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INNER RATES OF FINITE MORPHISMS

YENNI CHERIK

ABSTRACT. Let $(X, 0)$ be a complex analytic surface germ embedded in $(\mathbb{C}^n, 0)$ with an isolated singularity and $\Phi = (g, f) : (X, 0) \rightarrow (\mathbb{C}^2, 0)$ be a finite morphism. We define a family of analytic invariants of the morphism Φ , called *inner rates* of Φ . By means of the inner rates we study the polar curve associated to the morphism Φ when fixing the topological data of the curve $(gf)^{-1}(0)$ and the surface germ $(X, 0)$, allowing to address a problem called polar exploration. We also use the inner rates to study the geometry of the Milnor fibers of a non constant holomorphic function $f : (X, 0) \rightarrow (\mathbb{C}, 0)$. The main result is a formula which involves the inner rates and the polar curve alongside topological invariants of the surface germ $(X, 0)$ and the curve $(gf)^{-1}(0)$.

INTRODUCTION

Let $(X, 0)$ be a complex analytic surface germ with an isolated singularity and $\Phi = (g, f) : (X, 0) \rightarrow (\mathbb{C}^2, 0)$ be a finite morphism. The polar curve of the morphism Φ is the curve Π_Φ defined as the topological closure of the ramification locus of Φ . Polar curves play an important role in the study of the geometry and the topology of germs of singular complex varieties, see, e.g. [Tei82, GBT99a, HP03, BNP14, BdSFP22a].

In this paper we introduce and study a family of analytic invariants of the morphism Φ that we call *inner rates* of Φ , generalizing the notion of inner rates of a complex analytic surface germ $(X, 0)$ first introduced by Birbrair, Neumann and Pichon in [BNP14] as metric invariants, and later defined in [BdSFP22a] by Belotto, Fantini and Pichon. Our main result (Theorem A) establishes a formula, the *inner-rates formula*, which relates the inner rates with analytical data of the polar curve Π_Φ and, in particular, provides a concrete way to compute the inner rates. It has an equivalent version in terms of the laplacian of a certain graph (Corollary 5.7) which is a broad generalization of the *Laplacian formula* [BdSFP22a, Theorem 4.3]. An important motivation of our result concerns the study of Milnor fibers. Consider a non-constant holomorphic function $f : (X, 0) \rightarrow (\mathbb{C}, 0)$ and a generic linear form (Definition 10.7) $\ell : (X, 0) \rightarrow (\mathbb{C}, 0)$. We provide an interpretation of the inner rates of the morphism $\Phi = (\ell, f) : (X, 0) \rightarrow (\mathbb{C}^2, 0)$ in terms of metric properties of the Milnor fiber, Theorem D.

As an application of our methods, we study the problem of *polar exploration* associated to Φ , following [BdSFP22a, BdSFNP22]. Roughly speaking, it is the study of the relative position on $(X, 0)$ of the two curves Π_Φ and $(gf)^{-1}(0)$ i.e., of the relative positions of their strict transform by a resolution of $(X, 0)$. More precisely it is the problem of determining the embedded topological type (Definition 9.1) of the union

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$\Pi_\Phi \cup (gf)^{-1}(0)$ from that of the curve $(gf)^{-1}(0)$. This is a natural problem which has been studied by many authors such as Merle, García Barroso, Delgado, Maugendre, Kuo, Parusiński, Michel, Belotto, Fantini, Némethi, Pichon, among others (see, e.g., [Mer77, KP04, GB00, DM03, Mic08, MM20, DM21, BdSFNP22, BdSFP22b]). An important contribution in the general case was made by Michel in [Mic08] via the *Hironaka quotients* of the morphism Φ (Definition 7.1). New techniques and results involving inner rates were recently developed in [BdSFNP22] in the case where Φ is a generic linear projection of $(X, 0)$, and a complete answer to polar exploration was given in [BdSFP22b] in the case where $(X, 0)$ is a Lipschitz normally embedded surface germ. In the present paper, we address the case of a general finite morphism Φ . We show the relation between the inner rates and the Hironaka quotients (Theorem B(ii)), and we give an alternative proof of [Mic08, Theorem 4.9] based on our inner rates formula. Finally, we give a family of examples where the inner rates formula provides tighter restrictions on the relative position of the polar curve than the ones obtained from previous methods (Proposition C).

We now present our results in a sharp form. Let $\pi : (X_\pi, E) \rightarrow (X, 0)$ be a *good resolution of singularities* of $(X, 0)$, that is, a proper bimeromorphic map which is an isomorphism on the complementary of a simple normal crossing divisor $\pi^{-1}(0) = E$, called the *exceptional divisor*. Let E_v be an irreducible component of E . We call *curvette* of E_v a smooth curve germ which intersects transversely E_v at a smooth point of E .

Throughout the paper, we use the big-Theta asymptotic notation of Bachmann–Landau in the following form: given two function germs $h_1, h_2 : ([0, \infty), 0) \rightarrow ([0, \infty), 0)$ we say that h_1 is a *big-Theta* of h_2 and we write $h_1(t) = \Theta(h_2(t))$ if there exists real numbers $\eta > 0$ and $K > 0$ such that for all $t \in [0, \eta)$ we have $K^{-1}h_2(t) \leq h_1(t) \leq Kh_2(t)$.

Let $(u_1, u_2) = (g, f)$ be the coordinates of \mathbb{C}^2 . We prove (Proposition 2.3) the existence of a rational number $q_{g,v}^f$ such that for any smooth point p of E in E_v which does not meet the strict transforms of $f^{-1}(0), g^{-1}(0)$ or the polar curve Π_Φ and for any pair of disjoint curvettes γ_1^* and γ_2^* of E_v passing through points of E close enough to p ,

$$d(\gamma_1 \cap \{u_2 = \epsilon\}, \gamma_2 \cap \{u_2 = \epsilon\}) = \Theta(\epsilon^{q_{g,v}^f}),$$

where $\gamma_1 = \Phi \circ \pi(\gamma_1^*)$, $\gamma_2 = \Phi \circ \pi(\gamma_2^*)$ and d is the standard hermitian metric of \mathbb{C}^2 . We call the number $q_{g,v}^f$ *inner rate of f with respect to g along E_v* .

As already mentioned, the notion of inner rates associated to a finite morphism generalizes the inner rates of [BdSFP22a]. Indeed, suppose that $(X, 0)$ is embedded in \mathbb{C}^n . Assume that π is a good resolution which factors through the blowup of the maximal ideal and the *Nash transform* (see e.g [Spi90, Introduction] for the definition of the Nash transform) and that Φ is a "generic" linear projection in the sense of [BdSFP22a, Subsection 2.2]. Then, the inner rates associated to Φ coincide with the inner rates of the surface germ $(X, 0)$ (see [BdSFP22a, Definition 3.3]), which are metric invariants of $(X, 0)$.

Let us now state the main result of this paper, which is proved in section 3. Let Γ_π be the *dual graph* of the good resolution π , that is, the graph whose vertices are in bijection with the irreducible components of E and such that the edges between the

vertices v and v' corresponding to E_v and $E_{v'}$ are in bijection with $E_v \cap E_{v'}$. Each vertex v of this graph is weighted with the self intersection number E_v^2 and the genus g_v of the corresponding complex curve E_v . Let $\text{val}_{\Gamma_\pi}(v) := \left(\sum_{i \in V(\Gamma_\pi), i \neq v} E_i \right) \cdot E_v$ be the **valency** of v . We denote by $V(\Gamma_\pi)$ the set of vertices of Γ_π and $E(\Gamma_\pi)$ the set of edges. Let us denote by $m_v(f)$ the order of vanishing of the function $f \circ \pi$ along the irreducible component E_v of E and by f^* the strict transform of the curve $f^{-1}(0)$ by π .

Theorem A (The inner rates formula). *Let $(X, 0)$ be a complex surface germ with an isolated singularity and let $\pi : (X_\pi, E) \rightarrow (X, 0)$ be a good resolution of $(X, 0)$. Let $g, f : (X, 0) \rightarrow (\mathbb{C}, 0)$ be two holomorphic functions on X such that the morphism $\Phi = (g, f) : (X, 0) \rightarrow (\mathbb{C}^2, 0)$ is finite. Let $E_{v_1}, E_{v_2}, \dots, E_{v_n}$ be the irreducible components of E . Let $M_\pi = (E_{v_i} \cdot E_{v_j})_{i, j \in \{1, 2, \dots, n\}}$ be the intersection matrix of the dual graph Γ_π . Consider the four following vectors: $a_{g, \pi}^f := (m_{v_1}(f)q_{g, v_1}^f, \dots, m_{v_n}(f)q_{g, v_n}^f)$, $K_\pi := (\text{val}_{\Gamma_\pi}(v_1) + 2g_{v_1} - 2, \dots, \text{val}_{\Gamma_\pi}(v_n) + 2g_{v_n} - 2)$, the **F-vector** $F_\pi = (f^* \cdot E_{v_1}, \dots, f^* \cdot E_{v_n})$ and the **P-vector** $P_\pi = (\Pi_\Phi^* \cdot E_{v_1}, \dots, \Pi_\Phi^* \cdot E_{v_n})$. Then we have:*

$$M_\pi \cdot \underline{a}_{g, \pi}^f = \underline{K}_\pi + \underline{F}_\pi - \underline{P}_\pi.$$

Equivalently, for each irreducible component E_v of E we have the following:

$$\left(\sum_{i \in V(\Gamma_\pi)} m_i(f)q_{g, i}^f E_i \right) \cdot E_v = \text{val}_{\Gamma_\pi}(v) + f^* \cdot E_v - \Pi_\Phi^* \cdot E_v + 2g_v - 2,$$

where “ \cdot ” denotes the intersection number between curves.

The idea of the proof is to relate the inner rates to the canonical divisor of the complex surface X_π and then apply the classical adjunction formula on the irreducible components of the exceptional divisor $E = \pi^{-1}(0)$. This proof is quite shorter and different from the one provided in [BdSFP22a] which relies on topological tools.

Not only does theorem A gives us a concrete way to compute the inner rates in terms of a good resolution π of $(X, 0)$ just by computing the polar curve (see Example 9.2), but, since the intersection matrix M_π is negative definite by a result of Mumford [Mum61, §1], it also proves that given the dual graph Γ_π together with the F -vector, the inner rates $q_{g, v}^f$ determines and are determined by the \mathcal{P} -vector $(\Pi_\Phi^* \cdot E_1, \dots, \Pi_\Phi^* \cdot E_n)$. This fact allows us to study polar curves by means of the inner rates, more specifically, to address the problem of *polar exploration*, which we now describe. Suppose that $\pi : (X_\pi, E) \rightarrow (X, 0)$ is the minimal good resolution of $(X, 0)$ and $(gf)^{-1}(0)$, that is, the minimal good resolution of $(X, 0)$ such that $f^* \cup g^*$ is a disjoint union of curvettes of E . Following [BdSFP22a], polar exploration consists in answering the following question: is it possible to determine the \mathcal{P} -vector from the data of the dual graph Γ_π , the F -vector and the G -vector?

An important contribution to this problem was made by Michel ([Mic08, Theorem 4.9]) by means of the *Hironaka quotients*. The Hironaka quotient of the morphism Φ associated to an irreducible component E_v of E is the rational number $h_{g, v}^f = \frac{m_v(g)}{m_v(f)}$, $v \in V(\Gamma_\pi)$. In this paper we will improve this result of Michel by using the inner rates instead of the Hironaka quotients. The reason why the inner rates are more efficient than the Hironaka quotients for polar exploration comes from the following result.

Theorem B. *Let $(X, 0)$ be a complex surface germ and let $\Phi = (g, f) : (X, 0) \rightarrow (\mathbb{C}^2, 0)$ be a finite morphism. Let $\pi : (X_\pi, E) \rightarrow (X, 0)$ be a good resolution of $(X, 0)$. There exists a subgraph \mathcal{A}_π of Γ_π such that:*

- (i) *Call f -node any vertex w of Γ_π such that $f^* \cdot E_w \neq 0$. Let v be a vertex of Γ_π . There exists a path from an f -node to v in Γ_π along which the inner rates are strictly increasing.*
- (ii) *The inner rates and the Hironaka quotients of Φ coincide on \mathcal{A}_π .*
- (iii) *The Hironaka quotients are constant on $\Gamma_\pi \setminus \mathcal{A}_\pi$.*

Points (i) and (ii) are proved in the sections 6 and 7, while the point (iii) comes from [MM20, Theorem 1].

In Section 8, we show that [Mic08, Theorem 4.9] can be obtained as a consequence of Theorems A and B. We state it here taking account of [MM20, Theorem 1]:

Theorem ([Mic08, Theorem 4.9]). *Let $(X, 0)$ be a complex surface germ with an isolated singularity and let $g, f : (X, 0) \rightarrow (\mathbb{C}, 0)$ be two holomorphic functions on X such that the morphism $\Phi = (g, f) : (X, 0) \rightarrow (\mathbb{C}^2, 0)$ is finite. Let $\pi : (X_\pi, E) \rightarrow (X, 0)$ be a good resolution of $(X, 0)$. Let \mathcal{A}_π be a subgraph of Γ_π as in the statement of Theorem B. Let \mathcal{Z} be a connected component of $\overline{\Gamma_\pi \setminus \mathcal{A}_\pi}$ or a single vertex on the complementary of $\overline{\Gamma_\pi \setminus \mathcal{A}_\pi}$, then :*

$$\sum_{v \in \mathcal{Z}} m_v(f) \Pi_\Phi^* \cdot E_v = - \left(\sum_{v \in \mathcal{Z}} m_v(f) \chi'_v \right).$$

where $\chi'_v := 2 - 2g_v - \text{val}(v) - f^* \cdot E_v - g^* \cdot E_v$.

It is now natural to ask: can we get a better restriction for the value of the \mathcal{P} -vector when using the inner rates and their properties? We give a positive answer to this question via the following result (see proposition 9.4 for a more precise statement).

Proposition C. *There exists a sequence of graphs with arrows $(\Gamma_n)_{n \geq 2}$ such that for each n :*

- (i) *There exists a complex surface singularity $(X_n, 0)$ and a finite morphism $\Phi_n = (g_n, f_n) : (X_n, 0) \rightarrow (\mathbb{C}^2, 0)$ such that Γ_n is the dual graph of the minimal good resolution of $(X_n, 0)$ and $(g_n, f_n)^{-1}(0)$;*
- (ii) *The \mathcal{P} -vector of any such morphism Φ_n belongs to a set of $n + 5$ elements.*

If one performs a polar exploration on this family of examples using only [Mic08, Theorem 4.9], one obtains an exponential bound for the number of \mathcal{P} -vectors (see Remark 9.8), while Proposition C provides a linear bound.

Another important motivation to study the inner rates is to provide analytic invariants of the Milnor fibration. Let $(X, 0)$ be a complex analytic surface germ embedded in \mathbb{C}^n and let $f : (X, 0) \rightarrow (\mathbb{C}, 0)$ be a non constant holomorphic function. Let $\pi : (X_\pi, E) \rightarrow (X, 0)$ be a good resolution which factors through the blowup of the maximal ideal and the Nash transform relative to f (Definition 10.1). Let ℓ be a generic linear form with respect to π (Definition 10.7). The inner rates $q_{\ell, v}^f, v \in V(\Gamma_\pi)$ associated to the morphism (ℓ, f) do not depend of the choice of the generic linear form ℓ , we then denote them $q_v^f, v \in V(\Gamma_\pi)$. In this case the inner rates q_v^f gives informations on the inner metric of the Milnor fibers:

Theorem D (Theorem 10.8). *Let $(X, 0) \subset (\mathbb{C}^n, 0)$ be a complex surface germ with an isolated singularity at the origin of \mathbb{C}^n and let $f : (X, 0) \rightarrow (\mathbb{C}, 0)$ be a non constant holomorphic function. Let $\pi : (X_\pi, E) \rightarrow (X, 0)$ be a good resolution which factors through the Nash transform of X relative to f and the blowup of the maximal ideal.*

Let γ_1^ and γ_2^* be two curvettes of an irreducible component E_v of the exceptional divisor E such that $\gamma_i^* \cap f^* = \emptyset$ for $i \in \{1, 2\}$. Then there exists $q_v^f \in \mathbb{Q}_{>0}$ such that*

$$d_\epsilon(\gamma_1 \cap f^{-1}(\epsilon), \gamma_2 \cap f^{-1}(\epsilon)) = \Theta(\epsilon^{q_v^f})$$

where $\gamma_1 = \pi(\gamma_1^), \gamma_2 = \pi(\gamma_2^*)$ and d_ϵ is the Riemannian metric induced by \mathbb{C}^n on the Milnor fiber $f^{-1}(\epsilon)$. Furthermore we have $q_v^f = q_{\ell, v}^f$ whenever ℓ is a generic linear form with respect to f and π .*

Theorem D is a relative version (with respect to f) and a generalization of [BdSFP22a, Lemma 3.2]. Indeed, one obtains [BdSFP22a, Lemma 3.2] by taking f a generic linear form in the sense of [BdSFP22a, Subsection 2.2].

As an application of Theorems A and D we give a generalization of a result of García Barroso and Teissier on the asymptotic behavior of the integral of the Lipschitz-Killing curvature (Definition 11.1) along Milnor fibers. Let $\pi : (X_\pi, E) \rightarrow (X, 0)$ be a good resolution which factors through the blowup of the maximal ideal and the Nash transform relative to f . Let E_v be an irreducible component of E and let $\mathcal{N}(E_v, \epsilon), \epsilon > 0$ be a family of tubular neighborhoods of E_v in X_π such that $\lim_{\epsilon \rightarrow 0} \mathcal{N}(E_v, \epsilon) = E_v$ and such that $\text{Horn}(\epsilon, v) := \pi(\mathcal{N}(E_v, \epsilon))$ is included in B_ϵ . Consider the set $F_{\epsilon, t}^v = f^{-1}(t) \cap \text{Horn}(\epsilon, v)$ and let $\delta_\epsilon^\ell > 0$ be such that for any complex number t with $|t| \leq \delta_\epsilon^\ell$, we have $\text{Card}\{F_{\epsilon, t}^v \cap \Pi_\ell\} = m_v(f)\Pi_\ell^* \cdot E_v$ where ℓ is a generic linear form with respect to π . Set $\delta_\epsilon = \min_\ell \{\delta_\epsilon^\ell\}$.

