\}.\end{aligned}$$

We rewrite

$$(1-u)\left[G_2\Psi + \widetilde{\mathrm{NL}}_2\right]\Psi' = \frac{1-u}{u}\left[G_2\Psi + \widetilde{\mathrm{NL}}_2\right]u\Psi' = \left(\frac{b}{x}-1\right)\left[G_2\Psi + \widetilde{\mathrm{NL}}_2\right]u\Psi'$$

and thus $\mathrm{NL}(\Phi)$ is given, structurally, by

$$\mathrm{NL}(\Phi) = x\sum_{j=2}^{4} m_j^{(1)}\Phi^j + b\sum_{j=2}^{4} m_j^{(2)}\Phi^j + \left[x\sum_{j=1}^{3} m_j^{(3)}\Phi^j + b\sum_{j=1}^{3} m_j^{(4)}\Phi^j\right]u\Phi'$$

with

$$m_j^{(\ell)}(x) = \sum_{k=0}^{+\infty} m_{jk}^{(\ell)}x^k, \quad |m_{jk}^{(\ell)}| \le C^k.$$

Conclusion. We have obtained the conjugated nonlinear problem (6.22) with

$$\begin{aligned}\mathcal{G} &= \mathcal{G}_0 + b\left[(bh_1(x)+xh_2(x))\Phi + (bh_3(x)+xh_4(x))u\Phi'\right] \\ &+ x\sum_{j=2}^{4} m_j^{(1)}\Phi^j + b\sum_{j=2}^{4} m_j^{(2)}\Phi^j + \left[x\sum_{j=1}^{3} m_j^{(3)}\Phi^j + b\sum_{j=1}^{3} m_j^{(4)}\Phi^j\right] u\Phi'.\end{aligned} \tag{6.28}$$

step 3 Final change of variables. We now let

$$\Phi = bu\Theta = x\Theta \tag{6.29}$$

so that (6.22) becomes:

$$\begin{aligned}&\Phi' - \left[\frac{\gamma-1}{u} + \frac{\gamma+\nu_b+1}{1-u}\right]\Phi = -\frac{\mathcal{G}}{1-u} \\ \Leftrightarrow\ & bu\Theta' + b\Theta - \left[\frac{\gamma-1}{u} + \frac{\gamma+\nu_b+1}{1-u}\right] bu\Theta = -\frac{\mathcal{G}}{1-u} \\ \Leftrightarrow\ & \Theta' - \left[\frac{\gamma-2}{u} + \frac{\gamma+\nu_b+1}{1-u}\right]\Theta = -\frac{\mathcal{G}}{bu(1-u)} \\ \Leftrightarrow\ & u(1-u)\Theta' - [\gamma-2+(\nu_b+3)u]\,\Theta = -\frac{\mathcal{G}}{b}.\end{aligned} \tag{6.30}$$

We now express $\mathcal{G}$ in terms of Θ and track the orders of vanishing in x. We compute

$$u\Phi' = u[bu\Theta' + b\Theta] = x(u\Theta' + \Theta).$$

Then,

$$\begin{aligned}& b\left[(bh_1(x)+xh_2(x))\Phi + (bh_3(x)+xh_4(x))u\Phi'\right] \\ =\ & \left[b^2h_1 + bxh_2\right]x\Theta + (b^2h_3+bxh_4)[ux\Theta' + x\Theta] \\ =\ & \left[b^2x(h_1+h_3) + bx^2(h_2+h_4)\right]\Theta + \left[b^2xh_3 + x^2h_4\right]u\Theta' \\ =\ & \left[b^2x\tilde{h}_1 + bx^2\tilde{h}_2\right]\Theta + \left[b^2x\tilde{h}_3 + bx^2\tilde{h}_4\right]u\Theta'.\end{aligned}$$

For the nonlinear term:

$$\begin{aligned}& x\sum_{j=2}^{4} m_j^{(1)}\Phi^j + b\sum_{j=2}^{4} m_j^{(2)}\Phi^j + \left[x\sum_{j=1}^{3} m_j^{(3)}\Phi^j + b\sum_{j=1}^{3} m_j^{(4)}\Phi^j\right] u\Phi' \\ =\ & x\sum_{j=2}^{4} x^j m_j^{(1)}\Theta^j + b\sum_{j=2}^{4} m_j^{(2)}x^j\Theta^j + \left[x\sum_{j=1}^{3} m_j^{(3)}x^j\Theta^j + b\sum_{j=1}^{3} m_j^{(4)}x^j\Theta^j\right](xu\Theta' + x\Theta) \\ =\ & \left[\sum_{j=2}^{4} x^{j+1}m_j^{(1)}\Theta^j + \sum_{j=1}^{3} m_j^{(3)}x^{j+2}\Theta^{j+1} + b\sum_{j=2}^{4} m_j^{(2)}x^j\Theta^j + b\sum_{j=1}^{3} m_j^{(4)}x^{j+1}\Theta^{j+1}\right] \\ +\ & \left[\sum_{j=1}^{3} m_j^{(3)}x^{j+2}\Theta^j + b\sum_{j=1}^{3} m_j^{(4)}x^{j+1}\Theta^j\right](u\Theta') \\ =\ & \sum_{j=2}^{4} x^{j+1}\tilde{m}_j^{(1)}\Theta^j + b\sum_{j=2}^{4} m_j^{(2)}x^j\Theta^j + \left[\sum_{j=1}^{3}\tilde{m}_j^{(3)}x^{j+2}\Theta^j + b\sum_{j=1}^{3}\tilde{m}_j^{(4)}x^{j+1}\Theta^j\right](u\Theta').\end{aligned}$$

We now rewrite

$$\mathcal{G} = \mathcal{G}_0 + \left[b^2 x\tilde{h}_1 + bx^2\tilde{h}_2\right]\Theta + \left[b^2 x\tilde{h}_3 + bx^2\tilde{h}_4\right]u\Theta' \tag{6.31}$$
$$+ \sum_{j=2}^{4} x^{j+1}\tilde{m}_j^{(1)}\Theta^j + b\sum_{j=2}^{4} m_j^{(2)}x^j\Theta^j + \left[\sum_{j=1}^{3}\tilde{m}_j^{(3)}x^{j+2}\Theta^j + b\sum_{j=1}^{3}\tilde{m}_j^{(4)}x^{j+1}\Theta^j\right](u\Theta')$$

where, for some large enough universal constant $C = C_{\nu,a} > 0$ independent of $b < b^*$ and all $k \geq 0$,

$$|(\mathcal{G}_0)_k| + |(\tilde{h}_l)_k| + |(\tilde{m}_j^{(l)})_k| \leq C^k b^k \tag{6.32}$$

6.6. **Bounding the sequence θ_k and proof of Proposition 6.4.** We let[17]

$$\phi_k = \frac{1}{k!}\frac{d^k\Phi}{du^k}(0), \quad \theta_k = \frac{1}{k!}\frac{d^k\Theta}{du^k}(0), \quad g_k = \frac{1}{k!}\frac{d^k\mathcal{G}}{du^k}(0)$$

so that from (6.29):

$$\left|\begin{array}{l}\phi_0 = 0\\ \phi_k = b\theta_{k-1}, \quad k \geq 1.\end{array}\right. \tag{6.33}$$

Lemma 6.6 (Boundedness for the θ_k sequence)**.** *There exists $c_{\nu,a} > 0$ and $b^*(\nu, a)$ such that for all $0 < b < b^*$, for all $0 \leq k \leq K-1$,*

$$|\theta_k| \leq |w_{\gamma-1,\nu_b+1}(k)| \tag{6.34}$$

(This implies (6.15)*.) Moreover,*

$$\forall 0 \leq k \leq K, \quad |g_k| \leq c_{\nu,a}\frac{|w_{\gamma,\nu_b}(k)|}{1+k}. \tag{6.35}$$

Proof of Lemma 6.6. This is a direct consequence of the form of $\mathcal{G}$ given in (6.31) (6.32).

step 1 Small frequency universal bound.

Lemma 6.7 (Stability by multiplication)**.** *Let $h(u) = \sum_{k=0}^{+\infty} b^k h_k u^k$ with the holomorphic bound*

$$|h_k| \leq C^k.$$

Then there exists C_h and $0 < b^*(C_h) \ll 1$ such that for all $0 < b < b^*(C_h)$ and any $0 \leq k^* \leq K$,

$$\max_{0\leq k\leq k^*}\frac{|(h\phi)_k|}{w_{\gamma,\nu_b}(k)} \leq C_h \max_{0\leq k\leq k^*}\frac{|\phi_k|}{w_{\gamma,\nu_b}(k)}. \tag{6.36}$$

Proof of Lemma 6.7. First observe that (C.2) implies the lower bound

$$\forall 0 \leq k \leq K, \quad w_{\gamma,\nu_b}(k) = \frac{\Gamma(\gamma-1-k)}{\Gamma(\gamma-1)}\Gamma(\nu_b+k+2) \geq \frac{\Gamma(\nu_b+k+2)}{(\gamma-1)^k} \tag{6.37}$$

which then gives the upper bound[18]:

$$\frac{b^k C^k}{w_{\gamma,\nu_b}(k)} \leq \frac{(bC(\gamma-1))^k}{\Gamma(\nu_b+k+2)} \leq \frac{(2Ca)^k}{\Gamma(k+\nu_b+2)} \leq \frac{c(C)}{1+k^2}. \tag{6.38}$$

for all $0 \leq k \leq K$. Therefore,

$$b^k\frac{|h_k|}{|w_{\gamma,\nu_b}(k)|} \leq \frac{b^k C^k}{|w_{\gamma,\nu_b}(k)|} \leq \frac{C_h}{1+k^2}. \tag{6.39}$$

[17]Observe that the coefficients ϕ_k and θ_k appearing here should not be confused with the ones of section 5.4 for the formal limit problem.

[18]We use in particular $\nu_b > 0$ and the fact that Γ is increasing on $[2, +\infty)$.

for all $0 \le k \le K$. We then estimate using (C.5), for $k \le k^* \le K$:

$$\begin{aligned}\left|\frac{(h\phi)_k}{w_{\gamma,\nu_b}(k)}\right| &= \frac{1}{w_{\gamma,\nu_b}(k)}\left|\sum_{k_1+k_2=k} b^{k_1} h_{k_1}\phi_{k_2}\right| \\ &\le \sum_{k_1+k_2=k} b^{k_1}\frac{|h_{k_1}|}{w_{\gamma,\nu_b}(k_1)}\frac{|\phi_{k_2}|}{w_{\gamma,\nu_b}(k_2)}\frac{w_{\gamma,\nu_b}(k_1)w_{\gamma,\nu_b}(k_2)}{w_{\gamma,\nu_b}(k)} \\ &\le C_h \max_{0\le k<k^*}\frac{|\phi_k|}{w_{\gamma,\nu_b}(k)}\sum_{k_1+k_2=k}\frac{w_{\gamma,\nu_b}(k_1)w_{\gamma,\nu_b}(k_2)}{w_{\gamma,\nu_b}(k)} \le C_h \max_{0\le k<k^*}\frac{|\phi_k|}{w_{\gamma,\nu_b}(k)}\end{aligned}$$

and (6.36) is proved. □

We conclude from (6.1), (6.17) that there exists $c_{\nu,a}$ such that for all $k^* > 1$, there exists $0 < b^*(k^*) \ll 1$ such that for $0 < b < b^*(k^*)$ and $0 \le k \le k^*$,

$$|\phi_k| \le c_{\nu,a} w_{\gamma,\nu_b}(k). \tag{6.40}$$

Thus, from (6.33):

$$|\theta_k| = \frac{1}{b}|\phi_{k+1}| \le \frac{\gamma}{a} c_{\nu,a} w_{\gamma,\nu_b}(k+1).$$

Together with (6.13), we deduce the existence of $M_{\nu,a}$ such that for all $k^* \ge 1$, for all $0 < b < b^*(k^*)$,

$$\forall 0 \le k \le k^*, \quad \frac{|\theta_k|}{w_{\gamma-1,\nu_b+1}(k)} \le M_{\nu,a}. \tag{6.41}$$

step 2 Induction relation. We now compute from (6.30) the induction relation satisfied by derivatives at the origin. We formally expand

$$\Theta = \sum_{k=0}^{+\infty}\theta_k u^k, \quad \mathcal{G} = \sum_{k=0}^{+\infty} g_k u^k$$

and obtain from (6.30):

$$\begin{aligned} & u(1-u)\Theta' - [\gamma - 2 + (\nu_b+3)u]\,\Theta = -\frac{\mathcal{G}}{b} \\ \Leftrightarrow\ & \sum_{k=1}^{+\infty}(u-u^2)k\theta_k u^{k-1} - (\gamma-2)\sum_{k=0}^{+\infty}\theta_k u^k - (\nu_b+3)\sum_{k=0}^{+\infty}\theta_k u^{k+1} = -\frac{1}{b}\sum_{k=0}^{+\infty} g_k u^k \\ \Leftrightarrow\ & \sum_{k=1}^{+\infty} k\theta_k u^k - \sum_{k=2}^{+\infty}(k-1)\theta_{k-1}u^k - (\gamma-2)\sum_{k=0}^{+\infty}\theta_k u^k - (\nu_b+3)\sum_{k=1}^{+\infty}\theta_{k-1}u^k = -\frac{1}{b}\sum_{k=0}^{+\infty} g_k u^k \\ \Leftrightarrow\ & \left|\begin{array}{l} -(\gamma-2)\theta_0 = -\frac{g_0}{b} \\ (k-\gamma+2)\theta_k - (k+\nu_b+2)\theta_{k-1} = -\frac{g_k}{b}, \quad k \ge 1 \end{array}\right. \\ \Leftrightarrow\ & \left|\begin{array}{l} \theta_0 = \frac{\gamma}{a(\gamma-2)} g_0 \\ (\gamma-k-2)\theta_k + (k+\nu_b+2)\theta_{k-1} = \frac{g_k}{b}, \quad k \ge 1 \end{array}\right. \end{aligned} \tag{6.42}$$

Let

$$\zeta_k = \frac{\theta_k}{w_{\gamma-1,\nu_b+1}(k)},$$

then (6.14), (6.13) yield:

$$
\begin{aligned}
&(\gamma-k-2)w_{\gamma-1,\nu_b+1}(k)\zeta_k+(k+\nu_b+2)w_{\gamma-1,\nu_b+1}(k-1)\zeta_{k-1}=\frac{g_k}{b}\\
\Leftrightarrow\quad &(k+\nu_b+2)w_{\gamma-1,\nu_b+1}(k-1)(\zeta_k+\zeta_{k-1})=\frac{\gamma g_k}{a}\\
\Leftrightarrow\quad &\zeta_k+\zeta_{k-1}=\frac{\gamma}{a(\gamma-2)}\frac{g_k}{(k+\nu_b+2)w_{\gamma,\nu_b}(k)}.
\end{aligned}
\tag{6.43}
$$

step 3 Bootstrap bound. Let $M_{\nu,a}$ be the universal constant in (6.41), we now bootstrap the bound

$$
\forall k^*\le k\le K-1,\quad |\zeta_k|\le 2M_{\nu,a}
\tag{6.44}
$$

which, by (6.41), holds for $0\le k<k^*$ arbitrarily large and $0<b<b^*(k^*)$. We argue by induction, assuming the claim for $0\le j\le k_{\rm ind}-1$ and proving it for $k_{\rm ind}$ with $k_{\rm ind}\le K-1$. We claim the following crucial nonlinear bound for g_k

$$
\forall 0\le k\le k_{\rm ind},\quad \frac{|g_k|}{|w_{\gamma,\nu_b}(k)|}\le\frac{c_{\nu,a}M_{\nu,a}^4}{1+k}.
\tag{6.45}
$$

Source term. The estimates (6.38) and (6.32) yield the following uniform in $b,k\le K$:

$$
\frac{|(\mathcal{G}_0)_k|}{w_{\gamma,\nu_b}(k)}\le\frac{(bC)^k}{w_{\gamma,\nu_b}(k)}\le\frac{c_{\nu,a}}{1+k}.
$$

Small linear term. Let $1\le j\le 4$ and $j\le k\le k_{\rm ind}$, then from (6.41) (6.44):

$$
|\theta_{k-j}|\le M_{\nu,a}w_{\gamma-1,\nu_b+1}(k-j)
\tag{6.46}
$$

We now observe the relation for $0\le j\le 4$ from (6.13):

$$
\frac{b^j|w_{\gamma-1,\nu_b+1}(k-j)|}{|w_{\gamma,\nu_b}(k)|}=\frac{b^j(\gamma-2)|w_{\gamma,\nu_b}(k-j+1)|}{|w_{\gamma,\nu_b}(k)|}\le c_{\nu,a}\frac{b^{j-1}|w_{\gamma\nu_b}(k-(j-1))|}{|w_{\gamma,\nu_b}(k)|}
$$

and from (6.14) for $0\le j\le 4$:

$$
\begin{aligned}
w_{\gamma,\nu_b}(k) &= \frac{k+\nu_b+1}{\gamma-k-1}w_{\gamma,\nu_b}(k-1)=\frac{k-1+\nu_b+2}{\gamma-2-(k-1)}w_{\gamma,\nu_b}(k-1)\\
&= w_{\gamma,\nu_b}(k-j)\Pi_{m=1}^j\frac{k-m+\nu_b+2}{\gamma-2-(k-m)}
\end{aligned}
$$

which leads to the estimate:

$$
\frac{|w_{\gamma,\nu_b}(k-m)|}{|w_{\gamma,\nu_b}(k)|}\le\Pi_{j=1}^m\frac{\gamma-2-k+j}{|k-j+\nu_b+2|}\le c_\nu\frac{\gamma^m}{(1+k)^m}
\tag{6.47}
$$

and then

$$
\frac{b^j|w_{\gamma-1,\nu_b+1}(k-j)|}{|w_{\gamma,\nu_b}(k)|}\le c_{\nu,a}\frac{b^{j-1}|w_{\gamma,\nu_b}(k-(j-1))|}{|w_{\gamma,\nu_b}(k)|}\le c_{\nu,a}\frac{a^{j-1}}{(1+k)^{j-1}}.
\tag{6.48}
$$

Moreover, from (6.38):

$$
\frac{b^kC^k}{|w_{\gamma-1,\nu_b+1}(k)|}\le c_{\nu,a}.
\tag{6.49}
$$

We estimate the worst term using (6.32), (6.46), (6.48), (6.49), (C.5): for $k \le k_{\rm ind}$,

$$\begin{aligned}
&\left|\left(bx^2\tilde{h}_4 u\Theta'\right)_k\right| = bb^2\left|\left(u^2\tilde{h}_4 u\Theta'\right)_k\right| = bb^2\left|\left(\tilde{h}_4 u\Theta'\right)_{k-2}\right| = bb^2\left|\sum_{k_1+k_2=k-2}(\tilde{h}_4)_{k_1}(k_2\theta_{k_2})\right| \\
\le\ & bb^2 k c_{\nu,a}M_{\nu,a}\sum_{k_1+k_2=k-2}|w_{\gamma-1,\nu_b+1}(k_1)w_{\gamma-1,\nu_b+1}(k_2)| \le bkc_{\nu,a}M_{\nu,a}b^2 w_{\gamma-1,\nu_b+1}(k-2) \\
\le\ & \frac{c_{\nu,a}M_{\nu,a}|w_{\gamma,\nu_b}|}{1+k}
\end{aligned}$$

since $bk \le b\gamma = a$. We estimate similarly for $k \le k_{\rm ind}$:

$$\begin{aligned}
&\left|\left(b^2x\tilde{h}_3 u\Theta'\right)_k\right| = b^3\left|\left(\tilde{h}_3 u\Theta'\right)_{k-1}\right| \le b^3 k c_{\nu,a}M_{\nu,a}|w_{\gamma-1,\nu_b+1}(k-1)| \le b^2 kc_{\nu,a}M_{\nu,a}w_{\gamma,\nu_b}(k) \\
\le\ & \frac{c_{\nu,a}M_{\nu,a}|w_{\gamma,\nu_b}(k)|}{1+k}.
\end{aligned}$$

Remaining linear terms do not have the k loss of $(u\Phi')_k$ and are easier to estimate.

Nonlinear term. The worst nonlinear term is for $1 \le j \le 3$:

$$\left|\left(\tilde{m}_j^{(3)}x^{j+2}\Theta^j(u\Theta')\right)_k\right| = b^{j+2}\left|\left(u^{j+2}\tilde{m}_j^{(3)}\Theta^j(u\Theta')\right)_k\right| = b^{j+2}\left|\left(\tilde{m}_j^{(3)}\Theta^j(u\Theta')\right)_{k-(j+2)}\right|.$$

Then for all $\ell \le k_{\rm ind}-1$ from (6.32), (6.46), (6.49), (6.44) and (C.6):

$$\begin{aligned}
&\left|\left(\tilde{m}_j^{(3)}\Theta^j(u\Theta')\right)\right|_\ell = \left|\sum_{k_1+\cdots+k_{j+2}=\ell}\tilde{m}_{jk_1}^{(3)}\theta_{k_2}\dots\theta_{k_{j+1}}(k_{j+2}\theta_{k_{j+2}})\right| \\
\le\ & \ell c_{\nu,a}M_{\nu,a}^{j+1}\sum_{k_1+\cdots+k_{j+2}=\ell}\Pi_{i=1}^{j+2}|w_{\gamma-1,\nu_b+1}(k_i)| \le \ell c_{\nu,a}M_{\nu,a}^{j+1}|w_{\gamma-1,\nu_b+1}(\ell)|
\end{aligned}$$

Therefore, recalling (6.48), for $0 \le k \le k_{\rm ind}$:

$$\begin{aligned}
&\left|\left(\tilde{m}_j^{(3)}x^{j+2}\Theta^j(u\Theta')\right)_k\right| = b^{j+2}\left|\left(\tilde{m}_j^{(3)}\Theta^j(u\Theta')\right)_{k-(j+2)}\right| \\
\le\ & kc_{\nu,a}M_{\nu,a}^{j+1}b^{j+2}|w_{\gamma-1,\nu_b+1}(k-(j+2))| \le c_{\nu,a}M_{\nu,a}^{j+1}\frac{k}{(1+k)^{j+2-1}}|w_{\gamma,\nu_b}(k)| \\
\le\ & c_{\nu,a}M_{\nu,a}^{j+1}\frac{|w_{\gamma,\nu_b}(k)|}{(1+k)^j} \le \frac{c_{\nu,a}M_{\nu,a}^4}{1+k}|w_{\gamma,\nu_b}(k)|
\end{aligned}$$

since $j \ge 1$. Similarly, for $1 \le j \le 3$:

$$\begin{aligned}
&\left|\left(b\tilde{m}_j^{(4)}x^{j+1}\Theta^j(u\Theta')\right)_k\right| = bb^{j+1}\left|\left(\tilde{m}_j^{(4)}\Theta^j(u\Theta')\right)_{k-j-1}\right| \\
\le\ & bkc_{\nu,a}M_{\nu,a}^{j+1}b^{j+1}|w_{\gamma-1,\nu_b+1}(k-j-1)| \\
\le\ & bkc_{\nu,a}M_{\nu,a}^{j+1}\frac{|w_{\gamma,\nu_b}(k)|}{(1+k)^j} \le \frac{c_{\nu,a}M_{\nu,a}^4}{1+k}|w_{\gamma,\nu_b}(k)|
\end{aligned}$$

since $bk \le b\gamma = a$ and $j \ge 1$. The two remaining nonlinear terms in (6.31) do not have the k loss of $(u\Theta')_k$ and are thus better by a factor of $\frac{1}{1+k}$. The collection of above bounds concludes the proof of (6.45).

step 4 Closing (6.44). From (6.45), (6.43), we have for $k \leq k_{\rm ind}$:

$$|\zeta_k| < |\zeta_{k-1}| + \frac{c_{\nu,a} M_{\nu,a}^4}{1+k^2}.$$

We sum over $k \in \{k^*, k_{\rm ind}\}$ and conclude from (6.41):

$$|\zeta_{k_{\rm ind}}| \leq |\zeta_{k^*}| + c_{\nu,a} M_{\nu,a}^4 \sum_{k=k^*}^{k_{\rm ind}} \frac{1}{k^2} \leq M_{\nu,a} + \frac{c_{\nu,a} M_{\nu,a}^4}{k^*} < 2M_{\nu,a}$$

provided $k^* > k^*(M_{\nu,a})$ has been chosen large enough. (6.44) is proved. This also concludes the proof of (6.35) and of Lemma 6.6. □

The proof of Proposition 6.4 follows immediately from (6.33), (6.34), (6.13), (6.1), and Lemma 6.7.

7. Quantitative study of the $\mathcal{C}^\infty$ solution

We now turn to the qualitative of the $\mathcal{C}^\infty$ solution of (4.15). Understanding of the Taylor expansion at $u = 0$ is not sufficient to analyze the solution away from $u = 0$. Our main goal is to show that truncating the Taylor series at $k = K$ yields the dominant terms in the solution which, together with a remainder, can be computed and estimated thanks to an explicit integral representation.

We study the $\mathcal{C}^\infty$ solution. We define the operator

$$\mathcal{T}(\mathcal{G}) = \frac{u^{\gamma-2}}{(1-u)^{\gamma+\nu_b+1}} \int_0^u \frac{(1-v)^{\gamma+\nu_b}}{v^{\gamma-1}} \frac{\mathcal{G}}{b} dv. \tag{7.1}$$

Remark 7.1. Recall from Lemma 3.6 that there exists a unique solution curve which is $\mathcal{C}^\infty$ at P_2, i.e at $u = 0$. Recall also from Remark 3.7 that all other solutions have only finite regularity at $u = 0$. The operator $\mathcal{T}$ in (7.1) will be used in the fixed point formulation (7.20) to identify the unique $\mathcal{C}^\infty$ solution at $u = 0$. Note that the non smooth solutions exhibit a $u^{\gamma-2}$ singularity at $u = 0$.

7.1. **Remainder function.** We introduce several special functions defined via the integral operator (7.1). These will be fundamental in understanding the leading order terms which appear when the Taylor expansion no longer dominates.

Lemma 7.2 (Definition and properties of the first remainder function)**.** *Let*

$$M_0(u) \quad = \quad (K + \nu_b + 2) w_{\gamma-1,\nu_b+1} (K-1) \mathcal{T}(b u^K). \tag{7.2}$$

Then

$$M_0(u) \quad = \quad (1 + o_{K\to+\infty}(1)) \Gamma(\alpha_\gamma) K^{\nu_b+4-\alpha_\gamma} \mathcal{T}(b u^K). \tag{7.3}$$

Moreover, there exist universal constants $0 < c_{\nu,1} < c_{\nu,2}$ such that for all $0 < b < b^*(\nu)$, the following holds:

behavior for small u: for $0 \leq u \leq b$,

$$c_{\nu,1} \leq \frac{M_0(u)}{\Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma) K^{\nu_b+4-\alpha_\gamma} u^K} \leq c_{\nu,2} \tag{7.4}$$

behavior for large u: for $b \leq u < \frac{1}{2}$:

$$c_{\nu,1} \leq \frac{M_0(u)}{\Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma) K^{\nu_b+3} \left(\frac{u}{1-u}\right)^{K-1} u^{\alpha_\gamma}} \leq c_{\nu,2}. \tag{7.5}$$

control of the iterate: let $1 \leq j \leq 5$, then

$$\left\| \frac{\mathcal{T}(u^j M_0)}{M_0} \right\|_{L^\infty(u \leq \frac{1}{2})} \leq \frac{c_\nu}{b}. \tag{7.6}$$

control of the derivative:

$$\forall 0 \leq u \leq \frac{1}{2}, \quad \frac{|uM_0'|}{M_0} \leq \frac{c_\nu}{b}. \tag{7.7}$$

Proof of Lemma 7.2. This follows from the explicit integral representation (7.1).

step 1 Proof of (7.3). By definition:

$$\begin{aligned} M_0(u) &= (K+\nu_b+2) w_{\gamma-1,\nu_b+1}(K-1)\mathcal{T}(bu^K) \\ &= (K+\nu_b+2)\frac{\Gamma(\alpha_\gamma)\Gamma(K+\nu_b+2)}{\Gamma(\gamma-2)}\mathcal{T}(bu^K) \\ &= \frac{\Gamma(\alpha_\gamma)\Gamma(K+\nu_b+3)}{\Gamma(K-1+\alpha_\gamma)}\mathcal{T}(bu^K) \\ &= (1+o_{K\to+\infty}(1))\Gamma(\alpha_\gamma)K^{\nu_b+4-\alpha_\gamma}\mathcal{T}(bu^K) \end{aligned}$$

where we used (A.8) in the last step.

step 2 Estimate for $0 \leq u \leq b$. For $u \leq b$,

$$(1-u)^K = e^{K\log(1-u)} = e^{-Ku+O(Ku^2)} = e^{O(1)}.$$

Then, from (7.3):

$$\begin{aligned} & M_0(u) = (1+o_{K\to+\infty}(1))\Gamma(\alpha_\gamma)K^{\nu_b+4-\alpha_\gamma}\frac{u^{\gamma-2}}{(1-u)^{\gamma+\nu_b+1}}\int_0^u \frac{(1-v)^{\gamma+\nu_b}}{v^{\gamma-1}}v^K dv \\ = \;& e^{O(1)}\Gamma(\alpha_\gamma)K^{\nu_b+4-\alpha_\gamma}u^{K-1+\alpha_\gamma}\int_0^u \frac{dv}{v^{\alpha_\gamma}} = e^{O_\nu(1)}\frac{\Gamma(\alpha_\gamma)}{1-\alpha_\gamma}K^{\nu_b+4-\alpha_\gamma}u^{K-1+\alpha_\gamma}u^{1-\alpha_\gamma} \\ = \;& e^{O(1)}\Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)K^{\nu_b+4-\alpha_\gamma}u^K \end{aligned}$$

where we used the fact that[19]

$$\frac{1}{2} \leq x\Gamma(x) \leq 1 \text{ for } 0 < x \leq 1,$$

and (7.4) is proved.

[19]Indeed, we have $x\Gamma(x) = \Gamma(x+1)$ and the well known bound $0.88 \leq \Gamma(x) \leq 1$ on $1 \leq x \leq 2$.

step 3 Estimate for $b \le u \le \frac{1}{2}$. First, from (7.3), (A.1), (A.8), we have the global bound

$$\begin{aligned} M_0(u) &\le c_\nu \Gamma(\alpha_\gamma) K^{\nu_b+4-\alpha_\gamma} \frac{u^{\gamma-2}}{(1-u)^{\gamma+\nu_b+1}} \int_0^1 \frac{(1-v)^{\gamma+\nu_b}}{v^{\gamma-1}} v^K dv \\ &\le c_\nu \Gamma(\alpha_\gamma) K^{\nu_b+4-\alpha_\gamma} \frac{u^{\gamma-2}}{(1-u)^{\gamma+\nu_b+1}} \int_0^1 (1-v)^{K+1+\alpha_\gamma+\nu_b} v^{-\alpha_\gamma} dv \\ &\le c_\nu \Gamma(\alpha_\gamma) K^{\nu_b+4-\alpha_\gamma} \frac{u^{K-1+\alpha_\gamma}}{(1-u)^{K+1+\alpha_\gamma+\nu_b+1}} B(1-\alpha_\gamma, K+\alpha_\gamma+\nu_b+2) \\ &\le c_\nu \Gamma(\alpha_\gamma) K^{\nu_b+4-\alpha_\gamma} \frac{u^{K-1+\alpha_\gamma}}{(1-u)^{K+1+\alpha_\gamma+\nu_b+1}} \frac{\Gamma(1-\alpha_\gamma)\Gamma(K+\nu_b+\alpha_\gamma+2)}{\Gamma(K+\nu_b+3)} \\ &\le c_\nu \Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma) K^{\nu_b+4-\alpha_\gamma} \left(\frac{u}{1-u}\right)^{K-1} \frac{u^{\alpha_\gamma}}{K^{1-\alpha_\gamma}} \\ &\le c_\nu \Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma) K^{\nu_b+3} \left(\frac{u}{1-u}\right)^{K-1} u^{\alpha_\gamma}. \end{aligned}$$

This gives the upper bound in (7.5). For the lower bound, we use:

$$\begin{aligned} &\int_0^u (1-v)^{K+1+\alpha_\gamma+\nu_b} v^{-\alpha_\gamma} dv \ge \int_0^b (1-v)^{K+1+\alpha_\gamma+\nu_b} v^{-\alpha_\gamma} dv \\ = &\int_0^b e^{\left[-(K+1+\alpha_\gamma+\nu_b)v+O(Kv^2)\right]} \frac{dv}{v^{\alpha_\gamma}} \ge c_\nu \int_0^b \frac{dv}{v^{\alpha_\gamma}} \ge \frac{c_\nu b^{1-\alpha_\gamma}}{1-\alpha_\gamma} \ge \frac{c_\nu \Gamma(1-\alpha_\gamma)}{K^{1-\alpha_\gamma}} \end{aligned}$$

Then, (7.3) gives

$$\begin{aligned} M_0(u) &\ge c_\nu \Gamma(\alpha_\gamma) K^{\nu_b+4-\alpha_\gamma} \frac{u^{K-1+\alpha_\gamma}}{(1-u)^{K+1+\alpha_\gamma+\nu_b+1}} \frac{\Gamma(1-\alpha_\gamma)}{K^{1-\alpha_\gamma}} \\ &\ge c_\nu \Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma) K^{\nu_b+3} \left(\frac{u}{1-u}\right)^{K-1} u^{\alpha_\gamma} \end{aligned}$$

and (7.5) is proved.

step 4 Control of the iterate. For $u \le b$, and since $j \ge 1$, we estimate from (7.4):

$$\begin{aligned} \frac{\mathcal{T}(u^j M_0)}{M_0} &\le \frac{c_\nu}{u^K} \frac{u^{\gamma-2}}{(1-u)^{\gamma+\nu_b+1}} \int_0^u \frac{(1-v)^{\gamma+\nu_b}}{v^{\gamma-1}} \frac{v^{K+j}}{b} dv \le \frac{c_\nu}{u^{1-\alpha_\gamma}} \int_0^u \frac{v^{j-\alpha_\gamma} dv}{b} \\ &\le \frac{c_\nu u^j}{b(j+1-\alpha_\gamma)} \le \frac{c_\nu b^j}{b(2-\alpha_\gamma)} \le c_\nu \end{aligned}$$

For $b \le u \le \frac{1}{2}$, we estimate from (7.4):

$$\begin{aligned} &\int_0^b \frac{(1-v)^{\gamma+\nu_b}}{v^{\gamma-1}} v^j M_0(v) dv \le \int_0^b \Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma) K^{\nu_b+4-\alpha_\gamma} \frac{v^{K+j}}{v^{\gamma-1}} dv \\ \le\ &\Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma) K^{\nu_b+4-\alpha_\gamma} \frac{b^{j+1-\alpha_\gamma}}{j+1-\alpha_\gamma} \le b^j \Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma) K^{\nu_b+3} \end{aligned}$$

and from (7.5)

$$\begin{aligned} &\int_b^u \frac{(1-v)^{\gamma+\nu_b}}{v^{\gamma-1}} v^j M_0(v) dv \le c_\nu \int_b^u \frac{(1-v)^{\gamma+\nu_b}}{v^{\gamma-1}} \Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma) K^{\nu_b+3} \left(\frac{v}{1-v}\right)^{K-1} v^{\alpha_\gamma} v^j dv \\ \le\ &c_\nu \Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma) K^{\nu_b+3} \int_b^u v^{j-1} dv \le c_\nu \Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma) K^{\nu_b+3}. \end{aligned}$$

Thus,

$$\begin{aligned}\frac{\mathcal{T}(u^j M_0)}{M_0} &\leq \frac{c_\nu}{\Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)K^{\nu_b+3}\left(\frac{u}{1-u}\right)^{K-1}u^{\alpha_\gamma}} \times \frac{u^{\gamma-2}}{(1-u)^{\gamma+\nu_b+1}}\frac{1}{b}\Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)K^{\nu_b+3} \\ &\leq \frac{c_\nu}{b}\end{aligned}$$

and (7.6) is proved.

step 5 Control of the derivative. From (7.1):

$$u\left[\mathcal{T}(\mathcal{G})\right]' = \frac{1}{1-u}\left[\frac{\mathcal{G}}{b} + [(\gamma-2)+(\nu_b+3)u]\mathcal{T}(\mathcal{G})\right] \tag{7.8}$$

which yields, recalling (7.3):

$$\begin{aligned}uM_0' &= (1+o_{K\to+\infty}(1))\Gamma(\alpha_\gamma)K^{\nu_b+4-\alpha_\gamma}u\left[\mathcal{T}(bu^K)\right]' \\ &= \frac{(1+o_{K\to+\infty}(1))\Gamma(\alpha_\gamma)K^{\nu_b+4-\alpha_\gamma}}{1-u}\left[\frac{bu^K}{b} + [(\gamma-2)+(\nu_b+3)u]\mathcal{T}(bu^k)\right] \\ &= \frac{(1+o_{K\to+\infty}(1))\Gamma(\alpha_\gamma)K^{\nu_b+4-\alpha_\gamma}}{1-u}u^K + \frac{[(\gamma-2)+(\nu_b+3)u]M_0}{1-u}. \end{aligned} \tag{7.9}$$

From (7.4), (7.9), we estimate for $u \leq b$:

$$\frac{|uM_0'|}{M_0} \leq c_\nu\frac{\Gamma(\alpha_\gamma)K^{\nu_b+4-\alpha_\gamma}u^K}{\Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)K^{\nu_b+4-\alpha_\gamma}u^K} + \frac{c_\nu}{b} \leq \frac{c_\nu}{b}$$

and for $b \leq u \leq \frac{1}{2}$ from (7.5), (7.9):

$$\frac{|uM_0'|}{M_0} \leq \frac{\Gamma(\alpha_\gamma)K^{\nu_b+4-\alpha_\gamma}u^K}{\Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)K^{\nu_b+3}\left(\frac{u}{1-u}\right)^{K-1}u^{\alpha_\gamma}} + \frac{c_\nu}{b} \leq \frac{c_\nu}{b}$$

and (7.7) is proved. □

We now establish additional estimates for the remainder functions.

Lemma 7.3 (Holomorphic representation and bounds)**.** *Let $j \geq 0$ and*

$$M_j(u) = (K+j+\nu_b+2)(-1)^j w_{\gamma-1,\nu_b+1}(K-1+j)\mathcal{T}(bu^{K+j}), \tag{7.10}$$

then we have the following convergent series representation for $|u|<1$:

$$M_j(u) = \sum_{m=j+1}^{+\infty}(-1)^m w_{\gamma-1,\nu_b+1}(K-1+m)u^{K-1+m}. \tag{7.11}$$

Moreover, let $1 \leq j \leq 5$, then there exist universal constants $0 < c_{\nu,1} < c_{\nu,2}$ such that:
behavior for small u: for $0 \leq u \leq b$,

$$c_{\nu,1} \leq \frac{M_j(u)}{\Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)K^{\nu_b+j+4-\alpha_\gamma}u^{K+j}} \leq c_{\nu,2} \tag{7.12}$$

behavior for large u: for $b \leq u < \frac{1}{2}$:

$$c_{\nu,1} \leq \frac{M_j(u)}{\Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)K^{\nu_b+3}\left(\frac{u}{1-u}\right)^{K-1}u^{\alpha_\gamma}} \leq c_{\nu,2}. \tag{7.13}$$

Proof. **step 1** Holomorphic representation. Given a $\mathcal{C}^\infty$ function $\mathcal{G}$ with $\mathcal{G} = O_{u\to 0}(u^K)$, $\Theta = \mathcal{T}(\mathcal{G})$ satisfies the linear equation

$$u(1-u)\Theta' - [\gamma - 2 + (\nu_b + 3)u]\,\Theta = \frac{\mathcal{G}}{b} \tag{7.14}$$

which can be formally solved via a series representation:

$$\left| \begin{array}{l} \mathcal{G} = \sum_{k=0}^{+\infty} g_k u^k \\ \Theta = \sum_{k=0}^{+\infty} \theta_k u^k \end{array} \right.$$

i.e.,

$$\begin{aligned} & u(1-u)\left(\sum_{k=0}^{\infty}\theta_k u^k\right)' - (\gamma - 2 + (\nu_b+3)u)\sum_{k=0}^{+\infty}\theta_k u^k \\ = & \sum_{k=1}^{+\infty}(k\theta_k u^k - k\theta_k u^{k+1}) - \sum_{k=0}^{+\infty}(\gamma-2)\theta_k u^k - (\nu_b+3)\sum_{k=0}^{+\infty}\theta_k u^{k+1} \\ = & -(\gamma-2)\theta_0 + \sum_{k=1}^{+\infty}(k-\gamma+2)\theta_k u^k - \sum_{k=1}^{+\infty}(k+\nu_b+2)\theta_{k-1}u^k \\ = & -(\gamma-2)\theta_0 + \sum_{k=1}^{+\infty}\left[(k-\gamma+2)\theta_k - (k+\nu_b+2)\theta_{k-1}\right]u^k \end{aligned}$$

with the induction relation

$$\left| \begin{array}{l} \theta_0 = -\frac{g_0}{b(\gamma-2)} \\ (\gamma-k-2)\theta_k + (k+\nu_b+2)\theta_{k-1} = -\frac{g_k}{b}, \quad k \geq 1. \end{array} \right.$$

Let

$$\zeta_k = \frac{\theta_k}{w_{\gamma-1,\nu_b+1}(k)},$$

then as in (6.43):

$$\zeta_k + \zeta_{k-1} = -\frac{\gamma}{a(\gamma-2)}\frac{g_k}{(k+\nu_b+2)w_{\gamma,\nu_b}(k)}, \quad k \geq 1$$

which yields

$$\zeta_k = -\frac{\gamma}{a(\gamma-2)}(-1)^k\sum_{j=0}^{k}\frac{(-1)^j g_j}{(j+\nu_b+2)w_{\gamma,\nu_b}(j)}, \quad k \geq 1$$

and thus

$$\left| \begin{array}{l} \theta_k = (-1)^k w_{\gamma-1,\nu_b+1}(k)S_k, \quad k \geq 0 \\ S_k = -\frac{\gamma}{a(\gamma-2)}\sum_{j=0}^{k}\frac{(-1)^j g_j}{(j+\nu_b+2)w_{\gamma,\nu_b}(j)}. \end{array} \right. \tag{7.15}$$

Given $j \geq 0$ and

$$\mathcal{G} = u^{K+j}$$

this yields:

$$\theta_k = \left| \begin{array}{l} 0 \quad \text{for} \quad k \leq K+j-1 \\ (-1)^k w_{\gamma-1,\nu_b+1}(k)S_{K+j} \quad \text{for} \quad k \geq K+j \end{array} \right.$$

Therefore, the representation is a normally convergent series[20] for $|u|<1$. Using (6.13):

$$\begin{aligned}\mathcal{T}(u^{K+j}) &= S_{K+j}\sum_{k=K+j}^{+\infty}(-1)^k w_{\gamma-1,\nu_b+1}(k)u^k\\ &= -\frac{\gamma}{a(\gamma-2)}\frac{(-1)^{K+j}}{(K+j+\nu_b+2)w_{\gamma,\nu_b}(K+j)}\sum_{k=K+j}^{+\infty}(-1)^k w_{\gamma-1,\nu_b+1}(k)u^k\\ &= \frac{1}{b}\frac{(-1)^{K-1+j}}{(K+j+\nu_b+2)w_{\gamma-1,\nu_b+1}(K+j-1)}\sum_{k=K+j}^{+\infty}(-1)^k w_{\gamma-1,\nu_b+1}(k)u^k\end{aligned}$$

gives

$$M_j(u)=(-1)^{K-1}\sum_{k=K+j}^{+\infty}(-1)^k w_{\gamma-1,\nu_b+1}(k)u^k=\sum_{m=j+1}^{+\infty}(-1)^m w_{\gamma-1,\nu_b+1}(K-1+m)u^{K-1+m}$$

and (7.11) is proved. We now assume $j\geq 1$.

step 2 Estimate for $0\leq u\leq b$. From (6.12) (6.13) and (A.8), we have

$$\begin{aligned}&(-1)^j w_{\gamma-1,\nu_b+1}(K-1+j)=\Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)\frac{\Gamma(K+j+\nu_b+2)}{\Gamma(K-1+\alpha_\gamma)\Gamma(j+1-\alpha_\gamma)}\\ =\ &\frac{\Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)}{\Gamma(j+1-\alpha_\gamma)}\left[1+o_{b\to 0}(1)\right]K^{\nu_b+j+3-\alpha_\gamma}\\ =\ &e^{O(1)}\Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)K^{\nu_b+j+3-\alpha_\gamma}\end{aligned}\tag{7.16}$$

where we used $j\leq 5\ll\gamma$ in the second step and $j\geq 1$ in the last step.

For $u\leq b$,

$$(1-u)^K=e^{K\log(1-u)}=e^{-Ku+O(Ku^2)}=e^{O(1)}$$

and therefore:

$$\begin{aligned}M_j(u) &= e^{O(1)}(K+j+\nu_b+2)\Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)K^{\nu_b+j+3-\alpha_\gamma}\\ &\times\ \frac{u^{\gamma-2}}{(1-u)^{\gamma+\nu_b+1}}\int_0^u\frac{(1-v)^{\gamma+\nu_b}}{v^{\gamma-1}}v^{K+j}dv\\ &= e^{O(1)}\Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)K^{\nu_b+j+4-\alpha_\gamma}u^{K-1+\alpha_\gamma}\int_0^u\frac{v^j dv}{v^{\alpha_\gamma}}\\ &= e^{O(1)}\Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)K^{\nu_b+j+4-\alpha_\gamma}u^{K+j}.\end{aligned}$$

[20]From (6.12) (6.13) and (A.8), we have

$$w_{\gamma-1,\nu_b+1}(k)=O(k^{K+\nu_b+3+\alpha_\gamma})\text{ as }k\to+\infty$$

which implies that the series converges for $|u|<1$.

step 3 Estimate for $b \le u \le \frac{1}{2}$. We have the global bound from (7.16), (A.1), (A.8):

$$\begin{aligned} M_j(u) &\le c_\nu \Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma) K^{\nu_b+j+4-\alpha_\gamma} \frac{u^{\gamma-2}}{(1-u)^{\gamma+\nu_b+1}} \int_0^1 \frac{(1-v)^{\gamma+\nu_b}}{v^{\gamma-1}} v^{K+j} dv \\ &\le c_\nu \Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma) K^{\nu_b+j+4-\alpha_\gamma} \frac{u^{\gamma-2}}{(1-u)^{\gamma+\nu_b+1}} \int_0^1 (1-v)^{K+1+\alpha_\gamma+\nu_b} v^{j-\alpha_\gamma} dv \\ &\le c_\nu \Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma) K^{\nu_b+j+4-\alpha_\gamma} \frac{u^{K-1+\alpha_\gamma}}{(1-u)^{K+1+\alpha_\gamma+\nu_b+1}} B(j+1-\alpha_\gamma, K+\alpha_\gamma+\nu_b+2) \\ &\le c_\nu \Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma) K^{\nu_b+j+4-\alpha_\gamma} \frac{u^{K-1+\alpha_\gamma}}{(1-u)^{K+1+\alpha_\gamma+\nu_b+1}} \frac{\Gamma(j+1-\alpha_\gamma)\Gamma(K+\nu_b+\alpha_\gamma+2)}{\Gamma(K+j+\alpha_\gamma+\nu_b+3-\alpha_\gamma)} \\ &\le c_\nu \Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma) K^{\nu_b+j+4-\alpha_\gamma} \left(\frac{u}{1-u}\right)^{K-1} \frac{u^{\alpha_\gamma}}{K^{j+1-\alpha_\gamma}} \\ &\le c_\nu \Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma) K^{\nu_b+3} \left(\frac{u}{1-u}\right)^{K-1} u^{\alpha_\gamma} \end{aligned}$$

which yields the upper bound in (7.12). For the lower bound, we use:

$$\begin{aligned} &\int_0^u (1-v)^{K+1+\alpha_\gamma+\nu_b} v^{j-\alpha_\gamma} dv \ge \int_0^b (1-v)^{K+1+\alpha_\gamma+\nu_b} v^{j-\alpha_\gamma} dv \\ = &\int_0^b e^{\left[-(K+1+\alpha_\gamma+\nu_b)v+O(Kv^2)\right]} \frac{v^j dv}{v^{\alpha_\gamma}} \ge c_\nu \int_0^b \frac{v^j dv}{v^{\alpha_\gamma}} \ge c_\nu b^{j+1-\alpha_\gamma} \end{aligned}$$

(7.16) then gives the lower bound:

$$\begin{aligned} M_j(u) &\ge c_\nu \Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma) K^{\nu_b+j+4-\alpha_\gamma} \frac{u^{K-1+\alpha_\gamma}}{(1-u)^{K+1+\alpha_\gamma+\nu_b+1}} b^{j+1-\alpha_\gamma} \\ &\ge c_\nu \Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma) K^{\nu_b+3} \left(\frac{u}{1-u}\right)^{K-1} u^{\alpha_\gamma} \end{aligned}$$

and (7.12) is proved. □

7.2. **Fixed point formulation of the $\mathcal{C}^\infty$ solution.** For a given function F with sufficient regularity at the origin, we denote

$$r_F(u) = F(u) - \sum_{k=0}^{K-1} f_k u^k, \quad f_k = \frac{f^{(k)}(0)}{k!}. \tag{7.17}$$

Lemma 7.4 (Fixed point formulation for the $\mathcal{C}^\infty$ solution)**.** *Let $\mathcal{G}$ be given by* (6.31) *and consider the decomposition*

$$\left| \begin{array}{l} \mathcal{G} = \sum_{k=0}^{K-1} g_k u^k + r_{\mathcal{G}}, \\ \Theta = \sum_{k=0}^{K-1} \theta_k u^k + r_\Theta, \end{array} \right. \tag{7.18}$$

where

$$\left| \begin{array}{l} \theta_k = (-1)^k w_{\gamma-1,\nu_b+1}(k) S_k, \quad 0 \le k \le K-1, \\ S_k = \frac{\gamma}{a(\gamma-2)} \sum_{j=0}^k \frac{(-1)^j g_j}{(j+\nu_b+2) w_{\gamma,\nu_b}(j)}, \end{array} \right. \tag{7.19}$$

then the unique solution to the fixed point problem[21]

$$r_\Theta = (-1)^{K-1} S_{K-1} M_0(u) - \mathcal{T}(r_{\mathcal{G}}). \tag{7.20}$$

[21]The fixed point procedure based on (7.20) will be performed in Lemma 7.8, see (7.49) for the space on r_Θ which will be used.

generates the unique solution Θ *to* (6.30) *which satisfies:*

$$\forall\, 0 \le k \le K-1, \quad \lim_{u\downarrow 0} \frac{\Theta^{(k)}(u)}{k!} = \theta_k \tag{7.21}$$

where $(\theta_k)_{k\ge 0}$ *is computed by induction from* (6.42).

Proof of Lemma 7.4. Recall (6.30):

$$u(1-u)\Theta' - [\gamma - 2 + (\nu_b+3)u]\,\Theta = -\frac{\mathcal{G}}{b}.$$

We let

$$\left| \begin{array}{l} \mathcal{G} = \sum_{k=0}^{K-1} g_k u^k + r_{\mathcal{G}} \\ \Theta = \sum_{k=1}^{K-1} \theta_k u^k + r_\Theta \end{array} \right.$$

step 1 Polynomial cancellations. We compute:

$$\begin{aligned} & u(1-u)\left(\sum_{k=0}^{K-1}\theta_k u^k\right)' - (\gamma - 2 + (\nu_b+3)u)\sum_{k=0}^{K-1}\theta_k u^k \\ = & \sum_{k=1}^{K-1}(k\theta_k u^k - k\theta_k u^{k+1}) - \sum_{k=0}^{K-1}(\gamma-2)\theta_k u^k - (\nu_b+3)\sum_{k=0}^{K-1}\theta_k u^{k+1} \\ = & -(\gamma-2)\theta_0 + \sum_{k=1}^{K-1}(k-\gamma+2)\theta_k u^k - \sum_{k=1}^{K}(k+\nu_b+2)\theta_{k-1}u^k \\ = & -(\gamma-2)\theta_0 + \sum_{k=1}^{K-1}\left[(k-\gamma+2)\theta_k - (k+\nu_b+2)\theta_{k-1}\right]u^k - (K+\nu_b+2)\theta_{K-1}u^K \end{aligned}$$

Therefore,

$$\begin{aligned} & u(1-u)\left(\sum_{k=0}^{K-1}\theta_k u^k\right)' - (\gamma - 2 + (\nu_b+3)u)\sum_{k=0}^{K-1}\theta_k u^k + \sum_{k=0}^{K-1}\frac{g_k}{b}u^k \\ = & \sum_{k=1}^{K-1}\left[(k-\gamma+2)\theta_k - (k+\nu_b+2)\theta_{k-1} + \frac{g_k}{b}\right]u^k \\ - & (\gamma-2)\theta_0 + \frac{g_0}{b} - (K+\nu_b+2)\theta_{K-1}u^K \end{aligned}$$

which yields the induction relation

$$\left| \begin{array}{l} \theta_0 = \frac{g_0}{b(\gamma-2)} \\ (\gamma-k-2)\theta_k + (k+\nu_b+2)\theta_{k-1} = \frac{g_k}{b}, \quad 1 \le k \le K-1. \end{array} \right.$$

Let

$$\zeta_k = \frac{\theta_k}{w_{\gamma-1,\nu_b+1}(k)},$$

then equivalently from (6.43):

$$\zeta_k + \zeta_{k-1} = \frac{\gamma}{a(\gamma-2)}\frac{g_k}{(k+\nu_b+2)w_{\gamma,\nu_b}(k)}, \quad 1 \le k \le K-1$$

which yields

$$\zeta_k = \frac{\gamma}{a(\gamma-2)}(-1)^k \sum_{j=0}^{k}\frac{(-1)^j g_j}{(j+\nu_b+2)w_{\gamma,\nu_b}(j)}, \quad 0 \le k \le K-1$$

and (7.19) follows.

step 2 Equation for the remainder. We conclude:

$$\begin{aligned}
& u(1-u)\Theta' - [\gamma - 2 + (\nu_b+3)u]\,\Theta = -\frac{\mathcal{G}}{b} \\
\Leftrightarrow\quad & -(K+\nu_b+2)\theta_{K-1}u^K + u(1-u)r'_\Theta - [\gamma-2+(\nu_b+3)u]\,r_\Theta = -\frac{r_\mathcal{G}}{b} \\
\Leftrightarrow\quad & r'_\Theta - \frac{\gamma-2+(\nu_b+3)u}{u(1-u)}r_\Theta = \frac{1}{u(1-u)}\left[(K+\nu_b+2)\theta_{K-1}u^K - \frac{r_\mathcal{G}}{b}\right] \\
\Leftrightarrow\quad & r'_\Theta - \left[\frac{\gamma-2}{u} + \frac{\gamma+\nu_b+1}{1-u}\right] r_\Theta = \frac{1}{u(1-u)}\left[(K+\nu_b+2)\theta_{K-1}u^K - \frac{r_\mathcal{G}}{b}\right] \\
\Leftrightarrow\quad & \frac{d}{du}\left(\frac{(1-u)^{\gamma+\nu_b+1}}{u^{\gamma-2}}r_\Theta\right)' = \frac{(1-u)^{\gamma+\nu_b+1}}{u^{\gamma-2}}\frac{1}{u(1-u)}\left[(K+\nu_b+2)\theta_{K-1}u^K - \frac{r_\mathcal{G}}{b}\right]
\end{aligned}$$

and thus, any $\mathcal{C}^\infty$ solution must be the unique solution to the fixed point equation:

$$r_\Theta = \frac{u^{\gamma-2}}{(1-u)^{\gamma+\nu_b+1}}\int_0^u \frac{(1-v)^{\gamma+\nu_b}}{v^{\gamma-1}}\left[(K+\nu_b+2)\theta_{K-1}v^K - \frac{r_\mathcal{G}}{b}\right]dv$$

with (7.21) forced by the Taylor expansion[22]. We now recall (7.2):

$$\begin{aligned}
M_0(u) &= (K+\nu_b+2)w_{\gamma-1,\nu_b+1}(K-1)\mathcal{T}(bu^K) \\
&= (K+\nu_b+2)w_{\gamma-1,\nu_b+1}(K-1)\frac{u^{\gamma-2}}{(1-u)^{\gamma+\nu_b+1}}\int_0^u \frac{(1-v)^{\gamma+\nu_b}}{v^{\gamma-1}}v^K dv
\end{aligned}$$

and thus

$$r_\Theta = \frac{\theta_{K-1}}{w_{\gamma-1,\nu_b+1}(K-1)}M_0(u) - \mathcal{T}(r_\mathcal{G}) = (-1)^{K-1}S_{K-1}M_0(u) - \mathcal{T}(r_\mathcal{G}),$$

this is (7.20). □

7.3. **Convergence of the leading order Taylor coefficient.** The truncation of the Taylor series produces the leading order term, provided the last Taylor coefficient is non zero. This is the $S_\infty(d,\ell) \neq 0$ condition.

