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# Geometric representability of 1-cycles on rationally connected threefolds

Claire Voisin*

*À la mémoire d'Alberto Collino*

**Abstract**

We prove that for any rationally connected threefold $X$, there exists a smooth projective surface $S$ and a family of 1-cycles on $X$ parameterized by $S$, inducing an Abel-Jacobi isomorphism $\mathrm{Alb}(S) \cong J^3(X)$. This statement was previously known for some classes of smooth Fano threefolds.

## 0 Introduction

Non-representability phenomena for algebraic cycles on smooth projective complex varieties were discovered by Mumford [16], Griffiths [10] and Clemens [6]. If $X$ is a smooth complex projective variety, for each integer $k \geq 0$, the $k$-th Griffiths-Abel-Jacobi map constructed by transcendental methods is a group morphism

$$\Phi_X^k : \mathrm{CH}^k(X)_{\mathrm{hom}} \to J^{2k-1}(X),$$

where

$$J^{2k-1}(X) := H^{2k-1}(X, \mathbb{C})/(F^k H^{2k-1}(X) \oplus H^{2k-1}(X, \mathbb{Z})_{\mathrm{tf}})$$

is a (usually non-algebraic) complex torus and $\mathrm{CH}^k(X)_{\mathrm{hom}}$ is the group of codimension $k$ cycles homologous to zero on $X$ modulo rational equivalence. (We use above the notation $\Gamma_{\mathrm{tf}}$ for $\Gamma/\mathrm{Tors}(\Gamma)$, for any abelian group $\Gamma$.) Denoting by $\mathrm{CH}^k(X)_{\mathrm{alg}} \subset \mathrm{CH}^k(X)_{\mathrm{hom}}$ the subgroup of cycles algebraically equivalent to 0 on $X$, the image $J^{2k-1}(X)_{\mathrm{alg}} := \Phi_X^k(\mathrm{CH}^k(X)_{\mathrm{alg}})$ is an abelian variety (we refer to [2] for an algebraic construction of it in the spirit of Murre [17] for codimension 2 cycles and Serre [18] for 0-cycles). The group $\mathrm{CH}^k(X)_{\mathrm{alg}}$ is not an abstract group; it has a sort of algebraic structure since it is exhausted by the images of morphisms

$$Z_* : M \to \mathrm{CH}^k(X)_{\mathrm{alg}},\ m \mapsto Z_m - Z_{m_0}$$

induced by any codimension $k$-cycle $Z \subset M \times X$, with $M$ smooth projective and connected, equipped with a reference point $m_0$. For each such cycle, the composition $\Phi_X^k \circ Z_* : M \to J^{2k-1}(X)_{\mathrm{alg}}$ induces by the universal property of the Albanese map (see [18]) a morphism

$$\Phi_Z : \mathrm{Alb}(M) \to J^{2k-1}(X)_{\mathrm{alg}}$$

such that $\Phi_Z \circ \mathrm{alb}_M(m - m_0) = \Phi_X^k \circ Z_*(m)$.

**Remark 0.1.** Note that there exists a universal codimension $k$ cycle $Z_J$ *with* $\mathbb{Q}$*-coefficients* on $J^{2k-1}(X)_{\mathrm{alg}} \times X$, namely a cycle $Z_J \in \mathrm{CH}^k(J^{2k-1}(X)_{\mathrm{alg}} \times X)$ such that

$$\Phi_{Z_J} = N Id_{J^{2k-1}(X)_{\mathrm{alg}}} : \mathrm{Alb}(J^{2k-1}(X)_{\mathrm{alg}}) = J^{2k-1}(X)_{\mathrm{alg}} \to J^{2k-1}(X)_{\mathrm{alg}} \tag{1}$$

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for some integer $N > 0$. To see this, one uses the fact that, almost by definition of $J^{2k-1}(X)_{\mathrm{alg}}$, there exists a smooth projective variety $M$ and a codimension $k$ algebraic cycle $Z_M \in \mathrm{CH}^k(M \times X)$ such that

$$\psi := \Phi_{Z_M} \circ \mathrm{alb}_M : M \to J^{2k-1}(X)_{\mathrm{alg}}$$

is surjective. Replacing $M$ by a linear section, one can assume that $\psi$ is generically finite of degree $N$. Then

$$Z_J := (\psi, Id_X)_* Z_M$$

satisfies (1).

Since the seminal work of Mumford [16], we know that the group $\mathrm{CH}^k(X)_{\mathrm{alg}}$ is not representable in general, where by representability of $\mathrm{CH}^k(X)_{\mathrm{alg}}$, we mean here that there exists a connected projective manifold $M$ and a codimension $k$ cycle $Z \in \mathrm{CH}^k(M \times X)$ such that the induced map

$$M \to \mathrm{CH}^k(X)_{\mathrm{alg}},$$
$$m \mapsto Z_*(m - m_0)$$

is surjective, where $m_0$ is a chosen reference point of $M$. The typical Mumford examples of nonrepresentable Chow groups are given by the groups $\mathrm{CH}_0(S)$ of smooth projective surfaces $S$ with $p_g \neq 0$. Indeed, Mumford [16] shows that $\mathrm{CH}_0(S)$ is under this assumption infinite dimensional in the sense that it cannot be exhausted by countably many families of 0-cycles parameterized by varieties of bounded dimension. As an easy consequence, the Albanese map (or Abel-Jacobi map) is not injective on $\mathrm{CH}_0(S)_{\mathrm{alg}}$.

The Abel-Jacobi map $\Phi^k_X$ is also not surjective in general (see [10]) and the fact that the transcendental part of the Abel-Jacobi map, defined on cycles modulo algebraic equivalence, with value in $J^{2k-1}(X)/J^{2k-1}(X)_{\mathrm{alg}}$, can be nontrivial, is a celebrated discovery of Griffiths. Clemens [6] even proved that its image can be infinitely generated modulo torsion, illustrating another aspect of nonrepresentability of cycles of intermediate dimension. However, in certain cases, for example rationally connected varieties $X$, or more generally smooth projective varieties with trivial $\mathrm{CH}_0$ group, one can easily show using the Bloch-Srinivas decomposition of the diagonal (see [1]) that the two intermediate Jacobians $J^3(X)$, which correspond to codimension 2 cycles, and $J^{2n-3}(X)$, which correspond to 1-cycles, have no transcendental part. Nevertheless, the geometry of the Abel-Jacobi map remains mysterious even in these cases. We study in this paper the following notion of "geometric representability" for $J^{2k-1}(X)$, which implies and strengthens the condition that $J^{2k-1}(X) = J^{2k-1}(X)_{\mathrm{alg}}$.

**Definition 0.2.** *The group $J^{2k-1}(X)$ is geometrically representable if there exists a smooth projective variety $M$ and a codimension $k$ cycle $Z_M \in \mathrm{CH}^k(M \times X)$ such that*

$$\Phi_{Z_M} : \mathrm{Alb}(M) \to J^{2k-1}(X)$$

*is an isomorphism.*

One motivation for this definition is the following result proved in [22, Proposition 3.6]:

**Proposition 0.3.** *Let $X$ be stably rational of dimension $n$. Then $J^3(X)$ and $J^{2n-3}(X)$ are geometrically representable in the sense of Definition 0.2. More generally, the geometric representability of $J^3(X)$ (resp. $J^{2\dim(X)-3}(X)$) is a stably birationally invariant property of smooth projective varieties $X$.*

**Remark 0.4.** The intermediate Jacobian $J^{2n-3}(X)$ corresponds to 1-cycles on $X$. By blowing-up a codimension 2 subvariety $W$ of $X$, we add to the group $\mathrm{CH}_1(X)$ the group $\mathrm{CH}_0(W)$. If $\dim X \geq 4$, we have $\dim W \geq 2$, and $W$ can be chosen to be a surface with $p_g \neq 0$. It follows that the group $\mathrm{CH}_1(X)$, for a rational variety $X$ of dimension $\geq 4$ is not in general representable in the sense of Mumford.

**Remark 0.5.** We proved in [21] that, even in the case of 1-cycles on rationally connected threefolds, there does not necessarily exist a universal codimension $k$ cycle on $J^{2k-1}(X)_{\mathrm{alg}} \times X$, namely a cycle $Z_J \in \mathrm{CH}^k(J^{2k-1}(X)_{\mathrm{alg}} \times X)$ which satisfies (1) with $N = 1$. This shows that we cannot in general ask that $M = J^{2k-1}(X)$ in Definition 0.2. Furthermore, the existence of a universal codimension $k$ cycle is not a birationally invariant property for $k \geq 3$, and even for $k = n - 1$ (1-cycles), see Corollary 0.14.

We are going to focus in this paper on the case of 1-cycles on rationally connected threefolds. More generally the group of codimension 2 cycles on rationally connected manifolds (or manifolds with trivial $\mathrm{CH}_0$-group) always satisfies representability in the Mumford sense since we have

**Theorem 0.6.** *(Bloch-Srinivas [1].) The Abel-Jacobi map $\Phi^2_X : \mathrm{CH}^2(X)_{\mathrm{alg}} \to J^3(X)$ is an isomorphism for codimension 2 cycles on rationally connected smooth projective manifolds $X$.*

We prove in this paper the following result which answers a question asked by Clemens in [5] (see also [22]).

