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Convergence of the stochastic Navier-Stokes- α solutions toward the stochastic Navier-Stokes solutions

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Abstract

Loosely speaking, the Navier-Stokes- α model and the Navier-Stokes equations differ by a spatial filtration parametrized by a scale denoted α . Starting from a strong two-dimensional solution to the Navier-Stokes- α model driven by a multiplicative noise, we demonstrate that it generates a strong solution to the stochastic Navier-Stokes equations under the condition $\alpha \rightarrow 0$. The initially introduced probability space and the Wiener process are maintained throughout the investigation, thanks to a local monotonicity property that abolishes the use of Skorokhod's theorem. High spatial regularity a priori estimates for the fluid velocity vector field are carried out within periodic boundary conditions.

Keywords: Navier-Stokes- α , Navier-Stokes, multiplicative noise, cylindrical Wiener process, strong solutions

2020 MSC: 60H15, 60H30, 37L55, 35Q30, 35Q35, 76D05

1 Introduction

To circumvent most of the Navier-Stokes drawbacks, a reasonable amount of Large Eddy Simulation (LES) models have been created and introduced to the fluid mechanics' literature. Among them is the Navier-Stokes- α (NS- α) model, which made its appearance in [7, 15] and is known under the names: Lagrangian averaged Navier-Stokes (LANS- α) equations [18] or the viscous Camassa-Holm problem [3]. Given a solution to the stochastic NS- α model:

$$\begin{cases} \frac{\partial}{\partial t} (\bar{u} - \alpha^2 \Delta \bar{u}) - \nu \Delta (\bar{u} - \alpha^2 \Delta \bar{u}) - \bar{u} \times (\nabla \times (\bar{u} - \alpha^2 \Delta \bar{u})) + \nabla \bar{p} = g(\cdot, \bar{u}) \frac{\partial W}{\partial t}, \\ \operatorname{div}(\bar{u}) = 0, \\ \bar{u}(0, \cdot) = \bar{u}_0, \end{cases} \quad (1.1)$$

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the main interest in this paper is to check whether or not it converges toward a solution of the stochastic Navier-Stokes equations (NSEs)

$$\begin{cases} \frac{\partial u}{\partial t} + -\nu \Delta u + [u \cdot \nabla]u + \nabla p = g(\cdot, u) \frac{\partial W}{\partial t}, \\ \operatorname{div}(u) = 0, \\ u(0, \cdot) = \bar{u}_0, \end{cases} \quad (1.2)$$

when the spatial scale α tends to 0. Both equations are equipped with the same configurations, including the initial datum \bar{u}_0 to guarantee a similar fluid state at time $t = 0$. The two-dimensional vectors \bar{u} and u denote the fluid velocities, the \mathbb{R} -valued quantities p and \bar{p} represent the pressure fields, the positive constant ν symbolizes the kinematic fluid viscosity, α is a small positive spatial scale at which the fluid motion is filtered, g is a diffusion coefficient depending on the velocity vector field, and W is an infinite-dimensional (possibly cylindrical) Wiener process. On account of the poor uniqueness properties of three-dimensional solutions to the stochastic NSEs, the conducted study herein will be limited to two dimensions to guarantee that the unique solution of the stochastic NS- α equations converges toward a sole one as α goes to 0.

In this paper, the study is accomplished through periodic boundary conditions for the sake of investigating the effect of α on the space regularity of a solution and taking advantage of the nonlinearity's properties that occur within this framework. It could have been carried out within Dirichlet boundary conditions if only the typical solution's space regularity was intended. Observe that α is always multiplied by $\Delta \bar{u}$ in equations (1.1), meaning that the extra granted regularity that does not figure in problem (1.2) can be loosened through a particular assumption on α when dealing with a finite-dimensional system, namely a Faedo-Galerkin approximation. The pressure field will be eliminated from the corresponding weak formulation throughout this work through the null divergence criterion, and the focus will be turned toward the velocity vector. Equations (1.1) will be transformed into a coupled problem of second-order so that its form matches somehow that of system (1.2), and the spatial scale α will be controlled by the inverse of a specific eigenvalue of the Stokes operator for the sake of absorbing the extra space regularity that is delivered by equations (1.1).

Investigating the convergence of equations (1.1) toward system (1.2) is beneficial because the principal reason for which the NS- α model was introduced is to overcome most of the Navier-Stokes shortcomings. If the converse scenario took place, equations (1.1) would have become obsolete, but fortunately, it is not. This convergence was also conducted for the deterministic settings (i.e. when $g = 0$) in [4], where the convergence rate in terms of α is revealed. The theoretical study herein has the advantage of building efficient numerical schemes for the stochastic Navier-Stokes problem while considering minimal assumptions on the spatial scale α . Since α is solely involved with solutions' space regularity, any time discretization should not come into play in any further hypotheses upon α .

Equations (1.1) were first inspected in [5, 6], where the existence of a unique variational solution was proven. It is worth highlighting one drawback of this model relative to the pressure's regularity that appears after applying a generalization of the De Rham theorem [17], which links the velocity's smoothness to that of the pressure. In point of fact, it was shown (c.f. [6, Theorem 3.3]) that \bar{p} is H^{-1} -valued, meaning that it is lower than that of p , which is L^2 -valued. This inconvenience originates from the biharmonic operator that appears in the first identity of system (1.1) and might have an uncooperative effect on convergence rates of numerical schemes concerned with a non-null divergence of velocities. The same goes for other stochastic Navier-Stokes variants, such as the Leray- α model [11]. Further examinations of equations (1.1) were performed in [10, 12], including a splitting-up scheme in [9].

This paper is organized as follows: all preliminaries, assumptions and configurations are presented in Section 2, which allows the main theorem of this work to be stated in Section 3, followed by Section 4 where the Faedo-Galerkin approximation of equations (1.1) is exploited to acquire a finite-dimensional system, and a priori estimates are carried out within multiple spatial regularities. Section 5 provides

the convergence steps of the projected system, including the local monotonicity property, which is a prominent member of the demonstration. Finally, a conclusion regarding the accomplished analysis in the previous section, the relationship with the Navier-Stokes problem, and a few perspectives are given in Section 6.

2 Configuration and materials

Given a positive number L , the domain D represents a two-dimensional torus $(0, L)^2$, and for a given $T > 0$, the time interval reads $[0, T]$. Throughout this paper, the Lebesgue and Sobolev spaces are denoted L^p and H^m (or $W^{m,p}$) respectively, and for an arbitrary normed vector space X , its associated norm will be symbolized by $\|\cdot\|_X$. The notation X_{per} signifies that all its members are periodic functions whose mean is null. Regarding the small spatial scale α that is present in equations (1.1), a special norm $\|\cdot\|_\alpha$ is associated with it and defined by $\|\cdot\|_\alpha^2 := \|\cdot\|_{L^2}^2 + \alpha^2 \|\nabla \cdot\|_{L^2}^2$. The notation $\mathcal{L}_2(E, F)$ is the space of all Hilbert-Schmidt operators; with E and F being two given Banach spaces, \lesssim embodies a shorthand for the less or equal symbol \leq up to a universal non-negative constant, and C_D will denote throughout this paper a positive constant depending only on the domain D . The solely employed Gelfand triple herein is $(H_{per}^1(D), L_{per}^2(D), H_{per}^{-1}(D))$, where $H_{per}^{-1}(D)$ is the dual space of $H_{per}^1(D)$. The $L^2(D)$ space will be endowed with its standard inner product (\cdot, \cdot) , and the duality brackets $\langle \cdot, \cdot \rangle$ will represent the duality product between $H_{per}^1(D)$ and $H_{per}^{-1}(D)$. Following the mathematical notations for the Navier-Stokes framework, the function spaces that will be frequently encountered herein are

$$\begin{aligned} \mathcal{V} &:= \{u \in [C_{per}^\infty(D)]^2 \mid \operatorname{div}(u) = 0\}, \\ \mathbb{H} &:= \{u \in [L_{per}^2(D)]^2 \mid \operatorname{div}(u) = 0 \text{ a.e. in } D\}, \\ \mathbb{V} &:= \{u \in [H_{per}^1(D)]^2 \mid \operatorname{div}(u) = 0 \text{ a.e. in } D\}. \end{aligned}$$

Let A be the Stokes operator defined from $D(A) := [H^2(D)]^2 \cap \mathbb{V}$ into \mathbb{H} by $A := -\mathcal{P}\Delta$, where $\mathcal{P}: [L_{per}^2(D)]^2 \rightarrow \mathbb{H}$ is the Leray Projector. In two-dimensional domains and under periodic boundary conditions, it is well-known that the Laplace-Leray commutator $[\mathcal{P}, \Delta]$ vanishes; namely $\mathcal{P}\Delta = \Delta\mathcal{P}$. Recall that operator A is self-adjoint whose inverse is compact (c.f. [21, 24]). From now on, all Cartesian products of a sole linear space will be symbolized by blackboard bold letters with the domain D being omitted. For instance, the Sobolev space $[H_{per}^1(D)]^2$ will become \mathbb{H}_{per}^1 .

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ be a filtered complete probability space whose filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ is right-continuous. Given a separable Hilbert space K equipped with a complete orthonormal basis $\{w_k, k \geq 1\}$, the K -valued cylindrical Wiener process $W(t), t \in [0, T]$ reads

$$W(t) := \sum_{k \geq 1} \beta_k(t) w_k, \quad \forall t \in [0, T],$$

where $\{\beta_k, k \geq 1\}$ is a family of independent and identically distributed \mathbb{R} -valued Brownian motions on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$. For any $\phi \in L^2(\Omega; L^2(0, T; \mathcal{L}_2(K, \mathbb{L}^2)))$, its stochastic integral with respect to the Wiener process $\{W(t), t \in [0, T]\}$ is defined (c.f. [22]) as the unique continuous \mathbb{L}^2 -valued \mathcal{F}_t -martingale such that for all $\psi \in \mathbb{L}^2$,

$$\left(\int_0^t \phi(s) dW(s), \psi \right) = \sum_{k \geq 1} \int_0^t (\phi(s) w_k, \psi) d\beta_k(s), \quad \forall t \in [0, T].$$

For clarity's sake, the nonlinear term in equations (1.1) will be denoted \tilde{b} ; that is

$$\tilde{b}(u, v, w) = -\left(u \times (\nabla \times v), w \right)$$

for appropriate vector fields u, v and w , where $v = u - \alpha^2 \Delta u$ in the equations of interest. The bilinear operator that can be derived from \tilde{b} will be denoted \tilde{B} and it reads: $\tilde{B}(u, v) := -u \times (\nabla \times v)$, for all $u, v \in \mathbb{V}$. The below proposition lists a few useful properties of the bilinear operator \tilde{B} .

Proposition 2.1 *The following assertions are satisfied by the nonlinear term:*

- (i) For all $u, v, w \in \mathbb{H}^1$, $\langle \tilde{B}(u, v), w \rangle = -\langle \tilde{B}(w, v), u \rangle$. In particular, $\langle \tilde{B}(u, v), u \rangle = 0$.
- (ii) $\langle \tilde{B}(u, v), w \rangle = ([u \cdot \nabla]v, w) - ([w \cdot \nabla]v, u)$, for all $u, v, w \in \mathbb{H}_{per}^1$. If additionally, u and v are divergence-free then, $\langle \tilde{B}(u, v), v \rangle = -([v \cdot \nabla]v, u)$.
- (iii) $|\langle \tilde{B}(u, v), w \rangle| \leq C_D \|u\|_{\mathbb{L}^4} \|\nabla v\|_{\mathbb{L}^2} \|w\|_{\mathbb{L}^2}^{\frac{1}{2}} \|\nabla w\|_{\mathbb{L}^2}^{\frac{1}{2}}$, for all $u, v, w \in \mathbb{H}_{per}^1$.

