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On the use of stochastic Bouc-Wen model for simulating viscoelastic internal variables from a finite element approximation of steady-rolling tire

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Abstract

Reducing tire rolling resistance and energy loss is a topic of interest to the tire industry. Understanding and modeling these phenomena are essential to approach this problem and propose robust solutions. This work investigates alternative solutions and suggests a reduced-order model based on the Bouc-Wen hysteresis model to simulate viscoelastic internal variables from viscoelastic constitutive laws. Furthermore, this work aims to identify critical parameters in the reduced-order model and understand the effect of uncertainties on its responses. Thus, Sobol's indices are used to identify influential parameters of the reduced-order model that are later calibrated by Bayesian inference. Finally, the uncertainties are propagated, and the reduced-order model is validated using data of viscoelastic internal variables from the finite element approximation of a steady-rolling tire provided by Michelin. Satisfactory results are obtained, as the reduced-order model can simulate viscoelastic internal variables with a reduced computational cost for some branches of interest and its responses are in agreement with the experimental data.

Keywords

Viscoelastic internal variables, Bouc-Wen hysteresis model, Sobol' indices, Bayesian inference, Propagation of uncertainties

Introduction

It is vital to reduce environmental impacts to improve sustainable practices in both academic and productive sectors. Due to the rise of the sustainability mindset in businesses, state regulations, and consumer requirements, the automobile industry has sought to develop technologies that aim to improve vehicle performance and minimize fuel consumption, as well as reduce pollutant emissions (Committee for the National Tire Efficiency Study 2006). Improving the performance of essential parts of vehicles such as tires is one way to accomplish this task.

In a rolling tire, the mechanisms of energy loss are the hysteresis within the viscoelastic material, the friction between tire and road, and the friction of the tire with the air (Walter and Conant 1974). All these mechanisms of energy loss contribute to the tire rolling resistance, which can be considered as a force that opposes the vehicle motion and is responsible for a significant amount of fuel energy consumption (Hall and Moreland 2001). Most of the rolling resistance is due to hysteresis loss, which depends on the construction of the tire, its material, and external variables such as vehicle speed, wheel load, inflation pressure, etc. (Walter and Conant 1974). At the present moment, the tire industry, particularly Michelin, is still working on new solutions to reduce the rolling resistance while also improving other performance of the tire, e.g., improving handling, grip, comfort, and durability, and reducing wear noise. To archive this, it is crucial to modify the factors that affect the rolling resistance, and it is not worth designing

a new tire, doing some tests, and repeating this procedure. Therefore, a robust tire model can be helpful to investigate how the uncertainties in tire properties affect the rolling resistance in the model by increasing or decreasing its value.

In the literature, there are several approaches to model a tire (Pacejka 2006). Among these approaches, the finite element method is widely adopted for modeling rolling tires (Ghoreishy 2008). The article of Le Tallec and Rahler (1994) is an example of the finite element approximation of a pneumatic tire in a steady-state rolling motion. In this article, viscoelasticity is described by a set of constitutive laws, which are internal variables whose evolution is governed by nonlinear differential equations. Nowadays, it is still a challenge to adequately characterize the viscoelastic material

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considering its particularities such as time-varying properties and energy dissipation. Even with good results, the finite element method can be time-consuming, and the high computational cost motivates the investigation of alternative modeling methods, e.g., the article of Brancati et al. (2011) develops an analytical model of dissipated energy based on a hysteresis model.

Alternatively, in the present work, viscoelastic internal variables are simulated, solving a computationally lessexpensive reduced-order model based on their hysteretic behavior, saving computational time with minor loss of information thanks to a dataset provided by Michelin, from which the parameters of the reduced-order model can be inferred via Bayesian inference. The great advantage of this approach is to obtain a simple, but representative model to describe the hysteresis effect in tires in a Bayesian context.

This paper is organized as follows: first, a statement of the problem in question is performed. Then, the dataset containing right Cauchy–Green deformation tensor and viscoelastic internal variables data from the finite element approximation of a steady-rolling tire is described. Next, a reduced-order model is suggested to approximate the hysteretic behavior of the viscoelastic internal variables based on the Bouc-Wen model. Then, the methodologies to infer the parameters of the reduced-order model from the dataset and to perform a sensitivity analysis are presented. The results of this work are investigated by comparing the responses of the reduced-order model to the viscoelastic internal variables from the dataset. Finally, conclusions, contributions and perspectives for future inquire are addressed.

Problem statement

This work aims to simulate viscoelastic internal variables from a computationally expensive finite element approximation using a reduced-order model, which is less computationally expensive to evaluate at first, but with an accuracy penalty. To do so, data of viscoelastic internal variables from a finite element approximation of a steady-rolling tire provided by Michelin is used. To face this problem, the methodology of this work is structured as follows:

- First, the reduced-order model and an error measure to evaluate the difference between the reduced-order model responses and viscoelastic internal variables data are defined;
- Second, a global sensitivity analysis is performed to identify the parameters of the reduced-order model that influence the error measure the most. The sensitivity analysis is performed using Sobol' indices and the parameters whose Sobol' indices are negligible are considered as constant quantities;
- Third, a Bayesian inference is performed to calibrate the parameters of the reduced-order model that influence the error measure the most;
- Finally, uncertainties are propagated into both error measure and reduced-order model responses.

The reduced-order model is suppose to be less computationally expensive and obtaining viscoelastic internal variables becomes simpler than a finite element procedure in which nonlinear equations are involved. Fig. 1 shows a schematic of the methodology employed in this work.

