# On the Saito basis and the Tjurina number for plane branches

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# ON THE SAITO BASIS AND THE TJURINA NUMBER FOR PLANE BRANCHES

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Abstract. We introduce the concept of a good Saito basis for a plane curve and we explore it to obtain a formula for the minimal Tjurina number in a topological class. In particular, we give a lower bound for the Tjurina number in terms of the Milnor number that allow us to present a positive answer for a question of Dimca and Greuel.

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## 1. Introduction.

Let $S : \{f = 0\}$ be a germ of an irreducible analytic plane curve. An important analytic invariant of $S$ is the Tjurina number $\tau(S) = \dim_{\mathbb{C}} \frac{\mathbb{C}\{x,y\}}{(f)+J(f)}$ where $J(f)$ denotes the Jacobian ideal of $f$.
In general, the computation of $\tau(S)$ is not easy. For instance, we can obtain it considering a Gröbner basis for the ideal $(f) + J(f)$, or alternatively, it is possible to compute $\tau$ by the dimension of $\frac{J(f):(f)}{J(f)}$ (see Theorem 1 in [7]) that is related with the $\mathbb{C}\{x,y\}$-module $\Omega^1(S)$ of all germs of 1-holomorphic forms $\omega \in \mathbb{C}\{x,y\}\mathrm{d}x + \mathbb{C}\{x,y\}\mathrm{d}y$ such that $f$ divides $\omega \wedge \mathrm{d}f$. More precisely, according to K. Saito [9], $\Omega^1(S)$ is freely generated by two elements $\{\omega_1, \omega_2\}$. It will be shown that $\tau(S)$ can be expressed from, among other invariants, the codimension of the ideal $(g_1, g_2)$ where $\omega_i \wedge \mathrm{d}f = g_i f \mathrm{d}x \wedge \mathrm{d}y$.
If $L$ denotes a topological class of a plane curve - for instance, given by the characteristic exponents - then the Milnor number $\mu = \dim_{\mathbb{C}} \frac{\mathbb{C}\{x,y\}}{J(f)}$ is constant for any $S : \{f = 0\} \in L$ and $\tau_{\min} \leq \tau(S) \leq \mu$. Generically, an element $S \in L$ is such that $\tau(S) = \tau_{\min}$, so $\tau_{\min}$ can be expressed using the topological data that characterizes $L$. Delorme in [3], presented a formula to compute the generic dimension $d(\beta_0, \beta_1)$ of the moduli space for an irreducible plane curve with characteristic exponents $(\beta_0, \beta_1)$. As $d(2, \beta_1) = 0$ and $d(\beta_0, \beta_1) = \frac{(\beta_0-3)(\beta_1-3)}{2} + \left[\frac{\beta_0}{\beta_1}\right] - 1 - \mu + \tau_{\min}$ (see [6]) we can compute the minimal Tjurina number for this topological class. On the other hand, Peraire in [8] developed an algorithm to compute $\tau_{\min}$ by means of a flag of $J(f)$.

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In this paper we present a way to express the difference $\mu - \tau$ for a singular irreducible plane curve $S$ when $\Omega^1(S)$ admits a basis $\{\omega_1, \omega_2\}$ of special kind, that we call a *good* Saito basis (see Definition 2).
More specifically, we present a formula (see Theorem 11) to compute the difference between $\mu(S) - \tau(S)$ and $\mu(\widetilde{S}) - \tau(\widetilde{S})$ where $\widetilde{S}$ denotes the strict transform of $S$.
If $S$ is generic in $L$, then, according to [5], $S$ admits a good basis and this fact allows us to obtain a formula to compute $\tau_{\min}$ in $L$ by the sole topological data: the sequence of multiplicities in the canonical resolution or the characteristic exponents for instance. In particular, for irreducible plane curves, we are able to present a lower bound for the minimum Tjurina number in $L$ in terms of the Milnor number that allow us to give an affirmative answer to a question of Dimca and Greuel [4] about the inequality $4\tau > 3\mu$ and obtained simultaneously by Alberich-Carramiñana *et al.* in [1] published in ArXiv a few days before the first version of this paper.
The paper is organized as follows. In the section 2 we present some general properties of a Saito basis. The concept of a good Saito basis is introduced in the section 3 and its properties as well. The section 4 is devoted to the formula for the minimal Tjurina number, a lower bound for the Tjurina number using the Milnor number and consequently an answer to the Dimca-Greuel question.

## 2. The Saito basis.

Let $S : \{f = 0\}$ be a germ of an analytic plane curve and consider the $\mathbb{C}\{x, y\}$-module $\Omega^1(S)$ of all germs of 1-holomorphic forms $\omega \in \mathbb{C}\{x, y\}\mathrm{d}x + \mathbb{C}\{x, y\}\mathrm{d}y$ such that $f$ divides $\omega \wedge \mathrm{d}f$. It is equivalent to require that the foliation induced by $\omega$ lets invariant $S$. Saito in [9] shows that $\Omega^1(S)$ is a free module of rank 2 and a basis of $\Omega^1(S)$ is called a Saito basis.
It is not trivial to obtain a Saito basis, but there is a simple criterion to verify if $\{\omega_1, \omega_2\}$ is a basis for $\Omega^1(S)$ (see Theorem, page 270 in [9]).