Proof: Assertion (i) can be proven by a simple application of the identity $(u \times v) \cdot w = -(w \times v) \cdot u$. To demonstrate equality (ii), we need to employ the following property:

$$\langle \tilde{B}(u, v), w \rangle = ([u \cdot \nabla]v, w) + ((\nabla u)^T \cdot v, w) - (\nabla(u \cdot v), w), \quad (2.1)$$

which may be straightforwardly proven via the identity

$$[u \cdot \nabla]v + (\nabla u)^T \cdot v - \nabla(u \cdot v) = -u \times (\nabla \times v).$$

Indeed, the quantity $((\nabla u)^T \cdot v, w)$ of equation (2.1) turns into $-([w \cdot \nabla]v, u) + (\nabla(u \cdot v), w)$ after applying two consecutive integration by parts. Plugging it back in equation (2.1) completes the proof of (ii). Finally, the Hölder and Ladyzhenskaya (see [16, Lemma I.1]) inequalities applied to assertion (ii) yield estimate (iii). \square

The operator \tilde{b} can be readily expressed via the trilinear form associated with the Navier-Stokes equations, as mentioned in Proposition 2.1-(ii). For brevity's sake, we deploy the next proposition to grant a few corresponding properties. The reader may refer to [23, Remark 2.2] for further information.

Proposition 2.2 (i) $([u \cdot \nabla]v, v) = 0$ for all $u, v \in \mathbb{V}$.

(ii) $|([u \cdot \nabla]v, w)| \leq C_D \|u\|_{\mathbb{L}^2} \|\nabla v\|_{\mathbb{L}^2} \|w\|_{\mathbb{L}^2}^{\frac{1}{2}} \|Aw\|_{\mathbb{L}^2}^{\frac{1}{2}}$, for all $u \in \mathbb{H}$, $v \in \mathbb{V}$ and $w \in D(A)$.

Assumptions

(S₁) $\mathbb{E} \left[\|\bar{u}_0\|_{\mathbb{H}^1}^{2p} \right] < +\infty$, for some $p \in [1, +\infty)$,

(S₂) $g \in L^2(\Omega; L^2(0, T; \mathcal{L}_2(K, \mathbb{L}^2)))$ satisfies: for all $u \in \mathbb{V}$, $g(\cdot, u)$ is \mathcal{F}_t -progressively measurable, and almost everywhere in $\Omega \times (0, T)$, it holds that:

$$\begin{aligned} \|g(\cdot, u) - g(\cdot, v)\|_{\mathcal{L}_2(K, \mathbb{L}^2)} &\leq L_g \|u - v\|_\alpha, \quad \forall u, v \in \mathbb{V}, \\ \|g(\cdot, u)\|_{\mathcal{L}_2(K, \mathbb{H}^1)} &\leq K_1 + K_2 \|u\|_\alpha, \quad \forall u \in \mathbb{V}. \end{aligned}$$

for some real, nonnegative, time-independent constants L_g, K_1, K_2 .

Remark 2.1 *Inequality $\|g(\cdot, u)\|_{\mathcal{L}_2(K, \mathbb{H}^1)} \leq K_1 + K_2 \|u\|_\alpha$ of assumption (S₂) is imposed in \mathbb{H}^1 instead of \mathbb{L}^2 to be able to execute high space-regularity estimates for the velocity field.*

To reduce repetitions, the below proposition gathers a few properties that will be employed throughout this paper.

Proposition 2.3 (i) $x^p \leq 1 + x^q$ for all $x \geq 0$, and $1 \leq p \leq q < +\infty$.

(ii) $2(a, b) = \|a\|_{\mathbb{L}^2}^2 - \|b\|_{\mathbb{L}^2}^2 + \|a - b\|_{\mathbb{L}^2}^2$, for all $a, b \in \mathbb{L}^2$.

(iii) $|a + b|^p \leq 2^{p-1} (|a|^p + |b|^p)$, for all $a, b \in \mathbb{R}$ and $p \geq 1$.

2.1 Concept of solutions

The underlying equations consist of a fourth-order problem which might not be insightful. Therefore, a continuous differential filter shall be introduced allowing equations (1.1) to turn into a second-order coupled problem.

Definition 2.1 (Continuous differential filter) *Let $v \in \mathbb{L}^2$ be a given vector field. A continuous differential filter \bar{u} of v is defined as part of the unique solution $(\bar{u}, \bar{p}) \in \mathbb{V} \times L_0^2(D)$ to the problem.*

$$\begin{cases} -\alpha^2 \Delta \bar{u} + \bar{u} + \nabla \bar{p} = v, & \text{in } D, \\ \operatorname{div}(\bar{u}) = 0, & \text{in } D. \end{cases} \quad (2.2)$$

The notation \bar{v} (instead of \bar{u}) is widely spread in the literature of differential filters. However, to maintain a visible relationship between equations (1.1) and (2.2), \bar{v} will be substituted by the notation \bar{u} . Observe that system (2.2) represents a deterministic steady Stokes problem and that v plays the role of an outer force. Additionally, projecting system (2.2) using the Leray projector \mathcal{P} yields

$$\alpha^2 A \bar{u} + \bar{u} = \mathcal{P}v, \text{ in } D.$$

which has a unique solution \bar{u} according to [14, Subsection 8.2]. Thereby, when it comes to the process $\{\bar{u}(t), t \in [0, T]\}$ of problem (1.1), the multiplication in L^2 of the above equation by $\varphi \in \mathbb{V}$ returns for all $t \in [0, T]$,

$$(v(t), \varphi) = (\bar{u}(t), \varphi) + \alpha^2 (\nabla \bar{u}(t), \nabla \varphi). \quad (2.3)$$

Based on the above identity, we define v_0 as the solution of $(v_0, \varphi) = (\bar{u}_0, \varphi) + \alpha^2 (\nabla \bar{u}_0, \nabla \varphi)$, for all $\varphi \in \mathbb{V}$. Since \bar{u}_0 belongs to \mathbb{V} , it is straightforward that $\alpha^2 \mathbb{E}[(\nabla \bar{u}_0, \nabla \varphi)] \rightarrow 0$ as $\alpha \rightarrow 0$. Subsequently, $\mathbb{E}[(v_0, \varphi)] = \mathbb{E}[(\bar{u}_0, \varphi)]$ for all $\varphi \in \mathbb{V}$ as $\alpha \rightarrow 0$. As a result, $v_0 = \bar{u}_0$ \mathbb{P} -a.s. and a.e. in D when α vanishes. The next definition states the compound of a solution to equations (1.1) whose existence and uniqueness are illustrated in [6].

Definition 2.2 *Let $T > 0$ and assume (S_1) - (S_2) . A $\mathbb{V} \times \mathbb{H}$ -valued stochastic process $(\bar{u}(t), v(t))$, $t \in [0, T]$ is said to be a variational solution to problem (1.1) if it fulfills the following conditions:*

- (i) $\bar{u} \in L^2(\Omega; L^2(0, T; \mathbb{H}^2 \cap \mathbb{V})) \cap L^2(\Omega; L^\infty(0, T; \mathbb{V}))$,
- (ii) $v \in L^2(\Omega; L^2(0, T; \mathbb{V})) \cap L^2(\Omega; L^\infty(0, T; \mathbb{H}))$,
- (iii) \mathbb{P} -almost surely, \bar{u} is weakly continuous with values in \mathbb{V} , and v is continuous with values in \mathbb{H} ,
- (iv) for all $t \in [0, T]$, \bar{u} satisfies the following equation \mathbb{P} -almost surely

$$\begin{cases} (v(t), \varphi) + \nu \int_0^t (\nabla v(s), \nabla \varphi) ds + \int_0^t \tilde{b}(\bar{u}(s), v(s), \varphi) ds \\ \quad = (v_0, \varphi) + \left(\int_0^t g(s, \bar{u}(s)) dW(s), \varphi \right), \quad \forall \varphi \in \mathbb{V}, \\ (v(t), \psi) = (\bar{u}(t), \psi) + \alpha^2 (\nabla \bar{u}(t), \nabla \psi), \quad \forall \psi \in \mathbb{V}. \end{cases} \quad (2.4)$$

It is worth mentioning that the weak continuity of \bar{u} is related to the strong continuity of v . This fact emerges from the relationship (2.3).

Two-dimensional strong solutions to equations (1.2) were conducted in [13, 19]. An appropriate definition is given by:

Definition 2.3 Let $T > 0$ be fixed and assumptions (S_1) - (S_2) be fulfilled. A process $u(t)$, $t \in [0, T]$ on a stochastic filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ is said to be a strong solution to equations (1.2) if it belongs to $L^2(\Omega; C([0, T]; \mathbb{H}) \cap L^2(0, T; \mathbb{V}))$, and it satisfies \mathbb{P} -a.s. for all $t \in [0, T]$, the weak formulation

$$\begin{aligned} & (u(t), \varphi) + \nu \int_0^t (\nabla u(s), \nabla \varphi) ds + \int_0^t ([u(s) \cdot \nabla] u(s), \varphi) ds \\ & = (\bar{u}_0, \varphi) + \left(\int_0^t g(s, u(s)) dW(s), \varphi \right), \quad \forall \varphi \in \mathbb{V}. \end{aligned}$$

Equations (1.2) have a unique solution in the sense of Definition 2.3, see for instance [19, Proposition 3.2]. This fact will be evoked all this paper long.

3 Main result

Theorem 3.1 Let $T > 0$, $L > 0$, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ be a filtered probability space, $D = (0, L)^2$ be a two-dimensional torus subject to periodic boundary conditions, and $1 \leq p < +\infty$ be given. Let $\{e_k, k \geq 1\}$ be a complete orthonormal basis of \mathbb{H} consisting of eigenfunctions of the Stokes operator A , and $\{\mu_k, k \geq 1\}$ be the associated eigenvalues whose values diverge when $k \rightarrow +\infty$. Assume that hypotheses (S_1) - (S_2) are fulfilled, and that for all $N \in \mathbb{N} \setminus \{0\}$, the spatial scale follows the decreasing rate $C_{\min} \mu_N^{-3/4} \leq \alpha := \alpha_N \leq C_{\max} \mu_N^{-3/4}$, for some constants $C_{\min}, C_{\max} > 0$ independent of N . Then, a solution $(\bar{u}, v) := (\bar{u}(\alpha_N), v(\alpha_N))$ to equations (1.1) in the sense of Definition 2.2 for a given α converges toward the unique strong solution v_{NS} of equations (1.2) in the sense of Definition 2.3 when $N \rightarrow +\infty$, and it satisfies:

$$\begin{aligned} (i) \quad & \mathbb{E} \left[\sup_{t \in [0, T]} \|v_{NS}(t)\|_{\mathbb{L}^2}^{2p} + 2p\nu \int_0^T \|v_{NS}(t)\|_{\mathbb{L}^2}^{2(p-1)} \|\nabla v_{NS}(t)\|_{\mathbb{L}^2}^2 dt \right] \leq C_2, \\ (ii) \quad & \mathbb{E} \left[\sup_{t \in [0, T]} \|\nabla v_{NS}(t)\|_{\mathbb{L}^2}^{2p} + \left(\nu \int_0^T \|Av_{NS}(t)\|_{\mathbb{L}^2}^2 dt \right)^p \right] \leq C_4, \end{aligned}$$

where $C_2 > 0$ depends on constants C_{\max}, C_1 of Lemma 4.1 and its parameters, and $C_4 > 0$ depends on $C_1, \|\bar{u}_0\|_{L^{6p}(\Omega; \mathbb{V})}$ and C_{\max} .

