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Guanxiong Wang, Song Zhao, Pierre Boivin, Eric Serre, Pierre Sagaut. A new hybrid lattice-Boltzmann method for thermal flow simulations in low-Mach number approximation. *Physics of Fluids*, 2022, 34 (4), pp.046114. 10.1063/5.0091517 . hal-03796386

HAL Id: hal-03796386

<https://hal.science/hal-03796386v1>

Submitted on 4 Oct 2022

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1 **A new hybrid Lattice-Boltzmann method for thermal flow simulations in low-Mach**
2 **number approximation**

3 Guanxiong WANG (王冠雄),¹ Song ZHAO (赵崧),¹ Pierre BOIVIN,¹ Eric SERRE,¹ and
4 Pierre SAGAUT¹

5 *Aix-Marseille Univ., CNRS, Centrale Marseille, M2P2, Marseille,*
6 *France*

7 (*Electronic mail: guanxiong.wang.isae@hotmail.com)

8 (Dated: 11 April 2022)

9 A new low-Mach algorithm for the thermal Lattice Boltzmann method (LBM) is proposed
10 aiming at reducing the computational cost of thermal flow simulations in the low Mach
11 number limit. The well known Low Mach Number Approximation (LMNA) is adopted to
12 accelerate the simulations by enlarging the time-step through re-scaling the pseudo acous-
13 tic speed to the same order of the fluid motion velocity. This specific process is inspired by
14 the similarity between the artificial compressibility method and the isothermal LBM, and is
15 further extended to its thermal counterpart. It must be emphasized that such low-Mach ac-
16 celeration strategy is in a general form, thus can be easily applied to other compressible LB
17 methods. The present method overcomes the drawback of classical PGS (Pressure Gradi-
18 ent Scaling) method due to the pressure gradient changing. The new algorithm is validated
19 by various well-documented academic test cases in laminar (1D (one dimensional) grav-
20 ity column, 2D (two dimensional) rising thermal bubble, 2D differentially heated square
21 cavity) and turbulent (3D (three dimensional) Taylor-Green vortex and 3D heated cylin-
22 der) regimes. All the results show excellent agreement with the reference data and high
23 computational efficiency.

24 I. INTRODUCTION

25 Low-speed thermal flows (low-Mach number limit) have been of scientific interest for a long
26 time due to their fundamental importance in the study of buoyancy driven thermal flows and their
27 relevance with many technological and geophysical processes. Cooling systems for automobiles
28 and nuclear reactors, as well as heat exchangers for turbomachinery and electronic devices, are
29 examples of typical engineering applications. Flow simulations based on the Boussinesq approxi-
30 mation (BO) have proliferated in the literature since the seminal work of Gordon Draisey *et al.*¹.
31 However the BO approximation is only valid for small temperature differences² (i.e., $\Delta T/T_0 < 0.1$)
32 and all thermal properties such as the expansion coefficient, viscosity and heat conductivity are
33 held constant. Nevertheless, in many applications, non-Boussinesq conditions with variable ther-
34 modynamic properties prevail due to large density or temperature differences³. In this circum-
35 stance, the fully compressible Navier-Stokes system must to be considered. However the system's
36 inherent fast acoustic waves are expensive to model numerically and moreover irrelevant in most
37 convection flows at low speed. When compared to implicit schemes, explicit time discretization
38 requires a smaller time step to meet the Courant-Friedrich-Lewy (CFL) stability condition, while
39 implicit systems may have convergence issues due to poor conditioning of the convective fluxes
40 of Jacobian matrix. Furthermore, the solution of the discrete equations contains pressure fluctua-
41 tions of the order of the Mach number Ma , whereas the continuous equations only have pressure
42 fluctuations on the order of Ma^2 , resulting in a loss of accuracy⁴ Therefore, low Mach thermal
43 flow simulations remain challenging despite ongoing development due to the issue of finding the
44 appropriate trade-off between algorithm accuracy and numerical efficiency.

45 One common and simple strategy to improve the computational efficiency is the Pressure Gra-
46 dient Scaling (PGS) method⁵ with an acceleration coefficient $\alpha_{\text{PGS}} > 1$. This method artificially
47 reduces the effective speed of sound by a factor α_{PGS} , which automatically increases the time step
48 by nearly the same factor. However, the PGS method has an inherent disadvantage that stems from
49 the modification of one dominant term in the governing equation, which can be a serious issue,
50 particularly when the flow dynamics are controlled by external forces such as buoyancy⁶. This
51 is not the case of another well-suited approach for dealing with heat dominated flows at arbitrary
52 large density differences - the so-called low Mach number approximation (LMNA). This approxi-
53 mation filters the acoustic waves from the Navier-Stokes equations using an asymptotic expansion
54 in powers of the Mach number. In practice, at least two scaling methods are possible depending on

55 the temperature variation⁷⁻⁹, while the truncating order $O(Ma^2)$ is widely considered and adopted
 56 in this investigation^{4,10}. The pressure $p(x,t)$ is then splitted into a thermodynamic pressure $p(t)$,
 57 which varies with time but is spatially homogeneous, and a hydrodynamic pressure $p^h(x,t)$, i.e.,
 58 $p(x,t) = p(t) + O(Ma^2)p^h(x,t)$. The total pressure is substituted by the thermodynamic pressure
 59 $p(t)$, except in the momentum equation, which decouples the pressure and density fluctuations
 60 and filters the sound waves from the system. Particularly, only thermodynamic pressure is taken
 61 into account in the law of state. The velocity field is no longer divergence free, and the spatial
 62 and temporal variations of density introduce additional nonlinearities into the equations. Nonethe-
 63 less, similar numerical approaches can still be employed because these equations have the same
 64 mixed hyperbolic-parabolic character as the equations for incompressible flow. The low Mach
 65 equations have already been applied successfully to research on natural convection, combustion,
 66 astrophysical flows and others (see some relevant examples in Bouloumou *et al.*¹¹).

67 In this study, we try to include the LMNA strategy into the Lattice Boltzmann method (LBM)
 68 and use this approach to simulate low-Mach number thermal flows. The decision to choose LBM
 69 is straightforward because of its attractive inherent properties such as simplicity to implement
 70 and parallelize, which provides a high computational efficiency for unsteady complex flows^{12,13}.
 71 Originally designed for weakly compressible isothermal approaches, many attempts have been
 72 conducted to extend LBM capabilities to deal with more complex configurations such as high sub-
 73 sonic and supersonic flow dynamics¹⁴⁻¹⁹, compressible thermal problems²⁰⁻²², turbulent flows²³,
 74 reactive flows^{24,25} and others. For small temperature differences, thermal effects can be included
 75 using the BO approximation (see for example in litteratures²⁶⁻²⁸) which corresponds to a decou-
 76 pling model and resolves an incompressible system with a linearized buoyancy force. In other
 77 configurations (corresponding to the fully compressible Navier-Stokes system), thermal effects
 78 are included either using another distribution function (Double Distribution Function - DDF ap-
 79 proach) or either by solving an energy equation (hybrid approach) with a perfect gas law^{21,29}.
 80 Most of these approaches investigate laminar flows and have been rarely extended to turbulent
 81 complex flows in low-Mach number limit.

82 Being intrinsically based on explicit time marching schemes, LBM's time-stepping is con-
 83 strained by the Courant-Friedrich-Lewy (CFL) number (i.e., $CFL_{ac} \equiv \max \left[\frac{(|\mathbf{u}|+c)\Delta t}{\Delta x} \right] \leq CFL_{max}$).
 84 Thus, when using a compressible model to simulate a low-Mach number flow, the time-step is
 85 strongly constrained by the fast acoustic waves scale, which strongly penalizes the numerical effi-
 86 ciency of the solver³⁰. To overcome this issue, several LB algorithms have been developed in the

87 LMNA, by modifying classical isothermal LB algorithm, i.e., zeroth-order moment of the hydro-
88 dynamic pressure³¹⁻³⁶. In such algorithms, additional terms need to be added to correctly recover
89 Navier-Stokes equations in LMNA, thus making the resolved system complicated to understand
90 and difficult to extend to other applications.

91 This paper proposes a new algorithm for the hybrid LBM in the LMNA for simulating low-
92 Mach number thermal flows. This low-Mach number algorithm only involves minor changes in
93 the compressible LBM that makes it easy to implement and nearly independent of the thermal
94 LBM used. The present method is evaluated on well-documented test cases of the literature for
95 both simple laminar thermal flows and complex turbulent heat transfer problems. The 1D gravity
96 column and the 2D rising bubble are considered to show the superiority of the new algorithm on the
97 Pressure Gradient scaling commonly used to improve the computational efficiency of LBM. Then,
98 a classical 2D non-Boussinesq natural convection flow is considered in a differentially heated
99 square cavity at Rayleigh number 10^7 . The 3D Taylor-Green vortex flow at Reynolds number
100 $Re = 1.2 \times 10^4$ is chosen to assess the ability of the present method to evaluate viscous tensor. The
101 final test case is the flow around a 3D heated cylinder at $Re = 3900$, to evaluate the performance
102 of the present method to predict complex thermal turbulent flow.

103 The rest of the paper is organized as follows: §II introduces the general low-Mach number
104 acceleration strategy adopted in this study and details the incorporation of low-Mach approach
105 within the hybrid compressible LBM framework. Then, in §III the code is benchmarked with re-
106 spect to numerical data from the literature for five test cases. At last, a conclusion with a summary
107 of the speedup is drawn in §IV.

108 **II. THE HYBRID LBM SOLVER IN THE LOW-MACH NUMBER APPROXIMATION**

109 As mentioned in the introduction, the low-Mach number approximation (LMNA) of the Navier-
110 Stokes equations is a very popular and efficient way to simulate low-Mach number flows at a
111 macroscopic level (Appendix A). However, it involves a Poisson equation to resolve the hydrody-
112 namic pressure p^h which makes it not directly applicable in the LBM framework while keeping
113 the locality feature.

114 We consider here the hybrid LBM for solving compressible flows and its extension to the low-
115 Mach number approximation proposed in this paper.

116 A. The hybrid LBM solver for compressible flows

117 A complete description of the hybrid LBM is provided in literature.^{18,37,38} In this method,
 118 the mass and momentum equations are solved using LBM, while the entropy equation is resolved
 119 using Finite Difference method (FDM). Three major steps detailed below are needed to obtain the
 120 variables at time step $t + \Delta t$ from the variables at t : a *local* collision step, a *linear* streaming step
 121 and the updating of macroscopic variables.¹

122 Thus, the three steps of the algorithm write as:

123 • **Step I: Collision.** The collision distribution function is calculated as

$$124 f_i^{\text{col}}(\mathbf{x}, t) \equiv f_i^{\text{eq}}(\mathbf{x}, t) + \left(1 - \frac{\Delta t}{\tau}\right) f_i^{\text{neq}}(\mathbf{x}, t) + \frac{\Delta t}{2} F_i(\mathbf{x}, t), \quad (1)$$

125 where

126 **I.1** The equilibrium distribution f_i^{eq} is constructed in a regularized manner on the basis of
 127 the lattice, i.e.

$$128 f_i^{\text{eq}} = \omega_i \left\{ a^{(0),\text{eq}} + \frac{\mathcal{H}_{i,\alpha}^{(1)}}{c_s^2} a_{\alpha}^{(1),\text{eq}} + \frac{\mathcal{H}_{i,\alpha\beta}^{(2)}}{2c_s^4} a_{\alpha\beta}^{(2),\text{eq}} + \frac{\mathcal{H}_{i,\alpha\beta\gamma}^{(3)}}{6c_s^6} a_{\alpha\beta\gamma}^{(3),\text{eq}} + \frac{\mathcal{H}_{i,\alpha\beta\gamma\delta}^{(4)}}{24c_s^8} a_{\alpha\beta\gamma\delta}^{(4),\text{eq}} \right\}, \quad (2)$$

129 where ω_i is the weight for direction i , c_s the lattice sound speed ($\Delta x / (\sqrt{3}\Delta t)$ in D3Q19
 130 lattice), \mathcal{H} the Hermite polynomial basis with $\alpha, \beta, \gamma, \delta$ being the coordinate indices.
 131 They are all constants only depending on the lattice applied. a^{eq} are the coefficients of
 132 the equilibrium distribution projected on the basis \mathcal{H} . They can be different from one
 133 LBM kernel to another, but in any case, they only depend on local density ρ , velocity
 134 \mathbf{u} , and a normalized temperature $\theta \equiv \frac{p}{\rho c_s^2}$ where p represent the pressure. Detailed
 135 expressions of the basis as well as the coefficients can be found in Farag *et al.*^{18,37}.