Michelin's dataset

The dataset provided by Michelin for this work is generated during the computation of a finite element approximation of a pneumatic tire in a steady rolling motion. A standard finite element procedure approximates the continuous problem of the steady rolling tire by coupling the equilibrium equations governing the steady rolling motion and the constitutive laws governing the viscoelastic material behavior (Le Tallec and Rahler 1994). And, because nonlinear differential equations govern the evolution of internal variables, convenient choices of internal variables should be made in terms of differential operators during this standard finite element procedure. Fancello et al. (2006) provide a general framework for constitutive viscoelastic models in which the elastic and viscous potentials derive from a generalized Kelvin-Maxwell model. Le Tallec and Rahler (1994) describe viscoelasticity by choosing the specific free energy potential as a function of three states variables: temperature, right Cauchy-Green deformation tensor, and viscoelastic internal variables. In short, Michelin's dataset gathers data of two of these states variables: the right Cauchy-Green deformation tensor Cand the viscoelastic internal variables A from a generalized Maxwell model. The viscoelastic internal variables A are rank 2 symmetric positive-definite tensors (Le Tallec and Rahler 1994).

The theoretical representation of the generalized Maxwell model consists of the parallel association of various Maxwell branches. Each Maxwell branch consists of the series association of elastic and viscous elements. Figure 2 is a representation of the viscoelastic constitutive model in which C is the right Cauchy-Green tensor and A_j the viscoelastic internal variables at the Maxwell branch j. The complete viscoelastic constitutive model has N = 9 branches.

A data selection was performed to approximately represent the given dataset of C. The selection procedure was based on the visual identification of patterns and redundancies. This work is interested in Maxwell branches from j = 0 to 4 because there is no evolution of C in the other Maxwell branches.

The reduced-order model

For one of the Maxwell branches of interest in Fig. 2, the component C = C(t) of the right Cauchy-Green deformation tensor and its corresponding viscoelastic internal variable A = A(t) are both time series. Therefore, C(t) is assumed to be the sum of three terms: a linear elastic term k A(t) in which k is a elasticity coefficient; a viscous term $c \dot{A}(t)$ in which c is a damping coefficient; and the hysteresis output Z(t). In this work, Z(t) is represented by the Bouc-Wen model. A survey on the Bouc-wen model can be find in Ismail et al. (2009).

The system of first order differential equations in Eq. 1 defines the reduced-order model:

$$\begin{cases} \dot{A}(t) = \frac{1}{c} \left(C(t) - k A(t) - \mathcal{Z}(A, \dot{A}) \right) \\ \dot{\mathcal{Z}} = \alpha \dot{A} - \gamma |\dot{A}| |\mathcal{Z}|^{\nu - 1} \mathcal{Z} - \delta \dot{A} |\mathcal{Z}|^{\nu}, \end{cases}$$
(1)



Figure 1. Methodology adopted to simulate viscoelastic internal variables.



Figure 2. Generalized Maxwell model.

where the input C(t) is completely determined from the dataset. For certain initial conditions $A|_{t=0} = A_0$ (also from the dataset) and $\mathcal{Z}|_{t=0} = \mathcal{Z}_0$, the output responses $A(\theta)$ and \mathcal{Z} depend only on the set of parameter $\theta = \{c, k, \alpha, \gamma, \delta, \nu\}$. α, γ, δ and ν are the parameters of the Bouc-Wen model. Eq. 1 can be solved numerically by a classic Runge-Kutta method.

Parameters of the reduced-order model

Both damping coefficient c and elasticity coefficient k are strictly positive parameters. Furthermore, from empirical evaluations that consist of manually changing parameter values in numerical simulations, the value of k is close to the unit.

It is worth noting that the Bouc-Wen model is a phenomenological model, and its parameters α , γ , δ , and ν do not necessarily have a physical sense. Ismail et al. (2009) indicate conditions on the Bouc-Wen model parameters for physical and mathematical consistency, e.g., the Bouc-Wen model fulfill the second law of Thermodynamics if and only if $\nu > 0$, $\gamma > 0$ and $-\gamma \le \delta \le \gamma$. The called "class I" Bouc-Wen model is bounded input-bounded output stable, consistent with the motion of physical systems, passive, and consistent with the second law of Thermodynamics if $\alpha > 0$,

 $\gamma + \delta > 0$ and $\gamma - \delta \ge 0$. In addition, Eq. 1 has a unique solution if $\nu > 1$.

The parameters estimation methodology

An advantage of the Bouc-Wen model is that it accommodates real hysteresis loops by choosing an appropriate set of parameters (Ismail et al. 2009). For this reason, problems involving the Bouc-Wen model are typically parameter estimation problems, as the example described in Jiang et al. (2020).

In this work, groups of N_j selected inputs are considered. In each group, a selected input C(t) and its corresponding reduced-order model response $A(\theta)$ describes a hysteresis loop. The selected inputs differ in shape, magnitude level, etc., and their corresponding reduced-order model responses also differ from each other. Due to these differences, it is possible to find a different set of parameters θ for each C(t)after solving the parameter estimation problem. However, it is desirable to describe the reduced-order model responses with just a single or two Bouc-Wen models per Maxwell branch of interest in Fig. 2. Hence, an error measure that evaluates these differences is defined.