Lemma 7.5. *We have*

$$S_{K-1} = (1 + o_{b\to 0}(1))S_\infty \tag{7.22}$$

where S_∞ is given by

$$S_\infty(d,\ell) := \frac{1}{a}\sum_{j=0}^{+\infty}\frac{(-1)^j a^j g_j^\infty}{\Gamma(\nu+j+3)} \tag{7.23}$$

and where g_j^∞ corresponds to the limiting problem (5.4) *and is given by* (5.15).

Proof of Lemma 7.5. Since we have from (5.16)

$$\sum_{j=0}^{+\infty}\left|\frac{(-1)^j a^j g_j^\infty}{\Gamma(\nu+j+3)}\right| \leq c_\nu\left(\sum_{j=0}^{+\infty}\frac{1}{1+j^2}\right) \leq c_\nu < +\infty,$$

and from (6.35),

$$\sum_{j=0}^{+\infty}\left|\frac{(-1)^j g_j}{(j+\nu_b+2)w_{\gamma,\nu_b}(j)}\right| \leq c_\nu\left(\sum_{j=0}^{+\infty}\frac{1}{1+j^2}\right) \leq c_\nu < +\infty,$$

[22]The statement on existence and uniqueness of the fixed point, the fact that the corresponding solution is smooth, and the fact that (7.21) holds has in fact already been proved in a more general case in Lemma 3.6.

its suffices to prove the convergence term by term as $b \to 0$. Now, recall (6.18),

$$\lim_{b\to 0} \frac{g_j}{b^j} = g_j^\infty.$$

Since from the definition of $w_{\gamma,\nu_b}(j)$,

$$\begin{aligned}(j+\nu_b+2)w_{\gamma,\nu_b}(j) &= \Gamma(\nu_b+j+3)\frac{\Gamma(\gamma-1-j)}{\Gamma(\gamma-1)} = \frac{\Gamma(\nu+j+3)}{\gamma^j}(1+o_{b\to 0}(1)) \\ &= \Gamma(\nu+j+3)\frac{b^j}{a^j}(1+o_{b\to 0}(1))\end{aligned}$$

we obtain

$$\lim_{b\to 0}(j+\nu_b+2)\frac{w_{\gamma,\nu_b}(j)}{b^j} = \frac{\Gamma(\nu+j+3)}{a^j}$$

from which we deduce the convergence term by term when $b \to 0$, as desired. □

7.4. **The $\Theta_{\rm main}$ leading order term.** We may now extract the leading order terms in Θ. From (7.18), (7.19), (7.20) and (7.22),

$$\begin{aligned}\Theta(u) &= \sum_{k=0}^{K-2}\theta_k u^k + \theta_{K-1}u^{K-1} + (-1)^{K-1}S_{K-1}M_0(u) - \mathcal{T}(r_{\mathcal{G}}) \\ &= \sum_{k=0}^{K-2}\theta_k u^k + (-1)^{K-1}S_{K-1}\left[w_{\gamma-1,\nu_b+1}(K-1)u^{K-1} + M_0(u)\right] - \mathcal{T}(r_{\mathcal{G}}) \\ &= \sum_{k=0}^{K-2}\theta_k u^k + (-1)^{K-1}S_\infty\left[1+o_{b\to 0}(1)\right]\Theta_{\rm main}(u) - \mathcal{T}(r_{\mathcal{G}}) \qquad (7.24)\end{aligned}$$

and from (7.11)

$$\begin{aligned}\Theta_{\rm main}(u) &= w_{\gamma-1,\nu_b+1}(K-1)u^{K-1} + M_0(u) \\ &= w_{\gamma-1,\nu_b+1}(K-1)u^{K-1} + \sum_{m=1}^{+\infty}(-1)^m w_{\gamma-1,\nu_b+1}(K-1+m)u^{K-1+m} \\ &= \sum_{j=0}^{+\infty}(-1)^j w_{\gamma-1,\nu_b+1}(K-1+j)u^{K-1+j} \\ &= \Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)\sum_{j=0}^{+\infty}\frac{\Gamma(K+j+\nu_b+2)}{\Gamma(K-1+\alpha_\gamma)\Gamma(j+1-\alpha_\gamma)}u^{K-1+j}. \qquad (7.25)\end{aligned}$$

Equivalently, from (7.16) and (7.11):

$$\begin{aligned}&\Theta_{\rm main}(u) = \Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)\frac{\Gamma(K+\nu_b+2)}{\Gamma(K-1+\alpha_\gamma)\Gamma(1-\alpha_\gamma)}u^{K-1} \\ &+ \Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)\frac{\Gamma(K+\nu_b+3)}{\Gamma(K-1+\alpha_\gamma)\Gamma(2-\alpha_\gamma)}u^K + M_1(u) \\ &= \Gamma(\alpha_\gamma)\frac{\Gamma(K+\nu_b+2)}{\Gamma(K-1+\alpha_\gamma)}u^{K-1}\left\{1+\frac{K+\nu_b+2}{\Gamma(2-\alpha_\gamma)}\Gamma(1-\alpha_\gamma)u\right\} + M_1(u) \\ &= \Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)K^{\nu_b+3-\alpha_\gamma}u^{K-1} \qquad (7.26) \\ &\times \left\{[1+o_{b\to 0}(1)]\left[\frac{1}{\Gamma(1-\alpha_\gamma)}+\frac{K+\nu_b+2}{\Gamma(2-\alpha_\gamma)}u\right] + (uK)^{\alpha_\gamma}\frac{M_1(u)}{u^{K-1}\Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)K^{\nu_b+3}u^{\alpha_\gamma}}\right\}.\end{aligned}$$

We now turn to the study of $\Theta_{\rm main}$.

Lemma 7.6 (Properties of Θ_{main})**.** *The function Θ_{main} is positive and strictly increasing on $[0, \frac{1}{2}]$. Moreover, pick universal constants $\frac{1}{\delta}, \Theta^* \gg 1$ then for all $0 < b < b^*(\Theta^*, \delta)$, there exists a unique solution $u^*(\alpha_\gamma)$ to*

$$\Theta_{\text{main}}(u^*(\alpha_\gamma)) = \Theta^*, \quad u^*(\alpha_\gamma) \in \left(0, \frac{1}{2}\right) \tag{7.27}$$

which satisfies the following bounds:

first boundary close to integers. If α_γ is such that

$$\alpha_\gamma = \frac{K^{\nu_b+3}(\sigma b)^{K-1}}{\Theta^*}, \quad \text{with } \delta < \sigma < \frac{1}{\delta}$$

then

$$\Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)K^{\nu_b+3-\alpha_\gamma}(u^*(\alpha_\gamma))^{K-1} = \Theta^* e^{O_\nu(\sigma)} \tag{7.28}$$

and

$$u^*(\alpha_\gamma) = \sigma b(1 + o_{b\to 0}(1)). \tag{7.29}$$

second boundary close to integers. If α_γ is such that

$$\alpha_\gamma = 1 - \frac{K^{\nu_b+3}(\sigma b)^{K-1}}{\Theta^*}, \quad \text{with } \delta < \sigma < \frac{1}{\delta},$$

then (7.29) *holds and*

$$\Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)K^{\nu_b+3-\alpha_\gamma}\frac{K+\nu_b+2}{\Gamma(2-\alpha_\gamma)}(u^*(\alpha_\gamma))^K = \Theta^* e^{O_\nu(\sigma)}. \tag{7.30}$$

away from the integers. If α_γ is such that

$$\frac{K^{\nu_b+3}\left(\frac{b}{\delta}\right)^{K-1}}{\Theta^*} < \alpha_\gamma < 1 - \frac{K^{\nu_b+3}\left(\frac{b}{\delta}\right)^{K-1}}{\Theta^*}$$

then

$$\Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)K^{\nu_b+3}\left(\frac{u^*(\alpha_\gamma)}{1-u^*(\alpha_\gamma)}\right)^{K-1}(u^*(\alpha_\gamma))^{\alpha_\gamma} = \Theta^* e^{O_\nu(1)} \tag{7.31}$$

and

$$\frac{b}{2\delta} < u^*(\alpha_\gamma) < \frac{1}{2}. \tag{7.32}$$

Proof of Lemma 7.6. The strict monotonicity and positivity follows from (7.25), since Θ_{main} is given by a series with positive coefficients. From (7.26) (7.13),

$$\Theta_{\text{main}}\left(\frac{1}{2}\right) \geq M_1\left(\frac{1}{2}\right) \geq c_{\nu,1}\Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)K^{\nu_b+3} \geq c_{\nu,1}uK^3 \gg \Theta^*$$

which together with $\Theta_{\text{main}}(0) = 0$ ensures that that there is a unique solution of the equation

$$\Theta_{\text{main}}(u^*(\alpha_\gamma)) = \Theta^*, \quad 0 < u^*(\alpha_\gamma) < \frac{1}{2}.$$

We now aim at estimating the size of the unique solution $u^*(\alpha_\gamma)$ in various regimes of the parameter α_γ.

step 1 First boundary layer. We introduce the following function

$$M^*(u) := \Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)K^{\nu_b+3-\alpha_\gamma}u^{K-1}.$$

Assume that

$$\alpha_\gamma = \frac{K^{\nu_b+3}(\sigma b)^{K-1}}{\Theta^*} \quad \text{for some } \delta < \sigma < \frac{1}{\delta}.$$

Then $\alpha_\gamma|\log b| \ll 1$ for $0 < b < b^*(\delta)$ and we compute:

$$\begin{aligned} u &= \left(\frac{M^*(u)}{K^{\nu_b+3-\alpha_\gamma}\Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)}\right)^{\frac{1}{K-1}} = \left(\frac{\alpha_\gamma M^*(u)}{K^{\nu_b+3}(1+O(\alpha_\gamma|\log b|))}\right)^{\frac{1}{K-1}} \qquad (7.33)\\ &= \left(\frac{K^{\nu_b+3}(\sigma b)^{K-1}}{\Theta^*}\frac{M^*(u)}{K^{\nu_b+3}(1+o_{b\to 0}(1))}\right)^{\frac{1}{K-1}} = (1+o_{b\to 0}(1))\left(\frac{M^*(u)}{\Theta^*}\right)^{\frac{1}{K-1}}\sigma b. \end{aligned}$$

Next, we bound from below and above the function $\Theta_{\rm main}(u)/M^*(u)$ for u in the interval $b\delta \le u \le b/\delta$. First, in view of (7.26), we have

$$\begin{aligned} \frac{\Theta_{\rm main}(u)}{M^*(u)} &= [1+o_{b\to 0}(1)]\left[\frac{1}{\Gamma(1-\alpha_\gamma)} + \frac{K+\nu_b+2}{\Gamma(2-\alpha_\gamma)}u\right]\\ &\quad +(uK)^{\alpha_\gamma}\frac{M_1(u)}{u^{K-1}\Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)K^{\nu_b+3}u^{\alpha_\gamma}}. \qquad (7.34) \end{aligned}$$

Since all three terms in (7.34) are positive and in view of the range of α_γ, we deduce

$$\frac{\Theta_{\rm main}(u)}{M^*(u)} \ge [1+o_{b\to 0}(1)]\frac{1}{\Gamma(1-\alpha_\gamma)} \ge \frac{1}{2}.$$

From (7.13) with $j=1$, we estimate for $b \le u \le \frac{b}{\delta}$, using also the fact that $|\log(1-u)| \le u$:

$$\begin{aligned} (uK)^{\alpha_\gamma}\frac{M_1(u)}{M^*(u)} &\le \frac{c_\nu\Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)K^{\nu_b+3}\left(\frac{u}{1-u}\right)^{K-1}u^{\alpha_\gamma}}{u^{K-1}\Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)K^{\nu_b+3}u^{\alpha_\gamma}}\\ &\le c_\nu e^{(K-1)\frac{b}{\delta}} \le c_\nu e^{\frac{c_\nu}{\delta}}. \qquad (7.35) \end{aligned}$$

Using (7.12), the estimate also holds for $\delta b \le u \le b$. We conclude that for $\delta b \le u \le \frac{b}{\delta}$:

$$\frac{1}{2}\frac{M^*(u)}{\Theta_*} \le \frac{\Theta_{\rm main}(u)}{\Theta_*} \le e^{\frac{c_\nu}{\delta}}\frac{M^*(u)}{\Theta_*}.$$

Since $M^*((0,+\infty)) = (0,+\infty)$, there exists $0 < u_1 < u_2 < +\infty$ such that

$$\frac{1}{2}\frac{M^*(u_2)}{\Theta^*} \ge 1, \qquad e^{\frac{c_\nu}{\delta}}\frac{M^*(u_1)}{\Theta^*} \le 1.$$

Note also that for any u such that $M^*(u) = O_\nu(\delta^{-1})\Theta_*$, we have from (7.33)

$$u = (1+o_{b\to 0}(1))\left(\frac{M^*(u)}{\Theta^*}\right)^{\frac{1}{K-1}}\sigma b = \sigma b\left(e^{O_\nu(\delta^{-1})}\right)^{\frac{1}{K-1}} = \sigma b(1+o_{b\to 0}(1))$$

so that u belongs to the range $b \le u \le \frac{b}{\delta}$. This is in particular true for u_1 and u_2, and therefore

$$\frac{\Theta_{\rm main}(u_2)}{\Theta_*} \ge 1, \quad \frac{\Theta_{\rm main}(u_1)}{\Theta_*} \le 1, \quad b\delta \le u_1 < u_2 \le \frac{b}{\delta}$$

so by the mean value theorem, $u^*(\alpha_\gamma)$ belongs to the range $\delta b \le u \le b/\delta$, and satisfies

$$u^*(\alpha_\gamma) = \sigma b(1+o_{b\to 0}(1))$$

and (7.28), (7.29) are proved.

step 2 Second boundary layer. We introduce the following function

$$M^{**}(u) := \Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)K^{\nu_b+3-\alpha_\gamma}u^{K-1}\frac{K+\nu_b+2}{\Gamma(2-\alpha_\gamma)}u.$$

Assume that we have

$$\alpha_\gamma = 1 - \frac{K^{\nu_b+3}(\sigma b)^{K-1}}{\Theta^*} \quad \text{for some } \delta < \sigma < \frac{1}{\delta}$$

and compute:

$$\begin{aligned} u &= \left(\frac{M^{**}(u)}{\Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)\frac{K+\nu_b+2}{\Gamma(2-\alpha_\gamma)}K^{\nu_b+3-\alpha_\gamma}}\right)^{\frac{1}{K}} \\ &= \left(\frac{M^{**}(u)(1-\alpha_\gamma)(1+o_{b\to 0}(1))}{K^{\nu_b+3}}\right)^{\frac{1}{K}} = \left(\frac{M^{**}(u)}{\Theta^*}\right)^{\frac{1}{K}}(\sigma b)^{\frac{K-1}{K}}(1+o_{b\to 0}(1)) \\ &= \left(\frac{M^{**}(u)}{\Theta^*}\right)^{\frac{1}{K}}\sigma b(1+o_{b\to 0}(1)) \end{aligned} \tag{7.36}$$

Next, we bound from below and above the function $\Theta_{\text{main}}(u)/M^{**}(u)$ for u in the interval $b\delta \le u \le b/\delta$. First, in view of (7.26), we have

$$\begin{aligned} \frac{\Theta_{\text{main}}(u)}{M^{**}(u)} &= [1+o_{b\to 0}(1)]\left[1+\frac{\Gamma(2-\alpha_\gamma)}{\Gamma(1-\alpha_\gamma)(K+\nu_b+2)u}\right] \\ &\quad + \frac{\Gamma(2-\alpha_\gamma)}{(K+\nu_b+2)u}(uK)^{\alpha_\gamma}\frac{M_1(u)}{u^{K-1}\Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)K^{\nu_b+3}u^{\alpha_\gamma}}. \end{aligned} \tag{7.37}$$

Since all three terms in (7.37) are positive, we deduce

$$\frac{\Theta_{\text{main}}(u)}{M^{**}(u)} \ge [1+o_{b\to 0}(1)] \ge \frac{1}{2}.$$

For $\delta b \le u \le b/\delta$, we have, recalling (7.35):

$$M_1(u) \le c_\nu \Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)K^{\nu_b+3-\alpha_\gamma}u^{K-1}e^{\frac{c_\nu}{\delta}}$$

and, since

$$\frac{\Gamma(2-\alpha_\gamma)}{(K+\nu_b+2)u} \le \frac{c_\nu}{bu} \le \frac{c_\nu}{\delta}$$

we have

$$\frac{1}{2}\frac{M^{**}(u)}{\Theta_*} \le \frac{\Theta_{\text{main}}(u)}{\Theta_*} \le e^{\frac{c_\nu}{\delta}}\frac{M^{**}(u)}{\Theta_*}$$

We may conclude, as in step 1, that $u^*(\alpha_\gamma)$ belongs to the range $\delta b \le u \le b/\delta$, and (7.29) (7.30) hold.

step 3 Away from the integer boundary layer. We now rewrite (7.26) using (A.8):

$$\begin{aligned} &\Theta_{\text{main}}(u) = \Gamma(\alpha_\gamma)\frac{\Gamma(K+\nu_b+2)}{\Gamma(K-1+\alpha_\gamma)}u^{K-1}\left\{1+\frac{K+\nu_b+2}{\Gamma(2-\alpha_\gamma)}\Gamma(1-\alpha_\gamma)u\right\} + M_1(u) \\ &= \Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)K^{\nu_b+3}\left(\frac{u}{1-u}\right)^{K-1}u^{\alpha_\gamma}\left[\frac{M_1(u)}{\Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)K^{\nu_b+3}\left(\frac{u}{1-u}\right)^{K-1}u^{\alpha_\gamma}} + \right. \\ &+ \left. \frac{(1+o_{b\to 0}(1))(1-u)^{K-1}}{(uK)^{\alpha_\gamma}\Gamma(1-\alpha_\gamma)} + \frac{(1+o_{b\to 0}(1))uK(1-u)^{K-1}}{(uK)^{\alpha_\gamma}\Gamma(2-\alpha_\gamma)}\right]. \end{aligned}$$

For $u = \sigma b$, $\sigma \ge \frac{1}{\delta}$, we have using the bound $\log(1-u) \le -u$:

$$\frac{(uK)^{-\alpha_\gamma}(1-u)^{K-1}}{\Gamma(1-\alpha_\gamma)} + \frac{(uK)^{1-\alpha_\gamma}(1-u)^{K-1}}{\Gamma(2-\alpha_\gamma)} \le c_\nu \sigma^{1-\alpha_\gamma}e^{-(K-1)\sigma b} \le c_\nu \sigma^{1-\alpha_\gamma}e^{-c_\nu\sigma} = o_{\delta\to 0}(1)$$

We then obtain the representation formula in this zone

$$\begin{aligned}&\Theta_{\rm main}(u)\\ =\;&\Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)K^{\nu_b+3}\left(\frac{u}{1-u}\right)^{K-1}u^{\alpha_\gamma}\left[\frac{M_1(u)}{\Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)K^{\nu_b+3}\left(\frac{u}{1-u}\right)^{K-1}u^{\alpha_\gamma}}+o_{\delta\to 0}(1)\right].\end{aligned}$$

We conclude from (7.13):

$$\Theta_{\rm main}(u)=\Theta^*\Leftrightarrow\Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)K^{\nu_b+3}\left(\frac{u}{1-u}\right)^{K-1}u^{\alpha_\gamma}=\Theta^*e^{O_\nu(1)}$$

and (7.31) follows.
Also, we have

$$\frac{\Theta_*}{\Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)K^{\nu_b+3}}=\frac{\Theta_*\alpha_\gamma}{(\Gamma(1-\alpha_\gamma))^2K^{\nu_b+3}}=\frac{\Theta_*(1-\alpha_\gamma)}{\Gamma(\alpha_\gamma)\Gamma(2-\alpha_\gamma)K^{\nu_b+3}}$$

Thus, in view of the range of α_γ, we infer

$$\frac{\Theta_*}{\Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)K^{\nu_b+3}}\geq\frac{1}{\Gamma\left(\frac12\right)\max\left(\Gamma\left(\frac12\right),\Gamma\left(\frac32\right)\right)}\left(\frac{b}{\delta}\right)^{K-1}$$

so that

$$\left(\frac{u}{1-u}\right)^{K-1}u^{\alpha_\gamma}\geq\left(\frac{b}{\delta}\right)^{K-1}e^{O_\nu(1)}$$

and (7.32) easily follows. □

7.5. **Bilinear estimate for $\mathcal{T}$.** We now develop the set of nonlinear estimates to control the fixed point equation (7.20).

Lemma 7.7 (Pointwise bilinear estimate)**.** *Let*

$$F=\sum_{k=0}^{K-1}f_ku^k+r_F,\quad G=\sum_{k=0}^{K-1}g_ku^k+r_G,$$

and

$$A_F=\sup_{0\leq k\leq K-1}\frac{|f_k|}{|w_{\gamma-1,\nu_b+1}(k)|},\quad A_G=\sup_{0\leq k\leq K-1}\frac{|g_k|}{|w_{\gamma-1,\nu_b+1}(k)|},$$

then we have the following pointwise bounds for $0\leq u\leq\frac12$*:*

(1) *Bound for $\mathcal{T}r$. Let $1\leq j\leq 5$, then*

$$\begin{aligned}&\left|\frac{\mathcal{T}(r_{u^jFG})}{M_0}(u)\right|\leq c_{\nu,a}A_FA_G\left[1+u^{K+j-2}\Gamma(\alpha_\gamma)K^{\nu_b+3-\alpha_\gamma}\right]\qquad(7.38)\\ +\;&\frac{c_{\nu,a}}{b}\left(1+u^{K-1}\Gamma(\alpha_\gamma)K^{\nu_b+3-\alpha_\gamma}\right)\left[A_F\left\|\frac{r_G}{M_0}\right\|_{L^\infty(v\leq u)}+A_G\left\|\frac{r_F}{M_0}\right\|_{L^\infty(v\leq u)}\right]\\ +\;&\frac{c_{\nu,a}}{b}\left\|\frac{r_F}{M_0}\right\|_{L^\infty(v\leq u)}\left\|\frac{r_G}{M_0}\right\|_{L^\infty(v\leq u)}\|M_0\|_{L^\infty(v\leq u)}.\end{aligned}$$

(2) *Bound for r. Let $0 \leq j \leq 5$,*

$$\left|\frac{r_{u^jFG}}{M_0}(u)\right| \leq c_{\nu,a}A_FA_G\left[1+u^{K-2}\Gamma(\alpha_\gamma)K^{\nu_b+2-\alpha_\gamma}+u^{K-1}\Gamma(\alpha_\gamma)K^{\nu_b+3-\alpha_\gamma}\right]$$
$$+ \quad c_{\nu,a}\left(1+u^{K-1}\Gamma(\alpha_\gamma)K^{\nu_b+3-\alpha_\gamma}\right)\left[A_F\left\|\frac{r_G}{M_0}\right\|_{L^\infty(v\leq u)}+A_G\left\|\frac{r_F}{M_0}\right\|_{L^\infty(v\leq u)}\right]$$
$$+ \quad c_{\nu,a}\left\|\frac{r_F}{M_0}\right\|_{L^\infty(v\leq u)}\left\|\frac{r_G}{M_0}\right\|_{L^\infty(v\leq u)}\|M_0\|_{L^\infty(v\leq u)}. \tag{7.39}$$

Proof of Lemma 7.7. We compute

$$\begin{aligned} u^jFG &= \sum_{k=0}^{2(K-1)}\left[\sum_{k_1+k_2=k,0\leq k_1,k_2\leq K-1}f_{k_1}g_{k_2}\right]u^{k+j}+u^jr_G\left(\sum_{k=0}^{K-1}f_ku^k\right)\\ &\quad +u^jr_F\left(\sum_{k=0}^{K-1}g_ku^k\right)+u^jr_Fr_G. \end{aligned}$$

Then, recalling (7.17):

$$r_{u^jFG}=\sum_{k=K-j}^{2(K-1)}\left[\sum_{k_1+k_2=k,0\leq k_1,k_2\leq K-1}f_{k_1}g_{k_2}\right]u^{k+j}+u^j\left\{r_G\left(\sum_{k=0}^{K-1}f_ku^k\right)+r_F\left(\sum_{k=0}^{K-1}g_ku^k\right)+r_Fr_G\right\}.$$

We now apply $\mathcal{T}$ and estimate all the terms in the corresponding identity.

step 1 Polynomial terms. We compute from (7.2):

$$\begin{aligned} &\frac{1}{M_0}\mathcal{T}\left[\left(\sum_{k=K-j}^{2(K-1)}\sum_{k_1+k_2=k,0\leq k_1,k_2\leq K-1}f_{k_1}g_{k_2}\right)u^{k+j}\right]\\ =&\sum_{k=K-j}^{2(K-1)}\left[\sum_{k_1+k_2=k,0\leq k_1,k_2\leq K-1}f_{k_1}g_{k_2}\right]\frac{\mathcal{T}(u^{k+j})}{(K+\nu_b+2)w_{\gamma-1,\nu_b+1}(K-1)\mathcal{T}(bu^K)}. \end{aligned}$$

We now observe the bound for $m\geq 0$ and $0\leq u<1$:

$$0\leq\frac{\mathcal{T}(u^{K+m})}{\mathcal{T}(u^K)}=\frac{\int_0^u\frac{(1-v)^{\gamma+\nu_b}}{v^{\gamma-1}}v^{K+m}dv}{\int_0^u\frac{(1-v)^{\gamma+\nu_b}v^K}{v^{\gamma-1}}dv}\leq u^m \tag{7.40}$$

which gives the estimate

$$\begin{aligned} &\left|\frac{1}{M_0}\mathcal{T}\left(\sum_{k=K-j}^{2(K-1)}\sum_{k_1+k_2=k,0\leq k_1,k_2\leq K-1}f_{k_1}g_{k_2}\right)u^{k+j}\right|\\ \leq&\left(\sup_{0\leq k\leq K-1}\frac{|f_k|}{|w_{\gamma-1,\nu_b+1}(k)|}\right)\left(\sup_{0\leq k\leq K-1}\frac{|g_k|}{|w_{\gamma-1,\nu_b+1}(k)|}\right)\frac{1}{b(K+\nu_b+2)w_{\gamma-1,\nu_b+1}(K-1)}\\ \times&\sum_{k=K-j}^{2(K-1)}\left[\sum_{k_1+k_2=k,0\leq k_1,k_2\leq K-1}|w_{\gamma-1,\nu_b+1}(k_1)||w_{\gamma-1,\nu_b+1}(k_2)|\right]u^{k+j-K}\\ \leq&\frac{A_FA_G}{w_{\gamma-1,\nu_b+1}(K-1)}\sum_{k=K-j}^{2(K-1)}\left[\sum_{k_1+k_2=k,0\leq k_1,k_2\leq K-1}|w_{\gamma-1,\nu_b+1}(k_1)||w_{\gamma-1,\nu_b+1}(k_2)|\right]u^{k+j-K}. \end{aligned}$$

case $K-j \le k \le K-1$: we use the convolution bound (C.5) which gives the estimate

$$\begin{aligned} & \sum_{k=K-j}^{K-1} \left[\sum_{k_1+k_2=k, 0\le k_1,k_2\le K-1} |w_{\gamma-1,\nu_b+1}(k_1)||w_{\gamma-1,\nu_b+1}(k_2)| \right] u^{k+j-K} \\ \le \; & c_{\nu,a} \sum_{k=K-j}^{K-1} w_{\gamma-1,\nu_b+1}(k) u^{k+j-K} = c_{\nu,a} \sum_{m=0}^{j-1} w_{\gamma-1,\nu_b+1}(K-1-(j-1-m)) u^m. \end{aligned}$$

We now recall (6.13) (6.47) which ensures that for $m \le j-1 \le 5$:

$$\frac{w_{\gamma-1,\nu_b+1}(K-1-(j-1-m))}{w_{\gamma-1,\nu_b+1}(K-1)} = \frac{w_{\gamma,\nu_b}(K-(j-1-m))}{w_{\gamma,\nu_b}(K)} \le c_\nu \left(\frac{\gamma}{1+K} \right)^{j-1-m} \le c_\nu$$

and leads to the bound:

$$\begin{aligned} & \frac{1}{w_{\gamma-1,\nu_b+1}(K-1)} \sum_{k=K-j}^{K-1} \left[\sum_{k_1+k_2=k, 0\le k_1,k_2\le K-1} |w_{\gamma-1,\nu_b+1}(k_1)||w_{\gamma-1,\nu_b+1}(k_2)| \right] u^{k+j-K} \\ \le \; & c_{\nu,a} \sum_{m=0}^{j-1} u^j \le c_{\nu,a}. \end{aligned} \tag{7.41}$$

case $k \ge K$: we use (6.13) and the truncated convolution bound (C.17):

$$\begin{aligned} & \frac{1}{w_{\gamma-1,\nu_b+1}(K-1)} \sum_{k=K}^{2(K-1)} \left[\sum_{k_1+k_2=k, 0\le k_1,k_2\le K-1} |w_{\gamma-1,\nu_b+1}(k_1)||w_{\gamma-1,\nu_b+1}(k_2)| \right] u^{k+j-K} \\ \le \; & \frac{c_{\nu,a}}{w_{\gamma-1,\nu_b+1}(K-1)} \sum_{k=K}^{2(K-1)} w_{\gamma-1,\nu_b+1}(k-(K-1)) w_{\gamma-1,\nu_b+1}(K-1) u^{k+j-K} \\ \le \; & c_{\nu,a} u^j \sum_{m=0}^{K-2} w_{\gamma-1,\nu_b+1}(m+1) u^m \le c_{\nu,a} u^{j-1} \sum_{k=1}^{K-1} w_{\gamma-1,\nu_b+1}(k) u^k. \end{aligned} \tag{7.42}$$

We now claim the uniform bound for $u \le \frac{1}{2}$:

$$\sum_{k=0}^{K-2} w_{\gamma-1,\nu_b+1}(k) u^k \le c_{\nu,a} \tag{7.43}$$

which is proved below. Since from (A.8)

$$\begin{aligned} w_{\gamma-1,\nu_b+1}(K-1) &= \frac{\Gamma(\alpha_\gamma)\Gamma(K-1+\nu_b+1+2)}{\Gamma(\gamma-2)} = \frac{\Gamma(\alpha_\gamma)\Gamma(K+\nu_b+2)}{\Gamma(K-1+\alpha_\gamma)} \\ &= [1+o_{K\to\infty}(1)]\,\Gamma(\alpha_\gamma) K^{\nu_b+3-\alpha_\gamma}, \end{aligned} \tag{7.44}$$

we conclude:

$$u^{j-1} \sum_{k=1}^{K-1} w_{\gamma-1,\nu_b+1}(k) u^k \le c_{\nu,a} u^{j-1} \left[1 + u^{K-1} \Gamma(\alpha_\gamma) K^{\nu_b+3-\alpha_\gamma} \right], \tag{7.45}$$

and the collection of the above bounds yields, using also $j \geq 1$,

$$\begin{aligned}
&\left| \frac{1}{M_0} \mathcal{T} \left(\sum_{k=K-j}^{2(K-1)} \sum_{k_1+k_2=k, 0 \leq k_1, k_2 \leq K-1} f_{k_1} g_{k_2} \right) u^{k+j} \right| \\
\leq \quad & \frac{A_F A_G}{w_{\gamma-1,\nu_b+1}(K-1)} \sum_{k=K-j}^{2(K-1)} \left[\sum_{k_1+k_2=k, 0 \leq k_1, k_2 \leq K-1} |w_{\gamma-1,\nu_b+1}(k_1)| |w_{\gamma-1,\nu_b+1}(k_2)| \right] u^{k+j-K} \\
\leq \quad & c_{\nu,a} A_F A_G \left[1 + u^{K+j-2} \Gamma(\alpha_\gamma) K^{\nu_b+3-\alpha_\gamma} \right]. \qquad (7.46)
\end{aligned}$$

Proof of (7.43). From (6.13) and (C.3), for some large enough K_ν:

$$\sum_{k=0}^{K-K_\nu} w_{\gamma-1,\nu_b+1}(k) u^k \leq \sum_{k=0}^{K-K_\nu} w_{\gamma-1,\nu_b+1}(k) = (\gamma-2) \sum_{k=1}^{K+1-K_\nu} w_{\gamma,\nu_b}(k) \leq c_\nu$$

and from (6.13) and (C.4):

$$\begin{aligned}
\sum_{k=K-K_\nu+1}^{K-2} w_{\gamma-1,\nu_b+1}(k) u^k \quad &= \quad \sum_{k=K-K_\nu+1}^{K-2} (\gamma-2) w_{\gamma,\nu_b}(k+1) u^k \\
&\leq \quad c_\nu \sum_{k=K-K_\nu+1}^{K-2} (\gamma-2) \Gamma(\gamma-2-k) \gamma^{\nu_b+2-(\gamma-2-k)} \frac{1}{2^k} \leq \frac{K^{c_\nu}}{2^K} \leq c_\nu
\end{aligned}$$

and (7.43) is proved.

step 2 Cross terms. We estimate pointwise from (7.6), (7.43), (7.44):

$$\begin{aligned}
&\left| \frac{\mathcal{T}\left(u^j r_G (\sum_{k=0}^{K-1} f_k u^k) \right)}{M_0} \right| \leq A_F \left(\sup_{v \leq u} \sum_{k=0}^{K-1} w_{\gamma-1,\nu_b+1}(k) v^k \right) \left\| \frac{r_G}{M_0} \right\|_{L^\infty(v \leq u)} \frac{\mathcal{T}(u^j M_0)}{M_0} \\
\leq \quad & \frac{c_{\nu,a}}{b} A_F \left(1 + u^{K-1} \Gamma(\alpha_\gamma) K^{\nu_b+3-\alpha_\gamma} \right) \left\| \frac{r_G}{M_0} \right\|_{L^\infty(v \leq u)}
\end{aligned}$$

and similarly for the other cross term.

step 3 Nonlinear term. We estimate directly from (7.6):

$$\begin{aligned}
\left| \frac{\mathcal{T}(u^j r_F r_G)}{M_0}(u) \right| \quad &\leq \quad \left\| \frac{r_F}{M_0} \right\|_{L^\infty(v \leq u)} \left\| \frac{r_G}{M_0} \right\|_{L^\infty(v \leq u)} \|M_0\|_{L^\infty(v \leq u)} \frac{\mathcal{T}(u^j M_0)}{M_0} \\
&\leq \quad \frac{c_{\nu,a}}{b} \left\| \frac{r_F}{M_0} \right\|_{L^\infty(v \leq u)} \left\| \frac{r_G}{M_0} \right\|_{L^\infty(v \leq u)} \|M_0\|_{L^\infty(v \leq u)}.
\end{aligned}$$

step 4 Estimate for r. We now revisit the above estimates allowing for $j = 0$. We start with the polynomial term. We have:

$$\begin{aligned}
&\left| \sum_{k=K-j}^{2(K-1)} \left[\sum_{k_1+k_2=k, 0 \leq k_1, k_2 \leq K-1} f_{k_1} g_{k_2} \right] u^{k+j} \right| \\
\leq \quad & c_{\nu,a} A_F A_G u^K \sum_{k=K-j}^{2(K-1)} \left[\sum_{k_1+k_2=k, 0 \leq k_1, k_2 \leq K-1} |w_{\gamma-1,\nu_b+1}(k_1)| |w_{\gamma-1,\nu_b+1}(k_2)| \right] u^{k+j-K}.
\end{aligned}$$

For $j \geq 0$ and $K-j \leq k \leq K-1$, we use (7.41) and (7.44) which imply:

$$\begin{aligned} & u^K \sum_{k=K-j}^{K-1} \left[\sum_{k_1+k_2=k, 0\leq k_1,k_2\leq K-1} |w_{\gamma-1,\nu_b+1}(k_1)||w_{\gamma-1,\nu_b+1}(k_2)| \right] u^{k+j-K} \\ \leq \; & c_{\nu,a} w_{\gamma-1,\nu_b+1}(K-1) u^K \leq c_{\nu,a}\Gamma(\alpha_\gamma) K^{\nu_b+3-\alpha_\gamma} u^K. \end{aligned}$$

For $K \leq k \leq 2(K-1)$ and $j \geq 0$, we argue differently, depending on u.

<u>case $u \leq b$</u>. We recall (7.42) which yields:

$$\begin{aligned} & u^K \sum_{k=K}^{2(K-1)} \left[\sum_{k_1+k_2=k, 0\leq k_1,k_2\leq K-1} |w_{\gamma-1,\nu_b+1}(k_1)||w_{\gamma-1,\nu_b+1}(k_2)| \right] u^{k+j-K} \\ \leq \; & c_{\nu,a} w_{\gamma-1,\nu_b+1}(K-1) u^K u^j \sum_{m=0}^{K-2} w_{\gamma-1,\nu_b+1}(m+1) u^m. \end{aligned}$$

We now have the rough bound from (6.14):

$$w_{\gamma-1,\nu_b+1}(m+1) = \frac{m+\nu_b+3}{\gamma-m-3} w_{\gamma-1,\nu_b+1}(m) = \frac{m+\nu_b+3}{K+\alpha_\gamma-m-2} w_{\gamma-1,\nu_b+1}(m) \leq \frac{c_\nu}{b} w_{\gamma-1,\nu_b+1}(m).$$

for $m \leq K-3$. Then, from (7.43) and (7.44):

$$\begin{aligned} & u^K \sum_{k=K}^{2(K-1)} \left[\sum_{k_1+k_2=k, 0\leq k_1,k_2\leq K-1} |w_{\gamma-1,\nu_b+1}(k_1)||w_{\gamma-1,\nu_b+1}(k_2)| \right] u^{k+j-K} \\ \leq \; & \frac{c_{\nu,a}\Gamma(\alpha_\gamma) K^{\nu_b+3-\alpha_\gamma}}{b} u^{K+j} \sum_{m=0}^{K-3} w_{\gamma-1,\nu_b+1}(m) u^m \\ & + c_{\nu,a}\Gamma(\alpha_\gamma) K^{\nu_b+3-\alpha_\gamma} u^K u^j w_{\gamma-1,\nu_b+1}(K-1) u^{K-2} \\ \leq \; & \frac{c_{\nu,a}\Gamma(\alpha_\gamma) K^{\nu_b+3-\alpha_\gamma}}{b} u^{K+j} + c_{\nu,a}\Gamma(\alpha_\gamma) K^{\nu_b+3-\alpha_\gamma} u^K u^j \Gamma(\alpha_\gamma) K^{\nu_b+3-\alpha_\gamma} u^{K-2} \end{aligned}$$

which yields the bound:

$$\begin{aligned} & \left| \sum_{k=K-j}^{2(K-1)} \left[\sum_{k_1+k_2=k, 0\leq k_1,k_2\leq K-1} f_{k_1} g_{k_2} \right] u^{k+j} \right| \\ \leq \; & \frac{c_{\nu,a}}{b} A_F A_G \Gamma(\alpha_\gamma) K^{\nu_b+3-\alpha_\gamma} u^K + A_F A_G c_{\nu,a}\Gamma(\alpha_\gamma) K^{\nu_b+3-\alpha_\gamma} u^K \Gamma(\alpha_\gamma) K^{\nu_b+3-\alpha_\gamma} u^{K-2}. \end{aligned}$$

Therefore, from (7.4) for $u \leq b$:

$$\begin{aligned} & \frac{1}{M_0} \left| \sum_{k=K-j}^{2(K-1)} \left[\sum_{k_1+k_2=k, 0\leq k_1,k_2\leq K-1} f_{k_1} g_{k_2} \right] u^{k+j} \right| \\ \leq \; & \frac{\frac{c_{\nu,a}}{b} A_F A_G \Gamma(\alpha_\gamma) K^{\nu_b+3-\alpha_\gamma} u^K + A_F A_G c_{\nu,a}\Gamma(\alpha_\gamma) K^{\nu_b+3-\alpha_\gamma} u^K \Gamma(\alpha_\gamma) K^{\nu_b+3-\alpha_\gamma} u^{K-2}}{\Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma) K^{\nu_b+4-\alpha_\gamma} u^K} \\ \leq \; & c_\nu A_F A_G + c_\nu A_F A_G \Gamma(\alpha_\gamma) K^{\nu_b+2-\alpha_\gamma} u^{K-2} \leq c_\nu A_F A_G \big[1 + u^{K-2}\Gamma(\alpha_\gamma) K^{\nu_b+2-\alpha_\gamma}\big]. \end{aligned}$$

case $u \geq b$. We recall (7.42) (7.44) (7.45) which yield

$$\begin{aligned} & u^K \sum_{k=K}^{2(K-1)} \left[\sum_{k_1+k_2=k, 0\leq k_1,k_2\leq K-1} |w_{\gamma-1,\nu_b+1}(k_1)||w_{\gamma-1,\nu_b+1}(k_2)| \right] u^{k+j-K} \\ \leq \quad & c_{\nu,a} u^{j-1} \Gamma(\alpha_\gamma) K^{\nu_b+3-\alpha_\gamma} u^K u^{j-1} \left[1 + u^{K-1} \Gamma(\alpha_\gamma) K^{\nu_b+3-\alpha_\gamma} \right] \end{aligned}$$

and thus, for $b \leq u \leq \frac{1}{2}$, from (7.5):

$$\begin{aligned} & \frac{1}{M_0} \left| \sum_{k=K-j}^{2(K-1)} \left[\sum_{k_1+k_2=k, 0\leq k_1,k_2\leq K-1} f_{k_1} g_{k_2} \right] u^{k+j} \right| \\ \leq \quad & \frac{A_F A_G \Gamma(\alpha_\gamma) K^{\nu_b+3-\alpha_\gamma} u^K u^{j-1} \left[1 + u^{K-1} \Gamma(\alpha_\gamma) K^{\nu_b+3-\alpha_\gamma} \right]}{\Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma) K^{\nu_b+3} \left(\frac{u}{1-u} \right)^{K-1} u^{\alpha_\gamma}} \\ \leq \quad & c_\nu A_F A_G \left[1 + u^{K-1} \Gamma(\alpha_\gamma) K^{\nu_b+3-\alpha_\gamma} \right]. \end{aligned}$$

We estimate the cross term from (7.43), (7.44)

$$\begin{aligned} & \left| \frac{u^j r_G (\sum_{k=0}^{K-1} f_k u^k)}{M_0} \right| \leq A_F \left(\sup_{v\leq u} \sum_{k=0}^{K-1} w_{\gamma-1,\nu_b+1}(k) v^k \right) \left\| \frac{r_G}{M_0} \right\|_{L^\infty(v\leq u)} \\ \leq \quad & c_{\nu,a} A_F \left(1 + u^{K-1} \Gamma(\alpha_\gamma) K^{\nu_b+3-\alpha_\gamma} \right) \left\| \frac{r_G}{M_0} \right\|_{L^\infty(v\leq u)} \end{aligned}$$

and the nonlinear term

$$\left| \frac{u^j r_F r_G}{M_0}(u) \right| \quad \leq \quad \left\| \frac{r_F}{M_0} \right\|_{L^\infty(v\leq u)} \left\| \frac{r_G}{M_0} \right\|_{L^\infty(v\leq u)} \|M_0\|_{L^\infty(v\leq u)}.$$

□

7.6. **Controlling the final remainder.** We are now in position to prove the exit condition for the $\mathcal{C}^\infty$ solution for a large enough range of parameters.

Lemma 7.8 (Uniform control of the final remainder)**.** *Pick universal constants $\frac{1}{\delta}, \Theta^* \gg 1$, then for all $0 < b < b^*(\delta, \Theta^*) \ll 1$ small enough, the following holds. Let*

$$\frac{K^{\nu_b+3} (b\delta)^{K-1}}{\Theta^*} < \alpha_\gamma < 1 - \frac{K^{\nu_b+3} (b\delta)^{K-1}}{\Theta^*} \tag{7.47}$$

and let $u^*(\alpha_\gamma)$ be the solution to (7.27) described by Lemma 7.6. Then the $\mathcal{C}^\infty$ solution to (7.20) satisfies for all $0 \leq u \leq u^*(\alpha_\gamma)$ the bound:

$$\frac{|r_{\mathcal{G}}|}{M_0} + \frac{|\mathcal{T} r_{\mathcal{G}}|}{M_0} < \sqrt{b}. \tag{7.48}$$

Proof of Lemma 7.8. We recall the fixed point formulation (7.24) of the $\mathcal{C}^\infty$ solution and the expression (6.31) for the nonlinear term:

$$\begin{aligned} & \mathcal{G} = \mathcal{G}_0 + \left[b^2 x \tilde{h}_1 + b x^2 \tilde{h}_2 \right] \Theta + \left[b^2 x \tilde{h}_3 + b x^2 \tilde{h}_4 \right] u\Theta' \\ + \quad & \sum_{j=2}^{4} x^{j+1} \tilde{m}_j^{(1)} \Theta^j + b \sum_{j=2}^{4} m_j^{(2)} x^j \Theta^j + \left[\sum_{j=1}^{3} \tilde{m}_j^{(3)} x^{j+2} \Theta^j + b \sum_{j=1}^{3} \tilde{m}_j^{(4)} x^{j+1} \Theta^j \right] (u\Theta'). \end{aligned}$$

We now bootstrap the bound

$$\frac{|r_\Theta(u)|}{M_0} < \Theta^*. \tag{7.49}$$

Let us first check that (7.49) holds for u small enough. From (7.24), we have

$$\begin{aligned}\frac{|r_\Theta(u)|}{M_0} &\leq c_{\nu,a}+\frac{|\mathcal{T}(r_\mathcal{G})(u)|}{M_0}\leq c_{\nu,a}+\left(\sup_{v\leq u}\frac{|\mathcal{G}(v)|}{v^K}\right)\frac{|\mathcal{T}(u^K)|}{M_0}\\ &\leq c_{\nu,a}+\left(\sup_{v\leq u}\frac{|\mathcal{G}(v)|}{v^K}\right)\frac{1}{b(K+\nu_b+2)w_{\gamma-1,\nu_b+1}(K-1)}\end{aligned}$$

where we have used (7.2) in the last inequality. Also, we have, using (6.35) and (6.13),

$$\lim_{u\to 0}\sup_{v\leq u}\frac{|\mathcal{G}(v)|}{v^K}=|g_K|\leq c_{\nu,a}\frac{|w_{\gamma,\nu_b}(K)|}{1+K}\leq c_{\nu,a}\frac{|w_{\gamma-1,\nu_b+1}(K-1)|}{(1+K)(\gamma-2)}.$$

We infer for u small enough that

$$\frac{|r_\Theta(u)|}{M_0}\leq c_{\nu,a}<\Theta_*$$

so that (7.49) indeed holds for u small enough. We therefore work on the interval $u\in[0,u_{\rm boot}]$ with $0<u_{\rm boot}\leq u^*(\alpha\gamma)$ where (7.49) holds, and aim at improving (7.49).

step 1 Uniform bounds for $0\leq u\leq u^*(\alpha_\gamma)$. By definition (7.25):

$$\Theta_{\rm main}(u)=w_{\gamma-1,\nu_b+1}(K-1)u^{K-1}+M_0(u)$$

and hence, since $\Theta_{\rm main}$ is non decreasing:

$$\forall u\in[0,u^*(\alpha_\gamma)],\quad 0\leq M_0(u)\leq\Theta_{\rm main}(u)\leq\Theta_{\rm main}(u^*(\alpha_\gamma))=\Theta^*. \tag{7.50}$$

Observe that in the regime (7.28):

$$\Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)K^{\nu_b+3-\alpha_\gamma}(u^*(\alpha_\gamma))^{K-1}=C_{\Theta^*,\delta},$$

in the regime (7.30), recalling (7.29):

$$\Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)K^{\nu_b+3-\alpha_\gamma}(u^*(\alpha_\gamma))^{K-1}\leq\frac{\Gamma(2-\alpha_\gamma)\Theta^*e^{O_\nu(\sigma)}}{u^*(\alpha_\gamma)(K+\nu_b+2)}\leq\frac{\Theta^*e^{O_\nu(\sigma)}}{\delta}\leq C_{\Theta^*,\delta},$$

and in the regime (7.31), recalling (7.32):

$$\Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)K^{\nu_b+3-\alpha_\gamma}(u^*(\alpha_\gamma))^{K-1}\leq\frac{(1-u^*(\alpha_\gamma))^{K-1}}{(Ku^*(\alpha_\gamma))^{\alpha_\gamma}}\Theta^*e^{O_\nu(1)}\leq C_{\Theta^*,\delta}$$

In all three cases

$$\forall u\in[0,u^*(\alpha_\gamma)],\quad\Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)K^{\nu_b+3-\alpha_\gamma}u^{K-1}\leq C_{\Theta^*,\delta}. \tag{7.51}$$

Also, we have, using (7.51),

$$\Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)K^{\nu_b+2-\alpha_\gamma}u^{K-2}\leq\Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)K^{\nu_b+2-\alpha_\gamma}(u^*(\alpha_\gamma))^{K-2}\leq\frac{C_{\Theta^*,\delta}}{Ku^*(\alpha_\gamma)}$$

and thus, using (7.29) (7.32), we deduce

$$\forall u\in[0,u^*(\alpha_\gamma)],\quad\Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)K^{\nu_b+2-\alpha_\gamma}u^{K-2}\leq C_{\Theta^*,\delta}. \tag{7.52}$$

We now decompose $r_\mathcal{G}$ according to the decomposition (6.31) of $\mathcal{G}$ and estimate all the terms.

step 2 Source term. For any holomorphic function $H(x)$, we have from (7.17):

$$|r_H|=\left|\sum_{k=K}^{+\infty}h_ku^k\right|\leq\sum_{k=K}^{+\infty}(bC_hu)^k\leq(bC_h)^Ku^K.$$

We therefore estimate for $u \le b$ from (7.4):

$$\frac{|r_H|}{M_0} \le \frac{(bC_H)^K u^K}{\Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)K^{\nu_b+4-\alpha_\gamma}u^K} \le \frac{(bC_H)^K}{\Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)K^{\nu_b+3}}$$

and for $b \le u \le \frac{1}{2}$ from (7.5):

$$\frac{|r_H|}{M_0} \le \frac{(bC_H)^K u^K}{\Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)K^{\nu_b+3}\left(\frac{u}{1-u}\right)^{K-1}u^{\alpha_\gamma}} \le \frac{(bC_H)^K}{\Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)K^{\nu_b+3}}$$

It implies the rough bound

$$\left\| \frac{r_H}{M_0} \right\|_{L^\infty(0\le u\le \frac{1}{2})} \le b^4. \tag{7.53}$$

In view of (6.32) this bound can be applied to the source term $\mathcal{G}_0$.

step 3 Derivative term. Let

$$\Theta = \sum_{k=0}^{K-1} \theta_k u^k + r_\Theta,$$

then

$$u\Theta' = \sum_{k=1}^{K-1} k\theta_k u^k + ur'_\Theta.$$

Therefore,

$$\left| \begin{array}{l} (u\Theta')_k = k\theta_k, \quad 0 \le k \le K-1 \\ r_{u\Theta'} = ur'_\Theta. \end{array} \right. \tag{7.54}$$

Moreover, from (7.20):

$$ur'_\Theta = (-1)^{K-1} S_{K-1} u M_0'(u) - u\left[\mathcal{T}(r_\mathcal{G})\right]'$$

We recall from (7.8):

$$u\left[\mathcal{T}(\mathcal{G})\right]' = \frac{1}{1-u}\left[\frac{\mathcal{G}}{b} + [(\gamma-2)+(\nu_b+3)u]\mathcal{T}(\mathcal{G})\right]$$

which yields

$$r_{u\Theta'} = (-1)^{K-1} S_{K-1} u M_0'(u) - \frac{1}{1-u}\left[\frac{r_\mathcal{G}}{b} + [(\gamma-2)+(\nu_b+3)u]\mathcal{T}(r_\mathcal{G})\right]$$