136 **I.2** The off-equilibrium distribution f_i^{neq} is evaluated on second and third order basis as

$$137 f_i^{\text{neq}} = \omega_i \left[\frac{\mathcal{H}_{i,\alpha\beta}^{(2)}}{2c_s^4} a_{\alpha\beta}^{(2),\text{neq}} + \frac{\mathcal{H}_{i,\alpha\beta\gamma}^{(3)}}{6c_s^6} a_{\alpha\beta\gamma}^{(3),\text{neq}} \right]. \quad (3)$$

138 where the coefficients of the non-equilibrium population on the basis $a_{\alpha\beta}^{(2),\text{neq}}$ and
 139 $a_{\alpha\beta\gamma}^{(3),\text{neq}}$ is evaluated through a hybrid recursive procedure introduced by Jacob *et al.*²³.

¹ The equations in the algorithm are normalized by LBM units, i.e. $x_{\infty} \equiv \Delta x$, $t_{\infty} \equiv \Delta t$.

140 **I.3** The forcing term F_i is added to recover correct effects of viscous and external force (e.g.
 141 gravity),

$$142 \quad F_i \equiv F_i^E + F_i^g, \quad (4)$$

143 in which the force F_i^E correct the higher order defects of the lattice to maintain the
 144 required stress tensor as in Eq. (A3c). Its exact expression can be found in^{18,37,38}. The
 145 gravity is considered through

$$146 \quad F_i^g = \omega_i \left[\frac{\mathcal{H}_{i,\alpha}^{(1)}}{c_s^2} \rho g_\alpha + \frac{\mathcal{H}_{i,\alpha\beta}^{(2)}}{2c_s^4} (\rho u_\alpha g_\beta + \rho u_\beta g_\alpha) \right]. \quad (5)$$

147 • **Step II: Streaming.** The collision distribution is streamed to its corresponding neighbor,
 148 i.e.

$$149 \quad f_i(\mathbf{x}, t + \Delta t) = f_i^{\text{col}}(\mathbf{x} - \mathbf{c}_i \Delta t, t) \quad (6)$$

150 where \mathbf{c}_i is the vector pointing from the current node to its i^{th} neighbor.

151 • **Step III: Macroscopic variable update.**

152 **III.1** The entropy equation is discretized using a explicit Euler time stepping, together with
 153 the flux and viscous terms calculated with FDM.

III.2 The density and momentum fields are updated using the distributions

$$\rho(\mathbf{x}, t + \Delta t) = \rho(\mathbf{x}, t) + \sum_i \left[f_i(\mathbf{x}, t + \Delta t) - f_i^{\text{col}}(\mathbf{x}, t) \right] \quad (7a)$$

$$\rho u_\alpha(\mathbf{x}, t + \Delta t) = \sum_i c_{i\alpha} f_i(\mathbf{x}, t + \Delta t) + \frac{\Delta t}{2} \rho(\mathbf{x}, t + \Delta t) g_\alpha \quad (7b)$$

154 We propose now to adapt this hybrid LBM solver in the low-Mach number approximation.

155 **B. The hybrid LBM solver in the low-Mach number approximation**

156 The LMNA is adapted here to the compressible LBM by analyzing the classical iso-thermal
 157 LBM. In fact, in athermal LBM, the ‘‘pressure’’ is nothing but a scaled density such as

$$158 \quad p^{\text{LBM}} \equiv \rho c_s^2 \quad (8)$$

It is updated at each time step by

$$\begin{aligned}
p^{\text{LBM}}(\mathbf{x}, t + \Delta t) &\equiv c_s^2 \rho(\mathbf{x}, t + \Delta t) \\
&= c_s^2 \left\{ \rho(\mathbf{x}, t) + \sum_{i=1}^Q \left[f_i^{\text{col}}(\mathbf{x} - \mathbf{c}_i \Delta t, t) - f_i^{\text{col}}(\mathbf{x}, t) \right] \right\} \\
&\approx p^{\text{LBM}}(\mathbf{x}, t) - \Delta t c_s^2 \frac{\partial \rho u_\alpha}{\partial x_\alpha}(\mathbf{x}, t) \\
&= p^{\text{LBM}}(\mathbf{x}, t) - \Delta t \frac{\partial p^{\text{LBM}} u_\alpha}{\partial x_\alpha}(\mathbf{x}, t) \\
&= p^{\text{LBM}}(\mathbf{x}, t) - \Delta t \left\{ u_\alpha \frac{\partial p^{\text{LBM}}}{\partial x_\alpha}(\mathbf{x}, t) - \rho c_s^2 \frac{\partial u_\alpha}{\partial x_\alpha} \right\} \tag{9}
\end{aligned}$$

159 As pointed out by He *et al.*³⁹, under low-Mach assumption, this equation mimics the acoustic
160 pressure transport except that the physical speed of sound c in the last term is substituted by the
161 lattice sound speed c_s . This makes the isothermal LBM an artificial compressibility method⁴⁰
162 where the transport of acoustic pressure fields are modified to adapt the permitted time step of the
163 numerical schemes³⁹.

We now extend the same idea to thermal LBM. Unfortunately, the temperature/density gra-
dients in low-Mach thermal flow fields can be relatively large, such that the flow dilatation
 $\partial(\rho u_\alpha)/\partial x_\alpha$ might break the scale balance in equation (9). At this stage, we just remove its
“thermal” part ($-\partial\rho/\partial t$), and only apply its “hydrodynamic” part for the hydrodynamic pressure
transport, i.e.

$$\begin{aligned}
p^h(\mathbf{x}, t + \Delta t) &= p^h(\mathbf{x}, t) + c_s^2 \left\{ \sum_{i=1}^Q \left[f_i^{\text{col}}(\mathbf{x} - \mathbf{c}_i \Delta t, t) - f_i^{\text{col}}(\mathbf{x}, t) \right] - [\rho(\mathbf{x}, t + \Delta t) - \rho(\mathbf{x}, t)] \right\} \\
&\approx p^h(\mathbf{x}, t) - \Delta t c_s^2 \frac{\partial(\rho u_\alpha)^h}{\partial x_\alpha} \\
&\approx p^h(\mathbf{x}, t) - \Delta t \left[u_\alpha \frac{\partial \rho^h c_s^2}{\partial x_\alpha} + \rho^h c_s^2 \frac{\partial u_\alpha}{\partial x_\alpha} + u_\alpha^h \frac{\partial \rho c_s^2}{\partial x_\alpha} + \rho c_s^2 \frac{\partial u_\alpha^h}{\partial x_\alpha} \right] \tag{10}
\end{aligned}$$

164 Compared to the theoretical transport equation of the hydrodynamic pressure, one can find
165 again that the speed of sound is substituted by c_s . Similar to the athermal LBM, the CFL constrain
166 of such system is now related to c_s instead of c . In practice, we still want to conserve the low-Mach
167 property, i.e.,

$$c_s \equiv \frac{\Delta x}{\sqrt{3}\Delta t} \geq 3|\mathbf{u}|_{\max} . \tag{11}$$

169 Now the time step is constrained by the flow speed instead of the speed of sound. In low-Mach
170 number applications, this will give us a much larger Δt that will accelerate the calculation. Similar

171 to every other LMNA method, this approach only modifies small scale quantities under low-Mach
 172 conditions, so that it will not suffer from the problem of PGS method when external forces are
 173 under consideration.

174 Such strategy can be easily applied to any compressible LBM. The following algorithm shows
 175 how all variables are advanced from time step t to $t + \Delta t$

176 • **Step I: Collision.** It remains the same as for the compressible LBM, except that the reduced
 177 “temperature” for the low-Mach version is now defined as

$$178 \quad \theta \equiv \frac{p^h}{\rho c_s^2}. \quad (12)$$

179 • **Step II: Streaming.** It remains the same as the compressible LBM.

180 • **Step III: Macroscopic variable updating.**

181 **III.1** The entropy equation (A3d) is resolved in the same manner as in the compressible
 182 LBM.

183 **III.2** The thermodynamic pressure $p(t + \Delta t)$. In open systems, it can be set directly to the
 184 ambient pressure. For closed configurations, it can be evaluated through global mass
 185 conservation, see appendix B.

186 The density field is evaluated through the equation of state (EOS) (A3e), i.e.

$$187 \quad \rho(\mathbf{x}, t + \Delta t) = \left[\frac{p(t + \Delta t)}{e^{s(\mathbf{x}, t + \Delta t)/c_v}} \right]^{1/\gamma} \quad (13)$$

188 The momentum fields are updated the same as in compressible solver.

189 The hydrodynamic pressure field is updated using equation (9), i.e.

$$190 \quad p^h(\mathbf{x}, t + \Delta t) = p^h(\mathbf{x}, t) + c_s^2 \left\{ \sum_i \left[f_i(\mathbf{x}, t + \Delta t) - f_i^{\text{col}}(\mathbf{x}, t) \right] - [\rho(\mathbf{x}, t + \Delta t) - \rho(\mathbf{x}, t)] \right\} \quad (14)$$

191 This low-Mach algorithm involves three changes with respect on the compressible LBM of
 192 §II A:

193 • i) The hydrodynamic pressure is used instead of the pressure field equilibrium population
 194 evaluation.

- 195 • ii) The density is solved through EOS instead of using LBM populations.
- 196 • iii) The hydrodynamic pressure is transported in step III.

197 This means that it is almost independent of the original thermal LBM scheme. In practice, it
 198 will be successfully applied in this paper to the pressure-based, the modified density-based and the
 199 unified LBM framework^{18,37}. This is a major benefit of the current implementation compared to
 200 other low-Mach LBM approaches based on iso-thermal LBM^{34,35}. It also permits us to keep some
 201 ingredients of the original LBM without any modification. For instance, the turbulence and wall
 202 models^{22,41} keep working out in this low-Mach version.

203 III. VALIDATION TESTS

204 The proposed low-Mach thermal Lattice Boltzmann solver is now assessed by a variety of
 205 test cases related to low-Mach thermal flows. In order to show the efficiency of the present
 206 solver (LMNA), numerical results are compared to simulations obtained using a fully compressible
 207 solver, i.e., the classical Hybrid Recursive Regularization pressure-based HRR- p solver^{18,37}, and
 208 using the PGS method⁵ (The PGS method is realized by multiplying pressure by a factor $1/\alpha_{PGS}^2$
 209). The following validation tests are investigated:

- 210 • One-dimensional gravity column and 2D rising thermal bubble for which the fluid dynamics
 211 is dominated by pressure gradient and external force, i.e., buoyancy.
- 212 • 2D natural convection in a closed square cavity at Rayleigh number $Ra = 10^7$.
- 213 • 3D Taylor Green Vortex flow at $Re = 1.2 \times 10^4$.
- 214 • 3D turbulent flow around a heated cylinder at $Re = 3900$.

215 A. One-dimensional gravity column

216 The one-dimensional gravity column is firstly considered to evaluate the ability of LMNA on
 217 capturing the balance between hydrodynamic pressure gradient with external force. The configura-
 218 tion consists of a 1D vertical column filled by air with pressure uniformly distributed. The gravity
 219 drives the fluid motion in the beginning till a converged static state is attained with a constant
 220 pressure gradient $\nabla p = \rho_0 g$.

221 In order to emphasize the computational efficiency of LMNA, simulations are also performed
 222 using PGS method and the fully compressible (HRR- p) solver. For the latter, the time step is fixed
 223 at $\Delta t = 1.65 \times 10^{-4} s$, whereas LMNA and PGS allow to consider a time step ten times larger equal
 224 to $\Delta t = 1.65 \times 10^{-3}$ that speeds up the computations of the same factor.

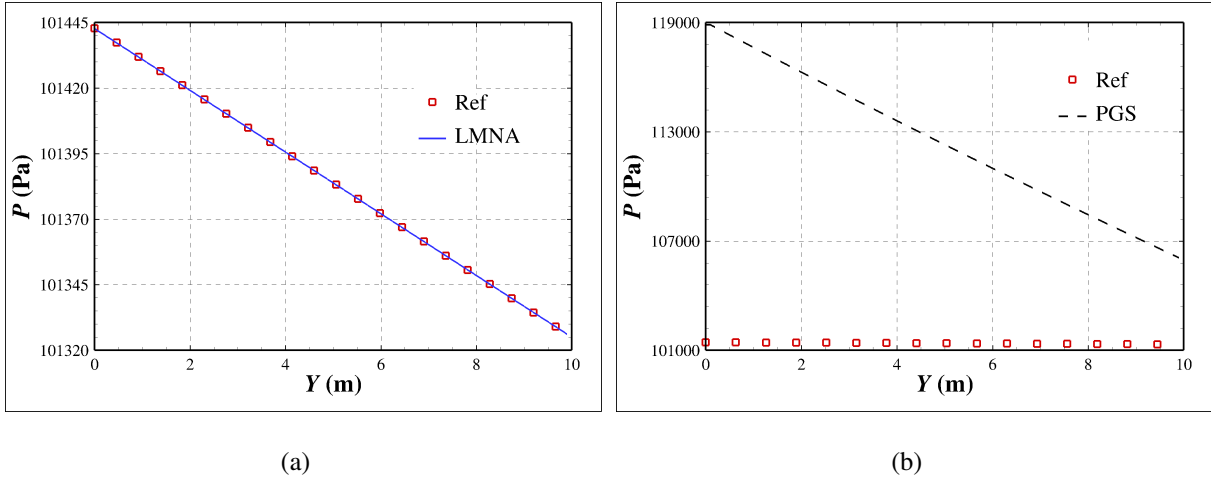


FIG. 1. Comparison of the pressure profiles obtained from LMNA (a) and PGS (b) simulations and with the fully compressible (HRR- p) solver which is considered as the reference.