Theorem E (Theorem 11.5). *Let $(X, 0)$ be a complex surface germ with an isolated singularity embedded in $(\mathbb{C}^n, 0)$ and let $f : (X, 0) \rightarrow (\mathbb{C}, 0)$ be a non constant holomorphic function germ. Let $\pi : (X_\pi, E) \rightarrow (X, 0)$ be a good resolution of $(X, 0)$ which factors through the Nash transform of X relative to f and through the blowup of the maximal ideal of $(X, 0)$. Let v be a vertex of Γ_π , then:*

$$\lim_{\epsilon \rightarrow 0, |t| < \delta_\epsilon} \int_{p \in F_{\epsilon, t}^v} K_{F_{\epsilon, t}^v}(p) dV = \frac{\pi\omega_2}{2\omega_{2n-1}} \text{Vol}(\mathbb{G}^{n-1}(\mathbb{C}^n)) C_f,$$

where

$$C_f = m_v(f) \left(2g_v - 2 + \text{Val}_{\Gamma_\pi}(v) + f^* \cdot E_v - \sum_{i \in V(\Gamma_\pi)} m_i(f) q_i^f E_i \cdot E_v \right).$$

and ω_i is the volume of the unit sphere S^i .

Theorem E generalizes the work of García Barroso and Teissier in [GBT99b] which treat the case of germs of holomorphic functions at the origin of \mathbb{C}^2 .

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1. RESOLUTION OF CURVES AND SURFACES

In this section we introduce some classical materials as they are presented in the introductions of [MM20],[Mic08] and [BdSFP22a].

Definition 1.1. Let $(X, 0)$ be a complex surface germ with an isolated singularity. A **resolution** of $(X, 0)$ is a proper bimeromorphic map $\pi : (X_\pi, E) \rightarrow (X, 0)$ such that X_π is a smooth complex surface and the restricted function $\pi|_{X_\pi \setminus E} : X_\pi \setminus E \rightarrow X \setminus 0$ is a biholomorphism. The curve $E = \pi^{-1}(0)$ is called the **exceptional divisor**.

The resolution $\pi : (X_\pi, E) \rightarrow (X, 0)$ is **good** if E is a simple normal crossing divisor i.e., it has smooth compact irreducible components and the singular points of E are transversal double points. Let E_v be an irreducible component of E . A **curvette** of E_v is a smooth curve germ which intersects E_v transversely at a smooth point of E .

Definition 1.2. Let $\pi : (X_\pi, E) \rightarrow (X, 0)$ be a resolution of $(X, 0)$ and let $(C, 0)$ be a curve germ in $(X, 0)$. The **strict transform** of C by π is the curve C^* in X_π defined as the topological closure of the set $\pi^{-1}(C \setminus \{0\})$. Let E_1, E_2, \dots, E_n be the irreducible components of E and $h : (X, 0) \rightarrow (\mathbb{C}, 0)$ be a holomorphic function. The **total transform** of h by π is the principal divisor $(h \circ \pi)$ on X_π , i.e.,

$$(h \circ \pi) = \sum_{i=1}^n m_i(h) E_i + h^*$$

where $m_i(h)$ is the order of vanishing of the holomorphic function $h \circ \pi$ on the irreducible component E_i of E and h^* is the strict transform of the curve $h^{-1}(0)$.

Proposition 1.3 ([Lau71, Theorem 2.6] or [N99, 2.6] for a topological proof). Let $\pi : (X_\pi, E) \rightarrow (X, 0)$ be a resolution of $(X, 0)$. Let $h : (X, 0) \rightarrow (\mathbb{C}, 0)$ be a holomorphic function, then we have the following:

$$(h \circ \pi) \cdot E_v = 0, \quad \forall v \in V(\Gamma_\pi).$$

where “ \cdot ” denote the intersection multiplicity between curves.

Definition 1.4. Let $(C, 0)$ be a curve germ in $(X, 0)$. A proper bimeromorphic map $\pi : (X_\pi, E) \rightarrow (X, 0)$ is a **good resolution of $(X, 0)$ and $(C, 0)$** if it is a good resolution of $(X, 0)$ such that the strict transform C^* is a disjoint union of curvettes.

Definition 1.5. The **dual graph** of a good resolution $\pi : (X_\pi, E) \rightarrow (X, 0)$ of $(X, 0)$ is the graph Γ_π whose vertices are in bijection with the irreducible components of E and such that the edges between the vertices v and v' corresponding to E_v and $E_{v'}$ are in bijection with $E_v \cap E_{v'}$, each vertex v of this graph is weighted with the self intersection number E_v^2 and the genus g_v of the corresponding curve E_v . We denote by $V(\Gamma_\pi)$ the set of vertices of Γ_π and $E(\Gamma_\pi)$ the set of edges. The **valency of a vertex v** is the number $\text{val}_{\Gamma_\pi}(v) := \left(\sum_{i \in V(\Gamma_\pi), i \neq v} E_i \right) \cdot E_v$. Let $\Phi = (g, f) : (X, 0) \rightarrow (\mathbb{C}, 0)$ be a finite morphism and let $E_v \subset E$ be an irreducible

component which meets the strict transform g^* (resp. f^*), following notations of [MM20], we attach to the vertex v corresponding to E_v a going-out arrow (resp. a going-in arrow) weighted with the intersection number $g^* \cdot E_v$ (resp. $f^* \cdot E_v$).

Example 1.6. Consider the finite morphism $\Phi = (g, f) : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$, where $g(x, y) = x + y$ and $f(x, y) = y^5 - x^{12}$. The minimal good resolution $\pi : (X_\pi, E) \rightarrow (\mathbb{C}^2, 0)$ of the curve $(gf)^{-1}(0)$ has the following dual graph: all the irreducible components of E have genus 0.

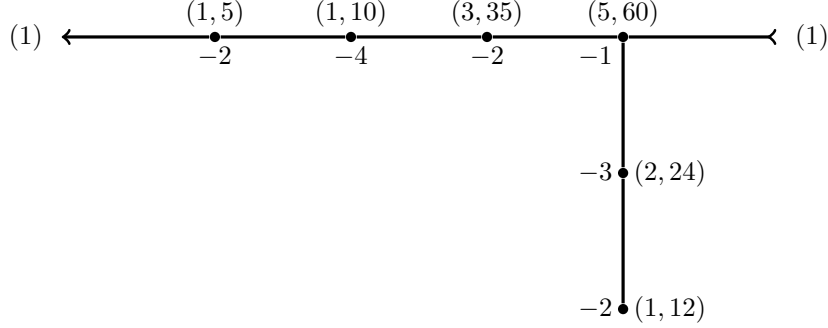


FIGURE 1. The numbers between parenthesis are the orders of vanishing $(m_v(g), m_v(f))$ of the functions $g \circ \pi$ and $f \circ \pi$ along the irreducible components of E , these numbers can be determined from the dual graph using Proposition 1.3.

2. INNER RATES OF A FINITE MORPHISM

Let $(X, 0)$ be a complex surface germ with an isolated singularity and $g, f : (X, 0) \rightarrow (\mathbb{C}, 0)$ two holomorphic functions such that the morphism $\Phi = (g, f) : (X, 0) \rightarrow (\mathbb{C}^2, 0)$ is finite. The aim of this section is to define the notion of **inner rates** associated to the morphism $\Phi = (g, f)$. This definition is a generalization of the **inner rates of a complex surface germ with an isolated singularity** first introduced in [BNP14] and later in [BdSFP22a, Definition 3.3].

Definition 2.1 (See e.g [KP04, Definition 0.1]). *The **polar curve** of the morphism Φ is the curve Π_Φ defined as the topological closure of the critical locus of the finite morphism $\Phi = (g, f)$, that is*

$$\Pi_\Phi = \overline{\{x \in X \setminus \{0\} \mid d_x \Phi : T_x X \rightarrow \mathbb{C}^2 \text{ is non surjective}\}}$$

Definition 2.2. (See e.g [BdSFP22a, section 3]) *Given two functions germs $h_1, h_2 : ([0, +\infty), 0) \rightarrow ([0, +\infty), 0)$, we say that h_1 is a **big-Theta** of h_2 and we write $h_1(t) = \Theta(h_2(t))$ if there exists real numbers $\eta > 0$ and $K > 0$ such that for all $t \in [0, \eta)$,*

$$\frac{1}{K}h_2(t) \leq h_1(t) \leq Kh_2(t)$$

Proposition 2.3. *Let $\pi : (X_\pi, E) \rightarrow (X, 0)$ be a good resolution of $(X, 0)$ and let E_v be an irreducible component of the exceptional divisor E . Let us denote by $(u_1, u_2) = (g, f)$ the coordinates of \mathbb{C}^2 and by d the standard hermitian metric of \mathbb{C}^2 . Then there exists a positive rational number $q_{g,v}^f$ such that for every smooth*

point p of E in $E_v \setminus (f^* \cup g^* \cup \Pi_{\Phi}^*)$, there exists an open neighborhood $O_p \subset E_v$ of p such that for every pair of curvettes γ_1^*, γ_2^* of E_v verifying:

$$\begin{cases} \gamma_1^* \cap \gamma_2^* = \emptyset \\ \gamma_i^* \cap O_p \neq \emptyset \text{ for } i = 1, 2. \end{cases} \quad (2.1)$$

we have:

$$d(\gamma_1 \cap \{u_2 = \epsilon\}, \gamma_2 \cap \{u_2 = \epsilon\}) = \Theta(\epsilon^{q_{g,v}^f}),$$

where $\gamma_1 = (\Phi \circ \pi)(\gamma_1^*)$, $\gamma_2 = (\Phi \circ \pi)(\gamma_2^*)$ and $\epsilon \in \mathbb{R}$. Furthermore the number $m_v(f)q_{g,v}^f$ is an integer.

Definition 2.4. We call $q_{g,v}^f$ the *inner rate of f with respect to g along E_v* .

Let us explain why the definition 2.4 is a generalization of the notion of inner rates of a complex surface germ with an isolated singularity as defined in [BdSFP22a, Definition 3.3]. Let $(X, 0)$ be a complex surface germ embedded in \mathbb{C}^n with an isolated singularity and let $\pi : (X_\pi, E) \rightarrow (X, 0)$ be a good resolution which factors through the blowup of the maximal ideal and the *Nash transform* (see e.g [Spi90, Introduction] for the definition of the Nash transform). Let ℓ_1 and ℓ_2 be two linear forms of \mathbb{C}^n such that the projection $\ell = (\ell_1, \ell_2)$ is a generic projection in the sense of [BdSFP22a, Subsection 2.2]. For every irreducible component E_v of E , by [BdSFP22a, Lemma 3.2], the inner rate $q_{\ell_1, v}^{\ell_2}$ is equal to the inner rate of the complex surface germ $(X, 0)$ associated with the component E_v .

Proof of Proposition 2.3. Let p be a smooth point of E_v which does not belong to the strict transforms f^* , g^* and Π_{Φ}^* . Let (x, y) be a local system of coordinates of X_π centered at p such that E_v has local equation $x = 0$ and such that $(f \circ \pi)(x, y) = x^{m_v(f)}$. Let U be the unit of $\mathbb{C}\{x, y\}$ such that $(g \circ \pi)(x, y) = x^{m_v(g)}U(x, y)$. We can write

$$U(x, y) = \sum_{i \geq 0} a_{i0}x^i + \sum_{j \geq 1} y^j \sum_{i \geq 0} a_{ij}x^i$$

Since Φ is a finite morphism the set $\{i \geq 0 \mid \exists j > 0, a_{ij} \neq 0\}$ is non empty. Let k be its minimal element. Then,

$$U(x, y) = g_0(x) + x^k \sum_{j \geq 1} y^j g_j(x),$$

where $g_0(x) = \sum_{i \geq 0} a_{i0}x^i$ and $g_j(x) = \sum_{i \geq 0} a_{ij}x^{i-k}$. Note that the set $\{j > 0 \mid g_j(0) \neq 0\}$ is non empty.

Setting $q_p := \frac{m_v(g)+k}{m_v(g)}$, we then have:

$$(g \circ \pi)(x, y) = x^{m_v(g)}g_0(x) + x^{q_p m_v(g)} \sum_{j \geq 1} y^j g_j(x). \quad (2)$$

Let γ_1^* and γ_2^* be two curvettes of E_v parametrized respectively by

$$t \mapsto (t, \alpha + th_1(t)), \quad t \mapsto (t, \beta + th_2(t)), \quad \alpha, \beta \in \mathbb{C}$$

where h_1 and h_2 are convergent power series. The curves $\gamma_1 = (\Phi \circ \pi)(\gamma_1^*)$ and $\gamma_2 = (\Phi \circ \pi)(\gamma_2^*)$ are then parametrized respectively by

$$t \mapsto (t^{m_v(g)}g_0(t) + t^{q_p m_v(g)} \sum_{j \geq 1} (\alpha + th_1(t))^j g_j(t), t^{m_v(f)})$$

and

$$t \mapsto (t^{m_v(g)} g_0(t) + t^{q_p m_v(g)} \sum_{j \geq 1} (\beta + t h_2(t))^j g_j(t), t^{m_v(f)}).$$

Therefore, for $\epsilon > 0$, we have

$$d(\gamma_1 \cap \{u_2 = \epsilon\}, \gamma_2 \cap \{u_2 = \epsilon\}) = \left| \epsilon^{\frac{q_p m_v(g)}{m_v(f)}} \right| H(\epsilon),$$

where

$$H(\epsilon) = \left| \sum_{j \geq 1} \left((\alpha + \epsilon^{\frac{1}{m_v(f)}} h_1(\epsilon^{\frac{1}{m_v(f)}}))^j - (\beta + \epsilon^{\frac{1}{m_v(f)}} h_2(\epsilon^{\frac{1}{m_v(f)}}))^j \right) g_j(\epsilon^{\frac{1}{m_v(f)}}) \right|.$$

We need to prove that

$$H(0) = \left| \sum_{j \geq 1} (\alpha^j - \beta^j) g_j(0) \right|.$$

is non zero 0 when α and β are distinct and in a small enough neighborhood of the origin of \mathbb{C} . Let $j_0 > 0$ be the minimal element of the set $\{j > 0 \mid g_j(0) \neq 0\}$ then

$$H(0) = \left| (\alpha^{j_0} - \beta^{j_0}) g_{j_0}(0) + \sum_{j \geq 1, j \neq j_0} (\alpha^j - \beta^j) g_j(0) \right|.$$

Now, let us prove that $j_0 = 1$. Let us compute in the coordinates (x, y) the equation of the total transform of the polar curve. The jacobian matrix of $\Phi \circ \pi$ is

$$\text{Jac}(\Phi \circ \pi)(x, y) = \begin{pmatrix} * & x^{q_p m_v(g)} (g_1(x) + 2y g_2(x) + \dots) \\ m_v(f) x^{m_v(f)-1} & 0 \end{pmatrix}$$

then $\text{Det}(\text{Jac}(\Phi \circ \pi)(x, y)) = m_v(f) x^{q_p m_v(g) + m_v(f) - 1} (g_1(x) + 2y g_2(x) + \dots) = 0$ is the equation of the total transform of $\Pi_{\mathbb{F}}$. Since $g_{j_0}(0) \neq 0$ it follows that the equation of the strict transform of the polar curve $\Pi_{\mathbb{F}}^*$ is

$$\sum_{j \geq 1} j y^{j-1} g_j(x) = 0.$$

By hypothesis $p \notin \Pi_{\mathbb{F}}^*$ which means that $g_1(0) \neq 0$ and then $j_0 = 1$. Thus

$$H(0) = |(\alpha - \beta) g_1(0) + \sum_{j > 1} (\alpha^j - \beta^j) g_j(0)|.$$

If α and β are distinct and in a small enough neighborhood of $0 \in \mathbb{C}$ we have $H(0) \neq 0$. With this, we can conclude that there exists a neighborhood O_p of p such that for every curvettes satisfying (2.1)

$$d(\gamma_1 \cap \{u_2 = \epsilon\}, \gamma_2 \cap \{u_2 = \epsilon\}) = \left| \epsilon^{\frac{q_p m_v(g)}{m_v(f)}} \right| H(\epsilon) = \Theta(\epsilon^{\frac{m_v(g) q_p}{m_v(f)}}).$$

Now we will see that the number $\frac{q_p m_v(g)}{m_v(f)}$ does not depend of the point p . Let p' be another smooth point of E in E_v such that $p' \notin f^* \cup g^* \cup \Pi_{\mathbb{F}}^*$. Let $c : [0, 1] \rightarrow E_v$ be a continuous injective path which does not pass through f^* , g^* or $\Pi_{\mathbb{F}}^*$ such that $c(0) = p$ and $c(1) = p'$. By compactness of E_v , we can choose a finite family of open sets $\{O_{p_i}, i \in \{1, \dots, s\}\}$ which covers the curve $c[0, 1]$ and such that for every pair

of curvettes $\gamma_{i,1}^*$ and $\gamma_{i,2}^*$ of E_v passing through two different points of O_{p_i} there exists $a_i \in \mathbb{R}$ which verifies

$$d(\gamma_{i,1} \cap \{u_2 = \epsilon\}, \gamma_{i,2} \cap \{u_2 = \epsilon\}) = \Theta(\epsilon^{a_i}),$$

where $\gamma_{i,1} = \Phi_\pi(\gamma_{i,1}^*)$ and $\gamma_{i,2} = \Phi_\pi(\gamma_{i,2}^*)$. We can suppose, even if it means refining and reordering the open cover $\{O_i\}$, that $a_1 = \frac{q_p m_v(g)}{m_v(f)}$, $a_s = \frac{q_p m_v(g)}{m_v(f)}$ and that $O_{p_i} \cap O_{p_{i+1}} \neq \emptyset$ for $i = 1, 2, \dots, s-1$.