Equation 2 defines the scale factor where A^{DS} is a viscoelastic internal variable from the dataset and N_{out} is the number of sampling points of the time series:

$$\bar{A} = \frac{1}{N_{\text{out}} - 1} \sum_{k=2}^{N_{\text{out}}} ||A_k^{\text{DS}} - A_{k-1}^{\text{DS}}||, \qquad (2)$$

and Eq. 3 defines E, the mean absolute scaled error (MASE) that measures the difference between the selected output A^{DS} from the dataset and $A(\theta)$:

$$E(\boldsymbol{\theta}) = \frac{1}{N_{\text{out}}} \sum_{k=1}^{N_{\text{out}}} \frac{\left|\left|A_k^{\text{DS}} - A_k(\boldsymbol{\theta})\right|\right|}{\bar{A}}.$$
 (3)

Hyndman and Koehler (2006) suggests that the MASE is the best available measure of forecast accuracy in situations where there are very different scales, including close to zero or negative data. Eq. 4 is simply the mean between the N_j error measures, where each $E_j(\theta)$ corresponds to the evaluation of a reduced-order model response:

$$\bar{E}(\boldsymbol{\theta}) = \frac{1}{N_j} \sum_{j=1}^{N_j} E_j(\boldsymbol{\theta}).$$
(4)

Equation 4 also defines the loss function of an optimization problem. The optimal solution of this optimization problem is indicated in Eq. 5:

$$\hat{E} = \bar{E}(\hat{\theta}), \quad \hat{\theta} = \arg\min_{\mathcal{B}} \bar{E}(\theta),$$
 (5)

where $\hat{\theta}$ is the optimal set of parameter in the set of feasible solutions \mathcal{B} that minimizes the error measure between the selected viscoelastic internal variables A^{DS} from the dataset and the reduced-order model responses $A(\theta)$. $\hat{\theta}$ can be determined by an optimization method such the Cross-Entropy method (Rubinstein and Kroese 2013).

The Cross-Entropy method

The Cross-Entropy (CE) method translates a rare event simulation problem into an optimization problem, and this method can be treated as a two-step iterative process:

- First, random samples are generated according to a given probability distribution is determined feasible region;
- Then, the statistics, i.e., the mean and variance of a set composed by the best performing samples, are used to refine the probability distribution parameters.

Among other applications, the CE method can be used to solve continuous optimization and combinatorial optimization problem (De Boer et al. 2005). Additional information about its theoretical framework and practical considerations about the optimization method can be found in Rubinstein and Kroese (2013).

The central idea of the CE method is based on importance sampling technique and variance minimization. At first, let $X \sim f$ be a random variable with probability density function (PDF) f and let H(X) be a function. The expected value of H(X) is:

$$\mu_f = \mathbb{E}_f\{H(X)\} = \int_{\mathcal{B}} H(x)f(x)dx, \tag{6}$$

Let g also be a PDF. The expected value of $H(X)\frac{f(X)}{q(X)}$ is:

$$\mu_g = \mathbb{E}_g \left\{ H(X) \frac{f(X)}{g(X)} \right\} = \int_{\mathcal{B}} H(x) \frac{f(x)}{g(x)} g(x) dx, \quad (7)$$

The importance sampling estimator $\hat{\mu}_q$ is:

$$\hat{\mu}_g = \frac{1}{N_k} \sum_{k=0}^{N_k} H(x_k) \frac{f(x_k)}{g(x_k)}, \quad X \sim g,$$
(8)

Where the term on the right defines $W(x) = \frac{f(x)}{g(x)}$ the likelihood ratio. The quality of the estimator $\hat{\mu}_g$ depends on

the PDF g. The optimal importance sampling PDF \hat{g}^* is the one in which the variance of $\hat{\mu}_g$ is minimal.

The Kullback-Leibler divergence denoted $\mathcal{D}(g, h)$ offers a measure of how different a chosen PDF h is with respect to the reference PDF g. It is defined as follows in Eq. 9:

$$\mathcal{D}(g,h) = \mathbb{E}_g \Big\{ \ln \frac{g(X)}{h(X)} \Big\},\tag{9}$$

The PDF $f(\cdot; \boldsymbol{v})$ determined by the hyper-parameters vector \boldsymbol{v} is chosen. As it is shown in Kroese et al. (2013), the minimization of $\mathcal{D}(\hat{g}^*, f(\cdot, \boldsymbol{v}))$ leads to Eq. 10, with $\hat{g}^* \propto Hf(\cdot; \boldsymbol{u})$:

$$\boldsymbol{v}^* = \arg\max_{\boldsymbol{w}} \mathbb{E}_{\boldsymbol{w}} \Big\{ H(x) W(x; \boldsymbol{u}, \boldsymbol{w}) \ln f(x; \boldsymbol{v}) \Big\}, \quad (10)$$

Finally, for $X_s \sim f(\cdot; \boldsymbol{w})$ and $W(x_s; \boldsymbol{u}, \boldsymbol{w}) = \frac{f(\cdot; \boldsymbol{u})}{f(\cdot; \boldsymbol{w})}$, $\hat{\boldsymbol{v}}$ is the hyper-parameter vector that approximates to the optimal importance estimator (minimal variance):

$$\hat{\boldsymbol{v}} = \arg \max_{\boldsymbol{w}} \frac{1}{N_k} \sum_{k=0}^{N_k} H(x_k) W(x_k; \boldsymbol{u}, \boldsymbol{w}) \ln f(x_k; \boldsymbol{v}),$$
(11)