225 Fig. 1 shows the pressure profiles obtained from LMNA, PGS and the fully compressible
 226 (HRR- p) solver simulations, the latter being considered as the reference solution. The LMNA
 227 result matches perfectly the reference profile, while the PGS result amplifies by a factor of about
 228 100 compared to the reference profile due to the re-scaled parameter $\alpha_{PGS} = 10$.

229 B. 2D rising thermal bubble

230 The rising thermal bubble configuration is dominated by gravity and pressure with thermal
 231 effect. The numerical configuration consists of a hot thermal bubble with a core temperature
 232 equal to 1200K that emerges in an atmosphere temperature of 300K. The width and height of the
 233 computational domain is respectively 20m and 15m. The bottom surface is configured to be a
 234 non-slip adiabatic wall, while the side and top boundaries are set to outlets with given pressure.

235 The maximum time step adopted by the fully compressible solver is equal to $\Delta t = 5.77 \times 10^{-5} s$,
 236 whereas LMNA and PGS allow to consider a time step around 25 times larger equal to $\Delta t = 1.44 \times$
 237 10^{-3} that speeds up the computations of the same factor. Buoyancy with a gravity $g = 9.81 m s^{-2}$
 238 drives the fluid motion, the bubble rises and deforms during the rising process.

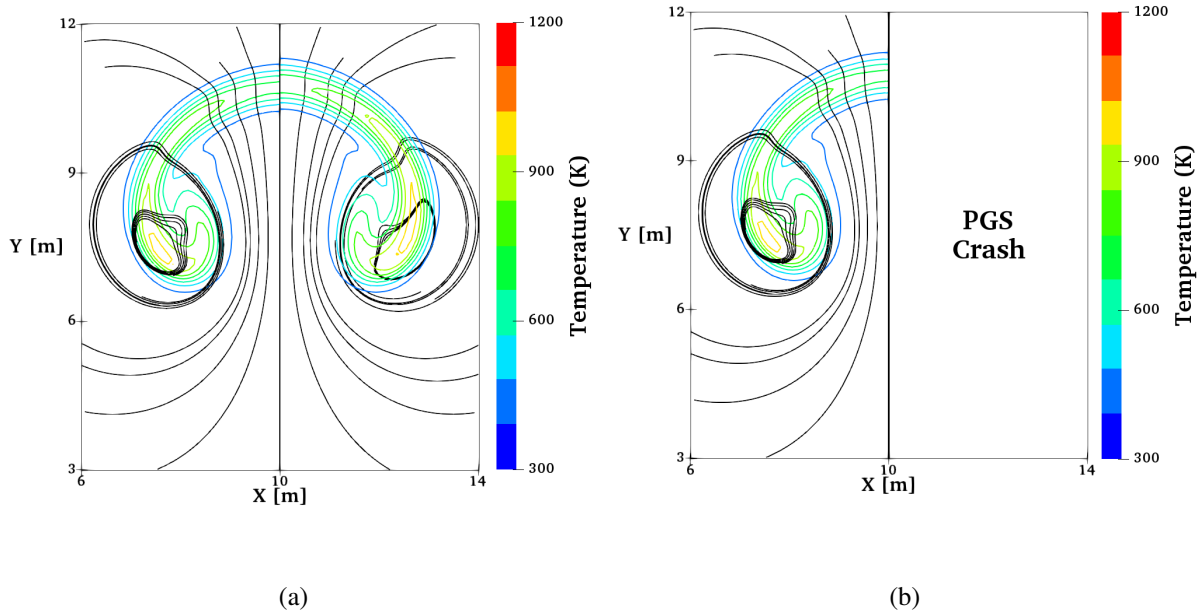


FIG. 2. Comparisons of 2D temperature contours together with streamlines showing the rising bubble. For each plot, the left hand side corresponds to the fully compressible (HRR- p) solver simulation considered as the reference. (a) on the right hand side, LMNA simulation. (b) on the right hand side, PGS simulation which is crashed.

239 The temperature contours and streamlines are shown on Fig. 2. The LMNA simulation pro-
 240 vides a result very similar to the reference solution with only negligible differences. However,
 241 the PGS simulation crashes using the same time step as LMNA. These results are consistent with
 242 the drawback of PGS as pressure plays a dominant role balancing the external force and viscous
 243 effects during the thermal bubble rising process. Consequently, the change of pressure gradient
 244 in PGS method leads to either incorrect results, i.e., one-dimensional gravity column in §III A or
 245 simulation crash as in this test case. By contrast, LMNA is able to provide the same results as with
 246 the fully compressible (HRR- p) solver, but with a time step between 10 and 25 times larger that
 247 allows much more efficient computations in these two low-Mach number configurations.

248 C. 2D natural convection inside a square cavity

249 This validation test has been intensively investigated in the literature both experimentally and
 250 numerically^{42–47}. Many benchmarks exist and solutions are available in the literature^{46,47}. The
 251 canonical configuration consists of a square cavity with two vertical differentially heated walls,

252 and two horizontal adiabatic ones ($\frac{\partial T}{\partial n} = 0$) as shown in Fig. 3. The fluid motion is driven by the
 253 buoyancy force ($g = 9.81\text{m s}^{-2}$), i.e., the fluid floats nearby the heat side wall whereas it sinks
 254 close to cold side wall thus generating a vortex inside the cavity. In this case, a large temperature

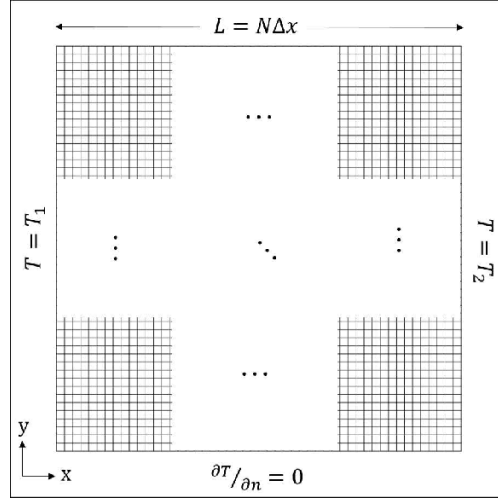


FIG. 3. Sketch of the 2D differentially heated square cavity together with boundary conditions and a Cartesian mesh.

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256

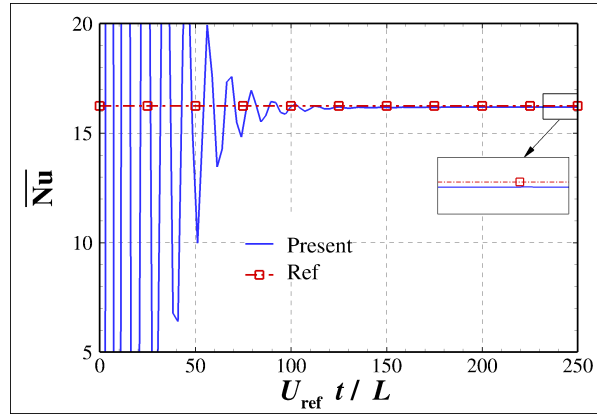


FIG. 4. Convergence in time of the average Nusselt number at $Ra = 10^7$. Present LMNA result ($Nu = 16.19$) and benchmark solution ($Nu = 16.24$)⁴⁷ as reference.

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258

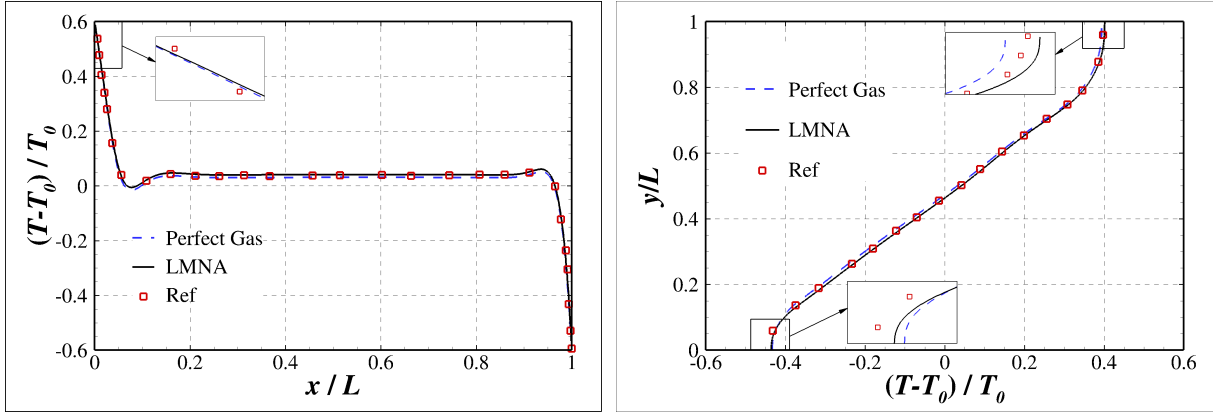
259 difference is considered to be beyond the Boussinesq validity limit, with the two vertical walls
 260 heated respectively at $T_1 = 240\text{K}$ and $T_2 = 960\text{K}$. This corresponds to a normalized temperature
 261 difference equal to $\varepsilon = \frac{T_1 - T_2}{T_1 + T_2} = 0.6$, which is indeed far beyond the Boussinesq validity limit $\varepsilon <$
 262 0.1 ^{44,45}. The fluid considered is air, and the dynamic viscosity is computed using the Sutherland's
 263 law $\mu(T) = \mu_0 \left(\frac{T}{T^*}\right)^{3/2} \frac{T^* + S}{T + S}$ with $T^* = 273\text{K}$, $S = 110.5\text{K}$, $\mu_0 = 1.68 \times 10^{-5} \text{kg m}^{-1}\text{s}^{-2}$. Then the

264 thermal conductivity is calculated by $\lambda(T) = \frac{\mu(T)C_p}{Pr}$, where C_p being the specific heat capacity of
 265 air, and Pr being the Prandtl number equal to 0.71.

266 The thermal physics is characterized by a dimensionless number:

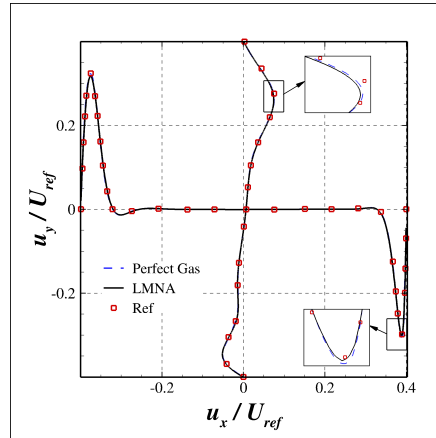
$$267 \quad Ra = \frac{g\rho_0^2\Delta TL^3Pr}{T_0\mu_0^2} \quad (15)$$

268 where ρ_0, μ_0 is the density and dynamic viscosity at mean temperature $T_0 = \frac{T_1+T_2}{2} = 600\text{K}$. The
 269 Rayleigh number $Ra = 10^7$ is chosen as it's related to the strongest heat convection effects over
 270 the range $10^3 - 10^7$ which is widely investigated^{46,47}.



(a) Temperature profiles at $x/L = 0.5$

(b) Temperature profiles at $y/L = 0.5$



(c) Velocity profiles

FIG. 5. Profiles of temperature ((a)-(b)) and velocity (c) at Rayleigh number $Ra = 10^7$. Comparison be-
 271 tween the present LMNA result (solid line), the fully compressible (HRR- p) result (dashed lines), and the
 272 benchmark solution⁴⁷ (markers).

273 The grid resolution is $N = \frac{L}{\Delta x} = 400$ in both directions, as proposed by Wang *et al.*²². The time
 274 steps adopted in the present LMNA and the fully compressible solver (HRR- p) are respectively

275 equal to $\Delta t = 2.32 \times 10^{-5}$ s and $\Delta t = 4.17 \times 10^{-7}$ s. There is thus a factor of about 55.6 in the time
 276 steps of the two simulations showing again the superior efficiency of the LMNA in this type of
 277 low-Mach number flow configuration.