Let $\gamma_{i,i+1}^*$ and $\gamma'_{i,i+1}$ be two curvettes passing through two different points of $O_{p_i} \cap O_{p_{i+1}}$, then, by definition of the open cover $\{O_{p_i}\}_i$ we have

$$d(\gamma_{i,i+1} \cap \{u_2 = \epsilon\}, \gamma'_{i,i+1} \cap \{u_2 = \epsilon\}) = \Theta(\epsilon^{a_i}) = \Theta(\epsilon^{a_{i+1}}),$$

where $\gamma_{i,i+1} = \Phi \circ \pi(\gamma_{i,i+1}^*)$, $\gamma'_{i,i+1} = \Phi \circ \pi(\gamma'_{i,i+1})$. Finally, we get that $\frac{q_p m_v(g)}{m_v(f)} = a_1 = a_2 = \dots = a_s = \frac{q_p m_v(g)}{m_v(f)} := q_{g,v}^f$. Furthermore $m_v(f) q_{g,v}^f = m_v(f) \frac{q_p m_v(g)}{m_v(f)}$ is an integer. \square

The following remark will be used in the proofs of the inner rates formula A and of Lemma 5.5.

Remark 2.5. *By equation (2), for any smooth point p of E in E_v with $p \notin f^* \cup g^* \cup \Pi_\Phi^*$, there exists a local system of coordinates (x, y) centered at p such that*

$$\begin{cases} (u_1 \circ \Phi \circ \pi)(x, y) &= x^{m_v(g)} g_0(x) + x^{q_{g,v}^f m_v(f)} \sum_{j \geq 1} y^j g_j(x) \\ (u_2 \circ \Phi \circ \pi)(x, y) &= x^{m_v(f)} \end{cases}$$

with $g_0(0) \neq 0, g_1(0) \neq 0$.

3. INNER RATES FORMULA

In this section we state and prove the main theorem of this article the **inner rates formula A**.

Theorem 3.1 (The inner rates formula). *Let $(X, 0)$ be a complex surface germ with an isolated singularity and let $\pi : (X_\pi, E) \rightarrow (X, 0)$ be a good resolution of $(X, 0)$. Let $g, f : (X, 0) \rightarrow (\mathbb{C}, 0)$ be two holomorphic functions on X such that the morphism $\Phi = (g, f) : (X, 0) \rightarrow (\mathbb{C}^2, 0)$ is finite. Let $M_\pi = (E_{v_i} \cdot E_{v_j})_{i,j \in \{1,2,\dots,n\}}$ be the **intersection matrix** of the dual graph Γ_π , $a_{g,\pi}^f := (m_{v_1}(f) q_{g,v_1}^f, \dots, m_{v_n}(f) q_{g,v_n}^f)$, $K_\pi := (\text{val}_{\Gamma_\pi}(v_1) + 2g_{v_1} - 2, \dots, \text{val}_{\Gamma_\pi}(v_n) + 2g_{v_n} - 2)$, $F_\pi = (f^* \cdot E_{v_1}, \dots, f^* \cdot E_{v_n})$ be the **F-vector** and $P_\pi = (\Pi_\Phi^* \cdot E_{v_1}, \dots, \Pi_\Phi^* \cdot E_{v_n})$ be the **polar vector or P-vector**. Then we have:*

$$M_\pi \cdot a_{g,\pi}^f = \underline{K}_\pi + \underline{F}_\pi - \underline{P}_\pi.$$

Equivalently, for each irreducible component E_v of E we have the following:

$$\left(\sum_{i \in V(\Gamma_\pi)} m_i(f) q_{g,i}^f E_i \right) \cdot E_v = \text{val}_{\Gamma_\pi}(v) + f^* \cdot E_v - \Pi_\Phi^* \cdot E_v + 2g_v - 2,$$

As mentioned in the introduction, this result is a generalization of the "laplacian formula" proved by Belotto, Fantini and Pichon [BdSFP22a, Theorem 4.3]. Their proof relies on topological tools. Here, we are going to prove the theorem 3.1 by using as main ingredient the classical adjunction formula. That we state now.

Definition 3.2 (See e.g [Sha13, Chapter 3. Subsection 6.3]). *Let S be a smooth complex surface. The **canonical divisor** of S , denoted K_S , is the divisor K_Ω associated to any meromorphic 2-form Ω defined on S . It is well defined up to linear equivalence i.e., for any pair of 2-forms Ω and Ω' of S there exists a meromorphic function h on S such that*

$$K_\Omega = (h) + K_{\Omega'}$$

where K_Ω and $K_{\Omega'}$ are respectively the divisors associated to the 2-forms Ω and Ω' .

Theorem 3.3 (Adjunction formula, see e.g [Sha13, Chapter 4. Subsection 2.3]). *Let S be a complex surface and $C \subset S$ be a compact Riemann surface embedded in S . Then*

$$(K_S + C) \cdot C = 2g_C - 2, \quad \text{where } g_C \text{ is the genus of } C.$$

Proof of Theorem 3.1. Let us set $\Phi_\pi = \Phi \circ \pi$. Let E_v be an irreducible component of E . Let us denote by $(u_1, u_2) = (g, f)$ the coordinates of \mathbb{C}^2 . By remark 2.5, given a smooth point p of E in E_v with $p \notin f^* \cup g^* \cup \Pi_\Phi^*$ there exists a local system coordinates (x, y) centered at p such that

$$\begin{cases} (u_1 \circ \Phi_\pi)(x, y) &= x^{m_v(g)} g_0(x) + x^{q_{g,v}^f m_v(f)} \sum_{j \geq 1} y^j g_j(x) \\ (u_2 \circ \Phi_\pi)(x, y) &= x^{m_v(f)} \end{cases}$$

with $g_0(0) \neq 0$ and $g_1(0) \neq 0$. Let us consider the holomorphic 2-form ω on \mathbb{C}^2 defined by $\omega = du_1 \wedge du_2$. Let $\Omega := \Phi_\pi^* \omega$ be the pullback of ω by the holomorphic function Φ_π . In the neighborhood of p , the 2-form Ω is given in the coordinates (x, y) by:

$$\Omega = -m_v(f) x^{q_{g,v}^f m_v(f) + m_v(f) - 1} \sum_{j \geq 1} j y^{j-1} g_j(x) dx \wedge dy.$$

Since $\Phi_\pi = \Phi \circ \pi$ is a local isomorphism on the complement of $E \cup \Pi_\Phi^*$ in X_π , the 2-form Ω does not vanish on this set. Therefore, the canonical divisor K_{X_π} of the smooth complex surface X_π is represented by

$$K_\Omega = \sum_{i \in V(\Gamma_\pi)} (q_{g,i}^f m_i(f) + m_i(f) - 1) E_i + \Pi_\Phi^*.$$

We now apply the adjunction formula 3.3 to the compact Riemann surface E_v

$$(K_\Omega + E_v) \cdot E_v = 2g_v - 2. \quad (3)$$

Replacing K_Ω by its value in the equation (3), we obtain:

$$\sum_{i \in V(\Gamma_\pi)} q_{g,i}^f m_i(f) E_i \cdot E_v + \sum_{i \in V(\Gamma_\pi)} m_i(f) E_i \cdot E_v - \sum_{i \in V(\Gamma_\pi), i \neq v} E_i \cdot E_v + \Pi_\Phi^* \cdot E_v = 2g_v - 2$$

Finally, by Proposition 1.3 we have:

$$\left(\sum_{i \in V(\Gamma_\pi)} m_i(f) E_i \cdot E_v \right) = (-f^* \cdot E_v).$$

Replacing this in the previous equation, we get the desired equality:

$$\sum_{i \in V(\Gamma_\pi)} q_{g,i}^f m_i(f) E_i \cdot E_v = \text{val}_{\Gamma_\pi}(v) + f^* \cdot E_v - \Pi_\Phi^* \cdot E_v + 2g_v - 2.$$

□

Remark 3.4. *Since the intersection matrix of the dual graph Γ_π associated with a good resolution $\pi : (X_\pi, E) \rightarrow (X, 0)$ is negative definite. Theorem 3.1 implies that given the dual graph Γ_π together with the F -vector $(f^* \cdot E_1, \dots, f^* \cdot E_n)$, the inner rates $q_{g,v}^f, v \in V(\Gamma_\pi)$ determines and are determined by the \mathcal{P} -vector $(\Pi_\Phi^* \cdot E_1, \dots, \Pi_\Phi^* \cdot E_n)$.*

Let us give an example where we compute the inner rates using the theorem 3.1.

Example 3.5. *Consider the case where $(X, 0) = (\mathbb{C}^2, 0)$. Let $f(x, y) = y^5 - x^{12}$ and $g(x, y) = x + y$ and consider the finite morphism $\Phi = (g, f) : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$. The polar curve Π_Φ of this morphism has equation $(\frac{\partial f}{\partial y} \cdot \frac{\partial g}{\partial x})(x, y) - (\frac{\partial f}{\partial x} \cdot \frac{\partial g}{\partial y})(x, y) = 5y^4 + 12x^{11} = 0$. Consider the minimal good resolution $\pi : (X_\pi, E) \rightarrow (\mathbb{C}^2, 0)$ of the curve $(gf)^{-1}(0)$. We attach each vertex v of Γ_π corresponding to an irreducible component E_v which meet the strict transform of the polar curve with a red going-out arrow weighted with the intersection number $\Pi_\Phi^* \cdot E_v$.*

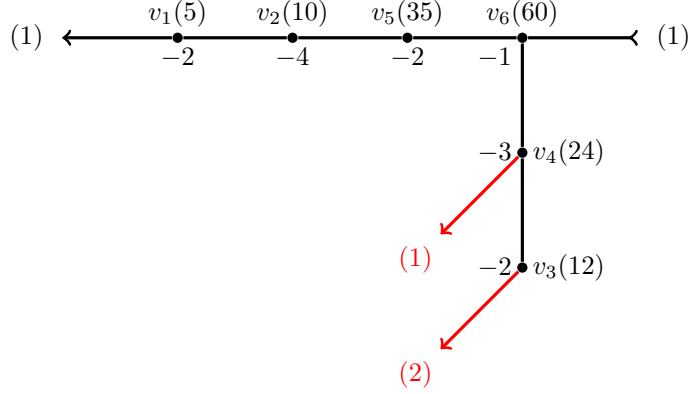


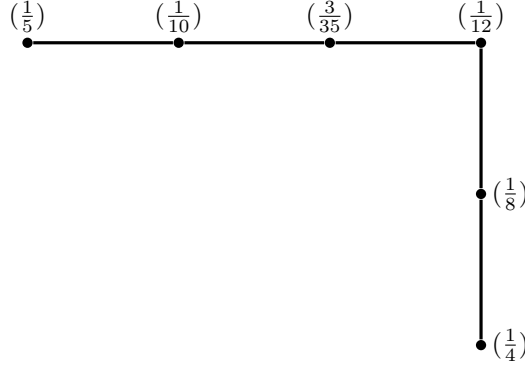
FIGURE 2. The graph Γ_π , decorated with the orders of vanishing of the function $f \circ \pi$ and red arrows corresponding to the components of the polar curve weighted with the intersection numbers.

For $i = 1, \dots, 6$, denote $q_i := q_{g,v_i}^f$. By applying Theorem 3.1 on the dual graph Γ_π we get the following system of equations:

$$\begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & -4 & 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \\ 0 & 1 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 5q_1 \\ 10q_2 \\ 12q_3 \\ 24q_4 \\ 35q_5 \\ 60q_6 \end{pmatrix} = \begin{pmatrix} 1-2 \\ 2-2 \\ 1-2 \\ 2-2 \\ 2-2 \\ 2-2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

whose solution is:

$$(q_1, q_2, q_3, q_4, q_5, q_6) = \left(\frac{1}{5}, \frac{1}{10}, \frac{1}{4}, \frac{1}{8}, \frac{3}{35}, \frac{1}{12} \right)$$

FIGURE 3. The dual graph weighted with the inner rates q_i .

4. NON-ARCHIMEDEAN LINK AND METRIC DUAL GRAPH

In this section we will follow closely [BdSFP22a, Preliminaries]. Let $(X, 0)$ be a complex surface germ with an isolated singularity. Denote by $\mathcal{O} = \widehat{\mathcal{O}_{X,0}}$ the completion of the local ring of X at 0 with respect to its maximal ideal.

Definition 4.1. A (rank 1) **semivaluation** on \mathcal{O} is a map $v : \mathcal{O} \rightarrow \mathbb{R} \cup \{+\infty\}$ such that, for every $f, g \in \mathcal{O}$ and every $\lambda \in \mathbb{C}^\times$

- $v(fg) = v(f) + v(g)$
- $v(f + g) \geq \min\{v(f), v(g)\}$
- $v(\lambda) = \begin{cases} +\infty & \text{if } \lambda = 0 \\ 0 & \text{if } \lambda \neq 0 \end{cases}$.

A **valuation** is semivaluation such that 0 is the only element sent to $+\infty$.

Example 4.2. Let $\pi : (X_\pi, E) \rightarrow (X, 0)$ be a good resolution of $(X, 0)$. Let E_v be an irreducible component of the exceptional divisor E . Let \mathfrak{M} be the maximal ideal of \mathcal{O} and set $m_v(\mathfrak{M}) = \inf\{m_v(f) \mid f \in \mathfrak{M}\}$. The map

$$\begin{aligned} \text{val}_{E_v} : \mathcal{O} &\rightarrow \mathbb{R}_+ \cup \{+\infty\}, \\ f &\mapsto \frac{m_v(f)}{m_v(\mathfrak{M})} \end{aligned}$$

is a valuation on \mathcal{O} . We call it the **divisorial valuation** associated with E_v .

Definition 4.3. The **non-archimedean link** $\text{NL}(X, 0)$ of $(X, 0)$ is the set

$$\text{NL}(X, 0) = \{v : \mathcal{O} \rightarrow \mathbb{R}_+ \cup \{+\infty\} \text{ semi-valuation} \mid v(\mathfrak{M}) = 1 \text{ and } v|_{\mathbb{C}^*} = 0\}$$

whose topology is induced from the product topology $(\mathbb{R}_+ \cup \{+\infty\})^{\mathfrak{M}}$.

Let $\pi : (X_\pi, E) \rightarrow (X, 0)$ be a good resolution of $(X, 0)$. There exists an embedding

$$i_\pi : \Gamma_\pi \rightarrow \text{NL}(X, 0)$$

and a continuous retraction

$$r_\pi : \text{NL}(X, 0) \rightarrow \Gamma_\pi$$

such that $r_\pi \circ i_\pi = \text{Id}_{\Gamma_\pi}$. The embedding i_π maps each vertex v of Γ_π to the divisorial valuation associated with the component E_v , and each edge $e_{v,v'}$ that corresponds

to a point p of the intersection $E_v \cap E_{v'}$ to the set of monomial valuations on X_π at p . We refer to [BdSFP22a, Preliminaries].

Theorem 4.4 (See e.g., [GR21, Theorem 2.27] and [Jon15, Theorem 7.9]). *The family of the continuous retractions $\{r_\pi \mid \pi \text{ is a good resolution of } (X, 0)\}$ induces a natural homeomorphism from $\text{NL}(X, 0)$ to the inverse limit of the dual graphs Γ_π*

$$\text{NL}(X, 0) \cong \varprojlim_{\pi} \Gamma_\pi.$$

Let $\Phi = (g, f) : (X, 0) \rightarrow (\mathbb{C}^2, 0)$ be a finite morphism. It induces a natural morphism $\tilde{\Phi} : \text{NL}(X, 0) \rightarrow (\mathbb{C}^2, 0)$. Indeed, we set $\Phi^\# : \widehat{\mathcal{O}}_{(\mathbb{C}^2, 0)} \rightarrow \widehat{\mathcal{O}}_{(X, 0)}$ defined by $\Phi^\#(h) = h \circ \Phi$. Hence

$$\begin{aligned} \tilde{\Phi} : \text{NL}(X, 0) &\rightarrow \text{NL}(\mathbb{C}^2, 0). \\ v &\mapsto v \circ \Phi^\# \end{aligned}$$

By using the description of $\text{NL}(X, 0)$ as inverse limit of dual graphs (Theorem 4.4), we have a way to compute the image by $\tilde{\Phi}$ of a divisorial valuation v :

- (i) Take a good resolution $\pi : (X_\pi, E) \rightarrow (X, 0)$ such that v is associated to some exceptional component E_v of E .
- (ii) Let p be a smooth point of E such that p does not belong to $f^* \cup g^* \cup \Pi_{\tilde{\Phi}}$.
- (iii) Take two curvettes γ_1^* and γ_2^* of E_v which passes through two distinct points of E_v which are in a neighborhood O_p of p .
- (iv) Consider the curves $\gamma_1 = \Phi \circ \pi(\gamma_1^*)$ and $\gamma_2 = \Phi \circ \pi(\gamma_2^*)$.
- (v) Let $\sigma : (Y, F) \rightarrow (\mathbb{C}^2, 0)$ be the minimal sequence of blowups such that the strict transforms of γ_1 and γ_2 are smooth and meet transversely an irreducible component $F_w \subset F$ at two different points. The morphism σ and the exceptional component F_w do not depend of the choice of the curvettes γ_1^* and γ_2^* as a consequence of Proposition 2.3.
- (vi) Finally, $\tilde{\Phi}(v) = w$ where w is the divisorial valuation of $\text{NL}(\mathbb{C}^2, 0)$ associated to F_w .