**Theorem 0.7.** *Let $X$ be a smooth projective rationally connected 3-fold over $\mathbb{C}$. Then there exists a smooth complex projective manifold $M$ and a family of 1-cycles*

$$Z \in \mathrm{CH}^2(M \times X)$$

*such that the induced Abel-Jacobi map*

$$\Phi_Z : \mathrm{Alb}(M) \to J^3(X) \tag{2}$$

*is an isomorphism. In other words, $J^3(X)$ is geometrically representable in the sense of Definition 0.2.*

**Remark 0.8.** If one replaces "isomorphism" by "isogeny" in Theorem 0.7, then the statement holds for codimension $k$-cycles and with $M = J^{2k-1}(X)$, under the only assumption on $X$ that $J^{2k-1}(X) = J^{2k-1}(X)_{\mathrm{alg}}$ , see Remark 0.1.

**Remark 0.9.** We can even assume in the conclusion of Theorem 0.7 that $M$ is a smooth projective surface. Indeed, by the Lefschetz theorem on hyperplane sections, if $S \subset M$ is a smooth surface which is a complete intersection of ample hypersurfaces, the natural map $\mathrm{Alb}(S) \to \mathrm{Alb}(M)$ is an isomorphism.

**Remark 0.10.** We cannot in general assume that $M$ is a smooth curve. In many cases, the intermediate Jacobian of a rationally connected threefold has a unique principal polarization, so if it were isomorphic to the Jacobian of a curve, it would be isomorphic as a principally polarized abelian variety to the Jacobian of a curve. This is the well-known obstruction to rationality discovered by Clemens and Griffiths [7], and it is already nontrivial in the case of the cubic threefold.

Combining Theorem 0.7 and Theorem 0.6, we conclude

**Corollary 0.11.** *If $X$ is a rationally connected threefold, there is a codimension 2 cycle $Z \in \mathrm{CH}^2(M \times X)$ such that*

$$Z_* : \mathrm{CH}_0(M)_{\mathrm{hom}} \to \mathrm{CH}^2(X)_{\mathrm{alg}}$$

*factors through an isomorphism*

$$\mathrm{Alb}(M) \cong \mathrm{CH}^2(X)_{\mathrm{alg}}.$$

Theorem 0.7 had been proved earlier by explicit constructions for some classes of Fano threefolds. Clemens-Griffiths [7] studied the case of the cubic threefold $X$ and proved that the family of lines in $X$, that is classically known to be parameterized by a smooth surface $\Sigma_X$, induces an Abel-Jacobi isomorphism between $\mathrm{Alb}(\Sigma_X)$ and $J^3(X)$. The paper [26] treats the case of the general quartic double solid $r : X \to \mathbb{P}^3$, and establishes a similar Abel-Jacobi isogeny for its surface of "lines", namely degree 1 curves on $X$ with respect to the ample line bundle $r^*\mathcal{O}_{\mathbb{P}^3}(1)$, and later on, this has been improved by Clemens [4] who proved that the Abel-Jacobi morphism for the family of lines in the quartic double solid is an isomorphism. Clemens' strategy has been then adapted successfully by Letizia in [14] to establish the corresponding statement for a general quartic threefold in $\mathbb{P}^4$ and its surface of conics, that had been shown to be smooth in [9]. A similar result was also obtained by Ceresa and Verra [3] who treated the case of the sextic double solid. Although it does not concern 1-cycles, we also mention a result of a similar shape by Collino, concerning 2-cycles in a general cubic 5-fold $X$. Collino proves in [8] that the surface $\Sigma_X$ of planes is smooth and induces an Abel-Jacobi isomorphism $\mathrm{Alb}(\Sigma_X) \to J^5(X)$.

Theorem 0.7 is an improvement of Theorem 0.2 of [22], which is the following statement

**Theorem 0.12.** *(Voisin [22].) Let $X$ be a rationally connected 3-fold. There exists a smooth projective curve $C$ and a cycle $Z \in \mathrm{CH}^2(C \times X)$ such that*

$$\Phi_Z : J(C) \to J^3(X)$$

*is surjective with connected fibers. (Equivalently, the induced morphism $\Phi_{X*} : H_1(C, \mathbb{Z}) \to H_3(X, \mathbb{Z})_{\mathrm{tf}}$ is surjective.)*

Indeed, Theorem 0.7 implies Theorem 0.12 since, assuming Theorem 0.7 and choosing a smooth curve $C \subset M$ which is a complete intersection of ample hypersurfaces, the natural morphism $J(C) \to \mathrm{Alb}(M)$ has connected fibers by the Lefschetz hyperplane section theorem which says that the morphism $H_1(C, \mathbb{Z}) \to H_1(M, \mathbb{Z})$ is surjective. Then the restricted cycle $Z_{M|C\times X} \in \mathrm{CH}^2(C \times X)$ satisfies the conclusion of Theorem 0.12.

One consequence of Theorem 0.7 is the fact that the obstruction to the existence of a universal codimension 2 cycle on a rationally connected threefold $X$ always reduces to the obstruction to the existence of a universal 0-cycle parameterized by $\mathrm{Alb}(M)$ for some smooth projective variety $M$:

**Corollary 0.13.** *(See Corollary 2.1.) There exists a smooth projective variety $M$ such that $M$ does not admit a universal 0-cycle.*

We note also the following

**Corollary 0.14.** *The existence of a universal codimension $k$-cycle is not a birationally invariant property for $k \geq 3$.*

Indeed, if we blow-up in $\mathbb{P}^4$ a surface $M$ as in Corollary 0.13, we get by the blow-up formulas that a universal 1-cycle on $\widetilde{\mathbb{P}^4}$ would provide a universal 0-cycle on $M$. Hence it does not exist in general.

We will also explain in Section 2.1 the following formal consequence concerning the Albanese motive, of the existence of smooth projective varieties with no universal 0-cycle, (for example the surface of lines of a general quartic double solid, and more generally, any variety $M$ as in Theorem 0.7, if $X$ does not admit a universal codimension 2 cycle). It is proved in [22] (see also Lemma 1.3) that given a smooth projective variety $M$ and an abelian variety $A$ contained in $\mathrm{Alb}(M)$, there exists a smooth projective variety $N$ and a 0-correspondence from $N$ to $M$ such that $\Gamma_* : \mathrm{Alb}(N) \to \mathrm{Alb}(M)$ is an isomorphism from $\mathrm{Alb}(N)$ to $A$. We will show that this result cannot be extended to direct sums, in the following sense

**Proposition 0.15.** *There exists a smooth projective variety $M$ such that*

*(1)* $\mathrm{Alb}(M)$ *decomposes as a direct sum* $\mathrm{Alb}(M) = A \oplus B$*,*

*(2) there does not exist a smooth projective variety* $N$ *and two* 0*-correspondences* $\Gamma_1$ *from* $N$ *to* $M$ *and* $\Gamma_2$ *from* $M$ *to* $N$ *such that* $\Gamma_{1*} : \mathrm{Alb}(N) \to \mathrm{Alb}(M)$ *has for image* $A$ *and*

$$\Gamma_{2*} \circ \Gamma_{1*} = Id : \mathrm{Alb}(N) \to \mathrm{Alb}(N).$$

In other words, a direct sum decomposition of the Albanese variety of $M$ does not always come from a projector in $\mathrm{CH}(M \times M)$.

We prove Theorem 0.7 in Section 1 starting from Theorem 0.12. (In fact, the proof of Theorem 0.7 follows closely the proof of Theorem 0.12 in [22].) Section 2.1 discusses two stronger notions of representability for 1-cycles, namely Shen's notion of universally generating family on one hand (we do not know whether it is always satisfied, see Question 2.8) and the existence of a universal codimension 2 cycle on the other hand (we know that it is not always satisfied, see [21]). One result that we will prove as a consequence of Theorem 0.7 is

**Theorem 0.16.** *(See Theorem 2.5) Let* $X$ *be a rationally connected threefold. Assume that* $X$ *admits a universally generating family* $(N, Z_N)$*. Then* $X$ *admits a universally generating family* $(M, Z_M)$ *such that* $\Phi_{Z_M} : \mathrm{Alb}(M) \to J^3(X)$ *is an isomorphism.*

# 1 Proof of Theorem 0.7

## 1.1 Preliminary results

Let $X$ be a rationally connected threefold. By Theorem 0.12, we know that there exists a smooth projective curve $C$ and a codimension 2-cycle $Z \in \mathrm{CH}^2(C \times X)$ such that the Abel-Jacobi map $\Phi_Z : J(C) \to J^3(X)$ has connected fibers, that is, its kernel $K := \mathrm{Ker}\,\Phi_Z$ is an abelian variety. We will prove the following result

**Theorem 1.1.** *There exists a smooth projective variety* $M$ *carrying a codimension* 2 *cycle* $Z_M \in \mathrm{CH}^2(M \times X)$*, with an inclusion* $j : C \hookrightarrow M$ *such that*

