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Lower bounds for the house in some radical extensions

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1 Introduction.

Let us consider the infinite extension

$$L = \mathbb{Q}(2^{1/2}, 2^{1/3}, 2^{1/4}, \dots).$$

Obviously, there exist algebraic numbers α in L of positive but arbitrary small (absolute, logarithmic) Weil height. Indeed, for a positive integer n we have:

$$h(2^{1/n}) = \frac{1}{n}h(2) = \frac{\log 2}{n}. \quad (1.1)$$

The following special (but still open) case of a conjecture of Rémond suggests that (1.1) are the only exceptions. To state this conjecture, we need a couple of standard definitions. Let $\Gamma \subset \overline{\mathbb{Q}}^*$ be a subgroup. We define its division group Γ^{div} as

$$\Gamma^{\text{div}} = \{\alpha \in \overline{\mathbb{Q}}^* \text{ such that } \exists n \in \mathbb{Z}_{\geq 1}, \alpha^n \in \Gamma\}.$$

We say that Γ is of finite rank if there exists a finitely generated subgroup Γ_0 of Γ such that Γ/Γ_0 is torsion, that is if $\Gamma \subseteq \Gamma_0^{\text{div}}$.

Conjecture 1.1 ([10], conjecture 3.4 and [1], conjecture 1.2). *Let $\Gamma \subset \overline{\mathbb{Q}}^*$ be a subgroup of finite rank. Then there exists a constant $c_\Gamma > 0$ such that for any $\alpha \in \mathbb{Q}(\Gamma)^* \setminus \Gamma^{\text{div}}$ we have $h(\alpha) \geq c_\Gamma$.*

A partial result in the direction of Conjecture 1.1 was proved in [1], where we give a positive answer for the rank 1 subgroup¹ $\langle \zeta_{3^t}, 2^{1/3^t} \rangle_{t \geq 1}$ of $\langle 2 \rangle_{\text{div}}$.

Conjecture 1.1 with $\Gamma = \langle 2^{1/2}, 2^{1/3}, 2^{1/4}, \dots \rangle$ predicts that for each $\alpha \in L^*$ we have either $\alpha = \pm 2^{k/n}$ for some integers k and $n \geq 1$, or $h(\alpha) \geq c$ for some absolute constant $c > 0$. Now we restrict ourselves to algebraic integers α in L

¹Here and in what follows, we denote by ζ_n a primitive n th root of unity.

and we consider instead of the height the so-called house of α , *i.e.* the maximum modulus $\overline{|\alpha|}$ of its algebraic conjugates. Since

$$h(\alpha) \leq \log \overline{|\alpha|}, \quad (1.2)$$

we again expect either $\alpha = \pm 2^{k/n}$ for some integers $k \geq 0$ and $n \geq 1$, or $\overline{|\alpha|} \geq C$ for some $C > 1$.

Note that lower bounds for the house can be easier to prove than lower bounds for the height, as the following fundamental example shows. Let $\gamma \in \overline{\mathbb{Q}}^*$ be an algebraic integer of degree d , not a root of unity. Lehmer conjecture (1933) predicts the lower bound $h(\gamma) \geq c/d$ for some absolute $c > 0$. Schinzel-Zassenhaus conjecture (1965) predicts the lower bound $\overline{|\gamma|} \geq C^{1/d}$ for some absolute $C > 1$. Thus Schinzel-Zassenhaus is a consequence of Lehmer, and for a long time it was (wrongly) believed that both were of the same difficulty. In spite of that, Dimitrov [7] recently proved in a very beautiful way Schinzel-Zassenhaus (with $C = 2^{1/4}$) while Lehmer is still open. This may suggest that, also in our setting, lower bounds of the house for integers in L can be more easily proved than lower bounds for the height. And indeed, for the special situation we are interested in, we get:

Theorem 1.2. *Let $\alpha \in \mathbb{Z}[2^{1/2}, 2^{1/3}, 2^{1/4}, \dots]$ be non-zero. Then either $\alpha = \pm 2^{k/n}$ for some integers $k \geq 0$ and $n \geq 1$, or*

$$\overline{|\alpha|} \geq \sqrt{2}.$$

Note that $\mathbb{Z}[2^{1/2}, 2^{1/3}, 2^{1/4}, \dots]$ is not the full ring of integers in L , and we are unable to extend the result of Theorem 1.2 to it, see Remark 2.1.