Since Φ is finite, the map $\tilde{\Phi}$ is a ramified covering. It is reflected at the level of the dual graphs as follows. Let $\pi : (X_\pi, E) \rightarrow (X, 0)$ be a good resolution of $(X, 0)$ and $\sigma : (Y, F) \rightarrow (\mathbb{C}^2, 0)$ be a sequence of blowups of the origin of \mathbb{C}^2 such that for any divisorial valuation v associated to an irreducible component E_v of E , the induced divisorial valuation $\tilde{\Phi}(v) = w$ is associated to an irreducible component F_w of F . Then the set $\{\tilde{\Phi}(\text{val}_{E_v}) \mid E_v \subset E\}$ defines a subgraph of Γ_σ and the restriction of $\tilde{\Phi}$ to Γ_π is a ramified covering of graphs onto its image.

Let $f : (X, 0) \rightarrow (\mathbb{C}, 0)$ be a holomorphic function germ. We now define a metric on $\text{NL}(X, 0)$ compatible with f following [BdSFP22a, Subsection 2.3]. We start by doing it on any graph Γ_π where $\pi : (X_\pi, E) \rightarrow (X, 0)$ is a good resolution. Let us endow the dual graph Γ_π with a metric by declaring the length of an edge $e_{v, v'}$ to be

$$\text{length}_f(e_{v, v'}) = \frac{1}{m_v(f)m_{v'}(f)}.$$

(See [BdSFP22a, Remark 2.5] for a geometric interpretation of this metric in terms of the Milnor fibers of f).

Now, let p be an intersection point between two irreducible components E_v and $E_{v'}$ of E and let $\tilde{\pi} : (X_{\tilde{\pi}}, \tilde{E}) \rightarrow (X, 0)$ be the good resolution obtained by blowing

up the point p . Then the exceptional component E_w that arises has multiplicity $m_w(f) = m_v(f) + m_{v'}(f)$. Since $\frac{1}{m_w(f)m_v(f)} + \frac{1}{m_w(f)m_{v'}(f)} = \frac{1}{m_v(f)m_{v'}(f)}$, the inclusion of Γ_π in $\Gamma_{\tilde{\pi}}$ is an isometry. Therefore, by passing to the limit, the metric on the dual graphs Γ_π defines a metric on the non-archimedean link $\text{NL}(X, 0)$ called the **skeletal metric on $\text{NL}(X, 0)$** with respect to f .

Example 4.5. Consider again the dual graph Γ_π of the minimal good resolution π of the curve of equation $f(x, y) = y^5 - x^{12} = 0$ in \mathbb{C}^2 introduced in Example 1.6. The following figure shows Γ_π decorated with the order of vanishing of the function $f \circ \pi$ and the length of each of its edges.

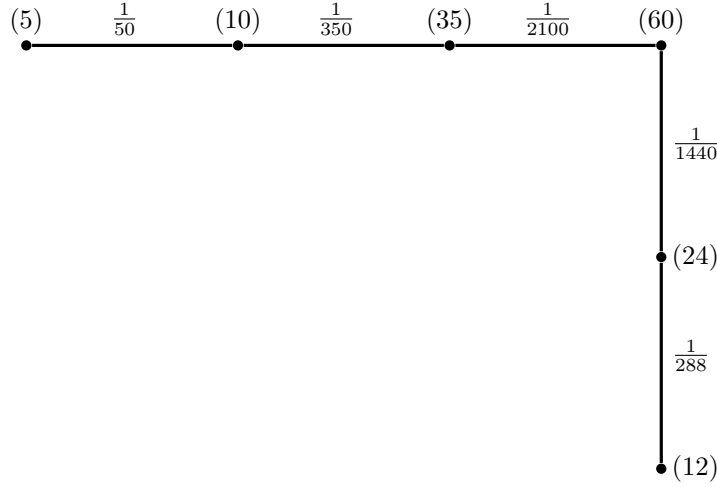


FIGURE 4. The graph Γ_π decorated with the orders of vanishing of the function $f \circ \pi$ and the length of each of its edges.

With the metric defined before on Γ_π we then have a metric graph

$$\Gamma_\pi = \left(V(\Gamma_\pi), E(\Gamma_\pi), \begin{array}{l} \text{length}_f : E(\Gamma_\pi) \rightarrow \mathbb{Q}_+ \\ e_{vv'} \mapsto \text{length}(e_{vv'}) = \frac{1}{m_v(f)m_{v'}(f)} \end{array} \right)$$

Definition 4.6. A function $F : \Gamma_\pi \rightarrow \mathbb{R}$ is **piecewise linear** if F is a continuous piecewise affine map with integral slopes with respect to the metric induced by length_f and F has only finitely many points of non-linearity on each edge of Γ_π .

A divisor $D = \sum_{v \in \Gamma_\pi} a_v v$ of Γ_π is a finite sum of points of Γ_π with integral coefficients $a_v \in \mathbb{Z}$. We denote by $\text{Div}(\Gamma_\pi)$ the group of divisors of Γ_π .

Definition 4.7. Let F be a piecewise linear map on the dual graph Γ_π . The **Laplacian** of F is the divisor $\Delta_{\Gamma_\pi}(F)$ of Γ_π whose coefficient at a point v of Γ_π is the sum of the **outgoing slopes** of F at v :

$$\Delta_{\Gamma_\pi}(F) = \sum_{v \in V(\Gamma_\pi)} \left(\sum_{v' \neq v} \frac{F(v') - F(v)}{\text{length}_f(v', v)} (E_v \cdot E_{v'}) \right) v.$$

5. THE INNER RATES FUNCTION

Let $(X, 0)$ be a complex surface germ with an isolated singularity and let $g, f : (X, 0) \rightarrow (\mathbb{C}, 0)$ be two holomorphic functions such that the morphism $\Phi = (g, f) : (X, 0) \rightarrow (\mathbb{C}^2, 0)$ is finite. The aim of this section is to define a continuous function on the non archimedean link whose values are the inner rates on the divisorial elements of $\text{NL}(X, 0)$.

Now, let us state the main proposition of this section which is a direct adaptation of [BdSFP22a, Lemma 3.8].

Proposition 5.1. *Denote by $(u_1, u_2) = (g, f)$ the coordinates of \mathbb{C}^2 . There exists a unique continuous function*

$$\mathcal{I}_g^f : \text{NL}(X, 0) \rightarrow \mathbb{R}_{>0} \cup \{+\infty\}$$

such that $\mathcal{I}_g^f(v) = q_{g,v}^f$ for every divisorial point v of $\text{NL}(X, 0)$ and such that we have the following commutative diagram

$$\begin{array}{ccc} \text{NL}(X, 0) & \xrightarrow{\tilde{\Phi}} & \text{NL}(\mathbb{C}^2, 0) \\ & \searrow \mathcal{I}_g^f & \downarrow \mathcal{I}_{u_1}^{u_2} \\ & & \mathbb{R}_{>0} \cup \{+\infty\} \end{array}$$

Furthermore, if π is a good resolution of $(X, 0)$ compatible with the finite morphism $\Phi = (g, f)$ then \mathcal{I}_g^f is piecewise linear on Γ_π with respect to the metric length_f .

Definition 5.2. *The function \mathcal{I}_g^f is called **the inner rates function with respect to the morphism $\Phi = (g, f)$.***

In order to prove Proposition 5.1 we will need the following lemmas.

Lemma 5.3. *Let $\Phi = (g, f) : (X, 0) \rightarrow (\mathbb{C}^2, 0)$ be a finite morphism and let $\pi : (X_\pi, E) \rightarrow (X, 0)$ be a good resolution of $(X, 0)$ and $(gf)^{-1}(0)$. Let E_v be an irreducible component of E such that it intersects the curve $f^* \cup g^*$. We have*

$$q_{g,v}^f = \frac{m_v(g)}{m_v(f)}$$

Proof. Let p be a point of g^* . Let (x, y) be a local system of coordinates centered at p such that E_v has local equation $x = 0$ and such that $(f \circ \pi)(x, y) = x^{m_v(f)}$. Let S_g be the element of $\mathbb{C}\{x, y\}$ such that $(g \circ \pi)(x, y) = x^{m_v(g)} S_g(x, y)$. Then, the equation of the strict transform g^* in the coordinates (x, y) is $S_g(x, y) = 0$. Since the curve g^* is transverse to the component E_v we can write $S_g(x, y) = y^{k_v} U_g(x, y)$ where U_g is a unit of $\mathbb{C}\{x, y\}$ and k_v is an element of \mathbb{N}^* .

Let γ_1^* and γ_2^* be two curvettes of E_v parametrized respectively by

$$t \mapsto (t, \alpha), \quad t \mapsto (t, \beta), \quad \alpha, \beta \in \mathbb{C}.$$

The curves $\gamma_1 = (\Phi \circ \pi)(\gamma_1^*)$ and $\gamma_2 = (\Phi \circ \pi)(\gamma_2^*)$ are then parametrized respectively by

$$\begin{aligned} t &\mapsto (\alpha^{k_v} t^{m_v(g)} U_g(t, \alpha), t^{m_v(f)}) \\ t &\mapsto (\beta^{k_v} t^{m_v(g)} U_g(t, \beta), t^{m_v(f)}). \end{aligned}$$

Therefore, for $\epsilon > 0$, we have

$$d(\gamma_1 \cap \{u_2 = \epsilon\}, \gamma_2 \cap \{u_2 = \epsilon\}) = \left| \epsilon^{\frac{m_v(g)}{m_v(f)}} \right| H(\epsilon),$$

where

$$H(\epsilon) = \left| \alpha^{k_v} U_g(\epsilon^{\frac{1}{m_v(f)}}, \alpha) - \beta^{k_v} U_g(\epsilon^{\frac{1}{m_v(f)}}, \beta) \right|.$$

There exists two numbers α and β such that $H(0) \neq 0$, we can then conclude that $q_{g,v}^f = \frac{m_v(g)}{m_v(f)}$. The case $p \in f^*$ is treated with similar arguments. \square

Definition 5.4. A good resolution $\pi : (X_\pi, E) \rightarrow (X, 0)$ of $(X, 0)$ is said to be **compatible with the finite morphism** $\Phi = (g, f)$ if it is a good resolution of the curve $\Pi_\Phi \cup (gf)^{-1}(0)$.

Lemma 5.5. Let $\Phi = (g, f) : (X, 0) \rightarrow (\mathbb{C}^2, 0)$ be a finite morphism and let $\pi : (X_\pi, E) \rightarrow (X, 0)$ be a good resolution of $(X, 0)$ compatible with the finite morphism Φ . Let E_v be an irreducible component of E . Let p be a point of E_v and let E_w be the exceptional component created by the blowup of X_π at p . Then:

- (i) if p is a smooth point of E , then
- if p does not lie in the strict transforms g^*, f^* or Π_Φ^* , then

$$m_w(g) = m_v(g), \quad m_w(f) = m_v(f), \quad \text{and} \quad q_{g,w}^f = q_{g,v}^f + \frac{1}{m_v(f)};$$

- if p lies in the strict transform g^* then:

$$m_w(g) = m_v(g) + g^* \cdot E_v, \quad m_w(f) = m_v(f), \quad \text{and} \quad q_{g,w}^f = \frac{m_w(g)}{m_w(f)} = q_{g,v}^f + \frac{g^* \cdot E_v}{m_v(f)};$$

- if p lies in the strict transform f^* then:

$$m_w(g) = m_v(g), \quad m_w(f) = m_v(f) + f^* \cdot E_v, \quad \text{and} \quad q_{g,w}^f = \frac{m_w(g)}{m_w(f)} = \frac{m_v(g)q_{g,v}^f}{m_v(f) + f^* \cdot E_v};$$

- if p lies in the strict transform Π_Φ^*

$$m_w(g) = m_v(g), \quad m_w(f) = m_v(f), \quad \text{and} \quad q_{g,w}^f = q_{g,v}^f + \frac{1 + \Pi_\Phi^* \cdot E_v}{m_v(f)};$$

- (ii) if p is a double point of E_v and $E_{v'}$, then

$$m_w(g) = m_v(g) + m_{v'}(g) \quad \text{and} \quad m_w(f) = m_v(f) + m_{v'}(f);$$

$$q_{g,w}^f = \frac{q_{g,v}^f m_v(f) + q_{g,v'}^f m_{v'}(f)}{m_v(f) + m_{v'}(f)}$$

Remark 5.6. The point (i) of this lemma is more general than needed for this paper. Indeed, we will only use it for the morphism identity of \mathbb{C}^2 , whose polar curve is empty.

Proof. Denote by e_p the blowup of X_π centered at p . Assume first that p is a smooth point of E_v which does not lie in the strict transform of f, g or the polar curve. By remark 2.5 there exists a local system of coordinates (x, y) centered at p such that:

$$\begin{cases} (g \circ \pi)(x, y) &= x^{m_v(g)} g_0(x) + x^{q_{g,v}^f m_v(f)} \sum_{j \geq 1} y^j g_j(x) \\ (f \circ \pi)(x, y) &= x^{m_v(f)} \end{cases}$$

with $g_0(0) \neq 0$, $g_1(0) \neq 0$.

Let e_p be the blowup of X_π at p . In the coordinates chart $(x, y) \mapsto (x', x'y')$ we have

$$\begin{cases} (g \circ \pi \circ e_p)(x', y') &= x'^{m_v(g)} g_0(x') + x'^{q_{g,v}^f m_v(f) + 1} \sum_{j \geq 1} y'^j x'^{j-1} g_j(x') \\ (f \circ \pi \circ e_p)(x', y') &= x'^{m_v(f)}. \end{cases}$$

Therefore $m_w(g) = m_v(g)$ and $m_w(f) = m_v(f)$. By comparing the equality of $g \circ \pi \circ e_p$ with the first equality given by the remark 2.5 for E_w , we get $q_{g,w}^f m_w(f) = q_{g,v}^f m_v(f) + 1$, which proves

$$q_{g,w}^f = q_{g,v}^f + \frac{1}{m_v(f)}.$$

Assume that p is in g^* . Using again the notations of the proof of Lemma 5.3. In the coordinates chart $(x, y) \mapsto (x', x'y')$, we have:

$$\begin{cases} (g \circ \pi \circ e_p)(x', y') &= x'^{m_v(g) + k_v} y'^{k_v} U_g(x', x'y') \\ (f \circ \pi \circ e_p)(x', y') &= x'^{m_v(f)} \end{cases}$$

where U_g is a unit of $\mathbb{C}\{x, y\}$ and k_v is a non zero integer. Thus, by noticing that $k_v = g^* \cdot E_v$, we have

$$m_w(g) = m_v(g) + g^* \cdot E_v \quad \text{and} \quad m_w(f) = m_v(f).$$

Since the component E_w intersects g^* , then, by the lemma 5.3, we get that

$$q_{g,w}^f = \frac{m_w(g)}{m_w(f)} = q_{g,v}^f + \frac{g^* \cdot E_v}{m_v(f)}$$

If p is in f^* , with similar arguments we get:

$$m_w(g) = m_v(g) \quad \text{and} \quad m_w(f) = m_v(f) + f^* \cdot E_v.$$

$$q_{g,w}^f = \frac{m_w(g)}{m_w(f)} = \frac{m_v(f) q_{g,v}^f}{m_v(f) + f^* \cdot E_v}$$

We will now suppose that p lies in the strict transform of the polar curve Π_Φ^* . Let (x, y) be a local system of coordinates centered at p such that E_v has local equation $x = 0$ and such that $(f \circ \pi)(x, y) = x^{m_v(f)}$. Let U be the unit of $\mathbb{C}\{x, y\}$ such that $(g \circ \pi)(x, y) = x^{m_v(g)} U(x, y)$. Let e_p be the blowup of X_π at p . in the coordinates chart $(x, y) \mapsto (x', x'y')$ we have

$$\begin{cases} (g \circ \pi \circ e_p)(x', y') &= x'^{m_v(g)} U(x', x'y') \\ (f \circ \pi \circ e_p)(x', y') &= x'^{m_v(f)}. \end{cases}$$

We then deduce that $m_w(f) = m_v(f)$, $m_w(g) = m_v(g)$ and it follows using the inner rates formula 3.1 on the vertex w that:

$$q_{g,w}^f = q_{g,v}^f + \frac{1 + \Pi_\Phi^* \cdot E_w}{m_v(f)}.$$

We now need to prove that $\Pi_\Phi^* \cdot E_w = \Pi_\Phi^* \cdot E_v$. The equation of the total transform of the polar curve in the coordinates (x, y) is $x^{m_v(g) + m_v(f) - 1} \frac{\partial U}{\partial y}(x, y) = 0$. Since

the polar curve is transverse to the exceptional divisor E_v we can write $\frac{\partial U}{\partial y}(x, y) = y^{k_v} V(x, y)$ where V is an element of $\mathbb{C}\{x, y\}$ such that y is not a divisor of V and k_v is a non zero positive integer. The total transform of the polar curve by $e_p \circ \pi$, in the coordinates chart $(x, y) \mapsto (x', x'y')$, is:

$$m_v(f)x'^{m_v(g)+m_v(f)+k_v-1}y'^{k_v}V(x', x'y') = 0.$$

Using this last equation, a direct computation shows that the intersection number between the curve E_w and the strict transform of the polar curve equals $k_v = \Pi_{\Phi}^* \cdot E_v$.

Assume now that p is an intersecting point between two irreducible components E_v and $E_{v'}$. In local coordinates (x, y) centered at p , we can assume without loss of generality that

$$(f \circ \pi)(x, y) = x^{m_v(f)}y^{m_{v'}(f)}.$$

In the coordinates chart $(x, y) \mapsto (x', x'y')$ we have

$$(f \circ \pi \circ e_p)(x', y') = x'^{m_v(f)+m_v(g)}y'^{m_{v'}(f)},$$

we then deduce that $m_w(f) = m_v(f) + m_{v'}(f)$. By symmetry of the roles of f and g we also have $m_w(g) = m_v(g) + m_{v'}(g)$. By applying the inner rates formula (Theorem 3.1) on the vertex w we get:

$$q_{g,w}^f = \frac{q_{g,v}^f m_v(f) + q_{g,v'}^f m_{v'}(f)}{m_v(f) + m_{v'}(f)}$$

□

Now we are ready to prove the proposition that shows the existence of the inner rates function.