278 The heat flux at the heated wall is estimated by the Nusselt number and by its average over the
 279 heated wall, defined as follows:

$$280 \quad Nu(y) = \frac{L}{\lambda_0 \Delta T} \lambda \frac{\partial T}{\partial x} \Big|_{wall} \quad \overline{Nu} = \frac{1}{L} \int_{y=0}^{y=L} Nu(y) dy \quad (16)$$

281 Time evolution of the average Nusselt number is plotted in Fig. 4, over a total dimensionless
 282 time $U_{ref} t / L = 250$, long enough to ensure the convergence as proposed by Wang *et al.*²², where
 283 $U_{ref} = \frac{Ra^{0.5} \mu(T_0)}{\rho_0 L} \approx 1.1 \text{ ms}^{-1}$ is the reference velocity at $Ra = 10^7$. As shown in Fig. 4, the average
 284 Nusselt number provided by the LMNA simulation ($Nu = 16.19$) converges to the value of the
 285 benchmark solution ($Nu = 16.24$)⁴⁷ that corresponds to a difference less than 0.3%.

286 The flow features including the temperature distribution and velocity profiles are displayed in
 287 Fig. 5. The boundary layer close to the heated wall is well captured by LMNA as shown in Fig.
 288 5(c). The high heat flux due to high temperature variations in the thermal boundary layer (see Fig.
 289 5(b)) is also correctly predicted compared to the benchmark solution⁴⁷ and the one obtained with
 290 the fully compressible (HRR- p) solver²².

291 D. Taylor Green Vortex

292 The investigation of Taylor Green Vortex (TGV) is aimed at assessing the ability of the present
 293 LMNA method of predicting the viscous tensor. The results and the computational efficiency
 294 of the classical pressure-based method reported by Tayyab *et al.*²⁵ are included as a reference.
 295 Pure fluid component (air) is considered and the Reynolds number investigated in this study is
 296 $Re = 1.2 \times 10^4$.

The flow field and hydrodynamic pressure field are initialized using an analytic N-S and Poisson
 solution respectively given by:

$$u_x(x, y, z) = u_0 \sin\left(\frac{2\pi x}{L}\right) \cos\left(\frac{2\pi y}{L}\right) \cos\left(\frac{2\pi z}{L}\right) \quad (17a)$$

$$u_y(x, y, z) = u_0 \cos\left(\frac{2\pi x}{L}\right) \sin\left(\frac{2\pi y}{L}\right) \cos\left(\frac{2\pi z}{L}\right) \quad (17b)$$

$$u_z(x, y, z) = 0 \quad (17c)$$

$$p^h(x, y, z) = \frac{1}{16} \rho_0 u_0^2 \left[\cos\left(\frac{4\pi x}{L}\right) + \sin\left(\frac{4\pi y}{L}\right) \right] \left[2 + \cos\left(\frac{4\pi z}{L}\right) \right] \quad (17d)$$

297 The initial velocity is $u_0 = 1 \text{ m s}^{-1}$, and the viscosity is set to $\mu = 0.625 \times 10^{-3} \text{ kg m}^{-1} \text{ s}^{-1}$. The
 298 thermodynamic pressure is the atmospheric pressure, i.e., $P_0 = 101325.0 \text{ Pa}$, and the temperature
 299 is at $T = 300 \text{ K}$. Thus, density is taken as $\rho = P_0/(rT) = 1.172 \text{ kg m}^{-3}$ where $r = 287.15 \text{ m}^2 \text{ s}^{-2}$
 300 K^{-1} is the specific gas constant. The fluid domain is periodic with for each side $L = 2\pi \text{ m}$. The
 301 time step allowed by LMNA is $\Delta t = 1.89 \times 10^{-3} \text{ s}$, compared to $\Delta t = 2.69 \times 10^{-5} \text{ s}$ used by the
 302 classical pressure-based method reported by Tayyab *et al.*²⁵. This leads to a speed up of 70 in the
 303 computation.

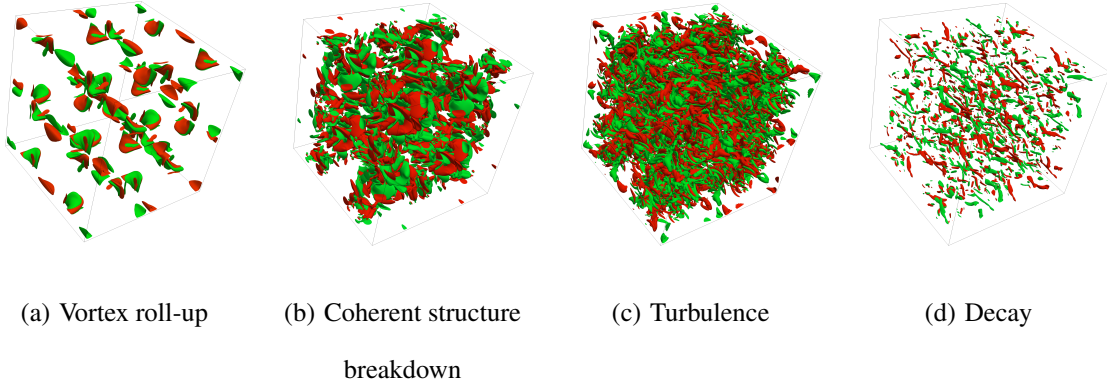


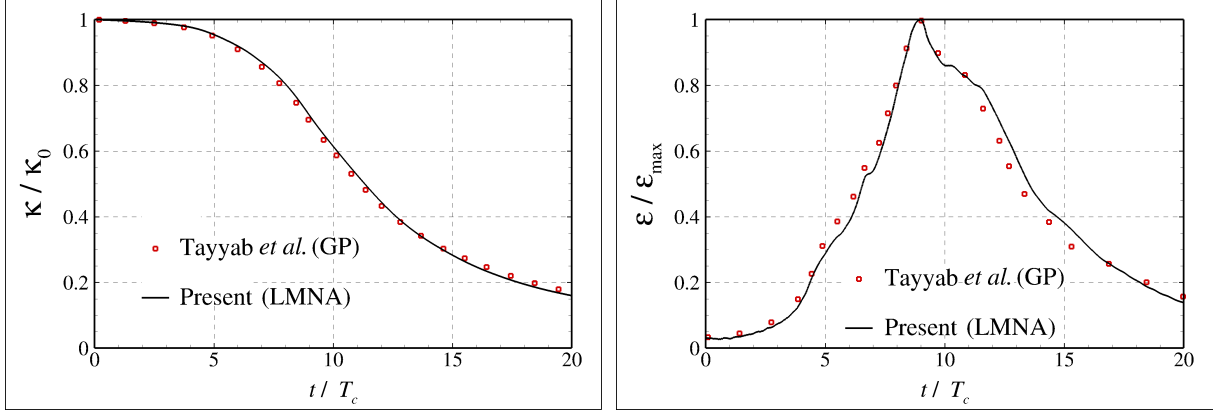
FIG. 6. Time slots of the iso-surfaces of the z-component of the vorticity at 5.0s (a), 8.0s (b), 12.1s (c),
 304 19.0s (d).
 305

306 The time evolution of the z-component of the vorticity is illustrated in Fig. 6. As can be
 307 seen, the vortex roll-up happens in the early stage followed by stretch and breakdown of these
 308 vortex while preserving laminar property. Laminar turbulence transition is then observed, and the
 309 coherent structures are developed to fully turbulence structures. At the end of the flow evolution,
 310 the turbulence is dissipated. These results compare very favourably to the results of Tayyab *et al.*²⁵
 311 which are performed using a classical pressure-based method.

312 In order to quantitatively analyze the viscous effect, the overall kinetic energy and its dissipation
 313 rate are evaluated in the next. The dissipation rate is of great importance because it's associated
 314 with the stress tensor. The overall kinetic energy is defined as the integral of the of kinetic energy
 315 over the whole domain, and the dissipation rate of kinetic energy is the temporal derivative of the
 316 overall kinetic energy, i.e.,

$$317 \quad \kappa(t) = \frac{1}{2} \int_V (u_x^2 + u_y^2 + u_z^2) dV \quad \varepsilon(t) = \frac{\partial \kappa(t)}{\partial t} \quad (18)$$

318 where the u_i are the velocity components with $i \in (x, y, z)$ and \int_V is the integration over the whole
 319 domain. Fig. 7 shows the time evolution of these two normalized quantities. Excellent agreements

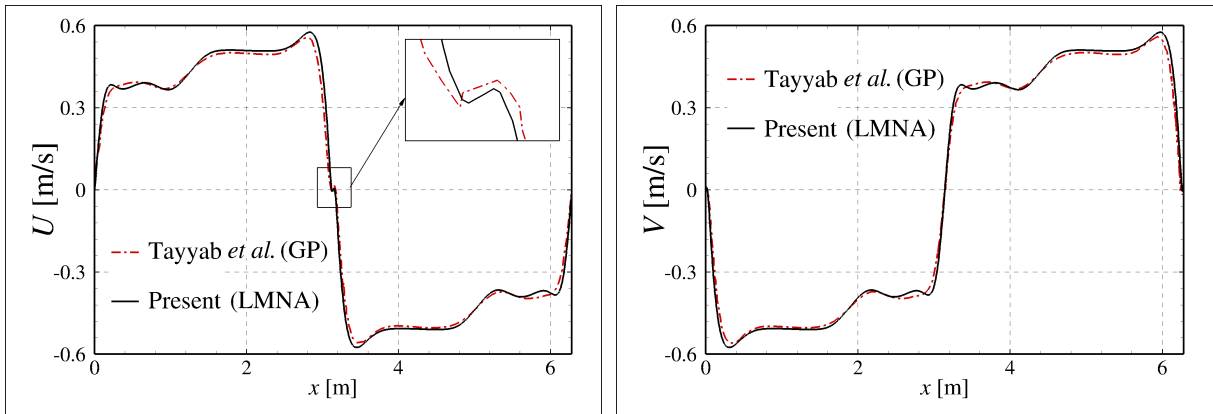


(a) Kinetic energy

(b) Kinetic dissipation rate

FIG. 7. Kinetic energy (a) and kinetic dissipation rate (b) evolution. $T_c = 1$ s, the kinetic energy and its dissipation rate are normalized respectively by the initial value and maximum value of corresponding quantity.

320 are obtained compared to the results of reference of Tayyab *et al.*²⁵. Furthermore, the x and y -
 321 components of the velocity profile at the central line are plotted at time $12.11T_c$ in Fig. 8. The
 322 LMNA results match very well the solution of reference²⁵. In addition, even the small turbulent
 323 structures are well captured by the LMNA, as shown on the x -component of the velocity in Fig.
 324 8(a). All these results show that the present LMNA method is able to simulate turbulent flow and
 325 laminar turbulence transition processes while keeping high computational efficiency.



(a) Velocity x component

(b) Velocity y component

FIG. 8. Profiles of x (a) and y (b) components of velocity at time $t = 12.11T_c$.

326

327

328 **E. Flow around a circular heated cylinder at Reynolds number 3900**

329 This section is dedicated to study the flow passing through a cylinder at Reynolds number
330 $Re = 3900$ with isothermal solid wall in order to assess the capability of the present method in
331 the LES (Large Eddy Simulation) framework⁴⁸. A mesh convergence study is conducted firstly on
332 an isothermal case where the temperature of solid wall is the same as the inflow. The results are
333 benchmarked with experiments⁴⁹ and DNS (Direct Numerical Simulation)⁵⁰. In addition, the LES
334 results reported by^{51,52} are also included to give a reference. Then, the thermal case is investigated
335 with temperature difference between the solid wall and the inflow of $\Delta T = 300K$. The thermal
336 effects on the flow features and turbulent heat flux are studied comparing with the LES results of
337 Sircar *et al.*⁵¹ and Jogee *et al.*⁵². In this study, we focus on the near wake region, for which a
338 variety of experimental and numerical results are available in the literature for validation.

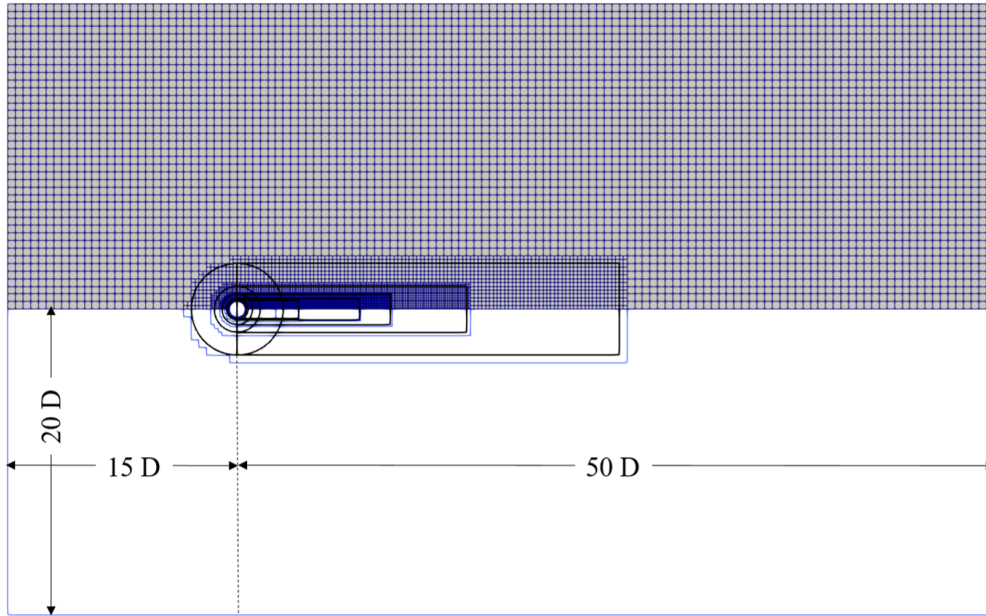


FIG. 9. Sketch of the grid refinement in the $x - y$ plane with seven levels of grid refinement. Black and blue lines show respectively the refinement boxes and the edges of each grid level.