We now state a general conjecture on lower bounds for the house in radical extensions.

Conjecture 1.3. *Let $\gamma_1, \dots, \gamma_r$ be non-zero algebraic numbers and let α be a non-zero algebraic integer in the infinite extension $\mathbb{Q}(\zeta_n, \gamma_1^{1/n}, \dots, \gamma_r^{1/n})_{n \geq 1}$. Then either $\alpha^n = \gamma_1^{k_1} \dots \gamma_r^{k_r}$ for some integers n, k_1, \dots, k_r with $n \geq 1$, or*

$$\overline{|\alpha|} \geq c$$

with $c = c(\gamma_1, \dots, \gamma_r) > 1$.

By inequality (1.2), this conjecture follows from Conjecture 1.1. We shall give a partially positive answer to it. As in Theorem 1.2, we cannot reach the full ring of integer of $\mathbb{Q}(\zeta_n, \gamma_1^{1/n}, \dots, \gamma_r^{1/n})_{n \geq 1}$, but we need to restrict ourselves to the subring $\mathbb{Z}[\zeta_n, \gamma_1^{1/n}, \dots, \gamma_r^{1/n}]_{n \geq 1}$, assuming $\gamma_1, \dots, \gamma_r$ algebraic integers. To simplify, we also assume $\gamma_j \in \mathbb{Z}$. It seems to be reasonable that this last restriction could be removed at the cost of some technical difficulties. On the other hand, we can consider roots of infinitely many integers.

Theorem 1.4. *Let $\alpha \in \mathbb{Z}[\zeta_n, a^{1/n}]_{a, n \geq 1}$ be non-zero. Then either there exists an integer $n \geq 1$ such that $\alpha^n \in \mathbb{Z}$, or*

$$|\alpha| \geq \sqrt{\frac{1 + \sqrt{5}}{2}}.$$

The plan of the paper is as follows. In section 2 we prove Theorem 1.2. The proof of Lemma 2.2, an elementary L_1 -mean estimate which is one of the ingredient of our method, is postponed at the the end of this section. In section 3 we prove our main technical result, Proposition 3.2, and we deduce Theorem 1.4 from it. We conclude this article with a more speculative section 4 where we state some questions and remarks on lower bounds for the house.

Acknowledgement. We are indebted to Gaël Rémond, who suggests us a simple proof of Lemma 2.2 with a better (and optimal) constant, and to Lukas Pottmeyer for Remark 4.2.

2 Proof of Theorem 1.2.

Take a non-zero $\alpha \in \mathbb{Z}[2^{1/2}, 2^{1/3}, 2^{1/4}, \dots]$ outside the multiplicative group generated by ± 1 and by the n th roots of 2. Thus $\alpha \in \mathbb{Z}[2^{1/N}]$ for some positive integer N and we can write $\alpha = a_0 + a_1 2^{1/N} + \dots + a_{N-1} 2^{(N-1)/N}$ for some integers a_0, \dots, a_{N-1} . We consider the trace $\text{Tr}: \mathbb{Q}(2^{1/N}) \rightarrow \mathbb{Q}$. Our result rests on this elementary remark²:

$$\frac{1}{N} \text{Tr}(2^{j/N}) = \begin{cases} 1 & \text{if } j = 0; \\ 0 & \text{if } 1 \leq j < N. \end{cases}$$

We distinguish two cases.

First case. There exists an index j such that $|a_j| \geq 2$. Since all the conjugates of $2^{-j/N}$ have absolute value $2^{-j/N} \leq 1$ we have:

$$2 \leq |a_j| = \frac{1}{N} |\text{Tr}(2^{-j/N} \alpha)| \leq |\overline{\alpha}|.$$

²This orthogonality relation is also one of the ingredient of an other result on height, see [5], proof of Theorem 1.

Thus we get in this case the better lower bound $|\alpha| \geq 2$.