Remark 3.1 Throughout this chapter, there will only be a single limit concept parameterized by N ; no successive double limits are intended within this context. In a more accurate way, we will neither treat the case $\alpha \rightarrow 0$ while fixing N nor the independent convergences of α and N . The whole study revolves around the convergence of N to $+\infty$, which leads α to vanish.

4 Faedo-Galerkin approximation and a priori estimates

It is well-known (c.f. [23, Lemma 3.1]) that the trilinear term of the Navier-Stokes equations $\int_D [z \cdot \nabla] z \Delta z dx$ vanishes if the configurations were set to two-dimensional domain with periodic boundary conditions. This property is unfortunately inapplicable to $\tilde{b}(z, z - \alpha^2 \Delta z, \Delta z)$. Therefore, we must find a way to achieve high spatial regularity estimates. To this purpose, let $N \in \mathbb{N} \setminus \{0\}$ be a large integer, $\{e_k, k \geq 1\}$ be a complete orthonormal basis of \mathbb{H} consisting of eigenfunctions of the Stokes operator A whose domain is $\mathbb{H}^2 \cap \mathbb{V}$, and $\{\mu_k, k \geq 1\}$ be the associated eigenvalues. Denote by $V_N :=$

$\text{span}\{e_1, \dots, e_N\}$ the finite-dimensional vector subspace of \mathbb{H} , and by $P_N: \mathbb{H} \rightarrow \mathbb{H}$ the projection operator of H onto V_N such that for all $v \in \mathbb{H}$, it holds that

$$\begin{aligned} (v, \pi) &= (P_N v, \pi), \quad \forall \pi \in V_N, \text{ and} \\ (\nabla v, \nabla \pi) &= (\nabla P_N v, \nabla \pi), \quad \forall \pi \in V_N. \end{aligned}$$

We will assume from now on that $C_{\min} \mu_N^{-3/4} \leq \alpha \leq C_{\max} \mu_N^{-3/4}$, for some constants $C_{\min}, C_{\max} > 0$ independent of N . That way, when N tends to $+\infty$, the spatial scale α goes to 0, thanks to the property $\mu_1 < \mu_2 < \dots < \mu_N \rightarrow +\infty$ as $N \rightarrow \infty$. N is opted to be significant to ensure that $1/\mu_N \leq 1$. Consequently, we introduce the following Faedo-Galerkin approximate system:

$$\begin{cases} (v_N(t), e_k) + \nu \int_0^t (\nabla v_N(s), \nabla e_k) ds + \int_0^t \tilde{b}(\bar{u}_N(s), v_N(s), e_k) ds \\ \quad = (v_0, e_k) + \left(\int_0^t g(s, \bar{u}_k(s)) dW(s), e_k \right), \\ (v_N(t), e_k) = (\bar{u}_N(t), e_k) + \alpha^2 (\nabla \bar{u}_N(t), \nabla e_k), \end{cases} \quad (4.1)$$

for all $t \in [0, T]$, $k \in \{1, \dots, N\}$, and \mathbb{P} -almost surely, with initial datum $\bar{u}_N(0) = P_N \bar{u}_0$ i.e. $v_N(0) = P_N v_0 = (P_N + \alpha^2 P_N A) \bar{u}_0$. System (4.1) converges to the unique strong solution of the stochastic Navier-Stokes equations when N tends to $+\infty$ in the sense of Definition 2.3 (see Section 5). We list down below all concerned a priori estimates for the projected couple (\bar{u}_N, v_N) .

Remark 4.1 Assumption $\alpha \leq C_{\max} \mu_N^{-3/4}$ could have been $\alpha \leq C_{\max} \mu_N^{-1/2}$ if only the convergence of solutions to equations (1.1) toward solutions to problem (1.2) was intended. The additional negative exponent on μ_N is solely required in this context to obtain high spacial regularity for the velocities v and \bar{u} .

Lemma 4.1 Let $T > 0$, $N \in \mathbb{N} \setminus \{0\}$, $p \geq 1$, and assumptions (S_1) - (S_2) be valid. Then, the finite-dimensional system (4.1) has a $\mathbb{V} \times \mathbb{H}$ -valued solution (\bar{u}_N, v_N) that satisfies the following estimates:

$$\begin{aligned} (i) \quad & \sup_{0 \leq t \leq T} \mathbb{E} \left[\|\bar{u}_N(t)\|_{\alpha}^{2p} \right] + 2p\nu \mathbb{E} \left[\int_0^T \|\bar{u}_N(t)\|_{\alpha}^{2(p-1)} \|\nabla \bar{u}_N(t)\|_{\alpha}^2 dt \right] \leq C_1, \\ (ii) \quad & \mathbb{E} \left[\sup_{0 \leq t \leq T} \|\bar{u}_N(t)\|_{\alpha}^{2p} \right] \leq C_1, \end{aligned}$$

for a certain constant $C_1 > 0$ depending only on $\mathbb{E} \left[\|\bar{u}_0\|_{\mathbb{H}^1}^{2p} \right]$, p, D, K_1, K_2 , and T . Moreover, if one assumes $\alpha \leq \mu_N^{-1/2}$ then, it holds that

$$(iii) \quad \mathbb{E} \left[\sup_{0 \leq t \leq T} \|v_N(t)\|_{\mathbb{L}^2}^{2p} \right] + 2p\nu \mathbb{E} \left[\int_0^T \|v_N(t)\|_{\mathbb{L}^2}^{2(p-1)} \|\nabla v_N(t)\|_{\mathbb{L}^2}^2 dt \right] \leq C_2,$$

where C_2 is a positive constant depending only on C_1 .

Proof: Problem (4.1) is a finite-dimensional system of ordinary differential equations subject to a polynomial nonlinearity. Therefore, it has a local solution (\bar{u}_N, v_N) . In order to apply the Itô formula, we need to define, for $n \in \mathbb{N} \setminus \{0\}$, the following stopping time:

$$\tau_N^n := \begin{cases} \inf \{t \in [0, T] : \|(I + \alpha^2 A)^{-1/2} v_N(t)\|_{\mathbb{L}^2} > n\} & \text{if the set is non-empty,} \\ +\infty & \text{otherwise.} \end{cases}$$

For $p \geq 1$, and $t \in [0, T]$, we define the process $F(v_N(t)) := \|(I + \alpha^2 A)^{-1/2} v_N(t)\|_{\mathbb{L}^2}^{2p}$. From equation (4.1)₂, and taking into account that $I + \alpha^2 A$ is self-adjoint and bijective from $D(A)$ to \mathbb{H} , it is straightforward that $F(v_N) = \|\bar{u}_N\|_\alpha^{2p}$. Moreover,

$$\begin{aligned} DF(v_N) &= 2p\|(I + \alpha^2 A)^{-1/2} v_N\|_{\mathbb{L}^2}^{2(p-1)} (I + \alpha^2 A)^{-1} v_N = 2p\|\bar{u}_N\|_\alpha^{2(p-1)} \bar{u}_N, \text{ and} \\ D^2 F(v_N) &= 4p(p-1)\|\bar{u}_N\|_\alpha^{2p-4} \bar{u}_N \otimes \bar{u}_N + 2p\|\bar{u}_N\|_\alpha^{2p-2} (I + \alpha^2 A)^{-1}, \end{aligned}$$

where the symbol \otimes denotes the usual dyadic product. Apply now the Itô formula to the process $F(v_N(t \wedge \tau_N^n))$:

$$\begin{aligned} \|\bar{u}_N(t \wedge \tau_N^n)\|_\alpha^{2p} &= \|\bar{u}_N(0)\|_\alpha^{2p} + 2p \int_0^{t \wedge \tau_N^n} \|\bar{u}_N(s)\|_\alpha^{2(p-1)} (\bar{u}_N(s), g(s, \bar{u}_N(s))) dW(s) \\ &+ 2p(p-1) \int_0^{t \wedge \tau_N^n} \|\bar{u}_N(s)\|_\alpha^{2p-4} \|(\bar{u}_N(s))^* g(s, \bar{u}_N(s))\|_K^2 ds \\ &+ p \int_0^{t \wedge \tau_N^n} \|\bar{u}_N(s)\|_\alpha^{2(p-1)} \left\| (I + \alpha^2 A)^{-1/2} g(s, \bar{u}_N(s)) \right\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 ds \\ &+ 2p \int_0^{t \wedge \tau_N^n} \|\bar{u}_N(s)\|_\alpha^{2(p-1)} \langle \bar{u}_N(s), -\nu A v_N(s) - \tilde{B}(\bar{u}_N(s), v_N(s)) \rangle ds. \end{aligned}$$

We have $\langle \bar{u}_N(s), A v_N(s) \rangle = \langle \nabla \bar{u}_N(s), \nabla (I + \alpha^2 A) \bar{u}_N(s) \rangle = \|\nabla \bar{u}_N(s)\|_\alpha^2$, and by Proposition 2.1-(i), the nonlinear term \tilde{B} in the last term on the right-hand side of the above equation vanishes so that

$$\begin{aligned} &\|\bar{u}_N(t \wedge \tau_N^n)\|_\alpha^{2p} + 2p\nu \int_0^{t \wedge \tau_N^n} \|\bar{u}_N(s)\|_\alpha^{2p-2} \|\nabla \bar{u}_N(s)\|_\alpha^2 ds \\ &\leq \|\bar{u}_N(0)\|_\alpha^{2p} + 2p \int_0^{t \wedge \tau_N^n} \|\bar{u}_N(s)\|_\alpha^{2p-2} (\bar{u}_N(s), g(s, \bar{u}_N(s))) dW(s) \\ &+ 2p(p-1) \int_0^{t \wedge \tau_N^n} \|\bar{u}_N(s)\|_\alpha^{2p-4} \|\bar{u}_N(s)\|_{\mathbb{L}^2}^2 \|g(s, \bar{u}_N(s))\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 ds \\ &+ p \int_0^{t \wedge \tau_N^n} \|\bar{u}_N(s)\|_\alpha^{2p-2} \left\| (I + \alpha^2 A)^{-1/2} g(s, \bar{u}_N(s)) \right\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 ds \\ &= \|\bar{u}_N(0)\|_\alpha^2 + I_1 + I_2 + I_3. \end{aligned} \tag{4.2}$$

Assumption (S₂) together with the stopping time τ_N^n yield $\mathbb{E}[I_1] = 0$. On the other hand, by virtue of Proposition 2.3-(i), assumption (S₂), and estimate $\|(I + \alpha^2 A)^{-1/2} z\|_{\mathbb{L}^2} \leq \|z\|_{\mathbb{L}^2}$, it holds that

$$\begin{aligned} I_2 + I_3 &\leq 2p(p-1) \int_0^{t \wedge \tau_N^n} \|\bar{u}_N(s)\|_\alpha^{2p-4} \|\bar{u}_N(s)\|_{\mathbb{L}^2}^2 (K_1 + K_2 \|\bar{u}_N(s)\|_\alpha)^2 ds \\ &+ p \int_0^{t \wedge \tau_N^n} \|\bar{u}_N(s)\|_\alpha^{2p-2} (K_1 + K_2 \|\bar{u}_N(s)\|_\alpha)^2 ds \\ &\leq p(2p-1) \int_0^{t \wedge \tau_N^n} \|\bar{u}_N(s)\|_\alpha^{2p-2} (K_1 + K_2 \|\bar{u}_N(s)\|_\alpha)^2 ds \\ &\leq 2p(2p-1) K_1^2 t \wedge \tau_N^n + 2p(2p-1) (K_1^2 + K_2^2) \int_0^{t \wedge \tau_N^n} \|\bar{u}_N(s)\|_\alpha^{2p} ds. \end{aligned}$$