339 **1. Numerical setup**

340 The sketch of the computational domain is shown in Fig. 9 together with the mesh. The
341 cylinder has a diameter of $D = 0.01m$, and its center is chosen as the origin. The x, y, z directions

342 are respectively the streamwise, transverse and spanwise direction regarding the inflow. The mesh
 343 in the transverse and spanwise directions as well as boundary conditions are symmetric with a
 344 spanwise extension of $L_z = 3D$ which is closed to the conventional $L_z = \pi D$ adopted by most
 345 LES studies^{49,51–54}. The transverse width is $L_y = 20D$ for each side, and the distances to the inlet
 346 and outlet are respectively equal to 15 and 50 diameters, which is sufficiently large to reduce the
 347 effects of spurious pressure wave reflecting from the boundary. The inlet velocity and temperature
 348 boundary conditions are constant equal to $U_\infty = 5.968 \text{ m s}^{-1}$ and $T_\infty = 300 \text{ K}$, respectively. Zero
 349 pressure gradient boundary is used for the outlet, and symmetric transverse top and bottom wall.
 350 Periodic boundary condition is provided in the spanwise direction. Isothermal no-slip boundary
 351 condition is imposed at the cylinder with two values of temperature equal to $T_w = 300 \text{ K}$ and
 352 $T_w = 600 \text{ K}$ to study the impact of the heat transfer.

353 Details on the grid refinement and related numerical parameters adopted in this study are given
 354 in Table. I. As Cartesian mesh is adopted by LBM, the mesh size is divided by a factor 2 for
 355 each grid refinement with the coarsest mesh size $\Delta x_{max}/D = 0.5$. The block refinement is the
 356 combination of cylinders and cuboids, as can be seen in Fig. 9 (cycle and rectangular for the 2D
 357 illustration). Three resolutions are chosen for this mesh convergence study with six, seven and
 358 eight grid levels. The number of mesh points are respectively equal to 1.88 millions, 6.52 millions
 359 and 25.16 millions. The finest grid sizes normalized by the cylinder diameter distributed along the
 360 isothermal cylinder wall are correspondingly $\Delta x_{min}/D = 1.56 \times 10^{-2}$, $\Delta x_{min}/D = 7.81 \times 10^{-3}$ and
 361 $\Delta x_{min}/D = 3.91 \times 10^{-3}$. The numbers of iterations required for a dimensionless time $U_\infty t^*/D = 1$
 362 are 1115, 2227 and 4464, respectively, that corresponds to CPU times (by hour) of 1.44, 30.7
 363 and 194.68, respectively. In order to obtain statistically converged results, 500 FOT (Flow Over
 364 Time), i.e., $t = 500t^*$ (around 100 vortex shedding cycles) is chosen as initial transient time. Then
 365 the turbulent statistics and mean flow validation are realized from data collected over 150 FOT,
 366 i.e., $t = 150t^*$ (around 30 vortex shedding cycles). In this section, brackets $\langle \cdot \rangle$ and (\cdot') denote
 367 time-averaged quantities and the corresponding fluctuations, respectively.

368 For compressible flows, the Favre-filtering is more convenient to avoid the SGS (Sub-Grid
 369 Scale) terms introduced in continuity equation^{55,56}. A Vreman sub-grid scale model is adopted⁵⁷
 370 with the model constant $C_s = 0.18$. A constant turbulent Prandtl number is chosen equal to $Pr_t =$
 371 0.85 as suggested^{51,52,58}.

TABLE I. Details of meshes and computational setup, $t^* = D/U_\infty = 1.68 \times 10^{-3}$ s is the FOT (Flow Over Time) and it's used as a dimensionless quantity.

	Mesh	Total cell points (M)	Grid level	$\Delta x_{min}/D$	$\Delta t_{min}/t^*$	CFL_{u+c}	$t_w(h)$
	Coarse	1.88	6	1.56×10^{-2}	8.97×10^{-4}	0.684	1.44
Isotherm	Middle	6.52	7	7.81×10^{-3}	4.49×10^{-4}	0.684	30.7
	Fine	25.16	8	3.91×10^{-3}	2.24×10^{-4}	0.684	194.68
Non-isotherm	Fine	25.16	8	3.91×10^{-3}	4.49×10^{-4}	0.84	44.26

372 2. Isothermal results at $\Delta T = 0$ K

373 It's observed by Parnaudeau *et al.*⁴⁹ that the main flow features and turbulent characteristics in
374 the wake region is determined by the re-circulation length L_{rc} , especially the transverse velocity
375 and Reynolds stress. The length of this re-circulation bubble is defined as the distance from the
376 rear stagnation point of the cylinder to the position where the sign change of the mean streamwise
377 velocity along the centerline in the wake region. So the grid convergence study is focused on the
378 prediction of this re-circulation zone.

379 The mean streamwise velocity $\langle u \rangle$ normalized by the inlet velocity U_∞ and the streamwise
380 Reynolds stress $\langle u'u' \rangle$ normalized by U_∞^2 are plotted in Fig. 10. The negative mean velocity corre-
381 sponds to the re-circulation zone as shown in Fig. 10(a). The overestimated re-circulation length
382 obtained by the coarse mesh simulation pushes the streamwise Reynolds stress to the downstream
383 direction compared to the experimental data by Parnaudeau *et al.*⁴⁹. The results obtained with the
384 middle mesh agree well with experimental data, except visible discrepancy of the maximum value
385 of mean velocity in the re-circulation region. Finally, an excellent agreement is achieved with the
386 fine mesh. In particular, the present LMNA result gives a re-circulation length L_{rc}/D equal to 1.46,
387 compared to 1.51 given by Parnaudeau *et al.*⁴⁹ and 1.47 given by Dong *et al.*⁵⁰ that corresponds
388 to an error of around 3% and less than 1%, respectively (see in Table II).

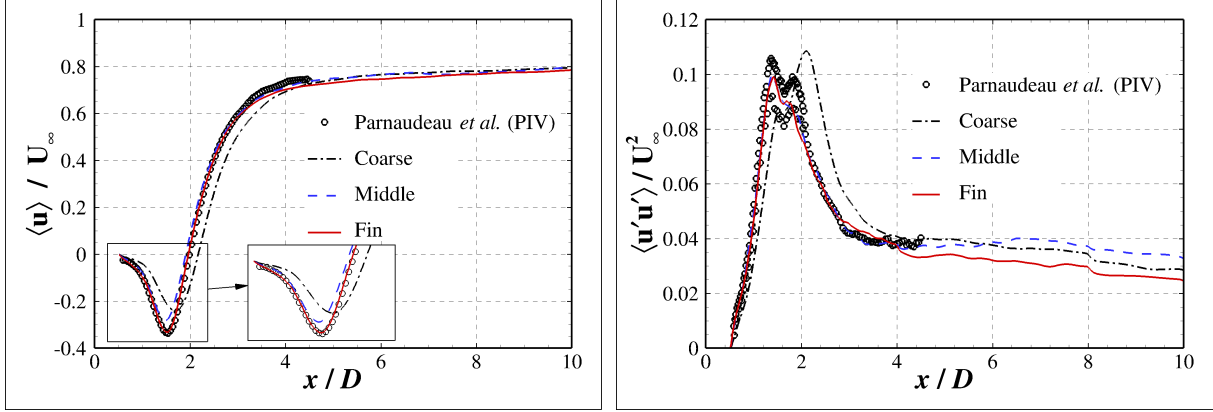
389 In the next, details of the flow features obtained for the finest mesh for the isothermal case

TABLE II. Experimental and numerical results of the literature together with present LMNA results. EXP stands for experiment.

Case	Model	$-C_{pb}$	$\langle C_d \rangle$	St	L_{rc}/D
Parnaudeau <i>et al.</i> ⁴⁹	EXP (PIV)	—	—	0.208	1.51
Dong <i>et al.</i> ⁵⁰	EXP (PIV)	—	—	0.212	1.47
Norberg <i>et al.</i> ^{59,60}	EXP (Multiple)	0.85-0.95	0.93-1.05	0.21	1.4-1.6
Ma <i>et al.</i> ⁶¹	DNS (case II)	0.84	—	0.219	1.59
Dong <i>et al.</i> ⁵⁰	DNS	0.93-1.04	—	0.206-0.210	1-1.18
Lysenko <i>et al.</i> ⁵³	LES	0.91	0.97	0.209	1.67
Mani <i>et al.</i> ⁵⁴	LES	0.86	0.99	0.206	—
Franke and Frank <i>et al.</i> ⁶²	LES	0.85-0.94	0.978-1.005	0.209	1.34-1.64
Kravchnko and Moin <i>et al.</i> ⁶³	LES	0.94	1.04	0.210	1.35
Parnaudeau <i>et al.</i> ⁴⁹	LES	—	—	0.208	1.56
Jogee <i>et al.</i> ⁴⁹	LES	—	—	—	1.68
Sircar <i>et al.</i> ⁵¹	LES	0.91	1.048	0.221	1.48
Coarse mesh	LES	0.81	1.06	0.212	1.67
Middle mesh	LES	0.86	1.04	0.223	1.41
Fine mesh	LES	0.84	1.04	0.223	1.46
Fine mesh (non-isothermal)	LES	0.79	1.00	0.212	1.69

390 are discussed including the mean flow, the turbulent characteristics and some aerodynamic coeffi-
391 cients.

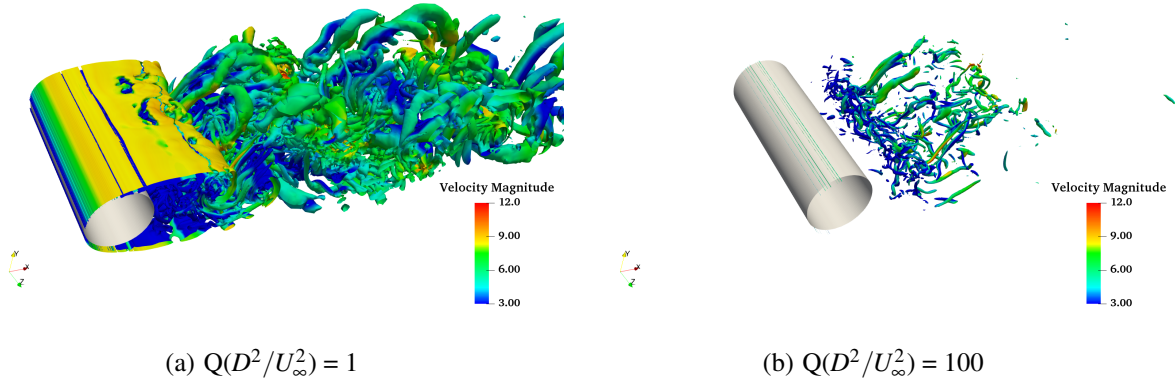
392 Fig. 11 shows the laminar and turbulent structures of the flow. At this Reynolds number, the
393 flow remains laminar along the cylinder wall and close to the cylinder just after the boundary layer
394 separation. Large scale vortices develop at around two diameters from the rear stagnation point
395 and are convected by the mean flow thus shedding alternatively from the upper and lower shear
396 layers forming the well-known Von Karman street. The instability of the shear layer contributes to



(a) Mean streamwise velocity profile

(b) Streamwise Reynolds stress profile

FIG. 10. Impact of grid refinement on the mean streamwise velocity (a) and streamwise Reynolds stress (b) distributions along the centerline in the wake region. PIV case I and II in the Ref.⁴⁹ are given in these figures.



(a) $Q(D^2/U_\infty^2) = 1$

(b) $Q(D^2/U_\infty^2) = 100$

FIG. 11. Instantaneous iso-surfaces of the Q criterion $Q^* = Q(D^2/U_\infty^2) = 1$ (a), and $Q(D^2/U_\infty^2) = 100$ (b), colored by the velocity magnitude.

397 the laminar-turbulent transition. Thus, the large scale vortices break down to small scale vortices
 398 in the wake region.

399 Fig. 12 shows the time-averaged velocity field and the associated streamlines. As expected,
 400 the mean flow is globally symmetric to the axial line in the xy -plane, indicating that the solution is
 401 convergence over the computational time. The re-circulation zone is confined within the top and
 402 bottom shear layers. Two transverse lines in the re-circulation region at positions $X/D = 1.06$ and
 403 $X/D = 1.54$ are probed where the shear layer dynamics and turbulent statistics are quantitatively
 404 analyzed.