**Theorem** (Saito criterion)**.** *The set $\{\omega_1, \omega_2\}$ is a Saito basis for $S : \{f = 0\}$ if and only if $\omega_1 \wedge \omega_2 = uf\mathrm{d}x \wedge \mathrm{d}y$, where $u$ is a unit in $\mathbb{C}\{x, y\}$.*

This criterion can be interpreted as follows : $\{\omega_1, \omega_2\}$ is a basis for $\Omega^1(S)$ if the tangency locus between the two forms reduces to $S$.
Below, we present some examples of Saito basis for $S : \{f = 0\}$. All of them will illustrate, in the sequel, various sensitivities of the Saito basis with respect to small perturbations of the curve $S$. In the whole article, we will keep the same numbering of the examples for the convenience of the reader.

**Example (1).** The simplest case is when $f = y^p - x^q$, that is $S_1 : \{f = 0\}$ is quasi-homogeneous. In fact, if $\omega_1 = qy\mathrm{d}x - px\mathrm{d}y$ and $\omega_2 = \mathrm{d}f$, then

$$\omega_1 \wedge \omega_2 = pqf\mathrm{d}x \wedge \mathrm{d}y$$

and $\{\omega_1, \omega_2\}$ is a basis for $\Omega^1(S_1)$.

**Example (2).** If $f = y^5 - x^6 + x^4y^3$ then $S_2 : \{f = 0\}$ is topologically quasi-homogeneous, that is, $S_2$ presents characteristic exponents $(5, 6)$, but not analytically equivalent to $y^5 - x^6 = 0$. One can show that the set $\{\omega_1, \omega_2\}$ where

$$\begin{aligned}\omega_1 &= \left(-6xy + \frac{16}{15}x^3y^2 - \frac{8}{5}xy^5\right)\mathrm{d}x + \left(5x^2 + \frac{4}{3}y^3 + \frac{4}{5}x^2y^4\right)\mathrm{d}y\\ \omega_2 &= \left(-6y^2 + \frac{8}{5}x^4 - \frac{12}{5}x^2y^3\right)\mathrm{d}x + \left(5xy + \frac{6}{5}x^3y^2\right)\mathrm{d}y\end{aligned}$$

satisfy $\omega_1 \wedge \omega_2 = 8f\mathrm{d}x \wedge \mathrm{d}y$, so $\{\omega_1, \omega_2\}$ is a Saito basis for $\Omega^1(S_2)$.

**Example (3).** The curve $S_3 : \{f = 0\}$ with $f = y^5 - x^{11} + x^6y^3$ is topologically equivalent to the any curve with characteristic exponents $(5, 11)$ and its strict transform is $S_2$. The set $\{\omega_1, \omega_2\}$ where

$$\begin{aligned}\omega_1 &= \left(605y^2 + 198xy^3 - 88x^6\right)\mathrm{d}x - \left(275xy + 66x^2y^2\right)\mathrm{d}y\\ \omega_2 &= \left(605x^4y + 150x^5y^2\right)\mathrm{d}x - \left(40y^3 + 275x^5 + 90x^6y\right)\mathrm{d}y\end{aligned}$$

satisfy $\omega_1 \wedge \omega_2 = (-24200 - 7920xy)f\mathrm{d}x \wedge \mathrm{d}y$, so $\{\omega_1, \omega_2\}$ is a Saito basis for $\Omega^1(S_3)$.

**Example (4).** The class of curve with characteristic exponents the form $(n, n+1)$ has been extensively studied by Zariski [11]. The curve $S_4$ given by

$$f = y^7 - x^8 - 7x^6y^2 - \frac{147}{8}x^4y^4$$

that, belongs to the latter class, will be shown of a peculiar interest. The forms

$$\begin{aligned}\omega_1 &= \left(8x^2y - \frac{147}{8}x^4 - \frac{3087}{4}x^2y^2 - \frac{21609}{16}y^4\right)\mathrm{d}x +\\ &\quad + \left(-7x^3 + \frac{7}{4}xy^2 + \frac{64827}{64}xy^3 + \frac{5145}{8}x^3y\right)\mathrm{d}y\\ \omega_2 &= \left(8xy^2 + \frac{1029}{8}x^3y\right)\mathrm{d}x + \left(-7x^2y + \frac{7}{4}y^3 - \frac{1029}{8}x^4\right)\mathrm{d}y.\end{aligned}$$

produce a Saito basis for $\Omega^1(S_4)$ because $\omega_1 \wedge \omega_2 = -\frac{151263}{64}f\mathrm{d}x \wedge \mathrm{d}y$.