Putting it all together and applying the mathematical expectation to equation (4.2) return

$$\begin{aligned} &\mathbb{E} [\|\bar{u}_N(t \wedge \tau_N^n)\|_\alpha^{2p}] + 2p\nu \mathbb{E} \left[\int_0^{t \wedge \tau_N^n} \|\bar{u}_N(s)\|_\alpha^{2p-2} \|\nabla \bar{u}_N(s)\|_\alpha^2 ds \right] \leq \mathbb{E} [\|\bar{u}_N(0)\|_\alpha^{2p}] \\ &+ 2p(2p-1) K_1^2 \mathbb{E} [t \wedge \tau_N^n] + 2p(2p-1) (K_1^2 + K_2^2) \int_0^{t \wedge \tau_N^n} \mathbb{E} [\|\bar{u}_N(s)\|_\alpha^{2p}] ds. \end{aligned}$$

The Grönwall inequality (c.f. [1]) finally implies

$$\begin{aligned} & \sup_{0 \leq t \leq T} \mathbb{E} \left[\|\bar{u}_N(t \wedge \tau_N^n)\|_\alpha^{2p} \right] + 2p\nu \mathbb{E} \left[\int_0^{t \wedge \tau_N^n} \|\bar{u}_N(s)\|_\alpha^{2p-2} \|\nabla \bar{u}_N(s)\|_\alpha^2 ds \right] \\ & \leq \left(\mathbb{E} \left[\|\bar{u}_N(0)\|_\alpha^{2p} \right] + 2p(2p-1)K_1^2 \mathbb{E} [t \wedge \tau_N^n] \right) \exp \left(2p(2p-1)(K_1^2 + K_2^2)t \wedge \tau_N^n \right). \end{aligned} \quad (4.3)$$

Taking into account that $\mathbb{E} \left[\|\bar{u}_N(0)\|_\alpha^{2p} \right] \leq \mathbb{E} \left[\|\bar{u}_0\|_\alpha^{2p} \right]$, and letting $n \rightarrow +\infty$ in equation (4.3) complete the proof of estimate (i). Now that we have illustrated that $\|\bar{u}_N\|_\alpha$ has finite moments, we can drop the stopping time in equation (4.2). whose supremum in time returns

$$\begin{aligned} & \mathbb{E} \left[\sup_{0 \leq t \leq T} \|\bar{u}_N(t)\|_\alpha^{2p} \right] \leq \mathbb{E} \left[\|\bar{u}_N(0)\|_\alpha^{2p} \right] \\ & + 2p \mathbb{E} \left[\sup_{0 \leq t \leq T} \left| \int_0^t \|\bar{u}_N(s)\|_\alpha^{2p-2} (\bar{u}_N(s), g(s, \bar{u}_N(s))) dW(s) \right| \right] \\ & + 2p(2p-1)K_1^2 T + 2p(2p-1)(K_1^2 + K_2^2) T \sup_{0 \leq t \leq T} \mathbb{E} \left[\|\bar{u}_N(t)\|_\alpha^{2p} \right]. \end{aligned} \quad (4.4)$$

By virtue of Proposition 2.3-(i), assumption (S_2) , the Burkholder-Davis-Gundy (c.f. [8]) and Young inequalities, the second term on the right-hand side can be bounded by

$$\begin{aligned} & \lesssim \mathbb{E} \left[\left(\int_0^T \|\bar{u}_N(t)\|_\alpha^{4p-2} \|g(t, \bar{u}_N(t))\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 dt \right)^{1/2} \right] \\ & \lesssim \mathbb{E} \left[\sup_{0 \leq t \leq T} \|\bar{u}_N(t)\|_\alpha^{\frac{2p-1}{2}} \|g(t, \bar{u}_N(t))\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^{1/2} \left(\int_0^T \|\bar{u}_N(t)\|_\alpha^{2p-1} \|g(t, \bar{u}_N(t))\|_{\mathcal{L}_2(K, \mathbb{L}^2)} dt \right)^{1/2} \right] \\ & \leq \frac{\varepsilon}{2} \mathbb{E} \left[K_1 + (K_1 + K_2) \sup_{0 \leq t \leq T} \|\bar{u}_N(t)\|_\alpha^{2p} \right] + \frac{1}{2\varepsilon} \left(K_1 T + (K_1 + K_2) T \sup_{0 \leq t \leq T} \mathbb{E} \left[\|\bar{u}_N(t)\|_\alpha^{2p} \right] \right), \end{aligned}$$

for some constant $\varepsilon > 0$ emerging from the Young inequality. Taking $\varepsilon = \frac{1}{K_1 + K_2}$, merging the above result into equation (4.4), and employing assertion (i) complete the proof of estimate (ii). Moving on to the inequality (iii), we have $(v_N, \psi) = (\bar{u}_N, \psi) + \alpha^2 (\nabla \bar{u}_N, \nabla \psi)$ \mathbb{P} -a.s. for all $\psi \in V_N$, thanks to equation (4.1)₂. Therefore, substituting ψ by $v_N(t)$ and employing the Cauchy-Schwarz inequality to get: $\|v_N(t)\|_{\mathbb{L}^2}^2 \leq \|\bar{u}_N(t)\|_{\mathbb{L}^2} \|v_N(t)\|_{\mathbb{L}^2} + \alpha^2 \|\nabla \bar{u}_N(t)\|_{\mathbb{L}^2} \|\nabla v_N(t)\|_{\mathbb{L}^2}$. On the other hand, the estimate $\|\nabla v_N(t)\|_{\mathbb{L}^2} \leq \sqrt{\mu_N} \|v_N(t)\|_{\mathbb{L}^2}$ together with the hypothesis $\alpha \leq \mu_N^{-1/2}$ and the Young inequality lead to

$$\|v_N(t)\|_{\mathbb{L}^2} \leq \sqrt{2} \|\bar{u}_N(t)\|_\alpha. \quad (4.5)$$

Following the same technique, but this time replacing ψ by $Av_N(t) \in V_N$, one obtains

$$\|\nabla v_N(t)\|_{\mathbb{L}^2} \leq \sqrt{2} \|\nabla \bar{u}_N(t)\|_\alpha. \quad (4.6)$$

It suffices now to raise inequality (4.5) to the power $2p$, take the supremum over $t \in [0, T]$, apply to it the mathematical expectation, and employ estimate (ii) to get $\mathbb{E} \left[\sup_{0 \leq t \leq T} \|v_N(t)\|_{\mathbb{L}^2}^{2p} \right] \lesssim C_1$. Similarly, $\|v_N(t)\|_{\mathbb{L}^2}^{2(p-1)} \|\nabla v_N(t)\|_{\mathbb{L}^2}^2 \lesssim \|\bar{u}_N(t)\|_\alpha^{2(p-1)} \|\nabla \bar{u}_N(t)\|_\alpha^2$, thanks to (4.5) and (4.6). Integrating over $[0, T]$, applying the mathematical expectation and employing estimate (i) terminate the proof. \square

The next lemma exhibits the regularity of v_0 with respect to \bar{u}_0 .

Lemma 4.2 *Let $1 \leq p < +\infty$, and assume (S_1) . If $C_{\min}\mu_N^{-1/2} \leq \alpha \leq C_{\max}\mu_N^{-1/2}$ for some constants $C_{\min}, C_{\max} > 0$, then $v_0 \in L^{2p}(\Omega; \mathbb{V})$, and*

$$\|\nabla v_0\|_{L^{2p}(\Omega; \mathbb{L}^2)} \leq \|\bar{u}_0\|_{L^{2p}(\Omega; \mathbb{V})}.$$

Proof: By equation (4.1)₂, we get

$$\|\nabla v_N(0)\|_{\mathbb{L}^2}^2 = (\nabla \bar{u}_N(0), \nabla v_N(0)) + \alpha^2 (A\bar{u}_N(0), Av_N(0)).$$

Taking into account the estimate $\|Az\|_{\mathbb{L}^2} \leq \sqrt{\mu_N}\|\nabla z\|_{\mathbb{L}^2}$ for all $z \in V_N$, apply it to $\|A\bar{u}_N(0)\|_{\mathbb{L}^2}$ and $\|Av_N(0)\|_{\mathbb{L}^2}$, and employ the Cauchy-Schwarz inequality, it follows $\|\nabla v_N(0)\|_{\mathbb{L}^2} \leq 2\|\nabla \bar{u}_N(0)\|_{\mathbb{L}^2}$. Subsequently, $\mathbb{E} \left[\|\nabla v_N(0)\|_{\mathbb{L}^2}^{2p} \right] \lesssim \mathbb{E} \left[\|\nabla \bar{u}_0\|_{\mathbb{L}^2}^{2p} \right] =: M$, which implies that $(v_N(0))_N$ is bounded in the reflexive Banach space $L^{2p}(\Omega; \mathbb{H}^1)$. Thus, there exists a subsequence $(v_{N_\ell}(0))_\ell$ that converges weakly in $L^{2p}(\Omega; \mathbb{H}^1)$ toward some limit ξ , and one gets $\mathbb{E} \left[\|\xi\|_{\mathbb{H}^1}^{2p} \right] \leq \liminf \mathbb{E} \left[\|v_{N_\ell}(0)\|_{\mathbb{H}^1}^{2p} \right] \leq C_D M$, thanks to the Poincaré inequality. It remains to identify ξ with v_0 . Indeed, since $L^{2p}(\Omega; \mathbb{H}^1) \hookrightarrow L^{2p}(\Omega; \mathbb{L}^2)$, the weak convergence of $(v_{N_\ell}(0))_\ell$ also takes place in $L^{2p}(\Omega; \mathbb{L}^2)$. Observe that $v_N(0) = P_N v_0$ converges strongly (and therefore weakly) toward v_0 in $L^{2p}(\Omega; \mathbb{L}^2)$ as $N \rightarrow +\infty$, thanks to the properties of the projector P_N . Consequently, by the weak limit uniqueness, $\xi = v_0$ \mathbb{P} -a.s. and a.e. in D , and the result follows. \square

Owing to Lemma 4.2, high space-regularity estimates are illustrated below for the process (\bar{u}_N, v_N) .

Lemma 4.3 *Let $N \in \mathbb{N} \setminus \{0\}$, and $p \in [1, +\infty)$. Assume that (S_1) - (S_2) are valid and that $\alpha \leq C_{\max}\mu_N^{-3/4}$, for some constant $C_{\max} > 0$ independent of N . Then, the solution (\bar{u}_N, v_N) of equation (4.1) satisfies*

$$(i) \quad \mathbb{E} \left[\sup_{t \in [0, T]} \|\nabla \bar{u}_N(t)\|_{\alpha}^{2p} + \left(\nu \int_0^T \|A\bar{u}_N(t)\|_{\alpha}^2 dt \right)^p \right] \leq C_3,$$

$$(ii) \quad \mathbb{E} \left[\sup_{t \in [0, T]} \|\nabla v_N(t)\|_{\mathbb{L}^2}^{2p} + \left(\nu \int_0^T \|Av_N(t)\|_{\mathbb{L}^2}^2 dt \right)^p \right] \leq C_4,$$

where $C_3 > 0$ depends on C_1 and $\|\bar{u}_0\|_{L^{6p}(\Omega; \mathbb{V})}$, and C_4 depends only on C_3 and C_{\max} .