Some assumptions are made in the sequence about the function H(x). For rare event simulation and optimization problems, $H(x) = \mathbb{1}_{J(x) \ge \epsilon}$ where $\mathbb{1}_{J(x) \ge \epsilon}$ is the indicator function and J(x) is an objetive function. Specifically, in this work:

$$H(\boldsymbol{\theta}) = \mathbb{1}_{\bar{E}(\boldsymbol{\theta}) \geq \epsilon} = \begin{cases} 1 & \text{if } \bar{E}(\boldsymbol{\theta}) \geq \epsilon \\ 0 & \text{if } \bar{E}(\boldsymbol{\theta}) < \epsilon, \end{cases}$$
(12)

The expected value of the indicator function is the probability of the event $\overline{E}(\theta) \ge \epsilon$ to occur.

$$\mu = \mathbb{E}\left\{\mathbb{1}_{\bar{E}(\boldsymbol{\theta}) \ge \epsilon}\right\} = \mathcal{P}\{\bar{E}(\boldsymbol{\theta}) \ge \epsilon\},\tag{13}$$

And its importance sampling estimator $\hat{\mu}_g$ is:

$$\hat{\mu}_{g} = \frac{1}{N_{k}} \sum_{k=0}^{N_{k}} \mathbb{1}_{\bar{E}(\boldsymbol{\theta}^{(k)})) \ge \epsilon}.$$
(14)

The computational algorithm of the CE method is described as follows:

- 1. Initialize: Choose initial hyper-parameters values $\hat{\mu}_0$ and $\hat{\sigma}_0^2$, and $\hat{v}_0 = {\hat{\mu}_0, \hat{\sigma}_0^2}$. Set level counter l = 1;
- 2. **Sampling:** Generate N_k independent and identically distributed (iid) samples from the standard multivariate Gaussian distribution:

$$\boldsymbol{\theta}^{(1)},\ldots,\boldsymbol{\theta}^{(N_k)}\sim \mathcal{N}(\hat{\mu}_{l-1},\hat{\sigma}_{l-1}^2);$$

3. Select: Evaluate the objective function for each sample and sort the N_k results in order:

$$\bar{E}(\boldsymbol{\theta}^{(\cdot)}) \leq \cdots \leq \bar{E}(\boldsymbol{\theta}^{(\cdot)}).$$

The called elite sample set \mathcal{E} gathers the $N_{\mathcal{E}} < N_k$ samples that better performed;

4. Update: Compute estimators:

$$\tilde{\mu}_l = \frac{1}{N_{\mathcal{E}}} \sum_{s=1}^{N_{\mathcal{E}}} \boldsymbol{\theta}^{(s)}$$
(15)

$$\hat{\sigma}_l^2 = \frac{1}{N_{\mathcal{E}}} \sum_{s=1}^{N_{\mathcal{E}}} \left(\boldsymbol{\theta}^{(s)} - \tilde{\mu}_l \right)^2 \tag{16}$$

5. Smooth: Apply the smooth updating schema:

$$\hat{\mu}_l := a \cdot \tilde{\mu}_l + (1-a) \cdot \hat{\mu}_{l-1};$$
 (17)

6. **Return** $\hat{\theta} = \hat{\mu}_l$ if the stopping criteria $\hat{\sigma}_l^2 < \epsilon_{\text{max}}$ is reached. Otherwise, increase level counter by 1 and return to the second step.

Operationally, the hyper-parameters vector \hat{v} in Eq. 11 can be estimated via maximum likelihood estimation method. The use of the standard multivariate Gaussian distribution (Botev 2016) on the sampling step of the CE method simplifies the estimation of \hat{v} and the hyper-parameters can be computed directly by Eqs. 15 and 16.

Despite its mathematical formulation, the CE method can be easily implemented, and only few parameters that are pretty intuitive are needed: given the PDF $f(\cdot; v)$, the total number of samples N_s , the number of samples $N_{\mathcal{E}}$ in the elite sample set, a stopping criterion ϵ_{max} and the maximum of iteration level l_{max} .

To the sequence, \hat{E} can be computed using Eq. 5 and the parameters of the reduced-order model are considered as random quantities. Their distributions are inferred via Bayesian inference.

Bayesian inference

Bayesian inference approach is a straightforward strategy to quantify uncertainties (Gelman et al. 2013). A set of parameter is considered as a random input vector x and the methodology to infer its distribution is based on the Bayes' theorem in Eq. 18:

$$\pi(\boldsymbol{x}|\hat{\boldsymbol{E}}) = \frac{\pi(\hat{\boldsymbol{E}}|\boldsymbol{x})\pi(\boldsymbol{x})}{\pi(\hat{\boldsymbol{E}})},$$
(18)

Where $\pi(x)$ is the prior distribution of the set of parameters x; $\pi(\hat{E}|x)$ is the likelihood function of \hat{E} given a set of parameters x at hand; and $\pi(x|\hat{E})$ is the posterior distribution of the set of parameters x given \hat{E} . The denominator $\pi(\hat{E})$ is the marginal likelihood: it is a normalized constant so the posterior distribution defines a probability density function with integral equal to the unity. In this case, Eq. 18 can be simplified into:

$$\pi(\boldsymbol{x}|\hat{\boldsymbol{E}}) \propto \pi(\hat{\boldsymbol{E}}|\boldsymbol{x}) \,\pi(\boldsymbol{x}),\tag{19}$$

The prior distribution $\pi(x)$ can be interpreted as the knowledge degree about x before any evidence. It is classified based on its influence on the posterior distribution, and an example of a diffuse – or non-informative – prior is a Uniform prior distribution. In this case, Eq. 19 can also be simplified into:

$$\pi(\boldsymbol{x}|\hat{\boldsymbol{E}}) \propto \pi(\hat{\boldsymbol{E}}|\boldsymbol{x}); \tag{20}$$