*(1)* $\Phi_{Z_M} \circ j_* = \Phi_Z$ *and*

*(2)* $\mathrm{Ker}\,\Phi_Z = \mathrm{Ker}\,(j_* : J(C) \to \mathrm{Alb}(M))$*.*

**Claim 1.2.** *Theorem 0.7 follows from Theorem 1.1.*

*Proof.* As $\mathrm{Ker}\,\Phi_Z = \mathrm{Ker}\,(j_* : J(C) \to \mathrm{Alb}(M))$, the image $\mathrm{Im}\,(j_* : J(C) \to \mathrm{Alb}(M))$ is isomorphic to $J^3(X)$. It follows that $J^3(X) \cong \mathrm{Im}\, j_* \subset \mathrm{Alb}(M)$ is realized as an abelian subvariety of $\mathrm{Alb}(M)$ and we can then apply the following

**Lemma 1.3.** *(See [22, Lemma 3.7].) Let* $M$ *be a smooth projective variety and let* $A \subset \mathrm{Alb}(M)$ *be an abelian subvariety. Then there exists a smooth projective variety* $M'$*, and a* 0*-correspondence* $\Gamma \in \mathrm{CH}^{\dim M}(M' \times M)$*, such that* $\Gamma_* : \mathrm{Alb}(M') \to \mathrm{Alb}(M)$ *is the composition of an isomorphism between* $\mathrm{Alb}(M')$ *and* $A$ *and the inclusion of* $A$ *in* $\mathrm{Alb}(M)$*.*

We apply Lemma 1.3 to $A := J^3(X) \subset \mathrm{Alb}(M)$. By Theorem 1.1, the morphism $\Phi_{Z_M} : \mathrm{Alb}(M) \to J^3(X)$ is an isomorphism when restricted to $J^3(X) \cong \mathrm{Im}\, j_* \subset \mathrm{Alb}(M)$, so we conclude that the cycle $Z_{M'} := Z_M \circ \Gamma \in \mathrm{CH}^2(M' \times X)$ satisfies the conclusion of Theorem 0.7. □

For the proof of Theorem 1.1, we will use the following lemma.

**Lemma 1.4.** *Let* $C \subset J(C)$ *be a smooth projective curve imbedded in its Jacobian, and let* $K \subset J(C)$ *be an abelian subvariety. Let* $x \in K$ *be a very general point. Let* $M$ *be a smooth projective variety containing the curve* $C$ *and let* $j : C \hookrightarrow M$ *be the inclusion, inducing* $j_* : J(C) \to \mathrm{Alb}(M)$*. Assume*

$$\mathrm{alb}_M \circ j_{0*}(x) = 0 \text{ in } \mathrm{Alb}\, M. \tag{3}$$

*Then* $\mathrm{alb}_M \circ j_*$ *vanishes on* $K$*.*

*Proof.* As $x$ is very general in $K$, the subgroup generated by $x$ is Zariski dense in $K$. As $\mathrm{alb}_M \circ j_{0*}$ vanishes on this subgroup, it vanishes on $K$. $\square$

## 1.2 Proof of Theorem 1.1

Let $X$ be a rationally connected threefold and let $C$ be a smooth projective curve, equipped with a a codimension 2-cycle $Z_C \in \mathrm{CH}^2(C \times X)$ such that the Abel-Jacobi map $\Phi_Z : J(C) \to J^3(X)$ has connected fibers. In order to construct a pair $(M, Z_M)$ satisfying the conclusion of Theorem 1.1, we will first use the following observation, already used in [23] and also in [22].

**Lemma 1.5.** *There exists a codimension 2-cycle $Z_{J(C)}$ on $J(C) \times X$, such that*

$$\Phi_{Z_{J(C)}} = \Phi_Z : J(C) \to J^3(X). \tag{4}$$

*Furthermore, for any $x \in K = \mathrm{Ker}\,(\Phi_Z : J(C) \to J^3(X))$, one has*

$$Z_{J(C)*}(\{x\} - \{0_J\}) = 0 \text{ in } \mathrm{CH}^2(X). \tag{5}$$

*Proof.* The cycle $Z_{J(C)}$ is obtained by seeing $J(C)$ as birational to $C^{(g)}$, $g := g(C)$, and by descending to $C^{(g)} \times X$ the cycle

$$\sum_i \mathrm{pr}_i^* Z_C$$

which is invariant under the natural action of the symmetric group $\mathfrak{S}_g$ on $C^g \times X$, where in the formula above, the $\mathrm{pr}_i : C^g \times X \to C \times X$ are the obvious projections. By construction of $Z_{J(C)}$, one has for a general element $x. = x_1 + \ldots + x_g \in C^{(g)}$,

$$Z_{J(C)*}(x.) = Z_{x_1} + \ldots + Z_{x_g} \text{ in } \mathrm{CH}^2(X). \tag{6}$$

By applying $\Phi_X$ to (6), we get (4). It follows from (4) that

$$\Phi_X(Z_{J(C)*}(\{x\} - \{0_J\})) \text{ in } J^3(X),$$

which implies (5) by Theorem 0.6. $\square$

We will prove the following theorem in Section 1.3. The proof is in fact very similar to that of Claim 2.13 of [22]. We choose as before a very general point $x \in K = \mathrm{Ker}\,\Phi_Z = \mathrm{Ker}\,\Phi_{Z_J}$.

**Theorem 1.6.** *There exists a singular quasiprojective variety $W := W_1 \cup W_2 \cup W_3$, where $W_1$ is a Zariski open set of $J(C)$ containing $x$ and $0_J$, and a diagram*

$$\begin{array}{ccc} \Sigma & \xrightarrow{f} & X \\ p \downarrow & & \\ W & & \end{array} \tag{7}$$

*with the following properties.*

1. *$p$ is proper flat of relative dimension 1 with semistable fibers.*
2. *The restriction to $W_1 \subset W$ of the correspondence (7) has for induced Abel-Jacobi map $W_1 \to J^3(X)$ the restriction to $W_1$ of the morphism $\Phi_{Z_J}$.*
3. *$W_2$ is isomorphic to a Zariski open set of $\mathbb{P}^1$, $W_3$ is Zariski open in a smooth projective connected and simply connected variety. $W_2 = \mathbb{P}^1$ intersects $J(C)$ in $0_J \in J(C)$ and $W_3$ in a point $w_2 \in W_3$, which correspond respectively to $0, \infty \in \mathbb{P}^1$. $W_3$ intersects $J(C)$ at the point $x \in J(C)$, which corresponds to $w_3 \in W_3$.*

4. *The fibers $f_t : \Sigma_t \to X$ are stable maps which are unobstructed at the three singular points $0_J$, $x$, $w_2$ of $W = W_1 \cup W_2 \cup W_3$.*

The statement and construction would be slightly clearer if $W_2 \cup W_3$ were a chain of rational curves meeting $W_1$ in the two points $0_J$ and $x$. Unfortunately we have not been able to achieve this. The heuristic proof of Theorem 1.6, 1 to 3 is the fact that, by Lemma 1.5, we know that $Z_{J(C)*}(\{x\}-\{0_J\}) = 0$ in $\mathrm{CH}_1(X)$. Assume naively that this vanishing translates into the existence of a family $Z'$ of curves parameterized by $\mathbb{P}^1$ such that $Z'_0 = Z_{0_J}$, $Z'_\infty = Z_x$, then we would take $W_2 = \mathbb{P}^1$, $W_3 = \emptyset$ and the correspondence $\Sigma$ would be given by $Z_J$ on $W_1$, and $Z'$ on $\mathbb{P}^1$, glued to $J$ via $0 = 0_J$, $x = \infty$. To achieve 4, one uses the Kollár-Miyaoka-Mori technic [13] which allows to make curves unobstructed by adding trees of very free rational curves. One has to show that this can be done in families at least generically, and without changing the Abel-Jacobi map.

We will explain the technical details of the proof of Theorem 1.6 in next section and we now prove Theorem 1.1 assuming Theorem 1.6.

*Proof of Theorem 1.1.* We construct the smooth projective variety $M$ and the cycle $Z_M$ starting from the universal deformation of the family of stable maps $f_t : \Sigma_t \to X$, $t \in W$ appearing in Theorem 1.6. More precisely, by Theorem 1.6, 4, we conclude that this deformation space $M_0$ is smooth near the three singular points $0_J$, $x$, $w_2$ and more precisely, using the fact that $J(C)$, $\mathbb{P}^1$ and $W_3$ are smooth and connected, $M_0$ has exactly one connected component $M_1$ which contains the image of $W$ under the classifying map associated with the family of stable maps (7) and is smooth near $0_J$, $x$, $w_2$. It follows that a smooth projective desingularization $M$ of $M_1$ contains a union

$$\widetilde{J(C)} \cup W_2 \cup \widetilde{W_3} \tag{8}$$

where the two varieties $\widetilde{J(C)}$ and $\widetilde{W_3}$ are smooth projective birational to $J(C)$ and $W_3$ respectively, and the union (8) is isomorphic to $J(C) \cup W_2 \cup W_3$ near the intersection points $0_J$, $x$ and $w_2$. The cycle $Z_M$ is given over a dense Zariski open set of $M$ by the universal evaluation map

$$\begin{array}{ccc} \Sigma_M & \stackrel{f_M}{\to} & X \\ p\downarrow & & \\ M & & . \end{array} \tag{9}$$

We now claim that the morphism $j_{1*} : J(C) = \mathrm{Alb}(\widetilde{J(C)} \to \mathrm{Alb}(M)$ induced by the inclusion $j_1$ of $\widetilde{J(C)}$ in $M$ constructed above satisfies

$$j_*(x) = 0. \tag{10}$$

Indeed, denoting by $j_2 : W_2 \to M$ and $j_3 : \widetilde{W_3} \to M$ the restrictions of $f$ we have $\mathrm{alb}_M(j_1(x) - j_3(w_2)) = 0$ because $j_1(x) = j_3(w_3)$ and $\widetilde{W_3}$ is simply connected. Similarly, $\mathrm{alb}_M(j_1(0_J) - j_3(w_2)) = 0$ because $j_1(0_J) = j_2(0))$, $j_3(w_2) = j_2(\infty)$ and $\mathbb{P}^1$ is simply connected. Thus $\mathrm{alb}_M(j_1(x) - j_1(0_J)) = 0$, which proves the claim.