Given a 1-form $\omega = A\mathrm{d}x + B\mathrm{d}y$ we denote by $\nu(\omega) = \min\{\nu(A), \nu(B)\}$ its algebraic multiplicity, where $\nu(H)$ indicates the multiplicity of $H \in \mathbb{C}\{x, y\}$ at $(0, 0) \in \mathbb{C}^2$. Among all the possible basis $\{\omega_1, \omega_2\}$ for $\Omega^1(S)$ we choose some that maximizes the sum $\nu(\omega_1) + \nu(\omega_2)$ that, following the Saito criterion, cannot be bigger than $\nu = \nu(f) = \nu(S)$. For such basis we denote

$$\nu_1 := \nu(\omega_1) \qquad \nu_2 := \nu(\omega_2).$$

The following result is immediate and identifies a new analytical invariant of $S$.

**Proposition 1.** *The couple $(\nu_1, \nu_2)$, up to order, is an analytical invariant of $S$.*

Remark that the pair $(\nu_1, \nu_2)$ is not a topological invariant. For instance, following the examples above, for $S_1$ with $p = 5$ and $q = 6$ we have $(\nu_1, \nu_2) = (1, 4)$. But the curve $S_3$ which is topological equivalent to $S_1$ has corresponding pair of multiplicities $(2, 2)$.

From now on, we consider $S:\{f=0\}$ singular and irreducible (a plane branch) with a Saito basis $\{\omega_1,\omega_2\}$ such that $\omega_i=A_i\mathrm{d}x+B_i\mathrm{d}y$ In particular, we have

$$A_1B_2-A_2B_1=uf \quad \text{and} \quad A_i\frac{\partial f}{\partial y}-B_i\frac{\partial f}{\partial x}=g_if \tag{2.1}$$

where $u(0,0)\neq 0$ and $g_i\in\mathbb{C}\{x,y\}$ is called the cofactor of $\omega_i$.
Applying a generic linear change of coordinates if necessary, we can suppose that for $i=1,2$, one has $\nu(A_i)=\nu(B_i)=\nu_i$ and in this coordinates $(x,y)$ the tangent cone of $f$, i.e. its $\nu$-jet, is $(y+\epsilon x)^\nu$. From now on, we denote $f^\nu=(y+\epsilon x)^\nu$.

**Example (1).** Consider the irreducible curve $S_1$. Suppose by symmetry that $p<q$, we have $\nu(A_1)=\nu(B_1)=\nu_1=1$ but $q-1=\nu(A_2)>p-1=\nu(B_2)=\nu_2$. Consider the change of coordinates $T(x,y)=(x,y-\epsilon x)$ with $\epsilon\neq 0$ we obtain $f_1=T^*(f)=(y-\epsilon x)^p-x^q$ and the Saito basis $\eta_1=T^*(\omega_1)$ and $\mathrm{d}f_1$

$$\begin{aligned}\eta_1 &= (q(y-\epsilon x)+\epsilon px)\mathrm{d}x-px\mathrm{d}y\\ \mathrm{d}f_1 &= (-\epsilon p(y-\epsilon x)^{p-1}-qx^{q-1})\mathrm{d}x+p(y-\epsilon x)^{p-1}\mathrm{d}y\end{aligned}$$

satisfying the above condition. In addition, $\eta_1\wedge\mathrm{d}f_1=pqf_1\mathrm{d}x\wedge\mathrm{d}y$, that is, $g_1=pq$ and $g_1=0$.

**Example (2).** For the curve $S_2$, we have

$$\omega_1\wedge\mathrm{d}f=(-30x-8xy^4)f\mathrm{d}x\wedge\mathrm{d}y \quad\text{and}\quad \omega_2\wedge\mathrm{d}f=(-30y-12x^2y^2)f\mathrm{d}x\wedge\mathrm{d}y,$$

that is, $g_1=-30x-8xy^4$ and $g_2=-30y-12x^2y^2$.

**Example (3).** Considering the curve $S_3$, we have $\nu(A_1)=\nu(B_1)=2$ but $5=\nu(A_2)>\nu(B_2)=3$. By the change of coordinates $T(x,y)=(x,x+y)$ we obtain $f_1=T^*(f)=(y+x)^5-x^{11}+x^6(y+x)^3$ and $\eta_i=T^*(\omega_i)=(A_i+B_i)\,\mathrm{d}x+B_i\mathrm{d}y$ with $\nu(A_1+B_1)=\nu(B_1)=2$ and $\nu(A_2+B_2)=\nu(B_2)=3$. In addition,

$$\eta_1\wedge\mathrm{d}f=(3025(x+y)+990x(y+x)^2)f_1\mathrm{d}x\wedge\mathrm{d}y$$

$$\eta_2\wedge\mathrm{d}f=(3025x^4+990x^5(y+x))f_1\mathrm{d}x\wedge\mathrm{d}y,$$

consequently, $g_1=3025(x+y)+990x(y+x)^2$ and $g_2=3025x^4+990x^5(y+x)$.

**Example (4).** Finally, for $S_4$ we find

$$\begin{aligned}\omega_1\wedge\mathrm{d}f &= \left(56x^2-\frac{151263}{16}y^3-\frac{21609}{4}x^2y\right)f\mathrm{d}x\wedge\mathrm{d}y\\ \omega_2\wedge\mathrm{d}f &= (56xy+1029x^3)f\mathrm{d}x\wedge\mathrm{d}y.\end{aligned}$$

Notice that any generator $\omega_i$ in a Saito basis $\{\omega_1,\omega_2\}$ has an isolated singularity, that is, $\gcd(A_i,B_i)=1$. In addition, by (2.1), we have that $\nu(g_i)\geq\nu_i-1$.