One assumes that \hat{E} is the MASE measure $\bar{E}(\theta)$ from the reduced-order model plus a discrepancy term ϵ that is the source of uncertainty:

$$\hat{E} = \bar{E}(\boldsymbol{\theta}) + \epsilon, \qquad (21)$$

It is supposed an additive Gaussian discrepancy with mean $\mu_{\epsilon} = 0$ and unknown variance σ_{ϵ}^2 . Thus, $\epsilon \sim \mathcal{N}(0, \sigma_{\epsilon}^2)$ and:

$$\pi(\hat{E}|\boldsymbol{x} = \{\boldsymbol{\theta}, \sigma_{\epsilon}^2\}) \sim \mathcal{N}(\hat{E}|\ \bar{E}(\boldsymbol{\theta}), \sigma_{\epsilon}^2), \qquad (22)$$

In the case of independent and identically distributed (iid) observations, the likelihood function is defined as:

$$\pi(\hat{\boldsymbol{E}}|\boldsymbol{x}) = \prod_{i=1}^{N} \pi(\hat{E}_{i}|\boldsymbol{x})$$
$$= \frac{1}{\sqrt{2\pi\sigma_{\epsilon}^{2}}} \exp\left(-\frac{1}{2}\sum_{i=1}^{N} \frac{\left(\hat{E}_{i} - \bar{E}(\boldsymbol{\theta})\right)^{2}}{\sigma_{\epsilon}^{2}}\right);$$
(23)

After Eqs. 20 and 23, the posterior distribution can be finally defined as follows:

$$\pi(\boldsymbol{x}|\hat{\boldsymbol{E}}) \propto \frac{1}{\sqrt{\sigma_{\epsilon}^2}} \exp\left(-\frac{1}{2} \sum_{i=1}^{N} \frac{\left(\hat{E}_i - \bar{E}(\boldsymbol{\theta})\right)^2}{\sigma_{\epsilon}^2}\right). \quad (24)$$

Markov chain Monte Carlo

To determine the posterior distribution $\pi(\boldsymbol{x}|\hat{\boldsymbol{E}})$ of parameters \boldsymbol{x} given $\hat{\boldsymbol{E}}$ is not always trivial. Usually, Monte Carlo simulation is used to approximate the solutions. This book is a review on Monte Carlo methods (Kroese et al. 2011).

The Metropolis-Hastings algorithm is a Markov chain Monte Carlo (MCMC) method based on the construction of a Markov chain such that the future state $x^{(k+1)}$ of the chain depends only on its current state $x^{(k)}$ and a transition probability distribution $\mathcal{T}(x^{(k+1)}|x^{(k)})$. In the presence of a sufficient number of unbiased samples, the sequence of random variables $X = \{x^{(0)}, x^{(1)}, x^{(2)}, ...\}$ represents the posterior distribution $\pi(x|\hat{E})$.

$$\frac{\pi(\boldsymbol{x}^{(k+1)}|\hat{\boldsymbol{E}})}{\pi(\boldsymbol{x}^{(k)}|\hat{\boldsymbol{E}})} = \frac{\mathcal{T}(\boldsymbol{x}^{(k+1)}|\boldsymbol{x}^{(k)})}{\mathcal{T}(\boldsymbol{x}^{(k)}|\boldsymbol{x}^{(k+1)})}$$
(25)

The procedure for generating the future state $x^{(k+1)}$ is a two-stage process: the first stage is to generate a candidate $x^{(*)}$ that depends only on the current state $x^{(k)}$ of the Markov chain; the second stage is to accept or reject $x^{(*)}$. To this end, it is necessary to compute the acceptance probability *a* according to Eq. 26 where $\mathcal{K}(x^{(*)}|x^{(k)})$ is the proposed probability distribution.

$$a = \min\left\{1, \frac{\pi\left(\boldsymbol{x}^{(*)}|\hat{\boldsymbol{E}}\right) \mathcal{K}\left(\boldsymbol{x}^{(*)}|\boldsymbol{x}^{(k)}\right)}{\pi\left(\boldsymbol{x}^{(k)}|\hat{\boldsymbol{E}}\right) \mathcal{K}\left(\boldsymbol{x}^{(k)}|\boldsymbol{x}^{(*)}\right)}\right\}$$
(26)

Then, a random number $u \sim \mathcal{U}(0, 1)$ is generated from a Uniform distribution with parameters 0 and 1. If u < a, the candidate $\mathbf{x}^{(*)}$ is accepted and $\mathbf{x}^{(k+1)} = \mathbf{x}^{(*)}$. Otherwise, if $u \ge a$, $\mathbf{x}^{(*)}$ is rejected and $\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)}$.

The random walk Metropolis algorithm is a particular case of the Metropolis-Hastings algorithm. In it, a symmetric Gaussian distribution with variance σ^2 is proposed to

generate the candidate $x^{(*)}$ (Saadi et al. 2011) yielding:

$$a = \min\left\{1, \frac{\pi(\boldsymbol{x}^{(*)}|\hat{\boldsymbol{E}})}{\pi(\boldsymbol{x}^{(k)}|\hat{\boldsymbol{E}})}\right\}$$
(27)

The following steps summarizes the random walk Metropolis algorithm:

- 1. Initialize the counter and assign initial value $x^{(0)}$;
- 2. Generate a candidate $\boldsymbol{x}^{(*)} \sim \mathcal{N}(\boldsymbol{x}^{(k)}, \sigma^2);$
- 3. Compute the acceptance probability $a(\mathbf{x}^{(*)}, \mathbf{x}^{(k)})$ and generate a random number $u \sim \mathcal{U}(0, 1)$;
- 4. Does u < a? If positive, accept the candidate and $\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(*)}$. If negative, reject and $\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)}$;
- 5. Increment the counter and return to the second step.