Proof: Define the stopping time

$$\tau_N^n := \begin{cases} \inf \{t \in [0, T] : \|A^{1/2}(I + \alpha^2 A)^{-1/2} v_N(t)\|_{\mathbb{L}^2} > n\} & \text{if the set is non-empty,} \\ +\infty & \text{otherwise,} \end{cases}$$

and the process $F(v_N) := \|A^{1/2}(I + \alpha^2 A)^{-1/2} v_N\|_{\mathbb{L}^2}^2$. By equation (4.1)₂, one gets $F(v_N(t \wedge \tau_N^n)) = \|\nabla \bar{u}_N(t \wedge \tau_N^n)\|_{\alpha}^2$. Moreover, $DF(x) = 2A(I + \alpha^2 A)^{-1}x$, and $D^2F(x) = 2A(I + \alpha^2 A)^{-1}$. In particular, $DF(v_N) = 2A\bar{u}_N$, thanks to equation (4.1)₂. By applying Itô's formula to the process $F(v_N(t \wedge \tau_N^n))$, it follows that

$$\begin{aligned} & \|\nabla \bar{u}_N(t \wedge \tau_N^n)\|_{\alpha}^2 + 2\nu \int_0^{t \wedge \tau_N^n} \|A\bar{u}_N(s)\|_{\alpha}^2 ds = \|\nabla \bar{u}_N(0)\|_{\alpha}^2 \\ & + 2 \int_0^{t \wedge \tau_N^n} (A\bar{u}_N(s), g(s, \bar{u}_N(s)) dW(s)) + \int_0^{t \wedge \tau_N^n} \|A^{1/2}(I + \alpha^2 A)^{-1/2} g(s, \bar{u}_N(s))\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 ds \\ & - 2 \int_0^{t \wedge \tau_N^n} \langle \tilde{B}(\bar{u}_N(s), v_N(s)), A\bar{u}_N(s) \rangle ds = \|\nabla \bar{u}_N(0)\|_{\alpha}^2 + I_1 + I_2 - I_3. \end{aligned}$$

On account of assumption (S_2) and the measurability of \bar{u}_N , we have $\mathbb{E}[I_1] = 0$. Now, the fact that for any $z \in \mathbb{L}^2$, the quantity $\|A^{1/2}(I + \alpha^2 A)^{-1/2}z\|_{\mathbb{L}^2}$ is optimally bounded by $\frac{1}{\alpha}\|z\|_{\mathbb{L}^2}$ justifies the opted assumption on $\|g(\cdot, z)\|_{\mathcal{L}_2(K, \mathbb{H}^1)}$. Therewith, (S_2) leads to

$$I_2 \leq \int_0^{t \wedge \tau_N^n} \|g(s, \bar{u}_N(s))\|_{\mathcal{L}_2(K, \mathbb{H}^1)}^2 ds \leq 2K_1^2 t \wedge \tau_N^n + 2K_2^2 \int_0^{t \wedge \tau_N^n} \|\bar{u}_N(s)\|_{\alpha}^2 ds.$$

Moreover, by virtue of equation $(4.1)_2$, the identity $v_N = \bar{u}_N + \alpha^2 A \bar{u}_N$ holds \mathbb{P} -a.s. and a.e. in $(0, T) \times D$. Thus, the integrand of I_3 can be amended to the following form

$$\left(\tilde{B}(\bar{u}_N(s), \bar{u}_N(s)), A \bar{u}_N(s) \right) + \alpha^2 \left(\tilde{B}(\bar{u}_N(s), A \bar{u}_N(s)), A \bar{u}_N(s) \right) =: B_1 + B_2.$$

Proposition 2.1-(ii) yields $B_1 = ([\bar{u}_N(s) \cdot \nabla] \bar{u}_N(s), A \bar{u}_N(s)) - ([A \bar{u}_N(s) \cdot \nabla] \bar{u}_N(s), \bar{u}_N(s))$. The first term vanishes thanks to [23, Lemma 3.1], as well as the second term (see Proposition 2.2-(i)). Thereby, $B_1 = 0$. On the other hand, by Proposition 2.1-(ii), it follows $B_2 = -\alpha^2 ([A \bar{u}_N(s) \cdot \nabla] A \bar{u}_N(s), \bar{u}_N(s))$. Hence,

$$\begin{aligned} |I_3| &\leq 2\alpha^2 C_D \int_0^{t \wedge \tau_N^n} \|A \bar{u}_N(s)\|_{\mathbb{L}^2} \|A^{3/2} \bar{u}_N(s)\|_{\mathbb{L}^2} \|\bar{u}_N(s)\|_{\mathbb{L}^2}^{1/2} \|A \bar{u}_N(s)\|_{\mathbb{L}^2}^{1/2} ds \\ &\leq 2\alpha^2 C_D \mu_N^{3/2} \int_0^{t \wedge \tau_N^n} \|A \bar{u}_N(s)\|_{\mathbb{L}^2}^{3/2} \|\bar{u}_N(s)\|_{\mathbb{L}^2}^{3/2} ds \\ &\leq \frac{4C_{max}^8 C_D^4}{\nu^3} \int_0^{t \wedge \tau_N^n} \|\bar{u}_N(s)\|_{\mathbb{L}^2}^6 ds + \frac{3\nu}{4} \int_0^{t \wedge \tau_N^n} \|A \bar{u}_N(s)\|_{\mathbb{L}^2}^2 ds. \end{aligned}$$

where Proposition 2.2-(ii), estimate $\|A^{3/2}z\|_{\mathbb{L}^2} \leq \mu_N^{3/2}\|z\|_{\mathbb{L}^2}$, for all $z \in V_N$, condition $\alpha \leq C_{max} \mu_N^{-3/4}$ together with the Young inequality with conjugate exponents $1/4$ and $3/4$ were taken advantage of. Observe that

$$\begin{aligned} \|\nabla \bar{u}_N(0)\|_{\alpha}^2 &= \|\nabla \bar{u}_N(0)\|_{\mathbb{L}^2}^2 + \alpha^2 \|A \bar{u}_N(0)\|_{\mathbb{L}^2}^2 \leq \|\nabla \bar{u}_N(0)\|_{\mathbb{L}^2}^2 + C_{max}^2 \|\nabla \bar{u}_N(0)\|_{\mathbb{L}^2}^2 \\ &\leq (1 + C_{max}^2) \|\nabla \bar{u}_0\|_{\mathbb{L}^2}^2, \end{aligned}$$

thanks to $\alpha \leq C_{max} / \mu_N^{3/4} \leq C_{max}$, and estimate $\|Az\|_{\mathbb{L}^2} \leq \sqrt{\mu_N} \|\nabla z\|_{\mathbb{L}^2}$ for all $z \in V_N$. Taking into account that $\sup_{0 \leq t \leq T} \|\bar{u}_N(t)\|_{\alpha}^{2q}$ is almost surely finite for all $q \geq 2$ on account of Lemma 4.1, the stopping time of last and first terms on the right-hand side of I_2 and I_3 can be omitted. Thereby,

$$\begin{aligned} \|\nabla \bar{u}_N(t \wedge \tau_N^n)\|_{\alpha}^2 &+ \frac{5\nu}{4} \int_0^{t \wedge \tau_N^n} \|A \bar{u}_N(s)\|_{\alpha}^2 ds \leq (1 + C_{max}^2) \|\nabla \bar{u}_0\|_{\mathbb{L}^2}^2 + 2K_1^2 t \wedge \tau_N^n \\ &+ 2 \int_0^{t \wedge \tau_N^n} (A \bar{u}_N(s), g(s, \bar{u}_N(s)) dW(s)) + 2K_2^2 \int_0^t \|\bar{u}_N(s)\|_{\alpha}^2 ds \\ &+ \frac{4C_{max}^8 C_D^4}{\nu^3} \int_0^t \|\bar{u}_N(s)\|_{\mathbb{L}^2}^6 ds. \end{aligned} \tag{4.7}$$

Subsequently, taking the mathematical expectation, employing Lemma 4.1 and letting $n \rightarrow +\infty$ imply

$$\begin{aligned} \mathbb{E} [\|\nabla \bar{u}_N(t)\|_{\alpha}^2] &+ \frac{5\nu}{4} \mathbb{E} \left[\int_0^t \|A \bar{u}_N(s)\|_{\alpha}^2 ds \right] \\ &\leq (1 + C_{max}^2) \mathbb{E} [\|\nabla \bar{u}_0\|_{\mathbb{L}^2}^2] + 2K_1^2 T + (2K_2^2 + \frac{4C_{max}^8 C_D^4}{\nu^3}) T C_1. \end{aligned} \tag{4.8}$$

We now raise equation (4.7) to the power p , use Proposition 2.3-(iii), and drop the stopping time τ_N^n , thanks to estimate (4.8). We obtain

$$\begin{aligned} & \sup_{0 \leq t \leq T} \|\nabla \bar{u}_N(t)\|_\alpha^{2p} + \left(\frac{5\nu}{4} \int_0^T \|A\bar{u}_N(t)\|_\alpha^2 dt \right)^p \lesssim \|\nabla \bar{u}_0\|_{\mathbb{L}^2}^{2p} + (K_1^2 T)^p \\ & + \left(\sup_{0 \leq t \leq T} \int_0^t (\nabla \bar{u}_N(s), \nabla g(s, \bar{u}_N(s)) dW(s)) \right)^p + (K_2^2 T)^p \sup_{0 \leq t \leq T} \|\bar{u}_N(t)\|_\alpha^{2p} \\ & + (C_{max}^8 C_D^4 T / \nu^3)^p \sup_{0 \leq t \leq T} \|\bar{u}_N(t)\|_{\mathbb{L}^2}^{6p}. \end{aligned} \quad (4.9)$$

We bound the third term on the right-hand side using the Burkholder-Davis-Gundy and Young inequalities, assumption (S_2) , and Proposition 2.3-(iii):

$$\begin{aligned} & \mathbb{E} \left[\left(\sup_{0 \leq t \leq T} \int_0^t (\nabla \bar{u}_N(s), \nabla g(s, \bar{u}_N(s)) dW(s)) \right)^p \right] \\ & \lesssim \mathbb{E} \left[\left(\int_0^T \|\nabla \bar{u}_N(t)\|_{\mathbb{L}^2}^2 \|\nabla g(t, \bar{u}_N(t))\|_{\mathcal{L}_2(K; \mathbb{L}^2)}^2 dt \right)^{p/2} \right] \\ & \leq \frac{1}{2} \mathbb{E} \left[\sup_{0 \leq t \leq T} \|\nabla \bar{u}_N(t)\|_\alpha^{2p} \right] + 2^{2p-2} T^p \mathbb{E} \left[K_1^{2p} + K_2^{2p} \sup_{0 \leq t \leq T} \|\bar{u}_N(t)\|_\alpha^{2p} \right], \end{aligned}$$