## 3. Good Saito basis and the Tjurina number for $S$.

As we mentioned before, given a Saito basis $\{\omega_1,\omega_2\}$ for $\Omega^1(S)$ we get $\nu_1+\nu_2\leq\nu$. In [5], the first author shows the following theorem:

**Theorem** (Generic Basis Theorem)**.** *In a fixed topological class $L$, generically any curve $S$ admits a Saito basis satisfying*

$$\nu_1 = \nu_2 = \tfrac{\nu}{2} \quad \textit{if } \nu = \nu(S) \textit{ is even}$$

$$\nu_1 = \nu_2 - 1 = \tfrac{\nu-1}{2} \quad \textit{if } \nu = \nu(S) \textit{ is odd.}$$

Notice that, generically $\nu_1 + \nu_2$ is maximum. Of course, Example 1 shows that we can obtain $\nu_1 + \nu_2 = \nu$ in other cases. This motives the following definition.

**Definition 2.** We say that $S$ (or $\Omega^1(S)$) admits a *good basis* if $\nu_1 + \nu_2 = \nu$.

This section is devoted to present some properties of a good basis. One of them is related with the index $\mathfrak{i}(S)$ we introduce in the sequel.
Let $E$ be the standard blowing-up of the origin in $\mathbb{C}^2$ with coordinates $(x, y)$ and suppose that, in the chart $(x_1, y_1)$ such that $E(x_1, y_1) = (x_1, x_1 y_1)$, the strict transform of $S$ goes through $(0, y_1)$.
As before, we consider $f^\nu = (y + \epsilon x)^\nu$.

**Definition 3.** For any $\omega = A\mathrm{d}x + B\mathrm{d}y \in \Omega^1(S)$, we denote by $i(\omega) \in \mathbb{N} \cup \{\infty\}$ the valuation given by

$$i(\omega) = \nu_{y_1=-\epsilon}\left(A^{(\nu(\omega))}(1, y_1) + y_1 B^{(\nu(\omega))}(1, y_1)\right)$$

where $\nu_{y_1=-\epsilon}(G)$ denotes de multiplicity of $G \in \mathbb{C}\{y_1\}$ at $-\epsilon \in \mathbb{C}$.
Moreover, we denote by $\mathfrak{i}(S) \in \mathbb{N}$ the integer

$$\mathfrak{i}(S) = \min_{\omega \in \Omega^1(S)} i(\omega).$$

The value $i(\omega)$ is nothing but the index $\mathrm{Ind}(\mathcal{F}, \mathrm{C}, 0)$ introduced in [2] for a germ of foliation $\mathcal{F}$ having $C$ as a smooth invariant curve.
Notice that for a given $\omega$, the index $i(\omega)$ is infinite if and only if $\omega$ is dicritical, that is, $A^{\nu(\omega)}(1, y_1) + y_1 B^{\nu(\omega)}(1, y_1) = 0$.
However, for any curve $\mathfrak{i}(S)$ is finite. Indeed, if $f$ is a reduced equation for $S$ then $\mathrm{d}f$ belongs to $\Omega^1(S)$ and it is not dicritical, thus $\mathfrak{i}(S) \leq i(\mathrm{d}f) < \infty$. In particular, if $\omega \in \Omega^1(S)$ is non dicritical, then $i(\omega) \leq \nu(\omega) + 1$.

**Example (1).** For $S_1$, with $p < q$, the considered Saito basis is a good basis. Moreover, $i(\omega_1) = 1$ and $i(\omega_2) = p$.

**Example (2).** Having a good basis is a property sensitive to perturbation. Indeed, for instance, the basis $\{\omega_1, \omega_2\}$ of $S_2$ computed in the example is not good. Besides that, we have $i(\omega_1) = 1$ and $i(\omega_2) = 2$.
Actually $S_2$ does not admit any good basis. In fact, if $S_2$ admits a good basis $\{\varpi_1, \varpi_2\}$, then we can suppose that $2 = \nu(\varpi_1), 3 = \nu(\varpi_2)$, $\omega_i^{(2)} = c_i \varpi_1^{(2)}$ where $c_i \in \mathbb{C} \setminus \{0\}$ and $\omega_i^{(2)} = A_i^2 \mathrm{d}x + B_i^2 \mathrm{d}y$ . In this way, $\omega_1^{(2)} = \frac{c_2}{c_1}\omega_2^{(2)}$, an absurd.

**Example (3).** Good basis is not preserved by blowing-up. In fact, $S_3$ has a good basis, but its strict transform is analytically equivalent to $S_2$ that does not admit good basis. For $S_3$ we have $i(\omega_1) = 2$ and $i(\omega_2) = 4$.