This article adopted some strategies while implementing the Metropolis-Hastings algorithm to accelerate the convergence of the Markov chain (Saadi et al. 2011). A first feature is the total number of generated samples: the Markov chain must be large enough to adequately represent the distributions of the parameters respecting the law of large numbers. The second one is the acceptance probability rate $\bar{a} = \frac{N_a}{N_b}$ that is the ratio between the number of accepted samples N_a and the total number of generated samples N_k . \bar{a} is controlled by the random walk step σ : on the one hand, if the jump from one sample to the other is too large, \bar{a} is small, and the chain keeps static; on the other hand, if the jump from one sample to the other is too small, \bar{a} is big, and the chain needs more time to go through the parameters space. Saadi et al. (2011) consider an optimal \bar{a} between 40% - 50%. The last feature is the burn-in: eliminating a defined number of initials samples from the final result. This feature is essential to eliminate biased results.

Global sensitivity analysis and Sobol' indices

Global sensitivity analysis (GSA) refers to a set of mathematical techniques that quantifies the influence of the input parameters on the response of interest of a system. Among other techniques, the GSA can be defined by the variance decomposition method, which aims to decompose the output variance as the sum of each input variable's contributions or combinations. Sobol' indices is a variance decomposition method defined in Sobol (1993) that engineers have recently explored. Its central idea is to determine the expansion of a computational model into the sum of terms of increasing dimensions and the rapport of the partial variances of these terms concerning the model's total variance.

Generally speaking, let \mathcal{M} be a mathematical model and X a random input vector gathering k independent input parameters. \mathcal{M} describes a scalar output of interest Y of a physical system:

$$Y = \mathcal{M}(X)$$
, $X = \{X_1, X_2, \dots, X_k\}$, (28)

According to Sobol (1993), Y can be defined by the decomposition into sums of different dimensions. For simplification purposes, the following equations will assume that the input parameters are uniformly distributed and the

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support of X is $\mathcal{D}_X = [0, 1]^k$. Therefore, Y can be also written as:

$$Y = \mathcal{M}_{0} + \sum_{i=1}^{k} \mathcal{M}_{i} (X_{i}) + \sum_{i < j}^{k} \mathcal{M}_{ij} (X_{i}, X_{j}) + \dots + \mathcal{M}_{1...k} (X_{1} \dots X_{k}),$$
(29)

All terms of this expansion can be computed through integrals. The first term \mathcal{M}_0 is a constant equal to the expected value:

$$\mathcal{M}_{0} = \int_{\mathcal{D}_{\boldsymbol{X}}} \mathcal{M}(\boldsymbol{X}) \, d\boldsymbol{X}$$
(30)

The other terms $\mathcal{M}_i(X_i)$ and $\mathcal{M}_{ij}(X_i, X_j)$ are the conditional mean values for the parameters *i* and *ij* $(i \neq j)$, respectively:

$$\mathcal{M}_{i}(X_{i}) = \int_{0}^{1} \cdots \int_{0}^{1} \mathcal{M}(\boldsymbol{X}) \ d\boldsymbol{X}_{\sim i} - \mathcal{M}_{0}, \qquad (31)$$

$$\mathcal{M}_{ij}(X_i, X_j) = \int_0^1 \cdots \int_0^1 \mathcal{M}(\mathbf{X}) \ d\mathbf{X}_{\sim ij} - \mathcal{M}_0 - \mathcal{M}_i(X_i) - \mathcal{M}_j(X_j).$$
(32)

The notation $\sim i$ indicates that parameter X_i is excluded. Eq. 29 has the property of orthogonality in terms of conditional means as defined by Homma (1996), and it is possible to define the Sobol decomposition in terms of conditional variances (Sobol 1993). Thus, one can compute the first-order Sobol' indices that quantify the additive effect of each input parameter separately concerning the total variance:

$$S_i = \frac{Var[\mathcal{M}_i(X_i)]}{Var[\mathcal{M}(\mathbf{X})]},\tag{33}$$

And the second-order Sobol' indices that quantify the interaction effects between two input parameters:

$$S_{ij} = \frac{Var[\mathcal{M}_{ij}(X_i, X_j)]}{Var[\mathcal{M}(\mathbf{X})]}.$$
(34)

Higher-order Sobol' indices are equally defined, and they take into account the interaction effects of various input parameters.

PCE-based Sobol' indices

Monte Carlo simulation can compute Sobol's indices, although it has a high computational cost due to the low convergence rate. An alternative way to calculate Sobol's indices is constructing a surrogate model based on Polynomial Chaos Expansion (PCE). Surrogate modeling consists of techniques to approximate a model to reduce processing time and computational cost while maintaining model accuracy. Crestaux et al. (2009) and Palar et al. (2018) compute Sobol' indices using PCE.

PCE was developed by Wiener (1938). In the engineering field, Ghanem and Spanos (1991) applied it in the spectral analysis of stochastic finite elements. Recently, PCE has been used in a wide variety of domains, e.g., in structural dynamics and uncertainty quantification. It is a non-intrusive method. It represents an uncertain quantity of interest through an expansion composed of deterministic coefficients

and orthogonal polynomials defined concerning the input probability density function. Sudret (2008) is a review on GSA using PCE.