Proof of Proposition 5.1. Let $\pi : (X_\pi, E) \rightarrow (X, 0)$ be a good resolution of $(X, 0)$ compatible with the morphism $\Phi = (g, f)$. We only need to show that the inner rates extend uniquely to a continuous map on Γ_π which is linear on its edges with integral slopes, because $\text{NL}(X, 0)$ is homeomorphic to the inverse limit of dual graphs.

Let $e_{vv'}$ be an edge of Γ_π . By definition, the slope on $e_{v,v'}$ is $\frac{q_{g,v'}^f - q_{g,v}^f}{\text{length}_f(e_{vv'})} = m_v(f)m_{v'}(f)(q_{g,v'}^f - q_{g,v}^f)$ and it is an integer by Proposition 2.3.

By density, it is sufficient to prove the linearity of \mathcal{I}_g^f along $e_{vv'}$ on the set of the divisorial valuations. On the other hand, since every divisorial point of that edge are obtained by successively blowing up its double points, it is sufficient to prove that \mathcal{I}_g^f is linear on the set $\{v, v', v''\}$, where v'' is obtained by blowing up the double point $E_v \cap E_{v'}$. Therefore, all we have to show is that

$$\frac{\mathcal{I}_g^f(v') - \mathcal{I}_g^f(v)}{\text{length}_f(e_{vv'})} = \frac{\mathcal{I}_g^f(v'') - \mathcal{I}_g^f(v)}{\text{length}_f(e_{vv''})}. \quad (4)$$

Since the good resolution π is compatible with the morphism Φ , the strict transforms f^*, g^* and Π_{Φ}^* do not pass through the double point $E_v \cap E_{v'}$. The equality (4) is then a direct consequence of the last part of Lemma 5.5.

□

5.1. Laplacian formula on the metric dual graph of a good resolution.

Let $(X, 0)$ be a complex surface germ with an isolated singularity and $\Phi = (g, f) : (X, 0) \rightarrow (\mathbb{C}^2, 0)$ be a finite morphism. Let $\pi : (X_\pi, E) \rightarrow (X, 0)$ be a good resolution of $(X, 0)$ compatible with the morphism Φ . We proved in Proposition 5.1 that the inner rate function \mathcal{I}_g^f is a piecewise linear map on the dual graph Γ_π with respect to the metric Length_f . Then, as an application of the inner rates formula 3.1 we will compute the laplacian of \mathcal{I}_g^f on the dual graph. We will now give three example of divisor on the dual graph Γ_π which are important for the statement of our result :

- (i) $K_{\Gamma_\pi} = \sum_{v \in \Gamma_\pi} m_v(f)(\text{val}_{\Gamma_\pi}(v) + 2g_v - 2) \cdot v$ it is a divisor because every point v which is not a vertex of Γ_π has valency 2 and genus equal 0.
- (ii) $F_{\Gamma_\pi} = \sum_{v \in \Gamma_\pi} (m_v(f) + m_v(g))(f^* \cdot E_v) \cdot v$ it is a divisor because there are finitely many branches of f^* .
- (iii) $P_{\Gamma_\pi} = \sum_{v \in \Gamma_\pi} m_v(f)(\Pi_\Phi^* \cdot E_v) \cdot v$ it is a divisor because there are finitely many branches of Π_Φ^* .

Now, we are ready to state the inner rates formula (Theorem 3.1) in terms of the laplacian of the function \mathcal{I}_g^f .

Corollary 5.7. *The laplacian of the inner rates function \mathcal{I}_g^f is given by the formula*

$$\Delta_{\Gamma_\pi}(\mathcal{I}_g^f) = K_{\Gamma_\pi} + F_{\Gamma_\pi} - P_{\Gamma_\pi}$$

Proof. Let v be a vertex of Γ_π and let us write the inner rates formula 3.1 on v :

$$\left(\sum_{i \in V(\Gamma_\pi)} m_i(f) q_{g,i}^f E_i \right) \cdot E_v = \text{val}_{\Gamma_\pi}(v) + f^* \cdot E_v - \Pi_\Phi^* \cdot E_v + 2g_v - 2.$$

Let v_1, v_2, \dots, v_k be the vertices of Γ_π adjacent to v . Then, the last equality is equivalent to

$$\sum_{i=1}^k m_{v_i}(f) q_{g,v_i}^f + m_v(f) q_{g,v}^f E_v^2 = \text{val}_{\Gamma_\pi}(v) + f^* \cdot E_v - \Pi_\Phi^* \cdot E_v + 2g_v - 2.$$

By Proposition 1.3 we have $m_v(f) E_v^2 = -\sum_{i=1}^k m_{v_i}(f) E_{v_i} - f^* \cdot E_v$ and we inject it on the last equation:

$$\sum_{i=1}^k m_{v_i}(f)(q_{g,v_i}^f - q_{g,v}^f) = (q_{g,v}^f + 1) f^* \cdot E_v + \text{val}_{\Gamma_\pi}(v) + 2g_v - 2 - \Pi_\Phi^* \cdot E_v.$$

If $f^* \cdot E_v \neq 0$, by Lemma 5.3, we have $q_{g,v}^f = \frac{m_v(g)}{m_v(f)}$. Then, by multiplying both sides of the equation by $m_v(f)$, we get

$$\sum_{i=1}^k m_v(f) m_{v_i}(f)(q_{g,v_i}^f - q_{g,v}^f) = (m_v(g) + m_v(f)) f^* \cdot E_v + m_v(f)(\text{val}_{\Gamma_\pi}(v) + 2g_v - 2) - m_v(f) \Pi_\Phi^* \cdot E_v.$$

Since $\text{length}_f(v, v_i) = \frac{1}{m_v(f) m_{v_i}(f)}$, we can then conclude that

$$\Delta_{\Gamma_\pi}(\mathcal{I}_g^f) = K_{\Gamma_\pi} + F_{\Gamma_\pi} - P_{\Gamma_\pi}$$

□

6. GROWTH BEHAVIOUR OF THE INNER RATES FUNCTION

The aim of this section is to prove Point (i) of Theorem B. For this aim, we will need the following lemma which is an adaptation of [BdSFP22a, Subsection 4.5].

Lemma 6.1. *Let $(X, 0)$ be a complex surface germ with an isolated singularity and let $\Phi = (g, f) : (X, 0) \rightarrow (\mathbb{C}^2, 0)$ be a finite morphism. Let $\pi : (X_\pi, E) \rightarrow (X, 0)$ be a good resolution of $(X, 0)$. There exists a finite sequence of blowups $\sigma : (Y, F) \rightarrow (\mathbb{C}^2, 0)$, a good resolution $\tilde{\pi} : (X_{\tilde{\pi}}, \tilde{E}) \rightarrow (X, 0)$, a finite sequence of blowups $\alpha_\pi : (X_{\tilde{\pi}}, \tilde{E}) \rightarrow (X_\pi, E)$ and a morphism $\Phi_{\tilde{\pi}} : (X_{\tilde{\pi}}, \tilde{E}) \rightarrow (Y, F)$ such that the following diagram commutes*

$$\begin{array}{ccccc} & & \tilde{\pi} & & \\ & & \curvearrowright & & \\ (X_{\tilde{\pi}}, \tilde{E}) & \xrightarrow{\alpha_\pi} & (X_\pi, E) & \xrightarrow{\pi} & (X, 0) \\ & \searrow \Phi_{\tilde{\pi}} & & & \downarrow \Phi \\ & & (Y, F) & \xrightarrow{\sigma} & (\mathbb{C}^2, 0) \end{array}$$

and the following properties are satisfied

- (i) For every vertex v of Γ_π , we have $\tilde{\Phi}(v) \in V(\Gamma_\sigma)$
- (ii) For every vertex w of Γ_σ , we have $\tilde{\Phi}^{-1}(w) \subset V(\Gamma_{\tilde{\pi}})$.

Definition 6.2. *We call the map $\tilde{\pi}$ defined as above the good resolution of $(X, 0)$ adapted to π and Φ .*

Remark 6.3. *This definition is a generalization of [BdSFP22a, Definition 4.13].*

Proof of Lemma 6.1. Let $\sigma : (Y, F) \rightarrow (\mathbb{C}^2, 0)$ be the minimal sequence of blowups of the origin of \mathbb{C}^2 such that $\tilde{\Phi}(V(\Gamma_\pi)) \subset V(\Gamma_\sigma)$. Consider the pullback by Φ of the resolution σ i.e.,

$$X' = \overline{\{(a, b) \in X \setminus 0 \times Y \setminus F \mid \Phi(a) = \sigma(b)\}} \subset X \times Y.$$

Let $p_1 : X' \rightarrow X$ and $p_2 : X' \rightarrow Y$ be the first and second projection respectively. Let $\pi' : X_{\tilde{\pi}} \rightarrow X'$ be the minimal good resolution of X' . We consider the morphisms

$$\tilde{\pi} := p_1 \circ \pi' : (X_{\tilde{\pi}}, \tilde{E}) \rightarrow (X, 0) \quad \text{and} \quad \Phi_{\tilde{\pi}} := p_2 \circ \pi' : (X_{\tilde{\pi}}, \tilde{E}) \rightarrow (Y, F).$$

The existence of the morphism $\Phi_{\tilde{\pi}}$ implies that for every vertex w of Γ_σ we have $\tilde{\Phi}^{-1}(w) \subset V(\Gamma_{\tilde{\pi}})$. Let $v \in \Gamma_\pi$. By definition of σ , we have $\tilde{\Phi}(v) \subset V(\Gamma_\sigma)$ and then $v \in V(\Gamma_{\tilde{\pi}})$. Therefore, $V(\Gamma_\pi) \subset V(\Gamma_{\tilde{\pi}})$, which means that the resolution $\tilde{\pi}$ factors through π . The resulting sequence of blowups is the desired $\alpha_\pi : (X_{\tilde{\pi}}, \tilde{E}) \rightarrow (X_\pi, E)$ \square

We are now ready to prove Point (i) of Theorem B that we restate in terms of the inner rates function:

Proposition 6.4. *Let $\pi : (X_\pi, E) \rightarrow (X, 0)$ be a good resolution of the complex surface germ $(X, 0)$. Let v be an element of $V(\Gamma_\pi)$. There exists a path from an f -node to v in Γ_π along which the function \mathcal{I}_g^f is strictly increasing.*

Remark 6.5. *This result is a generalization of [BdSFP22a, Proposition 3.9] which treated the case where the morphism Φ is a linear projection on \mathbb{C}^2 .*

Proof. The case where $(X, 0) = (\mathbb{C}^2, 0)$ and Φ is the identity of \mathbb{C}^2 and where π is a finite sequence of point blowups is a direct consequence of Lemma 5.5.

Let us now treat the general case. The statement when v is an f -node is trivial by taking the constant path. Assume now that v is not an f -node. Let $\tilde{\pi} : (X_{\tilde{\pi}}, \tilde{E}) \rightarrow (X, 0)$ be the good resolution of $(X, 0)$ adapted to π , we will use the notations of lemma 6.1. Let $\tilde{v} \in V(\Gamma_{\tilde{\pi}})$ such that $\alpha_{\tilde{\pi}}(\tilde{v}) = v$. Since v is not an f -node, it is also the case for \tilde{v} . Let (u_1, u_2) be the coordinates of \mathbb{C}^2 . Consider $w = \tilde{\Phi}(\tilde{v})$ and let β' be the simple path in $\Gamma_{\tilde{\sigma}}$ starting from u_2 -node and ending at w . By the particular case $(X, 0) = (\mathbb{C}^2, 0)$ treated at the beginning of the proof, the function $\mathcal{I}_{u_1}^{u_2}$ is strictly increasing along β' . The path β' can be lifted via the ramified covering $\tilde{\Phi}$ to a union of simple paths each of them joining an f -node to a vertex of $\tilde{\Phi}^{-1}(w) \subset \Gamma_{\tilde{\pi}}$. Let us choose one of them ending at \tilde{v} and name it $\tilde{\beta}$. Let $\tilde{\alpha}_{\tilde{\pi}} : \Gamma_{\tilde{\pi}} \rightarrow \Gamma_{\pi}$ be the continuous map induced by $\alpha_{\tilde{\pi}}$. Then, the function \mathcal{I}_g^f is strictly increasing along the path $\beta = \tilde{\alpha}_{\tilde{\pi}}(\tilde{\beta})$ which start from an f -node and ends at v . \square

7. PROOF OF POINTS (ii) AND (iii) OF THEOREM B

Let $(X, 0)$ be a complex surface germ with an isolated singularity and $\Phi = (g, f) : (X, 0) \rightarrow (\mathbb{C}^2, 0)$ be a finite morphism. We start by restating Theorem B in terms of the inner rates function defined in Section 5 and the Hironaka quotients function on $\text{NL}(X, 0)$ that we define now.

Definition 7.1 ([MM20, Definition 4]). *Let E_v be an irreducible component of E . The **Hironaka quotient** of E_v with respect to $\Phi = (g, f)$ is the number:*

$$h_{g,v}^f := \frac{m_v(g)}{m_v(f)}.$$

Using Lemma 5.5 and the same arguments of the proof of Proposition 5.1 we obtain:

Proposition 7.2. *There exists a unique continuous function*

$$\mathcal{H}_g^f : \text{NL}(X, 0) \rightarrow \mathbb{R}_{>0} \cup \infty$$

such that $\mathcal{H}_g^f(v) = h_{g,v}^f$ for every divisorial point v of $\text{NL}(X, 0)$ and we have the following commutative diagram

$$\begin{array}{ccc} \text{NL}(X, 0) & \xrightarrow{\tilde{\Phi}} & \text{NL}(\mathbb{C}^2, 0) \\ & \searrow \mathcal{H}_g^f & \downarrow \mathcal{H}_{u_1}^{u_2} \\ & & \mathbb{R}_{>0} \cup \infty \end{array}$$

If π is a good resolution of $(X, 0)$ compatible with the finite morphism $\Phi = (g, f)$ then \mathcal{H}_g^f is linear on the edges of Γ_{π} with integral slopes.

Proof. The commutative diagram of this proposition is a consequence of [MM20, Remark 5]. We get the rest with the arguments of the proof of Proposition 5.1 \square

Definition 7.3. *The function \mathcal{H}_g^f is called the **Hironaka quotients function** with respect to the morphism $\Phi = (g, f)$.*

Definition 7.4. We denote by \mathcal{A}_π the union of every simple paths joining an f -node to a g -node along which the inner rates function \mathcal{I}_g^f is strictly increasing.

Now, we can restate Theorem B in terms of the inner rates functions and the Hironaka quotients function.

Theorem 7.5. Let $(X, 0)$ be a complex surface germ with an isolated singularity and let $\Phi = (g, f) = (u_1, u_2) : (X, 0) \rightarrow (\mathbb{C}^2, 0)$ be a finite morphism. Let $\pi : (X_\pi, E) \rightarrow (X, 0)$ be a good resolution of $(X, 0)$. Then:

- (i) Let v be a vertex of Γ_π . There exists a path from an f -node to v in Γ_π along which the function \mathcal{I}_g^f is strictly increasing.
- (ii) The functions \mathcal{I}_g^f and \mathcal{H}_g^f coincide on \mathcal{A}_π .
- (iii) The Hironaka quotients function \mathcal{H}_g^f restricted to Γ_π is constant on every connected component of the topological closure of $\Gamma_\pi \setminus \mathcal{A}_\pi$.

Remark 7.6. Point (i) is exactly Proposition 6.4. It implies in particular that the inner rates function is not locally constant on Γ_π .

The rest of this section is devoted to the proof of Points (ii) and (iii).

Proof of Point (ii) of Theorem 7.5. The case where $(X, 0) = (\mathbb{C}^2, 0)$ and Φ is the identity map of $(\mathbb{C}^2, 0)$ is a direct consequence of Lemmas 5.5 and 5.3. Notice that in this case \mathcal{A}_π is a simple path joining the u_2 -node to the u_1 -node where (u_1, u_2) are the coordinates of \mathbb{C}^2 .