We obtain the estimate, using (7.7) and (6.34):

$$\left| \begin{array}{l} \frac{|r_{u\Theta'}|}{M_0} \le \frac{c_\nu}{b}\left[1 + \frac{|r_\mathcal{G}|}{M_0} + \frac{|\mathcal{T}(r_\mathcal{G})|}{M_0}\right] \\ \sup_{0\le k\le K-1} \frac{|(u\Theta')_k|}{w_{\gamma-1,\nu_b+1}(k)} \le \frac{c_\nu}{b}. \end{array} \right. \tag{7.55}$$

step 4 Linear term. Recall from (6.36) that for a holomorphic function:

$$\forall 0 \le k \le K-1, \quad |(H\Theta)_k| \le c_H w_{\gamma-1,\nu_b+1}(k).$$

no derivative term. We apply (7.38) with the bounds (7.53), (7.50), (7.51) (7.52) and (6.34) to derive for $1 \le j \le 5$:

$$\begin{aligned}
&\left|\frac{\mathcal{T}(r_{u^j H\Theta})}{M_0}(u)\right| \le c_{\nu,a} A_\Theta A_H \left[1 + u^{K+j-2}\Gamma(\alpha_\gamma) K^{\nu_b+3-\alpha_\gamma}\right] \\
+ \quad & \frac{c_{\nu,a}}{b}\left(1 + u^{K-1}\Gamma(\alpha_\gamma) K^{\nu_b+3-\alpha_\gamma}\right)\left[A_\Theta \left\|\frac{r_H}{M_0}\right\|_{L^\infty(v\le u)} + A_H \left\|\frac{r_\Theta}{M_0}\right\|_{L^\infty(v\le u)}\right] \\
+ \quad & \frac{c_{\nu,a}}{b}\left\|\frac{r_\Theta}{M_0}\right\|_{L^\infty(v\le u)}\left\|\frac{r_H}{M_0}\right\|_{L^\infty(v\le u)} \|M_0\|_{L^\infty(v\le u)} \le c_{\nu,a} + \frac{C_{\Theta^*,\delta}}{b}
\end{aligned}$$

where we used the bootstrap bound (7.49) in the last step. Now, writing

$$\left|\begin{array}{l} b^2 x\tilde{h}_1\Theta = b^3 u(\tilde{h}_1\Theta) \\ b x^2 \tilde{h}_2\Theta = b^3 u^2(\tilde{h}_2\Theta) \end{array}\right.$$

gives

$$\left|\frac{\mathcal{T} r_{\left[b^2 x\tilde{h}_1 + b x^2\tilde{h}_2\right]\Theta}}{M_0}\right| \le b^3\left[c_{\nu,a} + \frac{C_{\Theta^*,\delta}}{b}\right] \le b^2 C_{\Theta^*,\delta}.$$

Similarly, from (7.39), (7.49), (7.53), (7.50), (7.51) (7.52) and (6.34):

$$\left|\frac{r_{u^j H\Theta}}{M_0}(u)\right| \le C_{\Theta^*,\delta}$$

Therefore,

$$\left|\frac{r_{\left[b^2 x\tilde{h}_1 + b x^2\tilde{h}_2\right]\Theta}}{M_0}\right| \le b^3 C_{\Theta^*,\delta}.$$

derivative term. We first use (7.55), (7.39) to estimate for a holomorphic function H:

$$\begin{aligned}
&\left|\frac{r_{u^j H(u\Theta')}}{M_0}(u)\right| \le \frac{c_{\nu,a}}{b}\left[1 + u^{K-1}\Gamma(\alpha_\gamma) K^{\nu_b+3-\alpha_\gamma} + u^{K-2}\Gamma(\alpha_\gamma) K^{\nu_b+2-\alpha_\gamma}\right] \\
+ \quad & c_{\nu,a}\left(1 + u^{K-1}\Gamma(\alpha_\gamma) K^{\nu_b+3-\alpha_\gamma}\right)\left[1 + \left\|\frac{r_{u\Theta'}}{M_0}\right\|_{L^\infty(v\le u)}\right] \\
+ \quad & C_{\Theta^*,\delta}\left\|\frac{r_{u\Theta'}}{M_0}\right\|_{L^\infty(v\le u)} \le C_{\Theta^*,\delta}\left[\frac{1}{b} + \left\|\frac{r_{u\Theta'}}{M_0}\right\|_{L^\infty(v\le u)}\right]
\end{aligned}$$

and from (7.55), (7.38):

$$\left|\frac{\mathcal{T}(r_{u^j H(u\Theta')})}{M_0}(u)\right| \le C_{\Theta^*,\delta}\left[\frac{1}{b^2} + \frac{1}{b}\left\|\frac{r_{u\Theta'}}{M_0}\right\|_{L^\infty(v\le u)}\right].$$

We conclude, using (7.55):

$$\begin{aligned}
&\left|\frac{\mathcal{T} r_{\left[b^2 x\tilde{h}_1 + b x^2\tilde{h}_2\right]u\Theta'}}{M_0}\right| \le b^3 C_{\Theta^*,\delta}\left[\frac{1}{b^2} + \frac{1}{b}\left\|\frac{r_{u\Theta'}}{M_0}\right\|_{L^\infty(v\le u)}\right] \le C_{\Theta^*,\delta}\left[b + b^2\left\|\frac{r_{u\Theta'}}{M_0}\right\|_{L^\infty(v\le u)}\right] \\
\le \quad & C_{\Theta^*,\delta}\left[b + b\left\|\frac{r_{\mathcal{G}}}{M_0}\right\|_{L^\infty(v\le u)} + b\left\|\frac{\mathcal{T}(r_{\mathcal{G}})}{M_0}\right\|_{L^\infty(v\le u)}\right]
\end{aligned}$$

and

$$\left|\frac{r_{[b^2x\tilde{h}_1+bx^2\tilde{h}_2]u\Theta'}}{M_0}\right| \leq b^3 C_{\Theta^*,\delta}\left[\frac{1}{b}+\left\|\frac{r_{u\Theta'}}{M_0}\right\|_{L^\infty(v\leq u)}\right]$$
$$\leq C_{\Theta^*,\delta}\left[b^2+b^2\left\|\frac{r_{\mathcal{G}}}{M_0}\right\|_{L^\infty(v\leq u)}+b^2\left\|\frac{\mathcal{T}(r_{\mathcal{G}})}{M_0}\right\|_{L^\infty(v\leq u)}\right]$$

step 5 Nonlinear term.

no derivative term. First, we have in view of (6.34) (C.6) that for $1\leq m\leq 5$ and $0\leq k\leq K-1$:

$$|(\Theta^m)_k| \leq \sum_{k_1+\cdots k_m=k}|\Theta_{k_1}|\cdots|\Theta_{k_m}| \leq c_{\nu,a}\sum_{k_1+\cdots k_m=k} w_{\gamma-1,\nu_b+1}(k_1)\cdots w_{\gamma-1,\nu_b+1}(k_m)$$
$$\leq c_{\nu,a}w_{\gamma-1,\nu_b+1}(k)$$

We then estimate, using (7.39) iteratively in m for $2\leq m\leq 5$, $0\leq j\leq 5$, and using also (7.49), (7.53), (7.50), (7.51) (7.52):

$$\left|\frac{r_{u^j\Theta^m}}{M_0}(u)\right| \leq c_{\nu,a}A_{u^j}\left(A_\Theta+\left\|\frac{r_\Theta}{M_0}\right\|_{L^\infty(v\leq u)}\right)\left(\sum_{\ell=1}^m A_{\Theta^\ell}+\left(A_\Theta+\left\|\frac{r_\Theta}{M_0}\right\|_{L^\infty(v\leq u)}\right)^{m-2}\right)$$
$$\times\left[1+u^{K-1}\Gamma(\alpha_\gamma)K^{\nu_b+3-\alpha_\gamma}+u^{K-2}\Gamma(\alpha_\gamma)K^{\nu_b+2-\alpha_\gamma}\right]^{m-1}\leq C_{\Theta^*,\delta}$$

Similarly, for a holomorphic function H:

$$\left|\frac{r_{u^jH\Theta^m}}{M_0}(u)\right|\leq C_{\Theta^*,\delta} \tag{7.56}$$

which implies that

$$\left|\frac{r_{\sum_{j=2}^4 x^{j+1}\tilde{m}_j^{(1)}\Theta^j+b\sum_{j=2}^4 m_j^{(2)}x^j\Theta^j}}{M_0}\right|\leq C_{\Theta^*,\delta}b^3.$$

Similarly, for $1\leq j\leq 5$ from (7.49) (7.53), (7.50), (7.51):

$$\left|\frac{\mathcal{T}(r_{u^jH\Theta^j})}{M_0}(u)\right|\leq\frac{C_{\Theta^*,\delta}}{b}$$

gives

$$\left|\frac{\mathcal{T}r_{\sum_{j=2}^4 x^{j+1}\tilde{m}_j^{(1)}\Theta^j+b\sum_{j=2}^4 m_j^{(2)}x^j\Theta^j}}{M_0}\right|\leq b^3\frac{C_{\Theta^*,\delta}}{b}\leq C_{\Theta^*,\delta}b^2.$$

derivative term. We estimate from (7.56), (7.39), (7.55), (7.50), (7.51) (7.52):

$$\left|\frac{r_{u^jH\Theta^m(u\Theta')}}{M_0}(u)\right|\leq C_{\Theta^*,\delta}\left[\frac{1}{b}+\left\|\frac{r_{u\Theta'}}{M_0}\right\|_{L^\infty(v\leq u)}\right]$$
$$\leq C_{\Theta^*,\delta}\left[\frac{1}{b}+\frac{1}{b}\left(\left\|\frac{r_{\mathcal{G}}}{M_0}\right\|_{L^\infty(v\leq u)}+\left\|\frac{\mathcal{T}(r_{\mathcal{G}})}{M_0}\right\|_{L^\infty(v\leq u)}\right)\right].$$

Similarly,

$$\left|\frac{\mathcal{T} r_{u^j H\Theta^m(u\Theta')}}{M_0}(u)\right| \leq \frac{C_{\Theta^*,\delta}}{b}\left[\frac{1}{b}+\left\|\frac{r_{u\Theta'}}{M_0}\right\|_{L^\infty(v\leq u)}\right]$$
$$\leq \frac{C_{\Theta^*,\delta}}{b^2}\left[1+\left\|\frac{r_{\mathcal{G}}}{M_0}\right\|_{L^\infty(v\leq u)}+\left\|\frac{\mathcal{T}(r_{\mathcal{G}})}{M_0}\right\|_{L^\infty(v\leq u)}\right].$$

Therefore,

$$\frac{|r_{\left[\sum_{j=1}^3 \tilde{m}_j^{(3)} x^{j+2}\Theta^j + b\sum_{j=1}^3 \tilde{m}_j^{(4)} x^{j+1}\Theta^j\right](u\Theta')}|}{M_0}$$
$$\leq b^3 C_{\Theta^*,\delta}\left[\frac{1}{b}+\frac{1}{b}\left(\left\|\frac{r_{\mathcal{G}}}{M_0}\right\|_{L^\infty(v\leq u)}+\left\|\frac{\mathcal{T}(r_{\mathcal{G}})}{M_0}\right\|_{L^\infty(v\leq u)}\right)\right]$$
$$\leq C_{\Theta^*,\delta} b^2\left[1+\left\|\frac{r_{\mathcal{G}}}{M_0}\right\|_{L^\infty(v\leq u)}+\left\|\frac{\mathcal{T}(r_{\mathcal{G}})}{M_0}\right\|_{L^\infty(v\leq u)}\right]$$

and

$$\frac{|\mathcal{T} r_{\left[\sum_{j=1}^3 \tilde{m}_j^{(3)} x^{j+2}\Theta^j + b\sum_{j=1}^3 \tilde{m}_j^{(4)} x^{j+1}\Theta^j\right](u\Theta')}|}{M_0}$$
$$\leq b^3\frac{C_{\Theta^*,\delta}}{b^2}\left[1+\left\|\frac{r_{\mathcal{G}}}{M_0}\right\|_{L^\infty(v\leq u)}+\left\|\frac{\mathcal{T}(r_{\mathcal{G}})}{M_0}\right\|_{L^\infty(v\leq u)}\right]$$
$$\leq bC_{\Theta^*,\delta}\left[1+\left\|\frac{r_{\mathcal{G}}}{M_0}\right\|_{L^\infty(v\leq u)}+\left\|\frac{\mathcal{T}(r_{\mathcal{G}})}{M_0}\right\|_{L^\infty(v\leq u)}\right].$$

step 6 Conclusion. The collection of the above bounds yields:

$$\frac{|r_{\mathcal{G}}|}{M_0} \leq C_{\Theta^*,\delta} b^2\left[1+\left\|\frac{r_{\mathcal{G}}}{M_0}\right\|_{L^\infty(v\leq u)}+\left\|\frac{\mathcal{T}(r_{\mathcal{G}})}{M_0}\right\|_{L^\infty(v\leq u)}\right]$$

and

$$\frac{|\mathcal{T} r_{\mathcal{G}}|}{M_0} \leq bC_{\Theta^*,\delta}\left[1+\left\|\frac{r_{\mathcal{G}}}{M_0}\right\|_{L^\infty(v\leq u)}+\left\|\frac{\mathcal{T}(r_{\mathcal{G}})}{M_0}\right\|_{L^\infty(v\leq u)}\right]$$

which imply for $0<b<b^*(\Theta*,\delta)$ small enough

$$\frac{|r_{\mathcal{G}}|}{M_0}+\frac{|\mathcal{T} r_{\mathcal{G}}|}{M_0} < \sqrt{b}$$

which, reinserted into (7.20), yields:

$$\frac{|r_\Theta|}{M_0} \leq c_{\nu,a}+\sqrt{b} < \frac{\Theta^*}{2},$$

and (7.49), (7.48) are proved. □

7.7. **Exit on the left of P_2.** We are now in position to establish the fundamental exit property of the $\mathcal{C}^\infty$ solution to the left of P_2. Recall from Lemma 3.6 that the unique $\mathcal{C}^\infty$ solution has a slope c_- at P_2. Recall also from (2.49) that c_- is strictly between the slope of the middle root $w_2(\sigma)$ of Δ_1 at P_2 and the slope of the smallest root $w_2^-(\sigma)$ of Δ_2 at P_2. Thus, at the left of P_2, for u small enough, the $\mathcal{C}^\infty$ solution lies strictly above the middle root of Δ_1, and strictly below the smallest root of Δ_2. The goal of Lemma 7.9 below is to show that the $\mathcal{C}^\infty$ solution exists this region, i.e.

that there exist a value $0 < U_* < 1$ such that at $u = U_*$, the solution either touches the middle root of Δ_1, or touches the smallest root of Δ_2. Indeed, since at such a point we have respectively $w'(\sigma) = 0$ or $w'(\sigma) = -\infty$, and since $(w_2^-)'(\sigma) < 0$ and $w_2'(\sigma) < 0$ respectively by (2.7) and (2.10), touching either root implies that the $\mathcal{C}^\infty$ solution exists the region for $u > U_*$.

We assume without loss of generality that

$$S_\infty(d, \ell) > 0 \tag{7.57}$$

and need only to reverse the parity of K in Lemma 7.9 below if $S_\infty(d, \ell) < 0$.

Lemma 7.9 (Exit on the left)**.** *Pick universal constants $\frac{1}{\delta}, \Theta^* \gg 1$ large enough as in Lemma 7.8, then for all $0 < b < b^*(\Theta^*, \delta)$ small enough and α_γ in the range (7.47), the $\mathcal{C}^\infty$ solution exits on the left of P_2 at $u = U_*$, where $0 < U_* < \frac{3}{4}$, by crossing the smallest root of Δ_2 for K odd, and by crossing the middle root of Δ_1 for K even.*

Proof of Lemma 7.9. **step 1** Reaching Θ^*. Recall (7.24)

$$\Theta(u) = \sum_{k=0}^{K-2} \theta_k u^k + (-1)^{K-1} S_\infty \left[1 + o_{b\to 0}(1)\right] \Theta_{\text{main}}(u) - \mathcal{T}(r_{\mathcal{G}})$$

then, provided Θ^* has been chosen large enough, we conclude from (7.43), (7.48), (7.50) that for $0 < b < b^*(\Theta^*, \delta)$ small enough, for all $u \in [0, u^*(\alpha_\gamma)]$:

$$|\Theta(u) - (-1)^{K-1} S_\infty \Theta_{\text{main}}(u)| \leq \frac{S_\infty \Theta^*}{10}. \tag{7.58}$$

Therefore,

$$\frac{\Theta^*}{2} \leq \frac{(-1)^{K-1}\Theta(u^*(\alpha_\gamma))}{S_\infty} \leq 2\Theta^*. \tag{7.59}$$

step 2 Computation of Δ_1, Δ_2. We now unfold our changes of variables and show that, depending on the sign of $(-1)^{K-1}S_\infty$, we must have passed through either the green or the red curves. To this end, we examine Δ_1 and Δ_2 under the assumption

$$|u| \leq \frac{3}{4}, \quad |\Psi(u)| \leq 2\Theta_*. \tag{7.60}$$

From (6.29), (6.1), (4.39)

$$\left| \begin{array}{l} \widetilde{\Phi} = (1-u)\Psi \\ \Psi(u) = M_b(x)\Phi(u) \\ \Phi = bu\Theta \end{array} \right.$$

where M_b is bounded and given by (6.2). Moreover, from (3.15), (4.19), (4.31), (4.34), (4.22):

$$\begin{aligned} & \Delta_1 - c_-\Delta_2 = \mathcal{G}_2 = b^2\mathcal{F}_1 = -b^2|\lambda_-|\tilde{\psi}_{\backsim}\tilde{w}_{\backsim}(c_+ - c_-)uF_1 \\ = \; & -b^2|\lambda_-|\tilde{\psi}_{\backsim}\tilde{w}_{\backsim}(c_+ - c_-)u\left[u(1-u)bH_1(b,u) + (1+G_1(bu))\widetilde{\Phi} + \text{NL}_1(u,\widetilde{\Phi})\right] \\ = \; & -b^2\tilde{w}_{\backsim}^2\tilde{e}_{20}(1+O(b))u\left[u(1-u)bH_1(b,u) + (1+G_1(bu))bu(1-u)M_b\Theta + \text{NL}_1(u,\widetilde{\Phi})\right] \\ = \; & -b^3\tilde{w}_{\backsim}^2\tilde{e}_{20}(1+O(b))u^2(1-u)\left[H_1(b,u) + (1+G_1(bu))M_b\Theta + \frac{\text{NL}_1(u,\widetilde{\Phi})}{bu(1-u)}\right] \end{aligned}$$

and from (3.14), (4.18), (4.31), (4.36), (4.22):

$$\begin{aligned} & -\Delta_1 + c_+\Delta_2 = \mathcal{G}_1 = -b^2\mathcal{F}_2 = b^2\tilde{w}_{\infty}(c_+ - c_-)|\mu_+|uF_2 \\ = \;& b^2\tilde{w}_{\infty}(c_+ - c_-)|\mu_+|u\left[(1-u)\left[1+H_2(b,u)\right] + G_2(bu)\widetilde{\Phi} + \mathrm{NL}_2(u,\widetilde{\Phi})\right] \\ = \;& b^2|\tilde{d}_{20}|\tilde{w}_{\infty}^2(1+O(b))u(1-u)\left[1 + H_2(b,u) + G_2(bu)M_b\Phi + \frac{\mathrm{NL}_2(u,\widetilde{\Phi})}{1-u}\right] \end{aligned}$$

From (6.2), for $x = bu$ and $u \le 1$,

$$M_b(x) = 1 + O(b), \tag{7.61}$$

and from (4.35), (4.37), (4.38) and (7.60)

$$\left|\begin{array}{l} H_1(b,u) = -(\tilde{E}_{11} + \tilde{E}_{30}) + O(b) \\ G_1 = b\tilde{E}_{11}u + O(b^2) \\ \mathrm{NL}_1 = O(b^2u) \\ H_2(b,u) = O(b) \\ G_2(x) = O(b) \\ \mathrm{NL}_2 = O(b^2u) \end{array}\right.$$

It implies that, as long as $\Theta(u) \neq 0$, we have:

$$\left|\begin{array}{l} \Delta_1 - c_-\Delta_2 = -\Theta(u)bu\left\{\tilde{w}_{\infty}^2\tilde{e}_{20}b^2u(1-u)\left(1+O\left(\frac{1}{\Theta(u)}\right) + O(b)\right)\right\} \\ -\Delta_1 + c_+\Delta_2 = |\tilde{d}_{20}|\tilde{w}_{\infty}^2b^2u(1-u)(1+O(b)) \end{array}\right.$$

i.e.,

$$\left|\begin{array}{l} \Delta_1 = \frac{1}{c_+-c_-}\tilde{w}_{\infty}^2b^2u(1-u)\left(1+O\left(\frac{1}{\Theta(u)}+b\right)\right)\left[-c_+\tilde{e}_{20}\Theta(u)bu + c_-|\tilde{d}_{20}|\right] \\ \Delta_2 = \frac{1}{c_+-c_-}\tilde{w}_{\infty}^2b^2u(1-u)\left(1+O\left(\frac{1}{\Theta(u)}+b\right)\right)\left[-\tilde{e}_{20}\Theta(u)bu + |\tilde{d}_{20}|\right] \end{array}\right.$$

or, equivalently,

$$\left|\begin{array}{l} \Delta_1 = -\frac{1}{c_+-c_-}\tilde{w}_{\infty}^2b^2u(1-u)\left(1+O\left(\frac{1}{\Theta(u)}+b\right)\right)\left[c_+\tilde{e}_{20}\Phi(u) + |c_-||\tilde{d}_{20}|\right] \\ \Delta_2 = \frac{1}{c_+-c_-}\tilde{w}_{\infty}^2b^2u(1-u)\left(1+O\left(\frac{1}{\Theta(u)}+b\right)\right)\left[-\tilde{e}_{20}\Phi(u) + |\tilde{d}_{20}|\right]. \end{array}\right. \tag{7.62}$$

Note that we have used the signs, valid for all $d \ge 2$ and $0 \le \ell \le d$:

$$\tilde{d}_{20} < 0, \quad \tilde{e}_{20} > 0,$$

see (F.8).

step 3 Touching $\Delta_1 = 0$ or $\Delta_2 = 0$. At $u = u^*(\alpha_\gamma)$, by (7.59) we have

$$\Phi(u^*) = bu^*\Theta(u^*), \quad \frac{(-1)^{K-1}\Theta(u^*)}{S_\infty} \ge \frac{\Theta^*}{2}.$$

We now claim that there exists

$$u^* \le U^* \le \frac{3}{4}, \quad \Phi(U^*) = (-1)^{K-1}\Theta^*$$

Then, from (7.62), since $\tilde{d}_{20} \neq 0$ and $\tilde{e}_{20} > 0$ by (F.8), and since $\Phi(0) = 0$, we must have crossed $\Delta_1 = 0$ or $\Delta_2 = 0$, depending on wether K is even or odd.
Assume K odd, so that from (7.59) $\Theta(u_*) > 0$. The case K even can be treated similarly. Since $u \le \frac{3}{4}$ and $M_b(u) = 1 + O(b)$:

$$\Psi(u^*) = bu^*\Theta(u^*)(1+O(b)), \quad \Psi(U^*) = \Theta^*.$$

In view of (4.15)

$$\begin{aligned} & \left[1+H_2+G_2\Psi+\widetilde{\mathrm{NL}}_2\right]u(1-u)\Psi' \\ + & \left[(1-2u)(1+H_2+G_2\Psi+\widetilde{\mathrm{NL}}_2)-\gamma(1+G_1)+2uG_2\right]\Psi \\ = & \ u\left[\gamma b H_1-2(1+H_2)+\frac{\gamma b\widetilde{\mathrm{NL}}_1}{x}-2\widetilde{\mathrm{NL}}_2\right]. \end{aligned}$$

Then, on the interval $u\in[u^*,U^*]$ with $U^*\le\frac{3}{4}$ and

$$\frac{|S_\infty|bu^*\Theta^*}{4}<\Psi(u)\le 2\Theta^* \tag{7.63}$$

we have for $b<b^*(\Theta^*)$ from (G.2), (4.43), (4.38), (4.35):

$$\left|\begin{array}{l} \left|H_1(b,u)+(\tilde{E}_{11}+\tilde{E}_{30})\right|+|H_2|+|G_2|+|G_1|\le C_{\Theta^*}b \\ |\widetilde{\mathrm{NL}}_2|+|\frac{\widetilde{\mathrm{NL}}_1}{x}|\le C_{\Theta^*}b \end{array}\right.$$

We insert this into (4.15) and conclude from (7.63), provided $\Theta^*>0$ has been chosen large enough,

$$u\Psi'\ge\frac{\gamma}{4}\Psi. \tag{7.64}$$

Therefore,

$$\Psi(u)\ge\Psi(u^*)\left(\frac{u}{u^*}\right)^{\frac{\gamma}{4}}\ge\frac{|S_\infty|bu^*\Theta^*}{4}\left(\frac{u}{u^*}\right)^{\frac{\gamma}{4}}\ge\Theta^*$$

for

$$\left(\frac{u}{u^*}\right)^{\frac{\gamma}{4}}\ge\frac{4}{|S_\infty|bu^*\Theta^*},\quad u\ge u^*\left(\frac{4}{|S_\infty|bu^*\Theta^*}\right)^{\frac{4}{\gamma}}=u^*\left(1+O(b|\log b|)\right)$$

where we have used the fact that $u^*\ge b\delta$ in view of (7.29) (7.32). Since $u^*\le\frac{1}{2}$, we established that the contact happens before $u=\frac{3}{4}$. □

7.8. **Exit on the right.** Below, we obtain an analog of Lemma 7.9 on the right, albeit in a significantly more restricted range of α_γ in $(0,1)$. We assume (7.57).

Lemma 7.10 (Exit on the right)**.** *Pick universal constants $\frac{1}{\delta},\Theta^*\gg 1$ large enough as in Lemma 7.8, then for all $0<b<b^*(\Theta^*,\delta)$ small enough and α_γ given by*

$$\alpha_\gamma=\frac{K^{\nu_b+3}\left(b\sqrt{\delta}\right)^{K-1}}{\Theta^*} \tag{7.65}$$

or

$$\alpha_\gamma=1-\frac{K^{\nu_b+3}\left(b\sqrt{\delta}\right)^{K-1}}{\Theta^*}, \tag{7.66}$$

the $\mathcal{C}^\infty$ solution exits on the right of P_2 at $u=U_*$, where $0<U_*<\frac{3}{4}$, by crossing $\Delta_1=0$ in the case (7.65) *and by crossing $\Delta_2=0$ in the case* (7.66)*.*

Proof of Lemma 7.10. For α_γ given by (7.65) or (7.66) we only need to consider u in the the range $-b\le u\le 0$. In that range, the proof of Lemma 7.10 follows very closely the one of Lemma 7.9 for the case $0\le u\le b$. The $\mathcal{C}^\infty$ regularity at the right of P_2 all the way to $u=0$ together with the property (7.21) follow again from the explicit integral representation of the remainder function. We focus on the exit behavior.

step 1 Bounds on M_j. For $-b \le u \le b$, we have

$$(1-u)^K = e^{K\log(1-u)} = e^{O(1)}$$

so that the cases $0 \le u \le b$ and $-b \le u \le 0$ can be treated similarly in the definition of $\mathcal{T}$ and, as a consequence, in M_0 and M_j. In particular, the proof of (7.4), (7.6), (7.7) and (7.12) obtained for $0 \le u \le b$ immediately extends to the case $-b \le u \le 0$, i.e., we have for $-b \le u \le 0$

$$c_{\nu,1} \le \frac{M_0(u)}{\Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)K^{\nu_b+4-\alpha_\gamma}u^K} \le c_{\nu,2}, \tag{7.67}$$

for $1 \le j \le 5$,

$$\left\| \frac{\mathcal{T}(u^j M_0)}{M_0} \right\|_{L^\infty(-b\le u\le 0)} \le \frac{c_\nu}{b}, \tag{7.68}$$

$$\forall -b \le u \le 0, \quad \frac{|uM_0'|}{M_0} \le \frac{c_\nu}{b}. \tag{7.69}$$

and for $1 \le j \le 5$ and $-b \le u \le 0$

$$c_{\nu,1} \le \frac{M_j(u)}{\Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)K^{\nu_b+j+4-\alpha_\gamma}u^{K+j}} \le c_{\nu,2}. \tag{7.70}$$

step 2 Estimate on $\Theta_{\rm main}$. Recall from (7.26)

$$\begin{aligned}\Theta_{\rm main}(u) &= \Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)K^{\nu_b+3-\alpha_\gamma}u^{K-1} \\ \times &\left\{[1+o_{b\to 0}(1)]\left[\frac{1}{\Gamma(1-\alpha_\gamma)} + \frac{K+\nu_b+2}{\Gamma(2-\alpha_\gamma)}u\right] + (uK)^{\alpha_\gamma}\frac{M_1(u)}{u^{K-1}\Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)K^{\nu_b+3}u^{\alpha_\gamma}}\right\}.\end{aligned}$$

In view of (7.70), we deduce for $-b\delta^{\frac{1}{3}} \le u \le -b\delta$,

$$\begin{aligned}\Theta_{\rm main}(u) &= \Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)K^{\nu_b+3-\alpha_\gamma}u^{K-1} \\ &\times \left\{[1+o_{b\to 0}(1)]\left[\frac{1}{\Gamma(1-\alpha_\gamma)} + \frac{K+\nu_b+2}{\Gamma(2-\alpha_\gamma)}u\right] + O\left(c_\nu\delta^{\frac{2}{3}}\right)\right\}.\end{aligned} \tag{7.71}$$

step 3 Boundary layer. In view of (7.71), we easily obtain the following analog of the first two cases of Lemma 7.6 for α_γ given by (7.65) or (7.66). Pick universal constants $\frac{1}{\delta}, \Theta^* \gg 1$ then for all $0 < b < b^*(\Theta^*, \delta)$, we have:
first layer: if α_γ is given by (7.65), then there exist a solution to

$$\Theta_{\rm main}(u^*(\alpha_\gamma)) = (-1)^{K-1}\Theta^*, \tag{7.72}$$

satisfying the following bounds

$$\Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)K^{\nu_b+3-\alpha_\gamma}(u^*(\alpha_\gamma))^{K-1} = \Theta^* e^{O_\nu(\delta^{-1})} \tag{7.73}$$

and

$$u^*(\alpha_\gamma) = -\sqrt{\delta}b(1+o_{b\to 0}(1)). \tag{7.74}$$

second layer: if α_γ is given by (7.66), then there exist a solution to

$$\Theta_{\rm main}(u^*(\alpha_\gamma)) = (-1)^K\Theta^*, \tag{7.75}$$

satisfying (7.74) and

$$\Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma)K^{\nu_b+3-\alpha_\gamma}\frac{K+\nu_b+2}{\Gamma(2-\alpha_\gamma)}(u^*(\alpha_\gamma))^K = \Theta^* e^{O_\nu(\delta^{-1})}. \tag{7.76}$$

step 4 Estimate on $r_{\mathcal{G}}$. In view of (7.67) and (7.74), we have for $u^*(\alpha_\gamma)$ defined in step 3

$$\forall u \in [u^*(\alpha_\gamma), 0], \quad 0 \leq |M_0(u)| \leq c_\nu |M_0(u^*(\alpha_\gamma))| \leq \frac{c_\nu \Theta^*}{\sqrt{\delta}} = C_{\Theta^*,\delta}. \tag{7.77}$$

Also, proceeding as in the proof of (7.51) and (7.52), we obtain the following analogs

$$\forall u \in [u^*(\alpha_\gamma), 0], \quad \Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma) K^{\nu_b+3-\alpha_\gamma} |u|^{K-1} \leq C_{\Theta^*,\delta}. \tag{7.78}$$

and

$$\forall u \in [u^*(\alpha_\gamma), 0], \quad \Gamma(\alpha_\gamma)\Gamma(1-\alpha_\gamma) K^{\nu_b+2-\alpha_\gamma} |u|^{K-2} \leq C_{\Theta^*,\delta}. \tag{7.79}$$

The estimate (7.48) holds for $-b \leq u \leq 0$, i.e. for $-b \leq u \leq 0$, and we now claim:

$$\frac{|r_{\mathcal{G}}|}{M_0} + \frac{|\mathcal{T} r_{\mathcal{G}}|}{M_0} < \sqrt{b}. \tag{7.80}$$

Indeed, the proof of (7.48) for the range $0 \leq u \leq b$ does not use the sign of u, and all estimates hold by replacing everywhere u with $|u|$. Using also (7.77) (7.78) (7.79), the proof immediately extends to the range $-b \leq u \leq 0$. Thus, (7.80) holds for $-b \leq u \leq 0$.

step 6 Conclusion. Recall (7.24)

$$\Theta(u) = \sum_{k=0}^{K-2} \theta_k u^k + (-1)^{K-1} S_\infty \left[1 + o_{b\to 0}(1)\right] \Theta_{\text{main}}(u) - \mathcal{T}(r_{\mathcal{G}})$$

then, provided Θ^* has been chosen large enough, we conclude from (7.43) (with u replaced by $|u|$), (7.80), (7.77) that for $0 < b < b^*(\Theta^*, \delta)$ small enough, for all $u \in [u^*(\alpha_\gamma), 0]$:

$$|\Theta(u) - (-1)^{K-1} S_\infty \Theta_{\text{main}}(u)| \leq \frac{S_\infty \Theta^*}{10}.$$

Therefore, if α_γ is given by (7.65)

$$\frac{\Theta^*}{2} \leq \frac{\Theta(u^*(\alpha_\gamma))}{S_\infty} \leq 2\Theta^*, \tag{7.81}$$

and, if α_γ is given by (7.66),

$$\frac{\Theta^*}{2} \leq \frac{-\Theta(u^*(\alpha_\gamma))}{S_\infty} \leq 2\Theta^*. \tag{7.82}$$

We now note that (7.62) holds independently of the sign of $u \in (-1, 1)$. If α_γ is given by (7.65), then, by (7.81),

$$\Phi(u^*) = b u^* \Theta(u^*), \quad \frac{\Theta(u^*)}{S_\infty} \geq \frac{\Theta^*}{2},$$

and, if α_γ is given by (7.66), then, by (7.82),

$$\Phi(u^*) = b u^* \Theta(u^*), \quad \frac{-\Theta(u^*)}{S_\infty} \geq \frac{\Theta^*}{2}.$$

The rest of the argument of step 3 in the proof of Lemma 7.9 extends to the case $u < 0$. Therefore,
first layer: if α_γ is given by (7.65), there exists $U^*(\alpha_\gamma)$ such that

$$-\frac{3}{4} \leq U^* \leq u^*, \quad \Phi(U^*) = -\Theta^*,$$

and the smooth solution crosses $\Delta_1 = 0$ for $u \geq -3/4$.
second layer: if α_γ is given by (7.66), there exists $U^*(\alpha_\gamma)$ such that

$$-\frac{3}{4} \leq U^* \leq u^*, \quad \Phi(U^*) = \Theta^*,$$

and the smooth solution crosses $\Delta_2 = 0$ for $u \geq -3/4$. This concludes the proof of Lemma 7.10. □

7.9. **Proof of Theorem 1.3.** We are now in position to conclude the proof of Theorem 1.3.

step 1 Continuous deformation of α_γ. Let K be even and large enough, we claim that there exists α_γ^K in $(0,1)$ such that the $\mathcal{C}^\infty$ solution $\Phi[K, \alpha_\gamma^K](u)$ coincides with the unique $P_6 - P_2$ solution (to the right of P_2) and exits to the left of P_2 by crossing $\Delta_1 = 0$ before reaching P_5 (and, as a result, extends to P_4.).

Indeed, let $\frac{1}{\delta}, \Theta^* \gg 1$ large enough and $0 < b < b^*(\Theta^*, \delta)$ small enough as in Lemma 7.9, and α_γ in the range (7.47). Assume also that K is even. Then, in view of Lemma 7.9, the $\mathcal{C}^\infty$ solution exits on the left of P_2 by crossing $\Delta_1 = 0$ before $u = \frac{3}{4}$. Consider then the $\mathcal{C}^\infty$ solution $\Phi[K, \alpha_\gamma](u)$ on the right of P_2. Let also $\Phi^{(rad)}[K, \alpha_\gamma](u)$ denote the unique solution, constructed earlier, which corresponds to the $P_2 - P_6$ trajectory. Then, let

$$F(\alpha_\gamma) \quad := \quad \Phi[K, \alpha_\gamma]\left(-\frac{3}{4}\right) - \Phi^{(rad)}[K, \alpha_\gamma]\left(-\frac{3}{4}\right). \tag{7.83}$$

Then, for α_γ given by (7.65), since $\Phi^{(rad)}[K, \alpha_\gamma](u)$ is located between the $\Delta_1 = 0$ and $\Delta_2 = 0$ curves for any $u < 0$, and, since $\Phi[K, \alpha_\gamma](u)$ has crossed $\Delta_1 = 0$ before $u = -3/4$ and cannot cross $\Delta_1 = 0$ twice, we deduce

$$F\left(\frac{K^{\nu_b+3}\left(b\sqrt{\delta}\right)^{K-1}}{\Theta^*}\right) > 0. \tag{7.84}$$

Also, for α_γ given by (7.66), since $\Phi^{(rad)}[K, \alpha_\gamma](u)$ lives between $\Delta_1 = 0$ and $\Delta_2 = 0$ curves for any $u < 0$, and, since $\Phi[K, \alpha_\gamma](u)$ has crossed $\Delta_2 = 0$ before $u = -3/4$ and cannot cross $\Delta_2 = 0$ twice, we deduce

$$F\left(1 - \frac{K^{\nu_b+3}\left(b\sqrt{\delta}\right)^{K-1}}{\Theta^*}\right) < 0. \tag{7.85}$$

Continuous dependence of the ode on the parameter $\alpha_\gamma \in (0,1)$ implies the continuity of F. We then infer by the mean value theorem the existence of α_γ^K such that

$$F\left(\alpha_\gamma^K\right) = 0, \quad \alpha_\gamma^K \in \left(\frac{K^{\nu_b+3}\left(b\sqrt{\delta}\right)^{K-1}}{\Theta^*}, 1 - \frac{K^{\nu_b+3}\left(b\sqrt{\delta}\right)^{K-1}}{\Theta^*}\right). \tag{7.86}$$

Then, by the uniqueness of solutions to the ode at $u = 3/4$, $\Phi[K, \alpha_\gamma^K]$ must coincide with $\Phi^{(rad)}[K, \alpha_\gamma^K]$.

To the left of P_2, with α_γ^K in the range (7.47), we have that $\Phi[K, \alpha_\gamma^K]$ crosses $\Delta_1 = 0$ before reaching $P_\Leftrightarrow$.

Thus, we have obtained for any even K large enough the existence of α_γ^K in $(0,1)$ such that the smooth profile $\Phi[K,\alpha_\gamma^K](u)$ coincides with the P_2-P_6 solution to the right of P_2 and exits on the left of P_2 by crossing $\Delta_1=0$ before reaching $P5$. The constructed solution is $\mathcal{C}^\infty$ to the right and the left of P_2, with derivatives satisfying (7.21) on both sides.

step 2 Conclusion. Since the curve crosses $\Delta_1=0$ for $\sigma_2<\sigma<\sigma_5$, it is attracted to P_4 by Lemma 3.2. It remains to show that in the $(\sigma(x),w(x))$ parametrization, the point P_2 is reached in finite time. Indeed,

$$\lim_{\Sigma\to 0}\frac{W}{\Sigma}=c_-$$

and thus, from (3.17), (2.50), (F.3):

$$\frac{d\Sigma}{dx}=\frac{d\sigma}{dx}=-\frac{\Delta_2}{\Delta}=-\frac{(c_2c_-+c_4)\Sigma(1+o(1))}{-2\sigma_2(1+c_-)\Sigma(1+o(1))}=\frac{\lambda_+}{2\sigma_2(1+c_-)}(1+o(1))>0$$

which proves the claim. The resulting $\mathcal{C}^\infty$ solution corresponds to the global $P_6-P_2-P_4$ trajectory.

7.10. **Uniform control of the P_2-P_{ss} separatrix.** We conclude this section with a simple uniform estimate on the separatrix P_2-P_{ss} which will be required in the proof of the exterior positivity property, see Lemma 9.9.

Recall that all integral curves of (1.9) that have a slope c_- at P_2 are parametrized by Θ with respect to the u variable. In view of Lemma 3.5, the separatrix P_2-P_{ss} is one of these curves, and we denote by Θ_S its Θ parametrization. The following lemma provides a bound on Θ_S.

Lemma 7.11 (Computing Θ_S)**.** *There exist universal constants C,b^* such that for all $0<b<b^*$, the unique separatrix curve in the eye P_2-P_{ss} satisfies:*

$$\forall 0\le u\le 1,\quad \left|\Theta_S+\tilde{E}_{11}+\tilde{E}_{30}+\frac{2}{a}\right|<Cb. \tag{7.87}$$

Proof of Lemma 7.11. This follows from the fixed point representation of the separatrix.

step 1 Bound on the separatrix. We first claim for the separatrix

$$|\Theta_S|\lesssim 1 \tag{7.88}$$

uniformly as $b\to 0$. Indeed, there holds from (6.30):

$$\Theta'-\left[\frac{\gamma-2}{u}+\frac{\gamma+\nu_b+1}{1-u}\right]\Theta=-\frac{\mathcal{G}}{bu(1-u)} \tag{7.89}$$

$$\Leftrightarrow\quad \left[\frac{(1-u)^{\gamma+\nu_b+1}}{u^{\gamma-2}}\Theta\right]'=-\frac{\mathcal{G}(1-u)^{\nu_b+\gamma+1}}{bu(1-u)u^{\gamma-2}}$$

and the separatrix is the unique solution which reaches $u=1$:

$$\Theta_S=\frac{u^{\gamma-2}}{(1-u)^{\gamma+\nu_b+1}}\int_u^1\frac{\mathcal{G}}{b\sigma(1-\sigma)}\frac{(1-\sigma)^{\gamma+\nu_b+1}}{\sigma^{\gamma-2}}d\sigma. \tag{7.90}$$

Let the operator

$$T(\mathcal{G})(u)=\frac{u^{\gamma-2}}{(1-u)^{\gamma+\nu_b+1}}\int_u^1\frac{\mathcal{G}}{b\sigma(1-\sigma)}\frac{(1-\sigma)^{\gamma+\nu_b+1}}{\sigma^{\gamma-2}}d\sigma,$$

we claim, recalling $a = \gamma b$,

$$\|T(\mathcal{G})\|_{L^\infty[0,1]} \lesssim_a \|\mathcal{G}\|_{L^\infty([0,1])} \tag{7.91}$$

Assume (7.91), then the source term is given by (6.23):

$$\left| \begin{array}{l} \mathcal{G} = \mathcal{G}_0 + \mathcal{L}(\Phi) + \mathrm{NL}(\Phi) \\ \Phi = bu\Theta \\ \mathcal{G}_0 = \frac{\gamma b H_1 - 2(1+H_2)}{M_b(1+H_{20})} \end{array} \right.$$

which is just bounded and (7.88) follows by an elementary fixed point argument.

Proof of (7.91). We estimate for $\frac{1}{2} \le u \le 1$:

$$\begin{aligned} & \frac{u^{\gamma-2}}{(1-u)^{\gamma+\nu_b+1}} \int_u^1 \frac{1}{b\sigma(1-\sigma)} \frac{(1-\sigma)^{\gamma+\nu_b+1}}{\sigma^{\gamma-2}} d\sigma \\ \le & \frac{u^{\gamma-2}}{(1-u)^{\gamma+\nu_b+1}} \frac{1}{bu^{\gamma-1}} \int_u^1 (1-\sigma)^{\gamma+\nu_b} d\sigma \le \frac{2}{b(\gamma+\nu_b)} \le \frac{2}{a} \end{aligned}$$

and for $0 < u < \frac{1}{2}$:

$$\begin{aligned} & \frac{u^{\gamma-2}}{(1-u)^{\gamma+\nu_b+1}} \int_u^1 \frac{1}{b\sigma(1-\sigma)} \frac{(1-\sigma)^{\gamma+\nu_b+1}}{\sigma^{\gamma-2}} d\sigma \\ \le & \frac{u^{\gamma-2}}{(1-u)^{\gamma+\nu_b+1}} \frac{(1-u)^{\gamma+\nu_b}}{b} \int_u^1 \frac{d\sigma}{\sigma^{\gamma-1}} \le \frac{2}{b(\gamma-2)} \le \frac{3}{a} \end{aligned}$$

and (7.91) follows.

step 2 Next term. We now extract the main term. Since $H_2 = O(b)$ we have

$$\mathcal{G}_0 = \frac{\gamma b H_1 - 2(1+H_2)}{M_b(1+H_{20})} = g_0 + O(b), \quad \left| \begin{array}{l} g_0 = -a(\tilde{E}_{11} + \tilde{E}_{30}) - 2 \\ a = \gamma b = \frac{|\lambda_-|}{|\mu_+|}. \end{array} \right.$$

Let

$$\Theta = \frac{g_0}{a} + \tilde{\Theta}$$

then from (7.89):

$$\tilde{\Theta}' - \left[\frac{\gamma-2}{u} + \frac{\gamma+\nu_b+1}{1-u} \right] \tilde{\Theta} = -\frac{\tilde{\mathcal{G}}}{bu(1-u)}$$

with

$$\tilde{\mathcal{G}} = \mathcal{G} - bu(1-u) \left(\frac{\gamma-2}{u} + \frac{\gamma+\nu_b+1}{1-u} \right) \frac{g_0}{a} = \mathcal{G} - b\gamma \frac{g_0}{a} + O(b) = O(b)$$

and hence (7.91) ensures $\|\tilde{\Theta}\|_{L^\infty([0,1])} \lesssim b$ which is (7.87). □

8. **Interior positivity**

This section is devoted to the proof of Lemma 1.6. This is the first positivity property at the heart of the control of the linearized operator in [10]. In this section we fix (d, ℓ, r) and assume that we are in the range (1.5) for some sufficiently small $0 < \varepsilon(d, \ell) \ll 1$. We let $(\sigma(x), w(x))$ be the $P_6 - P_2$ trajectory given in Lemma 3.1, and aim at proving the positivity property (1.24) in the region $\sigma \ge \sigma_2$, where we recall the definition (1.23).

The proof relies on classical maximum principle arguments for ODEs along the $P_6 - P_2$ curve. We will need to compute various numerical constants depending on

(d, ℓ, r). Their signs will be essential and checked at $r = r_\circledcirc(d, \ell)$ only, using (1.22) and $\varepsilon(d, \ell)$ small enough. We will work in the variables

$$W = w - w_2, \quad \Sigma = \sigma - \sigma_2, \quad \Phi = \frac{W}{\Sigma}$$

and use the algebra which follows from (3.16):

$$\left| \begin{array}{l} \Delta_1 = \Sigma \left[c_1\Phi + c_3 + [d_{20}\Phi^2 + d_{11}\Phi + d_{02}]\Sigma + [\Phi^3 - d\Phi]\Sigma^2 \right] \\ \Delta_2 = \Sigma \left[c_2\Phi + c_4 + [e_{20}\Phi^2 + e_{11}\Phi + e_{02}]\Sigma + [e_{21}\Phi^2 - 1]\Sigma^2 \right] \\ \Delta = -\Sigma \left[2\sigma_2(1+\Phi) + \Sigma(1-\Phi^2) \right] \end{array} \right. \tag{8.1}$$

8.1. **Sharp bound on the slope function Φ.** We recall the notations

$$w_e = \frac{\ell(r-1)}{d}, \quad w_2 = 1 - \sigma_2.$$

Lemma 8.1 (Sharp bound on the slope function)**.** *Assume* (1.22)*. Let*

$$\left| \begin{array}{l} \alpha(r) = \frac{|c_-|}{w_2 - w_e} \\ \alpha^\infty = \alpha(r_\circledcirc) = \left| \begin{array}{ll} \frac{2d(\ell+\sqrt{d})}{d(\sqrt{d}-1)+\ell(\sqrt{d}+1)} & \textit{for} \quad r = r^* \\ \frac{d(1+\sqrt{\ell})^2}{d\sqrt{\ell}+\ell} & \textit{for} \quad r = r_+ \end{array} \right. \end{array} \right. \tag{8.2}$$

then for $\Sigma \geq 0$:

$$\frac{c_-}{1 + \alpha(r)\Sigma} \leq \Phi < 0. \tag{8.3}$$

Remark 8.2. The bound (8.3) is saturated at P_2.

Proof of Lemma 8.1. We compute

$$\begin{aligned} w_2(r^*) - w_e(r^*) &= 1 - \frac{\sqrt{d}}{\ell + \sqrt{d}} - \frac{\ell\left(\frac{d+\ell}{\ell+\sqrt{d}} - 1\right)}{d} = \frac{\ell}{\ell + \sqrt{d}} - \frac{\ell(\sqrt{d}-1)}{\sqrt{d}(\ell+\sqrt{d})} \\ &= \frac{\ell}{\sqrt{d}(\ell + \sqrt{d})} \end{aligned} \tag{8.4}$$

and

$$\begin{aligned} w_2(r_+) - w_e(r_+) &= \frac{\sqrt{\ell}}{1+\sqrt{\ell}} - \frac{\ell(d-1)}{d(1+\sqrt{\ell})^2} = \frac{\sqrt{\ell}d(1+\sqrt{\ell}) - \ell(d-1)}{d(1+\sqrt{\ell})^2} \\ &= \frac{d\sqrt{\ell} + \ell}{d(1+\sqrt{\ell})^2}. \end{aligned} \tag{8.5}$$

It implies that $w_2 > w_e$ for r close enough to $r_\circledcirc(\ell)$. For $0 < \alpha < \alpha^\infty$ let us consider the function $\widetilde{\Phi}$ of Σ given by

$$\widetilde{\Phi} = (1 + \alpha\Sigma)\Phi = \Phi + \alpha W$$

then

$$\left| \begin{array}{l} \widetilde{\Phi}(0) = c_- \\ \widetilde{\Phi}(+\infty) = -\alpha(w_2 - w_e) \end{array} \right. \tag{8.6}$$

From (8.2) and (F.3), (F.11) which imply that $c_- < 0$,

$$c_- < -\alpha(w_2 - w_e) \Leftrightarrow \alpha < \frac{|c_-|}{w_2 - w_e} = \alpha(r).$$

We compute:

$$\frac{dW}{d\Sigma} = \Sigma\frac{d\Phi}{d\Sigma} + \Phi = \frac{\Delta_1}{\Delta_2} \Leftrightarrow \frac{d\Phi}{d\Sigma} = \frac{\Delta_1 - \Phi\Delta_2}{\Sigma\Delta_2}$$

Thus,

$$\begin{aligned}\frac{d\widetilde{\Phi}}{d\Sigma} &= \frac{d\Phi}{d\Sigma}+\alpha\frac{dW}{d\Sigma}=\frac{\Delta_1-\Phi\Delta_2}{\Sigma\Delta_2}+\alpha\frac{\Delta_1}{\Delta_2}=\frac{\Delta_1-\Phi\Delta_2+\alpha\Sigma\Delta_1}{\Sigma\Delta_2}\\ &= \frac{(1+\alpha\Sigma)^2\Delta_1-\tilde{\Phi}\Delta_2}{\Sigma(1+\alpha\Sigma)\Delta_2}.\end{aligned}$$

step 1 Repulsivity. We assume that there exists a point Σ where $\tilde{\Phi}=c_-$ with $\Sigma>0$. First, we claim that

$$(1+\alpha\Sigma)^2\Delta_1-\tilde{\Phi}\Delta_2>0. \tag{8.7}$$

Assuming that, since $\Delta_2<0$ in this zone, we obtain

$$\frac{d\widetilde{\Phi}}{d\Sigma}<0.$$

Since the $\widetilde{\Phi}(\infty)>c_-$, this leads to a contradiction and implies that $\frac{c_-}{1+\alpha\Sigma}<\Phi<0$. (8.3) follows by passing to the limit $\alpha\to\alpha(r)$.
Proof of (8.7).

$$\begin{aligned}T &= \frac{(1+\alpha\Sigma)^2\Delta_1-c_-\Delta_2}{\Sigma}=(1+\alpha\Sigma)^2\left[c_1c_-+c_3+[d_{20}c_-^2+d_{11}c_-+d_{02}]\Sigma+[c_-^3-dc_-]\Sigma^2\right]\\ &- c_-\left[c_2c_-+c_4+[e_{20}c_-^2+e_{11}c_-+e_{02}]\Sigma+[e_{21}c_-^2-1]\Sigma^2\right]\end{aligned}$$

From (2.45), $c_1c_-+c_3-c_-(c_2c_-+c_4)=0$. At $r_\infty(d,\ell)$ we also have the additional cancellation $(c_2c_-+c_4)=\lambda_+=0$. Therefore,

$$\begin{aligned}&\frac{T}{\Sigma}=(1+\alpha\Sigma)^2\left[d_{20}c_-^2+d_{11}c_-+d_{02}+(c_-^3-dc_-)\Sigma\right]\\ &- c_-\left[e_{20}c_-^2+e_{11}c_-+e_{02}+(e_{21}c_-^2-1)\Sigma\right]\\ &= (1+2\alpha\Sigma+\alpha^2\Sigma^2)\left[d_{20}c_-^2+d_{11}c_-+d_{02}+(c_-^3-dc_-)\Sigma\right]\\ &- c_-\left[e_{20}c_-^2+e_{11}c_-+e_{02}+(e_{21}c_-^2-1)\Sigma\right]=\sum_{i=0}^{3}\tilde{A}_i^\infty(1+o_{b\to0}(1))\Sigma^i\end{aligned}$$

where $\tilde{A}_i^\infty$ corresponds to the limiting values at $r=r_\infty(d,\ell)$. We claim

$$\tilde{A}_i^\infty>0 \tag{8.8}$$

which concludes the proof of (8.7) for r close enough to $r^*(d,\ell)$.

step 2 $\tilde{A}_0^\infty$. We compute:

$$\begin{aligned}\tilde{A}_0^\infty &= d_{20}^\infty(c_-^\infty)^2+d_{11}^\infty c_-^\infty+d_{02}^\infty-c_-^\infty(e_{20}^\infty(c_-^\infty)^2+e_{11}^\infty c_-^\infty+e_{02}^\infty)\\ &= -e_{20}^\infty(c_-^\infty)^3+(c_-^\infty)^2(d_{20}^\infty-e_{11}^\infty)+(d_{11}^\infty-e_{02}^\infty)c_-^\infty+d_{02}^\infty=\tilde{e}_{20}^\infty>0\end{aligned}$$

from (F.8), (F.13).

step 3 $\tilde{A}_3^\infty$. We compute:

$$\tilde{A}_3^\infty=(\alpha^\infty)^2\left[(c_-^\infty)^3-dc_-^\infty\right]=-c_-^\infty(\alpha^\infty)^2(d-(c_-^\infty)^2).$$

case $r = r^*$:

$$\begin{aligned} d-(c_-^\infty)^2 &= d - \frac{4\ell^2 d}{\Big[d(\sqrt{d}-1)+\ell(\sqrt{d}+1)\Big]^2} \\ &= \frac{d}{\Big[d(\sqrt{d}-1)+\ell(\sqrt{d}+1)\Big]^2}\left(\Big[d(\sqrt{d}-1)+\ell(\sqrt{d}+1)\Big]^2 - 4\ell^2\right) \\ &= \frac{d\Big[d(\sqrt{d}-1)+\ell(\sqrt{d}+1)-2\ell\Big]\Big[d(\sqrt{d}-1)+\ell(\sqrt{d}+1)+2\ell\Big]}{\Big[d(\sqrt{d}-1)+\ell(\sqrt{d}+1)\Big]^2} \\ &= \frac{d\Big[d(\sqrt{d}-1)+\ell(\sqrt{d}-1)\Big]\Big[d(\sqrt{d}-1)+\ell(\sqrt{d}+3)\Big]}{\Big[d(\sqrt{d}-1)+\ell(\sqrt{d}+1)\Big]^2} > 0 \end{aligned}$$

case $r = r_+$:

$$\tilde{A}_3 = -c_-^\infty(\alpha^\infty)^2(d-(c_-^\infty)^2) = (\alpha^\infty)^2(d-1).$$

Hence $\tilde{A}_3^\infty > 0$.

step 4 $\tilde{A}_1^\infty$. We compute:

$$\begin{aligned} \tilde{A}_1^\infty &= (c_-^\infty)^3 - dc_-^\infty - c_-^\infty(e_{21}^\infty(c_-^\infty)^2 - 1) + 2\alpha^\infty(d_{20}^\infty(c_-^\infty)^2 + d_{11}^\infty c_-^\infty + d_{02}^\infty) \\ &= -(d-1)c_-^\infty - (c_-^\infty)^3\frac{d-1}{\ell} + 2\alpha^\infty(d_{20}^\infty(c_-^\infty)^2 + d_{11}^\infty c_-^\infty + d_{02}^\infty). \end{aligned}$$

case $r = r^*$.

$$\begin{aligned} & d_{20}^\infty c_-^\infty + d_{11}^\infty = \frac{\sqrt{d}+d-\ell}{\ell+\sqrt{d}}\frac{2\ell\sqrt{d}}{d(\sqrt{d}-1)+\ell(\sqrt{d}+1)} - \frac{2d\sqrt{d}}{\ell+\sqrt{d}} \\ = & \frac{2\sqrt{d}}{(\ell+\sqrt{d})(d(\sqrt{d}-1)+\ell(\sqrt{d}+1))}\Big\{-d\Big[d(\sqrt{d}-1)+\ell(\sqrt{d}+1)\Big]+\ell(\sqrt{d}+d-\ell)\Big\} \\ = & \frac{2\sqrt{d}}{(\ell+\sqrt{d})(d(\sqrt{d}-1)+\ell(\sqrt{d}+1))}\Big[-d^2(\sqrt{d}-1)-\ell\sqrt{d}(d-1)-\ell^2\Big] < 0 \end{aligned}$$

It implies

$$\begin{aligned} & d_{20}^\infty(c_-^\infty)^2 + d_{11}^\infty c_-^\infty + d_{02}^\infty = c_-^\infty(d_{20}^\infty c_-^\infty + d_{11}^\infty) - \frac{\ell\sqrt{d}}{\ell+\sqrt{d}} \\ = & -\frac{2\ell\sqrt{d}}{d(\sqrt{d}-1)+\ell(\sqrt{d}+1)}\frac{2\sqrt{d}}{(\ell+\sqrt{d})[d(\sqrt{d}-1)+\ell(\sqrt{d}+1)]}\Big[-d^2(\sqrt{d}-1)-\ell\sqrt{d}(d-1)-\ell^2\Big] \\ - & \frac{\ell\sqrt{d}}{\ell+\sqrt{d}} \\ = & \frac{\ell\sqrt{d}}{\Big[d(\sqrt{d}-1)+\ell(\sqrt{d}+1)\Big]^2(\ell+\sqrt{d})}\Big\{4\sqrt{d}\Big[d^2(\sqrt{d}-1)+\ell\sqrt{d}(d-1)+\ell^2\Big] \\ - & \Big[d(\sqrt{d}-1)+\ell(\sqrt{d}+1)\Big]^2\Big\} \end{aligned}$$

and

$$\begin{aligned} & 4\sqrt{d}\left[d^2(\sqrt{d}-1)+\ell\sqrt{d}(d-1)+\ell^2\right]-\left[d(\sqrt{d}-1)+\ell(\sqrt{d}+1)\right]^2 \\ = \;& d^2(\sqrt{d}-1)[4\sqrt{d}-(\sqrt{d}-1)]+\ell\left[4d(d-1)-2d(d-1)\right]+\ell^2\left[4\sqrt{d}-(d+2\sqrt{d}+1)\right] \\ = \;& d^2(\sqrt{d}-1)(3\sqrt{d}+1)+2d(d-1)\ell-\ell^2(d+1-2\sqrt{d}) \end{aligned}$$

Therefore,

$$\begin{aligned} & d_{20}^\infty(c_-^\infty)^2+d_{11}^\infty c_-^\infty+d_{02}^\infty \\ = \;& \frac{\ell\sqrt{d}[d^2(\sqrt{d}-1)(3\sqrt{d}+1)+2d(d-1)\ell-\ell^2(d+1-2\sqrt{d})]}{\left[d(\sqrt{d}-1)+\ell(\sqrt{d}+1)\right]^2(\ell+\sqrt{d})}. \end{aligned} \tag{8.9}$$

In the numerator, the polynomial in ℓ is increasing on $0<\ell<d$; it is >0 at $\ell=0$, so that it is strictly positive on $0<\ell<d$. Therefore,

$$d_{20}^\infty(c_-^\infty)^2+d_{11}^\infty c_-^\infty+d_{02}^\infty>0.$$

Hence $\tilde{A}_1>0$ near r^*.