Taking afterwards the mathematical expectation of equation (4.9) and employing Lemma 4.1 complete the proof of estimate (i). On the other hand, $\|\nabla v_N(t)\|_{\mathbb{L}^2}^2 \leq 2 \max(1, C_{max}^2) \|\nabla \bar{u}_N(t)\|_\alpha^2$ holds for all $t \in [0, T]$, thanks to equation (4.6) which is slightly amended here to fit the case $\alpha \leq C_{max} \mu_N^{-3/4}$. Furthermore, multiplying in \mathbb{L}^2 the identity $v_N(t) = \bar{u}_N(t) + \alpha^2 A \bar{u}_N(t)$ by $A^2 v_N$ and making use of Cauchy-Schwarz inequality give

$$\|Av_N(t)\|_{\mathbb{L}^2}^2 \leq \|A\bar{u}_N(t)\|_{\mathbb{L}^2} \|Av_N(t)\|_{\mathbb{L}^2} + \alpha^2 \|A^{3/2} \bar{u}_N(t)\|_{\mathbb{L}^2} \|A^{3/2} v_N(t)\|_{\mathbb{L}^2}.$$

We use $\alpha \leq C_{max} \mu_N^{-3/4}$, $\|A^{3/2} v_N\|_{\mathbb{L}^2} \leq \sqrt{\mu_N} \|Av_N\|_{\mathbb{L}^2}$, and simplify by $\|Av_N(t)\|_{\mathbb{L}^2}$ to obtain eventually

$$\|Av_N(t)\|_{\mathbb{L}^2} \leq \max(1, C_{max}) \left(\|A\bar{u}_N(t)\|_{\mathbb{L}^2} + \alpha \|A^{3/2} \bar{u}_N(t)\|_{\mathbb{L}^2} \right).$$

Squaring both sides offers $\|Av_N(t)\|_{\mathbb{L}^2}^2 \leq 2 \max(1, C_{max}^2) \|A\bar{u}_N(t)\|_\alpha^2$. The proof of inequality (ii) follows after applying estimate (i). \square

5 Convergence of system (4.1)

This section is devoted to proving the convergence of (\bar{u}_N, v_N) towards the unique strong solution of the stochastic Navier-Stokes equations. The followed steps are typical: we first need to bound each item of system (4.1) in a reflexive Banach space. Then, limits identification shall be carried out to match all Navier-Stokes problem's terms.

Boundedness and convergence: Now that all data are clear, we begin by bounding each term of equations (4.1) in a suitable reflexive Banach space. The bilinear operator $\{\tilde{B}(\bar{u}_N, v_N)\}_N$ is bounded in $L^2(\Omega; L^2(0, T; \mathbb{V}'))$. Indeed, Proposition 2.1-(iii), the embedding $\mathbb{H}^1 \hookrightarrow \mathbb{L}^4$, the Cauchy-Schwarz inequality, and Lemma 4.3 yield

$$\mathbb{E} \left[\int_0^T \left\| \tilde{B}(\bar{u}_N(t), v_N(t)) \right\|_{\mathbb{V}'}^2 dt \right] \leq C_D \mathbb{E} \left[\sup_{t \in [0, T]} \|\nabla \bar{u}_N(t)\|_{\mathbb{L}^2}^2 \int_0^T \|\nabla v_N(t)\|_{\mathbb{L}^2}^2 dt \right] \leq C_D C_3 C_4.$$

Therefore, setting $R(\bar{u}_N) := -\nu\Delta v_N + \tilde{B}(\bar{u}_N, v_N)$, we conclude from Lemma 4.1 that $\{R(\bar{u}_N)\}_N$ is bounded in $L^2(\Omega; L^2(0, T; \mathbb{V}'))$. Moreover, by virtue of Lemma 4.1 and assumption (S_2) , $\{v_N\}_N$, $\{\bar{u}_N\}_N$ are bounded in $L^2(\Omega; L^\infty(0, T; \mathbb{H}) \cap L^2(0, T; \mathbb{V}))$, and $\{g(\cdot, \bar{u}_N)\}_N$ too in the Hilbert space $L^2(\Omega; L^2(0, T; \mathcal{L}_2(K, \mathbb{L}^2)))$. This implies the existence of two subsequences $\{v_{N_\ell}\}_\ell$, $\{\bar{u}_{N_\ell}\}_\ell$ of $\{v_N\}_N$, $\{\bar{u}_N\}_N$ respectively, and four limiting functions $v_{NS}, u_{NS} \in L^2(\Omega; L^\infty(0, T; \mathbb{H}) \cap L^2(0, T; \mathbb{V}))$, $R_0 \in L^2(\Omega; L^2(0, T; \mathbb{V}'))$, and $g_0 \in L^2(\Omega; L^2(0, T; \mathcal{L}_2(K, \mathbb{L}^2)))$ such that

$$v_{N_\ell} \rightharpoonup v_{NS} \ \& \ \bar{u}_{N_\ell} \rightharpoonup u_{NS} \text{ (weakly)} \quad \text{in } L^2(\Omega; L^2(0, T; \mathbb{V})), \quad (5.1)$$

$$v_{N_\ell} \xrightarrow{*} v_{NS} \ \& \ \bar{u}_{N_\ell} \xrightarrow{*} u_{NS} \text{ (weakly-*)} \quad \text{in } L^2(\Omega; L^\infty(0, T; \mathbb{H})), \quad (5.2)$$

$$R(\bar{u}_{N_\ell}) \rightharpoonup R_0 \text{ (weakly)} \quad \text{in } L^2(\Omega; L^2(0, T; \mathbb{V}')), \quad (5.3)$$

$$g(\cdot, \bar{u}_{N_\ell}) \rightharpoonup g_0 \text{ (weakly)} \quad \text{in } L^2(\Omega; L^2(0, T; \mathcal{L}_2(K, \mathbb{L}^2))). \quad (5.4)$$

As a result, the limiting function v_{NS} satisfies \mathbb{P} -a.s. and for all $t \in [0, T]$ the equation:

$$(v_{NS}(t), \varphi) + \int_0^t \langle R_0(s), \varphi \rangle ds = (v_0, \varphi) + \left(\int_0^t g_0(s) dW(s), \varphi \right), \quad \forall \varphi \in \mathbb{V}, \quad (5.5)$$

where we recall the v_0 is the limit of $P_N v_0$ as $N \rightarrow +\infty$ in $L^4(\Omega; \mathbb{H})$. Making use of the classical approach in [20], and taking into account equation (5.5) which is fulfilled by v_{NS} , it is straightforward to show that $v_{NS} \in L^2(\Omega; C([0, T]; \mathbb{H}))$. Besides, identity $v_{N_\ell} = \bar{u}_{N_\ell} + \alpha^2 A \bar{u}_{N_\ell}$ grants equality between processes u_{NS} and v_{NS} . Indeed, for all $\varphi \in \mathbb{H}$, it holds that

$$\left| \alpha^2 \mathbb{E} \left[\int_0^T (A \bar{u}_{N_\ell}(t), \varphi) dt \right] \right| \leq \alpha \|\varphi\|_{\mathbb{L}^2} \mathbb{E} \left[\int_0^T \alpha^2 \|A \bar{u}_{N_\ell}(t)\|_{\mathbb{L}^2}^2 dt \right]^{1/2} \leq \alpha \|\varphi\|_{\mathbb{L}^2} C_1 \rightarrow 0$$

as $\ell \rightarrow +\infty$, thanks to the hypothesis $\alpha \leq C_{max} \mu_N^{-3/4}$. Subsequently, $\{\alpha^2 A \bar{u}_{N_\ell}\}_\ell$ converges weakly in $L^2(\Omega; L^2(0, T; \mathbb{L}^2))$ to 0, which offers, by the use of the aforementioned identity together with (5.1), the equality $u_{NS} = v_{NS}$ \mathbb{P} -a.s. and a.e. in $[0, T] \times D$. The only remaining task in this section consists in identifying R_0 and g_0 with their solution-dependent counterparts. To this purpose, we must first state one essential property that enables such an identification.

Proposition 5.1 *For $N \in \mathbb{N} \setminus \{0\}$, assume that $\alpha \leq C_{max} \mu_N^{-3/4}$. Let v_N^1, v_N^2 be two vector fields in V_N such that $v_N^1 = \bar{u}_N^1 + \alpha^2 A \bar{u}_N^1$ and $v_N^2 = \bar{u}_N^2 + \alpha^2 A \bar{u}_N^2$. If $L_g \leq \frac{\sqrt{\nu}}{C_P \sqrt{2}}$ then, there exists a constant $\mathcal{K} > 0$ depending only on D and C_{max} such that*

$$\begin{aligned} & \left\langle -\nu\Delta(v_N^1 - v_N^2) + \tilde{B}(\bar{u}_N^1, v_N^1) - \tilde{B}(\bar{u}_N^2, v_N^2) + \frac{\mathcal{K}}{\nu^3} \|\bar{u}_N^2\|_{\mathbb{L}^4}^4 w_N, w_N \right\rangle \\ & - \left\| g(\cdot, \bar{u}_N^1) - g(\cdot, \bar{u}_N^2) \right\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 \geq 0, \end{aligned}$$

where $C_P > 0$ is the Poincaré constant and $w_N := \bar{u}_N^1 - \bar{u}_N^2$.

Proof: $\langle -\nu\Delta(v_N^1 - v_N^2), w_N \rangle = \nu (A^{1/2}(I + \alpha^2 A)w_N, A^{1/2}w_N) = \nu \|\nabla w_N\|_\alpha^2$. Besides, Proposition 2.1-(i) and (iii) yield

$$\begin{aligned} & \left| \langle \tilde{B}(\bar{u}_N^1, v_N^1) - \tilde{B}(\bar{u}_N^2, v_N^2), w_N \rangle \right| = \left| \langle \tilde{B}(\bar{u}_N^2, v_N^1 - v_N^2), w_N \rangle \right| \\ & \leq C_D \|\bar{u}_N^2\|_{\mathbb{L}^4} \|\nabla(v_N^1 - v_N^2)\|_{\mathbb{L}^2} \|w_N\|_{\mathbb{L}^2}^{\frac{1}{2}} \|\nabla w_N\|_{\mathbb{L}^2}^{\frac{1}{2}}, \end{aligned} \quad (5.6)$$

where identity $v_N^1 - v_N^2 = w_N + \alpha^2 Aw_N$ implies $\nabla(v_N^1 - v_N^2) = \nabla w_N + \alpha^2 \nabla Aw_N$ and therefore, it follows that $\|\nabla(v_N^1 - v_N^2)\|_{\mathbb{L}^2} \leq (1 + C_{max})\|\nabla w_N\|_{\mathbb{L}^2}$, thanks to the condition $\alpha \leq C_{max}\mu_N^{-3/4}$. Plugging this result back into equation (5.6) and applying the Young inequality to get

$$\left| \langle \tilde{B}(\bar{u}_N^1, v_N^1) - \tilde{B}(\bar{u}_N^2, v_N^2), w_N \rangle \right| \leq \frac{\nu}{4} \|\nabla w_N\|_{\mathbb{L}^2}^2 + \frac{\mathcal{K}}{\nu^3} \|\bar{u}_N^2\|_{\mathbb{L}^4}^4 \|w_N\|_{\mathbb{L}^2}^2,$$

where $\mathcal{K} > 0$ depends only on C_D and C_{max} . Assumption (S_2) implies $\|g(\cdot, \bar{u}_N^1) - g(\cdot, \bar{u}_N^2)\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 \geq -L_g^2 \|w_N\|_{\alpha}^2$ in addition. Putting it all together and employing the Poincaré inequality, we obtain

$$\begin{aligned} & \langle -\nu \Delta(v_N^1 - v_N^2) + \tilde{B}(\bar{u}_N^1, v_N^1) - \tilde{B}(\bar{u}_N^2, v_N^2) + \frac{\mathcal{K}}{\nu^3} \|\bar{u}_N^2\|_{\mathbb{L}^4}^4 w_N, w_N \rangle - \|g(\cdot, \bar{u}_N^1) - g(\cdot, \bar{u}_N^2)\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 \\ & \geq \left(\frac{\nu}{2} - L_g^2 C_P^2\right) \|\nabla w_N\|_{\mathbb{L}^2}^2 + \alpha^2 (\nu - L_g^2 C_P^2) \|Aw_N\|_{\mathbb{L}^2}^2 \end{aligned}$$

which is nonnegative when $L_g \leq \frac{\sqrt{\nu}}{C_P \sqrt{2}}$. \square

Remark 5.1 The quantities \bar{u}_N^1 and \bar{u}_N^2 in the statement of Proposition 5.1 exist and are unique, thanks to the bijectivity of operator $I + \alpha^2 A$ from $D(A)$ to \mathbb{H} .