Second case. For each j we have $|a_j| \leq 1$. Then, by assumption, there exist two distinct indexes j_0, j_1 such that $a_{j_0} = \pm 1$ and $a_{j_1} = \pm 1$. Choosing the sign in a right way we get

$$\frac{1}{N} |\operatorname{Tr}((2^{-j_0/N} \pm 2^{-j_1/N})\alpha)| = 2.$$

We now prove an upper bound for the quantity on the left hand side. Let $k = N/\gcd(j_1 - j_0, N)$. We notice that $k \geq 2$ since $j_0 \neq j_1$. Thus

$$\frac{1}{N} |\operatorname{Tr}((2^{-j_0/N} \pm 2^{-j_1/N})\alpha)| \leq |\alpha| \frac{1}{k} \sum_{l=0}^{k-1} |2^{-j_0/N} \pm 2^{-j_1/N} \exp(2\pi li/k)|.$$

We quote the following elementary estimate (see Lemma 2.2 below), which holds since $2^{-j_0/N}, 2^{-j_1/N} \leq 1$ and $k \geq 2$:

$$\frac{1}{k} \sum_{l=0}^{k-1} |2^{-j_0/N} \pm 2^{-j_1/N} \exp(2\pi li/k)| \leq \sqrt{2}. \quad (2.1)$$

We get

$$|\alpha| \geq 2/\sqrt{2} = \sqrt{2}$$

as required.

Remark 2.1. *The above argument cannot be extended to the full ring of integers, since the trace of an arbitrary integer in $\mathbb{Q}(2^{1/N})$ could be 1. Consider the following example taken from [6, Theorem 5.1]. Let $p \geq 3$ be a Wieferich prime to base 2, that is a prime satisfying the congruence $2^{p-1} \equiv 1 \pmod{p^2}$, as for instance $p = 1093$. Then $\gamma = (2^{1/p} - 2)^{p-1}/p \notin \mathbb{Z}[2^{1/p}]$ is an integer of trace 2^{p-1} . Taking a suitable linear combination over \mathbb{Z} of γ and 1, we get an integer $\alpha \in \mathbb{Q}(2^{1/p})$ of trace 1.*

We now state and prove a lemma which generalizes the inequality (2.1) we have used in the proof of Theorem 1.2.

Lemma 2.2. *Let x, y be complex numbers and let $k \geq 2$ be an integer. Then*

$$\frac{1}{k} \sum_{l=0}^{k-1} |x + y \exp(2\pi li/k)| \leq \sqrt{|x|^2 + |y|^2}.$$

Proof. By Cauchy-Schwarz,

$$\frac{1}{k} \sum_{l=0}^{k-1} |x + y \exp(2\pi li/k)| \leq \frac{1}{k} \left(\sum_{l=0}^{k-1} |x + y \exp(2\pi li/k)|^2 \right)^{1/2} \times \sqrt{k}.$$

We have

$$|x + y \exp(2\pi li/k)|^2 = |x|^2 + |y|^2 + 2\Re(\bar{x}y \exp(2\pi li/k))$$

and $\sum_{l=0}^{k-1} \Re(\bar{x}y \exp(2\pi li/k)) = 0$ since $k \geq 2$. Hence

$$\sum_{l=0}^{k-1} |x + y \exp(2\pi li/k)|^2 = k(|x|^2 + |y|^2).$$

The conclusion follows. □

Remark 2.3. *The lemma is optimal: take $x/y = i$ and $k = 2$.*

3 Proof of Theorem 1.4.

Let $K \subset \mathbb{C}$ be a subfield of $\overline{\mathbb{Q}}$, and let \mathcal{O} be a subring of K such that the following holds.

There exists $C > 1$ such that for each non-zero $\gamma \in \mathcal{O}$

$$\text{either } \gamma \text{ is a root of unity, or } |\overline{\gamma}| \geq C. \quad (3.1)$$

Let $\gamma_1, \gamma_2, \dots \in \mathcal{O}$ be non-zero. We assume that for each j the archimedean absolute value of γ_j is ≥ 1 . We further assume that for each finite subset $\Gamma \subset \{\gamma_1, \gamma_2, \dots\}$ and for each integer $N \geq 1$ the extension $K(\gamma^{1/N} | \gamma \in \Gamma)/K$ is of degree $N^{|\Gamma|}$.

Remark 3.1. *By Kummer's Theory this last assertion holds if*

- i) K contains the roots of unity;*
- ii) None of the γ_j is a (non trivial) power in K ;*
- iii) For each integers $N \geq 1$, $\gamma_1, \gamma_2, \dots$ are multiplicatively independent modulo³ $(K^*)^N$.*

The following is our main result.