<u>case $r=r_+$</u>. We compute

$$\begin{aligned} & d_{11}^\infty-d_{20}^\infty-d_{02}^\infty=\frac{-2d}{1+\sqrt{\ell}}-\frac{\ell-\sqrt{\ell}-d-1}{(1+\sqrt{\ell})^2}+\frac{\ell+d\sqrt{\ell}}{(1+\sqrt{\ell})^2} \\ = \;& \frac{-2d(1+\sqrt{\ell})+\ell+d\sqrt{\ell}-\ell+\sqrt{\ell}+d+1}{(1+\sqrt{\ell})^2}=\frac{\sqrt{\ell}(-d+1)-d+1}{(1+\sqrt{\ell})^2}=\frac{-(d-1)}{1+\sqrt{\ell}} \end{aligned} \tag{8.10}$$

and hence

$$\begin{aligned} \tilde{A}_1^\infty &= -(d-1)c_-^\infty-(c_-^\infty)^3\frac{d-1}{\ell}+2\alpha^\infty(d_{20}^\infty(c_-^\infty)^2+d_{11}^\infty c_-^\infty+d_{02}^\infty) \\ &= d-1+\frac{d-1}{\ell}+2\alpha(d_{20}^\infty-d_{11}^\infty+d_{02}^\infty)>0. \end{aligned}$$

step 5 $\tilde{A}_2$. We compute

$$\begin{aligned} \tilde{A}_2^\infty &= 2\alpha^\infty\left[(c_-^\infty)^3-dc_-^\infty\right]+(\alpha^\infty)^2(d_{20}^\infty(c_-^\infty)^2+d_{11}^\infty c_-^\infty+d_{02}^\infty) \\ &= -2c_-^\infty\alpha^\infty(d-(c_-^\infty)^2)+(\alpha^\infty)^2(d_{20}^\infty(c_-^\infty)^2+d_{11}^\infty c_-^\infty+d_{02}^\infty)>0 \end{aligned}$$

Since $-1\le c_-^\infty<0$, $\alpha^\infty>0$ and $d_{20}^\infty(c_-^\infty)^2+d_{11}^\infty c_-^\infty+d_{02}^\infty>0$, we infer $\tilde{A}_2>0$. This concludes the proof of (8.7). □

8.2. **Formula for F.** The function F given by (1.23) has a special structure.

Lemma 8.3 (Formula for F)**.** *Assume* (1.22)*. Then,*

$$F=-\frac{(d-1)\sigma}{\ell\Delta}(w-w_-)(w-w_+) \tag{8.11}$$

where (w_-,w_+) are the w-coordinates of P_2,P_3.

Proof of Lemma 8.3. We compute

$$\begin{aligned} F &= \sigma + \frac{d\sigma}{dx} = \frac{1}{\Delta}\left[\sigma\Delta - \Delta_2\right] \\ &= \frac{\sigma}{\ell\Delta}\left[\ell(w^2 - 2w + 1 - \sigma^2) - (\ell + d - 1)w^2 + (\ell + d + \ell r - r)w - \ell r + \ell\sigma^2\right] \\ &= -\frac{\sigma}{\ell\Delta}\left[(d-1)w^2 - (d - r + \ell(r-1))w + \ell(r-1)\right] \\ &= -(d-1)\frac{\sigma}{\ell\Delta}(w - w_-)(w - w_+) \end{aligned} \tag{8.12}$$

where (w_-, w_+) are the ordinates of P_2, P_3 from (2.24). □

8.3. **Positivity to the right of P_2.** We claim the following *fundamental* lower bound at the right of P_2.

Lemma 8.4 (Positivity to the right of P_2)**.** *Assume* (1.22)*. Then, for* $\Sigma \geq 0$*:*

$$1 - w - \frac{dw}{dx} - F \geq c > 0. \tag{8.13}$$

Proof of Lemma 8.4. The key here is the sharp bound (8.3).

step 1 Value at P_2. We compute at P_2:

$$\begin{aligned} F &= \sigma + \frac{d\sigma}{dx} = \sigma - \frac{\Delta_2}{\Delta} \\ &= \sigma_2 + \Sigma - \frac{\Sigma\left[c_2\Phi + c_4 + [e_{20}\Phi^2 + e_{11}\Phi + e_{02}]\Sigma + [e_{21}\Phi^2 - 1]\Sigma^2\right]}{-\Sigma\left[2\sigma_2(1+\Phi) + \Sigma(1-\Phi^2)\right]} \end{aligned}$$

and therefore since $\Phi(P_2) = c_-$ and $-1 < c_- < 0$, $c_2 < c_4 < 0$ from (2.49):

$$F(P_2) = \sigma_2 + \frac{c_2 c_- + c_4}{2\sigma_2(1 + c_-)} = \frac{c_-(c_2 - c_4)}{2\sigma_2(1 + c_-)} > 0$$

and

$$1 - w_2 - F(P_2) = -\frac{\lambda_+}{2\sigma_2(1 + c_-)} > 0.$$

Then,

$$\frac{dw}{dx} = -\frac{\Delta_1}{\Delta} = \frac{c_1 c_- + c_3}{2\sigma_2(1 + c_-)} + O(\Sigma) = \frac{c_-\lambda_+}{2\sigma_2(1 + c_-)} + O(\Sigma)$$

and

$$\left(1 - w - \frac{dw}{dx} - F\right)(P_2) = -\frac{\lambda_+}{2\sigma_2(1 + c_-)} - \frac{c_-\lambda_+}{2\sigma_2(1 + c_-)} = -\frac{\lambda_+}{2\sigma_2} > 0. \tag{8.14}$$

step 2 Bad set and no contact condition. We now study the *null* equation:

$$\begin{aligned}
&\Delta\left(1-w-\frac{dw}{dx}-F\right)=0\Leftrightarrow\Delta\left[1-w+\frac{\Delta_1}{\Delta}-\sigma+\frac{\Delta_2}{\Delta}\right]=0\\
\Leftrightarrow\quad&\left[(1-w)^2-\sigma^2\right](1-w-\sigma)+\Delta_1+\Delta_2=0\\
\Leftrightarrow\quad&(1-w)^3-(1-w)^2\sigma-(1-w)\sigma^2+\sigma^3\\
+\quad&w(1-w)(r-w)-d(w-w_e)\sigma^2+\frac{\sigma}{\ell}\left[(\ell+d-1)w^2-w(\ell+d+\ell r-r)+\ell r-\ell\sigma^2\right]=0\\
\Leftrightarrow\quad&(1-w)\left[(1-w)^2+w(r-w)\right]+\left[\frac{(d+\ell-1)w^2}{\ell}-(1-w)^2+r-\frac{w}{\ell}(\ell+d+\ell r-r)\right]\sigma\\
&-\left[d(w-w_e)+1-w\right]\sigma^2=0\\
\Leftrightarrow\quad&\left[d(w-w_e)+1-w\right]\sigma^2-\left[\frac{(d-1)w^2+(\ell+r-d-\ell r)w+\ell(r-1)}{\ell}\right]\sigma\\
&-(1-w)\left[(1-w)^2+w(r-w)\right]=0.
\end{aligned}$$

To the right of P_2, between the red and green curves, we have $w_e<w<w_2<1$. As a result, the coefficients $\alpha(w),\gamma(w)$ of the above quadratic equation

$$\alpha(w)\sigma^2+\beta(w)\sigma-\gamma(w)=0$$

are positive and therefore there is exactly one positive root $\sigma_0(w)$. The dependence of $\sigma_0(w)$ on w is continuous and $\sigma_0(w_2)=\sigma_2$. In the (Σ,W) plane the null set is represented by the continuous null curve $(\Sigma_0(W),W)$ with $W\in[w_e-w_2,0]$.
We now compute the derivative at P_2 recalling (2.41):

$$\begin{aligned}
&(1-w-\sigma_0(w))\Delta+\Delta_1(w,\sigma_0(w))+\Delta_2(w,\sigma_0(w))=0\Rightarrow c_1+\sigma_0'c_3+c_2+\sigma_0'c_4=0\\
\Rightarrow\quad&\sigma_0'(w_2)=-\frac{c_1+c_2}{c_3+c_4}\Rightarrow w_0'(\sigma_2)=-\frac{c_3+c_4}{c_1+c_2}
\end{aligned}$$

and hence the local representation in the (Σ,W) plane of the null set:

$$W_0(\Sigma)=-\frac{c_3+c_4}{c_1+c_2}\Sigma+O(\Sigma^2).$$

On the other hand, from (2.45), (F.3):

$$\begin{aligned}
&-\frac{c_3+c_4}{c_1+c_2}<c_-\Leftrightarrow c_3+c_4<-(c_1+c_2)c_-\Leftrightarrow c_1c_-+c_3+c_2c_-+c_4<0\\
\Leftrightarrow\quad&(1+c_-)\lambda_+<0.
\end{aligned}$$

Therefore, the curve $\Phi=\frac{c_-}{1+\alpha(r)\Sigma}$, which is equivalent to $W=\frac{c_-\Sigma}{1+\alpha(r)\Sigma}$, lies for $\Sigma>0$ small, strictly above the null curve $(\Sigma_0(W),W)$.
We now claim that, if we can show that

$$\forall\Sigma>0,\quad\left(1-w-\frac{dw}{dx}-F\right)\left(\Sigma,\frac{c_-}{1+\alpha(r)\Sigma}\right)>0,\tag{8.15}$$

it will imply that $(1-w-w'-F)>0\ \forall\Sigma\geq0$ along the solution curve. We argue by contradiction. First, using (8.14), we could find a positive value Σ_0 such that $(1-w-w'-F)\left(\Sigma_0,W(\Sigma_0)\right)=0$. Therefore, the point $(\Sigma_0,W(\Sigma_0))$ belongs to the null set. Since along the solution curve $w_e<w(\sigma)<w_2$ for any $\sigma>\sigma_2$, the point $(\Sigma_0,W(\Sigma_0))$ must lie on the null curve $(\Sigma_0(W),W)$. That is $\Sigma_0(W(\Sigma_0))=\Sigma_0$. We now follow this curve from the value $W(\Sigma_0)$ to $W=0$. For W sufficiently small the null curve lies below the curve $W=\frac{c_-\Sigma}{1+\alpha(r)\Sigma}$, while at the point $(\Sigma_0,W(\Sigma_0))$, which belongs to the solution curve, it must be above $W=\frac{c_-\Sigma}{1+\alpha(r)\Sigma}$. Therefore, the curve

$W = \frac{c_-\Sigma}{1+\alpha(r)\Sigma}$ and the null curve must intersect, which is impossible in view of the claim (8.15).
From now on, we focus on the proof of (8.15).

step 3 Computation of $1 - w - w'$. We compute in (W, Σ) coordinates:

$$\begin{aligned}
& \Delta(1-w-w') = \Delta(1-w) + \Delta_1 = -(\sigma_2 - W)\Sigma\left[2\sigma_2(1+\Phi) + \Sigma(1-\Phi^2)\right] \\
+ \;& \Sigma\left[c_1\Phi + c_3 + [d_{20}\Phi^2 + d_{11}\Phi + d_{02}]\Sigma + [\Phi^3 - d\Phi]\Sigma^2\right] \\
= \;& \Sigma\Big\{ - \sigma_2\left[2\sigma_2(1+\Phi) + \Sigma(1-\Phi^2)\right] + \Phi\Sigma\left[2\sigma_2(1+\Phi) + \Sigma(1-\Phi^2)\right] \\
+ \;& c_1\Phi + c_3 + [d_{20}\Phi^2 + d_{11}\Phi + d_{02}]\Sigma + [\Phi^3 - d\Phi]\Sigma^2\Big\} \\
= \;& \Sigma\left\{B_0 + B_1\Sigma + B_2\Sigma^2\right\}
\end{aligned}$$

with

$$\left|\begin{array}{l} B_0 = c_1\Phi + c_3 - 2\sigma_2^2(1+\Phi) \\ B_1 = d_{20}\Phi^2 + d_{11}\Phi + d_{02} - \sigma_2(1-\Phi^2) + 2\sigma_2\Phi(1+\Phi) \\ B_2 = \Phi^3 - d\Phi + \Phi(1-\Phi^2) = -(d-1)\Phi \end{array}\right.$$

Let

$$K(\Sigma, \Phi) = B_0(\Phi) + B_1(\Phi)\Sigma + B_2(\Phi)\Sigma^2,$$

then we have obtained:

$$1 - w - w' = \frac{\Delta(1-w) + \Delta_1}{\Delta} = \frac{\Sigma K(\Sigma, \Phi)}{\Delta}.$$

To ease the notations, we let $\alpha = \alpha(r)$ and compute

$$\begin{aligned}
& K\left(\Sigma, \frac{c_-}{1+\alpha\Sigma}\right) = \frac{c_1c_-}{1+\alpha\Sigma} + c_3 - 2\sigma_2^2\left(1 + \frac{c_-}{1+\alpha\Sigma}\right) \\
+ \;& \left[(d_{20} + 3\sigma_2)\left(\frac{c_-}{1+\alpha\Sigma}\right)^2 + (d_{11} + 2\sigma_2)\left(\frac{c_-}{1+\alpha\Sigma}\right) + d_{02} - \sigma_2\right]\Sigma \\
- \;& (d-1)\frac{c_-}{1+\alpha\Sigma}\Sigma^2 \\
= \;& \frac{1}{(1+\alpha\Sigma)^2}\Big\{(c_1c_- - 2\sigma_2^2c_-)(1+\alpha\Sigma) + (c_3 - 2\sigma_2^2)(1+\alpha\Sigma)^2 \\
+ \;& (d_{20} + 3\sigma_2)c_-^2\Sigma + c_-(d_{11} + 2\sigma_2)\Sigma(1+\alpha\Sigma) + (d_{02} - \sigma_2)\Sigma(1+\alpha\Sigma)^2 \\
- \;& (d-1)c_-\Sigma^2(1+\alpha\Sigma)\Big\} = \frac{\sum_{i=0}^3 \tilde{B}_i\Sigma^i}{(1+\alpha\Sigma)^2}
\end{aligned}$$

which yields the formula:

$$(1 - w - w')\left(\Sigma, \frac{c_-}{1+\alpha\Sigma}\right) = \frac{\Sigma(\sum_{i=0}^3 \tilde{B}_i\Sigma^i)}{\Delta(1+\alpha\Sigma)^2}. \tag{8.16}$$

We now compute the coefficients $\tilde{B}_i$. For $\tilde{B}_0$, we need to keep the exact structure at r, but for $\tilde{B}_{1,2,3}$ is suffices to compute the approximate values at $r = r_\infty(d, \ell)$ which is done below.

step 4 $\tilde{B}_0$. We compute for all r:

$$\tilde{B}_0 \;=\; c_1c_- - 2\sigma_2^2c_- + c_3 - 2\sigma_2^2 = c_-\lambda_+ - 2\sigma_2^2(1+c_-).$$

step 5 $\tilde{B}_1^\infty$. We compute

$$\begin{aligned}\tilde{B}_1^\infty &= (c_1^\infty c_-^\infty - 2(\sigma_2^\infty)^2 c_-^\infty)\alpha^\infty + 2\alpha^\infty(c_3^\infty - 2(\sigma_2^\infty)^2) + (c_-^\infty)^2(d_{20}^\infty + 3\sigma_2^\infty)\\ &+ c_-^\infty(d_{11}^\infty + 2\sigma_2^\infty) + d_{02}^\infty - \sigma_2^\infty.\end{aligned}$$

<u>case $r = r^*$</u>. We compute using $c_1^\infty c_-^\infty + c_3^\infty = 0$:

$$\begin{aligned}&c_1^\infty c_-^\infty - 2(\sigma_2^\infty)^2 c_-^\infty + 2(c_3^\infty - 2(\sigma_2^\infty)^2) = c_3^\infty - 2(\sigma_2^\infty)^2(2 + c_-^\infty)\\ =\ & c_3^\infty - 2(\sigma_2^\infty)^2 - 2(\sigma_2^\infty)^2(1 + c_-^\infty)\\ =\ & -\frac{2d(\ell+1)}{(\ell+\sqrt{d})^2} - 2\sigma_2^2(1 + c_-) = -\frac{2d(\ell+1)}{(\ell+\sqrt{d})^2} - \frac{2d}{(\ell+\sqrt{d})^2}\frac{(\sqrt{d}-1)(d-\ell)}{d(\sqrt{d}-1)+\ell(\sqrt{d}+1)}\\ =\ & -\frac{2d}{(\ell+\sqrt{d})^2\left[d(\sqrt{d}-1)+\ell(\sqrt{d}+1)\right]}\left\{(\ell+1)[d(\sqrt{d}-1)+\ell(\sqrt{d}+1)] + (d-\ell)(\sqrt{d}-1)\right\}\\ =\ & -\frac{2d}{(\ell+\sqrt{d})^2\left[d(\sqrt{d}-1)+\ell(\sqrt{d}+1)\right]}\left\{2d(\sqrt{d}-1) + (d\sqrt{d}-d+2)\ell + (\sqrt{d}+1)\ell^2\right\}\end{aligned}$$

and hence

$$\begin{aligned}&(c_1^\infty c_-^\infty - 2(\sigma_2^\infty)^2 c_-^\infty)\alpha^\infty + 2\alpha^\infty(c_3^\infty - 2(\sigma_2^\infty)^2)\\ =\ & -\frac{2d}{(\ell+\sqrt{d})^2\left[d(\sqrt{d}-1)+\ell(\sqrt{d}+1)\right]}\left\{2d(\sqrt{d}-1) + (d\sqrt{d}-d+2)\ell + (\sqrt{d}+1)\ell^2\right\}\\ \times\ & \frac{2d(\ell+\sqrt{d})}{d(\sqrt{d}-1)+\ell(\sqrt{d}+1)} = -\frac{4d^2\left\{2d(\sqrt{d}-1) + (d\sqrt{d}-d+2)\ell + (\sqrt{d}+1)\ell^2\right\}}{(\ell+\sqrt{d})\left[d(\sqrt{d}-1)+\ell(\sqrt{d}+1)\right]^2}.\end{aligned}$$

Then,

$$\begin{aligned}&3(c_-^\infty)^2\sigma_2^\infty + 2\sigma_2^\infty c_-^\infty - \sigma_2^\infty = \sigma_2^\infty(3(c_-^\infty)^2 + 2c_-^\infty - 1) = \sigma_2^\infty(1 + c_-^\infty)(3c_-^\infty - 1)\\ =\ & -\frac{\sqrt{d}}{\ell+\sqrt{d}}\frac{(\sqrt{d}-1)(d-\ell)}{d(\sqrt{d}-1)+\ell(\sqrt{d}+1)}\left(1 + \frac{6\ell\sqrt{d}}{d(\sqrt{d}-1)+\ell(\sqrt{d}+1)}\right)\\ =\ & -\frac{\sqrt{d}(\sqrt{d}-1)(d-\ell)\left[d(\sqrt{d}-1)+\ell(7\sqrt{d}+1)\right]}{(\ell+\sqrt{d})\left[d(\sqrt{d}-1)+\ell(\sqrt{d}+1)\right]^2}.\end{aligned}$$

Now, recalling (8.9):

$$\tilde{B}_1^\infty = -\frac{P_{\tilde{B}_1}(\ell)}{(\ell+\sqrt{d})\left[d(\sqrt{d}-1)+\ell(\sqrt{d}+1)\right]^2}$$

with

$$\begin{aligned}&P_{\tilde{B}_1}(\ell) = 4d^2\left\{2d(\sqrt{d}-1) + (d\sqrt{d}-d+2)\ell + (\sqrt{d}+1)\ell^2\right\}\\ +\ & \sqrt{d}(\sqrt{d}-1)(d-\ell)\left[d(\sqrt{d}-1)+\ell(7\sqrt{d}+1)\right]\\ -\ & \ell\sqrt{d}[d^2(\sqrt{d}-1)(3\sqrt{d}+1) + 2d(d-1)\ell - \ell^2(d+1-2\sqrt{d})]\\ =\ & \sqrt{d}(\sqrt{d}-1)(d-\ell)\left[d(\sqrt{d}-1)+\ell(7\sqrt{d}+1)\right]\\ +\ & 8d^3(\sqrt{d}-1) + \ell\left[d^3\sqrt{d} - 2d^3 + d^2\sqrt{d} + 8d^2\right] + \ell^2\left[2d^2\sqrt{d} + 4d^2 + 2d\sqrt{d}\right] + \ell^3\sqrt{d}(d+1-2\sqrt{d})\\ >\ & 0\end{aligned}$$

and thus $\tilde{B}_1^\infty < 0$. We can fully expand:

$$(d-\ell)\left[d(\sqrt{d}-1)+\ell(7\sqrt{d}+1)\right] = d^2(\sqrt{d}-1)+(6d\sqrt{d}+2d)\ell-(7\sqrt{d}+1)\ell^2$$

and

$$\begin{aligned}
& P_{\tilde{B}_1}(\ell) = 8d^3(\sqrt{d}-1)+\sqrt{d}(\sqrt{d}-1)d^2(\sqrt{d}-1)\\
+ & \left[d^3\sqrt{d}-2d^3+d^2\sqrt{d}+8d^2+\sqrt{d}(\sqrt{d}-1)(6d\sqrt{d}+2d)\right]\ell\\
+ & \left[2d^2\sqrt{d}+4d^2+2d\sqrt{d}-\sqrt{d}(\sqrt{d}-1)(7\sqrt{d}+1)\right]\ell^2\\
+ & \sqrt{d}(d+1-2\sqrt{d})\ell^3\\
= & 9d^3\sqrt{d}-10d^3+d^2\sqrt{d}+\left[d^3\sqrt{d}-2d^3+7d^2\sqrt{d}+4d^2-2d\sqrt{d}\right]\ell\\
+ & \left[2d^2\sqrt{d}+4d^2-5d\sqrt{d}+6d+\sqrt{d}\right]\ell^2+\left[d\sqrt{d}-2d+\sqrt{d}\right]\ell^3.
\end{aligned}$$

<u>case $r = r_+$</u>. We compute using $c_1^\infty c_-^\infty + c_3^\infty = 0$:

$$\begin{aligned}
& c_1^\infty c_-^\infty - 2(\sigma_2^\infty)^2 c_-^\infty + 2(c_3^\infty - 2(\sigma_2^\infty)^2) = c_3^\infty - 2(\sigma_2^\infty)^2(2+c_-^\infty)\\
= & c_3^\infty - 2(\sigma_2^\infty)^2 - 2(\sigma_2^\infty)^2(1+c_-^\infty)\\
= & c_3^\infty - 2(\sigma_2^\infty)^2 = -\frac{2\sqrt{\ell}(d+\sqrt{\ell})}{(1+\sqrt{\ell})^3} - \frac{2}{(1+\sqrt{\ell})^2} = -\frac{2[\sqrt{\ell}(d+\sqrt{\ell})+1+\sqrt{\ell}]}{(1+\sqrt{\ell})^3}\\
= & \frac{-2\ell-2(d+1)\sqrt{\ell}-2}{(1+\sqrt{\ell})^3}
\end{aligned}$$

and hence

$$\begin{aligned}
& (c_1^\infty c_-^\infty - 2(\sigma_2^\infty)^2 c_-^\infty)\alpha^\infty + 2\alpha^\infty(c_3^\infty - 2(\sigma_2^\infty)^2) = \frac{-2\ell-2(d+1)\sqrt{\ell}-2}{(1+\sqrt{\ell})^3}\left(\frac{d(1+\sqrt{\ell})^2}{d\sqrt{\ell}+\ell}\right)\\
= & \frac{-2d\ell-2d(d+1)\sqrt{\ell}-2d}{(1+\sqrt{\ell})(d\sqrt{\ell}+\ell)}.
\end{aligned}$$

Then

$$3(c_-^\infty)^2\sigma_2^\infty + 2\sigma_2^\infty c_-^\infty - \sigma_2^\infty = \sigma_2^\infty(3(c_-^\infty)^2 + 2c_-^\infty - 1) = 0.$$

Finally recalling (8.10):

$$(c_-^\infty)^2 d_{20}^\infty + c_-^\infty d_{11}^\infty + d_{02}^\infty = d_{20}^\infty - d_{11}^\infty + d_{02}^\infty = \frac{d-1}{1+\sqrt{\ell}}$$

and hence

$$\begin{aligned}
\tilde{B_1^\infty} &= \frac{-2d\ell-2d(d+1)\sqrt{\ell}-2d}{(1+\sqrt{\ell})(d\sqrt{\ell}+\ell)} + \frac{d-1}{1+\sqrt{\ell}} = \frac{-2d\ell-2d(d+1)\sqrt{\ell}-2d+(d-1)(d\sqrt{\ell}+\ell)}{(1+\sqrt{\ell})(d\sqrt{\ell}+\ell)}\\
&= -\frac{(d+1)\ell+(d^2+3d)\sqrt{\ell}+2d}{(1+\sqrt{\ell})(d\sqrt{\ell}+\ell)} < 0.
\end{aligned}$$

step 6 $\tilde{B}_2$.
case $r = r^*$. We compute at $r = r^*(d,\ell)$:

$$\begin{aligned}
& \tilde{B}_2^\infty = (c_3^\infty - 2(\sigma_2^\infty)^2)(\alpha^\infty)^2 + c_-^\infty \alpha^\infty (d_{11}^\infty + 2\sigma_2^\infty) + 2\alpha^\infty (d_{02}^\infty - \sigma_2^\infty) - (d-1)c_-^\infty \\
= & \left(\frac{-2\ell d}{(\ell+\sqrt{d})^2} - \frac{2d}{(\ell+\sqrt{d})^2}\right)\alpha^2 + c_-\alpha\left[-\frac{2d\sqrt{d}}{\ell+\sqrt{d}} + \frac{2\sqrt{d}}{\ell+\sqrt{d}}\right] + 2\alpha\left[-\frac{(\ell+1)\sqrt{d}}{\ell+\sqrt{d}}\right] - (d-1)c_- \\
= & -\frac{(2d+2\ell d)4d^2}{\left[d(\sqrt{d}-1)+\ell(\sqrt{d}+1)\right]^2} - \frac{2\alpha\sqrt{d}}{\ell+\sqrt{d}}\left[\ell+1+c_-(d-1)\right] - (d-1)c_- \\
= & -\frac{(2d+2\ell d)4d^2}{\left[d(\sqrt{d}-1)+\ell(\sqrt{d}+1)\right]^2} - \frac{4d\sqrt{d}}{d(\sqrt{d}-1)+\ell(\sqrt{d}+1)}(\ell+1+c_-(d-1)) - (d-1)c_- \\
= & -\frac{(2d+2\ell d)4d^2}{\left[d(\sqrt{d}-1)+\ell(\sqrt{d}+1)\right]^2} - \frac{4d\sqrt{d}}{d(\sqrt{d}-1)+\ell(\sqrt{d}+1)}\left(\ell+1-\frac{2\ell(d-1)\sqrt{d}}{d(\sqrt{d}-1)+\ell(\sqrt{d}+1)}\right) \\
+ & \frac{2\ell\sqrt{d}(d-1)}{d(\sqrt{d}-1)+\ell(\sqrt{d}+1)} = -\frac{P_{\tilde{B}_2}(\ell)}{\left[d(\sqrt{d}-1)+\ell(\sqrt{d}+1)\right]^2}
\end{aligned}$$

with

$$\begin{aligned}
& P_{\tilde{B}_2}(\ell) = 8d^3(d+\ell) + 4d\sqrt{d}\left[(\ell+1)\left[d(\sqrt{d}-1)+\ell(\sqrt{d}+1)\right] - 2\ell(d-1)\sqrt{d}\right] \\
- & 2\ell\sqrt{d}(d-1)\left[d(\sqrt{d}-1)+\ell(\sqrt{d}+1)\right] \\
= & 8d^4 + 4d^2(d-\sqrt{d}) + \left[2d^3 - 2d^2\sqrt{d} + 14d^2 + 2d\sqrt{d}\right]\ell + \left[2d^2 + 2d\sqrt{d} + 2d + 2\sqrt{d}\right]\ell^2 \\
> & 0
\end{aligned}$$

which implies $P_{\tilde{B}_2}(\ell) > 0$ and $\tilde{B}_2^\infty < 0$.

case $r = r_+$. We compute at $r = r_+(\ell)$:

$$\begin{aligned}
& \tilde{B}_2^\infty = (c_3^\infty - 2(\sigma_2^\infty)^2)(\alpha^\infty)^2 + c_-^\infty \alpha^\infty (d_{11}^\infty + 2\sigma_2^\infty) + 2\alpha^\infty (d_{02}^\infty - \sigma_2^\infty) - (d-1)c_-^\infty \\
= & (c_3^\infty - 2(\sigma_2^\infty)^2)(\alpha^\infty)^2 + (2d_{02}^\infty - d_{11}^\infty - 4\sigma_2^\infty)\alpha^\infty + d - 1 \\
= & \left[-\frac{2\sqrt{\ell}(d+\sqrt{\ell})}{(1+\sqrt{\ell})^3} - \frac{2}{(1+\sqrt{\ell})^2}\right]\left(\frac{d(1+\sqrt{\ell})^2}{d\sqrt{\ell}+\ell}\right)^2 \\
+ & \left[2\left(\frac{-\ell-d\sqrt{\ell}}{(1+\sqrt{\ell})^2}\right) + \frac{2d-4}{1+\sqrt{\ell}}\right]\frac{d(1+\sqrt{\ell})^2}{d\sqrt{\ell}+\ell} + d - 1 \\
= & -\frac{d^2(1+\sqrt{\ell})}{(d\sqrt{\ell}+\ell)^2}\left[2\sqrt{\ell}(d+\sqrt{\ell}) + 2(1+\sqrt{\ell})\right] + \frac{d}{d\sqrt{\ell}+\ell}\left[-2\ell - 2d\sqrt{\ell} + (2d-4)(1+\sqrt{\ell})\right] + d - 1 \\
= & -\frac{d^2(1+\sqrt{\ell})[2\ell + (2d+2)\sqrt{\ell} + 2]}{(d\sqrt{\ell}+\ell)^2} + \frac{d(-2\ell - 4\sqrt{\ell} + 2d - 4)}{d\sqrt{\ell}+\ell} + d - 1 \\
= & -\frac{P_{\tilde{B}_2}(\ell)}{(d\sqrt{\ell}+\ell)^2}
\end{aligned}$$

with

$$\begin{aligned} & P_{\tilde{B}_2}(\ell) = d^2(1+\sqrt{\ell})[2\ell + (2d+2)\sqrt{\ell}+2] + d(2\ell + 4\sqrt{\ell} - 2d + 4)(d\sqrt{\ell}+\ell) - (d-1)(d\sqrt{\ell}+\ell)^2 \\ = \;& (d+1)\ell^2 + (2d^2+6d)\ell\sqrt{\ell} + (d^3+7d^2+4d)\ell + (8d^2)\sqrt{\ell} + 2d^2 \end{aligned}$$

and hence $P_{\tilde{B}_2}(\ell) > 0$ and $\tilde{B}_2^\infty < 0$.

step 7 $\tilde{B}_3$.
<u>case $r = r^*$</u>. We compute at $r = r^*(d,\ell)$:

$$\begin{aligned} \tilde{B}_3 &= (\alpha^\infty)^2(d_{02}^\infty - \sigma_2^\infty) - (d-1)c_-^\infty\alpha^\infty = \alpha^\infty\left[\frac{2d(\ell+\sqrt{d})}{d(\sqrt{d}-1)+\ell(\sqrt{d}+1)}\left(-\frac{\ell\sqrt{d}}{\ell+\sqrt{d}} - \frac{\sqrt{d}}{\ell+\sqrt{d}}\right)\right. \\ &+ \left.(d-1)\frac{2\ell\sqrt{d}}{d(\sqrt{d}-1)+\ell(\sqrt{d}+1)}\right] = -\frac{2\alpha\sqrt{d}}{d(\sqrt{d}-1)+\ell(\sqrt{d}+1)}\left[d(\ell+1)-(d-1)\ell\right] \\ &= -\frac{2\alpha(d+\ell)\sqrt{d}}{d(\sqrt{d}-1)+\ell(\sqrt{d}+1)} < 0. \end{aligned}$$

<u>case $r = r_+$</u>. We compute at $r = r_+(\ell)$:

$$\begin{aligned} \tilde{B}_3 &= (\alpha^\infty)^2(d_{02}^\infty - \sigma_2^\infty) - (d-1)c_-^\infty\alpha^\infty \\ &= \alpha\left[\left(\frac{d(1+\sqrt{\ell})^2}{d\sqrt{\ell}+\ell}\right)\left(\frac{-\ell - d\sqrt{\ell}}{(1+\sqrt{\ell})^2} - \frac{1}{1+\sqrt{\ell}}\right) + d - 1\right] \\ &= \alpha\left\{\frac{d(1+\sqrt{\ell})^2}{d\sqrt{\ell}+\ell}\left(\frac{-\ell - d\sqrt{\ell} - 1 - \sqrt{\ell}}{(1+\sqrt{\ell})^2}\right) + d - 1\right\} \\ &= \frac{\alpha}{d\sqrt{\ell}+\ell}\left[d\left(-\ell - (d+1)\sqrt{\ell} - 1\right) + (d-1)(d\sqrt{\ell}+\ell)\right] \\ &= \frac{\alpha}{d\sqrt{\ell}+\ell}[-\ell - 2d\sqrt{\ell} - d] = -\frac{d(1+\sqrt{\ell})^2(\ell + 2d\sqrt{\ell} + d)}{(d\sqrt{\ell}+\ell)^2} < 0 \end{aligned}$$

step 7 Computation of F.

$$\begin{aligned} & F = \sigma + \sigma' = \sigma - \frac{\Delta_2}{\Delta} = \frac{\sigma\Delta - \Delta_2}{\Delta} \\ = \;& -\frac{\Sigma}{\Delta}\left\{(\sigma_2+\Sigma)[2\sigma_2(1+\Phi) + \Sigma(1-\Phi^2)] + c_2\Phi + c_4 + [e_{20}\Phi^2 + e_{11}\Phi + e_{02}]\Sigma + [e_{21}\Phi^2 - 1]\Sigma^2\right\} \\ = \;& -\frac{\Sigma}{\Delta}\Big\{2\sigma_2^2(1+\Phi) + c_2\Phi + c_4 \\ + \;& \left[2\sigma_2(1+\Phi) + \sigma_2(1-\Phi^2) + e_{20}\Phi^2 + e_{11}\Phi + e_{02}\right]\Sigma + \left[1 - \Phi^2 + e_{21}\Phi^2 - 1\right]\Sigma^2\Big\} \\ = \;& \frac{\Sigma\Big\{(2\sigma_2^2 + c_2)\Phi + \left[(2\sigma_2 + e_{11})\Phi + (e_{20} - \sigma_2)\Phi^2\right]\Sigma + (e_{21}-1)\Phi^2\Sigma^2\Big\}}{(-\Delta)} \end{aligned}$$

where we used that for all r,

$$c_4 = -2\sigma_2^2, \quad e_{02} = -3\sigma_2.$$

We therefore obtain

$$\begin{aligned} & F\left(\Sigma, \frac{c_-}{1+\alpha\Sigma}\right) \\ = \ & \frac{\Sigma}{-\Delta}\Bigg\{(2\sigma_2^2+c_2)\frac{c_-}{1+\alpha\Sigma} + \left[(2\sigma_2+e_{11})\frac{c_-}{1+\alpha\Sigma} + (e_{20}-\sigma_2)\left(\frac{c_-}{1+\alpha\Sigma}\right)^2\right]\Sigma \\ + \ & (e_{21}-1)\left(\frac{c_-}{1+\alpha\Sigma}\right)^2\Sigma^2\Bigg\} \\ = \ & \frac{\Sigma}{-\Delta(1+\alpha\Sigma)^2}\Big\{(2\sigma_2^2+c_2)c_-(1+\alpha\Sigma) + (2\sigma_2+e_{11})c_-\Sigma(1+\alpha\Sigma) \\ + \ & (e_{20}-\sigma_2)(c_-)^2\Sigma + (e_{21}-1)(c_-)^2\Sigma^2\Big\} = \frac{\Sigma\sum_{i=0}^2 A_i\Sigma^i}{(-\Delta)(1+\alpha\Sigma)^2}. \end{aligned} \tag{8.17}$$

As for the $\tilde{B}_i$, we compute A_0 for all r but $A_{1,2}$ at $r = r^*(d,\ell)$ only.

step 8 A_0. We compute:

$$A_0 = (2\sigma_2^2+c_2)c_-.$$

step 9 A_1^∞. We compute

$$A_1^\infty = \alpha^\infty(2(\sigma_2^\infty)^2+c_2^\infty)c_-^\infty + (2\sigma_2^\infty+e_{11}^\infty)c_-^\infty + (e_{20}^\infty-\sigma_2^\infty)(c_-^\infty)^2$$

<u>case $r = r^*(d,\ell)$</u>. First,

$$\begin{aligned} & 2\sigma_2^\infty + e_{11}^\infty = \frac{2\sqrt{d}}{\ell+\sqrt{d}} - \frac{d(\sqrt{d}-1)+\ell(1+\sqrt{d})}{\ell(\ell+\sqrt{d})} = \frac{2\ell\sqrt{d}-d(\sqrt{d}-1)-\ell(1+\sqrt{d})}{\ell(\ell+\sqrt{d})} \\ = \ & \frac{-d(\sqrt{d}-1)+\ell(\sqrt{d}-1)}{\ell(\ell+\sqrt{d})} = -\frac{(\sqrt{d}-1)(d-\ell)}{\ell(\ell+\sqrt{d})} \end{aligned}$$

and

$$\begin{aligned} & \alpha^\infty(2(\sigma_2^\infty)^2+c_2^\infty) \\ = \ & \frac{2d(\ell+\sqrt{d})}{d(\sqrt{d}-1)+\ell(\sqrt{d}+1)}\left\{\frac{2d}{(\ell+\sqrt{d})^2} - \frac{\sqrt{d}}{\ell(\ell+\sqrt{d})^2}\left[d(\sqrt{d}-1)+\ell(\sqrt{d}+1)\right]\right\} \\ = \ & \frac{2d\sqrt{d}}{\ell(\ell+\sqrt{d})\left[d(\sqrt{d}-1)+\ell(\sqrt{d}+1)\right]}\left[2\ell\sqrt{d} - \Big(d(\sqrt{d}-1)+\ell(\sqrt{d}+1)\Big)\right] \\ = \ & -\frac{2d\sqrt{d}(\sqrt{d}-1)(d-\ell)}{\ell(\ell+\sqrt{d})\left[d(\sqrt{d}-1)+\ell(\sqrt{d}+1)\right]} \end{aligned}$$

As a result,

$$\begin{aligned}
&\alpha^\infty(2(\sigma_2^\infty)^2+c_2^\infty)c_-^\infty+(2\sigma_2^\infty+e_{11}^\infty)c_-^\infty\\
=&\left[-\frac{2d\sqrt{d}(\sqrt{d}-1)(d-\ell)}{\ell(\ell+\sqrt{d})\left[d(\sqrt{d}-1)+\ell(\sqrt{d}+1)\right]}-\frac{(\sqrt{d}-1)(d-\ell)}{\ell(\ell+\sqrt{d})}\right]c_-\\
=&\left[\frac{2\ell\sqrt{d}}{d(\sqrt{d}-1)+\ell(\sqrt{d}+1)}\right]\frac{(\sqrt{d}-1)(d-\ell)[2d\sqrt{d}+d(\sqrt{d}-1)+\ell(\sqrt{d}+1)]}{\ell(\ell+\sqrt{d})\left[d(\sqrt{d}-1)+\ell(\sqrt{d}+1)\right]}\\
=&\frac{2\sqrt{d}(\sqrt{d}-1)(d-\ell)[3d\sqrt{d}-d+\ell(\sqrt{d}+1)]}{(\ell+\sqrt{d})\left[d(\sqrt{d}-1)+\ell(\sqrt{d}+1)\right]^2}.
\end{aligned}$$

We expand

$$\begin{aligned}
&(d-\ell)[3d\sqrt{d}-d+\ell(\sqrt{d}+1)]=d^2(3\sqrt{d}-1)+\ell(d\sqrt{d}+d-3d\sqrt{d}+d)-(\sqrt{d}+1)\ell^2\\
=&\ d^2(3\sqrt{d}-1)+(-2d\sqrt{d}+2d)\ell-(\sqrt{d}+1)\ell^2
\end{aligned}$$

and obtain

$$\begin{aligned}
&\alpha^\infty(2(\sigma_2^\infty)^2+c_2^\infty)c_-^\infty+(2\sigma_2^\infty+e_{11}^\infty)c_-^\infty\\
=&\frac{2\sqrt{d}(\sqrt{d}-1)\left[d^2(3\sqrt{d}-1)+(-2d\sqrt{d}+2d)\ell-(\sqrt{d}+1)\ell^2\right]}{(\ell+\sqrt{d})\left[d(\sqrt{d}-1)+\ell(\sqrt{d}+1)\right]^2}\\
=&\frac{6d^3\sqrt{d}-8d^3+2d^2\sqrt{d}+\left[-4d^2\sqrt{d}+8d^2-4d\sqrt{d}\right]\ell-2\sqrt{d}(d-1)\ell^2}{(\ell+\sqrt{d})\left[d(\sqrt{d}-1)+\ell(\sqrt{d}+1)\right]^2}.
\end{aligned}$$

We now add

$$\begin{aligned}
&(e_{20}^\infty-\sigma_2^\infty)(c_-^\infty)^2=\left(\frac{\sqrt{d}(d-1+\ell)}{\ell(\ell+\sqrt{d})}-\frac{\sqrt{d}}{\ell+\sqrt{d}}\right)c_-^2=\frac{\sqrt{d}(d-1)}{\ell(\ell+\sqrt{d})}\left(\frac{2\ell\sqrt{d}}{d(\sqrt{d}-1)+\ell(\sqrt{d}+1)}\right)^2\\
=&\frac{4\ell d\sqrt{d}(d-1)}{(\ell+\sqrt{d})\left[d(\sqrt{d}-1)+\ell(\sqrt{d}+1)\right]^2}
\end{aligned}$$

and get the formula:

$$A_1^\infty=\frac{P_{A_1}}{(\ell+\sqrt{d})\left[d(\sqrt{d}-1)+\ell(\sqrt{d}+1)\right]^2}$$

with

$$\begin{aligned}
P_{A_1}&=6d^3\sqrt{d}-8d^3+2d^2\sqrt{d}+\left[-4d^2\sqrt{d}+8d^2-4d\sqrt{d}+4d^2\sqrt{d}-4d\sqrt{d}\right]\ell-2\sqrt{d}(d-1)\ell^2\\
&=6d^3\sqrt{d}-8d^3+2d^2\sqrt{d}+\left[8d^2-8d\sqrt{d}\right]\ell-2\sqrt{d}(d-1)\ell^2
\end{aligned}$$

<u>case $r=r_+(d,\ell)$</u>. Recall

$$A_1^\infty=\alpha^\infty(2(\sigma_2^\infty)^2+c_2^\infty)c_-^\infty+(2\sigma_2^\infty+e_{11}^\infty)c_-^\infty+(e_{20}^\infty-\sigma_2^\infty)(c_-^\infty)^2.$$

Observe

$$2\sigma_2^\infty+e_{11}^\infty=\frac{2}{1+\sqrt{\ell}}-\frac{2}{1+\sqrt{\ell}}=0$$

and

$$2(\sigma_2^\infty)^2 + c_2^\infty = \frac{2}{(1+\sqrt{\ell})^2} - \frac{2}{(1+\sqrt{\ell})^2} = 0$$

and hence

$$A_1^\infty = e_{20}^\infty - \sigma_2^\infty = \frac{\ell+d-1}{\ell(1+\sqrt{\ell})} - \frac{1}{1+\sqrt{\ell}} = \frac{d-1}{\ell(1+\sqrt{\ell})}.$$

step 10 A_2^∞. We have

$$A_2^\infty = \alpha^\infty(2\sigma_2^\infty + e_{11}^\infty)c_-^\infty + (e_{21}^\infty - 1)(c_-^\infty)^2.$$

case $r = r^*(d,\ell)$. First,

$$(e_{21}^\infty - 1)(c_-^\infty)^2 = \left(\frac{\ell+d-1}{\ell} - 1\right)\left(\frac{2\ell\sqrt{d}}{d(\sqrt{d}-1)+\ell(\sqrt{d}+1)}\right)^2 = \frac{4\ell d(d-1)}{[d(\sqrt{d}-1)+\ell(\sqrt{d}+1)]^2}$$

and

$$\begin{aligned} & \alpha^\infty(2\sigma_2^\infty + e_{11}^\infty)c_-^\infty \\ = \ & \frac{2d(\ell+\sqrt{d})}{d(\sqrt{d}-1)+\ell(\sqrt{d}+1)}\left[-\frac{(\sqrt{d}-1)(d-\ell)}{\ell(\ell+\sqrt{d})}\right]\left[-\frac{2\ell\sqrt{d}}{d(\sqrt{d}-1)+\ell(\sqrt{d}+1)}\right] \\ = \ & \frac{4d\sqrt{d}(\sqrt{d}-1)(d-\ell)}{[d(\sqrt{d}-1)+\ell(\sqrt{d}+1)]^2} \end{aligned}$$

which implies

$$A_2^\infty = \frac{P_{A_2}}{\left[d(\sqrt{d}-1)+\ell(\sqrt{d}+1)\right]^2}$$

with

$$P_{A_2} = 4\ell d(d-1) + 4d\sqrt{d}(\sqrt{d}-1)(d-\ell) = 4d^2(d-\sqrt{d}) + \ell(4d\sqrt{d}-4d).$$

case $r = r_+(d,\ell)$. Recall

$$A_2 = \alpha(2\sigma_2 + e_{11})c_- + (e_{21}-1)c_-^2$$

and hence

$$A_2^\infty = \alpha^\infty\left[\frac{2}{1+\sqrt{\ell}} - \frac{2}{1+\sqrt{\ell}}\right] + \frac{\ell+d-1}{\ell} - 1 = \frac{d-1}{\ell}.$$

step 11 Conclusion. We are now in position to prove (8.15). From (8.17), (8.16):

$$\begin{aligned} (1-w-w'-F)\left(\Sigma, \frac{c_-}{1+\alpha\Sigma}\right) &= \frac{\Sigma(\sum_{i=0}^3 \tilde{B}_i\Sigma^i)}{\Delta(1+\alpha\Sigma)^2} - \frac{\Sigma\sum_{i=0}^2 A_i\Sigma^i}{(-\Delta)(1+\alpha\Sigma)^2} \\ &= \frac{\Sigma}{\Delta(1+\alpha\Sigma)^2}\sum_{i=0^3}(\tilde{B}_i + A_i)\Sigma^i \end{aligned}$$

with $A_3 = 0$. We claim for $0 < r_\circledast(d,\ell) - r \ll 1$:

$$\tilde{B}_i + \tilde{A}_i < 0, \quad 0 \le i \le 3 \tag{8.18}$$

Since $\Delta < 0$ at the right of P_2 where $\Sigma > 0$, (8.18) gives

$$(1-w-w'-F)\left(\Sigma, \frac{c_-}{1+\alpha\Sigma}\right) = \frac{\Sigma}{\Delta(1+\alpha\Sigma)^2}\sum_{i=0}^3(\tilde{B}_i + A_i)\Sigma^i > 0$$

and (8.15) is proved.

step 12 Proof of (8.18) for $r_\infty = r^*$.
$\underline{\tilde{B}_0 + A_0}$. We compute using $c_4 = -2\sigma_2^2$ for all r

$$A_0 = (2\sigma_2^2 + c_2)c_- = 2\sigma_2^2(1 + c_-) + c_2c_- + c_4 = 2\sigma_2^2(1 + c_-) + \lambda_+$$

and hence

$$A_0 + \tilde{B}_0 = 2\sigma_2^2(1 + c_-) + \lambda_+ + c_-\lambda_+ - 2\sigma_2^2(1 + c_-) = (1 + c_-)\lambda_+ < 0.$$

$\underline{\tilde{B}_1^\infty + A_1^\infty}$. We compute at r^*

$$\begin{aligned}\tilde{B}_1^\infty + A_1^\infty &= -\frac{P_{\tilde{B}_1}(\ell)}{(\ell + \sqrt{d})\left[d(\sqrt{d} - 1) + \ell(\sqrt{d} + 1)\right]^2} + \frac{P_{A_1}}{(\ell + \sqrt{d})\left[d(\sqrt{d} - 1) + \ell(\sqrt{d} + 1)\right]^2} \\ &= -\frac{P_{\tilde{B}_1} - P_{A_1}}{(\ell + \sqrt{d})\left[d(\sqrt{d} - 1) + \ell(\sqrt{d} + 1)\right]^2}\end{aligned}$$

with

$$\begin{aligned}&P_{\tilde{B}_1} - P_{A_1} = 9d^3\sqrt{d} - 10d^3 + d^2\sqrt{d} + \left[d^3\sqrt{d} - 2d^3 + 7d^2\sqrt{d} + 4d^2 - 2d\sqrt{d}\right]\ell \\ +\ &\left[2d^2\sqrt{d} + 4d^2 - 5d\sqrt{d} + 6d + \sqrt{d}\right]\ell^2 + \left[d\sqrt{d} - 2d + \sqrt{d}\right]\ell^3 \\ -\ &\left[6d^3\sqrt{d} - 8d^3 + 2d^2\sqrt{d} + \left[8d^2 - 8d\sqrt{d}\right]\ell - 2\sqrt{d}(d - 1)\ell^2\right] \\ =\ &3d^3\sqrt{d} - 2d^3 - d^2\sqrt{d} + \left[d^3\sqrt{d} - 2d^3 + 7d^2\sqrt{d} - 4d^2 + 6d\sqrt{d}\right]\ell \\ +\ &\left[2d^2\sqrt{d} + 4d^2 - 3d\sqrt{d} + 6d - \sqrt{d}\right]\ell^2 + \left[d\sqrt{d} - 2d + \sqrt{d}\right]\ell^3 \\ >\ &0\end{aligned}$$

for $d \geq 2$ and therefore $\tilde{B}_1^\infty + A_1^\infty < 0$.
$\underline{\tilde{B}_2^\infty + A_2^\infty}$. We compute:

$$\tilde{B}_2 + A_2 = -\frac{P_{\tilde{B}_2} - P_{A_2}}{\left[d(\sqrt{d} - 1) + (\sqrt{d} + 1)\right]^2}$$

with

$$\begin{aligned}&P_{\tilde{B}_2} - P_{A_2} \\ =\ &8d^4 + 4d^2(d - \sqrt{d}) + \left[2d^3 - 2d^2\sqrt{d} + 14d^2 + 2d\sqrt{d}\right]\ell + \left[2d^2 + 2d\sqrt{d} + 2d + 2\sqrt{d}\right]\ell^2 \\ -\ &\left[4d^2(d - \sqrt{d}) + \ell(4d\sqrt{d} - 4d)\right] \\ =\ &8d^4 + \left[2d^3 - 2d^2\sqrt{d} + 14d^2 - 2d\sqrt{d} + 4d\right]\ell \\ +\ &\left[2d^2 + 2d\sqrt{d} + 2d + 2\sqrt{d}\right]\ell^2 > 0\end{aligned}$$

for $d \geq 2$, which implies $\tilde{B}_2 + A_2 < 0$.
$\underline{\tilde{B}_3^\infty + A_3^\infty}$. We have $\tilde{B}_3 + A_3 = \tilde{B}_3 < 0$.
This concludes the proof of (8.18) for $r_\infty = r^*$.

step 13 Proof of (8.18) for $r_\infty = r_+$.
$\underline{\tilde{B}_0 + A_0}$. We have verbatim as above

$$A_0 + \tilde{B}_0 = \lambda_+ < 0.$$

$\underline{\tilde{B}_1^\infty + A_1^\infty}$. We compute:

$$\tilde{B}_1^\infty + A_1^\infty = -\frac{(d+1)\ell + (d^2+3d)\sqrt{\ell} + 2d}{(1+\sqrt{\ell})(d\sqrt{\ell}+\ell)} + \frac{d-1}{\ell(1+\sqrt{\ell})}$$
$$= -\frac{\ell\Big[(d+1)\ell + (d^2+3d)\sqrt{\ell} + 2d\Big] - (d-1)(d\sqrt{\ell}+\ell)}{\ell(1+\sqrt{\ell})(d\sqrt{\ell}+\ell)} = -\frac{Q_1(\ell)}{\ell(1+\sqrt{\ell})(d\sqrt{\ell}+\ell)}$$

with

$$Q_1(\ell) = (d+1)\ell^2 + (d^2+3d)\ell\sqrt{\ell} + (d+1)\ell - d(d-1)\sqrt{\ell} > 0$$

for $\ell > d$, $d = 2, 3$.