**Example (4).** Finally, $S_4$ does not have a good basis. We find $i(\omega_1) = 1$ and $i(\omega_2) = 2$.

The next result shows that if $S$ admits a good basis, the index $\mathfrak{i}(S)$ is achieved for one of its elements.

**Proposition 4.** *If $S$ admits a good basis $\{\omega_1,\omega_2\}$ then*

$$\mathfrak{i}(S)=\min\{i(\omega_1),i(\omega_2)\}.$$

*Proof.* By Saito criterion, one has $\omega_1\wedge\omega_2=uf$ with $u(0,0)\neq 0$. Since $\nu_1+\nu_2=\nu$, one has $\omega_1^{(\nu_1)}\wedge\omega_2^{(\nu_2)}\neq 0$, where $\omega_i^{(\nu_i)}=A_i^{\nu_i}\mathrm{d}x+B_i^{\nu_i}\mathrm{d}y$. In particular, both forms $\omega_1$ and $\omega_2$ cannot be dicritical and therefore $\min\{i(\omega_1),i(\omega_2)\}<\infty$.
Now, consider any form $\omega=P_1\omega_1+P_2\omega_2\in\Omega^1(S)$ with $P_i\in\mathbb{C}\{x,y\}$ and $m_i=\nu(P_i)$. Since $P_1^{(m_1)}\omega_1^{(\nu_1)}+P_2^{(m_2)}\omega_2^{(\nu_1)}$ cannot vanish identically, it is the homogeneous part of smallest degree of $\omega$. Therefore

$$\begin{aligned}i(\omega)=\nu_{y_1=-\epsilon}\Big(&P_1^{(m_1)}(1,y_1)\left(A_1^{\nu_1}(1,y_1)+y_1B_1^{\nu_1}(1,y_1)\right)\\&+P_2^{(m_2)}(1,y_1)\left(A_2^{\nu_2}(1,y_1)+y_1B_2^{\nu_2}(1,y_1)\right)\Big)\\\geq\min&\{i(\omega_1),i(\omega_2)\}.\end{aligned}$$

□

In the previous section, we remark that for an element $\omega_i$ in a Saito basis we get $\nu(g_i)\geq\nu_i-1$ and $i(\omega_i)\leq\nu(\omega_i)+1$. For good basis it is possible to obtain the following result.

**Lemma 5.** *Given a good basis $\{\omega_1,\omega_2\}$ for $S$, if $\nu(g_i)\geq\nu_i$ then $i(\omega_i)=\nu_i+1$.*

*Proof.* By symmetry let us consider $i=1$ and suppose that $\nu(g_1)\geq\nu_1$. The $(\nu_1-1+\nu)$-jet of

$$A_1\frac{\partial f}{\partial y}-B_1\frac{\partial f}{\partial x}=g_1f\quad\text{ is }\quad A_1^{(\nu_1)}\nu(y+\epsilon x)^{\nu-1}-B_1^{(\nu_1)}\nu\epsilon(y+\epsilon x)^{\nu-1}=0,$$

thus $A_1^{(\nu_1)}=\epsilon B_1^{(\nu_1)}$. On the other hand the $\nu$-jet of $A_1B_2-A_2B_1=uf$ where $u(0,0)\neq 0$ reduces to

$$A_1^{(\nu_1)}B_2^{(\nu_2)}-A_2^{(\nu_2)}B_1^{(\nu_1)}=B_1^{(\nu_1)}\left(\epsilon B_2^{(\nu_2)}-A_2^{(\nu_2)}\right)=u(0,0)(y+\epsilon x)^{\nu}.$$

Thus, there exists some constant $c\neq 0$ such that $B_1^{(\nu_1)}=c(y+\epsilon x)^{\nu_1}$. Therefore, $\omega_1$ can be written

$$\omega_1=\frac{c}{\nu_1+1}\mathrm{d}\left((y+\epsilon x)^{\nu_1+1}\right)+\text{h.o.t.}$$

thus $i(\omega_1)=\nu_1+1$. □

Notice that the above proof ensures that the inequality $\nu(g_i)\geq\nu_i$ cannot hold for both elements in a good basis. Moreover, given a good basis for $\Omega^1(S)$ we can always get a good basis with some nice properties. To do this we present the following lemmas.

**Lemma 6.** *If $\Omega^1(S)$ admits a good basis $\{\omega_1,\omega_2\}$, then we can suppose that*

$$i(\omega_1)=\mathfrak{i}(S)\qquad\text{and}\quad\nu(g_1)=\nu_1-1.$$

*Proof.* By symmetry we can suppose that $i(\omega_1) = \mathfrak{i}(S)$.
**Case 1.** If $i(\omega_2) = i(\omega_1)$, then, as mentioned above, for $i = 1$ or 2, one has $\nu(g_i) = \nu_i - 1$. Switching maybe the two forms, we can suppose that $\omega_1$ satisfies the conclusion of the lemma.
**Case 2.** Suppose now that $i(\omega_1) < i(\omega_2)$.
**Subcase 2.a** if $\nu_1 \leq \nu_2$, we consider, the family $\{\omega_1, \overline{\omega_2}\}$, where $\overline{\omega_2} = \omega_2 + cx^{\nu_2-\nu_1}\omega_1$ and $c \in \mathbb{C}$. For a generic value of $c$, we still have a good basis for $S$. Moreover, the $\nu_2$-jet of $\overline{\omega_2}$ is

$$\left(A_2^{(\nu_2)} + cx^{\nu_2-\nu_1}A_1^{(\nu_1)}\right)\mathrm{d}x + \left(B_2^{(\nu_2)} + cx^{\nu_2-\nu_1}B_1^{(\nu_1)}\right)\mathrm{d}y.$$