Let $\tilde{\pi} : (X_{\tilde{\pi}}, \tilde{E}) \rightarrow (X, 0)$ be the good resolution of $(X, 0)$ adapted to π , we use again the commutative diagram of Lemma 6.1. We need to prove that $\tilde{\Phi}(\mathcal{A}_\pi) \subset \mathcal{A}_\sigma$. Let $\alpha : [0, 1] \rightarrow \mathcal{A}_{\tilde{\pi}}$ be a path from an f -node to a g -node along which the inner rates function \mathcal{I}_g^f is strictly increasing. The existence of the morphism $\tilde{\Phi}_{\tilde{\pi}}$ implies that every f -node (resp. g -node) is sent by $\tilde{\Phi}$ to the u_2 -node (resp. u_1 -node) of Γ_σ . Thus, $\beta(t) = \tilde{\Phi}(\alpha(t))$ is a path starting from the u_2 -node and ending in the u_1 -node. By Proposition 5.1 we have the following diagram

$$\begin{array}{ccc} \text{NL}(X, 0) & \xrightarrow{\tilde{\Phi}} & \text{NL}(\mathbb{C}^2, 0) \\ & \searrow \mathcal{I}_g^f & \downarrow \mathcal{I}_{u_1}^{u_2} \\ & & \mathbb{R}_{>0} \cup \{+\infty\} \end{array}$$

Then $\mathcal{I}_{u_1}^{u_2}(\beta(t)) = (\mathcal{I}_{u_1}^{u_2} \circ \tilde{\Phi})(\alpha(t)) = \mathcal{I}_g^f(\alpha(t))$ which means that the function $\mathcal{I}_{u_1}^{u_2}(\beta(t))$ is strictly increasing. By definition, the curve $\tilde{\Phi}(\alpha) = \beta$ is included in \mathcal{A}_σ and it follows that $\tilde{\Phi}(\mathcal{A}_\pi) \subset \mathcal{A}_\sigma$. Since α_π is a sequence of blowups of X_π then $\mathcal{A}_\pi \subset \mathcal{A}_{\tilde{\pi}}$ which implies that $\tilde{\Phi}(\mathcal{A}_\pi) \subset \tilde{\Phi}(\mathcal{A}_{\tilde{\pi}}) \subset \mathcal{A}_\sigma$. Now, let $v \in \mathcal{A}_\pi$, we know that the functions $\mathcal{I}_{u_1}^{u_2}$ and $\mathcal{H}_{u_1}^{u_2}$ coincide on \mathcal{A}_σ which implies that

$$\mathcal{I}_g^f(v) = (\mathcal{I}_{u_1}^{u_2} \circ \tilde{\Phi})(v) = (\mathcal{H}_{u_1}^{u_2} \circ \tilde{\Phi})(v). \quad (5)$$

On the other hand, by proposition 7.2, we have the following commutative diagram

$$\begin{array}{ccc} \text{NL}(X, 0) & \xrightarrow{\tilde{\Phi}} & \text{NL}(\mathbb{C}^2, 0) \\ & \searrow \mathcal{H}_g^f & \downarrow \mathcal{H}_{u_1}^{u_2} \\ & & \mathbb{R}_{>0} \cup \{+\infty\}, \end{array}$$

which implies that $(\mathcal{H}_{u_1}^{u_2} \circ \tilde{\Phi})(v) = \mathcal{H}_g^f(v)$. Finally, by equation (5), it follows that

$$\mathcal{I}_g^f(v) = \mathcal{H}_g^f(v).$$

□

Example 7.7. This example illustrates the statement of Theorem 7.5 in the case $(X, 0) = (\mathbb{C}^2, 0)$ and $\Phi(x, y) = (g(x, y), f(x, y)) = (x + y, y^5 - x^{12})$. The following graph is the dual graph of the minimal good resolution π of the curve $(gf)^{-1}(0)$

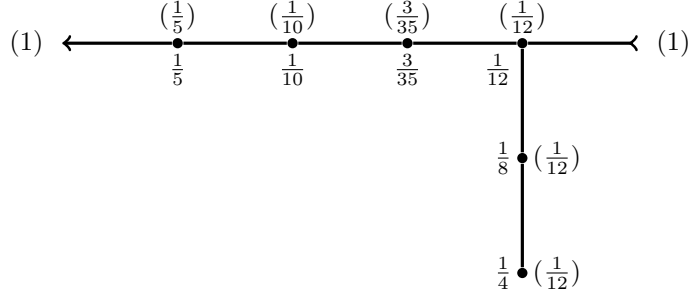


FIGURE 5. The graph Γ_π is weighted with the inner rates (without parenthesis) and the Hironaka quotients (between parenthesis).

We now recall the growth behaviour of the Hironaka quotients function which was proved by Maugendre and Michel in [MM20].

Theorem 7.8 ([MM20, Theorem 1]). *We denote by \mathcal{G}_π the union of every path joining an f -node to a g -node along which the Hironaka quotients function \mathcal{H}_g^f is strictly increasing. Let $(X, 0)$ be a complex surface germ with an isolated singularity and $\Phi = (g, f) : (X, 0) \rightarrow (\mathbb{C}^2, 0)$ be a finite morphism. Let $\pi : (X_\pi, E) \rightarrow (X, 0)$ be a good resolution of $(X, 0)$. The Hironaka quotients function \mathcal{H}_g^f restricted to Γ_π is constant on every connected component of the topological closure of $\Gamma_\pi \setminus \mathcal{G}_\pi$.*

Proof of Point (iii) of Theorem 7.5. By Theorem [MM20] it suffices to prove that $\mathcal{A}_\pi = \mathcal{G}_\pi$. Theorem 7.5(ii) implies directly the inclusion $\mathcal{A}_\pi \subset \mathcal{G}_\pi$. Then, we need to prove the inclusion $\mathcal{G}_\pi \subset \mathcal{A}_\pi$. For that aim, we prove that the functions \mathcal{I}_g^f and \mathcal{H}_g^f coincide on \mathcal{G}_π . In the case where $(X, 0) = (\mathbb{C}^2, 0)$, we get directly from Lemmas 5.5 and 5.3 that $\mathcal{I}_{u_1}^{u_2}$ and $\mathcal{H}_{u_1}^{u_2}$ coincide on \mathcal{G}_σ . The rest of the proof is easily adapted from the arguments of the proof of Theorem 7.5(ii) by using the two commutative diagrams of Propositions 5.1 and 7.2. □

8. A FIRST APPLICATION OF THE INNER RATES FORMULA

As a first application of the inner rates formula 3.1, we give a new proof of [Mic08, Theorem 4.9].

Theorem 8.1 ([Mic08, Theorem 4.9]). *Let $(X, 0)$ be a complex surface germ with an isolated singularity and let $g, f : (X, 0) \rightarrow (\mathbb{C}, 0)$ be two holomorphic functions on X such that the morphism $\Phi = (g, f) : (X, 0) \rightarrow (\mathbb{C}^2, 0)$ is finite. Let $\pi : (X_\pi, E) \rightarrow (X, 0)$ be a good resolution of $(X, 0)$. Let \mathcal{Z} be a connected component of $\overline{\Gamma_\pi \setminus \mathcal{A}_\pi}$ or a single vertex on the complementary of $\overline{\Gamma_\pi \setminus \mathcal{A}_\pi}$, then :*

$$\sum_{v \in \mathcal{Z}} m_v(f) \Pi_{\Phi}^* \cdot E_v = - \left(\sum_{v \in \mathcal{Z}} m_v(f) \chi'_v \right).$$

where $\chi'_v := 2 - 2g_v - \text{val}(v) - f^* \cdot E_v - g^* \cdot E_v$

Remark 8.2. *The theorem 8.1 is more general than the original statement [Mic08, Theorem 4.9] because we do not assume that π is a good resolution of the curve $(gf)^{-1}(0)$. Notice that without that hypothesis the number χ'_v is not, in general, equal to the euler characteristic of the surface $E'_v := E_v - \left(\left(\bigcup_{i \neq v} E_i \right) \cup f^* \cup g^* \right)$ which appears in the original statement.*

Proof of Theorem 8.1. Let us recall the inner rates formula 3.1 applied on an irreducible component E_v of the exceptional divisor

$$\left(\sum_{i \in V(\Gamma_\pi)} m_i(f) q_{g,i}^f E_i \right) \cdot E_v = \text{val}(v) + f^* \cdot E_v - \Pi_{\Phi}^* \cdot E_v + 2g_v - 2.$$

Let us add the term $g^* \cdot E_v$ to both sides of this equality knowing that

$$g^* \cdot E_v = - \left(\sum_{i \in V(\Gamma_\pi)} m_i(g) E_i \cdot E_v \right) \quad (\text{proposition 1.3}).$$

We get

$$\sum_{i \in V(\Gamma_\pi)} \left(m_i(f) q_{g,i}^f - m_i(g) \right) E_i \cdot E_v = \text{val}(v) + f^* \cdot E_v + g^* \cdot E_v - \Pi_{\Phi}^* \cdot E_v + 2g_v - 2,$$

by definition of χ'_v , the last equation becomes

$$\sum_{i \in V(\Gamma_\pi)} \left(m_i(f) q_{g,i}^f - m_i(g) \right) E_i \cdot E_v = -\Pi_{\Phi}^* \cdot E_v - \chi'_v.$$

Let \mathcal{Z} be as in the statement of the theorem. Let us multiply both side of the last equality by $m_v(f)$ and take the sum over v in $V(\mathcal{Z})$

$$\sum_{v \in V(\mathcal{Z})} \sum_{i \in V(\Gamma_\pi)} \left(m_v(f) m_i(f) q_{g,i}^f - m_v(f) m_i(g) \right) E_i \cdot E_v = - \sum_{v \in V(\mathcal{Z})} m_v(f) \Pi_{\Phi}^* \cdot E_v + m_v(f) \chi'_v.$$

It remains to prove that the left side

$$A = \sum_{v \in V(\mathcal{Z})} \sum_{i \in V(\Gamma_\pi)} \left(m_v(f) m_i(f) q_{g,i}^f - m_v(f) m_i(g) \right) E_i \cdot E_v$$

is equal zero. Let us inject the equality $m_v(f) E_v^2 = - \sum_{i \neq v} m_i(f) E_i \cdot E_v + f^* \cdot E_v$ (by proposition 1.3) in A :

$$A = \sum_{v \in V(\mathcal{Z})} \sum_{i \in V(\Gamma_\pi), i \neq v} \left(m_i(f) m_v(f) (q_{g,i}^f - q_{g,v}^f) + m_i(f) m_v(g) - m_i(g) m_v(f) \right) E_i \cdot E_v.$$

Let us notice that for every vertex v of $\mathcal{A}_\pi \cap \mathcal{Z}$ we have:

$$\sum_{i \in V(\mathcal{A}_\pi), i \neq v} \left(m_i(f)m_v(f)(q_{g,i}^f - q_{g,v}^f) + m_i(f)m_v(g) - m_i(g)m_v(f) \right) E_i \cdot E_v = 0,$$

because $q_{g,i}^f = \frac{m_i(g)}{m_i(f)}$ for every vertex i of \mathcal{A}_π . On the other hand, if v is a vertex of $\mathcal{Z} \setminus \mathcal{A}_\pi$, all its neighbor vertices are in \mathcal{Z} , this imply that

$$A = \sum_{v \in V(\mathcal{Z})} \sum_{i \in V(\mathcal{Z}), i \neq v} \left(m_i(f)m_v(f)(q_{g,i}^f - q_{g,v}^f) + m_i(f)m_v(g) - m_i(g)m_v(f) \right) E_i \cdot E_v.$$

By Theorem 7.5(ii), $\frac{m_i(g)}{m_i(f)} = \frac{m_j(g)}{m_j(f)}$, for every $i, j \in V(\mathcal{Z})$. Therefore

$$A = \sum_{v \in V(\mathcal{Z})} \sum_{i \in V(\mathcal{Z}), i \neq v} m_i(f)m_v(f)(q_{g,i}^f - q_{g,v}^f) E_i \cdot E_v.$$

By reordering the terms of A , we obtain:

$$A = \sum_{v, i \in V(\mathcal{Z}), v \neq i} (q_{g,i}^f - q_{g,v}^f)(m_i(f)m_v(g) - m_v(f)m_i(g)) E_i \cdot E_v.$$

Again, by Theorem 7.5(ii), all the terms of A are equal to zero therefore $A = 0$ as desired. \square

9. POLAR EXPLORATION

As an application of the inner rates formula (Theorem 3.1), we will perform *polar exploration* using the inner rates expanding the ideas of [BdSFP22a, BdSFNP22, BdSFP22b]. The main aim of this section is to prove Proposition C .

Definition 9.1. *Let $(X_1, 0)$ and $(X_2, 0)$ be two complex analytic surface germ with an isolated singularity. Let $(C_1, 0)$ and $(C_2, 0)$ be two germ of complex analytic curves embedded in $(X_1, 0)$ and $(X_2, 0)$ respectively. We say that the curves $(C_1, 0)$ and $(C_2, 0)$ have the **same embedded topological type** if there exists a germ of homeomorphism $\psi : (X_1, C_1) \rightarrow (X_2, C_2)$.*

Neumann gave in [Neu81] a complete invariant of the embedded topological type of a germ of complex analytic curve embedded in a complex surface with an isolated singularity as follows. Let $(C, 0)$ be a germ of analytic curve embedded in a germ of complex analytic surface $(X, 0)$. The embedded topological type of the curve $(C, 0)$ determines and is determined by the dual graph of the minimal good resolution π of $(X, 0)$ and $(C, 0)$ decorated with arrows corresponding to the irreducible components of the strict transform of C by π .

Let $(X, 0)$ be a complex analytic surface germ with an isolated singularity and let $\Phi = (g, f) : (X, 0) \rightarrow (\mathbb{C}^2, 0)$ be a finite morphism. Let $\pi : (X_\pi, E) \rightarrow (X, 0)$ be a good resolution of $(X, 0)$ and $(gf)^{-1}(0)$. Let $E_{v_1}, E_{v_2}, \dots, E_{v_n}$ be the irreducible components of E . Let us recall the inner rates formulas applied on morphisms (g, f) and (f, g) :

$$M_\pi \cdot \underline{a_{g,\pi}^f} = \underline{K_\pi} + \underline{F_\pi} - \underline{P_\pi}, \quad M_\pi \cdot \underline{a_{f,\pi}^g} = \underline{K_\pi} + \underline{G_\pi} - \underline{P_\pi}$$

where $M_\pi = (E_{v_i} \cdot E_{v_j})_{i,j \in \{1,2,\dots,n\}}$, $\underline{a_{g,\pi}^f} := (m_{v_1}(f)q_{g,v_1}^f, \dots, m_{v_n}(f)q_{g,v_n}^f)$, $\underline{K_\pi} := (\text{val}_{\Gamma_\pi}(v_1) + 2g_{v_1} - 2, \dots, \text{val}_{\Gamma_\pi}(v_n) + 2g_{v_n} - 2)$, $\underline{F_\pi} = (f^* \cdot E_{v_1}, \dots, f^* \cdot E_{v_n})$, $\underline{G_\pi} = (g^* \cdot E_{v_1}, \dots, g^* \cdot E_{v_n})$, and $\underline{P_\pi} = (\Pi_\Phi^* \cdot E_{v_1}, \dots, \Pi_\Phi^* \cdot E_{v_n})$. What we call

polar exploration consists of finding the \mathcal{P} -vector $P_\pi = (\Pi_\Phi^* \cdot E_{v_1}, \dots, \Pi_\Phi^* \cdot E_{v_n})$ knowing the vectors M_π, K_π, F_π and G_π in the case π is the minimal good resolution of $(X, 0)$ and $(gf)^{-1}(0)$. In other words, we want to find all the possible \mathcal{P} -vector for a fixed embedded topological type of the curve $(gf)^{-1}(0)$.

Let us give an example where for a given topological type the polar exploration gives three possible \mathcal{P} -vectors. We prove they are all realized by a morphism.

Example 9.2. Consider again Example 1.6.

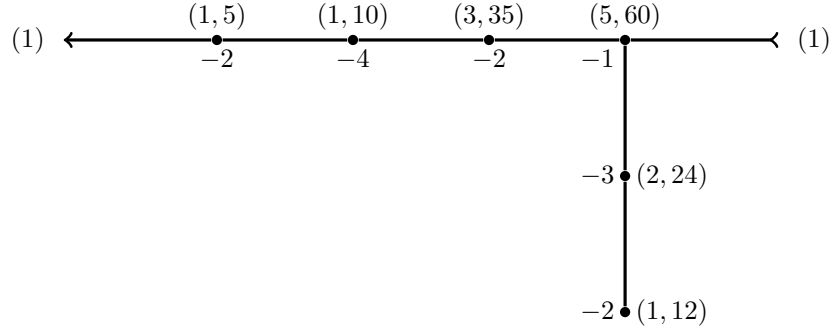


FIGURE 6. The numbers between parenthesis are the orders of vanishing $(m_v(g), m_v(f))$ of the functions $g \circ \pi$ and $f \circ \pi$ along the irreducible components of E , these numbers can be determined from the dual graph using Proposition 1.3.

Now, we will focus on determining the \mathcal{P} -vector of Φ using only the dual graph Γ_π . By applying [Mic08, Theorem 4.9] (Theorem 8.1) successively to each vertex v_1, v_2 and v_3 of \mathcal{A}_π , we get $\Pi_\Phi^* \cdot E_{v_i} = 0$ for $i = 1, 2, 3$. Now we apply the theorem on the set $\{v_4, v_5, v_6\}$ which is a connected component of $\overline{\Gamma_\pi \setminus \mathcal{A}_\pi}$ and we get the equation

$$60\Pi_\Phi^* \cdot E_{v_4} + 24\Pi_\Phi^* \cdot E_{v_5} + 12\Pi_\Phi^* \cdot E_{v_6} = 48.$$

this gives three possible \mathcal{P} -vectors

$$(\Pi_\Phi^* \cdot E_{v_1}, \Pi_\Phi^* \cdot E_{v_2}, \Pi_\Phi^* \cdot E_{v_3}, \Pi_\Phi^* \cdot E_{v_4}, \Pi_\Phi^* \cdot E_{v_5}, \Pi_\Phi^* \cdot E_{v_6}) \in \left\{ \begin{array}{l} (0, 0, 2, 1, 0, 0), \\ (0, 0, 4, 0, 0, 0), \\ (0, 0, 0, 2, 0, 0) \end{array} \right\}$$

We already know from the computations made in Example 3.5 that the \mathcal{P} -vector $(0, 0, 2, 1, 0, 0)$ corresponds to the morphism Φ . Now, a natural question is whether the two vectors $(0, 0, 4, 0, 0, 0)$, $(0, 0, 0, 2, 0, 0)$ can be realized by the \mathcal{P} -vectors of two other morphisms. The answer is yes, we will show it by using the following lemma:

Lemma 9.3 ([CA93, lemma 6.6.1]). Let γ be a germ of complex curve at the origin of \mathbb{C}^2 . The following assertions are equivalent:

(i) γ has equation

$$h(x, y) = a_{0,n}y^n + a_{m,0}x^m + \sum_{ni+mj > nm} a_{i,j}x^i y^j = 0$$

with $a_{0,n} \neq 0$ and $a_{m,0} \neq 0$.