$\underline{\tilde{B}_2^\infty + A_2^\infty}$. We compute:

$$\begin{aligned} & \tilde{B}_2^\infty + A_2^\infty \\ = \; & -\frac{(d+1)\ell^2 + (2d^2+6d)\ell\sqrt{\ell} + (d^3+7d^2+4d)\ell + (8d^2)\sqrt{\ell} + 2d^2}{(d\sqrt{\ell}+\ell)^2} + \frac{d-1}{\ell} \\ = \; & -\frac{Q_2(\ell)}{\ell(d\sqrt{\ell}+\ell)^2} \end{aligned}$$

with

$$\begin{aligned} & Q_2 = (d+1)\ell^3 + (2d^2+6d)\ell^2\sqrt{\ell} + (d^3+7d^2+4d)\ell^2 + (8d^2)\ell\sqrt{\ell} + 2d^2\ell \\ - \; & (d-1)(d^2\ell + 2d\ell\sqrt{\ell} + \ell^2) \\ = \; & (d+1)\ell^3 + (2d^2+6d)\ell^2\sqrt{\ell} + (d^3+7d^2+3d+1)\ell^2 + (6d^2+2d)\ell\sqrt{\ell} \\ + \; & (3d^2-d^3)\ell > 0. \end{aligned}$$

$\underline{\tilde{B}_3^\infty + A_3^\infty}$. We have $\tilde{B}_3^\infty + A_3^\infty = \tilde{B}_3^\infty < 0$.

This concludes the proof of (8.18) for $r_{\text{⌓}} = r_+$. □

8.4. **Proof of Lemma 1.6.** We are now in position to finish the proof of Lemma 1.6. We need to show (1.24). To the right of P_2, $w' > 0$ and $F > 0$ from (8.11). Therefore, the first statement in (1.24) follows from (8.13). For the second, to the right of P_2 we have $\sigma \geq \sigma_2$ and $\Delta < 0$, i.e., $\sigma > 1 - w$. These, together with (8.13), imply

$$1 - w - w' - \frac{1-w}{\sigma}F \geq 1 - w - w' - F \geq c > 0$$

The third statement follows from $F > 0$ at the right of P_2. (1.24) is proved.

9. **Exterior positivity**

We now turn to the proof of Lemma 1.7. We will establish the following result which gives a precise range of validity of (1.26) and enlarges the set of admissible parameters. Let

$$\ell_2(d) = d - 2\sqrt{d} \quad \text{for} \quad d \geq 5. \tag{9.1}$$

Also, recall the definition (1.20) of p. The following implies Lemma 1.7.

Lemma 9.1 (Necessary/sufficient conditions for (1.26))**.** *Assume $d \geq 3$ and (1.22). For $\ell > d$, $r_{\text{⌓}} = r^*(d,\ell)$, let $\ell_2(d)$ be given by (9.1), and assume moreover that (ℓ, d)*

satisfy:

$$\left|\begin{array}{lll} \sqrt{3} < \ell & \text{for} \ \ d = 3 & \\ 0.11 < \ell < \ell_2(5) = 0.53 & \text{for} \ \ d = 5 & \text{i.e.,} \ \ 9 \le p \le 37, \\ 0.2 < \ell < \ell_2(6) = 1.10 & \text{for} \ \ d = 6 & \text{i.e.,} \ \ 5 \le p \le 21, \\ 0.3 < \ell < \ell_2(7) = 1.71 & \text{for} \ \ d = 7 & \text{i.e.,} \ \ 4 \le p \le 14, \\ 0.45 < \ell < \ell_2(8) = 2.34 & \text{for} \ \ d = 8 & \text{i.e.,} \ \ 3 \le p \le 9, \\ 0.65 < \ell < \ell_2(9) = 3 & \text{for} \ \ d = 9 & \text{i.e.,} \ \ 3 \le p \le 7; \end{array}\right. \tag{9.2}$$

Then there exists $\varepsilon(d,\ell)$ *such that for all*

$$r_{\text{⌓}}(d,\ell) - \varepsilon(d,\ell) < r < r_{\text{⌓}}(d,\ell) \tag{9.3}$$

and for any $P_2 - P_4$ *trajectory with* c_- *slope at* P_2 *as in Lemma 3.2, there exists* $c_r > 0$ *such that*

$$\forall 0 < \sigma \le \sigma_2, \quad \left|\begin{array}{l} (1-w-w')^2 - F^2 > c_r \\ 1 - w - w' > c_r. \end{array}\right. \tag{9.4}$$

Remark 9.2. We note that

$$\ell < \ell_2(d) \Leftrightarrow r^*(d,\ell) > 2 \tag{9.5}$$

The latter is a fundamental property for the study of the defocusing (NLS) problem, [10]. The lower bounds in (9.2) are actually given by the condition

$$\ell > \ell_1(d) \tag{9.6}$$

where

$$\begin{aligned} & \ell_1(d) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (9.7) \\ = \ & \frac{-(d\sqrt{d} - d + 5\sqrt{d} - 1) + \sqrt{(d\sqrt{d} - d + 5\sqrt{d} - 1)^2 + 4(d\sqrt{d} - d - 2\sqrt{d})(\sqrt{d}+1)}}{2(\sqrt{d}+1)}, \end{aligned}$$

see (9.13). We will see below that (9.6) is a necessary condition for (9.4) to hold near $r^*(d,\ell)$.

Remark 9.3. From (8.11), we have that along the solution curve for $\sigma < \sigma_2$

$$F < 0 \text{ for } w < w_2, \qquad F > 0 \text{ for } w > w_2.$$

We will first treat the case $w < w_2$, in which case it suffices to prove $1 - w - w' + F > 0$, and then the case $w > w_2$, in which case it suffices to prove $1 - w - w' - F > 0$.

9.1. **Bound on the slope for the $P_2 - P_4$ separatrix.** The $P_2 - P_4$ separatrix[23], i.e., the unique solution connecting P_2 and P_4 with the slope c_+ at P_2 described in (3) of Lemma 3.5, furnishes a natural lower bound for the solution curves of Theorem 1.3. We start with a *rough* estimate on its slope[24].

Lemma 9.4 (Upper bound on the slope for the $P_2 - P_4$ separatrix)**.** *Under the assumptions of Lemma 9.1, the* $P_2 - P_4$ *separatrix given in (3) of Lemma 3.5 satisfies*

$$0 < \Phi_S = \frac{W}{\Sigma} \le c_+ \tag{9.8}$$

where we recall that $W = w - w_2$ *and* $\Sigma = \sigma - \sigma_2$.

[23]We will sometime denote by a subscript S, e.g. Φ_S

[24]we mean here the slope of the line segments from P_2 to points on the separatrix.

Proof of Lemma 9.4. We have $\sigma < \sigma_2$, i.e., $\Sigma < 0$ and $W = w - w_2 < 0$. Therefore,

$$\Phi = \frac{W}{\Sigma} > 0.$$

step 1 Setting up. We have

$$\Phi_S(P_2) = c_+, \quad \Phi_S(P_4) = \frac{-w_2}{-\sigma_2} = \frac{1-\sigma_2}{\sigma_2}$$

and hence at $r = r_{\circledcirc}$:

$$\Phi_S(P_2^\infty) - \Phi_S(P_4^\infty) = \left| \begin{array}{l} \ell - \frac{1-\sigma_2}{\sigma_2} = \ell + 1 - \frac{\ell+\sqrt{d}}{\sqrt{d}} = \frac{\ell(\sqrt{d}-1)}{\sqrt{d}} > 0 \quad \text{for} \quad r = r^* \\ \frac{\sqrt{\ell}(d+\sqrt{\ell})}{1+\sqrt{\ell}} - \sqrt{\ell} = \frac{(d-1)\sqrt{\ell}}{1+\sqrt{\ell}} > 0 \quad \text{for} \quad r = r_+ \end{array} \right.$$

which ensures

$$\Phi_S(P_2) - \Phi_S(P_4) > 0. \tag{9.9}$$

We now recall

$$\frac{d\Phi_S}{d\Sigma} = \frac{\Delta_1 - \Phi_S \Delta_2}{\Sigma \Delta_2}$$

and study the sign at a possible point of contact on the curve with the value

$$\Phi_S(\Sigma) = c_+, \quad \Sigma < 0.$$

From (8.1), at the point of contact:

$$\begin{aligned} \frac{\Delta_1 - c_+\Delta_2}{\Sigma} &= c_1 c_+ + c_3 - c_+(c_2 c_+ + c_4) + \left[d_{20}c_+^2 + d_{11}c_+ + d_{02} - c_+(e_{20}c_+^2 + e_{11}c_+ + e_{02})\right]\Sigma \\ &+ \left[c_+^3 - dc_+ - c_+(e_{21}c_+^2 - 1)\right]\Sigma^2 = A_0 + A_1\Sigma + A_2\Sigma^2. \end{aligned}$$

We have $A_0 = 0$ from (2.45). We now compute all coefficients at $r_{\circledcirc}(d,\ell)$, and the associated non degeneracy claim will follow for r close enough to $r_{\circledcirc}(d,\ell)$.

step 2 Sign of A_2^∞.
<u>case $r = r^*$</u>. We compute:

$$\begin{aligned} A_2^\infty &= \ell^3 - d\ell - \ell\left(\frac{\ell+d-1}{\ell}\ell^2 - 1\right) = \ell^3 - d\ell - \ell^2(\ell+d-1) + \ell \\ &= \ell\left[-d - \ell(d-1) + 1\right] = -\ell(d-1)(1+\ell) < 0. \end{aligned}$$

<u>case $r = r_+$</u>. We compute:

$$\begin{aligned} A_2^\infty &= (c_+^\infty)^3 - dc_+^\infty - c_+^\infty(e_{21}^\infty(c_+^\infty)^2 - 1) = c_+^\infty\left[(1-e_{21}^\infty)(c_+^\infty)^2 - (d-1)\right] \\ &= c_+^\infty\left[\left(1 - \frac{\ell+d-1}{\ell}\right)\frac{\ell(d+\sqrt{\ell})^2}{(1+\sqrt{\ell})^2} - (d-1)\right] = -(d-1)c_+^\infty\left[\frac{(d+\sqrt{\ell})^2}{(1+\sqrt{\ell})^2} + 1\right] < 0. \end{aligned}$$

step 3 Sign of A_1^∞.
<u>case $r = r^*$</u>. We have

$$A_1^\infty = d_{20}^\infty(c_+^\infty)^2 + d_{11}^\infty c_+^\infty + d_{02}^\infty - c_+^\infty(e_{20}^\infty(c_+^\infty)^2 + e_{11}^\infty c_+^\infty + e_{02}^\infty)$$

We compute:

$$e_{20}^{\infty}(c_+^{\infty})^2+e_{11}^{\infty}c_+^{\infty}+e_{02}^{\infty}=\frac{\sqrt{d}(d-1+\ell)}{\ell(\ell+\sqrt{d})}\ell^2-\frac{d(\sqrt{d}-1)+\ell(1+\sqrt{d})}{\ell(\ell+\sqrt{d})}\ell-\frac{3\sqrt{d}}{\ell+\sqrt{d}}$$

$$\begin{aligned}
&= \frac{\ell\sqrt{d}(d-1+\ell)-d\sqrt{d}+d-\ell(1+\sqrt{d})-3\sqrt{d}}{\ell+\sqrt{d}}\\
&= \frac{-d\sqrt{d}+d-3\sqrt{d}+\ell(d\sqrt{d}-\sqrt{d}-1-\sqrt{d})+\ell^2\sqrt{d}}{\ell+\sqrt{d}}\\
&= \frac{-d\sqrt{d}+d-3\sqrt{d}+\ell(d\sqrt{d}-2\sqrt{d}-1)+\ell^2\sqrt{d}}{\ell+\sqrt{d}}
\end{aligned}$$

and

$$d_{20}^{\infty}(c_+^{\infty})^2+d_{11}^{\infty}c_+^{\infty}+d_{02}^{\infty}=\frac{-\sqrt{d}-(d-\ell)}{\ell+\sqrt{d}}\ell^2-\frac{2d\sqrt{d}}{\ell+\sqrt{d}}\ell-\frac{\ell\sqrt{d}}{\ell+\sqrt{d}}$$

$$\begin{aligned}
&= -\frac{\ell}{\ell+\sqrt{d}}\left[\ell(\sqrt{d}+d-\ell)+2d\sqrt{d}+\sqrt{d}\right]\\
&= -\frac{\ell}{\ell+\sqrt{d}}\left[-\ell^2+\ell(\sqrt{d}+d)+2d\sqrt{d}+\sqrt{d}\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
&(\ell+\sqrt{d})A_1^{\infty}=\ell\left[\ell^2-\ell(\sqrt{d}+d)-2d\sqrt{d}-\sqrt{d}\right]\\
-\;&\ell\left[-d\sqrt{d}+d-3\sqrt{d}+\ell(d\sqrt{d}-2\sqrt{d}-1)+\ell^2\sqrt{d}\right]\\
=\;&\ell\left[\ell^2(1-\sqrt{d})-\ell(\sqrt{d}+d+d\sqrt{d}-2\sqrt{d}-1)-2d\sqrt{d}-\sqrt{d}+d\sqrt{d}-d+3\sqrt{d}\right]\\
=\;&-\ell\left[(\sqrt{d}-1)\ell^2+(d\sqrt{d}+d-\sqrt{d}-1)\ell+d\sqrt{d}+d-2\sqrt{d}\right]<0.
\end{aligned}$$

<u>case $r=r_+$</u>. We have

$$A_1^{\infty} = d_{20}^{\infty}(c_+^{\infty})^2+d_{11}^{\infty}c_+^{\infty}+d_{02}^{\infty}-c_+^{\infty}(e_{20}^{\infty}(c_+^{\infty})^2+e_{11}^{\infty}c_+^{\infty}+e_{02}^{\infty})$$

We compute:

$$e_{20}^{\infty}(c_+^{\infty})^2+e_{11}^{\infty}c_+^{\infty}+e_{02}^{\infty}=\frac{\ell+d-1}{\ell(1+\sqrt{\ell})}\left(\frac{\sqrt{\ell}(d+\sqrt{\ell})}{1+\sqrt{\ell}}\right)^2-\frac{2}{1+\sqrt{\ell}}\left(\frac{\sqrt{\ell}(d+\sqrt{\ell})}{1+\sqrt{\ell}}\right)-\frac{3}{1+\sqrt{\ell}}$$

$$\begin{aligned}
&= \frac{(\ell+d-1)(d+\sqrt{\ell})^2-2\sqrt{\ell}(d+\sqrt{\ell})(1+\sqrt{\ell})-3(1+\sqrt{\ell})^2}{(1+\sqrt{\ell})^3}\\
&= \frac{(\ell+d-1)(d^2+2d\sqrt{\ell}+\ell)-2\sqrt{\ell}(d+(d+1)\sqrt{\ell}+\ell)-3(1+2\sqrt{\ell}+\ell)}{(1+\sqrt{\ell})^3}\\
&= \frac{\ell^2+(2d-2)\ell\sqrt{\ell}+(d^2-d-6)\ell+(2d^2-4d-6)\sqrt{\ell}+d^3-d^2-3}{(1+\sqrt{\ell})^3}=\frac{Q_1(\ell)}{(1+\sqrt{\ell})^3}
\end{aligned}$$

and

$$\begin{aligned}
&d_{20}^{\infty}(c_+^{\infty})^2+d_{11}^{\infty}c_+^{\infty}+d_{02}^{\infty}\\
=\;&\frac{\ell-\sqrt{\ell}-d-1}{(1+\sqrt{\ell})^2}\left(\frac{\sqrt{\ell}(d+\sqrt{\ell})}{1+\sqrt{\ell}}\right)^2-\frac{2d}{1+\sqrt{\ell}}\left(\frac{\sqrt{\ell}(d+\sqrt{\ell})}{1+\sqrt{\ell}}\right)-\frac{\ell+d\sqrt{\ell}}{(1+\sqrt{\ell})^2}\\
=\;&\frac{\ell(\ell-\sqrt{\ell}-d-1)(d+\sqrt{\ell})^2-2d\sqrt{\ell}(d+\sqrt{\ell})(1+\sqrt{\ell})^2-(\ell+d\sqrt{\ell})(1+\sqrt{\ell})^2}{(1+\sqrt{\ell})^4}
\end{aligned}$$

and

$$\begin{aligned} & Q_2(\ell) = \ell(\ell - \sqrt{\ell} - d - 1)(d + \sqrt{\ell})^2 - 2d\sqrt{\ell}(d + \sqrt{\ell})(1 + \sqrt{\ell})^2 - (\ell + d\sqrt{\ell})(1 + \sqrt{\ell})^2 \\ = \;& [\ell^2 - \ell\sqrt{\ell} - (d+1)\ell](d^2 + 2d\sqrt{\ell} + \ell) - (2d^2\sqrt{\ell} + 2d\ell)(1 + 2\sqrt{\ell} + \ell) - (\ell + d\sqrt{\ell})(1 + 2\sqrt{\ell} + \ell) \\ = \;& \ell^3 + (2d-1)\ell^2\sqrt{\ell} + (d^2 - 5d - 2)\ell^2 + (-5d^2 - 7d - 2)\ell\sqrt{\ell} \\ + \;& (-d^3 - 5d^2 - 4d - 1)\ell + (-2d^2 - d)\sqrt{\ell}. \end{aligned}$$

Hence

$$A_1^\infty = \frac{Q_2(\ell)}{(1+\sqrt{\ell})^4} - \frac{\sqrt{\ell}(d+\sqrt{\ell})}{1+\sqrt{\ell}} \frac{Q_1(\ell)}{(1+\sqrt{\ell})^3} = -\frac{Q_3}{(1+\sqrt{\ell})^4}$$

with

$$\begin{aligned} & Q_3 = \sqrt{\ell}(d + \sqrt{\ell})Q_1 - Q_2 \\ = \;& (d\sqrt{\ell} + \ell)\Big\{\ell^2 + (2d-2)\ell\sqrt{\ell} + (d^2 - d - 6)\ell + (2d^2 - 4d - 6)\sqrt{\ell} + d^3 - d^2 - 3\Big\} \\ - \;& \ell^3 - (2d-1)\ell^2\sqrt{\ell} - (d^2 - 5d - 2)\ell^2 + (5d^2 + 7d + 2)\ell\sqrt{\ell} \\ + \;& (d^3 + 5d^2 + 4d + 1)\ell + (2d^2 + d)\sqrt{\ell} \\ = \;& (d-1)\ell^2\sqrt{\ell} + (2d^2 + 2d - 4)\ell^2 + (d^3 + 6d^2 - 3d - 4)\ell\sqrt{\ell} + (4d^3 - 2d - 2)\ell \\ + \;& (d^4 - d^3 + 2d^2 - 2d)\sqrt{\ell} > 0 \end{aligned}$$

and hence $A_1^\infty < 0$.

Derivative at P_2. Near P_2 we have the Taylor expansion:

$$\frac{d\Phi}{d\Sigma} = \frac{A_1\Sigma + A_2\Sigma^2}{\Sigma\left[c_2\Phi + c_4 + O(\Sigma)\right]} \to \frac{A_1}{c_2c_+ + c_4} = \frac{A_1}{\lambda_-} > 0. \tag{9.10}$$

Conclusion. At the point of contact, we therefore obtain:

$$\frac{d\Phi}{d\Sigma} = \frac{\Delta_1 - c_+\Delta_2}{\Sigma\Delta_2} = \frac{\Sigma(A_1 + A_2\Sigma)}{\Delta_2} = -\frac{|\Sigma|(-|A_1| + |A_2||\Sigma|)}{\Delta_2} = \frac{|\Sigma|(|A_1| - |A_2||\Sigma|)}{\Delta_2}.$$

Let

$$\tilde{\Sigma} = -\Sigma, \quad \widetilde{\Phi}(\tilde{\Sigma}) = \Phi(\Sigma),$$

then

$$\frac{d\widetilde{\Phi}}{d\tilde{\Sigma}} = -\frac{\tilde{\Sigma}(|A_1| - |A_2|\tilde{\Sigma})}{\Delta_2}.$$

Assume now that there exists $0 < \tilde{\Sigma}^* < \sigma_2$ with $\widetilde{\Phi}(\tilde{\Sigma}^*) > c_+$, then from (9.9), (9.10), the curve is strictly below c_+ for $0 < \tilde{\Sigma} \ll 1$ and for $|\tilde{\Sigma} - \sigma_2| \ll 1$. Therefore, there must exist $\tilde{\Sigma}_1 < \tilde{\Sigma}^* < \tilde{\Sigma}_2$ with

$$\left| \begin{array}{l} \widetilde{\Phi}(\tilde{\Sigma}_1) = \widetilde{\Phi}(\tilde{\Sigma}_2) = c_+ \\ \widetilde{\Phi}'(\tilde{\Sigma}_1) \geq 0 \\ \widetilde{\Phi}'(\tilde{\Sigma}_2) \leq 0 \end{array} \right.$$

Since $\Delta_2 > 0$ in this zone:

$$|A_1| - |A_2|\tilde{\Sigma}_1 \leq 0, \quad |A_1| - |A_2|\tilde{\Sigma}_2 \geq 0$$

which forces

$$\tilde{\Sigma}_2 \leq \frac{|A_1|}{|A_2|} \leq \tilde{\Sigma}_1,$$

a contradiction. □

9.2. **Study of F on the $P_2 - P_4$ separatrix.**

Lemma 9.5 (Value at P_2)**.** *Under the assumptions of Lemma 9.1, for the $P_4 - P_2$ separatrix and for $r_\infty(\ell) - \varepsilon(d,\ell) < r < r_\infty(d,\ell)$:*

$$F_S(P_2) < 0 \tag{9.11}$$

Moreover, in the case $\ell < d$, we have:

$$\begin{array}{ll} & \left[\exists 0 < \varepsilon(d,\ell) \ll 1 \quad \text{such that} \quad \forall 0 < \varepsilon < \varepsilon(d,\ell), \quad (1 - w - w' + F)(P_2) > 0\right] \\ \Leftrightarrow & \quad \ell > \ell_1(d) \end{array} \tag{9.12}$$

with ℓ_1 given by (9.7)*.*

Remark 9.6. The necessary admissible range near r^* is therefore

$$\ell_1(d) < \ell < \ell_2(d) = d - 2\sqrt{d}.$$

We compute numerically:

$$\left|\begin{array}{ll} \ell_1(5) = 0.1023, & \ell_2(5) = 0.5279, \\ \ell_1(6) = 0.1845, & \ell_2(6) = 1.101, \\ \ell_1(7) = 0.2525, & \ell_2(7) = 1.7085, \\ \ell_1(8) = 0.3098, & \ell_2(8) = 2.3431, \\ \ell_1(9) = 0.3589, & \ell_2(9) = 3. \end{array}\right. \tag{9.13}$$

Proof of Lemma 9.5. We compute all coefficients at $r_\infty(d,\ell)$ and argue by continuity for r close enough to $r_\infty(d,\ell)$. Near P_2 from (8.1), (2.45), (2.50):

$$\begin{aligned} F_S &= \sigma_2 - \frac{\Sigma\left[c_2\Phi + c_4 + [e_{20}\Phi^2 + e_{11}\Phi + e_{02}]\Sigma + [e_{21}\Phi^2 - 1]\Sigma^2\right]}{-\Sigma\left[2\sigma_2(1+\Phi) + \Sigma(1-\Phi^2)\right]} + O(\Sigma) \\ &= \sigma_2 + \frac{c_2 c_\pm + c_4}{2\sigma_2(1+c_\pm)} + O(\Sigma) = \sigma_2 + \frac{\lambda_-}{2\sigma_2(1+c_+)} + O(\Sigma) \end{aligned}$$

and

$$\begin{aligned} w' &= -\frac{\Delta_1}{\Delta} = -\frac{c_1\Phi + c_3 + [d_{20}\Phi^2 + d_{11}\Phi + d_{02}]\Sigma + [\Phi^3 - d\Phi]\Sigma^2}{-\left[2\sigma_2(1+\Phi) + \Sigma(1-\Phi^2)\right]} = \frac{c_1 c_+ + c_3}{2\sigma_2(1+c_+)} + O(\Sigma) \\ &= \frac{c_+\lambda_-}{2\sigma_2(1+c_+)} + O(\Sigma) \end{aligned}$$

and we compute these quantities at P_2 and $r = r_\infty$.

<u>case $r_\infty = r^*$</u>.

$$\begin{aligned} F_S &= \frac{2(\sigma_2^\infty)^2(1+\ell) + \lambda_-}{2\sigma_2(1+\ell)} \\ &= \frac{1}{2\sigma_2^\infty(1+\ell)(\ell+\sqrt{d})^2}\left[2d(1+\ell) - \left[d(d-\sqrt{d}) + 2d + \ell(d+\sqrt{d})\right]\right] \\ &= -\frac{(d-\ell)(d-\sqrt{d})}{2\sqrt{d}(1+\ell)(\ell+\sqrt{d})} < 0 \end{aligned}$$

and

$$\begin{aligned}
& 1-w-w'+F_S \\
= \;& \sigma_2^\infty - \frac{\ell\lambda_-}{2\sigma_2^\infty(1+\ell)} + \sigma_2^\infty + \frac{\lambda_-}{2\sigma_2^\infty(1+\ell)} = 2\left[\sigma_2^\infty + \frac{\lambda_-}{2\sigma_2^\infty(1+\ell)}\right] - \frac{(1+\ell)\lambda_-}{2\sigma_2^\infty(1+\ell)} \\
= \;& -\frac{2(d-\ell)(d-\sqrt{d})}{2\sqrt{d}(1+\ell)(\ell+\sqrt{d})} + \frac{1+\ell}{\frac{2\sqrt{d}(1+\ell)}{\ell+\sqrt{d}}}\frac{d(d-\sqrt{d})+2d+\ell(d+\sqrt{d})}{(\ell+\sqrt{d})^2} \\
= \;& \frac{1}{2\sqrt{d}(1+\ell)(\ell+\sqrt{d})}\left[-2(d-\ell)(d-\sqrt{d}) + (1+\ell)\left(d(d-\sqrt{d})+2d+\ell(d+\sqrt{d})\right)\right] \\
= \;& \frac{P_d(\ell)}{2\sqrt{d}(1+\ell)(\ell+\sqrt{d})} \qquad (9.14)
\end{aligned}$$

with

$$\begin{aligned}
& P_d(\ell) = -2(d-\ell)(d-\sqrt{d}) + (1+\ell)\left(d(d-\sqrt{d})+2d+\ell(d+\sqrt{d})\right) \\
= \;& -2d(d-\sqrt{d}) + d(d-\sqrt{d}) + 2d \\
+ \;& \ell\left[2(d-\sqrt{d}) + d(d-\sqrt{d}) + 2d + d + \sqrt{d}\right] + \ell^2(d+\sqrt{d}) \\
= \;& -d^2 + d\sqrt{d} + 2d + \left[d^2 - d\sqrt{d} + 5d - \sqrt{d}\right]\ell + (d+\sqrt{d})\ell^2 \\
= \;& \sqrt{d}\left\{-d\sqrt{d} + d + 2\sqrt{d} + \left[d\sqrt{d} - d + 5\sqrt{d} - 1\right]\ell + (\sqrt{d}+1)\ell^2\right\} \qquad (9.15)
\end{aligned}$$

Observe that

$$\begin{aligned}
P_d(1) &= -2(d-1)(d-\sqrt{d}) + 2\left(d(d-\sqrt{d})+2d+(d+\sqrt{d})\right) \\
&= 2(d-\sqrt{d}) + 2\left(2d+(d+\sqrt{d})\right) > 0
\end{aligned}$$

and

$$P_d(0) = -2d(d-\sqrt{d}) + d(d-\sqrt{d}) + 2d < 0$$

which implies the condition

$$\ell > \ell_1(d), \quad \ell_1(d) < 1$$

with ℓ_1 given by (9.7).

<u>case $r_\circledcirc = r_+$</u>. We compute

$$F_S = \sigma_2 + \frac{c_2c_+ + c_4}{2\sigma_2(1+c_+)} = \frac{-c_4(1+c_+) + c_2c_+ + c_4}{2\sigma_2(1+c_+)} = \frac{c_+(c_2-c_4)}{2\sigma_2(1+c_+)} < 0$$

from (2.49), and $F(P_2^\infty) = 0$ from (F.11). Then at $r_\circledcirc$ and P_2^∞:

$$1-w-w'+F_S = \sigma_2 - \frac{c_+\lambda_-}{2\sigma_2(1+c_+)} = \frac{2\sigma_2^2(1+c_+) - c_+\lambda_-}{2\sigma_2(1+c_+)} = -\frac{(1+c_+)\lambda_-}{2\sigma_2(1+c_+)} > 0$$

□

9.3. **Positivity in the region where $w \le w_2$.** We now consider *any solution curve* $P_2 - P_4$ with the c_- slope at P_2 given in Lemma 3.2. Recall that such curves cross the middle root of Δ_1 at $0 < \sigma_* < \sigma_2$. In view of the signs of Δ, Δ_1 and Δ_2, see figure 3, these curves are:

- decreasing on (σ_*, σ_2),
- increasing on (σ_0, σ_*), where $0 \le \sigma_0 < \sigma_*$ is such that $w(\sigma_0) = 0$,
- $w(\sigma) \le 0$ on $[0, \sigma_0]$.

In particular, on $[0,\sigma_2]$, $w(\sigma)$ can reach the value w_2 only on (σ_0,σ_*) where it is increasing, and hence there exists a unique $0<\sigma_1<\sigma_2$ such that we have

$$\left|\begin{array}{l} w>w_2 \ \text{ for } \ \sigma_1<\sigma<\sigma_2 \\ w<w_2 \ \text{ for } \ 0<\sigma<\sigma_1 \end{array}\right. \tag{9.16}$$

Lemma 9.7 (Positivity in the region where $\sigma<\sigma_1$ and $w<w_2$)**.** *Under the assumptions of Lemma 9.1, any P_2-P_4 curve with c_- slope at P_2 satisfies* (9.4) *in the region $w<w_2$ and $0<\sigma<\sigma_1$,*

Proof of Lemma 9.7. Using (8.11), we have along the solution curve in the region $w<w_2$:

$$F<0, \quad w<w_2<1 \ \text{ and } \ \Delta>0.$$

Note also that the solution curve has the slope c_- at P_2 and can not intersect the separatrix curve (strictly) between P_2 and P_4. Therefore, it must lie above the P_2-P_4 separatrix. As a consequence its function Φ also satisfies (9.8):

$$0<\Phi\le c_+,$$

where the lower bound follows from the fact that we consider the region $w\le w_2$. Thus we focus below on the region

$$0<\sigma<\sigma_1, \qquad w<w_2, \qquad 0<\Phi\le c_+.$$

step 1 Study of $1-w-w'+F$.

Remark 9.8. We will consider the expression for $1-w-w'+F$

$$1-w-w'+F=(1-w+\sigma)+\frac{\Delta_1-\Delta_2}{\Delta}$$

not just as a function on the solution curve but more generally as a function of σ and w.

We compute

$$\begin{aligned}
&(1-w-w')+F=-\frac{(d-1)\sigma}{\ell\Delta}(w-w_-)(w-w_+)+(1-w+\frac{\Delta_1}{\Delta}) \\
=\ & \frac{1}{\ell\Delta}\Big\{-(d-1)\sigma(w-w_-)(w-w_+) \\
+\ & \ell\Big[(1-w)[(1-w)^2-\sigma^2]+w(1-w)(r-w)-d(w-w_e)\sigma^2\Big]\Big\} \\
=\ & -\frac{1}{\ell\Delta}\Big\{\ell\left[1-w+d(w-w_e)\right]\sigma^2+(d-1)(w_--w)(w_+-w)\sigma-\ell(1-w)[(1-w)^2+w(r-w)]\Big\} \\
=\ & -\frac{1}{\ell\Delta}\Big\{\ell(d-1)(w-w_*)\sigma^2+(d-1)(w_--w)(w_+-w)\sigma-\ell(1-w)[(1-w)^2+w(r-w)]\Big\} \\
=\ & -\frac{1}{\ell\Delta}\Big\{\ell(d-1)(w-w_*)\sigma^2+(d-1)(w_--w)(w_+-w)\sigma-\ell(1-w)[1+(r-2)w]\Big\} \\
=\ & -\frac{P_\sigma(w)}{\ell\Delta}
\end{aligned} \tag{9.17}$$

with

$$P_\sigma(w)=[(d-1)\sigma+\ell(r-2)]w^2+a_1(\sigma)w+a_2(\sigma) \tag{9.18}$$

and

$$w_*=\frac{dw_e-1}{d-1}=\frac{\ell(r-1)-1}{d-1}. \tag{9.19}$$

We study the roots of $P_\sigma(w)$ for $0<w<w_2=w_-<w_+<1<r$ which ensures:

$$\left|\begin{array}{l} (1-w)[(1-w)^2+w(r-w)]>0 \\ (w_+-w)(w_--w)>0 \end{array}\right. \tag{9.20}$$

Then, at $r^*(d,\ell)$:

$$(d-1)(w_2^\infty-(w^*)^\infty)=(d-1)w_2^\infty+1-\ell(r^*-1)=(d-1)(1-\sigma_2^\infty)+1-\ell\left(\frac{d+\ell}{\ell+\sqrt{d}}-1\right)$$
$$=\quad d-(d-1)\frac{\sqrt{d}}{\ell+\sqrt{d}}-\frac{\ell(d-\sqrt{d})}{\ell+\sqrt{d}}=\frac{d(\ell+\sqrt{d})-d\sqrt{d}+\sqrt{d}-\ell d+\ell\sqrt{d}}{\ell+\sqrt{d}}=\frac{\sqrt{d}(\ell+1)}{\ell+\sqrt{d}}>0$$

and at $r_+(d,\ell)$:

$$(d-1)(w_2^\infty-(w_*)^\infty)=\frac{(d-1)\sqrt{\ell}}{1+\sqrt{\ell}}+1-\frac{\ell(d-1)}{(1+\sqrt{\ell})^2}$$
$$=\quad\frac{d\sqrt{\ell}+1}{1+\sqrt{\ell}}-\frac{\ell(d-1)}{(1+\sqrt{\ell})^2}=\frac{\ell+(d+1)\sqrt{\ell}+1}{(1+\sqrt{\ell})^2}>0$$

and hence for r close enough to r_∞:

$$w_2>w_*.$$

step 2 Root for $w_2\le w\le 1$. In this regime, we have from (9.20), since $w\ge w_2>w^*$, only one positive root

$$\sigma=\sigma_1(w)=\frac{1}{2\ell(d-1)(w-w_*)}\Big\{-(d-1)(w_--w)(w_+-w)\tag{9.21}$$
$$+\quad\sqrt{[(d-1)(w_--w)(w_+-w)]^2+4\ell(1-w)[(1-w)^2+w(r-w)]\ell(d-1)(w-w_*)}\Big\}.$$

The points $P_2=(\sigma_2,w_2)$, $P_3=(\sigma_+,w_+)$ and $P_1=(0,1)$ are among the roots of $P_\sigma(w)$, and hence the curve $\sigma_1(w)$ must pass through these points and connects continuously (σ_2,w_2) to $(0,1)$.

step 3 Positivity on $\Phi=c_+$. Let the line

$$(D)=\{W=c_+\Sigma,\quad 0\le\sigma\le\sigma_2\}\tag{9.22}$$

we claim that under the assumptions of Lemma 9.1:

$$1-w-w'+F>0\ \text{ on }\ (D)\tag{9.23}$$

Indeed, we compute:

$$\begin{aligned}&\Delta(1-w-w'+F)=(1-w+\sigma)\Delta+\Delta_1-\Delta_2\\=\;&(1-w_2-W+\sigma_2+\Sigma)\left\{-\Sigma\left[2\sigma_2(1+\Phi)+\Sigma(1-\Phi^2)\right]\right\}\\+\;&\Sigma\left[c_1\Phi+c_3+[d_{20}\Phi^2+d_{11}\Phi+d_{02}]\Sigma+[\Phi^3-d\Phi]\Sigma^2\right]\\-\;&\Sigma\left[c_2\Phi+c_4+[e_{20}\Phi^2+e_{11}\Phi+e_{02}]\Sigma+[e_{21}\Phi^2-1]\Sigma^2\right]\\=\;&-\Sigma G(\Sigma,\Phi)\end{aligned}$$

with

$$\begin{aligned}
& G(\Sigma,\Phi) = [2\sigma_2 + \Sigma(1-\Phi)][2\sigma_2(1+\Phi) + \Sigma(1-\Phi^2)] \\
- \;& c_1\Phi - c_3 - [d_{20}\Phi^2 + d_{11}\Phi + d_{02}]\Sigma - [\Phi^3 - d\Phi]\Sigma^2 \\
+ \;& c_2\Phi + c_4 + [e_{20}\Phi^2 + e_{11}\Phi + e_{02}]\Sigma + [e_{21}\Phi^2 - 1]\Sigma^2 \\
= \;& 4\sigma_2^2(1+\Phi) + \Sigma[2\sigma_2(1-\Phi^2) + 2\sigma_2(1-\Phi^2)] + \Sigma^2(1-\Phi)^2(1+\Phi) \\
- \;& c_1\Phi - c_3 - [d_{20}\Phi^2 + d_{11}\Phi + d_{02}]\Sigma - [\Phi^3 - d\Phi]\Sigma^2 \\
+ \;& c_2\Phi + c_4 + [e_{20}\Phi^2 + e_{11}\Phi + e_{02}]\Sigma + [e_{21}\Phi^2 - 1]\Sigma^2 \\
= \;& 4\sigma_2^2(1+\Phi) + c_2\Phi + c_4 - c_1\Phi - c_3 \\
+ \;& [4\sigma_2(1-\Phi^2) + e_{20}\Phi^2 + e_{11}\Phi + e_{02} - d_{20}\Phi^2 - d_{11}\Phi - d_{02}]\Sigma \\
+ \;& [(1-\Phi)^2(1+\Phi) + e_{21}\Phi^2 - 1 - \Phi^3 + d\Phi]\Sigma^2 \\
= \;& A_0(\Phi) + A_1(\Phi)\Sigma + A_2(\Phi)\Sigma^2.
\end{aligned}$$

and we now distinguish $r_\circledcirc = r^*$ and $r_\circledcirc = r_+$.

step 4 Proof of (9.22) for $r_\circledcirc = r^*$. We compute the A_i on (D) at $r = r^*(d,\ell)$ i.e., $\Phi = c_+ = \ell$ and evaluate the obtained sign of G.
Computation of A_0^∞. We have from (9.14), (9.12):

$$A_0^\infty = 4(\sigma_2^\infty)^2(1+\ell) + \lambda_-^\infty - \ell\lambda_-^\infty = 2\sigma_2^\infty(1+\ell)\frac{P_d(\ell)}{2\sqrt{d}(1+\ell)(\ell+\sqrt{d})} = \frac{P_d(\ell)}{(\ell+\sqrt{d})^2} > 0.$$

Computation of A_1^∞. We have

$$A_1^\infty = A_{11} + A_{12}$$

with

$$\begin{aligned}
A_{11} &= 4\sigma_2^\infty(1-\ell^2) + e_{20}^\infty\ell^2 + e_{11}^\infty\ell + e_{02}^\infty \\
&= \frac{4\sqrt{d}}{\ell+\sqrt{d}}(1-\ell^2) + \frac{\sqrt{d}(d-1+\ell)}{\ell(\ell+\sqrt{d})}\ell^2 - \frac{d(\sqrt{d}-1)+\ell(1+\sqrt{d})}{\ell(\ell+\sqrt{d})}\ell - \frac{3\sqrt{d}}{\ell+\sqrt{d}} \\
&= \frac{(4\sqrt{d} - 4\ell^2\sqrt{d} + \ell\sqrt{d}(d-1+\ell) - d(\sqrt{d}-1) - \ell(1+\sqrt{d}) - 3\sqrt{d}}{\ell+\sqrt{d}} \\
&= \frac{\sqrt{d} - d\sqrt{d} + d + \ell^2(-4\sqrt{d}+\sqrt{d}) + \ell(\sqrt{d}(d-1) - 1 - \sqrt{d})}{\ell+\sqrt{d}} \\
&= -\frac{3\sqrt{d}\ell^2 - [(d-2)\sqrt{d}-1]\ell + \sqrt{d}(d-\sqrt{d}-1)}{\ell+\sqrt{d}}
\end{aligned}$$

and

$$\begin{aligned}
A_{12} &= -d_{20}^\infty\ell^2 - d_{11}^\infty\ell - d_{02}^\infty = \left(\frac{\sqrt{d}+(d-\ell)}{\ell+\sqrt{d}}\right)\ell^2 + \left(\frac{2d\sqrt{d}}{\ell+\sqrt{d}}\right)\ell + \frac{\ell\sqrt{d}}{\ell+\sqrt{d}} \\
&= \frac{\ell(2d\sqrt{d}+\sqrt{d}) + (d-\ell+\sqrt{d})\ell^2}{\ell+\sqrt{d}}
\end{aligned}$$

which implies

$$\begin{aligned} A_1^\infty &= \frac{-3\sqrt{d}\ell^2+[(d-2)\sqrt{d}-1]\ell-\sqrt{d}(d-\sqrt{d}-1)+\ell(2d\sqrt{d}+\sqrt{d})+(d-\ell+\sqrt{d})\ell^2}{\ell+\sqrt{d}} \\ &= \frac{(d-2\sqrt{d}-\ell)\ell^2+(3d\sqrt{d}-\sqrt{d}-1)\ell-\sqrt{d}(d-\sqrt{d}-1)}{\ell+\sqrt{d}} \\ &= -\frac{Q_d(\ell)}{\ell+\sqrt{d}} \end{aligned}$$

with

$$Q_d(\ell)=\sqrt{d}(d-\sqrt{d}-1)-(3d\sqrt{d}-\sqrt{d}-1)\ell-(d-2\sqrt{d}-\ell)\ell^2.$$

Computation of A_2. We have

$$\begin{aligned} A_2(\Phi) &= (1-\Phi)^2(1+\Phi)+e_{21}\Phi^2-1-\Phi^3+d\Phi \\ &= (1-\Phi)(1-\Phi^2)+e_{21}\Phi^2-1-\Phi^3+d\Phi \\ &= 1-\Phi^2-\Phi+\Phi^3+e_{21}\Phi^2-1-\Phi^3+d\Phi \\ &= \Phi\left[(d-1)+(e_{21}-1)\Phi\right]=\Phi\left[d-1+\left(\frac{\ell+d-1}{\ell}-1\right)\Phi\right] \\ &= \frac{(d-1)\Phi(\ell+\Phi)}{\ell} \end{aligned} \tag{9.24}$$

which implies

$$A_2^\infty = 2\ell(d-1).$$

Discriminant Recall

$$G(\Sigma,\ell)=A_0+A_1\Sigma+A_2\Sigma^2,$$

we compute the discriminant at the critical value:

$$\mathrm{Discr}_d(\ell)=(A_1^\infty)^2-4A_0^\infty A_2^\infty=\frac{Q_d^2}{(\ell+\sqrt{d})^2}-\frac{8\ell(d-1)P_d}{(\ell+\sqrt{d})^2}=\frac{Q_d^2-8\ell(d-1)P_d}{(\ell+\sqrt{d})^2}.$$

We collect the values

$$\left|\begin{array}{l} P_d(\ell)=-d^2+d\sqrt{d}+2d+\left[d^2-d\sqrt{d}+5d-\sqrt{d}\right]\ell+(d+\sqrt{d})\ell^2 \\ Q_d(\ell)=\sqrt{d}(d-\sqrt{d}-1)-(3d\sqrt{d}-\sqrt{d}-1)\ell-(d-2\sqrt{d}-\ell)\ell^2 \\ \mathrm{test}(d,\ell)=Q_d^2-8\ell(d-1)P_d \end{array}\right. \tag{9.25}$$

and evaluate numerically:
for $5\le d\le 12$: we find $\mathrm{Discr}_d(\ell)<0$ in the range (9.2). It fails numerically for $d\ge 13$.

for $d=3$, $\ell_1(3)<0$ and for $\sqrt{3}=1.73<\ell<3$, $\mathrm{Discr}_3(\ell)<0$.

Conclusion. Since $A_2^\infty>0$, we conclude $G(\Sigma,\Phi)=A_0(\Phi)+A_1(\Phi)\Sigma+A_2(\Phi)\Sigma^2>0$ on (D) given by (9.22). Therefore,

$$1-w-w'+F=-\frac{\Sigma G(\Sigma,\Phi)}{\Delta}>0$$

on (D), and (9.23) is proved.

step 5 Proof of (9.22) for $r_\infty = r_+$.
Computation of A_0. We have since $\Phi = c_+$ and recalling (2.45), (2.50):

$$\begin{aligned} A_0 &= 4\sigma_2^2(1+c_+) + c_2c_+ + c_4 - c_1c_+ - c_3 = -2c_4(1+c_+) + c_2c_+ + c_4 - c_+\lambda_- \\ &= -c_4 - c_2c_+ - 2(c_4 - c_2)c_+ - c_+\lambda_- = -(1+c_+)\lambda_- - 2(c_4 - c_2)c_+ \end{aligned}$$

and hence

$$A_0^\infty = -(1+c_+^\infty)\lambda_-^\infty > 0.$$

We compute explicitly

$$\begin{aligned} A_0 &= \left(1 + \frac{\sqrt{\ell}(d+\sqrt{\ell})}{1+\sqrt{\ell}}\right)\frac{2(\ell + (d+1)\sqrt{\ell}+1}{(1+\sqrt{\ell})^3} \\ &= \frac{2(\ell+(d+1)\sqrt{\ell}+1)^2}{(1+\sqrt{\ell})^4} \end{aligned}$$

Computation of A_1^∞. We have

$$A_1^\infty = A_{11} + A_{12}$$

with

$$\begin{aligned} & A_{11} = 4\sigma_2^\infty(1-(c_+^\infty)^2) + e_{20}^\infty(c_+^\infty)^2 + e_{11}^\infty c_+^\infty + e_{02}^\infty \\ = \ & \frac{4}{1+\sqrt{\ell}}\left[1 - \frac{\ell(d+\sqrt{\ell})^2}{(1+\sqrt{\ell})^2}\right] + \frac{\ell+d-1}{\ell(1+\sqrt{\ell})}\frac{\ell(d+\sqrt{\ell})^2}{(1+\sqrt{\ell})^2} - \frac{2}{1+\sqrt{\ell}}\frac{\sqrt{\ell}(d+\sqrt{\ell})}{1+\sqrt{\ell}} - \frac{3}{1+\sqrt{\ell}} \\ = \ & \frac{Q_1(\ell)}{(1+\sqrt{\ell})^3} \end{aligned}$$

with

$$\begin{aligned} Q_1 &= 4\left[1 + 2\sqrt{\ell} + \ell - \ell(d^2 + 2d\sqrt{\ell} + \ell)\right] + (\ell + d - 1)(d^2 + 2d\sqrt{\ell} + \ell) \\ &- 2(\ell + d\sqrt{\ell})(1+\sqrt{\ell}) - 3(1 + 2\sqrt{\ell} + \ell) \\ &= \ell^2(-3) + \ell\sqrt{\ell}(-6d-2) + \ell(-3d^2 - d - 2) + \sqrt{\ell}(2d^2 - 4d + 2) + d^3 - d^2 + 1. \end{aligned}$$

Then

$$\begin{aligned} & A_{12} = -d_{20}^\infty(c_+^\infty)^2 - d_{11}^\infty c_+^\infty - d_{02}^\infty \\ = \ & -\frac{\ell - \sqrt{\ell} - d - 1}{(1+\sqrt{\ell})^2}\frac{\ell(d+\sqrt{\ell})^2}{(1+\sqrt{\ell})^2} + \frac{2d}{1+\sqrt{\ell}}\frac{\sqrt{\ell}(d+\sqrt{\ell})}{1+\sqrt{\ell}} + \frac{\ell + d\sqrt{\ell}}{(1+\sqrt{\ell})^2} = \frac{Q_2(\ell)}{(1+\sqrt{\ell})^4} \end{aligned}$$

with

$$\begin{aligned} & Q_2 = (d^2 + 2d\sqrt{\ell} + \ell)(-\ell^2 + \ell\sqrt{\ell} + (d+1)\ell) + (2d\ell + 2d^2\sqrt{\ell})(1 + 2\sqrt{\ell} + \ell) \\ + \ & (\ell + d\sqrt{\ell})(1 + 2\sqrt{\ell} + \ell) \\ = \ & -\ell^3 + (-2d+1)\ell^2\sqrt{\ell} + \ell^2(-d^2 + 5d + 2) + \ell\sqrt{\ell}(5d^2 + 7d + 2) \\ + \ & \ell(d^3 + 5d^2 + 4d + 1) + \sqrt{\ell}(d + 2d^2). \end{aligned}$$

Hence

$$A_1^\infty = \frac{Q_3}{(1+\sqrt{\ell})^4}$$

with

$$\begin{aligned} Q_3 &= \left[\ell^2(-3)+\ell\sqrt{\ell}(-6d-2)+\ell(-3d^2-d-2)+\sqrt{\ell}(2d^2-4d+2)+d^3-d^2+1\right](1+\sqrt{\ell}) \\ &- \ell^3+(-2d+1)\ell^2\sqrt{\ell}+\ell^2(-d^2+5d+2)+\ell\sqrt{\ell}(5d^2+7d+2) \\ &+ \ell(d^3+5d^2+4d+1)+\sqrt{\ell}(d+2d^2) \\ &= -\ell^3-(2d+2)\ell^2\sqrt{\ell}-(d^2+d+3)\ell^2+(2d^2-2)\ell\sqrt{\ell} \\ &+ (d^3+4d^2-d+1)\ell+(d^3+3d^2-3d+3)\sqrt{\ell}+d^3-d^2+1 \end{aligned}$$

Computation of A_2. We have $A_2(\Phi)>0$ from (9.24) and explicitely

$$A_2^\infty=\frac{(d-1)c_+(\ell+c_+)}{\ell}=\frac{d-1}{\ell}\frac{\sqrt{\ell}(d+\sqrt{\ell})}{1+\sqrt{\ell}}\left[\ell+\frac{\sqrt{\ell}(d+\sqrt{\ell})}{1+\sqrt{\ell}}\right]=\frac{(d-1)(d+\sqrt{\ell})(\ell+2\sqrt{\ell}+d)}{(1+\sqrt{\ell})^2}$$

Conclusion. We are in the case $d=3$. We numerically evaluate $Q_3(\ell)$ and obtain $Q_3(\ell)<0$ for $\ell\geq 4$ in which case $A_1^\infty<0$ and since $\Sigma<0$, $G(\Sigma,\Phi)=A_0(\Phi)+A_1(\Phi)\Sigma+A_2(\Phi)\Sigma^2>0$. For $3<\ell<4$, we form the discriminant

$$\begin{aligned} \text{Discr} &= (A^\infty)_1^2-4A_0^\infty A_2^\infty=\frac{Q_3^2}{(1+\sqrt{\ell})^8}-\frac{8(\ell+(d+1)\sqrt{\ell}+1)^2}{(1+\sqrt{\ell})^4}\frac{(d-1)(d+\sqrt{\ell})(\ell+2\sqrt{\ell}+d)}{(1+\sqrt{\ell})^2} \\ &= \frac{Q_4}{(1+\sqrt{\ell})^8} \end{aligned}$$

with

$$Q_4=Q_3^2-8(d-1)(1+\sqrt{\ell})^2(\ell+(d+1)\sqrt{\ell}+1)^2(d+\sqrt{\ell})(\ell+2\sqrt{\ell}+d)$$

and numerically evaluate $Q_4(\ell)<0$ for $3<\ell<4$, and hence $G(\Sigma,\Phi)>0$. Hence

$$1-w-w'+F=-\frac{\Sigma G(\Sigma,\Phi)}{\Delta}>0$$

on (D), and (9.23) is proved.

step 4 Proof of (9.4). Observe that

$$P_\sigma(w_2)=\ell(d-1)(w_2-w_*)\sigma^2-\ell(1-w_2)[1+(r-2)w_2]$$

is a second order polynomial in σ with $P_{\sigma_2}(w_2)=0$, positive highest order coeffieicent and such that at $r^*(d,\ell)$:

$$P_0(w_2)=-\ell(1-w_2)[1+(r-2)w_2]$$

with

$$\begin{aligned} 1+(r^*(d,\ell)-2)w_2 &= 1+\left(\frac{d+\ell}{\ell+\sqrt{d}}-2\right)\left(1-\frac{\sqrt{d}}{\ell+\sqrt{d}}\right)=1+\frac{\ell(d-\ell-2\sqrt{d})}{(\ell+\sqrt{d})^2} \\ &= \frac{\ell^2+2\ell\sqrt{d}+d+\ell(d-\ell-2\sqrt{d})}{(\ell+\sqrt{d})^2}=\frac{\ell(d+1)}{(\ell+\sqrt{d})^2} \end{aligned}$$

and

$$\begin{aligned} 1+(r_+(d,\ell)-2)w_2 &= 1+\left(1+\frac{d-1}{(1+\sqrt{\ell})^2}-2\right)\frac{\sqrt{\ell}}{1+\sqrt{\ell}} \\ &= 1-\frac{\sqrt{\ell}}{1+\sqrt{\ell}}\left[1-\frac{d-1}{(1+\sqrt{\ell})^2}\right]=1-\frac{\sqrt{\ell}[(1+\sqrt{\ell})^2-(d-1)]}{(1+\sqrt{\ell})^3} \\ &= \frac{1+(d+1)\sqrt{\ell}+\ell}{(1+\sqrt{\ell})^3}>0. \end{aligned}$$

Hence $P_0(w_2) < 0$ implies that

$$P_\sigma(w_2) < 0 \quad \text{for} \quad 0 \le \sigma < \sigma_2.$$

Let $w_3(\sigma)$ parametrize the line $(w_3(\sigma), \sigma) \in (D)$, then we also have $P_\sigma(w_3(\sigma)) < 0$ from (9.23). We now distinguish three cases:

case $(d-1)\sigma + \ell(r-2) > 0$. Since $P_\sigma(w)$ is second order in w with a positive highest order coefficient,

$$P_\sigma(w) < 0 \quad \text{for} \quad w_3(\sigma) \le w \le w_2$$

which, together with the fact that (9.8) holds on the solution curve, implies that

$$1 - w - w' + F > 0$$

along the trajectory. The function $1-w-w'+F$ is strictly positive at P_2, converges to 1 as $\sigma \to 0$ and can not vanish. It implies

$$1 - w - w' + F \ge c > 0.$$

Since, by the Remark 9.3, $F \le 0$ at the left of P_2 on the solution curve, (9.4) follows.

case $(d-1)\sigma + \ell(r-2) < 0$. In this case, for each $0 \le \sigma < \sigma_2$, $P_\sigma(w_2) < 0$. On the other hand, we also have the curve $\sigma_1(w)$ which, as we vary $w \in [w_2, 1]$ connects $\sigma = \sigma_2$ and $\sigma = 0$, and on which $P_\sigma(w) = 0$. Since the point $(0, 1)$ belongs to this curve and lies above the line $w = w_2$, the whole curve $\sigma_1(w)$ must lie above the line $w = w_2$. In particular, for each $0 \le \sigma < \sigma_2$ we can find a (possibly non-unique) value $w_1(\sigma)$ such that $P_\sigma(w_1) = 0$ and $w_1(\sigma) > w_2$. This implies $P_\sigma(w) < 0$ for $w < w_2$, thus, in particular, on the solution curve, and the conclusion follows as above.

case $(d-1)\sigma + \ell(r-2) = 0$. Since $P_\sigma(w)$ is first order in w, this implies

$$P_\sigma(w) < 0 \quad \text{for} \quad w_3(\sigma) \le w \le w_2$$

and we conclude as above. □

9.4. **Positivity in the region where $w \ge w_2$.** We now are in position to conclude the proof of (9.4) and of Lemma 9.1. To this end, it suffices to prove the following lemma concerning the region $w \ge w_2$.