We now apply Lemma 1.4 to the inclusion $j$ of $\widetilde{J(C)}$ in $M$ and conclude that $K = \mathrm{Ker}\,\Phi_Z$ is contained in $\mathrm{Ker}\,(j_* : J(C) \to \mathrm{Alb}(M))$. Finally, as $\Phi_{Z_M} \circ j_* = \Phi_{Z_J}$ by statement 2 of Theorem 1.6, we also have $\mathrm{Ker}\,(j_* : J(C) \to \mathrm{Alb}(M)) \subset K$ and the proof of Theorem 1.1 is concluded. □

## 1.3 Proof of Theorem 1.6

We describe in this section the proof of Theorem 1.6. It is obtained by a minor modification of the proof of Theorem 0.12 of [22]. This proof being rather intricate, we will only sketch

its main steps, explaining how it has to be modified to give a proof of Theorem 1.6. We recall that Lemma 1.5 provides us with a codimension 2 cycle $Z_{J(C)}$ on $J(C) \times X$ with the property that $\mathrm{Ker}\,(\Phi_{Z_{J(C)}} : J(C) \to J^3(X))$ is an abelian variety $K$. The cycle $Z_{J(C)}$ can be modified by adding a constant 1-cycle in $X$ and we can thus assume, by choosing a rationally equivalent representative, that over a Zariski open set $W_1$ of $J(C)$, the fiber $Z_t$, $t \in W_1$ is a smooth curve in $X$. We can also arrange that the Zariski open set $W_1$ contains the two points $x$, $0_J \in K \subset J(C)$. The cycle $Z_{J(C)}$ is thus represented over $W_1$ by a family of smooth curves (or stable maps)

$$\begin{array}{ccc} \mathcal{C}_1 & \stackrel{f_1}{\to} & X \\ p_1 \downarrow & & \\ W_1 & & \end{array} \tag{11}$$

in $X$.

**Step 1**. *Constructing $W_2$ and the family of stable maps over it.* By assumption, $x \in \mathrm{Ker}\,(\Phi_{Z_J} : \mathrm{Alb}(J(C)) = J(C) \to J^3(X))$. As by assumption

$$\Phi_{Z_J}(\mathrm{alb}_{J(C)}(\{x\} - \{0_J\})) = \Phi_X(\mathcal{C}_{1,x} - \mathcal{C}_{1,0_J}) = 0,$$

we get using Theorem 0.6 that the 1-cycle $\mathcal{C}_{1,x} - \mathcal{C}_{1,0_J}$ is rationally equivalent to 0 in $X$. This rational equivalence relation is analyzed in [22] and this leads to the following statement possibly after replacing $X$ by $X \times \mathbb{P}^r$ or a blow-up of $X$.

**Lemma 1.7.** *There exists a family of stable maps*

$$\begin{array}{ccc} \mathcal{C}_2 & \stackrel{f_2}{\to} & X \\ p_2 \downarrow & & \\ W_2 & & \end{array} \tag{12}$$

*parameterized by a Zariski open set $W_2 \subset \mathbb{P}^1$ containing 0 and $\infty$, satisfying the following properties :*

1. *The fiber of $(p_2, f_2)$ over $0 \in W_2$ is the union of the fiber $f_{1,0_J} : \mathcal{C}_{1,0_J} \to X$ of the family (11) over $0_J \in J(C)$ and a stable map $f_A := f_{2|A} : A \to X$ glued at finitely many points. The fiber of $(p_2, f_2)$ over $\infty \in W_2$ is the union of the fiber $f_{1,x} : \mathcal{C}_{1,x} \to X$ of the family (11) over $x \in J(C)$ and a stable map $f_B := f_{2|B} : B \to X$ glued at finitely many points.*

2. *The two maps $f_A : A \to X$ and $f_B : B \to X$ have the same image in $X$. More precisely, they are different partial normalizations of the same nodal curve $E$ contained in $X$.*

**Step 2.** *Deforming $\mathcal{C}_{1,x} \cup B$ to $\mathcal{C}_{1,x} \cup A$.* The two curves or rather morphisms

$$f_A : A \to X,\ f_B : B \to X$$

and their respective attachments to the curves $\mathcal{C}_{1,0_J}$ and $\mathcal{C}_{1,x}$ are not equal as stable maps, although they have the same support. They may differ combinatorially (for example the number of attachments points of $A$ and $B$ to the respective fibers $\mathcal{C}_{1,0_J}$ and $\mathcal{C}_{1,x}$ may differ) and this is why we cannot consider that the two fibers $f_{2,0} : \mathcal{C}_{2,0} \to X$ and $f_{2,\infty} : \mathcal{C}_{2,\infty} \to X$ are obtained by gluing the *same* curve to the respective fibers $f_{1,0_J} : \mathcal{C}_{1,0_J} \to X$, $f_{1,x} : \mathcal{C}_{1,x} \to X$ by the same number of points. This is corrected by deforming $\mathcal{C}_{1,x} \cup A$ to $\mathcal{C}_{1,x} \cup B$ as follows.

**Lemma 1.8.** *There exists a family of stable maps*

$$\begin{array}{ccc} \mathcal{C}_3 & \stackrel{f_3}{\to} & X \\ p_3 \downarrow & & \\ W_3 & & \end{array} \tag{13}$$

*parameterized by a Zariski open set $W_3$ in a smooth projective connected and simply connected variety, and two points $w_2$, $w_3 \in W_3$ satisfying the following properties :*

1. *The fiber $f_{3,w_2} : \mathcal{C}_{3,w_2} \to X$ of the family (13) over $w_2$ is the union of the fiber $f_{2,\infty} : \mathcal{C}_{2,\infty} = \mathcal{C}_{1,x} \cup B \to X$ of the family (12) over $\infty \in \mathbb{P}^1$ and a curve $R \hookrightarrow X$, glued at finitely many points.*

2. *The fiber $f_{3,w_3} : \mathcal{C}_{3,w_3} \to X$ of the family (13) over $w_3$ is the union of the fiber $f_{1,x} : \mathcal{C}_{1,x} \to X$ of the family (11) over $x \in J(C)$, the curve $f_A : A \to X$ glued at finitely many points, and the same curve $R \hookrightarrow X$ as in 1, which is glued to the union $\mathcal{C}_{1,x} \cup A$ at finitely many points.*

3. *The number of attachment points of $R \cup B$ to $\mathcal{C}_{1,x}$ in 1 (that is, inside the curve $\mathcal{C}_{3,w_2}$) equals the number of attachment points of $R \cup A$ to $\mathcal{C}_{1,x}$ in 2 (that is, inside the curve $\mathcal{C}_{3,w_3}$).*

4. *The number of attachment points of $R$ to $\mathcal{C}_{1,x} \cup B$ in 1 (that is, inside the curve $\mathcal{C}_{3,w_2}$) equals the number of attachment points of $R$ to $\mathcal{C}_{1,x} \cup A$ in 2 (that is, inside the curve $\mathcal{C}_{3,w_3}$).*

Note that we asked above that $W_3$ is Zariski open in a simply connected smooth projective variety. In [22], we just asked the property that the Abel-Jacobi map of the family (13 ) is trivial so that Lemma 1.8 was proved in [22] in this slightly weaker form. Lemma 1.8 may seem technical but it is an immediate application of the following simple fact.