³That is: $\gamma_1^{a_1}, \dots, \gamma_r^{a_r} \in (K^*)^N$ whenever $\gamma_1^{a_1} \cdots \gamma_r^{a_r} \in (K^*)^N$ for some integers a_i .

Proposition 3.2. *Let α be a non-zero algebraic number in*

$$\mathcal{O}[\gamma_1^{1/n}, \gamma_2^{1/n}, \dots]_{n \geq 1}.$$

Then either $\alpha^n = \gamma_1^{k_1} \cdots \gamma_r^{k_r}$ for some integers n, r, k_1, \dots, k_r with $n, r \geq 1$ and $k_j \geq 0$, or

$$|\alpha| \geq \min(C, \sqrt{2}).$$

More precisely, if $|\alpha| < C$ then the maximum absolute value of the conjugates of α over K is $\geq \sqrt{2}$.

Proof. The proof follows the same lines as the proof of Theorem 1.2. Take a non-zero

$$\alpha \in \mathcal{O}[\gamma_1^{1/n}, \gamma_2^{1/n}, \dots]_{n \geq 1}.$$

Thus $\alpha \in \mathcal{O}[\gamma_1^{1/N}, \dots, \gamma_r^{1/N}]$ for some positive integers r, N . Let $L = K(\gamma_1^{1/N}, \dots, \gamma_r^{1/N})$ and $\text{Tr} = \text{Tr}_K^L$ be the trace relative to K . We suppose that for each integers n, k_1, \dots, k_r with $n \geq 1$, α does not satisfy $\alpha^n = \gamma_1^{k_1} \cdots \gamma_r^{k_r}$. We write

$$\alpha = \sum_{\mathbf{j}} a_{\mathbf{j}} \gamma_1^{j_1/N} \cdots \gamma_r^{j_r/N} \quad \text{with } a_{\mathbf{j}} \in \mathcal{O},$$

where the sum is over the multi-indexes $\mathbf{j} = (j_1, \dots, j_r)$ with $0 \leq j_i < N$. We distinguish two cases.

First case. There exists an index \mathbf{j} such that $a_{\mathbf{j}}$ is neither zero nor a root of unity. We have

$$\frac{1}{N^r} \text{Tr}(\gamma^{-j_1/N} \cdots \gamma^{-j_r/N} \alpha) = a_{\mathbf{j}}.$$

By assumption (3.1), $|a_{\mathbf{j}}| \geq C$. Thus

$$C \leq |a_{\mathbf{j}}| = \frac{1}{N^r} \left| \text{Tr}(\gamma^{-j_1/N} \cdots \gamma^{-j_r/N} \alpha) \right| \leq |\alpha|$$

since all the conjugates of $\gamma^{-j/N}$ have absolute value ≤ 1 . We get in this case the lower bound $|\alpha| \geq C$.

Second case. For each \mathbf{j} the coefficient $a_{\mathbf{j}}$ is either zero or a root of unity. By assumption, there exist two distinct multi-indexes \mathbf{j}, \mathbf{j}' such that $a_{\mathbf{j}}$ and $a_{\mathbf{j}'}$ are both roots of unity. Otherwise, $\alpha = a_{\mathbf{j}} \gamma_1^{j_1/N} \cdots \gamma_r^{j_r/N}$ for some \mathbf{j} and in this case either $\alpha = 0$ or $\alpha^{Nk} = \gamma_1^{j_1 k} \cdots \gamma_r^{j_r k}$ if $a_{\mathbf{j}}$ is a k th root of unity.

Let $x = (a_{\mathbf{j}} \gamma_1^{j_1/N} \cdots \gamma_r^{j_r/N})^{-1}$ and $y = (a_{\mathbf{j}'} \gamma_1^{j_1'/N} \cdots \gamma_r^{j_r'/N})^{-1}$. Then

$$\frac{1}{N^r} \text{Tr}((x + y)\alpha) = 2. \tag{3.2}$$

By assumption L/K is of degree N^r . Thus the K -embeddings $L \hookrightarrow \mathbb{C}$ are given by $\sigma_\omega(\gamma_i^{1/N}) = \omega_i \gamma_i^{1/N}$ with $\omega_i \in \boldsymbol{\mu}_N$, the group of N th roots of unity. Thus