Thus, to evaluate its index, one writes

$$\begin{aligned} i(\overline{\omega_2}) &= \nu_{y=-\epsilon}\left(A_2^{(\nu_2)}(1,y) + cA_1^{(\nu_1)}(1,y) + y\left(B_2^{(\nu_2)}(1,y) + cB_1^{(\nu_1)}(1,y)\right)\right) \\ &= \nu_{y=-\epsilon}\left(A_2^{(\nu_2)}(1,y) + yB_2^{(\nu_2)}(1,y) + c\left(A_1^{(\nu_1)}(1,y) + yB_1^{(\nu_1)}(1,y)\right)\right) \quad = i(\omega_1). \end{aligned}$$

Thus we are led to the previous case (1).
**Subcase 2.b.** Finally, if $\nu_1 > \nu_2$, suppose that $\nu(g_1) \geq \nu_1$, then by Lemma 5 we have $i(\omega_1) = \nu_1 + 1$. Consequently $i(\omega_1) > \nu_2 + 1$ and then $i(\omega_2) > \nu_2 + 1$. If $\omega_2$ is not dicritical, the inequality above leads to a contradiction, thus $\omega_2$ is dicritical. Therefore, it can be seen that $\nu(g_2) = \nu_2 - 1$. Let us consider now $\overline{\omega}_1 = \omega_1 + x^{\nu_1-\nu_2}\omega_2$. Then, the family $\{\overline{\omega}_1, \omega_2\}$ is still a good basis and one has

$$\begin{aligned} \overline{\omega}_1 \wedge \mathrm{d}f &= \overline{g}_1 f \mathrm{d}x \wedge \mathrm{d}y \qquad \text{with } \nu(\overline{g}_1) = \nu_1 - 1 \\ i(\overline{\omega}_1) &= i(\omega_1) = \mathfrak{i}(S). \end{aligned}$$

□

In addition, from a basis for $\Omega^1(S)$ we can get a basis satisfying the following lemma.

**Lemma 7.** *Given a basis $\{\omega_1, \omega_2\}$ for $\Omega^1(S)$ with $i(\omega_1) \leq i(\omega_2)$ we can suppose that*

$$\gcd\left(B_i, \frac{\partial f}{\partial y}\right) = 1, \quad \textit{for } i = 1, 2.$$

*Proof.* Suppose that $H = \gcd\left(B_1, B_2, \frac{\partial f}{\partial y}\right)$. Since by (2.1) $A_1B_2 - A_2B_1 = uf$, $H$ would divide $f$. As $\frac{\partial f}{\partial y}$ and $f$ are relatively prime, we get

$$\gcd\left(B_1, B_2, \frac{\partial f}{\partial y}\right) = 1. \tag{3.1}$$

Now consider the family $\{\overline{\omega_1} = \omega_1 + P_1\omega_2, \overline{\omega_2} = \omega_2 + P_2\omega_1\}$ where $P_i \in \mathbb{C}\{x, y\}$ with $\nu(P_i) \gg 1$. Note that for $P_i$ of algebraic multiplicity big enough, the forms

$$\begin{aligned} \overline{\omega_1} &= (A_1 + P_1A_2)\mathrm{d}x + (B_1 + P_1B_2)\mathrm{d}y = \overline{A_1}\mathrm{d}x + \overline{B_1}\mathrm{d}y \\ \overline{\omega_2} &= (A_2 + P_2A_1)\mathrm{d}x + (B_2 + P_2B_1)\mathrm{d}y = \overline{A_2}\mathrm{d}x + \overline{B_2}\mathrm{d}y \end{aligned}$$

satisfy $\nu(\overline{\omega_i}) = \nu\left(\overline{A_i}\right) = \nu\left(\overline{B_i}\right) = \nu_i$, and $i(\omega_1) = i(\overline{\omega_1}) \leq i(\overline{\omega_2})$.
Moreover, $\{\overline{\omega_1}, \overline{\omega_2}\}$ is a basis for $\Omega^1(S)$. Now the relation (3.1) ensures that for a generic choice of the $P_i's, i = 1, 2$ - in the sense of Krull -, one has

$$\gcd\left(\overline{B_i}, \frac{\partial f}{\partial y}\right) = 1.$$

□

As a consequence we obtain the following.