(ii) γ has the same embedded topological type as the curve of equation

$$y^n + x^m = 0$$

By lemma 9.3 the dual graph of the minimal good resolution of the curves $(x, y^5 - x^{12}) = 0$ and $(x + y)(y^5 - x^{12} + x^{10}y) = 0$ is Γ_π . A direct computation shows that:

(i) The polar curve Π_{Φ_1} of the morphism $\Phi_1 = (x, y^5 - x^{12})$ has equation $y^4 = 0$, the \mathcal{P} -vector is

$$(0, 0, 4, 0, 0, 0).$$

(ii) The polar curve Π_{Φ_2} of the morphism $\Phi_2 = (x + y, y^5 - x^{12} + x^{10}y)$ has equation $5y^4 + x^{10} + 12x^{11} - 10x^9y = 0$, the \mathcal{P} -vector is

$$(0, 0, 0, 2, 0, 0)$$

In Example 9.2, Theorem 8.1 ([Mic08, Theorem 4.9]) is sufficient to find all the realized \mathcal{P} -vectors. We will now prove Proposition C stated in the introduction which provides a family of examples where the knowledge of the inner rates and their properties (Theorem 7.5) gives a better restriction on the possible \mathcal{P} -vectors than Theorem 8.1. In fact we will prove the following more precise version of Proposition C.

Proposition 9.4. *Let $n \in \mathbb{N}_{\geq 2}$ and consider the following graph with arrows Γ_n :*

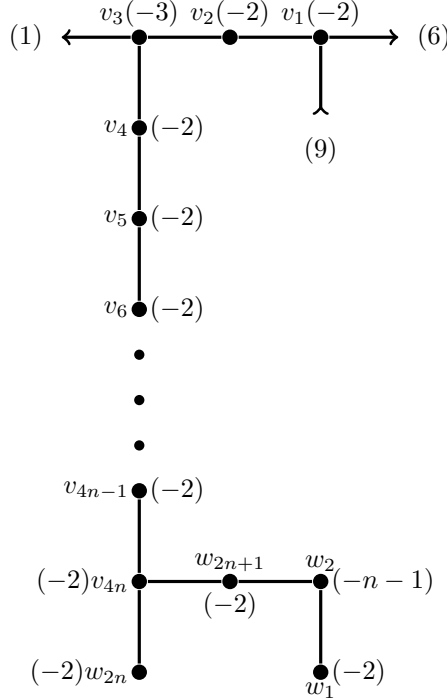


FIGURE 7. The genus is 0 for every vertex

(i) *There exists a complex surface singularity $(X_n, 0)$ and a finite morphism $\Phi_n = (g_n, f_n) : (X_n, 0) \rightarrow (\mathbb{C}^2, 0)$ such that Γ_n is the dual graph of the minimal good resolution of $(X_n, 0)$ and $(g_n, f_n)^{-1}(0)$;*

(ii) The \mathcal{P} -vector of any such morphism Φ_n belongs to a set of $n + 5$ elements.

More precisely:

$$\begin{aligned} & \Pi_{\Phi_n}^* \cdot E_{v_1} = 14 \text{ and } \Pi_{\Phi_n}^* \cdot E_{v_k} = 0 \text{ for every } k = 2, 3, \dots, 4n, \\ & (\Pi_{\Phi_n}^* \cdot E_{w_{2n}}, \Pi_{\Phi_n}^* \cdot E_{w_{2n+1}}, \Pi_{\Phi_n}^* \cdot E_{w_2}, \Pi_{\Phi_n}^* \cdot E_{w_1}) \in \left\{ \begin{array}{l} (1, 0, 1, 0), \\ (1, 0, 0, 2), \\ (0, 1, 0, 1), \\ (0, 0, k, k + 2), \\ 0 \leq k \leq n + 1 \end{array} \right\} \end{aligned}$$

In order to prove Proposition 9.4 we will need some tools we are going to present now. All the definitions and results of this subsection can be found e.g in [N99].

We call **cycle** on Γ_π every effective divisor of X_π which is supported on E . Let $h : (X, 0) \rightarrow (\mathbb{C}, 0)$ be a holomorphic function. Denote by (h_π) the compact part of the total transform of h by π i.e,

$$(h_\pi) = \sum_{i \in V(\Gamma_\pi)} m_i(h) E_i.$$

The cycle (h_π) is called analytic cycle and we denote by $Z_{an}(\Gamma_\pi)$ the set of **analytic cycles** on Γ_π . A **topological cycle** D is an effective cycle such that $D \cdot E_v \leq 0$ for every irreducible component E_v of E , we denote by $Z_{top}(\Gamma_\pi)$ the set of such cycles.

There is a natural ordering of the cycles: $D_1 = \sum n_i E_{v_i} \leq D_2 = \sum n'_i E_{v_i}$ if and only if $n_i \leq n'_i$ for all i . By [N99, Lemma 2.6] the set $Z_{top}(\Gamma_\pi)$ contains a minimal element denoted $Z_{min}(\Gamma_\pi)$ called the **minimal cycle**.

Definition 9.5 ([N99, Appendix 2]). *The Euler characteristic of an effective cycle D is defined by*

$$\chi(D) = -\frac{1}{2} D \cdot (D + K_\pi)$$

where K_{X_π} is the canonical divisor of X_π .

Definition 9.6 ([N99][Theorem 3.8], [Art66]). *The complex surface germ $(X, 0)$ is said to be **rational** if $\chi(Z_{min}(\Gamma_\pi)) = 1$ where π is a good resolution of $(X, 0)$.*

Theorem 9.7 ([N99, Theorem 3.14], [Art66]). *Let $(X, 0)$ be a complex surface germ with an isolated singularity. If $(X, 0)$ is rational then*

$$Z_{top}(\Gamma_\pi) = Z_{an}(\Gamma_\pi),$$

for every good resolution π of $(X, 0)$.

Now, we are ready to prove Proposition 9.4.

Proof of Proposition 9.4. Let us first prove the existence of the surfaces $(X_n, 0)$. By a classical result of Grauert ([Gra62]) every weighted graph without loops and with negative definite intersection matrix can be realized as the dual graph Γ_π associated with the minimal good resolution of some complex analytic surface germ $(X, 0)$ (See e.g [NP07, Section 2] for details). The intersection matrix of the dual graph Γ_n is negative definite, therefore, there exists a germ complex surface singularity $(X_n, 0)$ such that the dual graph of its minimal good resolution $\pi_n : (X_{\pi_n}, E_n) \rightarrow (X_n, 0)$ is Γ_n .

Now, we prove the existence of the morphisms $\Phi_n = (g_n, f_n)$. The minimal cycle of Γ_{π_n} denoted $Z_{min}(\Gamma_{\pi_n})$ is obtained by using *Laufer's algorithm* [Lau71] (see e.g, [N99, 2.10] for more details):

$$Z_{min}(\Gamma_{\pi_n}) = E_{v_1} + E_{v_2} + E_{v_3} + 2(E_{v_4} + E_{v_5} + \dots + E_{v_{4n-1}} + E_{v_{4n}} + E_{w_{2n+1}}) + E_{w_2} + E_{w_1} + E_{w_{2n}}.$$

A direct computation using the adjunction formula 3.3 shows that $\chi(Z_{min}(\Gamma_{\pi_n})) = 1$. By using Lemma 1.3 we now compute from the dual graph Γ_{π_n} the orders of vanishing of the functions $f_n \circ \pi_n$ and $g_n \circ \pi_n$ along the irreducible components of E_n :

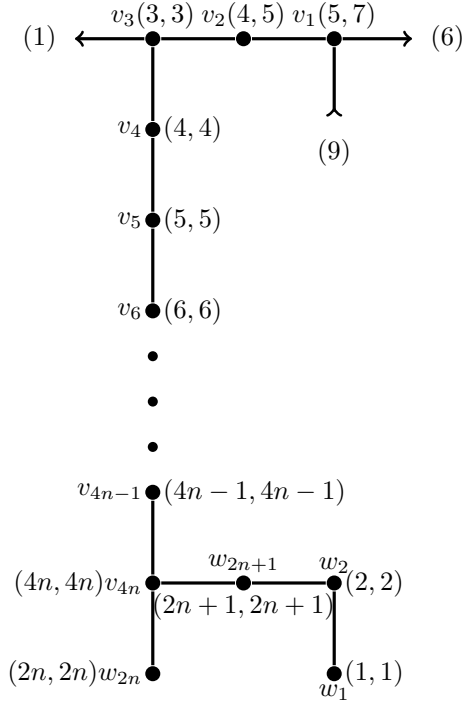


FIGURE 8. The numbers between parenthesis are the orders of vanishing $(m_v(g_n), m_v(f_n))$ of the functions $g_n \circ \pi_n$ and $f_n \circ \pi_n$ along the irreducible components of E_n , these numbers can be determined from the dual graph using Proposition 1.3.

We are going to construct two holomorphic function germs $g_n, f_n : (X_n, 0) \rightarrow (\mathbb{C}, 0)$ whose analytic cycles are respectively:

$$D_{n,1} = 7E_{v_1} + 5E_{v_2} + \sum_{k=3}^{4n} kE_{v_k} + 2nE_{w_{2n}} + (2n+1)E_{w_{2n+1}} + 2E_{w_2} + E_{w_1}$$

$$D_{n,2} = 5E_{v_1} + 4E_{v_2} + \sum_{k=3}^{4n} kE_{v_k} + 2nE_{w_{2n}} + (2n+1)E_{w_{2n+1}} + 2E_{w_2} + E_{w_1}.$$

and such that $f_n^* \cup g_n^*$ is a disjoint union of curvettes of E_n .

Let a_1, a_2, \dots, a_{15} be distinct smooth points of E_{v_1} . Let e_n be the composition of the blowups of the points blowup a_i and let $E_{v'_i} = e_n^{-1}(a_i)$ for $i = 1, \dots, 15$. Since $\chi(Z_{\min}(\Gamma_{\pi_n})) = 1$, by Theorem 9.7, there exists functions f_n and g_n whose analytic cycles on $\Gamma_{e_n \circ \pi_n}$ are

$$D'_{n,1} = D_{n,1} + 7(E_{v'_1} + \dots + E_{v'_6}) + 8(E_{v'_7} + \dots + E_{v'_{15}})$$

$$D'_{n,2} = D_{n,1} + 8(E_{v'_1} + \dots + E_{v'_7}) + 7(E_{v'_8} + \dots + E_{v'_{15}}).$$

Using Lemma 1.3 we can check that $f_n^* \cup g_n^*$ is a disjoint union of curvettes of the exceptional divisor of $e_n \circ \pi_n$ such that $f_n^* \cdot E_{v'_i} = 0$ and $g_n^* \cdot E_{v'_i} = 1$ for $i = 1, \dots, 6$, and $f_n^* \cdot E_{v'_i} = 1$ and $g_n^* \cdot E_{v'_i} = 0$ for $i = 7, \dots, 15$. Therefore blowing down the fifteen components $E_{v'_i}$ we obtain that the union of strict transforms of f_n and g_n by π_n is a disjoint union of curvettes of E_n .

Let us now prove (ii). Let Φ_n be a morphism as in (i). Let compute its possible \mathcal{P} -vectors. Until the end of the proof we will use the notations $p_i := \Pi_{\Phi_n}^* \cdot E_{v_i}$ and $p'_i := \Pi_{\Phi_n}^* \cdot E_{w_i}$.

By Theorem 7.5(ii), we know that $q_{g_n, v_1}^{f_n} = \frac{5}{7}$, $q_{g_n, v_2}^{f_n} = \frac{4}{5}$ and $q_{g_n, v_3}^{f_n} = 1$. Let us apply Theorem 3.1 on the vertices v_1 and v_2 , we then obtain $p_1 = 14$ and $p_2 = 0$. Apply again Theorem 3.1 on the vertex v_3 , we obtain $p_3 = 5 - 4q_{g_n, v_4}^{f_n}$. By Theorem 7.5(i) we have $q_{g_n, v_4}^{f_n} > q_{g_n, v_3}^{f_n} = 1$, then $p_3 = 5 - 4q_{g_n, v_4}^{f_n} < 1$ and it follows $p_3 = 0$, $q_{g_n, v_4}^{f_n} = \frac{5}{4}$.

We now prove by induction that for every $k \in \{3, 4, \dots, 4n\}$ we have $q_{g_n, v_k}^{f_n} = \frac{2k-3}{k}$ and $p_{k-1} = 0$. It is true for $k = 3$. Let us suppose it is true for a given $k < 4n$. By applying the inner rates formula on the vertex E_k we get

$$p_k = 2k - 1 - (k+1)q_{g_n, v_{k+1}}^{f_n}.$$

By Theorem 7.5(i) we know that $q_{g_n, v_{k+1}}^{f_n} > q_{g_n, v_k}^{f_n} = \frac{2k-3}{k}$. Then

$$p_k = 2k - 1 - (k+1)q_{g_n, v_{k+1}}^{f_n} < 2k - 1 - \frac{2k-3}{k}(k+1) = \frac{3}{k} < 1.$$

Therefore $p_k = 0$ and it follows that $q_{g_n, v_{k+1}}^{f_n} = \frac{2k-1}{k+1}$ as desired.

Let us now apply Theorem 8.1 on the vertices $\{v_1, \dots, v_{4n}, w_{2n}, w_{2n+1}, w_2, w_1\}$, we get the following equality:

$$2n + 2 = 4np_{4n} + 2np'_{2n} + (2n+1)p'_{2n+1} + 2p'_2 + p'_1.$$

Solving this last equation ends the proof. \square

Remark 9.8. *In this last proposition if we did not make use of the inner rates and only restricted ourselves to Theorem 8.1 we would have ended with the following equality*

$$3p_3 + 4p_4 + 5p_5 + \dots + 4np_{4n} + 2np'_{2n} + (2n+1)p'_{2n+1} + 2p'_2 + p'_1 = 2n + 2$$

where $p'_i := \Pi_{\Phi_n}^* \cdot E_{w_i}$ and $p_i := \Pi_{\Phi_n}^* \cdot E_{v_i}$.

The cases $p'_{2n+1} = 1$ and $p'_{2n} = 1$ were already treated so we can suppose that $p'_{2n+1} = p'_{2n} = 0$

By renaming $x_1 = p'_1, x_2 = p'_2, x_3 = p_3, x_4 = p_4$, etc., we end up with the equation

$$x_1 + 2x_2 + 3x_3 + \dots + (2n+2)x_{2n+2} = 2n + 2, \quad x_i \in \mathbb{N},$$

let $\Pi(n)$ be the cardinal of the set of solutions of this equation. The sequence $\Pi(n)$ is equivalent to

$$\Pi(n) \sim \frac{1}{(8n+8)\sqrt{3}} \exp\left(\pi\sqrt{\frac{4n+4}{3}}\right),$$

as was proved by Hardy and Ramanujan (see e.g [HW08]). On the other hand, by using the inner rates in the proof of Proposition 9.4, we end up with only $n+5$ possible cases.

10. GEOMETRIC INTERPRETATION OF THE INNER RATES

Let $(X, 0) \subset (\mathbb{C}^n, 0)$ be a complex surface germ with an isolated singularity at the origin of \mathbb{C}^n . The aim of this section is to give a geometrical interpretation of the inner rates as metric invariants of the Milnor fibers of a non constant holomorphic function f . First, we will need to define the notion of generic linear form and generic polar curve with respect to a resolution and a function.

10.1. Generic linear form and generic polar curve. All the results of this subsection are an adaption of of [BNP14, Section 3] and [BdSFP22a, Subsection 2.2].

Let $f : (X, 0) \rightarrow (\mathbb{C}, 0)$ be a non constant holomorphic function. We call **Gauss map relative to f** as defined in [Tei82, page 364] the map :

$$\begin{aligned} \gamma_f : X \setminus \{0\} &\rightarrow \mathbb{P}(\mathbb{C}^n). \\ p &\mapsto \ker(d_p f) \end{aligned}$$

Let us now consider the following complex surface

$$\mathcal{N}_f(X) = \overline{\text{Graph}(\gamma_f)} = \overline{\{(p, d) \in X \setminus \{0\} \times \mathbb{P}(\mathbb{C}^n) \mid d_p f(d) = 0\}} \subset \mathbb{C}^n \times \mathbb{P}(\mathbb{C}^n)$$

Definition 10.1 ([Tei82, Definition page 367]). *The **Nash transform** of X relative to f is the modification ν_f defined by*

$$\begin{aligned} \nu_f : \mathcal{N}_f(X) &\rightarrow X \\ (p, d) &\mapsto p \end{aligned}$$

Let $D \in \mathbb{P}(\mathbb{C}^n)$ and consider its dual linear form $\ell_D : \mathbb{C}^n \rightarrow \mathbb{C}$. Let Ω_f be the Zariski open set of $\mathbb{P}(\mathbb{C}^n)$ such that $\Phi_D = (\ell_D|_X, f) : (X, 0) \rightarrow (\mathbb{C}^2, 0)$ is a finite morphism. We will denote by Π_D the polar curve associated with the morphism Φ_D , whenever D is an element of Ω_f .

Definition 10.2 ([Spi90, Definition 1.1 Chapter III]). *Let $\alpha : (\tilde{X}, E) \rightarrow (X, 0)$ be a modification and let Ω be an open Zariski set of $\mathbb{P}(\mathbb{C}^n)$. Let $\{(\gamma_D, 0) \subset (X, 0)\}_{D \in \Omega}$ be a family of complex curve germs. We say that a point p of E is a **base point** of α for the family of curves $\{(\gamma_D, 0)\}_{D \in \Omega}$ if there exists an open Zariski set Ω_p of $\mathbb{P}(\mathbb{C}^n)$ contained in Ω such that $p \in \gamma_D^*$ for all D in Ω_p , where γ_D^* is the strict transform of the curve γ_D by the modification α .*

We will use the following properties of the Nash transform. For (i) see [Tei82, 1.1 page 417] and (ii) is a direct consequence of the third point of [Tei82, Corollary 1.3.2 page 420].