**Sublemma 1.9.** *Let $\Gamma \subset S$ be a nodal curve in a smooth simply connected surface. Let $g_1 : \Gamma_1 \to S$, $g_2 : \Gamma_2 \to S$ be the two stable maps obtained by normalizing $\Gamma$ along a subset $N_1 \subset \Gamma$ (resp. $N_2 \subset \Gamma$) of nodes, where $N_1$ and $N_2$ have the same cardinality $n$. Then, up to replacing $\Gamma$ by a union $\Gamma \cup R$, for some smooth curve $R \overset{i}{\hookrightarrow} S$ in general position, there is a family of stable maps parameterized by a Zariski open set in a connected simply connected projective variety, which has $g_1' = (g_1, i) : \Gamma_1 \cup R \to S$ and $g_2' = (g_2, i) : \Gamma_2 \cup R \to S$ as fibers.*

*Proof.* We prove this result by noting that the universal family of curves in $S$ with $n$ specified nodes in the class $\Gamma \cup R$, for $R$ ample enough, is parameterized by a projective bundle $P$ over a Zariski open set in $S^{[n]}$. As $S$ is simply connected, $S^{[n]}$ is simply connected, and $P$ admits a simply connected projective compactification. The desired family is obtained by normalizing the curves of the universal family over $P$ at the specified set of $n$ nodes, which specializes to $N_1$ as set of specified nodes for $\Gamma_1 \cup R$ and to $N_2$ as set of specified nodes for $\Gamma_2 \cup R$. □

We deduce Lemma 1.8 from Sublemma 1.9 by choosing for $S$ a smooth surface complete intersection of ample hypersurfaces in $X$, containing the curves $\mathcal{C}_{1,x}$ and $E$. We may need to blow-up $X$ to guarantee that the union $\mathcal{C}_{1,x} \cup E$ is nodal in $X$ so that such a smooth surface $S$ exists. The simple connectedness of $S$ follows from that of $X$ which is rationally connected. Assume first that, in Lemma 1.7, the number of attachments of $A$ to $\mathcal{C}_{1,0_J}$ and $B$ to $\mathcal{C}_{1,x}$ are the same. The curves $\mathcal{C}_{1,0_J}$ and $\mathcal{C}_{1,x}$ have the same arithmetic genus since they are fibers of the family (11), and the curves $\mathcal{C}_{2,0} = \mathcal{C}_{1,0_J} \cup A$ and $\mathcal{C}_{2,\infty} = \mathcal{C}_{1,x} \cup B$ have the same arithmetic genus since they are fibers of the family (12). It follows that, in that case, the two partial normalization maps $f_A : A \to E$ and $f_B : B \to E$ normalize the same number of points of $E$, so that Sublemma 1.9 applies directly to $A = \Gamma_1$, $B = \Gamma_2$, $E = \Gamma$. In general, we reduce to the previous case by gluing an extra component to both $\mathcal{C}_{2,0}$ and $\mathcal{C}_{2,\infty}$ and by modifying in an adequate way the family $\mathcal{C}_2$ of (12) using Lemma 1.10 below.

**Step 3.** *Attaching curves to glue the families.* We now have three families $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ respectively over $W_1$, $W_2$ and $W_3$, with

- respective special fibers $\mathcal{C}_{1,0_J}$, $\mathcal{C}_{1,x}$ over $0_J$, $x \in W_1$,

- respective special fibers $\mathcal{C}_{1,0_J} \cup A$, $\mathcal{C}_{1,x} \cup B$ over $0, \infty \in W_2$, and
- respective special fibers $\mathcal{C}_{1,x} \cup B \cup R$, $\mathcal{C}_{1,x} \cup A \cup R$ over $w_2, w_3 \in W_3$.

In other words, if we make correspond 0 with $0_J$, $\infty$ with $w_2$, and $w_3$ with $x$, the corresponding special fibers of the families $\mathcal{C}_1$ and $\mathcal{C}_2$ coincide if we glue $A$ to the fibers of $f_1$, the corresponding special fibers of the families $\mathcal{C}_2$ and $\mathcal{C}_3$ coincide if we glue $R$ to the fibers of $f_2$, and the corresponding special fibers of the families $\mathcal{C}_1$ and $\mathcal{C}_3$ coincide if we glue $A \cup R$ to the fibers of $f_2$. Furthermore, by items 3 and 4 of Lemma 1.8, we know that the gluings for the corresponding special fibers are made at the same number of points.

We now explain how we can modify the families above in order to make them actually having the same corresponding special fibers. A key point is the following result which allows to attach curves in families, without changing the Abel-Jacobi map.

**Lemma 1.10.** *Let $Y$ be a smooth projective threefold and let $\phi : \mathcal{C} \to Y$, $p : \mathcal{C} \to B$ be a family of stable maps with value in $Y$, that we assume to be injective for simplicity, parameterized by a connected basis $B$, let $T \subset Y$ be a curve meeting two fibers $\mathcal{C}_{b_1}$, $\mathcal{C}_{b_2}$ along given sets $N_1$, resp. $N_2$ of points of the same cardinality. Then for an adequate choice of curve $D \subset Y$ there exists a family $\phi' : \mathcal{C}' \to Y$, $p' : \mathcal{C}' \to B^0$, where $B^0$ is a Zariski open set of $B$ containing $b_1$ and $b_2$, with the following properties.*

1. *The fibers of the family $\mathcal{C}'$ are obtained by gluing curves to the fibers of the family $\mathcal{C}$.*
2. *The glued curves have constant rational equivalence class. In particular, the Abel-Jacobi maps of the families $\mathcal{C}'$ and $\mathcal{C}$ are equal.*
3. *Over $b_1$, resp. $b_2$, the fiber $\mathcal{C}'_{b_1}$, resp. $\mathcal{C}'_{b_2}$, is a union $\mathcal{C}_{b_1} \cup_{N_1} T \cup D$, resp. $\mathcal{C}_{b_2} \cup_{N_2} T \cup D$ (the maps $\phi'_{b_i}$ coincide with $\phi_{b_i}$ on $\mathcal{C}_{b_i}$ and are the inclusion maps on the other components).*

A key point for us is the fact that the same curve $D$ can be used for different families, as it is here just to give more positivity.

*Proof of Lemma 1.10.* Let $S \subset Y$ be a smooth projective surface containing $T$ and in general position otherwise. We write

$$\mathcal{C}_{b_1} \cap S = N_1 \cup N'_1,\ \mathcal{C}_{b_2} \cap S = N_2 \cup N'_2,$$

and we choose a very ample linear system of curves on $S$, which contains a curve of the form $T \cup D$, intersecting $\mathcal{C}_{b_1}$ along $N_1 \cup N'_1$ (and possibly other points), and $\mathcal{C}_{b_2}$ along $N_2 \cup N'_2$ (and possibly other points). The general curve $\mathcal{C}_b$, $b \in B$, intersects $S$ along a set $N''_b$ deforming $N_1 \cup N'_1$, and, as the linear system $|T + D|$ is chosen ample enough on $S$, there is a projective bundle over $B$ parameterizing curves $D'_b$ in $|T + D|$ passing through $N''_b$. We choose a rational section $b \mapsto D'_b$ of this projective bundle which is well-defined on a Zariski open set of $B$ containing $b_1$ and $b_2$ and passes through the point parameterizing $T \cup D$ at both points, and attach $D'_b$ to $\mathcal{C}_b$ along $N''_b$. As all the glued components belong to the same linear system on $S$, they all have the same rational equivalence class in $Y$. □

We apply Lemma 1.10 to the three families already constructed. The curve $R$ being as in 2 of Lemma 1.8, this lemma provides us with a family of stable maps

$$p'_1 : \mathcal{C}'_1 \to W_1,\ f'_1 : \mathcal{C}'_1 \to X$$

parameterized by $W_1$ (or a Zariski dense open set of it containing its two special fibers), with special fibers $\mathcal{C}_{1,0_J} \cup A \cup R \cup D_1$ over $0_J$ and $\mathcal{C}_{1,x} \cup A \cup R \cup D_1$ over $x$.

Similarly, Lemma 1.10 provides us with a family of stable maps

$$p'_2 : \mathcal{C}'_2 \to W_2,\ f'_2 : \mathcal{C}'_1 \to X$$

parameterized by $W_2$ (or a Zariski dense open set of it containing its two special fibers), with special fibers $\mathcal{C}_{2,0} \cup R \cup D_2 = \mathcal{C}_{1,0_J} \cup A \cup R \cup D_2$ over $0 \in \mathbb{P}^1$ and $\mathcal{C}_{2,\infty} \cup R \cup D_2 = \mathcal{C}_{1,x} \cup B \cup R \cup D_2$ over $x$.

Finally, Lemma 1.10 provides us with a family of stable maps

$$p'_3 : \mathcal{C}'_3 \to W_3,\ f'_3 : \mathcal{C}'_3 \to X$$

parameterized by $W_3$ (or a Zariski dense open set of it containing its two special fibers), with special fibers $\mathcal{C}_{3,w_2} \cup D_3 = \mathcal{C}_{1,x} \cup B \cup R \cup D_3$ over $w_2 \in W_3$ and $\mathcal{C}_{3,w_3} \cup D_3 = \mathcal{C}_{1,x} \cup A \cup R \cup D_3$ over $w_3 \in W_3$.

There is a lot of freedom in choosing the curves $D_i$, as shows the proof of Lemma 1.10, hence we can do $D_1 = D_2 = D_3$. The three families $\mathcal{C}'_1$, $\mathcal{C}'_2$, $\mathcal{C}'_3$ then have identical corresponding special fibers, hence they glue to give a family of stable maps parameterized by the union

$$B = W_1 \cup_{0_J=0,\, x=w_3} (W_2 \cup_{\infty=w_2} W_3).$$

Theorem 1.6 is then almost proved, except for property 4 (unobstructedness), which we will achieve now by a further modification of the family.