405 As shown in Fig. 13(a) and Table. II, compared to the experimental results of Parnaudeau

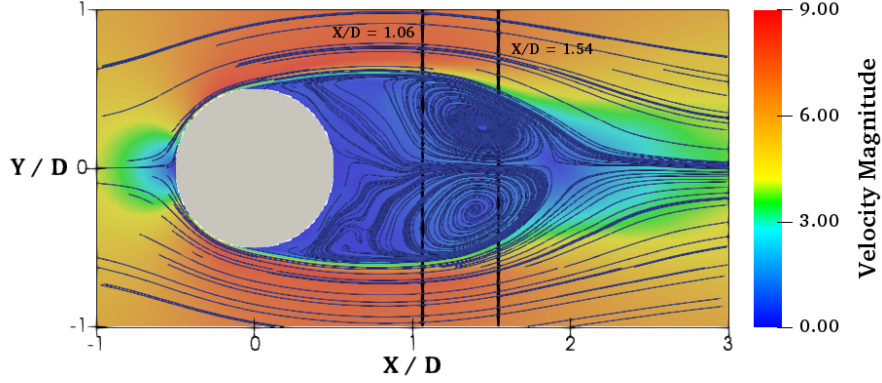
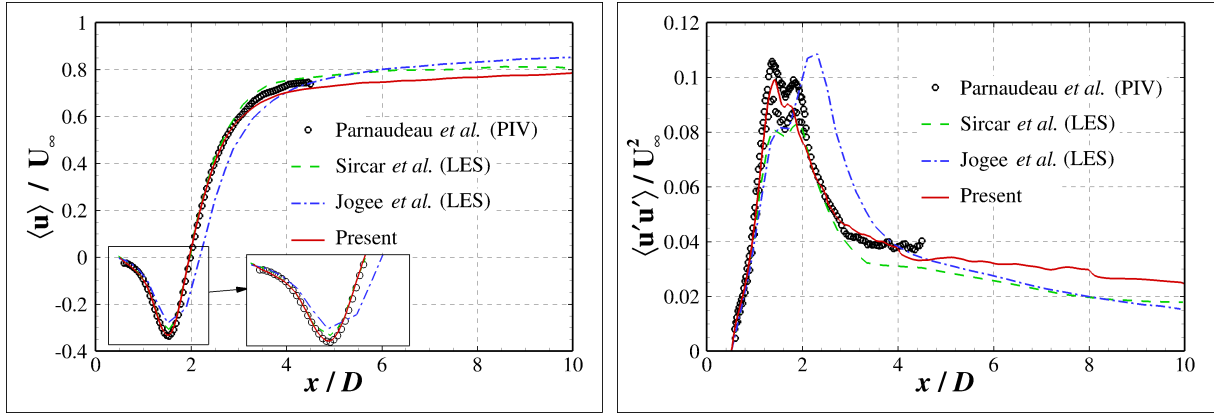


FIG. 12. Streamlines and velocity field of the mean flow at $Re = 3900$. Isothermal case.

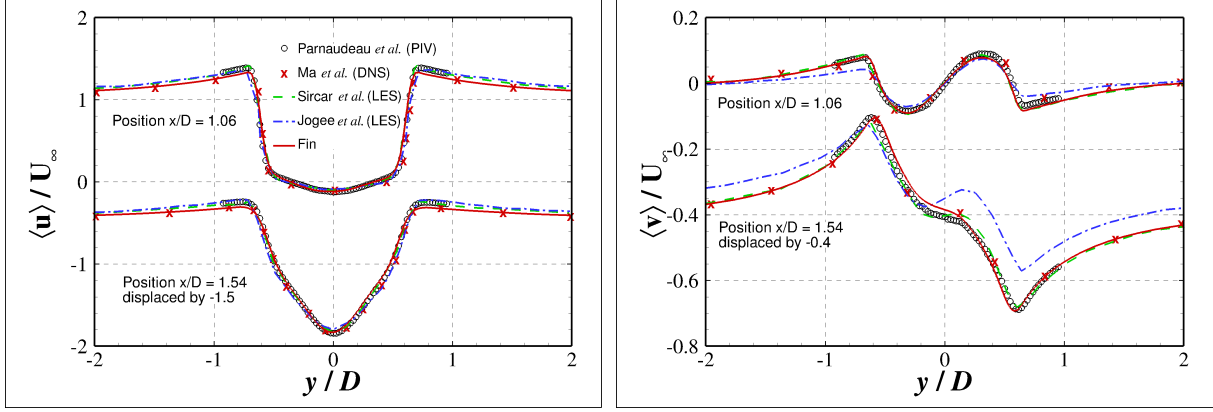


(a) Mean streamwise velocity profile

(b) Streamwise Reynolds stress profile

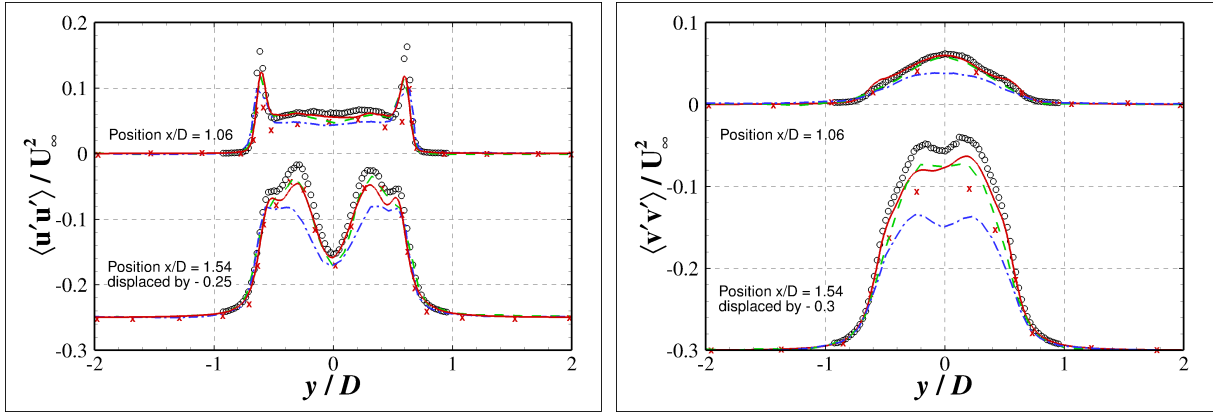
FIG. 13. Mean streamwise velocity (a) and streamwise Reynolds stress (b) distributions along the centerline in the wake region. PIV case I and II are provided from ref.⁴⁹. LES results of Sircar *et al.*⁵¹ and Jogee *et al.*⁵² are also included.

406 *et al.*⁴⁹, the re-circulation length is correctly estimated by Sircar *et al.*⁵¹ with $L_{rc,sircar}/D = 1.48$
 407 and by the present LMNA with $L_{rc,present}/D = 1.46$. However, the overestimation of Jogee *et al.*⁵²
 408 results in overestimating the peak value of the mean streamwise Reynolds stress $\langle u \rangle / U_\infty$, and in
 409 shifting the profiles of these quantities to the downstream direction. Moreover, the peak values
 410 of the mean streamwise velocity and corresponding Reynolds stress given by the present LMNA
 411 are closest to the experimental data. Furthermore, even in the relatively far wake region, i.e.,
 412 $x/D \in [2.5, 4.5]$, present method gives best agreement with the experimental data. The far wake
 413 region, i.e., $x/D > 4.5$, won't be intensively studied in this investigation because of the lack of
 414 experimental results.



(a) Mean streamwise velocity profile

(b) Mean transverse velocity profile



(c) Streamwise Reynolds stress distribution

(d) Transverse Reynolds stress distribution

FIG. 14. (a, b) Mean streamwise and transverse velocity components. (c, d) Streamwise and transverse Reynolds stress components at position $x/D = 1.06$ and 1.54 . The reference data from experiments PIV case II of Parnaudeau *et al.*⁴⁹ and DNS case II of Ma *et al.*⁶¹ are included for comparison.

415 The normalized mean streamwise and transverse velocity $\langle u \rangle / U_\infty$ and $\langle v \rangle / U_\infty$ profiles and
 416 normalized streamwise and transverse Reynolds stress $\langle u'u' \rangle / U_\infty^2$ and $\langle v'v' \rangle / U_\infty^2$ at the two probe lines
 417 in the near wake region are shown in Fig. 14. As can be seen, the variation of these profiles are
 418 concentrated on the re-circulation region, i.e., $y/D \in [-0.8, 0.8]$. The disturbance outside this re-
 419 gion is negligible. Hence the velocity tends to the inflow velocity profile and the turbulent statistics
 420 tends to zero. The mean streamwise velocity has the well-known U-shaped profile at $x/D = 1.06$
 421 close to the cylinder boundary, and tends to a V-shaped profile as the probe moves to downstream
 422 direction at $x/D = 1.54$. Contrary to the symmetric form of mean streamwise velocity profile, the
 423 mean transverse velocity profile is anti-symmetric with respect to the axial line. Regarding the
 424 transverse quantities like the transverse mean velocity component (Fig. 14(b)) and the transverse

425 Reynolds stress component (Fig. 14(d)), the discrepancy with the results of the literature is larger.
 426 The deviation of Jogee’s results are very significant compared to the experimental⁴⁹ and DNS⁶¹
 427 data. However, the discrepancy between the present LMNA results and the LES results of Sircar
 428 *et al.*⁵¹, or the experimental⁴⁹ and DNS⁶¹ results is rather small.

429 The streamwise Reynolds stress profiles show two peaks (Fig. 14(c)) which correspond to the
 430 detached shear layers. The instability induced by shear phenomena produces highest perturbation
 431 near the shear layers. The streamwise Reynolds stress are three times larger at the shear layer
 432 than the values in the center of the re-circulation bubble. The two peaks show a diffusive trend as
 433 the probe moves downstream, which is probably influenced by the oscillations of the alternative
 434 vortex shedding phenomena. The values of the transverse Reynolds stress components are around
 435 three times larger than the streamwise ones in the center of the re-circulation bubble because of
 436 the oscillations in the transverse direction of the vortex shedding phenomena.

437 **3. Non-isothermal results at $\Delta T = 300K$**

438 The cylinder is now heated to $T_w = 600$ K. In order to investigate the influence of heat transfer
 439 on the flow features, the turbulent heat flux and temperature fluctuations are analyzed quantita-
 440 tively at the same probe locations as in previous section. The non-isothermal LES results of Sircar
 441 *et al.*⁵¹ and Jogee *et al.*⁵² at the same temperature difference $\Delta T = 300$ K are included as refer-
 442 ences. The fine mesh defined in former section is used for the non-isothermal simulations here.
 443 The time step is fixed at $\Delta t^* = 4.49 \times 10^{-4}$. Let’s notice that similar computations with the fully
 444 compressible solver requires a time step 16.4 times smaller equal to $\Delta t^* = 2.74 \times 10^{-5}$.

TABLE III. Recirculation lengths for the isothermal and the non-isothermal cases. Present LMNA results are given together with LES predictions of the literature by Sircar *et al.*⁵¹ and Jogee *et al.*⁵².

Case	L_{rc}/D		Bubble length change (%)
	Isothermal	$\Delta T = 300K$	
Jogee <i>et al.</i> ⁵²	1.68	2.04	21.4
Sircar <i>et al.</i> ⁵¹	1.480	1.606	8.5
Current	1.46	1.69	15.8

445 Table III shows that the re-circulation bubble length becomes larger when the cylinder is heated.
 446 This is probably caused by the temperature increase in the detached shear layer. As can be seen
 447 in the time-averaged temperature field in Fig. 15(b), the temperatures of the shear layer and of the
 448 re-circulation zone are higher than elsewhere. These local temperature elevations lead to a change
 449 of density and viscosity, and hence results in a lower local Reynolds number. Consequently, the
 450 laminar laminar-turbulent transition is delayed downstream that leads to a larger re-circulation
 451 bubble.

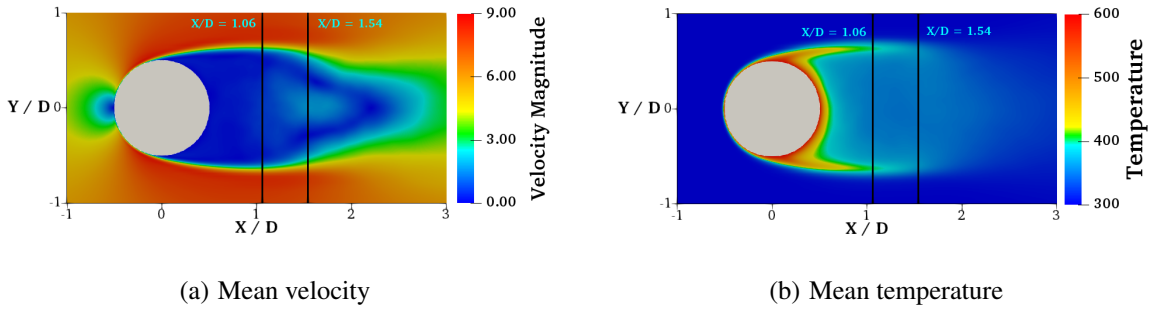


FIG. 15. Color maps of the mean velocity magnitude (a) and temperature (b).