Proposition 10.3. *The Nash transform has the following properties:*

- (i) *The map $(\gamma_f \circ \nu_f) : \mathcal{N}_f(X) \setminus \nu_f^{-1}(0) \rightarrow \mathbb{P}(\mathbb{C}^n)$ extends to a holomorphic map $\tilde{\gamma}_f$ on $\mathcal{N}_f(X)$.*

(ii) *The Nash transform has no base point for the family of the polar curves.*

The following definition is an adaptation of the definition of the local bilipchitz constant introduced in [BNP14, Section 3].

Definition 10.4. *Let v_p be a unit vector of $T_p X$ such that $d_p f(v_p) = 0$. The **local bilipschitz constant of the morphism** $\Phi_D = (\ell_{D|X}, f)$ is the map defined by:*

$$K_D : X \setminus 0 \rightarrow \mathbb{R} \cup \{\infty\}$$

$$p \mapsto \begin{cases} \frac{1}{\|d_p \Phi_D(v_p)\|} & \text{if } p \notin \Pi_D \\ \infty & \text{if } p \in \Pi_D \end{cases}$$

Lemma 10.5. *Given any neighborhood U of $\Pi_D^* \cap \nu_f^{-1}(B_\epsilon \cap X)$ in $\mathcal{N}_f(X) \cap \nu_f^{-1}(B_\epsilon \cap X)$, the local bilipschitz constant K_D of the morphism Φ_D is bounded on $B_\epsilon \cap (X \setminus \nu_f(U))$.*

Proof. We have

$$(K_D \circ \nu_f)(p, d) = K_D(p) = \frac{1}{\|(0, \ell_D(v_p))\|}$$

Let us consider the map

$$\kappa_D : \mathbb{P}(\mathbb{C}^n) \rightarrow \mathbb{R} \cup \{\infty\}$$

$$d \mapsto \begin{cases} \frac{1}{\|(0, \ell_D(\frac{d}{\|d\|}))\|} & \text{if } d \notin D^\perp \\ \infty & \text{if } d \in D^\perp \end{cases}$$

By definition of the Nash transform we have $d_p f(d) = 0$ for all $(p, d) \in \mathcal{N}_f(X)$. We then have the equality

$$(K_D \circ \nu_f)(p, d) = (\kappa_D \circ \tilde{\gamma}_f)(p, d), \quad \forall (p, d) \notin \Pi_D^* \quad (6)$$

where $\tilde{\gamma}_f$ is the holomorphic extension of the map $\gamma_f \circ \nu_f$ to $\mathcal{N}_f(X)$ (Remark 10.3). The map $\kappa_D \circ \tilde{\gamma}_f$ is continuous and takes finite values outside Π_D^* . This implies that given any open neighbourhood U of $\Pi_D^* \cap \nu_f^{-1}(B_\epsilon \cap X)$ in $\mathcal{N}_f(X) \cap \nu_f^{-1}(B_\epsilon \cap X)$ the map $\kappa_D \circ \tilde{\gamma}_f$ is bounded on the compact set $\nu_f^{-1}(B_\epsilon \cap X) \setminus U$. Then, by Equality (6), the local bilipschitz constant K_D of Φ_D is bounded on $B_\epsilon \cap (X \setminus \nu_f(U))$. \square

Let $\pi : (X_\pi, E) \rightarrow (X, 0)$ be a good resolution which factors through the Nash transform $\nu_f : \mathcal{N}_f(X) \rightarrow X$ and the blowup of the maximal ideal $b_0 : \text{BL}_0(X) \rightarrow X$ i.e. there exists two modifications $\mu_\pi : (X_\pi, E) \rightarrow (\mathcal{N}_f(X), \nu_f^{-1}(0))$ and $\mu'_\pi : (X_\pi, E) \rightarrow (\text{BL}_0(X), b_0^{-1}(0))$ such that $\nu_f \circ \mu_\pi = \pi$ and $b_0 \circ \mu'_\pi = \pi$. Let us denote by E_f the strict transform of $\nu_f^{-1}(0)$ by the modification μ_π and by E_0 the strict transform of $b_0^{-1}(0)$ by the modification μ'_π . By Remark 10.3 there exists an open Zariski set Ω_f^π included in Ω_f such that for any element D of Ω_f^π , the strict transforms of the curves Π_D and $\ell_D^{-1}(0)$ by π meet E_f and E_0 respectively at smooth points of E . With this, we have the following result as direct consequence of Lemma 10.5

Corollary 10.6. *Given any element D of Ω_f^π and any neighborhood U of $\Pi_D^* \cap \pi^{-1}(B_\epsilon \cap X)$ in $X_\pi \cap \pi^{-1}(B_\epsilon \cap X)$, the local bilipschitz constant K_D of the morphism Φ_D is bounded on $B_\epsilon \cap (X \setminus \pi(U))$.*

□

Now, we can define the notion of generic linear form and generic polar curve with respect to f and π . The following definition is an adaptation of the notion of generic projection defined in [BdSFP22a, Subsection 2.2].

Definition 10.7. *We say that a linear form $\ell : (X, 0) \rightarrow (\mathbb{C}, 0)$ is **generic** with respect to f and π if it is the dual linear form of an element D of Ω_f^π . In which case we say that Π_D is a generic polar curve.*

10.2. Inner rates of a Milnor fibration. We are now ready to state and prove the main theorem of this section which allows us to see the inner rates as metric invariants of the Milnor fibration of the function f . It is also a relative version of [BdSFP22a, Lemma 3.2].

Theorem 10.8. *Let $(X, 0) \subset (\mathbb{C}^n, 0)$ be a complex surface germ with an isolated singularity and let $f : (X, 0) \rightarrow (\mathbb{C}, 0)$ be a non constant holomorphic function. Let $\pi : (X_\pi, E) \rightarrow (X, 0)$ be a good resolution which factors through the Nash transform of X relative to f and the blowup of the maximal ideal of $(X, 0)$.*

There exists a rational number $q_v^f \in \mathbb{Q}_{>0}$ such that for any pair of curvettes γ_1^ and γ_2^* of an irreducible component E_v of the exceptional divisor E verifying $\gamma_i^* \cap f^* = \emptyset$ for $i \in \{1, 2\}$ we have*

$$d_\epsilon(\gamma_1 \cap f^{-1}(\epsilon), \gamma_2 \cap f^{-1}(\epsilon)) = \Theta(\epsilon^{q_v^f}),$$

where $\gamma_1 = \pi(\gamma_1^*), \gamma_2 = \pi(\gamma_2^*)$ and d_ϵ is the Riemannian metric induced by \mathbb{C}^n on the Milnor fiber $f^{-1}(\epsilon)$. Furthermore we have $q_v^f = q_{\ell, v}^f$ whenever ℓ is a generic linear form with respect to f and π .

Proof. Let p be a smooth point of E in $E_v \setminus f^*$. Since π factors through the Nash transform of X relative to f and the blowup of the maximal ideal, it has no base point for the family of the generic polar curves and the family of the generic hyperplane sections (by remark 10.3). Thus, there exists a generic linear form ℓ such that $p \notin \ell^* \cup \Pi_\Phi^*$ where $\Phi = (\ell, f)$. By proposition 2.3 there exists an open neighborhood $O_p \subset E_v$ of p such that for every pair of curvettes γ_1^*, γ_2^* of E_v verifying:

- (i) $\gamma_1^* \cap \gamma_2^* = \emptyset$
- (ii) $\gamma_i^* \cap O_p \neq \emptyset$, for $i = 1, 2$,

we have:

$$d(\Phi(\gamma_1) \cap \{u_2 = \epsilon\}, \Phi(\gamma_2) \cap \{u_2 = \epsilon\}) = \Theta(\epsilon^{q_{\ell, v}^f}),$$

where $\gamma_1 = \pi(\gamma_1^*), \gamma_2 = \pi(\gamma_2^*)$ and $\epsilon \in \mathbb{R}$. By using Corollary 10.5 we deduce that

$$d_\epsilon(\gamma_1 \cap f^{-1}(\epsilon), \gamma_2 \cap f^{-1}(\epsilon)) = \Theta(\epsilon^{q_{\ell, v}^f}).$$

This means that the number $q_{\ell, v}^f$ does not depend on the choice of the generic linear form ℓ , it will be then denoted q_v^f .

Let us now take any pair of curvettes γ^* and γ'^* meeting E_v at two distinct smooth points p and p' of E . Since $d_\epsilon(\gamma \cap f^{-1}(\epsilon), \tilde{\gamma} \cap f^{-1}(\epsilon)) = \Theta(\epsilon^{q_v^f})$ for any neighbor curve germ $\tilde{\gamma}$ of $\gamma = \pi(\gamma^*)$ we have

$$d_\epsilon(\gamma \cap f^{-1}(\epsilon), \gamma' \cap f^{-1}(\epsilon)) = \Theta(\epsilon^{q_v(\gamma, \gamma')}) \text{ with } q_v(\gamma, \gamma') \leq q_v^f,$$

where $\gamma = \pi(\gamma^*)$ and $\gamma' = \pi(\gamma'^*)$.

By compactness of E , we can choose a finite sequence of smooth points $p = p_1, p_2, \dots, p_s = p'$ and a curvette γ_i^* passing through each p_i such that:

$$d_\epsilon(\gamma_i \cap f^{-1}(\epsilon), \gamma_{i+1} \cap f^{-1}(\epsilon)) = \Theta(\epsilon^{q_v^f})$$

We then have

$$K' \epsilon^{q_v(\gamma, \gamma')} \geq d_\epsilon(\gamma \cap f^{-1}(\epsilon), \gamma' \cap f^{-1}(\epsilon)) \geq \sum_{i=1}^s d_\epsilon(\gamma_i \cap f^{-1}(\epsilon), \gamma_{i+1} \cap f^{-1}(\epsilon)) \geq K \epsilon^{q_v^f}.$$

where K and K' are constants. We deduce that $q_v(\gamma, \gamma') \geq q_v^f$, which ends the proof. \square

11. APPLICATION TO LIPCHITZ-KILLING CURVATURE

As an application of Theorem 10.8 we will prove theorem E which generalizes the result of García Barosso and Teissier in [GBT99b].

11.1. Lipchitz-Killing curvature. Let X be a real smooth submanifold of \mathbb{R}^n of dimension m . Let $T_p X$ be the tangent space of X at a point p . Let \vec{n}_p be a normal vector of $T_p X$ in \mathbb{R}^n . Consider $V_{\vec{n}_p}$ the vector space generated by $T_p X$ and \vec{n}_p . Let $P_{\vec{n}_p}$ be the orthogonal projection on $V_{\vec{n}_p}$ and $X_{\vec{n}_p} = P_{\vec{n}_p}(X)$.

Definition 11.1. [Lan80a] *The Lipchitz-Killing curvature of X at the point p is:*

$$K_X(p) = \frac{\omega_m}{2\omega_{n-1}} \int_{\vec{n}_p \in \nu(p)} k_{X_{\vec{n}_p}}(P_{\vec{n}_p}(p)),$$

where ω_i is the volume of the unitary sphere \mathbb{S}^i of \mathbb{R}^{i+1} , $\nu(p)$ is the set of normal vectors at p and $k_{X_{\vec{n}_p}}$ is the classical Gaussian curvature.

Now let us state the **exchange formula** due to Rémi Langevin which provides an easy way to compute the integral of the Lipshitz-Killing curvature in the case of a compact complex submanifold of \mathbb{C}^n possibly with boundaries .

Theorem 11.2. [Lan80b, Theorem A.III.3'] *Let X be a compact complex submanifold embedded in \mathbb{C}^n of dimension m then*

$$(-1)^m \int_{p \in M} K_X(p) dV = \frac{\pi \omega_{2m}}{2\omega_{2n-1}} \int_{H \in \mathbb{G}^{n-1}(\mathbb{C}^n)} \text{Card}\{z \in X \mid T_z X \subset H\} dH,$$

where dV and dH are respectively the volume forms of X and the grassmannian of hyperplanes $\mathbb{G}^{n-1}(\mathbb{C}^n)$.

11.2. The Lipshitz-Killing curvature of a Milnor fiber. Let $(X, 0)$ be a complex surface germ with an isolated singularity embedded in \mathbb{C}^n and $f : (X, 0) \rightarrow (\mathbb{C}, 0)$ be a non constant holomorphic function. For any element H of $\mathbb{G}^{n-1}(\mathbb{C}^n)$ denote by ℓ_H a linear form whose kernel is H . Consider the morphism $\Phi_H = (\ell_H, f)$ and denote by Π_H the associated polar curve. Let us now apply Theorem 11.2 on the Milnor fiber $F_{\epsilon, t} := f^{-1}(t) \cap B_\epsilon$:

$$\int_{p \in F_{\epsilon, t}} K_{F_{\epsilon, t}}(p) dV = \frac{-\pi \omega_2}{2\omega_{2n-1}} \int_{H \in \mathbb{G}^{n-1}(\mathbb{C}^n)} \text{Card}\{\Pi_H \cap F_{\epsilon, t}\} dH.$$

On the other hand, for every element H of $\mathbb{G}^{n-1}(\mathbb{C}^n)$ there exists a positive real number δ_ϵ^H such that for every t in \mathbb{C} whose module is less than or equal to δ_ϵ^H we have

$$\text{Card}\{F_{\epsilon,t} \cap \Pi_H\} = \Pi_H \cdot f^{-1}(0).$$

Let $\delta_\epsilon := \inf\{\delta_\epsilon^H \mid H \in \mathbb{G}^{n-1}(\mathbb{C}^n)\}$. Thus, by taking the limit we get

$$\lim_{\epsilon \rightarrow 0, |t| < \delta_\epsilon} \int_{p \in F_{\epsilon,t}} K_{F_{\epsilon,t}}(p) dV = \frac{-\pi\omega_2}{2\omega_{2n-1}} \int_{H \in \mathbb{G}^{n-1}(\mathbb{C}^n)} \Pi_H \cdot f^{-1}(0) dH. \quad (7)$$

As a direct consequence of (7) and Theorem 8.1 we obtain the following result.

Theorem 11.3. *Let $(X, 0)$ be a complex surface germ with an isolated singularity embedded in $(\mathbb{C}^n, 0)$ and let $f : (X, 0) \rightarrow (\mathbb{C}, 0)$ be a non constant holomorphic function. Let $\pi : (X_\pi, E) \rightarrow (X, 0)$ be a good resolution of $(X, 0)$, then:*

$$\lim_{\epsilon \rightarrow 0, |t| < \delta_\epsilon} \int_{p \in F_{\epsilon,t}} K_{F_{\epsilon,t}}(p) dV = \frac{\pi\omega_2}{2\omega_{2n-1}} \text{Vol}(\mathbb{G}^{n-1}(\mathbb{C}^n)) \sum_{v \in V(\Gamma_\pi)} m_v(f) \chi'_v,$$

where $\chi'_v := 2 - 2g_v - \text{val}(v) - f^* \cdot E_v - \ell_H^* \cdot E_v$.

□

Remark 11.4. *Theorem 11.3 was proved in [Lan79] when $(X, 0) = (\mathbb{C}^k, 0)$.*

Let $\pi : (X_\pi, E) \rightarrow (X, 0)$ be a good resolution of $(X, 0)$ and let E_v be an irreducible component of E . Let $\mathcal{N}(E_v, \epsilon)$, $\epsilon > 0$ be a family of tubular neighborhoods of E_v in X_π such that

$$\lim_{\epsilon \rightarrow 0} \mathcal{N}(E_v, \epsilon) = E_v,$$

and such that the set $\text{Horn}(\epsilon, v) := \pi(\mathcal{N}(E_v, \epsilon))$ is included in B_ϵ . For example one can choose a Riemannian metric on X_π and consider the set of points which are close enough to E_v with respect to that metric. Consider now the intersection of the Milnor fiber of f with this set

$$F_{\epsilon,t}^v = f^{-1}(t) \cap \text{Horn}(\epsilon, v).$$

Let $\delta_\epsilon^H > 0$ be such that for any complex number t whose module is less than or equal to δ_ϵ^H we have

$$\text{Card}\{F_{\epsilon,t}^v \cap \Pi_H\} = m_v(f) \Pi_H^* \cdot E_v,$$

where Π_H^* is the strict transform of Π_H by π .

Theorem 11.5. *Let $(X, 0)$ be a complex surface germ with an isolated singularity embedded in $(\mathbb{C}^n, 0)$ and let $f : (X, 0) \rightarrow (\mathbb{C}, 0)$ be a non constant holomorphic function. Let $\pi : (X_\pi, E) \rightarrow (X, 0)$ be a good resolution of $(X, 0)$ which factors through the Nash transform of X relative to f and the blowup of the maximal ideal of $(X, 0)$. Let v be a vertex of Γ_π , then:*

$$\lim_{\epsilon \rightarrow 0, |t| < \delta_\epsilon} \int_{p \in F_{\epsilon,t}^v} K_{F_{\epsilon,t}^v}(p) dV = \frac{\pi\omega_2}{2\omega_{2n-1}} \text{Vol}(\mathbb{G}^{n-1}(\mathbb{C}^n)) C_f,$$

where

$$C_f = m_v(f) \left(2g_v - 2 + \text{Val}_{\Gamma_\pi}(v) + f^* \cdot E_v - \sum_{i \in V(\Gamma_\pi)} m_i(f) q_i^f E_i \cdot E_v \right).$$

Proof. It is a direct consequence of Theorems 3.1, 10.8 and 11.2. \square

Remark 11.6. *García Barroso and Teissier gave in [GBT99b] an analogous result to Theorem 11.5 when f is a germ of non constant holomorphic function at the origin of \mathbb{C}^2 .*

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