**Step 4.** *Realizing unobstructedness.* Unobstructedness is obtained by applying the following

**Proposition 1.11.** *Let $X$ be a rationally connected smooth projective variety and let*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & X \\ p \downarrow & & \\ W & & \end{array} \tag{14}$$

*be a family of stable maps to $X$ parameterized by a quasiprojective variety $W$. Let $w_1, \ldots, w_N$ be points of $W$. Then there exists a family of stable maps*

$$\begin{array}{ccc} \mathcal{C}' & \xrightarrow{f'} & X \\ p' \downarrow & & \\ W^0 & & \end{array} \tag{15}$$

*parameterized by a Zariski open set $W^0$ of $W$ containing the points $w_i$, with the following properties.*

1. *For any $t \in W^0$, the stable map $f'_t : \mathcal{C}'_t \to X$ is unobstructed.*
2. *For any $t \in W^0$, the stable map $f'_t : \mathcal{C}'_t \to X$ is of the form*

$$\mathcal{C}'_t = \mathcal{C}_t \cup \mathcal{E}_t,\ f'_{t|\mathcal{C}_t} = f_t,$$

*and the 1-cycles $f'_{t*}\mathcal{E}_t$ are all rationally equivalent in $X$.*

*Proof.* This proposition is essentially proved in [22] although maybe not stated with this degree of generality, so we just sketch the proof. First of all, we can assume without loss of generality (we can indeed replace $X$ by $X \times \mathbb{P}^r$) that the maps $f_t$ are generically injective and attach to the generic curve $\mathcal{C}_t$ some complete intersections $C_1, \ldots, C_M$ of ample hypersurfaces in $X$ at the $M$ points $x_1, \ldots, x_m$ in the intersection $f_t(\mathcal{C}_t) \cap Y$, where $Y$ is a sufficiently ample hypersurface in $X$. These complete intersection curves have constant rational equivalence class. It is proved by Hartshorne and Hirschowitz in [11] that this operation, which can be done over a Zariski open set of $W$ containing the $N$ given points, provides for $M$ large enough and for generic choices of attaching directions, local complete intersection curves $f''_t : \mathcal{C}''_t \hookrightarrow X$ with the property that the normal bundle $N_{\mathcal{C}''_t/X}$ restricted to $\mathcal{C}_t$ is arbitrarily positive, hence satisfies $H^1(\mathcal{C}_t, N_{\mathcal{C}''_t/X|\mathcal{C}_t}) = 0$. Unfortunately, this is not enough to guarantee

that the curves $\mathcal{C}_t''$ are unobstructed, since we need for this the stronger property that the normal bundle $N_{\mathcal{C}_t''/X}$ has $H^1(\mathcal{C}_t'', N_{\mathcal{C}_t''/X}) = 0$. This stronger condition is achieved by gluing as in [13] very free rational curves $R_{ij}$ to each of the complete intersection components $C_i$. We can perform first this second construction over the function field of the variety parameterizing curves complete intersections of very ample hypersurfaces of given degree. This variety is rational so the union $\cup_j R_{ij}$ of the rational legs we add to $C_i$ is a 1-cycle of constant rational equivalent class in $X$. By a slight modification of [13], the curve

$$\mathcal{C}_t' = \mathcal{C}_t'' \cup_{ij} R_{ij} = \mathcal{C}_t \cup C_1 \cup \ldots \cup C_m \cup_{ij} R_{ij}$$

is unobstructed if we add enough free rational curves in general position, so 1 is satisfied. By construction, this curve also satisfies 2, proving the proposition. □

By applying Proposition 1.11 to the family constructed in Step 3, we finally get a family satisfying all the properties stated in Theorem 1.6, which is thus proved.

# 2 Further results and questions

## 2.1 Stronger representability results

Recall from [23], [21] that a codimension 2-cycle $Z_J \in \mathrm{CH}^2(J^3(X) \times X)$ is a universal codimension 2 cycle if it satisfies

$$\Phi_{Z_J} = Id_{J^3(X)} : \mathrm{Alb}(J^3(X)) = J^3(X) \to J^3(X).$$

Similarly, for any smooth projective variety $M$ of dimension $n$, a universal 0-cycle parameterized by $A := \mathrm{Alb}(M)$ is a codimension $n$ cycle

$$Z_A \in \mathrm{CH}^n(A \times M)$$

with $Z_{A*}(a) \in \mathrm{CH}_0(M)_{\mathrm{hom}}$ for any $a \in \mathrm{Alb}(M)$ and such that

$$\mathrm{alb}_M \circ Z_{A*} : \mathrm{Alb}(A) = \mathrm{Alb}(M) \to \mathrm{Alb}(M)$$

is the identity. We observed in [23] that the existence of a universal codimension 2-cycle on $X$ is a necessary criterion for the existence of a cohomological decomposition of the diagonal of $X$, and hence for the stable rationality of $X$. We proved in [21] that some rationally connected threefolds do not admit a universal codimension 2 cycle (this is known to happen for a very general quartic double solid with seven nodes by [21] and it follows that this also holds for a very general quartic double solid). We get the following consequence of Theorem 0.7.

**Corollary 2.1.** *Let $X$ be a rationally connected threefold and let $M$, $Z_M$ be as in Theorem 0.7. Then $X$ admits a universal codimension 2 cycle if $M$ admits a universal 0-cycle.*

*In particular, for a very general quartic double solid $X$ and any smooth projective variety $M$ equipped with a codimension 2 cycle $Z_M \in \mathrm{CH}^2(M \times X)$ such that $\Phi_{Z_M} : \mathrm{Alb}(M) \to J^3(X)$ is an isomorphism, $M$ does not admit a universal 0-cycle.*

**Remark 2.2.** Whether the converse holds for an adequate choice of pair $(M, Z_M)$ is related to the question whether we can choose $(M, Z_M)$ universally generating, that we are going to discuss below.

*Proof of Corollary 2.1.* The proof is immediate since a universal 0-cycle

$$Z_{\mathrm{univ},M} \in \mathrm{CH}^{\dim M}(\mathrm{Alb}(M) \times M)$$

gives rise to a codimension 2-cycle

$$Z_{\mathrm{univ},X} := Z_M \circ Z_{\mathrm{univ},M} \in \mathrm{CH}^2(\mathrm{Alb}(M) \times X)$$

which is universal for $X$ using the fact that $\Phi_{Z_M}$ induces an isomorphism between $\mathrm{Alb}(M)$ and $J^3(X)$. □

We now prove Corollary 0.15 stated in the introduction.

*Proof of Corollary 0.15.* Let $M_0$ be a smooth projective variety not admitting a universal 0-cycle. Consider the variety

$$M = M_0 \times \mathrm{Alb}(M_0).$$

As $\mathrm{Alb}(M) = \mathrm{Alb}(M_0) \oplus \mathrm{Alb}(M_0)$, it contains the abelian subvariety $\Delta = \mathrm{Diag}(\mathrm{Alb}(M_0)) \subset \mathrm{Alb}(M_0) \oplus \mathrm{Alb}(M_0)$ which is obviously a direct summand of $\mathrm{Alb}(M)$ since

$$(\mathrm{Alb}(M_0) \times \{0\}) \oplus \Delta = \mathrm{Alb}(M).$$

We claim that there does not exist a smooth projective variety $N$ and a pair of correspondences $\Gamma_1 \in \mathrm{CH}^d(N \times M)$, $\Gamma_2 \in \mathrm{CH}^{d'}(M \times N)$, where $d = \dim M$, $d' = \dim N$, such that

$$\Gamma_{2*} \circ \Gamma_{1*} = Id_{\mathrm{Alb}(N)}$$

and $\mathrm{Im}\,(\Gamma_{1*} : \mathrm{Alb}(N) \to \mathrm{Alb}(M)) = \Delta$. Indeed, denote by $i$ the inclusion $\mathrm{Alb}(M_0) = \{m_0\} \times \mathrm{Alb}(M_0) \hookrightarrow M$. Assume $N$, $\Gamma_1$, $\Gamma_2$ as above exist. Then consider the correspondence

$$\Gamma := pr_1 \circ \Gamma_1 \circ \Gamma_2 \circ i \in \mathrm{CH}^d(\mathrm{Alb}(M_0) \times M_0),\ d = \dim M_0.$$

Then $\Gamma_* : \mathrm{Alb}\, M_0 \to \mathrm{Alb}\, M_0$ is the identity, which provides a universal 0-cycle for $M_0$ and is a contradiction. □

To compare various notions of representability for $J^3(X)$, note that, if $X$ admits a universal codimension 2 cycle, then we can take for $M$ in Theorem 0.7 the intermediate Jacobian $J^3(X)$ itself, and for $Z_M$ any universal codimension 2 cycle. Thus the existence of a universal codimension 2 cycle is a stronger statement than Theorem 0.7, but it is too strong and not satisfied in general. This shows that Theorem 0.7 is almost optimal. However, it still leaves open a slightly stronger statement, coming from Shen's notion of universal generation [19], namely

**Definition 2.3.** *(M. Shen [19].) A codimension $k$ cycle $Z_M \in \mathrm{CH}^k(M \times X)$, where $M$ is smooth projective of dimension $d$, is universally generating if it has the property that for any smooth projective variety $Y$ and codimension $k$ cycle $Z_Y \in \mathrm{CH}^k(Y \times X)$, there exists a codimension $d$ cycle $W_Y \in \mathrm{CH}^d(M \times Y)$ such that*

$$Z_{Y*} = Z_{M*} \circ W_{Y*} : \mathrm{CH}_0(Y)_{\mathrm{hom}} \to \mathrm{CH}^k(X)_{\mathrm{alg}}.$$

We now consider the case where $k = 2$ and $X$ is rationally connected, so that the Abel-Jacobi map $\Phi_X : \mathrm{CH}^2(X)_{\mathrm{alg}} \to J^3(X)$ is an isomorphism by Theorem 0.6. An easy consequence of Theorem 0.7 is the following result.