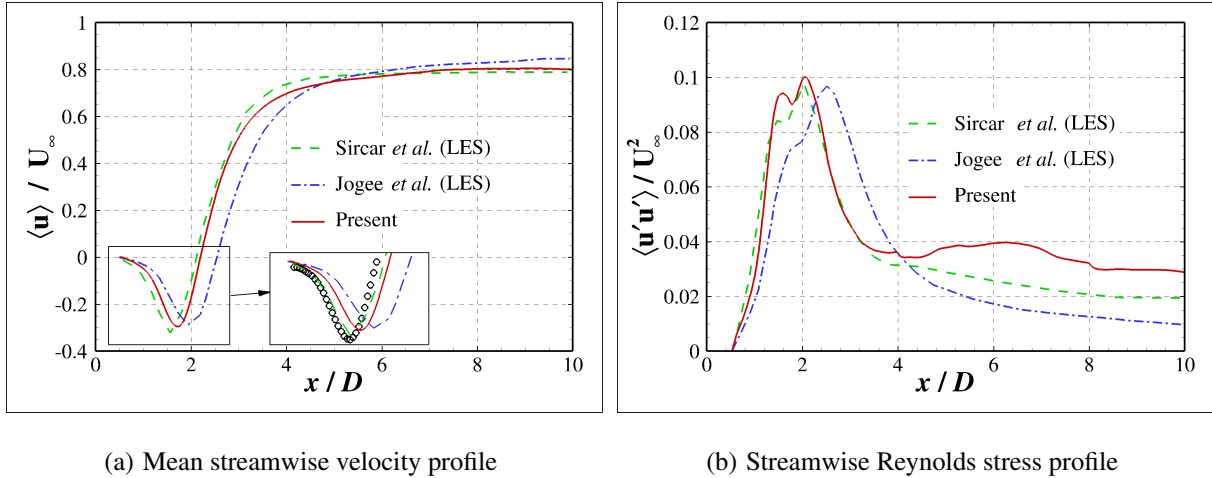
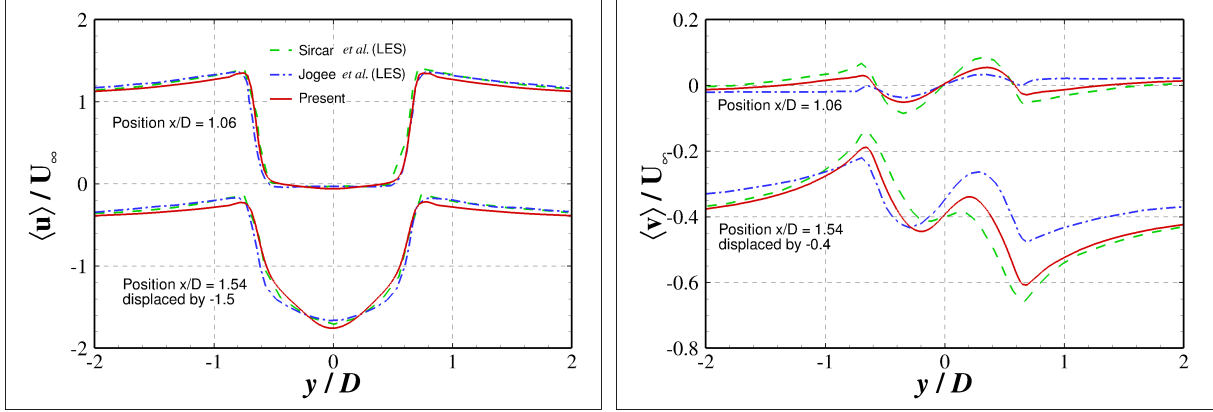


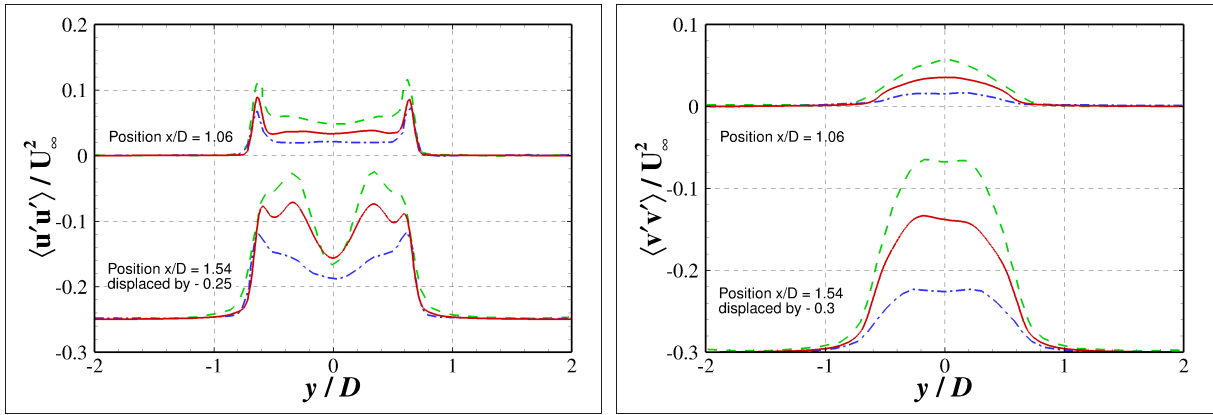
FIG. 16. Profiles along the centerline in the wake region of the mean streamwise velocity (a) and streamwise Reynolds stress (b) components. Non-isothermal LES results of Sircar *et al.*⁵¹ and Jogee *et al.*⁵² at the same temperature difference $\Delta T = 300$ K are included as references.

452 Compared to the isothermal simulations, the bubble length is increased of 21.4 % and 8.5 %
 453 for Jogee *et al.*⁵² and Sircar *et al.*⁵¹, respectively, indicating that the Jogee's LES results are more
 454 sensitive to the thermal effects than the Sircar ones. In the present simulation the increase is of
 455 15.8%. However, as the isothermal bubble length predicted by the Jogee's LES results was already



(a) Mean streamwise velocity profile

(b) Mean transverse velocity profile



(c) Streamwise Reynolds stress distribution

(d) Transverse Reynolds stress distribution

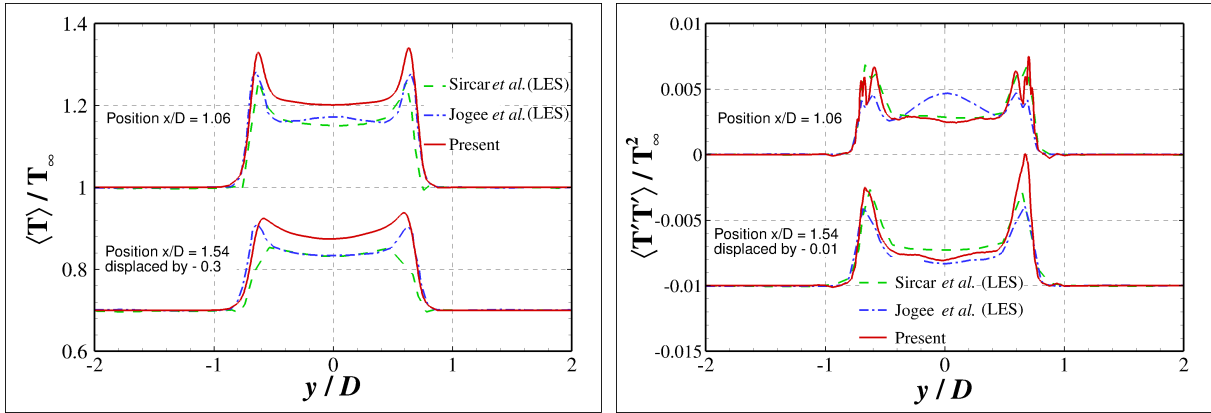
FIG. 17. Mean streamwise and transverse velocity components (a, b), streamwise and transverse Reynolds stress components (c, d) at $x/D = 1.06$ and 1.54 . Non-isothermal LES results of Sircar *et al.*⁵¹ and Jogee *et al.*⁵² at the same temperature difference $\Delta T = 300$ K are included as references.

456 11% larger than the experimental data of reference, the bubble length predicted by the Jogee's non-
 457 isothermal LES is closed to $L_{rc,Jogee}/D = 2.04$, which is much larger than the values predicted by
 458 Sircar *et al.* and by the present method. This difference in the bubble length is clearly visible
 459 on Fig. 16. Despite the two times larger bubble length change predicted by the present method
 460 compared to Sircar *et al.*, the thermal re-circulation length is quite close and the discrepancy of
 461 the turbulent characteristics is also small in the near wake region.

462 The mean flow profiles and Reynolds stress at the same probe locations show the turbulent
 463 flow feature modification caused by thermal effects in Fig. 17. As can be seen, by comparing
 464 the velocity profiles of Fig. 14, the mean streamwise velocity profiles at $x/D = 1.06$ are quite
 465 close to the isothermal ones. However the profiles at $x/D = 1.54$ change from a V-shape to a

466 U-shape because the relative positions of the probe lines with respect to the bubble length are
 467 pushed upstream due to the growth of the bubble length. This change is magnified for the mean
 468 transverse velocity profiles shown in Fig. 17(b), especially at $x/D = 1.54$. As can be seen, the
 469 mean transverse velocity component in the isothermal case at $x/D = 1.54$ shows a monotonous
 470 drop from $y/D = -0.6$ to $y/D = 0.6$. However, all the transverse mean velocity profiles of the
 471 thermal cases change sign three times of which these profiles are more similar to the ones at
 472 $x/D = 1.06$.

473 The sensitivity with thermal effect of the turbulent statistics is shown in Fig. 17(c) and Fig.
 474 17(d). Both the streamwise and transverse Reynolds stress profiles obtained by Sircar *et al.*⁵¹ are
 475 very close to the isothermal results of Parnaudeau *et al.*⁴⁹. This means that the results of Sircar
 476 *et al.*⁵¹ are rather insensitive to thermal effects. In contrast, the results of Jogee *et al.*⁵² exhibit
 477 significant variations with respect to their isothermal results. Present results are between those of
 478 Sircar *et al.*⁵¹ and Jogee *et al.*⁵² but closer to Sircar *et al.*⁵¹ results. This is consistent with the
 479 re-circulation bubble length change due to thermal effect, Table. III.



(a) Temperature profile

(b) RMS of Temperature fluctuation distribution

FIG. 18. Mean temperature profiles (a) and RMS of temperature fluctuations (b) at $x/D = 1.06$ and $x/D = 1.54$. Non-isothermal LES results of Sircar *et al.*⁵¹ and Jogee *et al.*⁵² at the same temperature difference $\Delta T = 300$ K are included as references.

480 In the next, the thermal characteristics including the temperature distribution and turbulent
 481 heat flux will be discussed. Fig. 18 shows the mean temperature profiles and the RMS (Root
 482 Mean Square) of temperature fluctuation $\langle T'T' \rangle / T_8^2$ distribution at the same probe lines. Similar
 483 to the streamwise Reynolds stress profiles, the mean temperature profiles and RMS distribution
 484 display the same features. Two peaks are located at the detached heated shear layers as shown in

485 Fig. 15(b). The temperature fluctuations peaks correspond to the peaks on the mean temperature
 486 profiles due to the temperature amplitude and the strong instability at the shear layers. In the
 487 present LMNA results, the mean temperature at the re-circulation bubble is slightly higher than
 488 the LES results of Sircar *et al.* and Jogee *et al.*, whereas the temperature fluctuations are very
 489 close. Fig. 19 shows the color map of the streamwise $\langle u'T' \rangle / U_\infty T_\infty$ and the transverse turbulent
 490 heat flux $\langle v'T' \rangle / U_\infty T_\infty$ where the positive (red) and negative (blue) zones are highlighted.

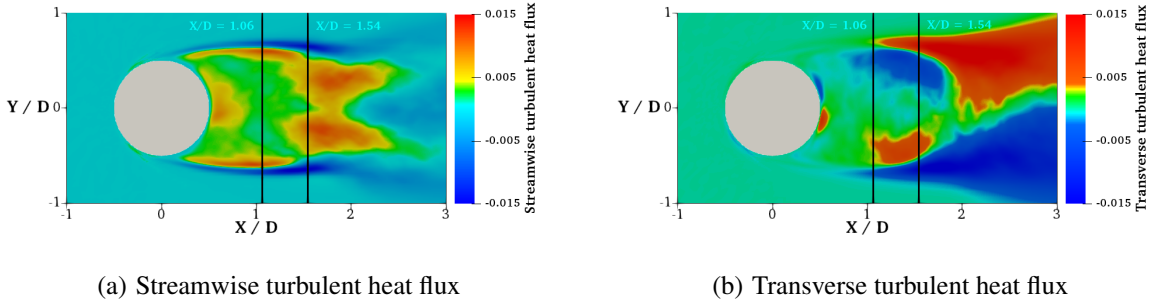


FIG. 19. 2D color maps of the streamwise (a) and transverse (b) turbulent heat flux obtained with the present LMNA.