Limits identification: For clarity's sake, the subsequences' subscript N_ℓ will be henceforth denoted N . Let $0 < m < N$ be a fixed integer, and $z, \bar{z} \in L^\infty(\Omega \times (0, T); V_m)$ be such that $z = \bar{z} + \alpha^2 A\bar{z}$. For $t \in [0, T]$, define the real valued process $\rho(\omega, t) := \frac{2\mathcal{K}}{\nu^3} \int_0^t \|z(\omega, s)\|_{\mathbb{L}^4}^4 ds$, where the constant \mathcal{K} is that of Proposition 5.1. Due to the properties of z , the process ρ is clearly time-continuous and adapted. By application of Itô's formula to the process $t \mapsto e^{-\rho(t)} \|v_N(t)\|_{\mathbb{L}^2}^2$, it follows that

$$\begin{aligned} e^{-\rho(t)} \|v_N(t)\|_{\mathbb{L}^2}^2 &= \|v_N(0)\|_{\mathbb{L}^2}^2 + 2 \int_0^t e^{-\rho(s)} (v_N(s), g(s, \bar{u}_N(s))) dW(s) \\ &\quad - \frac{2\mathcal{K}}{\nu^3} \int_0^t e^{-\rho(s)} \|z(s)\|_{\mathbb{L}^4}^4 \|v_N(s)\|_{\mathbb{L}^2}^2 ds - 2 \int_0^t e^{-\rho(s)} (v_N(s), R(\bar{u}_N(s))) ds \\ &\quad + \int_0^t e^{-\rho(s)} \|P_N g(s, \bar{u}_N(s))\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 ds, \end{aligned}$$

where we recall that $R(\bar{u}_N) = \nu Av_N + \tilde{B}(\bar{u}_N, v_N)$. The mathematical expectation of the second term on the right-hand side is null, thanks to assumption (S_2) and the measurability of v_N . Therefore, the above equation transforms into

$$\begin{aligned} & \mathbb{E} \left[e^{-\rho(T)} \|v_N(T)\|_{\mathbb{L}^2}^2 - \|v_N(0)\|_{\mathbb{L}^2}^2 \right] \\ &= \frac{2\mathcal{K}}{\nu^3} \mathbb{E} \left[\int_0^T e^{-\rho(t)} \|z(t)\|_{\mathbb{L}^4}^4 \{ \|z(t)\|_{\mathbb{L}^2}^2 - 2(v_N(t), z(t)) \} dt \right] \\ &\quad - 2\mathbb{E} \left[\int_0^T e^{-\rho(t)} \left(R(\bar{u}_N(t)) - R(\bar{z}(t)) + \frac{\mathcal{K}}{\nu^3} \|z(t)\|_{\mathbb{L}^4}^4 (v_N(t) - z(t)), v_N(t) - z(t) \right) dt \right] \\ &\quad - 2\mathbb{E} \left[\int_0^T e^{-\rho(t)} (R(\bar{u}_N(t)) - R(\bar{z}(t)), z(t)) dt \right] - 2\mathbb{E} \left[\int_0^T e^{-\rho(t)} (R(\bar{z}(t)), v_N(t)) dt \right] \quad (5.7) \\ &\quad + \mathbb{E} \left[\int_0^T e^{-\rho(t)} \|P_N g(t, \bar{u}_N(t)) - P_N g(t, \bar{z}(t))\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 dt \right] \\ &\quad + 2\mathbb{E} \left[\int_0^T e^{-\rho(t)} (P_N g(t, \bar{u}_N(t)), P_N g(t, \bar{z}(t)))_{\mathcal{L}_2(K, \mathbb{L}^2)} dt \right] \\ &\quad - \mathbb{E} \left[\int_0^T e^{-\rho(t)} \|P_N g(t, \bar{z}(t))\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 dt \right] =: I_1 + \dots + I_7, \end{aligned}$$

where the notation $(\cdot, \cdot)_{\mathcal{L}_2(K, \mathbb{L}^2)}$ represents the $\mathcal{L}_2(K, \mathbb{L}^2)$ -scalar product. By convergence 5.1, I_1 converges toward $\frac{2\mathcal{K}}{\nu^3} \mathbb{E} \left[\int_0^T e^{-\rho(t)} \|z(t)\|_{\mathbb{L}^4}^4 \{ \|z(t)\|_{\mathbb{L}^2}^2 - 2(v_{NS}(t), z(t)) \} dt \right]$ as $N \rightarrow +\infty$. Moreover,

$$\begin{aligned} I_2 = & -2\mathbb{E} \left[\int_0^T e^{-\rho(t)} \left(R(\bar{u}_N(t)) - R(\bar{z}(t)) + \frac{\mathcal{K}}{\nu^3} \|z(t)\|_{\mathbb{L}^4}^4 (\bar{u}_N(t), \bar{z}(t)), \bar{u}_N(t) - \bar{z}(t) \right) dt \right] \\ & - 2\alpha^2 \mathbb{E} \left[\int_0^T e^{-\rho(t)} \left(R(\bar{u}_N(t)) - R(\bar{z}(t)) + \frac{\mathcal{K}}{\nu^3} \|z(t)\|_{\mathbb{L}^4}^4 (\bar{u}_N(t) - \bar{z}(t)), A\bar{u}_N(t) - A\bar{z}(t) \right) dt \right] \\ & - \frac{2\mathcal{K}\alpha^2}{\nu^3} \mathbb{E} \left[\int_0^T e^{-\rho(t)} \|z(t)\|_{\mathbb{L}^4}^4 (A\bar{u}_N(t) - A\bar{z}(t), v_N(t) - z(t)) dt \right] =: I_{2,1} + I_{2,2} + I_{2,3}. \end{aligned}$$

Proposition 5.1 implies that $I_{2,1} + I_5 \leq 0$. Additionally, by turning (\cdot, \cdot) into $\langle \cdot, \cdot \rangle$, it follows that

$$|I_{2,2}| \leq 2\alpha^2 \mathbb{E} \left[\int_0^T \left(\|R(\bar{u}_N)\|_{\mathbb{H}^{-1}} + \|R(\bar{z})\|_{\mathbb{H}^{-1}} + \frac{\mathcal{K}}{\nu^3} \|z\|_{\mathbb{L}^4}^4 \|\bar{u}_N - \bar{z}\|_{\mathbb{H}^{-1}} \right) \|A\bar{u}_N - A\bar{z}\|_{\mathbb{H}^1} dt \right].$$

By the definition of operator R , one gets

$$\begin{aligned} \|R(\bar{u}_N(t))\|_{\mathbb{H}^{-1}} & \leq \nu \|A\bar{u}_N(t)\|_{\mathbb{H}^{-1}} + \|\tilde{B}(\bar{u}_N(t), v_N(t))\|_{\mathbb{H}^{-1}} \\ & \leq \nu \|\nabla \bar{u}_N(t)\|_{\mathbb{L}^2} + C_D \|\bar{u}_N(t)\|_{\mathbb{L}^2}^{\frac{1}{2}} \|\nabla \bar{u}_N(t)\|_{\mathbb{L}^2}^{\frac{1}{2}} \|\nabla v_N(t)\|_{\mathbb{L}^2}, \end{aligned}$$

thanks to Proposition 2.1-(iii) and the Gagliardo-Nirenberg inequality. Therefore,

$$\begin{aligned} & 2\alpha^2 \mathbb{E} \left[\int_0^T \|R(\bar{u}_N(t))\|_{\mathbb{H}^{-1}} \|A\bar{u}_N(t) - A\bar{z}(t)\|_{\mathbb{H}^1} dt \right] \\ & \leq 2\alpha\nu \mathbb{E} \left[\sup_{t \in [0, T]} \|\nabla \bar{u}_N(t)\|_{\mathbb{L}^2}^2 \right]^{\frac{1}{2}} \mathbb{E} \left[\int_0^T \alpha^2 \|A\bar{u}_N(t) - A\bar{z}(t)\|_{\mathbb{H}^1}^2 dt \right]^{\frac{1}{2}} \\ & \quad + 2\alpha C_D \mathbb{E} \left[\sup_{t \in [0, T]} \|\bar{u}_N(t)\|_{\mathbb{L}^2} \|\nabla \bar{u}_N(t)\|_{\mathbb{L}^2} \|\nabla v_N(t)\|_{\mathbb{L}^2}^2 \right]^{\frac{1}{2}} \mathbb{E} \left[\int_0^T \alpha^2 \|A\bar{u}_N(t) - A\bar{z}(t)\|_{\mathbb{H}^1}^2 dt \right]^{\frac{1}{2}} \\ & \lesssim 2\alpha\nu C_3 + 2\alpha C_D C_3 C_4 \rightarrow 0 \text{ as } N \rightarrow +\infty, \end{aligned}$$

thanks to Lemma 4.3 and the assumption $\alpha \leq C_{max} \mu_N^{-3/4}$. The same goes for the remaining terms of $I_{2,2}$, which are easier to handle. Thus, $I_{2,2} \rightarrow 0$ as $N \rightarrow +\infty$. Moving on to $I_{2,3}$, we have

$$\begin{aligned} |I_{2,3}| & \leq \frac{2\mathcal{K}\alpha}{\nu^3} \mathbb{E} \left[\sup_{t \in [0, T]} \|z(t)\|_{\mathbb{L}^4}^8 \|v_N(t) - z(t)\|_{\mathbb{L}^2}^2 \right]^{\frac{1}{2}} \mathbb{E} \left[\int_0^T \alpha^2 \|A\bar{u}_N(t) - A\bar{z}(t)\|_{\mathbb{L}^2}^2 dt \right]^{\frac{1}{2}} \\ & \lesssim \frac{2\mathcal{K}\alpha}{\nu^3} C_2 C_1 \rightarrow 0 \text{ as } N \rightarrow +\infty, \end{aligned}$$

by virtue of Lemma 4.1 and $\alpha \leq C_{max} \mu_N^{-3/4}$. It is straightforward to show that when $N \rightarrow +\infty$, z and \bar{z} become equal \mathbb{P} -a.s. and a.e. in $[0, T] \times D$. We exploit this fact and convergence 5.3 to obtain $I_3 \rightarrow -2\mathbb{E} \left[\int_0^T e^{-\rho(t)} \langle R_0(t) - R(z(t)), z(t) \rangle dt \right]$ as $N \rightarrow +\infty$, and convergence 5.1 to accomplish $I_4 \rightarrow -2\mathbb{E} \left[\int_0^T e^{-\rho(t)} \langle R(z(t)), v_{NS}(t) \rangle dt \right]$. Similarly, $I_6 \rightarrow 2\mathbb{E} \left[\int_0^T e^{-\rho(t)} (g_0(t), g(t, z(t)))_{\mathcal{L}_2(K, \mathbb{L}^2)} dt \right]$, thanks to result (5.4), the continuity of g with respect to its second variable, and the properties of projector P_N which also grant the convergence of I_7 i.e. $I_7 \rightarrow -\mathbb{E} \left[\int_0^T e^{-\rho(t)} \|g(t, z(t))\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 dt \right]$.