**Corollary 2.4.** *Let $X$ be a rationally connected threefold. If $X$ admits a universally generating family $(M, Z_M)$, then $J^3(X)$ is a direct summand of* $\mathrm{Alb}(M)$.

*Proof.* We know by Theorem 0.7 that there also exists a smooth projective variety $N$ and a codimension 2 cycle $Z_N \in \mathrm{CH}^2(N \times X)$ such that $\Phi_{Z_N} : \mathrm{Alb}(N) \to J^3(X)$ is an isomorphism. By the universal generation property, there exists a cycle $\Gamma \in \mathrm{CH}^d(N \times M)$, $d = \dim M$, such that

$$\Phi_{Z_N} = \Phi_{Z_M} \circ \Gamma_* : \mathrm{Alb}(N) \to J^3(X).$$

It thus follows that $\Gamma_*(\mathrm{Alb}(N)) \subset \mathrm{Alb}(M)$ is isomorphic to $J^3(X)$ and $\Phi_{Z_M}$ then provides a retraction of $\mathrm{Alb}(M)$ onto $J^3(X)$, making it into a direct summand. □

In fact, we can prove a better statement:

**Theorem 2.5.** *Assume a rationally connected threefold $X$ admits a universally generating family $(N, Z_N)$. Then $X$ admits a universally generating family $(M, Z_M)$ such that $\Phi_{Z_M} : \mathrm{Alb}(M) \to J^3(X)$ is an isomorphism.*

*Proof.* The proof will be done in several steps.

**Step 1**. We first prove by mimicking the proof of Theorem 0.7 the following

**Proposition 2.6.** *Let $(N, Z_N)$ be a pair consisting of a smooth projective variety $N$ and a codimension 2 cycle $Z_N \in \mathrm{CH}^2(N \times X)$. Then there exists a smooth projective variety $N'$, a codimension 2 cycle $Z_{N'} \in \mathrm{CH}^2(N' \times X)$, and a rational map*

$$\phi : N \dashrightarrow N'$$

*with the following properties.*

*(1)* $\Phi_{Z_N} = \Phi_{Z_{N'}} \circ \phi : N \dashrightarrow J^3(X)$.

*(2)* $\mathrm{Ker}\,(\Phi_{Z_N} : \mathrm{Alb}(N) \to J^3(X)) \subset \mathrm{Ker}\,(\phi_* : \mathrm{Alb}(N) \to \mathrm{Alb}(N'))$.

Note that (1) and (2) in fact imply that

$$\mathrm{Ker}\,(\Phi_{Z_N} : \mathrm{Alb}(N) \to J^3(X)) = \mathrm{Ker}\,(\phi_* : \mathrm{Alb}(N) \to \mathrm{Alb}(N')).$$

**Step 2**. Assume now that $(N, Z_N)$ is universally generating. Then for any smooth projective variety $Y$ and correspondence $\Gamma \in \mathrm{CH}^2(Y \times X)$, there exists a correspondence $\Gamma' \in \mathrm{CH}^n(Y \times N)$, $n = \dim N$, such that

$$\Phi_\Gamma = \Phi_{Z_N} \circ \Gamma'_* : \mathrm{Alb}(Y) \to J^3(X).$$

Let $\Gamma'' := \phi \circ \Gamma' \in \mathrm{CH}^{n'}(Y \times N)$, $n' = \dim N'$. Then by (1) above, we have

$$\Phi_\Gamma = \Phi_{Z_N} \circ \Gamma'_* = \Phi_{Z_{N'}} \circ \phi_* \circ \Gamma'_* = \Phi_{Z_{N'}} \circ \Gamma''_* : \mathrm{Alb}(Y) \to J^3(X).$$

It follows that $(N', Z_{N'})$ is also universally generating. Unfortunately, we can have $\mathrm{Alb}(N') \neq J^3(X)$ so we have to modify the construction in an adequate way. We observe that, as proved in Corollary 2.4, the image of $\phi_* : \mathrm{Alb}(N) \to \mathrm{Alb}(N')$ is a direct summand $A$ of $\mathrm{Alb}(N')$ which is isomorphic to $J^3(X)$. By definition of $A$, the composite $\mathrm{alb}_{N'} \circ \phi : N \to \mathrm{Alb}(N')$ takes value in $A$. Our next proposition is the following complement to Lemma 1.3.

**Proposition 2.7.** *Let $N$, $N'$ be two smooth projective varieties and $\phi : N \to N'$ be a morphism (or rational map). Let $A := \mathrm{Im}\,(\phi_* : \mathrm{Alb}(N) \to \mathrm{Alb}(N')) \subset \mathrm{Alb}(N')$. Then there exists a smooth projective variety $N''$, a rational map $\psi : N \dashrightarrow N''$, and a correspondence $\gamma \in \mathrm{CH}^{n'}(N'' \times N')$ with the following properties:*

*(3)* $\gamma_* \circ \psi_* = \phi_* : \mathrm{Alb}(N) \to \mathrm{Alb}(N')$.

*(4)* $\gamma_* : \mathrm{Alb}(N'') \to \mathrm{Alb}(N')$ *is injective with image $A$.*

Let now $M = N''$, $Z_M := Z_{N'} \circ \gamma \in \mathrm{CH}^2(M \times X)$. By Proposition 2.6, (1) and Proposition 2.7 (3), we get by the same argument as above that $(M, Z_M)$ is universally generating. Using Proposition 2.7 (4), we find that $\Phi_{Z_M} : \mathrm{Alb}(M) \to J^3(X)$ is injective. This completes the proof of Theorem 2.5 admitting Proposition 2.7. □

*Proof of Proposition 2.7.* Let $n' = \dim N'$ and let $L$ be a very ample line bundle on $N'$. A choice of $n'$ general sections of $L$ gives us a dominant rational map $f : N' \dashrightarrow \mathbb{P}^{n'-1}$ whose general fiber $C_t$, $t \in \mathbb{P}^{n'-1}$, is a smooth complete intersection of $n'-1$ hypersurfaces in $|L|$. Let $r$ be any integer and let $P_r$ be any smooth projective model of the relative Picard variety $\mathrm{Pic}^r(N'/\mathbb{P}^{n'-1})$. Let $x_0 \in N'$ be one of the base points of the linear system. We thus have a rational map $a_r : P_r \dashrightarrow \mathrm{Alb}(N')$ (which in fact is a morphism) given on the general point $d_t \in \mathrm{Pic}^r(C_t)$ by

$$a_r(d_t) = \mathrm{alb}_{N'}(j_{t*}(d_t - rx_{0,t})),$$

where $j_t$ is the inclusion of $C_t$ in $N'$ and $x_{0,t}$ is the point $x_0$ seen as a point of $C_t$.

Let now $N'' := a_r^{-1}(A_s)$, where $A_s \subset \mathrm{Alb}(N')$ is a general translate of $A$. We first claim that $\mathrm{Alb}(N'') = A$. This follows indeed from the fact that the general fiber of $a_r$ is a fibration over $\mathbb{P}^{n'-1}$ with fiber the abelian variety $\mathrm{Ker}\,(j_{t*} : J(C_t) \to \mathrm{Alb}(N'))$. Indeed, by Lefschetz theorem on hyperplane sections, $\mathrm{alb}_{N'} \circ j_{t_*} : J(C_t) \to \mathrm{Alb}(N')$ is surjective

(with connected fibers) with kernel the abelian variety built on the weight 1 Hodge structure $H_1(C_t,\mathbb{Z})_{\mathrm{van}} := \mathrm{Ker}\,(j_{t*} : H_1(C_t,\mathbb{Z}) \to H_1(N',\mathbb{Z}))$. Furthermore the corresponding local system of vanishing degree 1 cohomology defined on a Zariski open set of $\mathbb{P}^{n'-1}$ has no nonzero global section by Deligne's invariant cycle theorem.

The isomorphism between $\mathrm{Alb}(N'')$ and $A$ is induced by the natural correspondence $\gamma$ between $N''$ and $N'$ constructed using the fact that $N''$ parameterizes divisors on curves in $N'$. It now remains to construct the desired rational map $\psi$ from $N$ to $N''$ for an adequate choice of $s$. We observe that for two general points $x_1$, $x_2$ in $N'$, the element $\mathrm{alb}(x_1-x_2)$ of $\mathrm{Alb}(N')$ has the property that the set of points $m\mathrm{alb}_{N'}(x_1-x_2)$ is Zariski dense in $\mathrm{Alb}(N')$. If $L$ is sufficiently ample, we can thus in the above construction choose $s = m\mathrm{alb}_{N'}(x_1-x_2)$ for some $m$, where $x_1$, $x_2$ are two base-points of our linear system, since the right hand side is a general point of $A$. Let now $D = \deg L^{n'}$ be the degree of $L_t := L_{|C_t}$. For an integer $k$ large enough, we set $r = kD+1$ and we get a rational embedding

$$f : N' \dashrightarrow P_r,$$

$$x \mapsto x + kL_t + m(x_{1,t} - x_{2,t})_t \in \mathrm{Pic}^r(X_t),\ t := f(x),$$

where $m$, $x_1$, $x_2$ are as above and $x_{i,t}$ for $i = 1, 2$ is the point $x_i$ seen as a point of the curve $C_t$.