$$\begin{aligned} \frac{1}{N^r} |\operatorname{Tr}((x+y)\alpha)| &= \frac{1}{N^r} \left| \sum_{\omega \in \boldsymbol{\mu}_N^r} (\sigma_\omega(x) + \sigma_\omega(y)) \sigma_\omega(\alpha) \right| \\ &\leq \overline{|\alpha|}_K \frac{1}{N^r} \sum_{\omega \in \boldsymbol{\mu}_N^r} |x + \omega_1^{j_1 - j'_1} \dots \omega_r^{j_r - j'_r} y| \\ &= \overline{|\alpha|}_K \frac{1}{k} \sum_{\omega \in \boldsymbol{\mu}_k} |x + \omega y| \end{aligned}$$

with

$$k = \operatorname{lcm} \left(\frac{N}{\gcd(j'_1 - j_1, N)}, \dots, \frac{N}{\gcd(j'_r - j_r, N)} \right)$$

and where

$$\overline{|\alpha|}_K = \max\{|\sigma\alpha|, \sigma \in \operatorname{Gal}(\overline{\mathbb{Q}}/K)\} \leq \overline{|\alpha|}$$

is the maximum of the absolute value of the conjugates of α over K .

Note that $k \geq 2$ since for at least one of the indexes i we have $j'_i \not\equiv j_i \pmod{N}$. Note also that $|x|, |y| \leq 1$ (since $|\gamma_j| \geq 1$). Thus we can apply Lemma 2.2:

$$\frac{1}{N^r} |\operatorname{Tr}((x+y)\alpha)| \leq \sqrt{2} \times \overline{|\alpha|}_K. \quad (3.3)$$

By (3.2) and (3.3) we get the desired conclusion $\overline{|\alpha|}_K \geq \sqrt{2}$. \square

Proof of Theorem 1.4 We apply Proposition 3.2, choosing $K = \mathbb{Q}^{\text{ab}}$ the maximal cyclotomic extension and $\mathcal{O} = \mathcal{O}_K = \mathbb{Z}[\zeta_n]_{n \in \mathbb{N}}$. Then (3.1) holds with

$$C = \sqrt{\frac{1 + \sqrt{5}}{2}} < \sqrt{2}$$

by a result of Schinzel (apply [11], Corollary 1', p. 386, to the linear polynomial $P(z) = z - \alpha$; see the next section for details). Notice also that the squares of the integers are in K . We choose $\gamma_i = \sqrt{p_i}$ where $(p_i)_{i \in \mathbb{N}}$ is the sequence of the rational primes. We still have to show that for each finite subset $\Gamma \subset \{\gamma_1, \gamma_2, \dots\}$ and for each integer $N \geq 1$ the extension $K(\gamma^{1/N} | \gamma \in \Gamma)/K$ is of degree $N^{|\Gamma|}$.

We make use of Remark 3.1. Assertion i) is satisfied by the choice of K . Assertion ii) is satisfied since K^* does not contain nontrivial k th roots of rationals for $k \geq 3$, since they generate over \mathbb{Q} a non-Galois extension. More precisely, let us assume that ii) does not hold. Then there exists a prime

p and an integer $n \geq 2$ such that $p^{1/2n} \in K = \mathbb{Q}^{\text{ab}}$, which is absurd since $\mathbb{Q}(p^{1/2n})/\mathbb{Q}$ is not Galois.

It is thus enough to prove assertion iii), *i.e.* that $\gamma_1, \gamma_2, \dots$ are multiplicatively independent modulo $(K^*)^N$. Let $\alpha := \prod_p p^{a_p/2}$ where the product is over a finite set of primes and where a_p are integers. Suppose $\alpha \in (K^*)^N$ for some $N \geq 2$. Then $\alpha^2 \in (K^*)^{2N} \cap \mathbb{Q}^*$. Since K^* does not contain nontrivial k th roots of rationals for $k \geq 3$, $\alpha^2 \in (\mathbb{Q}^*)^N$ which in turn implies $N \mid a_p$ for all p , and thus $p^{a_p/2} \in (K^*)^N$.