Consequently, we pass to the limit in equation (5.7) while taking advantage of all generated results to achieve eventually:

$$\begin{aligned}
& \mathbb{E} \left[e^{-\rho(T)} \|v_{NS}(T)\|_{\mathbb{L}^2}^2 - \|v_{NS}(0)\|_{\mathbb{L}^2}^2 \right] \leq \liminf_{N \rightarrow +\infty} \mathbb{E} \left[e^{-\rho(T)} \|v_N(T)\|_{\mathbb{L}^2}^2 - \|v_N(0)\|_{\mathbb{L}^2}^2 \right] \\
& \leq \frac{2\mathcal{K}}{\nu^3} \mathbb{E} \left[\int_0^T e^{-\rho(t)} \|z(t)\|_{\mathbb{L}^4}^4 \{ \|z(t)\|_{\mathbb{L}^2}^2 - 2 \langle v_{NS}(t), z(t) \rangle \} dt \right] \\
& - 2\mathbb{E} \left[\int_0^T e^{-\rho(t)} \langle R_0(t) - R(z(t)), z(t) \rangle dt \right] - 2\mathbb{E} \left[\int_0^T e^{-\rho(t)} \langle R(z(t)), v_{NS}(t) \rangle dt \right] \\
& + 2\mathbb{E} \left[\int_0^T e^{-\rho(t)} \langle g_0(t), g(t, z(t)) \rangle_{\mathcal{L}_2(K, \mathbb{L}^2)} dt \right] - \mathbb{E} \left[\int_0^T e^{-\rho(t)} \|g(t, z(t))\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 dt \right].
\end{aligned} \tag{5.8}$$

Next, we apply Itô's formula to the process $t \mapsto e^{-\rho(t)} \|v_{NS}(t)\|_{\mathbb{L}^2}^2$, where we recall that v_{NS} satisfies equation (5.5). It holds that

$$\begin{aligned}
& \mathbb{E} \left[e^{-\rho(T)} \|v_{NS}(T)\|_{\mathbb{L}^2}^2 - \|v_{NS}(0)\|_{\mathbb{L}^2}^2 \right] = -\frac{2\mathcal{K}}{\nu^3} \mathbb{E} \left[\int_0^T e^{-\rho(t)} \|z(t)\|_{\mathbb{L}^4}^4 \|v_{NS}(t)\|_{\mathbb{L}^2}^2 dt \right] \\
& - 2\mathbb{E} \left[\int_0^T e^{-\rho(t)} \langle R_0(t), v_{NS}(t) \rangle dt \right] + \mathbb{E} \left[\int_0^T e^{-\rho(t)} \|g_0(t)\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 dt \right].
\end{aligned} \tag{5.9}$$

Plugging result (5.9) in equation (5.8) grants:

$$\begin{aligned}
& \frac{2\mathcal{K}}{\nu^3} \mathbb{E} \left[\int_0^T e^{-\rho(t)} \|z(t)\|_{\mathbb{L}^4}^4 \|v_{NS}(t) - z(t)\|_{\mathbb{L}^2}^2 dt \right] \\
& + 2\mathbb{E} \left[\int_0^T e^{-\rho(t)} \langle R_0(t) - R(z(t)), v_{NS}(t) - z(t) \rangle dt \right] \\
& \geq \mathbb{E} \left[\int_0^T e^{-\rho(t)} \|g_0(t) - g(t, z(t))\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 dt \right], \quad \forall z \in L^\infty(\Omega \times (0, T); V_m).
\end{aligned} \tag{5.10}$$

Arguing by density, the above inequality holds for all $z \in L^4(\Omega; L^\infty(0, T; \mathbb{H})) \cap L^2(\Omega; L^2(0, T; \mathbb{V}))$. Setting $z = v_{NS}$ in equation (5.10) yields $g(\cdot, v_{NS}) = g_0$ \mathbb{P} -a.s. and a.e. in $(0, T) \times D$. Furthermore, for an arbitrary $w \in L^4(\Omega; L^\infty(0, T; \mathbb{H})) \cap L^2(\Omega; L^2(0, T; \mathbb{V}))$ and $\theta \in \mathbb{R}_+^*$, we set $z = v_{NS} + \theta w$, and make use of equation (5.10) once again to obtain:

$$\begin{aligned}
& \frac{\mathcal{K}\theta}{\nu^3} \mathbb{E} \left[\int_0^T e^{-\rho(t)} \|v_{NS}(t) + \theta w(t)\|_{\mathbb{L}^4}^4 \|w(t)\|_{\mathbb{L}^2}^2 dt \right] \\
& - \mathbb{E} \left[\int_0^T e^{-\rho(t)} \langle R_0(t) - R(v_{NS}(t) + \theta w(t)), w(t) \rangle dt \right] \geq 0.
\end{aligned}$$

Letting θ go to 0 and using the hemi-continuity of the operator R lead to

$$\mathbb{E} \left[e^{-\rho(t)} \langle R_0(t) - R(v_{NS}(t)), w(t) \rangle dt \right] \leq 0, \quad \forall w \in L^4(\Omega; L^\infty(0, T; \mathbb{H})) \cap L^2(\Omega; L^2(0, T; \mathbb{V})),$$

which eventually implies $R_0 = R(v_{NS})$ in $L^2(\Omega; L^2(0, T; \mathbb{H}^{-1}))$. The acquired limiting function v_{NS} satisfies the following lemma.

Lemma 5.1 *Let $T > 0$, $1 \leq p < +\infty$, and $N \in \mathbb{N} \setminus \{0\}$ be given. Assume that hypotheses (S_1) - (S_2) are fulfilled, and that for some constant $C_{max} > 0$ independent of N , the spatial scale $\alpha \leq C_{max} \mu_N^{-3/4}$. Then, the process $\{v_{NS}(t), t \in [0, T]\}$ fulfills:*

$$(i) \mathbb{E} \left[\sup_{t \in [0, T]} \|v_{NS}(t)\|_{\mathbb{L}^2}^{2p} + 2p\nu \int_0^T \|v_{NS}(t)\|_{\mathbb{L}^2}^{2(p-1)} \|\nabla v_{NS}(t)\|_{\mathbb{L}^2}^2 dt \right] \leq C_2,$$

$$(ii) \mathbb{E} \left[\sup_{t \in [0, T]} \|\nabla v_{NS}(t)\|_{\mathbb{L}^2}^{2p} + \left(\nu \int_0^T \|Av_{NS}(t)\|_{\mathbb{L}^2}^2 dt \right)^p \right] \leq C_4,$$

where $C_2 > 0$ depends on constants C_{max} , C_1 of Lemma 4.1 and its parameters, and $C_4 > 0$ depends on C_1 , $\|\bar{u}_0\|_{L^{6p}(\Omega; \mathbb{V})}$ and C_{max} .

Proof: We only illustrate here the proof of estimate (ii) as (i) can be concluded from (ii). Let $p \geq 1$. On account of Lemma 4.3-(ii), the sequence $(v_N)_N$ is bounded in $L^{2p}(\Omega; L^\infty(0, T; \mathbb{V}))$ which implies the existence of a function $\xi \in L^{2p}(\Omega; L^\infty(0, T; \mathbb{V}))$ such that for some subsequence $(v_{N_\ell})_\ell$, it holds that $v_{N_\ell} \xrightarrow{*} \xi$ in $L^{2p}(\Omega; L^\infty(0, T; \mathbb{V})) \hookrightarrow L^2(\Omega; L^\infty(0, T; \mathbb{H}))$, and

$$\mathbb{E} \left[\sup_{t \in [0, T]} \|\xi(t)\|_{\mathbb{V}}^{2p} \right] \leq \liminf \mathbb{E} \left[\sup_{t \in [0, T]} \|v_{N_\ell}(t)\|_{\mathbb{V}}^{2p} \right] \leq C_4,$$

thanks to Lemma 4.3-(ii). By convergence 5.2 and the weak limit uniqueness, we infer that $\xi = v_{NS}$ \mathbb{P} -a.s. and a.e. in $(0, T) \times D$. This is valid because v_{NS} is the unique solution to equations (1.2) which means that the whole sequence $(v_N)_N$ is convergent. Arguing in a similar fashion, and owing to Lemma 4.3-(ii), $(v_N)_N$ is bounded in the reflexive Banach space $L^{2p}(\Omega; L^2(0, T; D(A)))$, which signifies that for some $(v_{N_\ell})_\ell$ and $\eta \in L^{2p}(\Omega; L^2(0, T; D(A)))$, we have $v_{N_\ell} \rightharpoonup \eta$ in $L^{2p}(\Omega; L^2(0, T; D(A)))$ and

$$\mathbb{E} \left[\left(\nu \int_0^T \|A\eta(t)\|_{\mathbb{L}^2}^2 dt \right)^p \right] \leq \liminf \mathbb{E} \left[\left(\nu \int_0^T \|Av_{N_\ell}\|_{\mathbb{L}^2}^2 dt \right)^p \right] \leq C_4.$$

As done earlier in this proof, one obtains $\eta = v_{NS}$ \mathbb{P} -a.s. and a.e. in $(0, T) \times D$. \square

6 Conclusion

Owing to Section 5, the limiting function v_{NS} satisfies for all $t \in [0, T]$, and \mathbb{P} -a.s. the following equation:

$$\begin{aligned} & (v_{NS}(t), \varphi) + \nu \int_0^t (\nabla v_{NS}(s), \nabla \varphi) + \int_0^t \tilde{b}(v_{NS}(s), v_{NS}(s), \varphi) ds \\ & = (v_0, \varphi) + \left(\int_0^t g(s, v_{NS}(s)) dW(s), \varphi \right), \quad \forall \varphi \in \mathbb{V}. \end{aligned}$$

By virtue of Proposition 2.1, one gets

$$\tilde{b}(v_{NS}(s), v_{NS}(s), \varphi) = -\tilde{b}(\varphi, v_{NS}(s), v_{NS}(s)) = ([v_{NS}(s) \cdot \nabla] v_{NS}(s), \varphi).$$

Moreover, as mentioned in Section 5, v_{NS} belongs to $L^2(\Omega; C([0, T]; \mathbb{H}))$. Besides the latter fact, Lemma 5.1 guarantees that $v_{NS} \in L^2(\Omega; L^2(0, T; \mathbb{V}))$. Consequently, collecting all results and comparing them with Definition 2.3, it follows that v_{NS} is the unique solution to equations (1.2) in the sense of Definition 2.3.

The convergence analysis followed in this paper could have been carried out differently. For instance, instead of controlling the spatial scale α with a quantity that vanishes at the limit, a convergence rate of

the difference $\|v_{NS} - \bar{u}\|$ in terms of α could have made up an alternative approach, as conducted in [2] for the stochastic Leray- α equations. We emphasize the uselessness of the imposed periodic boundary conditions if high spatial regularities of the solution were not utilized. In this case, Dirichlet boundary conditions are required.

The demonstration techniques employed in this paper are only functional for two-dimensional domains. In three dimensions, another approach must be applied to acquire a solution to the stochastic Navier-Stokes problem from the stochastic LANS- α model, as performed in article [12].

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