The composite map

$$\psi = f \circ \phi$$

is well-defined for a general choice of linear system, and we observe that $\psi(N) \subset N'' = a_r^{-1}(A_s)$ since $A = \phi(\mathrm{Alb}(M))$. □

We conclude this section with the following questions.

**Question 2.8.** *Does any rationally connected threefold admit a universally generating family?*

**Question 2.9.** *Does there exist a rationally connected threefold $X$ which admits a universally generating family of 1-cycles, but does not admit a universal codimension 2 cycle?*

By [21], an affirmative answer to Question 2.8 implies an affirmative answer to Question 2.9.

## 2.2 The case of cubic threefolds and quartic double solids

Let $X$ be a smooth cubic threefold. One has $\dim J^3(X) = 5$ and it is proved in [23] that if the minimal class $\frac{\theta^4}{4!} \in H^8(J^3(X),\mathbb{Z})$ is algebraic, then $X$ admits a universal codimension 2 cycle. This implication is very specific to cubic threefolds and uses the fact that, by [15], for any smooth cubic threefold $X$, there exists a smooth projective variety $M$, and a codimension 2 cycle $Z \in \mathrm{CH}^2(M \times X)$ such that $\Phi_Z : M \to J^3(X)$ is surjective with rationally connected fibers.

Next it is proved in [23] that the two conditions together (the algebraicity of the minimal class and the existence of a universal codimension 2 cycle) imply the existence of a cohomological decomposition of the diagonal for a rationally connected threefold with no torsion in $H^3(X,\mathbb{Z})$. Hence, in the case of the cubic threefold, the algebraicity of the minimal class implies the existence of a cohomological decomposition of the diagonal. We used in turn in [21] these two implications to show that a general desingularized quartic double solid with 7 nodes does not admit a parameterization

$$\Phi_Z : M \to J^3(X)$$

of its intermediate Jacobian with rationally connected fibers. Indeed, its minimal class is algebraic since it has dimension 3, it has no torsion in $H^3$, and it does not admit a cohomological decomposition of the diagonal by the main theorem of [21].

Coming back to the case of the cubic threefold $X$, it is even proved in [24] that if the minimal class $\frac{\theta^4}{4!} \in H^8(J^3(X), \mathbb{Z})$ is algebraic, then $X$ admits a Chow decomposition of the diagonal.

Conversely, the algebraicity of the minimal class of $J^3(X)$ is implied by the existence of a cohomological decomposition of the diagonal of a rationally connected threefold $X$ (see [24]), hence we conclude that for a cubic threefold, the algebraicity of the minimal class is equivalent to the existence of a cohomological (and even Chow-theoretic) decomposition of the diagonal. Despite these results, the following question remains open:

**Question 2.10.** *Let $X$ be a general cubic threefold. Does $X$ admit a universal codimension 2 cycle?*

Turning to the property of the existence of a universally generating family, Mingmin Shen proves the following.

**Theorem 2.11.** *(M. Shen, [19].) Let $X$ be a smooth cubic threefold and $\Sigma_X$ the surface of lines in $X$. Then the universal family of lines in $X$ parameterized by $\Sigma_X$ is universally generating.*

The argument is quite interesting by its nonlinear character. Mingmin Shen associates to a 1-cycle $Z \in \mathrm{CH}^2(X)$, the 0-cycle of bisecants lines to $Z$ and computes the class of the 1-cycle of these lines in $X$.

In the case of a quartic double solid $X$, although we know that $X$ does not admit in general a universal codimension 2 cycle, we do not know if $X$ admits a universally generating family (see Question 2.8). In particular, the following question seems to be open.

**Question 2.12.** *Let $X$ be a smooth quartic double solid. Is the universal family of lines in $X$ universally generating?*

## 2.3 The case of 1-cycles on higher dimensional rationally connected manifolds

We now turn to the case of 1-cycles on higher dimensional rationally connected manifolds. There are two important differences with the case of dimension 3, namely Theorem 0.6 is not true for two reasons. First of all, as mentioned in the introduction, and can be seen when we blow-up a surface inside a fourfold, the $\mathrm{CH}_0$ group of a surface can be present in $\mathrm{CH}_1(X)$, and by [16], we do not expect a Mumford representability result for $\mathrm{CH}_1(X)$, which can be infinite dimensional even when $X$ is rational. In particular, the Bloch-Srinivas injectivity theorem (Theorem 0.6) for the Abel-Jacobi map on codimension 2 cycles algebraically equivalent to 0 is not true for 1-cycles on rationally connected $n$-folds. This was our starting point in the proof of Theorem 1.6. A second phenomenon is the fact that the Abel-Jacobi map

$$\Phi_X : \mathrm{CH}^{n-1}(X) \to J^{2n-3}(X) = J_3(X)$$

for 1-cycles on rationally connected manifolds is a priori not universal (in the sense of [17]) anymore, as it admits the Walker factorization [25], which one can describe as follows: Recall that, inside $H^{2n-3}(X, \mathbb{Z})_{\mathrm{tf}}$, we have the subgroup $N^{n-2}H^{2n-3}(X, \mathbb{Z})_{\mathrm{tf}}$ of homology classes supported on a possibly singular surface $S \subset X$. For a rationally connected $n$-fold, one has

$$H^{2n-3}(X, \mathbb{Q}) = N^{n-2}H^{2n-3}(X, \mathbb{Q})$$

as shows an argument involving the decomposition of the diagonal with $\mathbb{Q}$-coefficients [1], and it follows that

$$N^{n-2}H^{2n-3}(X, \mathbb{Z})_{\mathrm{tf}} \subset H^{2n-3}(X, \mathbb{Z})_{\mathrm{tf}}$$

is a finite index subgroup. This determines a modified intermediate Jacobian

$$J^{2n-3}(X)_{\mathrm{Walker}} := H^{2n-3}(X, \mathbb{C})/(F^{n-1}H^{2n-3}(X) \oplus N^{n-2}H^{2n-3}(X, \mathbb{Z})_{\mathrm{tf}}),$$

which is obviously isogenous to $J^{2n-3}(X)$, and it is proved in [25] that the Abel-Jacobi map $\Phi_X^{n-1}$ factors through

$$\Phi^{n-1}_{X,\mathrm{Walker}} : \mathrm{CH}^{n-1}(X)_{\mathrm{alg}} \to J^{2n-3}(X)_{\mathrm{Walker}}.$$

Note that a factorization of this type holds for the Abel-Jacobi map for cycles on any codimension on any smooth projective variety (see [25]). The following result is the general version of Theorem 0.12.

**Theorem 2.13.** *(Voisin [22].) Let $X$ be a rationally connected $n$-fold. There exists a smooth projective curve $C$ and a cycle $Z \in \mathrm{CH}^{n-1}(C \times X)$ such that*

$$\Phi_{Z,\mathrm{Walker}} : J(C) \to J^{2n-3}(X)_{\mathrm{Walker}}$$

*is surjective with connected fibers. Equivalently the induced morphism $[Z]_* : H^1(C, \mathbb{Z}) \to N^{n-2}H^{2n-3}(X, \mathbb{Z})_{\mathrm{tf}}$ is surjective, so $N^{n-2}H^{2n-3}(X, \mathbb{Z})_{\mathrm{tf}}$ is of strong coniveau $\geq n-2$.*

As mentioned above, Theorem 0.6 is not true anymore for 1-cycles on rationally connected manifolds for $n \geq 4$ but one has the following result for torsion 1-cycles.

**Theorem 2.14.** *(Suzuki [20].) The Walker Abel-Jacobi map $\Phi^{n-1}_{X,\mathrm{Walker}}$ is injective on the torsion subgroup of 1-cycles algebraically equivalent to zero modulo rational equivalence of a rationally connected $n$-fold.*

Theorem 0.7 and Theorem 2.14 suggest the following question

**Question 2.15.** *Let $X$ be a smooth projective rationally connected $n$-fold over $\mathbb{C}$. Does there exist a smooth complex projective manifold $M$ and a family of 1-cycles*

$$Z \in \mathrm{CH}^{n-1}(M \times X)$$

*such that the induced Walker Abel-Jacobi map*

$$\Phi_Z^{\mathrm{Walker}} := \Phi^{n-1}_{X,\mathrm{Walker}} \circ Z_* : \mathrm{Alb}(M) \to J^{2n-3}(X)_{\mathrm{Walker}} \tag{16}$$

*is an isomorphism?*

Theorem 0.7 answers affirmatively this question when $n = 3$. As explained in [22], the stated property is satisfied by any $X$ admitting an integral cohomological decomposition of the diagonal and this is thus an obstruction to stable rationality (see [23]). Furthermore, if $X$ has an integral cohomological decomposition of the diagonal, its integral cohomology of positive degree is of strong coniveau $\geq 1$, and the Walker Abel-Jacobi map coincides with the Griffiths Abel-Jacobi map.

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