**Corollary 8.** *For any basis $\{\omega_1, \omega_2\}$ for $\Omega^1(S)$ satisfying the previous Lemma we have*

$$\gcd(B_i, g_i) = \gcd\left(\frac{\partial f}{\partial y}, g_i\right) = 1.$$

*Proof.* As $A_i\frac{\partial f}{\partial y} - B_i\frac{\partial f}{\partial x} = g_i f$, if $1 \neq H = \gcd(B_i, g_i)$ then $H$ must divide $A_i\frac{\partial f}{\partial y}$. By the previous lemma, $\gcd(B_i, \frac{\partial f}{\partial y}) = 1$ so $H$ divides $A_i$, a contradiction because $\omega_i$ has an isolated singularity.
Suppose $H' = \gcd\left(\frac{\partial f}{\partial y}, g_i\right)$, so $H'$ divides $B_i\frac{\partial f}{\partial x}$. As $\gcd\left(\frac{\partial f}{\partial y}, \frac{\partial f}{\partial x}\right) = \gcd(B_i, g_i) = 1$, we must have $H' = 1$. □

In particular, the above lemma allow us to consider a good Saito basis $\{\omega_1, \omega_2\}$ with $\mathfrak{i}(S) = i(\omega_1)$ and $\gcd\left(B_i, \frac{\partial f}{\partial y}\right) = \gcd(B_i, g_i) = \gcd\left(\frac{\partial f}{\partial y}, g_i\right) = 1$.

**Lemma 9.** *If $S : \{f = 0\}$ admits a good basis satisfying the previous conditions, then the intersection of the tangent cone of*

(1) *$g_1$ and $g_2$,*
(2) *$B_i$ and $g_i$, for $i = 1, 2$,*
(3) *$B_i$ and $\frac{\partial f}{\partial y}$, for $i = 1, 2$*

*is empty or equal to $y + \epsilon x = 0$.*

*Proof.* The $\nu$-jet of (2.1) is

$$A_1^{(\nu_1)}B_2^{(\nu_2)} - A_2^{(\nu_2)}B_1^{(\nu_1)} = c\,(y + \epsilon x)^{\nu}\,. \tag{3.2}$$

where $c \neq 0$ and $\epsilon \in \mathbb{C}$. Now, for $i = 1, 2$, both following relations $A_i^{(\nu_i)} - \epsilon B_i^{(\nu_i)} = 0$ cannot be true all together since it would yield a contradiction with the relation (3.2). Suppose the relation above is not true for at least $i = 1$, then the cofactor relations ensures that

$$A_1^{(\nu_1)} - \epsilon B_1^{(\nu_1)} = \frac{1}{\nu} g_1^{(\nu(g_1))}\,(y + \epsilon x)\,.$$

Combining the above relations yields

$$g_1^{(\nu(g_1))}B_2^{(\nu_2)} - g_2^{(\nu(g_2))}B_1^{(\nu_1)} = c\nu\,(y + \epsilon x)^{\nu-1}\,, \quad \text{or} \quad g_1^{(\nu(g_1))}B_2^{(\nu_2)} = c\nu\,(y + \epsilon x)^{\nu-1}$$

from which is derived (1) and (2). The point (3) follows from the fact that the tangent cone of $\frac{\partial f}{\partial y}$ and $f$ are the same. □

In what follows we denote by $I_P(G,H)$ the intersection multiplicity of $G,H \in \mathbb{C}\{x,y\}$ at the point $P \in \mathbb{C}^2$. If $P=(0,0)$ then we write $I(G,H) := I_P(G,H)$, that is, $I(G,H) = \dim_{\mathbb{C}} \frac{\mathbb{C}\{x,y\}}{(G,H)}$.
An important topological invariant for $S:\{f=0\}$ is the Milnor number $\mu$ which can be computed by

$$\mu := I\left(\frac{\partial f}{\partial y}, \frac{\partial f}{\partial x}\right) = \sum_{i=1}^{N} \nu_{(i)}(\nu_{(i)}-1) \tag{3.3}$$

where $\nu_{(i)};\ i=1,\ldots,N$ denote the sequence of multiplicities in the canonical resolution of $S$. In addition, by Zariski (see (2.4) in [11]), we have

$$I\left(\frac{\partial f}{\partial y}, f\right) = \mu + \nu - 1. \tag{3.4}$$

Combining the Lemma 6 and the above result we can obtain an expression for $I(g_1,g_2)$.

**Lemma 10.** *If $g_1$ and $g_1$ are the cofactors for a good basis for $\Omega^1(S)$, then $I(g_1,g_2)$ is finite and*

$$I\left(g_1,g_2\right) = I\left(\frac{\partial f}{\partial y}, B_1\right) - I\left(B_1,g_1\right) - \nu + 1.$$

*Proof.* By Lemma 6 we have $\nu(g_1) = \nu_1 - 1 < \nu$. As $f$ is irreducible it follows that $\gcd(f,g_1)=1$ and $I\left(f\frac{\partial f}{\partial y}, g_1\right) < \infty$. So, from (2.1) that

$$I\left(f\frac{\partial f}{\partial y}, g_1\right) = I\left(A_1B_2\frac{\partial f}{\partial y} - A_2B_1\frac{\partial f}{\partial y}, g_1\right) = I\left(B_1B_2\frac{\partial f}{\partial x} - A_2B_1\frac{\partial f}{\partial y}, g_1\right) = I\left(B_1g_2f, g_1\right).$$