Lemma 9.9 (Positivity in the region where $\sigma_1 \le \sigma \le \sigma_2$ and $w \ge w_2$)**.** *Under the assumptions of Lemma 9.1, any $P_2 - P_4$ curve with c_- slope at P_2 satisfies* (9.4) *in the region $\sigma_1 \le \sigma \le \sigma_2$ and $w \ge w_2$.*

Proof. For $w \ge w_2$, in the region $\Delta_1 \ge 0$, we have

$$\Delta(1 - w - w' - F) \;=\; (1 - w - \sigma)\Delta + \Delta_1 + \Delta_2 > 0$$

since $\Delta_2 > 0$, $\Delta > 0$ and $1 - w - \sigma > 0$ on the solution curve for $\Sigma < 0$. Thus, it suffices to consider the region $\Sigma < 0$ before the solution curve crosses the middle root of $\Delta_1 = 0$. Note in particular that this region is included in

$$\{\sigma_5 - \sigma_2 < \Sigma < 0\} \cap \{\Delta_2 > 0\} \cap \{\Delta_1 < 0\} \cap \{1 - w - \sigma > 0\}.$$

step 1 Reduction to the control of the $P_2 - P_\infty$ separatrix. We compute

$$\begin{aligned}
&\Delta(1-w-w'-F) = (1-w-\sigma)\Delta + \Delta_1 + \Delta_2 \\
= \;& (1-w_2-W-\sigma_2-\Sigma)\left\{-\Sigma\left[2\sigma_2(1+\Phi)+\Sigma(1-\Phi^2)\right]\right\} \\
+ \;& \Sigma\left[c_1\Phi + c_3 + [d_{20}\Phi^2 + d_{11}\Phi + d_{02}]\Sigma + [\Phi^3 - d\Phi]\Sigma^2\right] \\
+ \;& \Sigma\left[c_2\Phi + c_4 + [e_{20}\Phi^2 + e_{11}\Phi + e_{02}]\Sigma + [e_{21}\Phi^2 - 1]\Sigma^2\right] \\
= \;& -\Sigma H(\Sigma,\Phi)
\end{aligned}$$

with

$$\begin{aligned}
&H(\Sigma,\Phi) = -\Sigma(1+\Phi)[2\sigma_2(1+\Phi)+\Sigma(1-\Phi^2)] \\
- \;& c_1\Phi - c_3 - [d_{20}\Phi^2 + d_{11}\Phi + d_{02}]\Sigma - [\Phi^3 - d\Phi]\Sigma^2 \\
- \;& c_2\Phi - c_4 - [e_{20}\Phi^2 + e_{11}\Phi + e_{02}]\Sigma - [e_{21}\Phi^2 - 1]\Sigma^2 \\
= \;& -c_2\Phi - c_4 - c_1\Phi - c_3 \\
+ \;& [-2\sigma_2(1+\Phi)^2 - e_{20}\Phi^2 - e_{11}\Phi - e_{02} - d_{20}\Phi^2 - d_{11}\Phi - d_{02}]\Sigma \\
+ \;& [-(1-\Phi)(1+\Phi)^2 - e_{21}\Phi^2 + 1 - \Phi^3 + d\Phi]\Sigma^2.
\end{aligned}$$

We introduce the notation

$$\widetilde{\Phi} \;:=\; \Phi - c_-.$$

We infer, using (2.45), (2.50):

$$\begin{aligned}
H(\Sigma,\Phi) &= -(c_2+c_1)\widetilde{\Phi} - c_2c_- - c_4 - c_1c_- - c_3 \\
&+ [-2\sigma_2 - e_{02} - d_{02} + (-4\sigma_2 - e_{11} - d_{11})\Phi + (-2\sigma_2 - e_{20} - d_{20})\Phi^2]\Sigma \\
&+ [(d-1)\Phi + (1-e_{21})\Phi^2]\Sigma^2 \\
&= -\lambda_+(1+c_-) - (c_2+c_1)\widetilde{\Phi} \\
&+ [-2\sigma_2 - e_{02} - d_{02} + (-4\sigma_2 - e_{11} - d_{11})c_- + (-2\sigma_2 - e_{20} - d_{20})c_-^2 \\
&\quad + (-4\sigma_2 - e_{11} - d_{11})\widetilde{\Phi} + 2c_-(-2\sigma_2 - e_{20} - d_{20})\widetilde{\Phi} + (-2\sigma_2 - e_{20} - d_{20})\widetilde{\Phi}^2]\Sigma \\
&+ [(d-1)c_- + (1-e_{21})c_-^2 + (d-1)\widetilde{\Phi} + 2c_-(1-e_{21})\widetilde{\Phi} + (1-e_{21})\widetilde{\Phi}^2]\Sigma^2 \\
&= F_0(\Sigma) + F_1(\Sigma)\widetilde{\Phi} + F_2(\Sigma)\widetilde{\Phi}^2
\end{aligned}$$

where

$$\begin{aligned}
F_0(\Sigma) \;:=\; & -\lambda_+(1+c_-) + [-2\sigma_2 - e_{02} - d_{02} + (-4\sigma_2 - e_{11} - d_{11})c_- + (-2\sigma_2 - e_{20} - d_{20})c_-^2]\Sigma \\
& + [(d-1)c_- + (1-e_{21})c_-^2]\Sigma^2,
\end{aligned}$$

$$\begin{aligned}
F_1(\Sigma) \;:=\; & -(c_2+c_1) + [(-4\sigma_2 - e_{11} - d_{11}) + 2c_-(-2\sigma_2 - e_{20} - d_{20})]\Sigma \\
& + [d - 1 + 2c_-(1-e_{21})]\Sigma^2,
\end{aligned}$$

$$F_2(\Sigma) \;:=\; (-2\sigma_2 - e_{20} - d_{20})\Sigma + (1-e_{21})\Sigma^2.$$

Sign of F_1. Since $-(c_2+c_1) > 0$ and does not degenerate as $r \to r_\infty$, we infer

$$F_1(\Sigma) = |c_1 + c_2| + O(\Sigma) > 0.$$

Sign of F_2. We have

$$F_2(\Sigma) = (2\sigma_2 + e_{20} + d_{20})|\Sigma|(1 + O(\Sigma)) \tag{9.26}$$

In the case $r_\circledcirc = r^*$, we have

$$2\sigma_2 + e_{20} + d_{20} = \frac{2\sqrt{d}}{\ell+\sqrt{d}} + \frac{\sqrt{d}(d-1+\ell)}{\ell(\ell+\sqrt{d})} + \frac{-\sqrt{d}-(d-\ell)}{\ell+\sqrt{d}} \tag{9.27}$$
$$= \frac{2\sqrt{d}\ell + \sqrt{d}(d-1+\ell) - \sqrt{d}\ell - \ell(d-\ell)}{\ell(\ell+\sqrt{d})} = \frac{\ell^2 + (2\sqrt{d}-d)\ell + \sqrt{d}(d-1)}{\ell(\ell+\sqrt{d})}$$

The discriminant of the second order polynomial $\ell^2 + (2\sqrt{d}-d)\ell + \sqrt{d}(d-1)$ is given by

$$\begin{aligned}(2\sqrt{d}-d)^2 - 4\sqrt{d}(d-1) &= \sqrt{d}\Big(\sqrt{d}(\sqrt{d}-2)^2 - 4(d-1)\Big)\\ &= \sqrt{d}\Big(d\sqrt{d} - 8d + 4\sqrt{d} + 4\Big)\end{aligned}$$

which is negative for $3 \le d \le 9$, so that the second order polynomial in ℓ has the sign of the main term. Therefore,

$$\ell^2 + (2\sqrt{d}-d)\ell + \sqrt{d}(d-1) > 0$$

which implies $F_2(\Sigma) > 0$ in the case $r_\circledcirc = r^*$.
In the case $r_\circledcirc = r_+$, we have

$$2\sigma_2 + e_{20} + d_{20} = \frac{\ell^2 + 2\ell^{\frac{3}{2}} + 2\ell - \sqrt{\ell} + d(1+\sqrt{\ell}-\ell) - 1}{\ell(1+\sqrt{\ell})^2} > 0$$

in the case $d = 3$, $\ell > d$, which implies $F_2(\Sigma) > 0$ in that case as well.

Computation of the roots. We have $F_0(\Sigma) = |\lambda_+||1+c_-| + O(\Sigma) = O(|r - r^*(\ell)|)$. Thus, the discriminant

$$F_1(\Sigma)^2 - 4F_2(\Sigma)F_0(\Sigma) = |c_2 + c_1| + O(r - r^*(\ell)) > 0.$$

Using also $F_2(\Sigma) > 0$ and $F_1(\Sigma) > 0$, the roots are given by

$$\widetilde{\Phi}_\pm = \frac{-|F_1(\Sigma)| \pm \sqrt{F_1(\Sigma)^2 - 4|F_2(\Sigma)|F_0(\Sigma)}}{2|F_2(\Sigma)|}.$$

We rewrite

$$\widetilde{\Phi}_+ = -\frac{2F_0(\Sigma)}{|F_1(\Sigma)| + \sqrt{F_1(\Sigma)^2 - 4|F_2(\Sigma)|F_0(\Sigma)}}. \tag{9.28}$$

Conclusion. Since $F_2(\Sigma) > 0$ and $\widetilde{\Phi}_- \ll -1$, we have $1 - w - w' - F > 0$ if and only if

$$\widetilde{\Phi} > \widetilde{\Phi}_+.$$

Since the curve connects P_2 to P_4 with c_- slope, the solution must lie strictly below the $P_2 - P_\circledcirc$ separatrix and hence $\Phi > \Phi_S$ and the conclusion follows from the lower bound along the separatrix curve:

$$\Phi_S > c_- + \widetilde{\Phi}_+ \ , \quad \forall u \in \left[0, \frac{3}{4}\right]. \tag{9.29}$$

Assuming (9.29), we have $1 - w - w' - F > 0$, and (9.4) is proved.

step 2 Proof of (9.29). We reexpress (9.29) using the renormalization (4.14), $u \in [0,1]$. Let

$$\left| \begin{array}{l} f_{01} = -2\sigma_2 - e_{02} - d_{02} + (-4\sigma_2 - e_{11} - d_{11})c_- + (-2\sigma_2 - e_{20} - d_{20})c_-^2 \\ f_{10} = -(c_2 + c_1) \\ B_0 = -\frac{b(1+c_-)}{c_1+c_2} \\ B_1 = \frac{c_+ - c_-}{\tilde{d}_{20}} \left[-\frac{\tilde{e}_{20}}{\lambda_-} + \frac{f_{01}}{f_{10}} \right] \end{array} \right. \tag{9.30}$$

We claim that there exist $C(d,\ell) > 0$ and $B_3(d,\ell), B_4(d,\ell)$ given by (H.10) such that for all $0 < b < b^*(d,\ell) \ll 1$ small enough, $\forall u \in [0,1]$, for $r_{\text{⌓}} = r^*$:

$$|\Phi_S - \Phi_+ - [B_0 + bB_1 u]| \leq Cb^2 \tag{9.31}$$

and for $r_{\text{⌓}} = r_+$:

$$\left|\Phi_S - \Phi_+ - \left[B_0 + bB_1 u + b^2(B_3 u + B_4 u^2)\right]\right| \leq Cb^3 \tag{9.32}$$

Then, (9.29) follows from the statement: $\exists c(d,\ell) > 0$, $\exists 0 < b^*(d,\ell) \ll 1$ such that for $r_{\text{⌓}} = r^*$:

$$\forall 0 < b < b^*, \quad \forall u \in \left[0, \frac{3}{4}\right], \quad B_0 + bB_1 u > cb. \tag{9.33}$$

and for $r_{\text{⌓}} = r_+$:

$$\forall 0 < b < b^*, \quad \forall u \in \left[0, \frac{3}{4}\right], \quad B_0 + bB_1 u + b^2(B_3 u + B_4 u^2) > cb^2. \tag{9.34}$$

The proof of (9.32) is detailed in Appendix H together with the explicit computation of the constants B_0, B_1, B_3, B_4 given by (H.10) which allows us to conclude the proof of (9.34) below.

case $r_{\text{⌓}} = r^*(d,\ell)$. We have

$$B_0 = -\frac{b(1+c_-)}{c_1+c_2} = \frac{b|1+c_-^\infty|}{|c_1^\infty + c_2^\infty|}(1 + O(b)).$$

Let

$$\beta = 2\sigma_2^\infty + e_{02}^\infty + d_{02}^\infty + (4\sigma_2^\infty + e_{11}^\infty + d_{11}^\infty)c_-^\infty + (2\sigma_2^\infty + e_{20}^\infty + d_{20}^\infty)(c_-^\infty)^2 > 0.$$

The inequality above follows by a direct check. Then, uniformly in b small enough and $u \in [0,1]$:

$$\begin{aligned} B_0 + bB_1 u &= b\left[\frac{1+c_-^\infty}{-(c_1^\infty + c_2^\infty)} + \frac{c_+^\infty - c_-^\infty}{\tilde{d}_{20}^\infty}\left(-\frac{\tilde{e}_{20}^\infty}{\lambda_-^\infty} + \frac{\beta}{c_1^\infty + c_2^\infty}\right)u\right] + O(b^2) \\ &= b\left[\frac{|1+c_-^\infty|}{|c_1^\infty + c_2^\infty|} + \frac{c_+^\infty - c_-^\infty}{|\tilde{d}_{20}^\infty|}\left(-\frac{\tilde{e}_{20}^\infty}{|\lambda_-^\infty|} + \frac{\beta}{|c_1^\infty + c_2^\infty|}|\right)u\right] + O(b^2) \\ &= \frac{b}{|c_1^\infty + c_2^\infty|}\left[|1+c_-^\infty| + \frac{c_+^\infty - c_-^\infty}{|\tilde{d}_{20}^\infty|}\left(|\beta| - \frac{\tilde{e}_{20}^\infty |c_1^\infty + c_2^\infty|}{|\lambda_-^\infty|}\right)u\right] + O(b^2). \end{aligned}$$

We compute:

$$\frac{(c_+^\infty - c_-^\infty)}{|\tilde{d}_{20}^\infty|}\left[|\beta| - \frac{|\tilde{e}_{20}^\infty||c_1^\infty + c_2^\infty|}{|\lambda_-^\infty|}\right] = -\frac{(\sqrt{d}-1)^2(d-\ell)(\ell + 2\sqrt{d} + d)}{2\sqrt{d}(d+\ell)(d(\sqrt{d}-1) + \ell(\sqrt{d}+1))}$$

and

$$1 + c_-^\infty = \frac{(\sqrt{d}-1)(d-\ell)}{d(\sqrt{d}-1) + \ell(\sqrt{d}+1)}$$

so that

$$|1+c_-^\infty|+\frac{c_+^\infty-c_-^\infty}{|\tilde{d}_{20}^\infty|}\left(|\beta|-\frac{\tilde{e}_{20}^\infty|c_1^\infty+c_2^\infty|}{|\lambda_-^\infty|}\right)u=\frac{(\sqrt{d}-1)(d-\ell)}{2\sqrt{d}(d+\ell)(d(\sqrt{d}-1)+\ell(\sqrt{d}+1))}Q(d,\ell,u)$$

with

$$Q(d,\ell,u)=2\sqrt{d}(d+\ell)-(\sqrt{d}-1)(\ell+2\sqrt{d}+d)u.$$

We have that $Q(d,\ell,u)$ is strictly positive on $0\le u<u(\ell,d)$ where

$$u(\ell,d) \;:=\; \frac{2\sqrt{d}(d+\ell)}{(\sqrt{d}-1)(\ell+2\sqrt{d}+d)}.$$

Also, since $\ell\to u(\ell,d)$ is increasing for $\ell\ge 0$, we have

$$u(\ell,d)\ge u(0,d)=\frac{2d^{\frac{3}{2}}}{(\sqrt{d}-1)(2\sqrt{d}+d)} \text{ for } \ell\ge 0.$$

Now, $d\to u(0,d)$ attains its minimum for $d=16$ and we find

$$\frac{2d^{\frac{3}{2}}}{(\sqrt{d}-1)(2\sqrt{d}+d)}\ge\frac{16}{9}>1 \text{ for } d\ge 1.$$

Therefore, $u(\ell,d)>1$ for all $d\ge 1$ and thus $\inf_{u\in[0,1]}Q(d,\ell)(u)>c(d,\ell)>0$ which ensures

$$B_0+bB_1u\ge\frac{b}{|c_1^\infty+c_2^\infty|}\left[c(d,\ell)+O(b)\right]>0$$

and (9.33) is proved.

case $r_\infty=r_+(d,\ell)$. This case is degenerate since an explicit computation shows that

$$\left|\begin{array}{l} B_0(b=0)=\frac{dB_0}{db}|_{b=0}=0\\ B_1(b=0)=0\end{array}\right.$$

and, as a result, we need to consider the order b^2. We compute, using Appendix H:

$$B_0+bB_1u+b^2(B_3u+B_4u^2)=b^2F(\ell,d,u)+O(b^3)$$

with

$$F(\ell,d,u)=\frac{\sqrt{d-1}\big(1+\sqrt{\ell}\big)^5(1+\ell)}{2\ell^{\frac{3}{4}}\Big(1+(1+d)\sqrt{\ell}+\ell\Big)^2}(1-u)+\frac{3\sqrt{d-1}\big(1+\sqrt{\ell}\big)^5(1+\ell)}{\ell^{\frac{3}{4}}\Big(1+(1+d)\sqrt{\ell}+\ell\Big)^2}u(1-u)$$

Then (9.34) follows from

$$\forall\ell>0,\quad \forall d>1,\quad \inf_{u\in[0,\frac{3}{4}]}F(\ell,d,u)>0. \tag{9.35}$$

which is immediate from the above formula for $F(\ell,d,u)$. □

Appendix A. Facts related to the Γ function

We collect various classical facts about the Γ function.

Euler β function. $B(x,y)$ is defined for all (x,y) with $\Re(x)>0$ and $\Re(y)>0$ by

$$B(x,y)=\int_0^1 u^{x-1}(1-u)^{y-1}du=\int_0^{+\infty}\frac{Y^{x-1}}{(1+Y)^{x+y}}dY=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{A.1}$$

$$\begin{aligned}\int_0^{+\infty}\frac{dz}{(1+z)^{K+2}z^{\alpha_\gamma}}dz &= \int_0^{+\infty}\frac{z^{-\alpha_\gamma}}{(1+z)^{K+2}}dz = B(1-\alpha_\gamma, K+1+\alpha_\gamma)\\ &= \frac{\Gamma(1-\alpha_\gamma)\Gamma(K+1+\alpha_\gamma)}{\Gamma(K+2)}. \end{aligned} \tag{A.2}$$

Recurrence formulas. We compute for $k_1 \le k_2$:

$$\Pi_{j=k_1}^{k_2}(\gamma-j) = \Pi_{j=k_1}^{k_2}(\alpha_\gamma+K+1-j) = \Pi_{\ell=K+1-k_2}^{K+1-k_1}(\alpha_j+\ell) = \frac{\Pi_{\ell=1}^{K+1-k_1}(\alpha_j+\ell)}{\Pi_{\ell=1}^{K-k_2}(\alpha_j+\ell)}.$$

We now recall

$$\left|\begin{array}{l}\Gamma(x+1) = x\Gamma(x), \quad x\in\mathbb{C}\backslash\mathbb{Z}_-\\ \Gamma(k+1) = k!\end{array}\right. \tag{A.3}$$

where $\mathbb{Z}_-$ denotes the set of negative integers, from which

$$\Pi_{\ell=1}^{m}(x+\ell) = \frac{\Gamma(x+m+1)}{\Gamma(x+1)}, \quad x\in\mathbb{C}\backslash\mathbb{Z}_- \tag{A.4}$$

which yields

$$\Pi_{j=k_1}^{k_2}(\gamma-j) = \frac{\Gamma(\alpha_\gamma+K+2-k_1)}{\Gamma(\alpha_\gamma+K-k_2+1)} = \frac{\Gamma(\gamma+1-k_1)}{\Gamma(\gamma-k_2)} \tag{A.5}$$

Therefore,

$$\Pi_{j=0}^{K-1}(\gamma-j-2) = \Pi_{j=2}^{K+1}(\gamma-j) = \frac{\Gamma(\alpha_\gamma+K)}{\Gamma(\alpha_\gamma)}. \tag{A.6}$$

Asymptotics. We recall Stirling's formula

$$\Gamma(x+1) = (1+o_{x\to+\infty}(1))\left(\frac{x}{e}\right)^x\sqrt{2\pi x}. \tag{A.7}$$

Let

$$|x| \lesssim 1 \ll \gamma,$$

this yields:

$$\begin{aligned}\frac{\Gamma(\gamma+x+1)}{\Gamma(\gamma+1)} &= (1+o_{\gamma\to+\infty}(1))\frac{\left(\frac{\gamma+x}{e}\right)^{\gamma+x}\sqrt{2\pi(\gamma+x)}}{\left(\frac{\gamma}{e}\right)^{\gamma}\sqrt{2\pi\gamma}}\\ &= (1+o_{\gamma\to+\infty}(1))\frac{1}{e^x}e^{(\gamma+x)\left[\log\gamma+\frac{x}{\gamma}+O\left(\frac{1}{\gamma^2}\right)\right]-\gamma\log\gamma}\\ &= (1+o_{\gamma\to+\infty}(1))\frac{1}{e^x}e^{x\log\gamma+x+O_x\left(\frac{1}{\gamma}\right)} = (1+o_{\gamma\to+\infty}(1))\gamma^x. \end{aligned} \tag{A.8}$$

Moreover,

$$\begin{aligned}\Gamma(\gamma+1+x) &= (1+o_{\gamma\to+\infty}(1))\left(\frac{\gamma+x}{e}\right)^{\gamma+x}\sqrt{2\pi(\gamma+x)}\\ &= (1+o_{\gamma\to+\infty}(1))\frac{\sqrt{2\pi\gamma}}{e^{\gamma+x}}e^{(\gamma+x)\left[\log\gamma+\frac{x}{\gamma}+O_x\left(\frac{1}{\gamma^2}\right)\right]}\\ &= (1+o_{\gamma\to+\infty}(1))\frac{\sqrt{2\pi\gamma}}{e^{\gamma+x}}e^{\gamma\log\gamma+x+x\log\gamma+O_x\left(\frac{1}{\gamma}\right)}\\ &= \sqrt{2\pi}(1+o_{\gamma\to+\infty}(1))\frac{\gamma^{\gamma+x+\frac{1}{2}}}{e^\gamma}. \end{aligned} \tag{A.9}$$

Value on $\mathbb{R}\backslash\mathbb{N}_-$.

Lemma A.1 (Value of $\Gamma(x)$ for $x \in \mathbb{R}\backslash\mathbb{N}_-$)**.** *Let*

$$x = -K_x + \alpha_x, \quad K_x \in \mathbb{N}^*, \quad 0 < \alpha_x < 1$$

then

$$\Gamma(x) = (-1)^{K_x} \frac{\Gamma(\alpha_x)\Gamma(1-\alpha_x)}{\Gamma(1-x)}. \tag{A.10}$$

Proof. By definition

$$\Gamma(x) = \frac{\Gamma(x+1)}{x} = \frac{\Gamma(x+J+1)}{\Pi_{j=0}^{J}(x+j)}$$

and thus, with $J = K_x - 1$ and using (A.4),

$$\begin{aligned}\Gamma(x) &= \frac{\Gamma(-K_x+\alpha_x+K_x-1+1)}{\Pi_{j=0}^{K_x-1}\Gamma(\alpha_x-K_x+j)} = \frac{\Gamma(\alpha_x)}{\Pi_{m=1}^{K_x}(\alpha_x-m)} = (-1)^{K_x}\frac{\Gamma(\alpha_x)}{\Pi_{m=1}^{K_x}(-\alpha_x+m)} \\ &= (-1)^{K_x}\frac{\Gamma(\alpha_x)}{\frac{\Gamma(-\alpha_x+K_x+1)}{\Gamma(-\alpha_x+1)}} = (-1)^{K_x}\frac{\Gamma(\alpha_x)\Gamma(1-\alpha_x)}{\Gamma(1-x)}.\end{aligned}$$

□

Appendix B. Uniform convolution bounds for $b = 0$

Lemma B.1 (Lower bound on the Stirling phase)**.** *Let $\nu > 0$. Pick a large enough universal constant $R > R_\nu \gg 1$. Then there exist $c_\nu > 1$ and $k^* = k^*(\nu, R) \gg 1$ such that for all $k > k^*$, for all*

$$k_1 = kx, \quad k_2 = k(1-x), \quad 0 \le x \le \frac{1}{2},$$

there holds the bound

$$\frac{\Gamma(k_1+\nu+2)\Gamma(k_2+\nu+2)}{\Gamma(k+\nu+2)} \le c_\nu\sqrt{1+k_1}e^{-k\left[\Phi_k^{(2)}(x)\right]} \tag{B.1}$$

where the phase function satisfies

$$\Phi_k^{(2)}(x) \ge 0, \quad (\Phi_k^{(2)})'(x) \ge 0.$$

Moreover, for $\frac{R}{k} < x \le \frac{1}{2}$,

$$\Phi_k^{(2)}(x) \ge \frac{1}{2}x|\log x|. \tag{B.2}$$

Proof of Lemma B.1. We assume $k \ge k^*(R,\nu)$ large enough and apply Stirling's formula

$$\Gamma(x+1) = (1+o_{x\to+\infty}(1))\left(\frac{x}{e}\right)^x\sqrt{2\pi x}$$

to upper bound

$$\begin{aligned}&\frac{\Gamma(k_1+\nu+2)\Gamma(k_2+\nu+2)}{\Gamma(k+\nu+2)} \\ \lesssim_\nu\ &\frac{\left(\frac{k_1+\nu+1}{e}\right)^{k_1+\nu+1}\sqrt{k_1+\nu+1}\left(\frac{k_2+\nu+1}{e}\right)^{k_2+\nu+1}\sqrt{k_2+\nu+1}}{\left(\frac{k+\nu+1}{e}\right)^{k+\nu+1}\sqrt{k+\nu+1}} \\ \lesssim_\nu\ &\sqrt{k_1+\nu+1}e^{-[(k+\nu+1)\log(k+\nu+1)-(k_1+\nu+1)\log(k_1+\nu+1)-(k_2+\nu+1)\log(k_2+\nu+1)]}\end{aligned}$$

step 1 Study of the Stirling phase. We compute:

$$\begin{aligned}
& (k+\nu+1)\log(k+\nu+1) - (k_1+\nu+1)\log(k_1+\nu+1) - (k_2+\nu+1)\log(k_2+\nu+1) \\
= \;& k\left(1+\frac{\nu+1}{k}\right)\left[\log k + \log\left(1+\frac{\nu+1}{k}\right)\right] \\
- \;& k\left(x+\frac{\nu+1}{k}\right)\left[\log k + \log\left(x+\frac{\nu+1}{k}\right)\right] \\
- \;& k\left((1-x)+\frac{\nu+1}{k}\right)\left[\log k + \log\left((1-x)+\frac{\nu+1}{k}\right)\right] \\
= \;& -(\nu+1)\log k + k\left(1+\frac{\nu+1}{k}\right)\log\left(1+\frac{\nu+1}{k}\right) \\
- \;& k\left[\left(x+\frac{\nu+1}{k}\right)\log\left(x+\frac{\nu+1}{k}\right) + \left((1-x)+\frac{\nu+1}{k}\right)\log\left((1-x)+\frac{\nu+1}{k}\right)\right] \\
= \;& -(\nu+1)\log(\nu+1) + (\nu+1)\log\left(\frac{\nu+1}{k}\right) + k\left(1+\frac{\nu+1}{k}\right)\log\left(1+\frac{\nu+1}{k}\right) \\
- \;& k\left[\left(x+\frac{\nu+1}{k}\right)\log\left(x+\frac{\nu+1}{k}\right) + \left((1-x)+\frac{\nu+1}{k}\right)\log\left((1-x)+\frac{\nu+1}{k}\right)\right] \\
= \;& -(\nu+1)\log(\nu+1) + k\Phi_k^{(2)}(x)
\end{aligned}$$

which yields

$$\frac{\Gamma(k_1+\nu+2)\Gamma(k_2+\nu+2)}{\Gamma(k+\nu+2)} \le c_\nu\sqrt{1+k_1}e^{-k\Phi_k^{(2)}(x)}. \tag{B.3}$$

We have

$$\Phi_k^{(2)}(0) = \Phi_k^{(2)}(1) = 0$$

and

$$\begin{aligned}
& \partial_x\Phi_k^{(2)}(x) = -\log\left(x+\frac{\nu+1}{k}\right) + \log\left((1-x)+\frac{\nu+1}{k}\right) \\
= \;& \log\left(\frac{(1-x)+\frac{\nu+1}{k}}{x+\frac{\nu+1}{k}}\right) = \log\left(1+\frac{(1-2x)}{x+\frac{\nu+1}{k}}\right) \ge 0
\end{aligned} \tag{B.4}$$

for $x \in [0, \frac{1}{2}]$. We conclude that $\Phi_k^{(2)}(x)$ is a non negative non decreasing function of $x \in [0, \frac{1}{2}]$.

step 4 Small x lower bound. Assume

$$\frac{R}{k} < x \le \frac{1}{2}$$

where $R > R_\nu \gg 1$, then

$$\begin{aligned}
\Phi_k^{(2)}(x) &= \frac{\nu+1}{k}\log\left(\frac{\nu+1}{k}\right) + \left(1+\frac{\nu+1}{k}\right)\log\left(1+\frac{\nu+1}{k}\right) \\
&- \left(x+\frac{\nu+1}{k}\right)\log\left(x+\frac{\nu+1}{k}\right) - \left((1-x)+\frac{\nu+1}{k}\right)\log\left((1-x)+\frac{\nu+1}{k}\right) \\
&= -\frac{\nu+1}{k}\left|\log\left(1+\frac{kx}{\nu+1}\right)\right| + \left(1+\frac{\nu+1}{k}\right)\left|\log\left(1+\frac{\nu+1}{k}\right)\right| + x\left|\log\left(x+\frac{\nu+1}{k}\right)\right| \\
&\quad + \left((1-x)+\frac{\nu+1}{k}\right)\left|\log\left((1-x)+\frac{\nu+1}{k}\right)\right| \\
&\geq -\frac{\nu+1}{k}\left|\log\left(1+\frac{R}{\nu+1}\right)\right| + x\left|\log x + \log\left(1+\frac{\nu+1}{kx}\right)\right| \\
&\geq x|\log x|\left\{1+O_\nu\left(\frac{\nu+1}{R}\left|\log\left(\frac{\nu+1}{R}\right)\right|\right)\right\} \\
&\geq \frac{1}{2}x|\log x|.
\end{aligned}$$

□

Lemma B.2 (Uniform convolution bound)**.** *Let $\nu > 0$, then there exists $C_\nu > 0$ such that for all $k \geq 1$, for all $j \geq 2$,*

$$\sum_{k_1+\cdots+k_j=k} \frac{\Pi_{i=1}^j \Gamma(k_i+\nu+2)}{\Gamma(k+\nu+2)} \leq C_\nu^j. \tag{B.5}$$

Proof. We argue by induction on j.

step 1 Case $j = 2$. We apply Lemma B.1 with some large enough $R_\nu > 1$ and may without loss of generality assume $k \geq k_\nu^*$, with k_ν^* large enough universal constant. This leads to

$$\frac{\Gamma(k_1+\nu+2)\Gamma(k_2+\nu+2)}{\Gamma(k+\nu+2)} \leq c_\nu\sqrt{1+k_1}e^{-k\left[\Phi_k^{(2)}(x)\right]},$$

where $x = k_1/k$. We may assume $k_1 \leq k_2$ so that $x \leq 1/2$. For $x \leq \frac{R_\nu}{k}$, we use the lower bound $\Phi_k^{(2)}(x) \geq 0$. For $x \geq \frac{R_\nu}{k}$ the monotonicity of $\Phi_k^{(2)}$ on $[0, 1/2]$ implies

$$\Phi_k^{(2)}(x) \geq \Phi_k^{(2)}\left(\frac{R_\nu}{k}\right) \geq +\frac{1}{2}\frac{R_\nu}{k}\left|\log\left(\frac{R_\nu}{k}\right)\right| \geq \frac{R_\nu}{4k}\log k, \tag{B.6}$$

where we required that $R_\nu \leq \sqrt{k_\nu^*}$, from which

$$\begin{aligned}
&\sum_{k_1+k_2=k} \frac{\Gamma(k_1+\nu+2)\Gamma(k_2+\nu+2)}{\Gamma(k+\nu+2)} \\
\lesssim_\nu\ & 1 + \sum_{k_1+k_2=k,\, \frac{R_\nu}{k}\leq x\leq\frac{1}{2}} \sqrt{1+k_1}e^{-\frac{R_\nu\log k}{4}} + \sum_{k_1+k_2=k, 0\leq x\leq\frac{R_\nu}{k}} \sqrt{1+k_1} \\
\lesssim_\nu\ & 1 + R_\nu^{\frac{3}{2}} + \frac{k^{\frac{3}{2}}}{k^{\frac{R_\nu}{4}}} \lesssim_\nu 1.
\end{aligned}$$

since the second sum has $k_1 = xk \leq R_\nu$ terms.

step 2 Induction. We assume j and prove $j+1$:

$$\begin{aligned}
&\sum_{k_1+\cdots+k_{j+1}=k} \frac{\Pi_{i=1}^{j+1}\Gamma(k_i+\nu+2)}{\Gamma(k+\nu+2)} \\
= &\sum_{k_1+m=k} \frac{\Gamma(k_1+\nu+2)\Gamma(m+\nu+2)}{\Gamma(k+\nu+2)} \sum_{k_2+\cdots+k_{j+1}=m} \frac{\Pi_{i=2}^{j+1}\Gamma(k_i+\nu+2)}{\Gamma(m+\nu+2)} \\
\le &\ C_\nu^j \sum_{k_1+m=k} \frac{\Gamma(k_1+\nu+2)\Gamma(m+\nu+2)}{\Gamma(k+\nu+2)} \le C_\nu^{j+1}
\end{aligned}$$

and (B.5) follows. □

Appendix C. Study of the weight w_{γ,ν_b} for $k \le K$

We derive estimates and convolution bounds for the weight $w_{\gamma,\nu_b}(k)$ given by (6.11). We start with estimating the weight.

Lemma C.1 (Estimates on the weight)**.** *There exist a universal constant $c>0$ and $\gamma^* \gg 1$ such that for al $\gamma > \gamma^*$, the following holds. Let $0 \le k < \gamma - 1$, then:*

$$\frac{w_{\gamma,\nu}(k)}{\Gamma(k+\nu+2)} \ge \frac{c}{\gamma^k}. \tag{C.1}$$

Proof. From (6.11), we have

$$\frac{w_{\gamma,\nu}(k)}{\Gamma(k+\nu+2)} = \frac{\Gamma(\gamma-1-k)}{\Gamma(\gamma-1)}.$$

Observe

$$\begin{aligned}
\Gamma(\gamma-1) &= (\gamma-2)\Gamma(\gamma-2) = \left[\Pi_{j=2}^{k+1}(\gamma-j)\right]\Gamma(\gamma-1-k) \\
&= \left[\Pi_{j=1}^{k}(\gamma-1-j)\right]\Gamma(\gamma-1-k)
\end{aligned}$$

and thus

$$\frac{(\gamma-1)^k\Gamma(\gamma-1-k)}{\Gamma(\gamma-1)} = \Pi_{j=1}^k\left(\frac{\gamma-1}{\gamma-1-j}\right) \ge 1. \tag{C.2}$$

□

Lemma C.2 (Summation bound)**.** *For some $c_{\nu,a}>0$, $K_\nu \gg 1$ and all $0<b<b^*(\nu)$ we have the following uniform bounds:*

$$\sum_{k=1}^{K-K_\nu} w_{\gamma,\nu_b}(k) \le c_{\nu,a} b. \tag{C.3}$$

and for $K-K_\nu+1 \le k \le K-1$:

$$w_{\gamma,\nu_b}(k) = e^{O_\nu(1)}\Gamma(\gamma-1-k)\gamma^{\nu_b+2-(\gamma-1-k)}. \tag{C.4}$$

Proof. **step 1** Away from the boundary. Pick a large enough universal integer $K_\nu \gg 1$ and assume first $k \le K-K_\nu$. Then,

$$\gamma-k-2 = K+1+\alpha_\gamma-k-2 = \alpha_\gamma+(K-k)-1 \ge K_\nu-1 \gg 1.$$

From Stirling's formula:

$$\begin{aligned} w_{\gamma,\nu_b}(k) &= \frac{\Gamma(\gamma-1-k)}{\Gamma(\gamma-1)}\Gamma(k+\nu_b+2) \\ &\leq c_\nu \frac{\left(\frac{\gamma-2-k}{e}\right)^{\gamma-2-k}\left(\frac{k+\nu_b+1}{e}\right)^{k+\nu_b+1}}{\left(\frac{\gamma-2}{e}\right)^{\gamma-2}}\left(\frac{(\gamma-2-k)(k+\nu_b+1)}{\gamma-2}\right)^{\frac{1}{2}} \\ &\leq c_\nu e^{-\Phi_\gamma(k)}\left(\frac{(\gamma-2-k)(k+\nu_b+1)}{\gamma-2}\right)^{\frac{1}{2}} \end{aligned}$$

with

$$\Phi_\gamma(x) = (\gamma-2)\log(\gamma-2) - (\gamma-2-x)\log(\gamma-2-x) - (x+\nu_b+1)\log(x+\nu_b+1).$$

We compute

$$\begin{aligned} \Phi'_\gamma(x) &= \log(\gamma-2-x)+1-\log(x+\nu_b+1)-1 = \log\left(\frac{\gamma-2-x}{x+\nu_b+1}\right) \\ &= \log\left(1+\frac{\gamma-\nu_b-3-2x}{x+\nu_b+1}\right)\left|\begin{array}{ll} \geq 0 & \text{for } 1\leq x\leq \frac{\gamma-\nu_b-3}{2} \\ \leq 0 & \text{for } \frac{\gamma-\nu_b-3}{2}\leq x\leq K-2. \end{array}\right. \end{aligned}$$

Pick a large enough universal constant $R_\nu \gg 1$, then for $1 \leq x < R_\nu$,

$$\begin{aligned} \Phi_\gamma(x) &= (\gamma-2)\log(\gamma-2) - (\gamma-2)\left(1-\frac{x}{\gamma-2}\right)\left[\log(\gamma-2)+\log\left(1-\frac{x}{\gamma-2}\right)\right] \\ &- (x+\nu_b+1)\log(x+\nu_b+1) \\ &= \frac{x}{\gamma-2} + O\left(\frac{x^2}{\gamma-2}\right) + x\log(\gamma-2) - (x+\nu_b+1)\log(x+\nu_b+1) \\ &= x\log(\gamma-2)\left[1+O_{R_\nu}\left(\frac{1}{\log(\gamma-2)}\right)\right] + O_\nu(1). \end{aligned}$$

This yields for $1 \leq k \leq R_\nu$: $\Phi_\gamma(x) \geq \log(\gamma-2) - C_\nu$ and, as a result,

$$w_{\gamma,\nu_b}(k) \leq c_\nu e^{-\log(\gamma-2)}\left(\frac{(\gamma-2-k)(k+\nu+1)}{\gamma-2}\right)^{\frac{1}{2}} \leq \frac{c_\nu}{\gamma}.$$

For $R_\nu < x \leq \frac{\gamma-\nu_b-3}{2}$

$$\Phi_\gamma(x) \geq \Phi_\gamma(R_\nu) \geq \frac{R_\nu}{2}\log(\gamma-2)$$

and thus,

$$w_\gamma(k) \leq \frac{c_\nu}{\gamma^{\frac{R_\nu}{4}}}.$$

We obtain the bound

$$\sum_{1\leq k\leq \frac{\gamma-\nu_b-3}{2}} w_{\gamma,\nu_b}(k) \leq c_{\nu,a} b.$$

On the other hand,

$$\begin{aligned} &\Phi_\gamma(\gamma-R_\nu) = (\gamma-2)\log(\gamma-2) - (R_\nu-2)\log(R_\nu-2) \\ -\ &(\gamma-R_\nu+\nu_b+1)\log(\gamma-R_\nu+\nu_b+1) \\ =\ &(\gamma-2)\log(\gamma-2) - (\gamma-2)\left[1-\frac{R_\nu-\nu_b-3}{\gamma-2}\right]\left[\log(\gamma-2)-\log\left(1-\frac{R_\nu-\nu_b-3}{\gamma-2}\right)\right] \\ +\ &O(R_\nu) \geq \frac{R_\nu}{4}\log(\gamma-2). \end{aligned}$$

The monotonicity of Φ_γ then implies

$$\Phi_\gamma(k) \geq \Phi_\gamma(\gamma - R_\nu) \geq \frac{R_\nu}{4}\log(\gamma-2) \ \text{ for } \ \frac{\gamma-\nu_b-3}{2} \leq k \leq \gamma - R_\nu$$

which yields

$$\sum_{\frac{\gamma-\nu_b-3}{2}\leq k\leq \gamma-K_\nu} w_{\gamma,\nu_b}(k) \leq c_{\nu,a} b$$

and concludes the proof of (C.3).

step 2 From Stirling's formula, for $K - K_\nu + 1 \leq k \leq K-1$:

$$\frac{\Gamma(k+\nu_b+2)}{\Gamma(\gamma-1)} = (1+o_\nu(1))\frac{\left(\frac{k+\nu_b+1}{e}\right)^{k+\nu_b+1}}{\left(\frac{\gamma-2}{e}\right)^{\gamma-2}}\left(\frac{k+\nu_b+1}{\gamma-2}\right)^{\frac{1}{2}} = e^{O_\nu(1)}e^{-\Phi_\gamma(k)}$$

with

$$\Phi_\gamma(k) = (\gamma-2)\log(\gamma-2) - (k+\nu_b+1)\log(k+\nu_b+1).$$

Let

$$k = \gamma - x = K+1+\alpha_\gamma - x, \quad 2+\alpha_\gamma \leq x \leq K_\nu + \alpha_\gamma,$$

then

$$\begin{aligned}
\Phi_\gamma(k) &= (\gamma-2)\log(\gamma-2) - (\gamma-x+\nu_b+1)\log(\gamma-x+\nu_b+1) \\
&= (\gamma-2)\log(\gamma-2) - (\gamma-2)\left[1+\frac{\nu_b+3-x}{\gamma-2}\right]\left[\log(\gamma-2)+\log\left(1+\frac{\nu_b+3-x}{\gamma-2}\right)\right] \\
&= (x-\nu_b-3)\log(\gamma-2) + O_\nu(1)
\end{aligned}$$

and we obtain the formula

$$\begin{aligned}
w_{\gamma,\nu_b}(k) &= \Gamma(\gamma-1-k)\frac{e^{O_\nu(1)}}{(\gamma-2)^{x-\nu_b-3}} = e^{O_\nu(1)}\Gamma(\gamma-1-k)\gamma^{\nu_b+3-(\gamma-k)} \\
&= e^{O_\nu(1)}\Gamma(\gamma-1-k)\gamma^{\nu_b+2-(\gamma-1-k)}.
\end{aligned}$$

□

We now turn to the convolution type estimate on the weight.

Lemma C.3 (Convolution estimate)**.** *There exist universal constants $c_{\nu,a} > 0$, $0 < b^*(\nu,a) \ll 1$ such that the following holds: for all $0 < b < b^*$, for all $0 \leq k \leq K$,*

$$\sum_{k_1+k_2=k} w_{\gamma,\nu}(k_1)w_{\gamma,\nu}(k_2) \leq c_{\nu,a} w_{\gamma,\nu}(k). \tag{C.5}$$

More generally,

$$\sum_{k_1+\cdots+k_j=k} w_{\gamma,\nu}(k_1)\dots w_{\gamma,\nu}(k_j) \leq c_{\nu,a}^j w_{\gamma,\nu}(k). \tag{C.6}$$

Proof of Lemma C.3. We prove (C.5). The proof of (C.6) follows by induction and is left to the reader.