Based on PCE expression, the scalar output of interest Y can be rewritten as:

$$Y \approx \sum_{\alpha \in \mathcal{A}} y_{\alpha} \psi_{\alpha}(X), \tag{35}$$

In Eq. 35, ψ_{α} are multivariate polynomials that are orthonormal in relation to the joint probability density function $f_{\mathbf{X}}$, y_{α} are unknown deterministic coefficients and \mathcal{A} is a truncation criterion, where $\mathcal{A} \subset \mathbb{N}^M$ is the set of selected multi-indices of multivariate polynomials. The coefficients can be determined through the least angle regression method.

The statistics of an uncertain output response Y can be determined using PCE. Therefore, the mean and the variance are defined, respectively, by:

$$\mathcal{M}_0 = y_0 \text{ and } \widehat{Var}(Y) = \sum_{\alpha \in \mathcal{A}} y_{\alpha}^2.$$
 (36)

Therefore, the Sobol indices can be directly determined with minor computational effort through the PCE coefficients. Thus, the first and second-order Sobol' indices are, respectively:

$$S_i^{PCE} = \sum_{\substack{\alpha \in \mathcal{A}_i \\ \alpha \neq 0}} y_{\alpha}^2 / \sum_{\alpha \in \mathcal{A}} y_{\alpha}^2.$$
(37)

$$S_{ij}^{PCE} = \sum_{\substack{\alpha \in \mathcal{A}_{ij} \\ \alpha \neq 0}} y_{\alpha}^2 \Big/ \sum_{\alpha \in \mathcal{A}} y_{\alpha}^2.$$
(38)

In this work, a global sensitivity analysis is performed. The Sobol indices are computed via a PCE-based surrogate model to verify the influence of the reduced-order model parameters on the error measure.

Results and Discussion

At first, the results in this section refer to the first Maxwell branch in Fig. 2. Later, the same procedure to infer the reduced-order model parameters is adopted to other Maxwell branches.

Initially, it is assumed that each reduced-order model parameter follows an Uniform distribution $\mathcal{U}(a, b)$ with *a* its minimum and *b* its maximum and they are chosen so that the Bouc-Wen model is "class I" defined. Their value are indicated in Tab. 1.

A prior global sensitivity analysis is performed in order to verify which reduced-order model parameter is influential. To do so, a PCE-based surrogate model $\tilde{E}(\theta)$ is first built and the PCE coefficients were computed using UQLab metamodeling module based on sparse least angle regression (Marelli and Sudret 2021). A comparison between responses $\bar{E}(\theta)$ and $\tilde{E}(\theta)$ can be seen in Fig. 3: the closer to the black diagonal line, the more reliable the PCE-based surrogate model is. In Fig. 3, both $\bar{E}(\theta)$ and $\tilde{E}(\theta)$ were evaluated at 100 cross-validation sample points. It is worth mentioning that the reduced-order model is not yet calibrated and it is



Figure 3. Comparison between responses $\overline{E}(\theta)$ and $\overline{E}(\theta)$. The disposition of the 100 cross-validation samples + indicates that the PCE-based surrogate model is adequate.

natural to observe in Fig. 3 high error measure values, e.g., $\bar{E}(\theta) > 1$. Tab. 2 contains additional information about the validation of the PCE-based surrogate model.

Once the PCE-based surrogate model is built and validated, the Sobol indices are computed with minor computational effort using the UQLab sensitivity analysis module (Marelli et al. 2021). This procedure allows drawing some conclusions about the reduced-order model parameters before inferring their distributions. Fig. 4 shows the total and first-order Sobol' indices. On the one side, these results indicate that parameters γ and δ have little influence on the error measure of the reduced-order model, i.e., having less or more variability on these parameters does not mean that the reduced-order model simulates viscoelastic internal variables more accurately. For this reason, thy are considered as completely determined quantities and their values are set to $\gamma = 1,000$ and $\delta = 1,000$ so $\frac{\gamma}{\gamma + \delta} = 0.5$. On the other side, parameters c, α, k , and ν have considerable influences, their variability requires attention and they are considered as unknowns.



Figure 4. Prior global sensitivity analysis. Total and first order Sobol' indices of parameters γ and δ are negligible. Therefore, they can be considered as determined quantities.

In the sequence, the optimization problem defined in Eq. 5 is solved through the CE method. The CE method parameters used in this step are $N_s = 100$ samples, $N_{\mathcal{E}} = 4$ samples,

Table 1. Uniform prior distribution $\mathcal{U}(a,b)$ of the reduced-order model parameters for global sensitivity analysis.

Support	с	k	α	$\gamma \; [imes 10^3]$	$\delta [\times 10^3]$	ν
a	0	0.999	0	1	-1	1
b	0.01	1.001	1	10	1	3

Table 2. Validation of the PCE-based surrogate model.

PCE degree	Exp. Design	LOO error
14	2,000	$5.7 \cdot 10^{-3}$

 $l_{\rm max} = 500$ iterations and $\epsilon_{\rm max} = 10^{-6}$. In addition, a smooth updating schema of 0.8 is also used. Tab. 3 contains the optimal reduced-order model parameters values and they can be used for deterministic simulation of viscoelastic internal variables. In this condition, $\hat{E} = 0.3168$.

Table 3. Reduced-order model optimal parameters valuesgiven by the CE method.