491 Fig. 20 shows the temperature fluctuations, and the streamwise heat flux distribution along the
 492 center line. As can be seen, the RMS of temperature fluctuations and turbulent heat flux attain a
 493 peak value near the rear point of the cylinder, showing that the thermal fluctuations are most signif-
 494 icant at this position. The turbulent heat flux is positive in the re-circulation region, and switches
 495 to a negative value outside the re-circulation region. This means that the heat transfer is from
 496 the heated cylinder to the cold fluid in the re-circulation zone. The position where the turbulent
 497 heat flux becomes zero is very close to the margin of re-circulation zone, and this phenomenon is
 498 consistent for all the three LES results: $L_{rc,Sircar} < L_{rc,current} < L_{rc,Jogee}$.

499 The streamwise and transverse turbulent heat flux profiles along the two probe lines are plotted
 500 in Fig. 21. The heat transfer direction is represented by the sign of the turbulent heat flux. The
 501 steep gradient of temperature at the shear layer produces the four peaks of each profile with oppo-
 502 site sign near each other. This is consistent with the color map of Fig. 19, where the highlighted
 503 blue zone and red zone are distributed alternatively near the shear layer. Moreover, the transverse
 504 turbulent heat flux at center line ($y/D = 0$) is zero because the transverse velocity along the center
 505 line is zero inside the re-circulation region. Lastly, the discrepancy between the results of Sircar *et*
 506 *al.* and the present ones is not significant for both the peak values and the profile shapes. However,
 507 the deviation with the LES results of Jogee *et al.* is considerably larger, which is consistent with

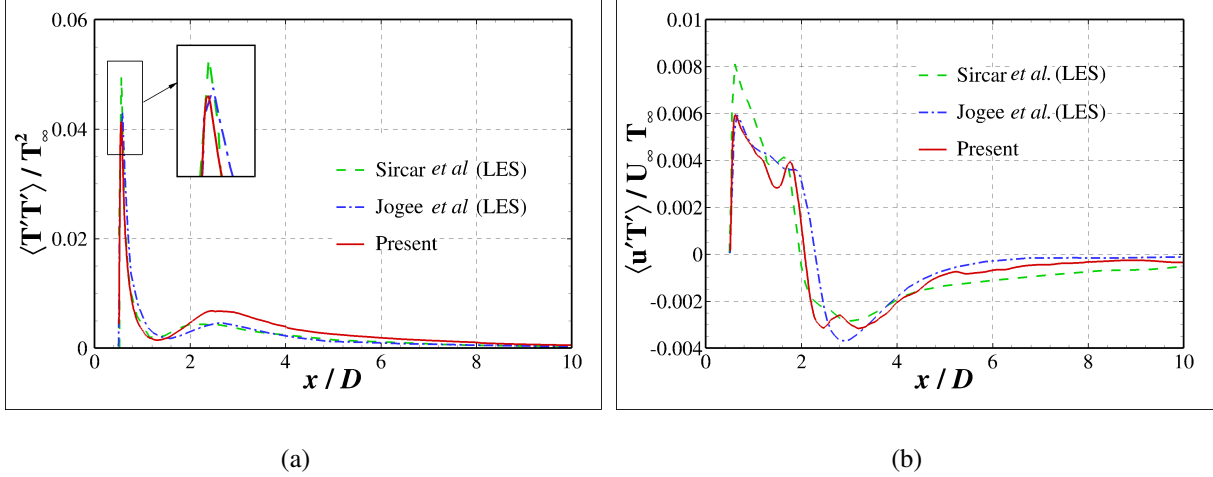


FIG. 20. RMS of temperature fluctuations (a) and turbulent heat flux (b) profiles along the centerline in the wake.

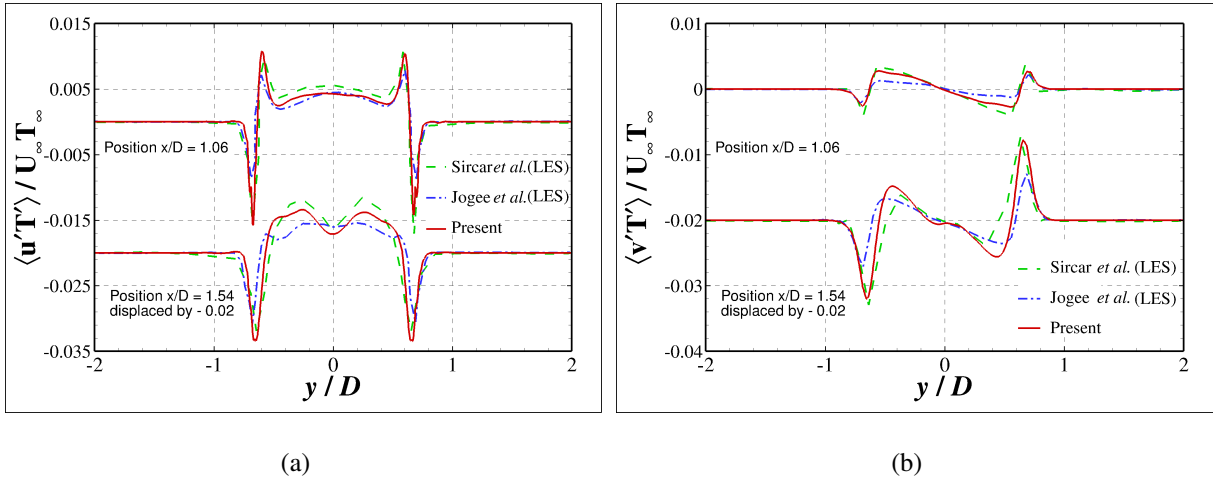


FIG. 21. Streamwise (a) and transverse (b) turbulent heat flux profiles at $x/D = 1.06$ and $x/D = 1.54$.

508 previous description. It is probably due to the weaker influence of the thermal effects on the flow
 509 features.

510 These results show that the new LMNA provides promising results for complex turbulent
 511 isothermal configurations and consistent explanation for heat transfer problems with large tem-
 512 perature differences. Moreover, with a time step more than 15 times larger than for a fully com-
 513 pressible LBM solver, the computational efficiency is strongly improved.

514 **IV. CONCLUSION**

515 In this paper, we have proposed a new hybrid Lattice Boltzmann method for dealing efficiently
516 with Low Mach thermal flows. To overcome the time step constraint of the fully compressible
517 LBM solvers due to acoustic waves, the low-Mach number approximation has been considered and
518 adapted to the compressible hybrid LBM solver developed by the team³⁷. The present algorithm
519 overcomes the drawback of the PGS method related to the modification of the pressure gradient. In
520 addition, the present algorithm can be easily implemented to improve any density based or pressure
521 based compressible solvers, which makes it more flexible than other existing LB algorithms based
522 on LMNA.

523 The performance of present method are assessed by two classes of cases. The first category of
524 test scenarios includes flows in laminar regime, i.e., one dimensional gravity column, 2D rising
525 thermal bubble and 2D natural convection for which all the results agree well with the reference.
526 The second category consists of 3D Taylor Green Vortex and 3D flow passing through a heated
527 cylinder in turbulent regime. Excellent agreements are obtained especially on the average flow
528 features with respect to reference results of the literature. Some discrepancies are observed for
529 the turbulent quantities of 3D heated cylinder compared to other LES results. The results of
530 present method are between the LES results of Sircar *et al.*⁵¹ and Jogee *et al.*⁵² and closer to
531 the one of Sircar. This indicates that current solution is more sensible to the turbulent thermal
532 effect than Sircar *et al.*⁵¹ and less sensible than Jogee *et al.*⁵². This disparity is most likely due to
533 the differences in turbulent models and mesh configurations adopted in these LES investigations.
534 Furthermore, table IV summarizes the computing speed-ups of all the test cases employed in this
535 investigation. As can be seen, the computational efficiency is increased by at least a factor of ten.

536 **ACKNOWLEDGEMENTS**

537 This work was performed within ProLB solver. Centre de Calcul Intensif d’Aix-Marseille
538 is acknowledged for granting access to its high performance computing resources. We also ac-
539 knowledge the supports of ANR, Renault, Airbus and SafranTech by the Industrial Chair Program
540 ALBUMS (ANR-CHIND-18-ALBUMS).

TABLE IV. Computations speedup for the different test cases of the paper. The speedup is evaluated with respect to fully compressible LBM computations.

Test cases	Speedup
1D Gravity column at $\text{Ma} = \frac{\sqrt{gL}}{c} \approx 0.029$	10.0
2D Rising bubble at $\text{Ma} = 0.018$	25.0
2D Natural convection at $\text{Ma} = 0.0035$	55.6
3D Taylor Green Vortex at $\text{Ma} = 0.0029$	70.3
3D Heated cylinder at $\text{Ma} = 0.017$	16.4

541 Appendix A: The low Mach number approximation of the Navier-Stokes equations

542 The low-Mach number acceleration strategy based on the scaling of fully compressible Navier-
543 Stokes equations in low-Mach number limit is presented in this section. One of the main chal-
544 lenges when simulating low-Mach number thermal flows using a compressible solver is to keep
545 high computational efficiency despite the non-dominant acoustic scale fluctuations. A method to
546 cure this issue is based on a scale analysis, which divides the fully compressible Navier-Stokes
547 equations by a leading order fluctuations thermal dynamics and a non-dominant acoustic scale.

548 Based on the asymptotic analysis, any variable ϕ appears in the fully compressible Navier-
549 Stokes system will be expanded as

$$550 \quad \phi = \phi^0 + \varepsilon \phi^h + \mathcal{O}(\varepsilon^2) \quad (\text{A1})$$

551 with

$$552 \quad \varepsilon \equiv \gamma_\infty \text{Ma}_\infty^2. \quad (\text{A2})$$

Then, the leading order Navier-Stokes equations read²

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_\alpha}{\partial x_\alpha} = 0, \quad (\text{A3a})$$

$$\frac{\partial p}{\partial x_\alpha} = 0, \quad (\text{A3b})$$

$$\frac{\partial \rho u_\alpha}{\partial t} + \frac{\partial \rho u_\alpha u_\beta}{\partial x_\beta} = -\frac{\partial p^h}{\partial x_\alpha} + \frac{1}{\text{Re}_\infty} \frac{\partial \Pi_{\alpha\beta}}{\partial x_\beta} + \frac{1}{\text{Fr}_\infty^2} \rho g_\alpha, \quad (\text{A3c})$$

$$\rho T \left(\frac{\partial s}{\partial t} + u_\alpha \frac{\partial s}{\partial x_\alpha} \right) = -\frac{1}{\text{Re}_\infty \text{Pr}_\infty} \frac{\partial q_\alpha}{\partial x_\alpha}, \quad (\text{A3d})$$

$$p = \rho r T, s = c_v \ln \frac{p}{\rho^\gamma}. \quad (\text{A3e})$$

553 Together with the reference Reynolds, Froude and Prandtl numbers

$$554 \quad \text{Re}_\infty \equiv \frac{\rho_\infty u_\infty x_\infty}{\mu_\infty}, \quad \text{Fr}_\infty \equiv \frac{u_\infty}{\sqrt{g_\infty x_\infty}}, \quad \text{Pr}_\infty \equiv \frac{c_{p_\infty} \mu_\infty}{\lambda_\infty}, \quad (\text{A4})$$

555 In the above system, the thermodynamic pressure p is spatial independent according to equa-
556 tion (A3b). Also, one additional equation for the hydrodynamic pressure p^h is necessary to resolve
557 this set of equations. Typically, this is achieved through a Poisson equation constructed from the
558 mass and momentum equations⁶⁴.

559 **Appendix B: Thermodynamic pressure in a closed system**

560 The equation of state $s = c_v \ln \frac{p}{\rho^\gamma}$ gives the following derivative relationship

$$561 \quad ds = \frac{c_v}{p} dp - \frac{c_p}{\rho} d\rho. \quad (\text{B1})$$

562 In a closed system, the global mass of the system should be conserved, i.e.

$$563 \quad \int \frac{\partial \rho}{\partial t} dV = 0, \quad (\text{B2})$$

564 together with equation (B1), the spatial independent thermodynamic pressure can be calculated by

$$565 \quad \frac{dp}{dt} = \frac{\int \left(\frac{\rho}{c_p} \frac{\partial s}{\partial t} \right) dV}{\int \frac{\rho}{\gamma p} dV}. \quad (\text{B3})$$

² The superscript 0 for the variables with 0th order is neglected here.

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