We parametrize

$$\left|\ k_1 = kx,\ k_2 = (1-x)k \right. \quad \Leftrightarrow \left| \begin{array}{l} \gamma-1-k = (\gamma-1)(1-\alpha) \\ \gamma-1-k_1 = (\gamma-1)(1-\alpha x) \\ \gamma-1-k_2 = (\gamma-1)(1-\alpha(1-x)) \end{array}\right.$$

For $k_1 = 0$ or $k_2 = 0$, the claim is trivial and we therefore assume without loss of generality

$$1 \le k_1 \le k_2 < \gamma - 2.$$

We recall

$$\gamma - 1 - k = \alpha_\gamma + (K - k), \quad 0 \le k \le K.$$

step 1 Generic case. Assume first

$$k \le K - 2 \tag{C.7}$$

which ensures:

$$\gamma - 2 - k = \alpha_\gamma + (K - k - 1) \ge 1. \tag{C.8}$$

Representation formula. We have

$$\frac{w_{\gamma,\nu}(k_1) w_{\gamma,\nu}(k_2)}{w_{\gamma,\nu}(k)} = \frac{\Gamma(\gamma - 1 - k_1)\Gamma(\gamma - 1 - k_2)}{\Gamma(\gamma - 1 - k)\Gamma(\gamma - 1)} \frac{\Gamma(k_1 + \nu + 2)\Gamma(k_2 + \nu + 2)}{\Gamma(k + \nu + 2)}.$$

The function $\Gamma(\gamma - 1 - z)$ is holomorphic and bounded for $0 \le \mathcal{R}(z) \le \gamma - 1$. Using the three line theorem, we obtain

$$\Gamma(\gamma - 1 - k_1) \le \Gamma(\gamma - 1)^{1-x} \Gamma(\gamma - 1 - k)^x,$$
$$\Gamma(\gamma - 1 - k_2) \le \Gamma(\gamma - 1)^{x} \Gamma(\gamma - 1 - k)^{1-x}$$

which implies that

$$\frac{\Gamma(\gamma - 1 - k_1)\Gamma(\gamma - 1 - k_2)}{\Gamma(\gamma - 1 - k)\Gamma(\gamma - 1)} \le 1$$

Since the Γ function is strictly positive and bounded on $[1, R]$ for all $R > 0$, we may use Stirling's formula to upper bound:

$$\begin{aligned} & \frac{\Gamma(\gamma - 1 - k_1)\Gamma(\gamma - 1 - k_2)}{\Gamma(\gamma - 1 - k)\Gamma(\gamma - 1)} \\ \le\ & c_\nu \frac{\left(\frac{\gamma-2-k_1}{e}\right)^{\gamma-2-k_1} \sqrt{2\pi(\gamma - 2 - k_1)} \left(\frac{\gamma-2-k_2}{e}\right)^{\gamma-2-k_2} \sqrt{2\pi(\gamma - 2 - k_2)}}{\left(\frac{\gamma-2-k}{e}\right)^{\gamma-2-k} \sqrt{2\pi(\gamma - 2 - k)} \left(\frac{\gamma-2}{e}\right)^{\gamma-2} \sqrt{2\pi(\gamma - 2)}}. \end{aligned}$$

We compute the Stirling phase:

$$\begin{aligned} & (\gamma - 2)\log(\gamma - 2) + (\gamma - 2 - k)\log(\gamma - 2 - k) \\ - \ & (\gamma - 2 - k_1)\log(\gamma - 2 - k_1) - (\gamma - 2 - k_2)\log(\gamma - 2 - k_2) \\ = \ & (\gamma - 2)\log(\gamma - 2) + (\gamma - 2 - k)\log(\gamma - 2 - k) \\ - \ & (\gamma - 2 - kx)\log(\gamma - 2 - kx) - (\gamma - 2 - k(1 - x))\log(\gamma - 2 - k(1 - x)) \\ = \ & \gamma \Phi^{(1)}_{k,\gamma}(x) \end{aligned} \tag{C.9}$$

Monotonicity for $\Phi^{(1)}_{k,\gamma}(x)$. We compute

$$\Phi^{(1)}_{k,\gamma}(0) = 0$$

and

$$\begin{aligned} & \left[\Phi^{(1)}_{k,\gamma}\right]'(x) = k\log(\gamma - 2 - kx) + k - k\log(\gamma - 2 - k(1 - x)) - k \\ = \ & k\log\left(\frac{\gamma - 2 - xk}{\gamma - 2 - (1 - x)k}\right) = k\log\left(1 + \frac{(1 - 2x)k}{\gamma - 2 - (1 - x)k}\right) > 0 \end{aligned} \tag{C.10}$$

for $0 \le x < \frac{1}{2}$. In particular,

$$\Phi^{(1)}_{k,\gamma}(x) \ge 0. \tag{C.11}$$

case k_1 large. Pick a large enough universal constant $R_\nu \gg 1$ as in Lemma B.1, then for $x \geq \frac{R_\nu}{k}$, from (B.1), (B.6):

$$\frac{\Gamma(k_1+\nu+2)\Gamma(k_2+\nu+2)}{\Gamma(k+\nu+2)} \leq c_\nu \sqrt{1+k_1} e^{-k\Phi_k^{(2)}(x)} \leq \frac{c_\nu}{k^{\frac{R_\nu}{4}}}$$

and therefore, from (C.11):

$$\begin{aligned} \frac{w_{\gamma,\nu}(k_1)w_{\gamma,\nu}(k_2)}{w_{\gamma,\nu}(k)} &= \frac{\Gamma(\gamma-1-k_1)\Gamma(\gamma-1-k_2)}{\Gamma(\gamma-1-k)\Gamma(\gamma-1)} \frac{\Gamma(k_1+\nu+2)\Gamma(k_2+\nu+2)}{\Gamma(k+\nu+2)} \\ &\leq \frac{c_\nu}{k^{\frac{R_\nu}{4}}} \left(\frac{(\gamma-2-k_1)(\gamma-2-k_2)}{(\gamma-2-k)(\gamma-2)} \right)^{\frac{1}{2}} \end{aligned}$$

If $k \leq \frac{\gamma-2}{2}$ then $k_1, k_2 \leq \frac{\gamma-2}{2}$ and

$$\frac{(\gamma-2-k_1)(\gamma-2-k_2)}{(\gamma-2-k)(\gamma-2)} \lesssim 1.$$

Otherwise, $\gamma - 1 > k = k_1 + k_2 \geq 2k_1$ and, using (C.8):

$$\left(\frac{(\gamma-2-k_1)(\gamma-2-k_2)}{(\gamma-2-k)(\gamma-2)} \right) \leq \gamma - 2 - k_1 \leq \gamma \leq 2k$$

which implies

$$\frac{w_{\gamma,\nu}(k_1)w_{\gamma,\nu}(k_2)}{w_{\gamma,\nu}(k)} \leq \frac{c_\nu}{k^{\frac{R_\nu}{8}}}$$

and gives

$$\sum_{k_1+k_2=k, R_\nu \leq k_1 \leq \frac{k}{2}} \frac{w_{\gamma,\nu}(k_1)w_{\gamma,\nu}(k_2)}{w_\gamma(k)} \leq c_\nu. \tag{C.12}$$

case k_1 small. for $x < \frac{R_\nu}{k}$ i.e. $k_1 < R_\nu$, using $\Phi_{k,\gamma}^{(1)} \geq 0$, $\Phi_k^{(2)} \geq 0$, we have the bound

$$\frac{w_{\gamma,\nu}(k_1)w_{\gamma,\nu}(k_2)}{w_{\gamma,\nu}(k)} \leq c_\nu \sqrt{1+k_1} \left(\frac{(\gamma-2-k_1)(\gamma-2-k_2)}{(\gamma-2-k)(\gamma-2)} \right)^{\frac{1}{2}},$$

and using (C.8):

$$\frac{(\gamma-2-k_1)(\gamma-2-k_2)}{(\gamma-2-k)(\gamma-2)} \leq \frac{\gamma-2-k_2}{\gamma-2-k} = \frac{\gamma-2-k+k_1}{\gamma-2-k} \leq 1 + \frac{R_\nu}{\gamma-2-k} \leq 1 + R_\nu \leq c_\nu$$

Since the sum has at most $k_1 \leq R_\nu$ terms, we obtain

$$\sum_{k_1+k_2=k, 0 \leq k_1 \leq R_\nu} \frac{w_{\gamma,\nu}(k_1)w_{\gamma,\nu}(k_2)}{w_\gamma(k)} \leq c_\nu. \tag{C.13}$$

This concludes the proof of (C.5) in the regime (C.7).

step 2 Boundary terms. We treat separately the cases

$$k \in \{K-1, K\}. \tag{C.14}$$

For $k_1 = 1$, we have

$$\begin{aligned} &\frac{\Gamma(\gamma-1-k_1)\Gamma(\gamma-1-k_2)}{\Gamma(\gamma-1-k)\Gamma(\gamma-1)} = \frac{\Gamma(\gamma-2)\Gamma(\gamma-1-(k-1))}{\Gamma(K-k+\alpha_\gamma)\Gamma(\gamma-1)} \\ \leq\ & \frac{c}{\gamma} \frac{\Gamma(K-k+\alpha_\gamma+1)}{\Gamma(K-k+\alpha_\gamma)} \leq c \end{aligned}$$

Then, using $\Phi_k^{(2)} \geq 0$:

$$\begin{aligned}\frac{w_{\gamma,\nu}(k_1)w_{\gamma,\nu}(k_2)}{w_{\gamma,\nu}(k)} &= \frac{\Gamma(\gamma-1-k_1)\Gamma(\gamma-1-k_2)}{\Gamma(\gamma-1-k)\Gamma(\gamma-1)}\frac{\Gamma(k_1+\nu+2)\Gamma(k_2+\nu+2)}{\Gamma(k+\nu+2)} \\ &\leq c_\nu\sqrt{1+k_1}e^{-k\Phi_k^{(2)}(x)} \leq c_\nu.\end{aligned}$$

We may therefore assume

$$k_1 \geq 2 \tag{C.15}$$

and therefore,

$$\gamma - 2 - k_2 \geq \gamma - 2 - (k-2) = \alpha_\gamma + (K-k) - 1 + 2 \geq 1$$

$k_1 \leq k_2$ implies $k_1 \leq \frac{k}{2} \leq \frac{\gamma-1}{2}$ and

$$\gamma - 1 - k_1 \gg 1.$$

Moreover,

$$\Gamma(\gamma-1-k) = \Gamma((K-k)+\alpha_\gamma) \gtrsim 1$$

independent of γ. As a result, (B.1) yields the upper bound:

$$\begin{aligned}\frac{w_{\gamma,\nu}(k_1)w_{\gamma,\nu}(k_2)}{w_{\gamma,\nu}(k)} &= \frac{\Gamma(\gamma-1-k_1)\Gamma(\gamma-1-k_2)}{\Gamma(\gamma-1-k)\Gamma(\gamma-1)}\frac{\Gamma(k_1+\nu+2)\Gamma(k_2+\nu+2)}{\Gamma(k+\nu+2)} \\ &\leq c_\nu\frac{\Gamma(\gamma-1-k_1)\Gamma(\gamma-1-k_2)}{\Gamma(\gamma-1)}\sqrt{1+k_1}e^{-k\Phi_k^{(2)}(x)}\end{aligned}$$

We compute the modified Stirling phase:

$$\begin{aligned}&\frac{\Gamma(\gamma-1-k_1)\Gamma(\gamma-1-k_2)}{\Gamma(\gamma-1)} \\ \leq\ & c\frac{\left(\frac{\gamma-2-k_1}{e}\right)^{\gamma-2-k_1}\sqrt{2\pi(\gamma-2-k_1)}\left(\frac{\gamma-2-k_2}{e}\right)^{\gamma-2-k_2}\sqrt{2\pi(\gamma-2-k_2)}}{\left(\frac{\gamma-2}{e}\right)^{\gamma-2}\sqrt{2\pi(\gamma-2)}} \\ \leq\ & c\left(\frac{(\gamma-2-k_1)(\gamma-2-k_2)}{\gamma-2}\right)^{\frac{1}{2}}e^{-\gamma\Phi_\gamma^{(3)}(x)}\end{aligned} \tag{C.16}$$

with

$$\begin{aligned}&\gamma\Phi_{k,\gamma}^{(3)}(x) \\ =\ & (\gamma-2)\log(\gamma-2) - (\gamma-2-k_1)\log(\gamma-2-k_1) - (\gamma-2-k_2)\log(\gamma-2-k_2) \\ =\ & (\gamma-2)\log(\gamma-2) - (\gamma-2-kx)\log(\gamma-2-kx) - (\gamma-2-k(1-x))\log(\gamma-2-k(1-x)).\end{aligned}$$

We have verbatim as above

$$\left[\Phi_\gamma^{(3)}\right]'(x) \geq 0 \ \text{ for } \ 0 \leq x \leq \frac{1}{2}$$

and, since $x = \frac{k_1}{k} \geq \frac{2}{k}$:

$$\begin{aligned}&\Phi_\gamma^{(3)}(x) \geq \Phi_\gamma^{(3)}\left(\frac{2}{k}\right) \\ =\ & (\gamma-2)\log(\gamma-2) - (\gamma-2-2)\log(\gamma-2-2) - (\gamma-2-k+2))\log(\gamma-2-k+2)) \\ =\ & (\gamma-2)\log(\gamma-2) - (\gamma-4)\log(\gamma-4) - (K-k+\alpha_\gamma+1)\log(K-k+\alpha_\gamma+1) \geq -c\end{aligned}$$

for some universal constant C independent of γ from (C.14). We obtain the bound

$$\frac{w_{\gamma,\nu}(k_1)w_{\gamma,\nu}(k_2)}{w_{\gamma,\nu}(k)} \leq c\left(\frac{(\gamma-2-k_1)(\gamma-2-k_2)}{\gamma-2}\right)^{\frac{1}{2}}\sqrt{1+k_1}e^{-k\Phi_k^{(2)}(x)}.$$

case k_1 large. Arguing verbatim as before, we pick a large enough universal constant $R_\nu \gg 1$ as in Lemma B.1 and estimate for $x \geq \frac{R_\nu}{k}$,

$$\frac{w_{\gamma,\nu}(k_1)w_{\gamma,\nu}(k_2)}{w_{\gamma,\nu}(k)} \leq c\left(\frac{(\gamma-2-k_1)(\gamma-2-k_2)}{\gamma-2}\right)^{\frac{1}{2}}\frac{1}{k^{\frac{R_\nu}{4}}}$$

and

$$\frac{(\gamma-2-k_1)(\gamma-2-k_2)}{\gamma-2} \leq \gamma-2 \leq 2k$$

(C.12) follows again.

case k_1 small. Since $k_1 \leq R_\nu$, and in view of (C.14):

$$\frac{(\gamma-2-k_1)(\gamma-2-k_2)}{\gamma-2} \leq \gamma-2-k_2 = \gamma-2-(k-k_1) \leq R_\nu+2$$

and (C.13) follows again. This concludes the proof of (C.5). □

Lemma C.4 (Truncated convolution estimate)**.** *For $K+1 \leq k \leq 2K$:*

$$\sum_{k_1+k_2=k, 0\leq k_1,k_2\leq K} w_{\gamma,\nu_b}(k_1)w_{\gamma,\nu_b}(k_2) \leq c_{\nu,a}w_{\gamma,\nu_b}(k-K)w_{\gamma,\nu_b}(K). \tag{C.17}$$

Proof of Lemma C.4. Let

$$\left|\begin{array}{l} K+1 \leq k \leq 2K \\ k_1+k_2=k \\ 0 \leq k_1 \leq k_2 \leq K \end{array}\right.$$

then

$$k-K \leq k_1 \leq \frac{k}{2}.$$

We compute

$$\begin{aligned}\frac{\Gamma(\nu_b+2+k_1)\Gamma(\nu_b+2+k_2)}{\Gamma(k-K)\Gamma(K)} &= \frac{\Pi_{j=0}^{k_1}(\nu_b+2+j)\Pi_{j=0}^{k_2}(\nu_b+2+j)}{\Pi_{j=0}^{K}(\nu_b+2+j)\Pi_{j=0}^{k-K}(\nu_b+2+j)} \\ &= \Pi_{j=k-K+1}^{k_1}\left(\frac{\nu_b+2+j}{\nu_b+2+j+(K-k_1)}\right) \leq 1\end{aligned}$$

since $k_1 \leq K$. It follows that

$$\frac{w_{\gamma,\nu_b}(k_1)w_{\gamma,\nu_b}(k_2)}{w_{\gamma,\nu_b}(k-K)w_{\gamma,\nu_b}(K)} \leq \frac{\Gamma(\gamma-1-k_1)\Gamma(\gamma-1-k_2)}{\Gamma(K+\alpha_\gamma-(k-K))\Gamma(\alpha_\gamma)} \tag{C.18}$$

For $k=2K$, we have $k_1=k_2=K$ and from (C.18):

$$\frac{w_{\gamma,\nu_b}(k_1)w_{\gamma,\nu_b}(k_2)}{w_{\gamma,\nu_b}(k-K)w_{\gamma,\nu_b}(K)} \leq \frac{\Gamma(\gamma-1-k_1)\Gamma(\gamma-1-k_2)}{\Gamma(\alpha_\gamma)\Gamma(\alpha_\gamma)} = \frac{\Gamma(\alpha_\gamma)\Gamma(\alpha_\gamma)}{(\Gamma(\alpha_\gamma))^2} = 1.$$

We therefore assume

$$k \leq 2K-1$$

and

$$k_2 \leq K-1, \quad k_1 \geq k-K+1$$

Using (C.18):

$$\frac{\Gamma(\gamma-1-k_1)\Gamma(\gamma-1-k_2)}{\Gamma(K+\alpha_\gamma-(k-K))\Gamma(\alpha_\gamma)} = \frac{\Gamma(\alpha_\gamma+K-k_1)\Gamma(\alpha_\gamma+K-k_2)}{\Gamma(2K+\alpha_\gamma-k)\Gamma(\alpha_\gamma)} \leq \frac{1}{\Gamma(\alpha_\gamma)}$$

$$\times \quad \frac{\left(\frac{\alpha_\gamma+K-k_1-1}{e}\right)^{\alpha_\gamma+K-k_1-1}\left(\frac{\alpha_\gamma+K-k_2-1}{e}\right)^{\alpha_\gamma+K-k_2-1}}{\left(\frac{\alpha_\gamma+2K-k-1}{e}\right)^{\alpha_\gamma+2K-k-1}}\left(\frac{(\alpha_\gamma+K-k_1-1)(\alpha_\gamma+K-k_2-1)}{\alpha_\gamma+2K-k-1}\right)^{\frac{1}{2}}$$

$$\leq \quad \frac{c}{\Gamma(\alpha_\gamma)}\left(\frac{(\alpha_\gamma+K-k_1-1)(\alpha_\gamma+K-k_2-1)}{\alpha_\gamma+2K-k-1}\right)^{\frac{1}{2}} e^{-\Phi^{(5)}_{\gamma,k}(k_1)}$$

where

$$\Phi^{(5)}_{\gamma,k}(x) = (\alpha_\gamma+2K-k-1)\log(\alpha_\gamma+2K-k-1)$$
$$- \quad (\alpha_\gamma+K-x-1)\log(\alpha_\gamma+K-x-1) - (\alpha_\gamma+K-(k-x)-1)\log(\alpha_\gamma+K-(k-x)-1).$$

satisfies

$$\left[\Phi^{(5)}_{\gamma,k}\right]'(x) = \log\left(\frac{\alpha_\gamma+K-x-1}{\alpha_\gamma+K-(k-x)-1}\right)$$
$$= \quad \log\left(1+\frac{k-2x}{\alpha_\gamma+K-(k-x)-1}\right) \geq 0.$$

Moreover, for $A>0$ and $|y| \ll A$:

$$\begin{aligned} A\log A-(A+y)\log(A+y) &= A\log A - A\left[1+\frac{y}{A}\right]\left[\log A+\frac{y}{A}+O\left[\left(\frac{y}{A}\right)^2\right]\right] \\ &= -y\left[\log A+1+O\left(\frac{y}{A}\right)\right] \end{aligned} \tag{C.19}$$

Pick a large enough universal constant $K_\nu \gg 1$. For

$$x = k-K+y, \quad 1 \leq y \leq K_\nu \tag{C.20}$$

we use (C.19) to estimate:

$$\Phi^{(5)}_{\gamma,k}(x) = (\alpha_\gamma+2K-k-1)\log(\alpha_\gamma+2K-k-1)$$
$$- \quad (\alpha_\gamma+2K-k-1-y)\log(\alpha_\gamma+2K-k-1-y) - (\alpha_\gamma-1+y)\log(\alpha_\gamma-1+y)$$
$$= \quad y\left[\log(\alpha_\gamma+2K-k-1)+O(1)\right] - (\alpha_\gamma-1+y)\log(\alpha_\gamma-1+y)$$
$$\geq \quad \frac{y}{2}\log\gamma \tag{C.21}$$

Similarly,

$$\frac{\Gamma(\nu_b+k_1+2)\Gamma(\nu_b+k_2+2)}{\Gamma(k-K+\nu_b+2)\Gamma(K+\nu_b+2)}$$
$$\leq \quad \left(\frac{(\nu_b+k_1)(\nu_b+k_2)}{(k-K+\nu_b+1)(K+\nu_b+2)}\right)^{\frac{1}{2}} e^{-\Phi^{(6)}_{\gamma,k}(k_1)}$$

where

$$\Phi^{(6)}_{\gamma,k}(x) = (k-K+\nu_b+1)\log(k-K+\nu_b+1) + (K+\nu_b+1)\log(K+\nu_b+1)$$
$$- \quad (\nu_b+x+1)\log(\nu_b+x+1) - (\nu_b+k-x+1)\log(\nu_b+k-x+1)$$

satisfies

$$\left[\Phi^{(6)}_{\gamma,k}\right]'(x) = \log\left[\frac{\nu_b+k-x+1}{\nu_b+x+1}\right] = \log\left(1+\frac{k-2x}{\nu_b+x+1}\right) \geq 0.$$

Moreover

$$\begin{aligned}
&\Phi^{(6)}_{\gamma,k}(x) = (k-K+\nu_b+1)\log(k-K+\nu_b+1) + (K+\nu_b+1)\log(K+\nu_b+1)\\
-\ &(\nu_b+x+1)\log(\nu_b+x+1) - (\nu_b+k-x+1)\log(\nu_b+k-x+1)\\
=\ &(k-K+\nu_b+1)\log(k-K+\nu_b+1) + (K+\nu_b+1)\log(K+\nu_b+1)\\
-\ &(\nu_b+k-K+1+y)\log(\nu_b+k-K+1+y) - (\nu_b+K+1-y)\log(\nu_b+K+1-y)
\end{aligned}$$

Let

$$\phi(z) = (z+\nu_b+1)\log(z+\nu_b+1) - (\nu_b+z+1+y)\log(\nu_b+z+1+y)$$

then

$$\phi'(z) = \log\left(\frac{z+\nu_b+1}{\nu_b+z+1+y}\right) = \log\left(1-\frac{y}{\nu_b+z+1+y}\right) < 0$$

so that ϕ is deceasing. Then, $\phi(k-K) \geq \phi(K)$, which implies

$$\begin{aligned}
&\Phi^{(6)}_{\gamma,k}(x) \geq (K+\nu_b+1)\log(K+\nu_b+1) + (K+\nu_b+1)\log(K+\nu_b+1)\\
-\ &(\nu_b+K+1+y)\log(\nu_b+K+1+y) - (\nu_b+K+1-y)\log(\nu_b+K+1-y)
\end{aligned}$$

Thus, in the regime (C.20), recalling (C.19):

$$\Phi^{(6)}_{\gamma,k}(x) \geq y\left[\log\left(\frac{K+\nu_b+1}{K+\nu_b+1}\right) + O\left(\frac{y}{K}\right)\right] = O\left(\frac{y^2}{K}\right). \tag{C.22}$$

Also, we have the bound

$$\begin{aligned}
&\frac{w_{\gamma,\nu_b}(k_1)w_{\gamma,\nu_b}(k_2)}{w_{\gamma,\nu_b}(k-K)w_{\gamma,\nu_b}(K)}\\
\leq\ &\frac{c}{\Gamma(\alpha_\gamma)}\left(\frac{(\alpha_\gamma+K-k_1-1)(\alpha_\gamma+K-k_2-1)}{\alpha_\gamma+2K-k-1}\frac{(\nu_b+k_1)(\nu_b+k_2)}{(k-K+\nu_b+1)(K+\nu_b+2)}\right)^{\frac{1}{2}}\\
\times\ &e^{-(\Phi^{(5)}_{\gamma,k}(x)+\Phi^{(6)}_{\gamma,k}(x))}.
\end{aligned}$$

Pick a large enough universal constant $K_\nu \gg 1$ and recall $x = k-K+y$.

Case $y \geq K_\nu$. Using the monotonicity of the Stirling phases and (C.21), (C.22) we have the lower bound:

$$\left[\Phi^{(5)}_{\gamma,k} + \Phi^{(6)}_{\gamma,k}\right](x) \geq \left[\Phi^{(5)}_{\gamma,k} + \Phi^{(6)}_{\gamma,k}\right](y = K_\nu) \geq \frac{K_\nu}{4}\log\gamma$$

which implies, for K_ν universal large enough:

$$\sum_{k_1+k_2=k, y\geq K_\nu, 0\leq k_1\leq k_2\leq K} \frac{w_{\gamma,\nu_b}(k_1)w_{\gamma,\nu_b}(k_2)}{w_{\gamma,\nu_b}(k-K)w_{\gamma,\nu_b}(K)} \leq c_\nu \frac{\gamma}{\gamma^{\frac{K_\nu}{8}}} \leq c_\nu.$$

Case $1 \leq y \leq K_\nu$. We estimate:

$$\begin{aligned}
&\frac{(\alpha_\gamma+K-k_1-1)(\alpha_\gamma+K-k_2-1)}{\alpha_\gamma+2K-k-1}\frac{(\nu_b+k_1)(\nu_b+k_2)}{(k-K+\nu_b+1)(K+\nu_b+2)}\\
=\ &\frac{(\alpha_\gamma+2K-k-y)(\alpha_\gamma+y-1)}{\alpha_\gamma+2K-k-1}\frac{(\nu_b+k-K+y)(\nu_b+K-y)}{(k-K+\nu_b+1)(K+\nu_b+2)}\\
\leq\ &\frac{c_\nu}{\alpha_\gamma}
\end{aligned}$$

where we used $2K-k-1\geq 0$. We then conclude, using the positivity of the total Stirling phase, since there are finitely many terms:

$$\sum_{k_1+k_2=k, y\leq K_\nu, 0\leq k_1\leq k_2\leq K} \frac{w_{\gamma,\nu_b}(k_1)w_{\gamma,\nu_b}(k_2)}{w_{\gamma,\nu_b}(k-K)w_{\gamma,\nu_b}(K)} \leq \frac{c_\nu}{\alpha_\gamma\Gamma(\alpha_\gamma)} \leq \frac{c_\nu}{\Gamma(1+\alpha_\gamma)} \leq c_\nu$$

The collection of the above bounds yields (C.17). □

Appendix D. Uniform convolution bounds for $b=0$ with ν complex

This appendix is the extension of Appendix B to the case where ν is complex. We use the standard formulas

$$\begin{aligned} \log(re^{i\theta}) &= \log(r)+i\theta, \\ \arg(x+iy) &= 2\arctan\left(\frac{y}{\sqrt{x^2+y^2}+x}\right) \quad \text{if} \quad x>0 \quad \text{or} \quad y\neq 0 \end{aligned}$$

where we have chosen the principal branch of the logarithm, i.e., on $\mathbb{C}\setminus\mathbb{R}^-$.

Lemma D.1 (Lower bound on the Stirling phase)**.** *Let $\nu\in\mathbb{C}\setminus\mathbb{Z}^-$. Pick a large enough universal constant $R>R_\nu\gg 1$. Then there exist $c_\nu>1$ and $k^*=k^*(\nu,R)\gg 1$ such that for all $k>k^*$, for all*

$$k_1=kx, \quad k_2=k(1-x), \quad 0\leq x\leq \frac{1}{2},$$

we have

$$\left|\frac{\Gamma(k_1+\nu+2)\Gamma(k_2+\nu+2)}{\Gamma(k+\nu+2)}\right| \leq c_\nu\sqrt{1+k_1}e^{-k\left[\widetilde{\Phi}_k^{(2)}(x)\right]} \tag{D.1}$$

where the phase function satisfies[25]

$$\widetilde{\Phi}_k^{(2)}(x)\geq 0, \quad (\widetilde{\Phi}_k^{(2)})'(x)\geq 0.$$

Moreover, for $\frac{R}{k}<x\leq\frac{1}{2}$,

$$\widetilde{\Phi}_k^{(2)}(x)\geq \frac{1}{2}x|\log x|. \tag{D.2}$$

Proof of Lemma D.1. We assume $k\geq k^*(R,\nu)$ large enough, and consider two cases: $k_1+\Re(\nu)\leq 0$ and $k_1+\Re(\nu)>0$.

step 1 We consider first the case $k_1+\Re(\nu)\leq 0$. In this case, we have

$$x=\frac{k_1}{k}\leq -\frac{\Re(\nu)}{k}, \qquad \Re(\nu)\leq 0.$$

Note that, since $k\geq k^*(R,\nu)$ is large enough, we have $k_2+\Re(\nu)>0$ and $k+\Re(\nu)>0$. We may thus apply Stirling's formula[26]

$$\Gamma(z+1)=(1+o_{z\to+\infty}(1))\left(\frac{z}{e}\right)^z\sqrt{2\pi z}$$

[25] In the case where $\Re(\nu)>0$, $\widetilde{\Phi}_k^{(2)}$ is smooth. In the case $\Re(\nu)\leq 0$, $\widetilde{\Phi}_k^{(2)}$ is smooth except at the point $x=-\frac{\Re(\nu)}{k}$ where it is only continuous, see Remark D.2. In particular, at that point, $(\widetilde{\Phi}_k^{(2)})'(x)\geq 0$ means that the left and right derivative, although not equal, are both positive.

[26] In what follows, we apply Stirling's formula for z satisfying $\Re(z)>0$. In particular, $\sqrt{z}$ and $\log(z)$ are both defined.

to upper bound, using also $0 \leq k_1 \leq |\nu|$ and $k_1 + \nu + 2 \in \mathbb{C} \setminus \mathbb{Z}^-$,

$$\begin{aligned} & \left| \frac{\Gamma(k_1+\nu+2)\Gamma(k_2+\nu+2)}{\Gamma(k+\nu+2)} \right| \\ \lesssim_\nu & \left| \frac{\left(\frac{k_2+\nu+1}{e}\right)^{k_2+\nu+1} \sqrt{k_2+\nu+1}}{\left(\frac{k+\nu+1}{e}\right)^{k+\nu+1} \sqrt{k+\nu+1}} \right| \\ \lesssim_\nu & e^{-\Re[(k+\nu+1)\log(k+\nu+1)-(k_2+\nu+1)\log(k_2+\nu+1)]} \end{aligned}$$

We compute:

$$\begin{aligned} & (k+\nu+1)\log(k+\nu+1) - (k_2+\nu+1)\log(k_2+\nu+1) \\ = & k\left(1+\frac{\nu+1}{k}\right)\left[\log k + \log\left(1+\frac{\nu+1}{k}\right)\right] \\ - & k\left((1-x)+\frac{\nu+1}{k}\right)\left[\log k + \log\left((1-x)+\frac{\nu+1}{k}\right)\right] \\ = & kx\log k + k\left(1+\frac{\nu+1}{k}\right)\log\left(1+\frac{\nu+1}{k}\right) \\ - & k\left((1-x)+\frac{\nu+1}{k}\right)\log\left((1-x)+\frac{\nu+1}{k}\right). \end{aligned}$$

Therefore,

$$\Re\left[(k+\nu+1)\log(k+\nu+1) - (k_2+\nu+1)\log(k_2+\nu+1)\right] = k\widetilde{\Phi}_k^{(2)}(x)$$

which yields, using also that $k_1 \geq 0$,

$$\left| \frac{\Gamma(k_1+\nu+2)\Gamma(k_2+\nu+2)}{\Gamma(k+\nu+2)} \right| \leq c_\nu \sqrt{1+k_1} e^{-k\widetilde{\Phi}_k^{(2)}(x)}. \tag{D.3}$$

We have

$$\widetilde{\Phi}_k^{(2)}(0) = 0$$

and

$$\begin{aligned} & \partial_x \widetilde{\Phi}_k^{(2)}(x) = \log(k) + 1 + \Re\left[\log\left((1-x)+\frac{\nu+1}{k}\right)\right] \\ = & \log(k) + 1 + \log\left|(1-x)+\frac{\nu+1}{k}\right| \\ = & \log(k) + 1 + \log\left(\sqrt{\left(1-x+\frac{\Re(\nu)+1}{k}\right)^2 + \left(\frac{\Im(\nu)}{k}\right)^2}\right) \\ \geq & 0 \end{aligned} \tag{D.4}$$

for $x \in [0, -\Re(\nu)/k]$. We conclude, in the case $k_1 + \Re(\nu) \leq 0$, that $\widetilde{\Phi}_k^{(2)}(x)$ is a non-negative non-decreasing function of $x \in [0, -\Re(\nu)/k]$. Also, note that for $k_1 + \Re(\nu) \leq 0$ and provided we choose $R > R_\nu$ large enough, we have $kx = k_1 \leq |\nu| < R$ so that we do not have to prove (D.2) in that case. This concludes the proof of the case $k_1 + \Re(\nu) \leq 0$.

step 2 From now on, we focus on the case $k_1 + \Re(\nu) > 0$. In this case, we have

- either $\Re(\nu) > 0$ and $x \in [0, \frac{1}{2}]$,
- or $\Re(\nu) \leq 0$ and $x \in (-\frac{\Re(\nu)}{k}, \frac{1}{2}]$.

Since we also have $k_2+\Re(\nu)>0$ and $k+\Re(\nu)>0$, we may thus apply Stirling's formula to upper bound

$$\begin{aligned}
& \left|\frac{\Gamma(k_1+\nu+2)\Gamma(k_2+\nu+2)}{\Gamma(k+\nu+2)}\right| \\
\lesssim_\nu & \left|\frac{\left(\frac{k_1+\nu+1}{e}\right)^{k_1+\nu+1}\sqrt{k_1+\nu+1}\left(\frac{k_2+\nu+1}{e}\right)^{k_2+\nu+1}\sqrt{k_2+\nu+1}}{\left(\frac{k+\nu+1}{e}\right)^{k+\nu+1}\sqrt{k+\nu+1}}\right| \\
\lesssim_\nu & \sqrt{|k_1+\nu+1|}e^{-\Re[(k+\nu+1)\log(k+\nu+1)-(k_1+\nu+1)\log(k_1+\nu+1)-(k_2+\nu+1)\log(k_2+\nu+1)]}
\end{aligned}$$

We compute:

$$\begin{aligned}
& (k+\nu+1)\log(k+\nu+1)-(k_1+\nu+1)\log(k_1+\nu+1)-(k_2+\nu+1)\log(k_2+\nu+1) \\
= \; & k\left(1+\frac{\nu+1}{k}\right)\left[\log k+\log\left(1+\frac{\nu+1}{k}\right)\right] \\
- \; & k\left(x+\frac{\nu+1}{k}\right)\left[\log k+\log\left(x+\frac{\nu+1}{k}\right)\right] \\
- \; & k\left((1-x)+\frac{\nu+1}{k}\right)\left[\log k+\log\left((1-x)+\frac{\nu+1}{k}\right)\right] \\
= \; & -(\nu+1)\log k+k\left(1+\frac{\nu+1}{k}\right)\log\left(1+\frac{\nu+1}{k}\right) \\
- \; & k\left[\left(x+\frac{\nu+1}{k}\right)\log\left(x+\frac{\nu+1}{k}\right)+\left((1-x)+\frac{\nu+1}{k}\right)\log\left((1-x)+\frac{\nu+1}{k}\right)\right].
\end{aligned}$$

Therefore,

- if $\Re(\nu)>0$, then

$$\begin{aligned}
& \Re\left[(k+\nu+1)\log(k+\nu+1)-(k_2+\nu+1)\log(k_2+\nu+1)\right] \\
= \; & -\Re\left[(\nu+1)\log(\nu+1)\right]+k\widetilde{\Phi}_k^{(2)}(x),
\end{aligned}$$

  with the choice

$$\begin{aligned}
\widetilde{\Phi}_k^{(2)}(x) = \; & \Re\Bigg\{\frac{\nu+1}{k}\log\left(\frac{\nu+1}{k}\right)+\left(1+\frac{\nu+1}{k}\right)\log\left(1+\frac{\nu+1}{k}\right) \\
- \; & \left[\left(x+\frac{\nu+1}{k}\right)\log\left(x+\frac{\nu+1}{k}\right)+\left((1-x)+\frac{\nu+1}{k}\right)\log\left((1-x)+\frac{\nu+1}{k}\right)\right]\Bigg\},
\end{aligned}$$

- if $\Re(\nu)\leq 0$, then

$$\begin{aligned}
& \Re\left[(k+\nu+1)\log(k+\nu+1)-(k_2+\nu+1)\log(k_2+\nu+1)\right] \\
= \; & -\log k-k\Re\left[\left(\frac{-\Re(\nu)+\nu+1}{k}\right)\log\left(\frac{-\Re(\nu)+\nu+1}{k}\right)\right]+k\widetilde{\Phi}_k^{(2)}(x) \\
= \; & -\Re\Big[\left(-\Re(\nu)+\nu+1\right)\log\left(-\Re(\nu)+\nu+1\right)\Big]+k\widetilde{\Phi}_k^{(2)}(x)
\end{aligned}$$

with the choice

$$\begin{aligned}\widetilde{\Phi}_k^{(2)}(x) &= -\frac{\Re(\nu)\log k}{k} + \Re\left[\left(\frac{-\Re(\nu)+\nu+1}{k}\right)\log\left(\frac{-\Re(\nu)+\nu+1}{k}\right)\right] \\ &\quad +\Re\Bigg\{\left(1+\frac{\nu+1}{k}\right)\log\left(1+\frac{\nu+1}{k}\right) \\ &- \left[\left(x+\frac{\nu+1}{k}\right)\log\left(x+\frac{\nu+1}{k}\right)+\left((1-x)+\frac{\nu+1}{k}\right)\log\left((1-x)+\frac{\nu+1}{k}\right)\right]\Bigg\}.\end{aligned}$$

Remark D.2. Note that the above choice for $\Re(\nu)\le 0$ is such that $\widetilde{\Phi}_k^{(2)}(x)$, which is defined in that case on $x\in(-\frac{\Re(\nu)}{k},\frac12]$, and in Step 1 on $x\in[0,-\frac{\Re(\nu)}{k}]$, is smooth on $x\in[0,\frac12]\setminus\{-\frac{\Re(\nu)}{k}\}$ and continuous at $x=-\frac{\Re(\nu)}{k}$.

The above choice for $\widetilde{\Phi}_k^{(2)}(x)$ yields

$$\left|\frac{\Gamma(k_1+\nu+2)\Gamma(k_2+\nu+2)}{\Gamma(k+\nu+2)}\right| \le c_\nu\sqrt{1+k_1}e^{-k\widetilde{\Phi}_k^{(2)}(x)}. \tag{D.5}$$

Recall that we are in the case $k_1+\Re(\nu)>0$, i.e. $x>-\Re(\nu)/k$. We have

$$\begin{aligned}&\partial_x\widetilde{\Phi}_k^{(2)}(x) = \Re\left[-\log\left(x+\frac{\nu+1}{k}\right)+\log\left((1-x)+\frac{\nu+1}{k}\right)\right] \\ =\ &\Re\left[\log\left(\frac{(1-x)+\frac{\nu+1}{k}}{x+\frac{\nu+1}{k}}\right)\right] = \log\left|1+\frac{(1-2x)}{x+\frac{\nu+1}{k}}\right| \\ =\ &\log\left(\sqrt{\left(1+\frac{(1-2x)\left(x+\frac{\Re(\nu)+1}{k}\right)}{\left(x+\frac{\Re(\nu)+1}{k}\right)^2+\left(\frac{\Im(\nu)}{k}\right)^2}\right)^2+\left(\frac{(1-2x)\frac{\Im(\nu)}{k}}{\left(x+\frac{\Re(\nu)+1}{k}\right)^2+\left(\frac{\Im(\nu)}{k}\right)^2}\right)^2}\right) \\ \ge\ &0\end{aligned}$$

for $x\in[0,\frac12]$ if $\Re(\nu)>0$ and for $x\in(-\frac{\Re(\nu)}{k},\frac12]$ if $\Re(\nu)\le 0$. Also, we have $\widetilde{\Phi}_k^{(2)}(0)=0$ if $\Re(\nu)>0$, and $\widetilde{\Phi}_k^{(2)}(-\frac{\Re(\nu)}{k})\ge 0$ if $\Re(\nu)\le 0$ in view of the continuity of $\widetilde{\Phi}_k^{(2)}$ at $-\frac{\Re(\nu)}{k}$ and its positivity on $x\in[0,-\frac{\Re(\nu)}{k}]$ established in step 1. Together with step 1, we conclude that $\widetilde{\Phi}_k^{(2)}(x)$ is a non-negative non-decreasing function of $x\in[0,\frac12]$.

step 3 Lower bound for small x. Assume

$$\frac{R}{k} < x \le \frac12$$

where $R > R_\nu \gg 1$. We start with the case $\Re(\nu) > 0$ for which we have in view of the definition of $\widetilde{\Phi}_k^{(2)}$ in that case

$$\begin{aligned}
\widetilde{\Phi}_k^{(2)}(x) &= \Re\left[\frac{\nu+1}{k}\log\left(\frac{\nu+1}{k}\right) + \left(1+\frac{\nu+1}{k}\right)\log\left(1+\frac{\nu+1}{k}\right)\right. \\
&- \left.\left(x+\frac{\nu+1}{k}\right)\log\left(x+\frac{\nu+1}{k}\right) - \left((1-x)+\frac{\nu+1}{k}\right)\log\left((1-x)+\frac{\nu+1}{k}\right)\right] \\
&= \frac{\Re(\nu)+1}{k}\log\left|\frac{\nu+1}{k}\right| + \left(1+\frac{\Re(\nu)+1}{k}\right)\log\left|1+\frac{\nu+1}{k}\right| \\
&- \left(x+\frac{\Re(\nu)+1}{k}\right)\log\left|x+\frac{\nu+1}{k}\right| - \left((1-x)+\frac{\Re(\nu)+1}{k}\right)\log\left|(1-x)+\frac{\nu+1}{k}\right| \\
&- \frac{\Im(\nu)}{k}\Im\left[\log\left(\frac{\nu+1}{k}\right) + \log\left(1+\frac{\nu+1}{k}\right) - \log\left(x+\frac{\nu+1}{k}\right) - \log\left((1-x)+\frac{\nu+1}{k}\right)\right] \\
&= -\frac{\Re(\nu)+1}{k}\log\left(\frac{\left|x+\frac{\nu+1}{k}\right|}{\left|\frac{\nu+1}{k}\right|}\right) + \left(1+\frac{\Re(\nu)+1}{k}\right)\log\left|1+\frac{\nu+1}{k}\right| \\
&- x\log\left|x+\frac{\nu+1}{k}\right| - \left((1-x)+\frac{\Re(\nu)+1}{k}\right)\log\left|(1-x)+\frac{\nu+1}{k}\right| + O(1)\frac{\Im(\nu)}{k}
\end{aligned}$$

where we used in the last inequality that $|\Im(\log(z))| \le \pi$ for $\Re(z) > 0$ or $\Im(z) \neq 0$. We deduce

$$\begin{aligned}
\widetilde{\Phi}_k^{(2)}(x) &= -\frac{\Re(\nu)+1}{k}\left|\log\left(\sqrt{\frac{(kx+\Re(\nu)+1)^2+(\Im(\nu))^2}{(\Re(\nu)+1)^2+(\Im(\nu))^2}}\right)\right| \\
&+ \left(1+\frac{\Re(\nu)+1}{k}\right)\left|\log\left(\sqrt{\left(1+\frac{\Re(\nu)+1}{k}\right)^2+\left(\frac{\Im(\nu)}{k}\right)^2}\right)\right| \\
&+ x\left|\log\left(\sqrt{\left(x+\frac{\Re(\nu)+1}{k}\right)^2+\left(\frac{\Im(\nu)}{k}\right)^2}\right)\right| \\
&+ \left((1-x)+\frac{\Re(\nu)+1}{k}\right)\left|\log\left(\sqrt{\left(1-x+\frac{\Re(\nu)+1}{k}\right)^2+\left(\frac{\Im(\nu)}{k}\right)^2}\right)\right| + O(1)\frac{\Im(\nu)}{k} \\
&\ge x\left[-\frac{\Re(\nu)+1}{kx}\left|\log\left(\sqrt{\frac{(R+\Re(\nu)+1)^2+(\Im(\nu))^2}{(\Re(\nu)+1)^2+(\Im(\nu))^2}}\right)\right|\right. \\
&+ \left.\left|\log(x)+\log\left(\sqrt{\left(1+\frac{\Re(\nu)+1}{kx}\right)^2+\left(\frac{\Im(\nu)}{kx}\right)^2}\right)\right| + O\left(\frac{\Im(\nu)}{R}\right)\right] \\
&\ge x|\log x|\left\{1+O_\nu\left(\frac{|\nu+1|}{R}\left|\log\left(\frac{|\nu+1|}{R}\right)\right|\right)\right\} \\
&\ge \frac{1}{2}x|\log x|
\end{aligned}$$

as desired.

It remains to treat the case $\Re(\nu) \leq 0$. Recall that since $x > \frac{R}{k}$, we are in the case $k_1 + \Re(\nu) > 0$. Then, in view of the definition of $\widetilde{\Phi}_k^{(2)}$ in that case, we have

$$\begin{aligned}
\widetilde{\Phi}_k^{(2)}(x) &= -\frac{\Re(\nu)\log k}{k} + \Re\left[\left(\frac{-\Re(\nu)+\nu+1}{k}\right)\log\left(\frac{-\Re(\nu)+\nu+1}{k}\right)\right] \\
&\quad +\Re\Bigg\{\left(1+\frac{\nu+1}{k}\right)\log\left(1+\frac{\nu+1}{k}\right) \\
&- \left[\left(x+\frac{\nu+1}{k}\right)\log\left(x+\frac{\nu+1}{k}\right) + \left((1-x)+\frac{\nu+1}{k}\right)\log\left((1-x)+\frac{\nu+1}{k}\right)\right]\Bigg\} \\
&= -\frac{\Re(\nu)\log k}{k} + \frac{1}{k}\log\left|\frac{-\Re(\nu)+\nu+1}{k}\right| + \left(1+\frac{\Re(\nu)+1}{k}\right)\log\left|1+\frac{\nu+1}{k}\right| \\
&- \left(x+\frac{\Re(\nu)+1}{k}\right)\log\left|x+\frac{\nu+1}{k}\right| - \left((1-x)+\frac{\Re(\nu)+1}{k}\right)\log\left|(1-x)+\frac{\nu+1}{k}\right| \\
&\quad -\frac{\Im(\nu)}{k}\Im\Bigg[\log\left(\frac{-\Re(\nu)+\nu+1}{k}\right) + \log\left(1+\frac{\nu+1}{k}\right) \\
&\quad -\log\left(x+\frac{\nu+1}{k}\right) - \log\left((1-x)+\frac{\nu+1}{k}\right)\Bigg] \\
&= -\frac{\Re(\nu)}{k}\log\left(\frac{\left|x+\frac{\nu+1}{k}\right|}{\frac{1}{k}}\right) - \frac{1}{k}\log\left(\frac{\left|x+\frac{\nu+1}{k}\right|}{\left|\frac{-\Re(\nu)+\nu+1}{k}\right|}\right) + \left(1+\frac{\Re(\nu)+1}{k}\right)\log\left|1+\frac{\nu+1}{k}\right| \\
&- x\log\left|x+\frac{\nu+1}{k}\right| - \left((1-x)+\frac{\Re(\nu)+1}{k}\right)\log\left|(1-x)+\frac{\nu+1}{k}\right| + O(1)\frac{\Im(\nu)}{k}
\end{aligned}$$

where we used in the last inequality that $|\Im(\log(z))| \le \pi$ for $\Re(z) > 0$ or $\Im(z) \neq 0$. We deduce

$$\begin{aligned}
\widetilde{\Phi}_k^{(2)}(x) &= -\frac{\Re(\nu)}{k}\left|\log\left(\sqrt{(kx+\Re(\nu)+1)^2+(\Im(\nu))^2}\right)\right| - \frac{1}{k}\left|\log\left(\sqrt{\frac{(kx+\Re(\nu)+1)^2+(\Im(\nu))^2}{1+(\Im(\nu))^2}}\right)\right| \\
&\quad + \left(1+\frac{\Re(\nu)+1}{k}\right)\left|\log\left(\sqrt{\left(1+\frac{\Re(\nu)+1}{k}\right)^2+\left(\frac{\Im(\nu)}{k}\right)^2}\right)\right| \\
&+ x\left|\log\left(\sqrt{\left(x+\frac{\Re(\nu)+1}{k}\right)^2+\left(\frac{\Im(\nu)}{k}\right)^2}\right)\right| \\
&\quad + \left((1-x)+\frac{\Re(\nu)+1}{k}\right)\left|\log\left(\sqrt{\left(1-x+\frac{\Re(\nu)+1}{k}\right)^2+\left(\frac{\Im(\nu)}{k}\right)^2}\right)\right| + O(1)\frac{\Im(\nu)}{k} \\
&\ge x\Bigg[-\frac{\Re(\nu)}{kx}\left|\log\left(\sqrt{(R+\Re(\nu)+1)^2+(\Im(\nu))^2}\right)\right| \\
&\quad -\frac{1}{kx}\left|\log\left(\sqrt{\frac{(R+\Re(\nu)+1)^2+(\Im(\nu))^2}{1+(\Im(\nu))^2}}\right)\right| \\
&+ \left|\log(x)+\log\left(\sqrt{\left(1+\frac{\Re(\nu)+1}{kx}\right)^2+\left(\frac{\Im(\nu)}{kx}\right)^2}\right)\right| + O\left(\frac{\Im(\nu)}{R}\right)\Bigg] \\
&\ge x|\log x|\left\{1+O_\nu\left(\frac{|\nu|+1}{R}\left|\log\left(\frac{|\nu|+1}{R}\right)\right|\right)\right\} \\
&\ge \frac{1}{2}x|\log x|
\end{aligned}$$

as desired. □

Lemma D.3 (Uniform convolution bound)**.** *Let $\nu \in \mathbb{C} \setminus \mathbb{Z}^-$. Then there exists $C_\nu > 0$ such that for all $k \ge 1$, for all $j \ge 2$,*

$$\sum_{k_1+\cdots+k_j=k}\left|\frac{\Pi_{i=1}^j \Gamma(k_i+\nu+2)}{\Gamma(k+\nu+2)}\right| \le C_\nu^j. \tag{D.6}$$

Proof. We argue by induction on j.

step 1 Case $j = 2$. We apply Lemma D.1 with some large enough $R_\nu > 1$ and assume, without loss of generality, $k \ge k_\nu^*$, with k_ν^* a universal, large enough constant. This leads to

$$\left|\frac{\Gamma(k_1+\nu+2)\Gamma(k_2+\nu+2)}{\Gamma(k+\nu+2)}\right| \le c_\nu\sqrt{1+k_1}e^{-k\left[\widetilde{\Phi}_k^{(2)}(x)\right]},$$

where $x = k_1/k$. We may assume $k_1 \le k_2$ so that $x \le 1/2$. For $x \le \frac{R_\nu}{k}$, we use the lower bound $\widetilde{\Phi}_k^{(2)}(x) \ge 0$. For $x \ge \frac{R_\nu}{k}$ the monotonicity of $\widetilde{\Phi}_k^{(2)}$ on $[0, 1/2]$ implies

$$\widetilde{\Phi}_k^{(2)}(x) \ge \widetilde{\Phi}_k^{(2)}\left(\frac{R_\nu}{k}\right) \ge +\frac{1}{2}\frac{R_\nu}{k}\left|\log\left(\frac{R_\nu}{k}\right)\right| \ge \frac{R_\nu}{4k}\log k$$

for $R_\nu \le \sqrt{k^*_\nu}$, from which

$$\begin{aligned}
&\sum_{k_1+k_2=k} \left| \frac{\Gamma(k_1+\nu+2)\Gamma(k_2+\nu+2)}{\Gamma(k+\nu+2)} \right| \\
\lesssim_\nu \; & 1 + \sum_{k_1+k_2=k,\, \frac{R_\nu}{k} \le x \le \frac12} \sqrt{1+k_1} e^{-\frac{R_\nu \log k}{4}} + \sum_{k_1+k_2=k, 0 \le x \le \frac{R_\nu}{k}} \sqrt{1+k_1} \\
\lesssim_\nu \; & 1 + R_\nu^{\frac32} + \frac{k^{\frac32}}{k^{\frac{R_\nu}{4}}} \lesssim_\nu 1,
\end{aligned}$$

since the second sum has $k_1 = xk \le R_\nu$ terms.

step 2 Induction. We assume j and prove $j+1$:

$$\begin{aligned}
&\sum_{k_1+\cdots+k_{j+1}=k} \left| \frac{\Pi_{i=1}^{j+1}\Gamma(k_i+\nu+2)}{\Gamma(k+\nu+2)} \right| \\
= \; & \sum_{k_1+m=k} \left| \frac{\Gamma(k_1+\nu+2)\Gamma(m+\nu+2)}{\Gamma(k+\nu+2)} \right| \sum_{k_2+\cdots+k_{j+1}=m} \left| \frac{\Pi_{i=2}^{j+1}\Gamma(k_i+\nu+2)}{\Gamma(m+\nu+2)} \right| \\
\le \; & C_\nu^j \sum_{k_1+m=k} \left| \frac{\Gamma(k_1+\nu+2)\Gamma(m+\nu+2)}{\Gamma(k+\nu+2)} \right| \le C_\nu^{j+1}
\end{aligned}$$

and (D.6) is proved. □

Appendix E. Proof of Lemma 1.5

This Appendix is devoted to the proof of Lemma 1.5 which is a direct consequence of analyticity. We study the function $S_\infty(d,\ell)$ from (7.23) which arises from the limiting problem (5.4). We recall that there are in fact two different limiting problems which correspond to the parameters $r = r_*, r^+$ associated to the respective ranges $0 < \ell < d$ and $\ell > d$. Each of the limiting problem gives rise to a collections of coefficients: $\nu_\infty, \mu_+, c_-, , c_+, ...$ each of which is a function of ℓ. In fact, each of them is a different function of ℓ depending on the case r_* or r^+, see Appendix F. Let us associate superscript $*$ to the former and $+$ to the later. Since r_+ corresponds to the range $\ell > d$, the coefficients $\nu_\infty^+, ...$ are originally defined for the same range of ℓ but, by the direct examination of Appendix F, can be extended *through the same formula* to the interval $\ell \in (0,d)$. In fact, they can be similarly extended as holomorphic functions to a small complex neighborhood of $\mathbb{R}^+ \setminus \{d\}$. In the $*$ case, the function ν_∞^* is originally positive on the subset $\mathcal{O}_d^* \subset (0,d)$. Again, by examining the formulas in Appendix F, we can conclude that $\nu_\infty^*, ...$ are holomorphic functions of ℓ in a small complex neighborhood of $\mathcal{O}_d^*$ (we do not need to extend them to $\ell > d$).

We now similarly define $S_\infty^*(d,\ell)$ and $S_\infty^+(d,\ell)$, originally on their respective sets $\mathcal{O}_d^*$ and $\ell > d$, and then argue that $S_\infty^*(d,\ell)$ is actually holomorphic in a neighborhood of $\mathcal{O}_d^*$, while for $d = 2,3$ $S_\infty^+(d,\ell)$ is holomorphic in a small complex neighborhood of $\mathbb{R}^+ \setminus \{d\}$. More precisely,

Lemma E.1. *We have*

(1) *if $d \ge 4$, each function $S_\infty^*(d,\ell)$ extends holomorphically in ℓ to a complex neighborhood of $\mathcal{O}_d^*$,*

(2) *if $d \geq 4$, each function $S^+_\infty(d,\ell)$ extends holomorphically in ℓ to a complex neighborhood of (d,∞),*
(3) *if $d = 2,3$, $S^*_\infty(d,\ell)$ extends holomorphically in ℓ to a complex neighborhood of $\{0 < \ell < d\}$,*
(4) *if $d = 2,3$, $S^+_\infty(d,\ell)$ extends holomorphically in ℓ to an open connected set in $\mathbb{C}$ containing neighborhoods of $\{0 < \ell < d\}$ and $\{d < \ell < +\infty\}$.*

We emphasize that the extensions above are not abstract but follow from extending the values of ℓ to the complex plane in *explicit* formulas.

Proof. Note from their definition that all the constants appearing in the definition of the holomorphic functions μ_0, μ_j and ν_j, i.e.,

$$r_\infty,\ \ \sigma_2,\ \ \lambda_-,\ \ \mu_+,\ \ c_-,\ \ c_+,\ \ c_j,\ \ d_{ij},\ \ e_{ij},\ \ \tilde{d}_{ij},\ \ \tilde{e}_{ij},\ \ \tilde{w}_\infty,\ \ \tilde{\psi}_\infty,$$
$$\tilde{D}_{ij},\ \ \tilde{E}_{ij},\ \ a,\ \ \nu,$$

are rational functions of ℓ, $\sqrt{\ell}$ and $\ell^{\frac{1}{4}}$, and hence are holomorphic in ℓ for $\ell \in \mathbb{C}\setminus\mathbb{R}^-$ wherever the denominators do not vanish. These denominators are given by the following list

$$\ell,\ \ \ell+\sqrt{d},\ \ \ 1+\sqrt{\ell},\ \ \ 1+(d+1)\sqrt{\ell}+\ell,\ \ \ \mu_+,\ \ \ c_+-c_-,\ \ \ \lambda_-,\ \ \ \tilde{d}_{20}.$$

For real values of ℓ, these denominators vanish at $\ell = 0$, $\ell = d$, as well as at certain negative values of ℓ which explicitly depend on d. In particular, we deduce that all the above constants are holomorphic in ℓ in a small neighborhood of $\mathbb{R}^+\setminus\{d\}$. This applies both to the $*$ and the $+$ case and immediately implies that all the Taylor coefficients $(\mu_j)_k$, $(\nu_j)_k$ and $(\mu_0)_k$ of μ_j, ν_j and μ_0 are holomorphic in ℓ on the same set.

We now consider the set Ω^+_d obtained the intersection of the above small neighborhood of $\mathbb{R}^+\setminus\{d\}$ with the set $(\nu^+_\infty)^{-1}\big(\mathbb{C}\setminus\mathbb{Z}^-\big)$. Note that since by (F.15) the function $\nu^+_\infty > 0$ for all $\ell > 0$, the set Ω^+_d contains $\mathbb{R}^+\setminus\{d\}$.

In the $*$ case, we define the set Ω^*_d to be simply a small complex neighborhood of $\mathcal{O}^*_d$. Note that since $\nu^*_\infty > 0$ on $\mathcal{O}^*_d$, the condition that $\nu \in \mathbb{C}\setminus\mathbb{Z}^-$ is automatically satisfied on Ω^*_d, provided the neighborhood is chosen to be sufficiently small.

Our goal is now is to show that S^*_∞ and S^+_∞ are holomorphic respectively on Ω^*_d and Ω^+_d and that $\mathbb{R}^+\setminus\{d\}$ belongs to the same connected component of Ω^+_d. The next argument applies to both, so will simply use the notations S_∞ and Ω_d.

In view of the definitions of Ω_d, in particular, for any k, $\nu+k+3 \in \mathbb{C}\setminus\mathbb{Z}^-$ and hence $\Gamma(\nu+k+3)$ is holomorphic in ℓ on Ω_d. We deduce that all the Taylor coefficients $(h_j)_k$, $(\tilde{h}_j)_k$ and $(h_0)_k$ of h_j, $\tilde{h}_j$ and h_0 are holomorphic in ℓ on Ω_d. In view of the definition of w_k, we conclude that for any k, w_k is holomorphic in ℓ on Ω_d. In particular, this is also true for $(-1)^k w_k$. As a result, for any k, S_k is holomorphic in ℓ on Ω_d.

To conclude that $S_\infty(d,\ell)$ is holomorphic in ℓ on Ω_d, it suffices to prove that $S_k(d,\ell)$ converges uniformly to $S_\infty(d,\ell)$ on any compact of Ω_d. Note that Lemma 5.4 implies that $S_k(d,\ell)$ converges uniformly to $S_\infty(d,\ell)$ on any compact of $0 < \ell < d$ in $\mathbb{R}$, so it suffices to prove that the conclusion of Lemma 5.4 still holds on Ω_d. Now, a quick inspection of the proof of Lemma 5.4 reveals that the only obstruction is that the conclusions of Appendix B do not hold if $\nu \in \mathbb{C}\setminus\mathbb{Z}^-$. The fact that these conclusions hold even if $\nu \in \mathbb{C}\setminus\mathbb{Z}^-$ has been checked in Appendix D. Thus $S_\infty(d,\ell)$ is holomorphic on Ω_d.