\hat{c}	\hat{k}	\hat{lpha}	$\hat{ u}$
0.0018	1.0000	0.1152	1.0220

However, it is desirable to verify how the uncertainties propagate in the reduced-order model response. For this, parameters c, k, α , ν and the discrepancy variance σ_{ϵ}^2 are calibrated through Bayesian inference, according to the following procedure:

- Minimum and maximum values a and b of the Uniform prior distribution U(a, b) were redefined because it was empirically verified that the ranges of values were too wide. The new bounds are indicated in Tab. 4;
- A MCMC sample was generated with $N_k = 10^5$ samples. The random walk step σ was manually adjusted, so the acceptance rate is $\bar{a} \approx 40 50\%$;
- No MCMC sample was eliminated in the burn-in.

Fig. 5 shows the trace plot of parameters c, k, α, ν and σ_{ϵ}^2 . It is clear in this figure that there is no need to eliminate burn-in samples. Moreover, Fig. 6 shows the densities and cumulative densities of each one of these parameters, where the continuous black line is the probability density estimates (EPDF) based on a normal kernel function, the black dashed line is the Uniform prior distribution, and the marked red line is the empirical cumulative distribution function (ECDF).

- It is possible to notice that the samples are centered around 0.002 and 1 for parameters c and k, respectively;
- There is a higher tendency for parameter α to get values between 0 and 0.2;
- There is a higher tendency for parameter ν to get values close to 1.

Similarly, Fig. 7 shows the density and cumulative density function of the error measure from Eq. 4, where the continuous black line is the probability density estimates (EPDF) based on a normal kernel function and the marked red line is the empirical cumulative distribution function (ECDF). In addition, the vertical dashed magenta line is the optimal error measure value \bar{E} determined by the CE method. Two cases are presented in Fig. 7: the first one refers to the error measure evaluated at the first Maxwell branch (Branch 1) in Fig. 2; and the second one to the fourth Maxwell branch (Branch 4). On the one hand, from their ECDF, there is a high probability of having $E(\theta) \leq 1$ in Branch 1. On the other hand, Branch 4 optimal error measure value \hat{E} is already higher than 1, and, adopting this criterion, the reduced-order model is not suitable for simulating viscoelastic internal variables from Branch 4.

Finally, Fig. 8 shows some Branch 1 responses of the reduced-order model with uncertainties. The inputs C(t) and initial conditions $A|_{t=0}$ used to validate in Fig. 8 are not the same that was used during the reduced-order model calibration procedure.

Conclusions

This work suggested a reduced-order model simulating viscoelastic internal variables from the constitutive model of a viscoelastic material used in the finite element formulation of a steady-rolling tire. To carry out this work, Michelin furnished data of right Cauchy-Green deformation and viscoelastic internal variables from a generalized Maxwell model composed of various Maxwell branches.

The suggested reduced-order model is based on the Bouc-Wen model due to the hysteretic behavior of the right Cauchy-Green deformation and viscoelastic internal variables. The reduced-order model gets selected data of right Cauchy-Green deformation, initial conditions, and a set of parameters, and it returns the response of the viscoelastic internal variable. An advantage of the reduced-order model is that it can be computed by numerically solving a less computationally expensive nonlinear system of first-order differential equations rather than a finite element model.

An error measure based on the mean absolute scaled error (MASE) is defined to evaluate the discrepancy between the reduced-order model responses and the data. Then, a global sensitivity analysis is performed by computing Sobol' indices after a PCE-based surrogate model to quantify the influence of the reduced-order model parameters into the error measure. Some of the parameters have a small influence, and they are considered as completely determined quantities. The distribution of the parameters that have non-negligible influence is inferred thanks to the Bayesian inference procedure. The distribution of the error measure is also estimated, and uncertainties are propagated through the viscoelastic internal variable responses.

The main efforts on this process were to calibrate the reduced-order model and, after this process, the reducedorder model was capable of simulating the components of viscoelastic internal variable considering uncertainties with reduced computational cost. By evaluating the error measure, the reduced-order model is adequate for simulating viscoelastic internal variables for the three initial Maxwell branches, and its responses are in accordance with the

Table 4. Uniform prior distribution U(a, b) of the reduced-order model parameters and discrepancy variance for Bayesian inference.

Support	С	k	α	ν	σ^2_{ϵ}
a	0	0.9995	0	1	0
b	0.005	1.0005	1	3	0.1

data. Obtaining a robust and cheaper computational model facilitates simulations and research to obtain a more robust tire. In this sense, the results of this paper contribute to the advancement of research on computational models that are more robust and with a reduced computational cost.

Availability of data and material

The MATLAB scripts used in the numerical simulations are available in the following GitHub repository: https://github.com/rafaelraqueti/UQ_ Bouc-Wen_calibration.git. The dataset used in this work belongs to Michelin and is not available in this repository.

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Declaration of conflicting interests

The authors declared no potential conflicts of interest concerning the research, authorship and publication of this article.

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Figure 5. Reduced-order model parameters and discrepancy variance trace plots: $\bar{a} \approx 40\%$. There is no need to burn-in samples.

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Figure 6. Reduced-order model parameters and discrepancy variance estimated probability density functions (EPDF), Uniform prior distributions and empirical cumulative distribution functions (ECDF).



Figure 7. Error measure estimated probability density function (EPDF) and empirical cumulative distribution function (ECDF). The minimum error measure value \hat{E} given by the CE method is indicated.



Figure 8. Reduced-order model responses related to different inputs C(t) and initial conditions $A|_{t=0}$, where \blacksquare is the 95% confidence interval (CI), -- the reduced-order model mean and - the viscoelastic internal variable from the dataset.