□

4 Concluding remarks

Dimitrov's proof of Schinzel-Zassenhaus conjecture and our results in this paper suggest that lower bounds for the house are sensibly simpler to prove than lower bound for the height. This opens a large spectre of conjectures, problems and results by considering the house of an algebraic number instead of the more familiar Weil's height.

Let \mathcal{A} be a set of algebraic numbers. Following [5], we say that \mathcal{A} has the *Bogomolov Property* (B) if there exists a real number $c = c(\mathcal{A}) > 0$ such that the set of non-zero $\alpha \in \mathcal{A}$ of height $< c$ consists of roots of unity. Several examples of fields with property (B) are known. For instance the field \mathbb{Q}^{tr} of all totally real algebraic numbers ([11]), the fields with bounded local degrees at some finite place ([5]), the maximal abelian extension \mathbb{Q}^{ab} of \mathbb{Q} ([3]). We refer the interested reader to the introduction of [2] for details and for other examples.

Property (B) for fields is not stable by finite extensions. For instance⁴ the field $\mathbb{Q}^{\text{tr}}(i)$, which is the compositum of all CM-field, does not have property (B), see again [2], section 5. However, the ring of integers of $\mathbb{Q}^{\text{tr}}(i)$ has property (B) by a result of Schinzel (apply [11], Corollary 1', p. 386, to the linear polynomial $P(z) = z - \alpha$), and the same is true for any finite extension of \mathbb{Q}^{tr} by [9], Theorem 1. Since $h(\alpha) \leq \log \overline{|\alpha|}$, *a fortiori* this ring satisfies the *House Property* below:

Definition 4.1. *Let \mathcal{A} be a set of algebraic integers. We say that \mathcal{A} has the House Property if there exists a real number $C = C(\mathcal{A}) > 1$ such that the set of non-zero $\alpha \in \mathcal{A}$ of house $< C$ consists of roots of unity.*

⁴note that the extension $\mathbb{Q}^{\text{tr}}(i)/\mathbb{Q}^{\text{tr}}$ is essentially the only known example of this phenomenon, see [9].

By abuse of notation we say that a field of algebraic numbers has the House Property if its ring of integers has it. In this context, some questions arise naturally.

By the remark above, fields with (B) satisfy the House Property, and moreover this property is satisfied by $\mathbb{Q}^{\text{tr}}(i)$, which does not satisfy (B) . Are there other significant examples of fields which satisfy the House Property but do not have (B) , or at least for which we cannot prove that they satisfy (B) ?

Related to this question, we might ask if the House Property is stable by finite extensions. A positive answer could be suggested by the example $\mathbb{Q}^{\text{tr}}(i)/\mathbb{Q}^{\text{tr}}$ above, where the House Property is satisfied even if $\mathbb{Q}^{\text{tr}}(i)$ does not satisfy (B) . In spite of that we have:

Remark 4.2 (Pottmeyer). *Let α be any Salem number⁵ and $K = \mathbb{Q}(\alpha)$. Let β be any non-real conjugate of α . Then $\beta^{1/n} \in K^{\text{tr}}(i)$ for each positive integer n . In particular, $K^{\text{tr}}(i)$ does not satisfy the House Property.*

Proof. Let n be a positive integer. Then all Galois conjugates of $\beta^{1/n}$ over K have absolute value 1. By [9, Lemma 1], $\beta^{1/n} \in K^{\text{tr}}(i)$. \square

By the already quoted result of Schinzel, K^{tr} satisfies (B) and hence the House property. Thus $K^{\text{tr}}(i)/K^{\text{tr}}$ is a degree two extension which does not satisfy the House property even if the ground field does.

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⁵Thus $\alpha > 1$ is an algebraic integer, α^{-1} is a conjugate of α , and the other conjugates lie on the unit circle.

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5 After

Given $\alpha \in \overline{\mathbb{Q}}^*$ and let K be a number field which contains α . We set $\widehat{\text{Tr}}(\alpha) = \frac{1}{[K:\mathbb{Q}]} \text{Tr}_{K/\mathbb{Q}}(\alpha)$. It is clear that the definition does not depend on K .

Theorem 5.1. *Let $1 = \gamma_1, \dots, \gamma_n \in \overline{\mathbb{Q}}^*$. We assume:*

1. $\widehat{\text{Tr}}(\gamma_i \gamma_j^{-1}) = \delta_{ij}$;
2. $1 = |\gamma_1|_v \leq |\gamma_2|_v \leq \dots \leq |\gamma_n|_v$ for any archimedean place v of $\mathbb{Q}(\gamma_1, \dots, \gamma_n)$.