Finally, it remains to prove that, for $d = 2, 3$, in the case $r = r_+$, $\{0 < \ell < d\}$ and $\{d < \ell < +\infty\}$ belong to the same connected component of Ω_d^+. First, recall from Lemma F.4 that, for $d = 2, 3$, in the case $r = r_+$, $\nu > 0$ on $\{0 < \ell < d\}$ and $\{d < \ell < +\infty\}$ so that $\{0 < \ell < d\}$ and $\{d < \ell < +\infty\}$ belong to Ω_d. It thus remains to prove that they belong to the same connected component of Ω_d. To this end, Ω_d^+ being open, it suffices to exhibit a continuous curve in Ω_d^+ with one end on $\{0 < \ell < d\}$ and the other on $\{d < \ell < +\infty\}$. Now, from the explicit formula (F.15), we have the following asymptotic as $\ell \to d$

$$\nu \;=\; \frac{\nu_0(d)}{(\ell-d)^2}\Big(1 + a_1(d)(\ell-d) + a_2(d)(\ell-d)^2 + O\big((\ell-d)^3\big)\Big), \qquad \nu_0(d) > 0.$$

We then choose the suitable curve in the complex plane as

$$\gamma_{r_0} = \left\{ d + \sqrt{\frac{\nu_0(d)}{r_0}} e^{i\left(\frac{\pi}{2}+\theta\right)}, \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \right\}, \qquad r_0 \gg 1.$$

For r_0 large enough, γ_{r_0} clearly has one end on $\{0 < \ell < d\}$ and the other on $\{d < \ell < +\infty\}$ and is also a curve in $\mathbb{C} \setminus \big(\{d\} \cup \mathbb{R}^-\big)$. So, in view of the definition of Ω_d^+, it suffices to prove that $\nu \notin \mathbb{Z}^-$ for any $\ell \in \gamma_{r_0}$ for a suitable choice of r_0. Now, in view of the above asymptotic for ν, we have for r_0 sufficiently large

$$\begin{aligned} & \nu\left(d + \sqrt{\frac{\nu_0(d)}{r_0}} e^{i\left(\frac{\pi}{2}+\theta\right)} \right) \\ =\; & -e^{-2i\theta}\Big(r_0 + i a_1(d)\sqrt{\nu_0(d)} e^{i\theta}\sqrt{r_0} - a_2(d)\nu_0(d) e^{2i\theta} + O\big(r_0^{-\frac{1}{2}}\big)\Big). \end{aligned}$$

Taking the imaginary part, we see that the above expression crosses $\mathbb{R}^-$ for $\theta = \theta_0$ with θ_0 satisfying

$$r_0 \sin(2\theta_0) - a_1(d)\sqrt{\nu_0(d)}\cos(\theta_0)\sqrt{r_0} \;=\; O\big(r_0^{-\frac{1}{2}}\big)$$

and hence

$$\sin(\theta_0) \;=\; \frac{a_1(d)\sqrt{\nu_0(d)}}{2\sqrt{r_0}} + O\big(r_0^{-\frac{3}{2}}\big).$$

Plugging back in the above expression, we infer at $\theta = \theta_0$

$$\begin{aligned} & \nu\left(d + \sqrt{\frac{\nu_0(d)}{r_0}} e^{i\left(\frac{\pi}{2}+\theta_0\right)} \right) \\ =\; & -r_0\cos(2\theta_0) - a_1(d)\sqrt{\nu_0(d)}\sin(\theta_0)\sqrt{r_0} + a_2(d)\nu_0(d) + O\big(r_0^{-\frac{1}{2}}\big) \\ =\; & -r_0 + a_2(d)\nu_0(d) + O\big(r_0^{-\frac{1}{2}}\big). \end{aligned}$$

Thus, choosing

$$r_0(k) \;:=\; k + a_2(d)\nu_0(d) - \frac{1}{2}, \qquad k \in \mathbb{N}, \qquad k \gg 1,$$

we obtain

$$\nu\left(d + \sqrt{\frac{\nu_0(d)}{r_0(k)}} e^{i\left(\frac{\pi}{2}+\theta_0(k)\right)} \right) \;=\; -k + \frac{1}{2} + O\big(k^{-\frac{1}{2}}\big)$$

and hence $\nu \notin \mathbb{Z}^-$ for any $\ell \in \gamma_{r_0(k)}$ provided $k \in \mathbb{N}$ is chosen large enough. $\square$

Appendix F. Slopes and eigenvalues at P_2 near the critical value

In this appendix we collect the values of the parameters which appear in the computations near P_2 for $r < r_{\infty}$ close to r_{∞}.

F.1. **Values of the parameters.** For $1 < r < r_+(d,\ell)$:

$$\left|\begin{array}{l} c_1 = 3w_2^2 - 2(r+1)w_2 + r - d\sigma_2^2 \\ c_2 = \frac{\sigma_2}{\ell}[2w_2(\ell+d-1) - (\ell+d+\ell r - r)] \\ c_3 = -2d\sigma_2 w_2 + 2\ell(r-1)\sigma_2 \\ c_4 = -2\sigma_2^2 \\ d_{20} = 3w_2 - (r+1) \\ d_{11} = -2d\sigma_2 \\ d_{02} = \ell(r-1) - dw_2 \\ e_{20} = \frac{\sigma_2(\ell+d-1)}{\ell} \\ e_{11} = \frac{2w_2(\ell+d-1)-(\ell+d+\ell r-r)}{\ell} \\ e_{02} = -3\sigma_2 \\ e_{21} = \frac{\ell+d-1}{\ell}. \end{array}\right. \tag{F.1}$$

Moreover,

$$\left|\begin{array}{l} \tilde{d}_{20} = (c_+e_{20} - d_{20})c_-^2 + (c_+e_{11} - d_{11})c_- + c_+e_{02} - d_{02} \\ \tilde{d}_{11} = 2c_-c_+(c_+e_{20} - d_{20}) + (c_- + c_+)(c_+e_{11} - d_{11}) + 2(c_+e_{02} - d_{02}) \\ \tilde{d}_{02} = (c_+e_{20} - d_{20})c_+^2 + (c_+e_{11} - d_{11})c_+ + c_+e_{02} - d_{02} \\ \tilde{d}_{30} = -(c_-^3 - dc_-) + c_+\left(\frac{\ell+d-1}{\ell}c_-^2 - 1\right) \\ \tilde{d}_{21} = -(3c_-c_+^2 - dc_- - 2dc_+) + c_+\left(\frac{\ell+d-1}{\ell}(2c_-c_+ + c_+^2) - 3\right) \\ \tilde{d}_{12} = -(3c_-c_+^2 - dc_- - 2dc_+) + c_+\left(\frac{\ell+d-1}{\ell}(2c_-c_+ + c_+^2) - 3\right) \\ \tilde{e}_{20} = (d_{20} - c_-e_{20})c_-^2 + (d_{11} - c_-e_{11})c_- + d_{02} - c_-e_{02} \\ \tilde{e}_{11} = 2c_-c_+(d_{20} - c_-e_{20}) + (c_- + c_+)(d_{11} - c_-e_{11}) + 2(d_{02} - c_-e_{02}) \\ \tilde{e}_{02} = (d_{20} - c_-e_{20})c_+^2 + (d_{11} - c_-e_{11})c_+ + d_{02} - c_-e_{02} \\ \tilde{e}_{21} = (3c_-^2c_+ - 2dc_- - dc_+) - c_-\left(\frac{\ell+d-1}{\ell}(c_-^2 + 2c_-c_+) - 3\right). \end{array}\right. \tag{F.2}$$

F.2. **Slopes and eigenvalues at P_2 for $r = r^*(d,\ell)$.** We compute the slopes and eigenvalues at P_2 for the critical speed $r = r^*(d,\ell)$, $0 < \ell < d$.

Lemma F.1 (Critical values of the slopes at P_2)**.** *Let*

$$0 < \ell < d, \quad r = r^*(d,\ell) = \frac{d+\ell}{\ell + \sqrt{d}},$$

then at P_2:

$$\left|\begin{array}{l} \sigma_2^\infty = \frac{\sqrt{d}}{\ell+\sqrt{d}} \\ \lambda_-^\infty = -\frac{d(d-\sqrt{d})+2d+\ell(d+\sqrt{d})}{(\ell+\sqrt{d})^2} < 0 \\ \lambda_+^\infty = 0 \\ 1 + c_-^\infty = \frac{(\sqrt{d}-1)(d-\ell)}{d(\sqrt{d}-1)+\ell(\sqrt{d}+1)} \\ c_+^\infty = \ell \\ c_1^\infty = -\frac{\sqrt{d}(d(\sqrt{d}-1)+\ell(\sqrt{d}+1)}{(\ell+\sqrt{d})^2} \\ c_2^\infty = -\frac{\sqrt{d}[d(\sqrt{d}-1)+\ell(\sqrt{d}+1)]}{\ell(\ell+\sqrt{d})^2} \\ c_3^\infty = -\frac{2\ell d}{(\ell+\sqrt{d})^2} \\ c_4^\infty = -\frac{2d}{(\ell+\sqrt{d})^2}. \end{array}\right. \tag{F.3}$$

Moreover,

$$\mu_+^\infty(r^*) = -\partial_r \lambda_+(r^*) = -\frac{4d}{-d + d^{\frac{3}{2}} + \ell + \sqrt{d}(2+\ell)} < 0. \tag{F.4}$$

Proof of Lemma F.1. The proof of (F.3) follows from (2.17), (2.32) which ensure

$$\sigma_3^\infty < \sigma_2^\infty = \sigma_5^\infty = \frac{r^*(d,\ell)\sqrt{d}}{d+\ell} = \frac{\sqrt{d}}{\ell+\sqrt{d}}.$$

Plugging this into (2.41) yields (F.3) via a direct computation. The computation of (F.4) can also be done analytically but is more involved and has been computed with Mathematica. □

Observe that $r = r^*(d,\ell)$ for $0 < \ell < d$ corresponds to *parallel slopes* at P_2:

$$c_-^\infty = -\frac{c_3^\infty}{c_1^\infty} = -\frac{c_4^\infty}{c_2^\infty}. \tag{F.5}$$

We observe the formulas

$$c_-^\infty = \frac{-2\ell\sqrt{d}}{d(\sqrt{d}-1) + \ell(\sqrt{d}+1)} \tag{F.6}$$

and the algebraic relation at $r^*(d,\ell)$:

$$2(\sigma_2^\infty)^2 \ell = c_-^\infty (2(\sigma_2^\infty)^2 + \lambda_-). \tag{F.7}$$

F.3. **Signs of $\tilde{d}_{20}$, $\tilde{e}_{30}$ and ν at r^*.** We compute explicitly the sign of the coefficients $\tilde{d}_{20}$, $\tilde{e}_{30}$ and ν at $r^*(\ell)$.

Lemma F.2. *For all $d \geq 2$ and $0 < \ell < d$, we have*

$$\tilde{d}_{20}^\infty < 0, \quad \tilde{e}_{20}^\infty > 0. \tag{F.8}$$

Also recalling (1.20) $p = 1 + \frac{4}{\ell}$, (5.2)*:*

$$\nu_\infty(d,\ell) > 0 \quad \textit{for} \quad \left| \begin{array}{l} d=2, \quad 0<\ell<2 \\ d=3, \quad 0<\ell<3 \\ d=5, \quad p \leq 10 \\ d=6, \quad p \leq 6 \\ d=7, \quad p \leq 4 \\ d=8, \quad p \leq 3 \\ d=9, \quad p \leq 3. \end{array} \right. \tag{F.9}$$

Proof. **step 1** Quadratic terms. We compute the values at $r^*(\ell)$ from (F.1):

$$\left| \begin{array}{l} d_{20}^\infty = \frac{-\sqrt{d}-(d-\ell)}{\ell+\sqrt{d}} \\ d_{11}^\infty = -\frac{2d\sqrt{d}}{\ell+\sqrt{d}} \\ d_{02}^\infty = \frac{-d\sqrt{d}+(d-\ell)\sqrt{d}}{\ell+\sqrt{d}} \\ e_{20}^\infty = \frac{(2d-1)\sqrt{d}-(d-\ell)\sqrt{d}}{\ell(\ell+\sqrt{d})} \\ e_{11}^\infty = \frac{-2d\sqrt{d}+(1+\sqrt{d})(d-\ell)}{\ell(\ell+\sqrt{d})} \\ e_{02}^\infty = \frac{-3\sqrt{d}}{\sqrt{d}+\ell}. \end{array} \right.$$

Also, recall that we have

$$c_+^\infty = \ell, \quad c_-^\infty = \frac{-2\ell\sqrt{d}}{d(\sqrt{d}-1)+\ell(\sqrt{d}+1)}, \quad \lambda_-^\infty = -\frac{d(d-\sqrt{d})+2d+\ell(d+\sqrt{d})}{(\ell+\sqrt{d})^2},$$

step 2 Computation of $\tilde{d}_{20}^{\infty}$, $\tilde{e}_{20}^{\infty}$ and ν_∞. Recall (F.2) which together with (F.3), (F.6) yields at $r^*(\ell)$:

$$\begin{aligned}
\tilde{d}_{20}^{\infty} &= -\frac{4\ell d(\sqrt{d}-1)(d-\ell)(d+\ell)}{(\ell+\sqrt{d})(d(\sqrt{d}-1)+\ell(1+\sqrt{d}))^2},\\
\tilde{e}_{20}^{\infty} &= -\frac{\ell(\sqrt{d}-1)\sqrt{d}(1+\sqrt{d})(\ell+d)\Big(\ell^2(\sqrt{d}-1)-2\ell\sqrt{d}(1+\sqrt{d})^2+d^{\frac{3}{2}}(-2+5\sqrt{d}-3d)\Big)}{(\ell+\sqrt{d})(d(\sqrt{d}-1)+\ell(1+\sqrt{d}))^3}.
\end{aligned}$$

We now compute from (5.2), (G.1), (4.22):

$$\begin{aligned}
\nu_\infty &= -\gamma b(\tilde{D}_{11}+\tilde{D}_{30}-\tilde{E}_{11})\\
&= -\frac{1}{(\tilde{d}_{20})^2}\Big[\tilde{e}_{20}\tilde{d}_{11}+(c_+-c_-)|\lambda_-|\tilde{d}_{30}-\tilde{e}_{11}|\tilde{d}_{20}|\Big]\\
&= -(1+\sqrt{d})\Big(2(\sqrt{d}-1)(\ell+d)d(d-\ell)^2\Big)^{-1}\\
&\quad\times\Big\{\ell^4(\sqrt{d}-1)^2-\ell^3 d(1+\sqrt{d})^2-4\ell^2 d(1-\sqrt{d}+2d-d^{\frac{3}{2}}+d^2)\\
&\quad+\ell d^2(4-12\sqrt{d}+3d+2d^{\frac{3}{2}}-d^2)+(\sqrt{d}-1)^2d^3(d-4)\Big\}. \qquad \text{(F.10)}
\end{aligned}$$

step 3 Sign of $\tilde{d}_{20}^{\infty}$, $\tilde{e}_{20}^{\infty}$ and ν_∞. The sign $\tilde{d}_{20}^{\infty}<0$ follows from its formula. Concerning $\tilde{e}_{20}^{\infty}$, we see from its formula that $-\tilde{e}_{20}^{\infty}$ has the same sign as

$$\mathcal{E}_{20} := \ell^2(\sqrt{d}-1)-2\ell\sqrt{d}(1+\sqrt{d})^2+d^{\frac{3}{2}}(-2+5\sqrt{d}-3d).$$

Now, we have

$$\mathcal{E}_{20}'(\ell) = 2(\sqrt{d}-1)\left(\ell-\frac{\sqrt{d}(1+\sqrt{d})^2}{\sqrt{d}-1}\right)$$

and since for $d\geq 2$ we have

$$\frac{\sqrt{d}(1+\sqrt{d})^2}{\sqrt{d}-1}>d,$$

we infer for $\ell\leq d$ that $\mathcal{E}_{20}$ is decreasing and hence for $0\leq\ell\leq d$,

$$\mathcal{E}_{20}(\ell)\leq\mathcal{E}_{20}(0)=-d^{\frac{3}{2}}(2-5\sqrt{d}+3d)=-3d^{\frac{3}{2}}(\sqrt{d}-1)\left(\sqrt{d}-\frac{2}{3}\right)<0$$

so that $\tilde{e}_{20}>0$ for all $d\geq 2$ and $0\leq\ell\leq d$.
Concerning ν, we see from its formula that ν has the opposite sign to

$$\begin{aligned}
N_0 := {}& \ell^4(\sqrt{d}-1)^2-\ell^3 d(1+\sqrt{d})^2-4\ell^2 d(1-\sqrt{d}+2d-d^{\frac{3}{2}}+d^2)\\
&+\ell d^2(4-12\sqrt{d}+3d+2d^{\frac{3}{2}}-d^2)+(\sqrt{d}-1)^2d^3(d-4).
\end{aligned}$$

We then check the sign of N_0 numerically and confirm that $\nu>0$ for the above claimed range. □

F.4. **Slopes and eigenvalues at P_2 for $r=r_+(d,\ell)$.**

Lemma F.3 (Critical values of the slopes at P_2)**.** *Let*

$$\ell>d,\quad r=r_+(d,\ell)=1+\frac{d-1}{(1+\sqrt{\ell})^2},$$

then at P_2:

$$\left|\begin{array}{l} \sigma_2^\infty = \frac{1}{1+\sqrt{\ell}} \\ w_2^\infty = \frac{\sqrt{\ell}}{1+\sqrt{\ell}} \\ c_1^\infty = -\frac{2\sqrt{\ell}(d+\sqrt{\ell})}{(1+\sqrt{\ell})^3} \\ c_2^\infty = -\frac{2}{(1+\sqrt{\ell})^2} \\ c_3^\infty = -\frac{2\sqrt{\ell}(d+\sqrt{\ell})}{(1+\sqrt{\ell})^3} \\ c_4^\infty = -\frac{2}{(1+\sqrt{\ell})^2} \\ c_-^\infty = -1 \\ c_+^\infty = \frac{\sqrt{\ell}(d+\sqrt{\ell})}{1+\sqrt{\ell}} \\ \lambda_+^\infty = 0 \\ \lambda_-^\infty = -\frac{2\left[\ell+(d+1)\sqrt{\ell}+1\right]}{(1+\sqrt{\ell})^3}. \end{array}\right. \tag{F.11}$$

Moreover,

$$\mu_+^\infty = (\partial_b \lambda_+)_{b=0} = -\frac{2\sqrt{d-1}(\ell-d)}{(1+\sqrt{\ell})\ell^{\frac{1}{4}}(1+(d+1)\sqrt{\ell}+\ell)} < 0. \tag{F.12}$$

Proof. This follows from a direct computation. (F.12) has been computed with Mathematica. □

F.5. **Signs of $\tilde{d}_{20}$, $\tilde{e}_{30}$ and ν at r_+.** We compute explicitly the sign of the coefficients $\tilde{d}_{20}$, $\tilde{e}_{30}$ and ν at $r^*(\ell)$.

Lemma F.4. *For all $d \geq 2$ and $\ell > d$, we have*

$$\tilde{d}_{20}^\infty < 0, \quad \tilde{e}_{20}^\infty > 0, \quad \nu_\infty > 0. \tag{F.13}$$

Proof. We collect the values

$$\left|\begin{array}{l} d_{20}^\infty = \frac{\ell-\sqrt{\ell}-d-1}{(1+\sqrt{\ell})^2} \\ d_{11}^\infty = \frac{-2d}{1+\sqrt{\ell}} \\ d_{02}^\infty = \frac{-\ell-d\sqrt{\ell}}{(1+\sqrt{\ell})^2} \\ e_{20}^\infty = \frac{\ell+d-1}{\ell(1+\sqrt{\ell})} \\ e_{11}^\infty = \frac{-2}{1+\sqrt{\ell}} \\ e_{02}^\infty = -\frac{3}{1+\sqrt{\ell}} \\ e_{21}^\infty = \frac{\ell+d-1}{\ell}. \end{array}\right. \tag{F.14}$$

We compute directly

$$\left|\begin{array}{l} \tilde{d}_{20}^\infty = -\frac{(d-1)(\ell-d)}{\sqrt{\ell}(1+\sqrt{\ell})^2} \\ \tilde{e}_{20}^\infty = \frac{(d-1)(\ell+1)}{\ell(1+\sqrt{\ell})}. \end{array}\right.$$

and

$$\begin{aligned} \nu_\infty \;=\; & \frac{2}{(d-1)(\ell-d)^2}\Big[d^3(\ell+2)+d^2(\ell^2+2\ell^{\frac{3}{2}}-2\ell+2\sqrt{\ell}-2)+d(2\ell^{\frac{5}{2}}+5\ell^2+4\ell^{\frac{3}{2}}+7\ell+2\sqrt{\ell}+1) \\ & +\ell(\ell^2+2\ell^{\frac{3}{2}}+\ell+2\sqrt{\ell}+1)\Big] > 0 \end{aligned} \tag{F.15}$$

and the claim is proved. We note here that this function ν_∞ is positive for *every* value of $\ell > 0$. □

Appendix G. Expansion of the functionals in (4.15)

In this Appendix we collect all the formulas for all the terms appearing in (4.15). We recall $x = bu$.

$\tilde{D}_{ij}, \tilde{E}_{ij}$ coefficients

$$\left|\begin{array}{l} \tilde{D}_{20} = \frac{\tilde{d}_{20}\tilde{w}_\infty}{|\mu_+|(c_+-c_-)} \\ \tilde{D}_{11} = \frac{d_{11}\tilde{\psi}_\infty\tilde{w}_\infty}{|\mu_+|(c_+-c_-)} \\ \tilde{D}_{02} = \frac{d_{02}\tilde{\psi}_\infty^2\tilde{w}_\infty}{|\mu_+|(c_+-c_-)} \\ \tilde{D}_{30} = \frac{d_{30}\tilde{w}_\infty^2}{|\mu_+|(c_+-c_-)} \\ \tilde{D}_{21} = \frac{d_{21}\tilde{\psi}_\infty\tilde{w}_\infty^2}{|\mu_+|(c_+-c_-)} \\ \tilde{D}_{12} = \frac{d_{12}\tilde{\psi}_\infty^2\tilde{w}_\infty^2}{|\mu_+|(c_+-c_-)} \\ \tilde{D}_{03} = \frac{d_{03}\tilde{\psi}_\infty^3\tilde{w}_\infty^2}{|\mu_+|(c_+-c_-)} \end{array}\right. , \quad \left|\begin{array}{l} \tilde{E}_{20} = \frac{\tilde{e}_{20}\tilde{w}_\infty}{(c_+-c_-)|\lambda_-|\tilde{\psi}_\infty} \\ \tilde{E}_{11} = \frac{\tilde{e}_{11}\tilde{w}_\infty}{(c_+-c_-)|\lambda_-|} \\ \tilde{E}_{02} = \frac{\tilde{e}_{02}\tilde{\psi}_\infty\tilde{w}_\infty}{(c_+-c_-)|\lambda_-|} \\ \tilde{E}_{30} = \frac{\tilde{e}_{30}\tilde{w}_\infty^2}{(c_+-c_-)|\lambda_-|\tilde{\psi}_\infty} \\ \tilde{E}_{21} = \frac{\tilde{e}_{21}\tilde{w}_\infty^2}{(c_+-c_-)|\lambda_-|} \\ \tilde{E}_{12} = \frac{\tilde{e}_{12}\tilde{\psi}_\infty\tilde{w}_\infty^2}{(c_+-c_-)|\lambda_-|} \\ \tilde{E}_{03} = \frac{\tilde{e}_{03}\tilde{\psi}_\infty^2\tilde{w}_\infty^2}{(c_+-c_-)|\lambda_-|} \end{array}\right. \tag{G.1}$$

Remark G.1. Note that the coefficients $\tilde{D}_{ij}, \tilde{E}_{ij}$ have a well defined limit $\tilde{D}_{ij}^\infty, \tilde{E}_{ij}^\infty$ as $b \to 0$ from (4.16).

Polynomials H_1, H_2

$$\left|\begin{array}{l} H_1(b,u) = \sum_{j=0}^3 b^j H_{1,j}(x) \\ H_2(b,u) = \sum_{j=0}^3 b^j H_{2,j}(x) \end{array}\right. \tag{G.2}$$

with

$$\left|\begin{array}{l} H_{1,0}(x) = -(\tilde{E}_{11}+\tilde{E}_{30}) + (\tilde{E}_{02}+\tilde{E}_{21})x - \tilde{E}_{12}x^2 + \tilde{E}_{03}x^3 \\ H_{1,1}(x) = (\tilde{E}_{02}+\tilde{E}_{21}) - \tilde{E}_{12}x + \tilde{E}_{03}x^2 \\ H_{1,2}(x) = -\tilde{E}_{12} + \tilde{E}_{03}x \\ H_{1,3}(x) = \tilde{E}_{03}, \\ H_{2,0}(x) = (\tilde{D}_{11}+\tilde{D}_{30})x - (\tilde{D}_{02}+\tilde{D}_{21})x^2 + \tilde{D}_{12}x^3 - \tilde{D}_{03}x^4 \\ H_{2,1}(x) = -(\tilde{D}_{02}+\tilde{D}_{21})x + \tilde{D}_{12}x^2 - \tilde{D}_{03}x^3 \\ H_{2,2}(x) = \tilde{D}_{12}x - \tilde{D}_{03}x^2 \\ H_{2,3} = -\tilde{D}_{03}x. \end{array}\right. \tag{G.3}$$

Polynomials G_1, G_2

$$\left|\begin{array}{l} G_1(b,u) = \tilde{E}_{11}x - (2\tilde{E}_{02}+\tilde{E}_{21})x^2 + 2\tilde{E}_{12}x^3 - 3\tilde{E}_{03}x^4 \\ G_2(b,u) = -\tilde{D}_{11}x + (2\tilde{D}_{02}+\tilde{D}_{21})x^2 - 2\tilde{D}_{12}x^3 + 3\tilde{D}_{03}x^4 \end{array}\right. \tag{G.4}$$

Nonlinear terms

$$\left|\begin{array}{l} \widetilde{\mathrm{NL}}_1 = \sum_{j=0}^2 b^j \widetilde{\mathrm{NL}}_{1j} \\ \widetilde{\mathrm{NL}}_2 = \sum_{j=0}^2 b^j \widetilde{\mathrm{NL}}_{2j} \end{array}\right. \tag{G.5}$$

with

$$\left|\begin{array}{l} \widetilde{\mathrm{NL}}_{10} = -xM_{11}\Psi^2 + x^2M_{12}\Psi^3 \\ \widetilde{\mathrm{NL}}_{11} = M_{11}\Psi^2 - 2xM_{12}\Psi^3 \\ \widetilde{\mathrm{NL}}_{12} = M_{12}\Psi^3 \end{array}\right. , \quad \left|\begin{array}{l} M_{11} = -\tilde{E}_{02}x + \tilde{E}_{12}x^2 - 3\tilde{E}_{03}x^3 \\ M_{12} = \tilde{E}_{03}x^2 \end{array}\right.$$

$$\left|\begin{array}{l} \widetilde{\mathrm{NL}}_{20} = -xM_{21}\Psi^2 + x^2M_{22}\Psi^3 \\ \widetilde{\mathrm{NL}}_{21} = M_{21}\Psi^2 - 2xM_{22}\Psi^3 \\ \widetilde{\mathrm{NL}}_{22} = M_{22}\Psi^3 \end{array}\right. , \quad \left|\begin{array}{l} M_{21} = \tilde{D}_{02}x - \tilde{D}_{12}x^2 + 3\tilde{D}_{03}x^3 \\ M_{22} = \tilde{D}_{03}x^2. \end{array}\right.$$

Appendix H. Proof of (9.31) and (9.32)

This section is devoted to the derivation of the estimates (9.31) and (9.32) by computing both the $P_2 - P_\infty$ separatrix Φ_S and the root Φ_+ in the variables of the renormalization (4.14).

H.1. Computation of Φ_S.

Lemma H.1 (Computation the separatrix Φ_S in u)**.** *Let*

$$\left| \begin{array}{l} a = \gamma b = \frac{|\lambda_-|}{|\mu_+|} \\ \Theta_0 = -\tilde{E}_{11} - \tilde{E}_{30} - \frac{2}{a} \\ a_1 = -\frac{(c_+ - c_-)\mu_+}{\tilde{d}_{20}} \\ a_2 = -\frac{1}{a_1}\frac{1}{\tilde{d}_{20}}\left(-\frac{\tilde{d}_{11}\tilde{e}_{20}}{(c_+-c_-)\lambda_-} + \tilde{d}_{30}\right)\left(\frac{(c_+-c_-)\mu_+}{\tilde{d}_{20}}\right)^2 \\ g_1 = \frac{a_1\tilde{e}_{20}}{(c_+-c_-)\lambda_-} = -\frac{\tilde{e}_{20}\mu_+}{\lambda_-\tilde{d}_{20}} \\ g_2 = \frac{a_1\tilde{e}_{20}}{(c_+-c_-)\lambda_-}\left[a_2 + \left(\frac{\tilde{e}_{30}}{\tilde{e}_{20}} - \frac{\tilde{e}_{11}}{(c_+-c_-)\lambda_-}\right)a_1\right]. \end{array} \right. \tag{H.1}$$

then

$$\begin{aligned} \Phi_S &= c_- - b\left[(c_+ - c_-)g_1 u\right] \\ &+ b^2\left\{-(c_+ - c_-)(g_2 + g_1\Theta_0)u + \left[(c_+ - c_-)g_1\Theta_0 - (c_+ - c_-)\, g_1^2\right] u^2\right\} + O(b^3) \end{aligned} \tag{H.2}$$

where the $O(b^3)$ is uniform in $0 < b < b^* \ll 1$ small enough and $u \in [0,1]$.

Proof of Lemma H.1. **step 1** Unfolding the change of variables. Recall:

$$\left| \begin{array}{l} \tilde{W} = -b\tilde{w} \\ \tilde{\Sigma} = b^2\tilde{\sigma} \end{array} \right. , \quad \left| \begin{array}{l} \tilde{\psi} = \frac{\tilde{\sigma}}{\tilde{w}} = \tilde{\psi}_\infty\phi \\ \tilde{w} = \tilde{w}_\infty u \end{array} \right. ,$$

and

$$\left| \begin{array}{l} W \\ \Sigma \end{array} \right. = P \left| \begin{array}{l} \tilde{W} \\ \tilde{\Sigma} \end{array} \right. = \left(\begin{array}{cc} c_- & c_+ \\ 1 & 1 \end{array} \right) \left| \begin{array}{l} \tilde{W} \\ \tilde{\Sigma} \end{array} \right. = \left| \begin{array}{l} c_-\tilde{W} + c_+\tilde{\Sigma} \\ \tilde{W} + \tilde{\Sigma}. \end{array} \right.$$

Then,

$$\left| \begin{array}{l} \tilde{W} = -b\tilde{w} = -b\tilde{w}_\infty u \\ \tilde{\Sigma} = b^2\tilde{\sigma} = b^2\tilde{w}_\infty\tilde{\psi}_\infty u\phi \end{array} \right. , \quad \left| \begin{array}{l} W = -c_- b\tilde{w}_\infty u + c_+ b^2\tilde{w}_\infty\tilde{\psi}_\infty u\phi \\ \Sigma = -b\tilde{w}_\infty u + b^2\tilde{w}_\infty\tilde{\psi}_\infty u\phi \end{array} \right. \tag{H.3}$$

which yields

$$\Phi = \frac{W}{\Sigma} = \frac{-c_- b\tilde{w}_\infty u + c_+ b^2\tilde{w}_\infty\tilde{\psi}_\infty u\phi}{-b\tilde{w}_\infty u + b^2\tilde{w}_\infty\tilde{\psi}_\infty u\phi} = \frac{c_- - bc_+\tilde{\psi}_\infty\phi}{1 - b\tilde{\psi}_\infty\phi}.$$

For $0 \le u \le 1$, recalling (7.87), (6.29), (6.1), (4.14), (6.2), (6.10):

$$\left| \begin{array}{l} \phi = u + (1-u)\Psi = u + bM_b u(1-u)\Theta \\ M_b = 1 + O(b) \end{array} \right. \tag{H.4}$$

such that for the separatrix

$$|\Theta_S| \le c(d,\ell)$$

uniformly as $b \to 0$ from (7.87). Therefore, $\tilde{\psi}_\infty\phi$ is bounded on $[0,1]$ uniformly as $b \to 0$, and we may thus expand in b uniformly for $u \in [0,1]$, to obtain for the separatrix curve:

$$\begin{aligned} \Phi &= \frac{c_- - bc_+\tilde{\psi}_\infty\phi}{1 - b\tilde{\psi}_\infty\phi} = (c_- - bc_+\tilde{\psi}_\infty\phi)(1 + b\tilde{\psi}_\infty\phi + b^2\tilde{\psi}_\infty^2\phi^2 + O(b^3)) \\ &= c_- - b\tilde{\psi}_\infty(c_+ - c_-)\phi - b^2\phi^2\tilde{\psi}_\infty^2\,(c_+ - c_-) + O(b^3). \end{aligned} \tag{H.5}$$

step 2 Computation of $\tilde{\psi}_{\infty}$. Recall (4.13):

$$\left| \begin{array}{l} \tilde{W}_{\infty} = -b\frac{(c_+-c_-)\mu_+}{\tilde{d}_{20}} + O(b^2) \\ \tilde{\Sigma}_{\infty} = -\frac{\tilde{e}_{20}\tilde{W}_{\infty}^2}{(c_+-c_-)\lambda_-} + O(b^3) \end{array} \right.$$

and

$$\begin{aligned} \mathcal{G}_1 &= (c_+ - c_-)\lambda_+\tilde{W} + \tilde{d}_{20}\tilde{W}^2 + \tilde{d}_{11}\tilde{W}\tilde{\Sigma} + \tilde{d}_{02}\tilde{\Sigma}^2 + \tilde{d}_{30}\tilde{W}^3 + \tilde{d}_{21}\tilde{W}^2\tilde{\Sigma} + \tilde{d}_{12}\tilde{W}\tilde{\Sigma}^2 + \tilde{d}_{03}\Sigma^3 \\ &= -\Delta_1 + c_+\Delta_2, \end{aligned}$$

$$\begin{aligned} \mathcal{G}_2 &= (c_+ - c_-)\lambda_-\tilde{\Sigma} + \tilde{e}_{20}\tilde{W}^2 + \tilde{e}_{11}\tilde{W}\tilde{\Sigma} + \tilde{e}_{02}\tilde{\Sigma}^2 + \tilde{e}_{30}\tilde{W}^3 + \tilde{e}_{21}\tilde{W}^2\tilde{\Sigma} + \tilde{e}_{12}\tilde{W}\tilde{\Sigma}^2 + \tilde{e}_{03}\Sigma^3 \\ &= \Delta_1 - c_-\Delta_2. \end{aligned}$$

This yields

$$\begin{aligned} & \tilde{\Sigma}_{\infty}[(c_+ - c_-)\lambda_- + \tilde{e}_{11}\tilde{W}_{\infty}] = -\tilde{e}_{20}\tilde{W}_{\infty}^2 - \tilde{e}_{30}\tilde{W}_{\infty}^3 + O(b^4) \\ \Rightarrow \quad & \tilde{\Sigma}_{\infty} = \frac{-\tilde{e}_{20}\tilde{W}_{\infty}^2 - \tilde{e}_{30}\tilde{W}_{\infty}^3 + O(b^4)}{(c_+ - c_-)\lambda_- + \tilde{e}_{11}\tilde{W}_{\infty}} \\ = \quad & -\frac{1}{(c_+ - c_-)\lambda_-}\left[\tilde{e}_{20}\tilde{W}_{\infty}^2 + \tilde{e}_{30}\tilde{W}_{\infty}^3 + O(b^4)\right]\left(1 - \frac{\tilde{e}_{11}\tilde{W}_{\infty}}{(c_+ - c_-)\lambda_-} + O(b^2)\right) \\ = \quad & -\frac{\tilde{e}_{20}\tilde{W}_{\infty}^2}{(c_+ - c_-)\lambda_-}\left[1 + \frac{\tilde{e}_{30}}{\tilde{e}_{20}}\tilde{W}_{\infty} + O(b^2)\right]\left(1 - \frac{\tilde{e}_{11}\tilde{W}_{\infty}}{(c_+ - c_-)\lambda_-} + O(b^2)\right) \\ = - \quad & \frac{\tilde{e}_{20}\tilde{W}_{\infty}^2}{(c_+ - c_-)\lambda_-}\left[1 + \left(\frac{\tilde{e}_{30}}{\tilde{e}_{20}} - \frac{\tilde{e}_{11}}{(c_+ - c_-)\lambda_-}\right)\tilde{W}_{\infty} + O(b^2)\right] \end{aligned}$$

and

$$\begin{aligned} & (c_+ - c_-)\mu_+ b + \tilde{d}_{20}\tilde{W}_{\infty} + \tilde{d}_{11}\tilde{\Sigma}_{\infty} + \tilde{d}_{30}\tilde{W}_{\infty}^2 + O(b^3) = 0 \\ \Rightarrow \quad & \tilde{W}_{\infty} = -\frac{1}{\tilde{d}_{20}}\left[(c_+ - c_-)\mu_+ b - \tilde{d}_{11}\frac{\tilde{e}_{20}\tilde{W}_{\infty}^2}{(c_+ - c_-)\lambda_-} + \tilde{d}_{30}\tilde{W}_{\infty}^2\right] + O(b^3) \\ = \quad & -\frac{1}{\tilde{d}_{20}}\left[(c_+ - c_-)\mu_+ b + \left(-\frac{\tilde{d}_{11}\tilde{e}_{20}}{(c_+ - c_-)\lambda_-} + \tilde{d}_{30}\right)\left(b\frac{(c_+ - c_-)\mu_+}{\tilde{d}_{20}}\right)^2\right] + O(b^3) \\ = \quad & \left(-\frac{(c_+ - c_-)\mu_+}{\tilde{d}_{20}}\right)b - \frac{1}{\tilde{d}_{20}}\left(-\frac{\tilde{d}_{11}\tilde{e}_{20}}{(c_+ - c_-)\lambda_-} + \tilde{d}_{30}\right)\left(\frac{(c_+ - c_-)\mu_+}{\tilde{d}_{20}}\right)^2 b^2 + O(b^3) \\ = \quad & a_1 b\left(1 + a_2 b + O(b^2)\right) \end{aligned}$$

with (a_1, a_2) given by (H.1). This yields

$$\begin{aligned} \tilde{\psi}_{\infty} &= -\frac{\tilde{\Sigma}_{\infty}}{b\tilde{W}_{\infty}} = \frac{1}{b\tilde{W}_{\infty}}\frac{\tilde{e}_{20}\tilde{W}_{\infty}^2}{(c_+ - c_-)\lambda_-}\left[1 + \left(\frac{\tilde{e}_{30}}{\tilde{e}_{20}} - \frac{\tilde{e}_{11}}{(c_+ - c_-)\lambda_-}\right)\tilde{W}_{\infty} + O(b^2)\right] \\ &= \frac{\tilde{e}_{20}\tilde{W}_{\infty}}{b(c_+ - c_-)\lambda_-}\left[1 + \left(\frac{\tilde{e}_{30}}{\tilde{e}_{20}} - \frac{\tilde{e}_{11}}{(c_+ - c_-)\lambda_-}\right)\tilde{W}_{\infty} + O(b^2)\right] \\ &= \frac{a_1\tilde{e}_{20}}{(c_+ - c_-)\lambda_-}\left(1 + a_2 b + O(b^2)\right)\left[1 + \left(\frac{\tilde{e}_{30}}{\tilde{e}_{20}} - \frac{\tilde{e}_{11}}{(c_+ - c_-)\lambda_-}\right)a_1 b + O(b^2)\right] \\ &= g_1 + b g_2 + O(b^2) \end{aligned}$$

with (g_1, g_2) given by (H.1).

step 3 Computation of Φ_S. We have for $0 \leq u \leq 1$ recalling (H.1), (H.4):

$$\phi_S = u + bu(1-u)M_b\Theta_S(u) = u + bu(1-u)\Theta_0 + O(ub^2)$$

and hence from (H.5):

$$\begin{aligned} & \Phi_S = c_- - b\tilde{\psi}_{\text{👁}}(c_+ - c_-)\phi_S - b^2\phi_S^2\tilde{\psi}_{\text{👁}}^2\,(c_+ - c_-) + O(b^3) \\ = \;& c_- - b(c_+ - c_-)\left[g_1 + bg_2 + O(b^2)\right]\left\{u + bu(1-u)\Theta_0 + O(ub^2)\right\} \\ - \;& b^2\,(c_+ - c_-)\,g_1^2u^2 + O(b^3) \\ = \;& c_- - b(c_+ - c_-)u\left\{g_1 + b\,(g_2 + g_1\Theta_0(1-u)) + O(b^2)\right\} - b^2\,(c_+ - c_-)\,g_1^2u^2 + O(b^3) \\ = \;& c_- - b\left[(c_+ - c_-)g_1u\right] + b^2\left[-(c_+ - c_-)[(g_2 + g_1\Theta_0)u - g_1\Theta_0u^2] - (c_+ - c_-)\,g_1^2u^2\right] + O(b^3) \\ = \;& c_- - b\left[(c_+ - c_-)g_1u\right] + b^2\left\{-(c_+ - c_-)(g_2 + g_1\Theta_0)u + \left[(c_+ - c_-)g_1\Theta_0 - (c_+ - c_-)\,g_1^2\right]u^2\right\} \\ + \;& O(b^3) \end{aligned}$$

and (H.2) is proved. □

H.2. **Computation of Φ_+.** We now use the separatrix curve $\Phi_S(u)$ to parametrize Σ in the eye and compute the root $\widetilde{\Phi}_+$.

Lemma H.2 (Computation of $\widetilde{\Phi}_+$)**.** *Let $\widetilde{\Phi}_+$ be given by* (9.28)*. Let*

$$\left|\begin{array}{l} f_{00} = -\lambda_+(1 + c_-) \\ f_{01} = -2\sigma_2 - e_{02} - d_{02} + (-4\sigma_2 - e_{11} - d_{11})c_- + (-2\sigma_2 - e_{20} - d_{20})c_-^2 \\ f_{02} = (d-1)c_- + (1 - e_{21})c_-^2 \\ f_{10} = -(c_2 + c_1) \\ f_{11} = (-4\sigma_2 - e_{11} - d_{11}) + 2c_-(-2\sigma_2 - e_{20} - d_{20}) \\ f_{21} = -2\sigma_2 - e_{20} - d_{20} \end{array}\right. \tag{H.6}$$

then, uniformyl for $0 < b < b^* \ll 1$ small enough and $0 \leq u \leq 1$ for $r_{\text{👁}} = r^*$:

$$\widetilde{\Phi}_+ \;=\; -\frac{f_{00}}{f_{10}} - \left(\frac{f_{01}}{f_{10}}a_1\right)bu + O(b^2) \tag{H.7}$$

and for $r_{\text{👁}} = r_+$:

$$\begin{aligned} & \widetilde{\Phi}_+ \\ = \;& -\frac{f_{00}}{f_{10}} - \left(\frac{f_{01}}{f_{10}}a_1\right)bu + \left[-\frac{f_{01}a_1a_2}{f_{10}}u + \left(\frac{f_{01}f_{11} - f_{02}f_{10}}{f_{10}^2}a_1^2 + \frac{f_{01}}{f_{10}}a_1g_1\right)u^2\right]b^2 + O(b^3). \end{aligned} \tag{H.8}$$

Proof of Lemma H.2. We recall the formulas using the notation (H.6)

$$\left|\begin{array}{l} F_0 = f_{00} + f_{01}\Sigma + f_{02}\Sigma^2 \\ F_1 = f_{10} + f_{11}\Sigma + O(\Sigma^2) \\ F_2 = f_{21}\Sigma + O(\Sigma^2) \end{array}\right.$$

and

$$\widetilde{\Phi}_\pm = \frac{-|F_1(\Sigma)| \pm \sqrt{F_1(\Sigma)^2 - 4|F_2(\Sigma)|F_0(\Sigma)}}{2|F_2(\Sigma)|}.$$

The difference between the case $r_{\text{👁}} = r^*$ and $r_{\text{👁}} = r_+$ is that $f_{00} = O(b)$ in the first case while $f_{00} = O(b^2)$ in the second case. The case $r_{\text{👁}} = r^*$ being similar and simpler is left to the reader. From now on, we focus the case $r_{\text{👁}} = r_+$ for which we need to prove (H.8).

We compute explicitely

$$\left|\begin{array}{l} f_{00} = O(b^2) \\ f_{10} = |c_1^\infty + c_2^\infty| + O(b) > 0 \\ f_{21} = -2\sigma_2^\infty - e_{20}^\infty - d_{20}^\infty < 0 \end{array}\right.$$

Then, remembering $\Sigma < 0$ and for $0 < b < b^*$ small enough, yields $F_2 > 0$, $F_1 < 0$ for $u \in [0,1]$.

step 1 Taylor expansion. We compute using $f_{00} = O(b^2)$:

$$\begin{aligned} & F_1(\Sigma)^2 - 4F_2(\Sigma)F_0(\Sigma) = (f_{10} + f_{11}\Sigma + O(\Sigma^2))^2 - 4(f_{21}\Sigma + O(\Sigma^2))\left(f_{00} + f_{01}\Sigma + O(\Sigma^2)\right) \\ = \;& f_{10}^2 + 2f_{10}f_{11}\Sigma + O(b^2) = f_{10}^2\left[1 + \left(\frac{2f_{11}}{f_{10}}\right)\Sigma + O(b^2)\right] \end{aligned}$$

and

$$\begin{aligned} & F_1(\Sigma) + \sqrt{F_1(\Sigma)^2 - 4F_2(\Sigma)F_0(\Sigma)} = f_{10} + f_{11}\Sigma + O(b^2) + f_{10}\left[1 + \left(\frac{2f_{11}}{f_{10}}\right)\Sigma + O(b^2)\right]^{\frac{1}{2}} \\ = \;& f_{10} + f_{11}\Sigma + O(b^2) + f_{10}\left(1 + \frac{f_{11}}{f_{10}}\Sigma\right) + O(b^2) = 2f_{10} + 2f_{11}\Sigma + O(b^2) \\ = \;& 2f_{10}\left(1 + \frac{f_{11}}{f_{10}}\Sigma + O(b^2)\right). \end{aligned}$$

We conclude

$$\begin{aligned} \widetilde{\Phi}_+ &= -\frac{2F_0(\Sigma)}{F_1(\Sigma) + \sqrt{F_1(\Sigma)^2 - 4F_2(\Sigma)F_0(\Sigma)}} = \frac{-2(f_{00} + f_{01}\Sigma + f_{02}\Sigma^2)}{2f_{10}\left[1 + \frac{f_{11}}{f_{10}}\Sigma + O(b^2)\right]} \\ &= -\frac{1}{f_{10}}\left[f_{00} + f_{01}\Sigma + f_{02}\Sigma^2\right]\left[1 - \frac{f_{11}}{f_{10}}\Sigma + O(b^2)\right] \\ &= -\frac{1}{f_{10}}\left[f_{00} + f_{01}\Sigma + \left(-\frac{f_{01}f_{11}}{f_{10}} + f_{02}\right)\Sigma^2 + O(b^3)\right] \\ &= -\frac{f_{00}}{f_{10}} - \frac{f_{01}}{f_{10}}\Sigma + \frac{f_{01}f_{11} - f_{02}f_{10}}{f_{10}^2}\Sigma^2 + O(b^3). \end{aligned}$$

step 2 Expression in terms of u. We recall from (H.3):

$$\begin{aligned} \Sigma &= -b\tilde{w}_\infty u + b^2\tilde{w}_\infty\tilde{\psi}_\infty u\phi = \tilde{W}_\infty u - b\tilde{W}_\infty\tilde{\psi}_\infty u\phi \\ &= a_1 b\left(1 + a_2 b + O(b^2)\right)u - b^2 a_1 g_1 u^2 + O(b^3) \\ &= [a_1 u]b + \left[a_1 a_2 u - a_1 g_1 u^2\right]b^2 + O(b^3) \end{aligned}$$

and hence

$$\begin{aligned} \widetilde{\Phi}_+ &= -\frac{f_{00}}{f_{10}} - \frac{f_{01}}{f_{10}}\left\{[a_1 u]b + \left[a_1 a_2 u - a_1 g_1 u^2\right]b^2\right\} + \frac{f_{01}f_{11} - f_{02}f_{10}}{f_{10}^2}a_1^2 b^2 u^2 + O(b^3) \\ &= -\frac{f_{00}}{f_{10}} - \left(\frac{f_{01}}{f_{10}}a_1\right)bu + \left[\frac{f_{01}f_{11} - f_{02}f_{10}}{f_{10}^2}a_1^2 u^2 - \frac{f_{01}}{f_{10}}(a_1 a_2 u - a_1 g_1 u^2)\right]b^2 + O(b^3) \\ &= -\frac{f_{00}}{f_{10}} - \left(\frac{f_{01}}{f_{10}}a_1\right)bu + \left\{-\frac{f_{01}a_1 a_2}{f_{10}}u + \left[\frac{f_{01}f_{11} - f_{02}f_{10}}{f_{10}^2}a_1^2 + \frac{f_{01}}{f_{10}}a_1 g_1\right]u^2\right\}b^2 + O(b^3) \end{aligned}$$

and (H.8) is proved. □

H.3. **The positivity condition and proof of** (9.31) **and** (9.32)**.** Again, the case $r_\circledcirc = r^*$ being similar and simpler is left to the reader, we focus on the case $r_\circledcirc = r_+$ for which we need to prove (9.32). We study the positivity condition

$$\Phi_S > \Phi_+$$

in the zone $0 \le u \le 1$ which becomes

$$\begin{aligned} & -b\left[(c_+ - c_-)g_1 u\right] + b^2\left\{-(c_+ - c_-)(g_2 + g_1\Theta_0)u + \left[(c_+ - c_-)g_1\Theta_0 - (c_+ - c_-)\, g_1^2\right] u^2\right\} \\ > \quad & -\frac{f_{00}}{f_{10}} - \left(\frac{f_{01}}{f_{10}}a_1\right) bu + \left\{-\frac{f_{01}a_1a_2}{f_{10}}u + \left[\frac{f_{01}f_{11} - f_{02}f_{10}}{f_{10}^2}a_1^2 + \frac{f_{01}}{f_{10}}a_1g_1\right] u^2\right\} b^2 + O(b^3) \\ \Leftrightarrow \quad & A_0 + bA_1u + b^2(A_3u + A_4u^2) + O(b^3) > 0 \end{aligned}$$

with

$$\left|\begin{array}{l} A_0 = \frac{f_{00}}{f_{10}} \\ A_1 = -(c_+ - c_-)g_1 + \frac{f_{01}}{f_{10}}a_1 \\ A_3 = \frac{f_{01}}{f_{10}}a_1a_2 - (c_+ - c_-)(g_2 + g_1\Theta_0) \\ A_4 = -\frac{f_{01}}{f_{10}}a_1g_1 + \frac{f_{02}f_{10} - f_{01}f_{11}}{f_{10}^2}a_1^2 - (c_+ - c_-)\, g_1^2 + (c_+ - c_-)g_1\Theta_0 \end{array}\right.$$

We compute

$$A_0 = \frac{f_{00}}{f_{10}} = \frac{-\lambda_+(1 + c_-)}{-(c_1 + c_2)} = b\mu_+\frac{1 + c_-}{c_1 + c_2}$$

and

$$\begin{aligned} & A_1 = -(c_+ - c_-)g_1 + \frac{f_{01}}{f_{10}}a_1 = (c_+ - c_-)\left(\frac{\tilde{e}_{20}\mu_+}{\lambda_-\tilde{d}_{20}}\right) + \frac{f_{01}}{f_{10}}\left(-\frac{(c_+ - c_-)\mu_+}{\tilde{d}_{20}}\right) \\ = \quad & \frac{(c_+ - c_-)\mu_+}{\tilde{d}_{20}}\left[\frac{\tilde{e}_{20}}{\lambda_-} - \frac{f_{01}}{f_{10}}\right] \end{aligned}$$

We therefore divide by $|\mu_+| = -\mu_+$ and obtain the condition

$$B_0 + bB_1u + b^2(B_3u + B_4u^2) + O(b^3) > 0 \tag{H.9}$$

with

$$\left|\begin{array}{l} B_0 = -\frac{b(1+c_-)}{c_1+c_2} \\ B_1 = \frac{c_+ - c_-}{\tilde{d}_{20}}\left[-\frac{\tilde{e}_{20}}{\lambda_-} + \frac{f_{01}}{f_{10}}\right] \\ B_3 = \frac{A_3}{|\mu_+|} \\ B_4 = \frac{A_4}{|\mu_+|}, \end{array}\right. \tag{H.10}$$

this is (9.32).

Appendix I. Numerical computation of specific values of $S_\infty(d, \ell)$

Let μ_0, μ_j and ν_j, for $j = 1, 2, 3, 4$ the holomorphic functions introduced in (5.7) and

$$\left|\begin{array}{l} (h_0)_k = \frac{a^k}{\Gamma(\nu+k+3)}(\mu_0)_k \\ h_k = \frac{a^k}{\Gamma(\nu+k+3)}\mu_k \\ \tilde{h}_k = \frac{a^k}{\Gamma(\nu+k+3)}\nu_k, \end{array}\right.$$

introduced in (5.20). Let $(w_k)_{k\ge 0}$ be the sequence defined by

$$\begin{aligned} & w_{k+1} + w_k = \frac{1}{a}(h_0)_{k+1} \\ + \quad & \frac{a}{(k+\nu+3)(k+\nu+2)}\sum_{j=1}^{4}\sum_{k_1+\ldots k_{j+1}=k-1} h_{jk_1}w_{k_2}\ldots w_{k_{j+1}}\frac{\Pi_{i=1}^{j+1}\Gamma(\nu+k_i+3)}{\Gamma(k-1+\nu+3)} \\ + \quad & \frac{a^2}{\Pi_{j=1}^{3}(k+\nu+j)}\sum_{j=1}^{4}\sum_{k_1+\ldots k_{j+2}=k-2} \tilde{h}_{jk_1}w_{k_2}\ldots w_{k_{j+1}}(k_{j+2}w_{k_{j+2}})\frac{\Pi_{i=1}^{j+2}\Gamma(\nu+k_i+3)}{\Gamma(k-2+\nu+3)} \end{aligned}$$

and let

$$S_k \quad = \quad \frac{1}{a}(-1)^k w_k.$$

Then, according to Lemma 5.4, S_k converges to a limit as $k \to \infty$ which is

$$S_\infty(d,\ell) \quad = \quad \frac{1}{a} \lim_{k\to+\infty} (-1)^k w_k.$$

We check numerically that $S_\infty(d,\ell) \neq 0$ in the following cases:

cases $\ell < d$.

(1) For $d = 2$, $\ell = 0.1$, $S_\infty = 0.1236$.
(2) For $d = 3$, $\ell = 0.1$, $S_\infty = 0.0948$.
(3) For $d = 5$, $p = 9$, $S_\infty = -0.0098$.
(4) For $d = 5$, $p = 10$, $S_\infty = -0.0119$.
(5) For $d = 6$, $p = 5$, $S_\infty = 0.0012$.
(6) For $d = 7$, $p = 4$, $S_\infty = -0.0006$.
(7) For $d = 8$, $p = 3$, $S_\infty = 6.1871 \times 10^{-6}$.
(8) For $d = 9$, $p = 3$, $S_\infty = -0.00024$.

cases $\ell > d$.

(1) For $d = 2$, $\ell = 0.1$, $S_\infty = 3.0557 \times 10^{-8}$.
(2) For $d = 3$, $\ell = 0.1$, $S_\infty = 2.8518 \times 10^{-4}$.

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