Then for any nonzero $\alpha \in \mathbb{Z}\gamma_1 + \dots + \mathbb{Z}\gamma_n$ with $\alpha \neq \gamma_i$ we have $|\alpha| \geq \sqrt{2}$.

Proof. We write $\alpha = a_1\gamma_1 + \cdots + a_n\gamma_n$ for some integers a_1, \dots, a_n . We distinguish two cases.

First case. There exists an index j such that $|a_j| \geq 2$. By assumption (??), $\widehat{\text{Tr}}(\gamma_j^{-1}\alpha) = a_j$. By assumption (??) $|\gamma_j| \geq 1$. Thus

$$2 \leq |a_j| = |\widehat{\text{Tr}}(\gamma_j^{-1}\alpha)| \leq \lceil \alpha \rceil$$

and we get in this case the better lower bound $\lceil \alpha \rceil \geq 2$.

Second case. For each j we have $|a_j| \leq 1$. Then, by assumption, there exist two indexes $j_0 < j_1$ such that $a_{j_0} = \pm 1$ and $a_{j_1} = \pm 1$. Choosing the signs in a right way we get, by assumption (??),

$$|\text{Tr}((\gamma_{j_0}^{-1} \pm \gamma_{j_1}^{-1})\alpha)| = 2. \quad (5.1)$$

We want an upper bound for the quantity on the left hand side. We have

$$|\text{Tr}((\gamma_{j_0}^{-1} \pm \gamma_{j_1}^{-1})\alpha)| \leq \lceil \alpha \rceil \frac{1}{d} \sum_{\sigma} |\sigma\gamma_{j_0}^{-1} \pm \sigma\gamma_{j_1}^{-1}|. \quad (5.2)$$

where the sum is over the d embeddings $\sigma: \mathbb{Q}(\gamma_1, \dots, \gamma_n) \hookrightarrow \mathbb{C}$.

By Cauchy-Schwarz,

$$\frac{1}{d} \sum_{\sigma} |\sigma\gamma_{j_0}^{-1} \pm \sigma\gamma_{j_1}^{-1}| \leq \left(\frac{1}{d} \sum_{\sigma} |\sigma\gamma_{j_0}^{-1} \pm \sigma\gamma_{j_1}^{-1}|^2 \right)^{1/2}. \quad (5.3)$$

By assumption ??, $|\sigma\gamma_{j_0}|^{-1} \leq 1$ and $|\sigma\gamma_{j_0}/\sigma\gamma_{j_1}| \leq 1$. Thus

$$\begin{aligned} |\sigma\gamma_{j_0}^{-1} \pm \sigma\gamma_{j_1}^{-1}|^2 &= |\sigma\gamma_{j_0}|^{-1} |1 \pm \sigma\gamma_{j_0}/\sigma\gamma_{j_1}|^2 \leq |1 \pm \sigma\gamma_{j_0}/\sigma\gamma_{j_1}|^2 \\ &= 1 \pm 2\Re(\sigma\gamma_{j_0}/\sigma\gamma_{j_1}) + |\sigma\gamma_{j_0}/\sigma\gamma_{j_1}|^2 \leq 2 \pm 2\Re(\sigma\gamma_{j_0}/\sigma\gamma_{j_1}). \end{aligned} \quad (5.4)$$

By assumption ??,

$$\frac{1}{d} \sum_{\sigma} 2\Re(\sigma\gamma_{j_0}/\sigma\gamma_{j_1}) = \frac{2}{d} \Re\left(\sum_{\sigma} \sigma\gamma_{j_0}/\sigma\gamma_{j_1} \right) = 2\Re\left(\widehat{\text{Tr}}(\gamma_{j_0}/\gamma_{j_1}) \right) = 0. \quad (5.5)$$

Combining (??), (??), (??) and (??) we get the upper bound

$$|\text{Tr}((\gamma_{j_0}^{-1} \pm \gamma_{j_1}^{-1})\alpha)| \leq \lceil \alpha \rceil \times \sqrt{2}.$$

Comparing this upper bound with (??) we get the desired lower bound for $\lceil \alpha \rceil$. □