Hence,

$$I(g_1,g_2) = I\left(\frac{\partial f}{\partial y}, g_1\right) - I(B_1,g_1). \tag{3.5}$$

The Corollary 8 insures that $\frac{\partial f}{\partial y}$ and $g_1$ are coprime. So, by (3.5) and using (3.4) we obtain

$$\begin{aligned} I\left(g_1,g_2\right) &= I\left(\frac{\partial f}{\partial y}, g_1\right) + I\left(\frac{\partial f}{\partial y}, f\right) - I\left(\frac{\partial f}{\partial y}, f\right) - I\left(B_1,g_1\right) \\ &= I\left(\frac{\partial f}{\partial y}, g_1 f\right) - I\left(\frac{\partial f}{\partial y}, f\right) - I\left(B_1,g_1\right) \\ &= I\left(\frac{\partial f}{\partial y}, A_1\frac{\partial f}{\partial y} - B_1\frac{\partial f}{\partial x}\right) - (\mu+\nu-1) - I\left(B_1,g_1\right) \\ &= I\left(\frac{\partial f}{\partial y}, B_1\right) - \nu + 1 - I\left(B_1,g_1\right). \end{aligned}$$

□

Let us consider the Tjurina number $\tau$ of a plane curve $S:\{f=0\}$, that is,

$$\tau := \dim_{\mathbb{C}} \frac{\mathbb{C}\{x,y\}}{\left(f, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial x}\right)}.$$

Zariski (see Theorem 1 in [10]) considered the torsion submodule $T\Omega^1_{\mathcal{O}/\mathbb{C}}$ of the Kähler differential module $\Omega^1_{\mathcal{O}/\mathbb{C}}$ over $\mathcal{O}=\frac{\mathbb{C}\{x,y\}}{(f)}$ and he showed that $\tau=\dim_{\mathbb{C}} T\Omega^1_{\mathcal{O}/\mathbb{C}}$. On the other hand, Michler (Theorem 1 in [7]) proved that $T\Omega^1_{\mathcal{O}/\mathbb{C}}$ is isomorphic as $\mathcal{O}$-module, to $\frac{\left(\frac{\partial f}{\partial y},\frac{\partial f}{\partial x}\right):(f)}{\left(\frac{\partial f}{\partial y},\frac{\partial f}{\partial x}\right)}$. As $\left(\frac{\partial f}{\partial y},\frac{\partial f}{\partial x}\right):(f)$ is precisely the cofactor ideal of $S$, that is, $(g_1,g_2)$, one has

$$\tau=\dim_{\mathbb{C}}\frac{(g_1,g_2)}{\left(\frac{\partial f}{\partial y},\frac{\partial f}{\partial x}\right)}=\dim_{\mathbb{C}}\frac{\mathbb{C}\{x,y\}}{\left(\frac{\partial f}{\partial y},\frac{\partial f}{\partial x}\right)}-\dim_{\mathbb{C}}\frac{\mathbb{C}\{x,y\}}{(g_1,g_2)}=\mu-I(g_1,g_2),$$

that is,

$$\mu-\tau=I(g_1,g_2).$$

Denoting $\widetilde{\mu}$ the Milnor number of $\widetilde{S}$, the strict transform of $S$ under a standard blowing up of the origin of $\mathbb{C}^2$, we provide in the next theorem a precise relation between $\mu-\tau$ and $\widetilde{\mu}-\widetilde{\tau}$ by means of the analytic invariants we have introduced previously for curves that admit a good basis.

**Theorem 11.** *If $S$ admits a good basis, then*

$$\mu-\tau=\widetilde{\mu}-\widetilde{\tau}+(\nu_1-1)(\nu_2-1)+\mathfrak{i}(S)-1.$$

*Proof.* By symmetry, one can suppose $\mathfrak{i}(S)=\min\{i(\omega_1),i(\omega_2)\}=i(\omega_1)$. By Lemma 9 and the Max-Noether formula one has,

$$\mu-\tau=I(g_1,g_2)=I_{(0,-\epsilon)}(\tilde{g}_1,\tilde{g}_2)+\nu(g_1)\nu(g_2),$$

where $\widetilde{H}:=E^*(H)$ and $E$ denotes the standard blowing-up of the origin in $\mathbb{C}^2$. In addition, the previous lemma and Lemma 9, yield

$$\begin{aligned}I(g_1,g_2)&=I\left(\frac{\partial f}{\partial y},B_1\right)-I(B_1,g_1)-\nu+1\\&=I_{(0,-\epsilon)}\left(\frac{\tilde{\partial f}}{\partial y},\tilde{B}_1\right)-I_{(0,-\epsilon)}\left(\tilde{B}_1,\tilde{g}_1\right)+\nu\left(\frac{\partial f}{\partial y}\right)\nu(B_1)-\nu(B_1)\nu(g_1)-\nu+1.\end{aligned}$$

If $\tilde{\omega}_i=\frac{E^*\omega_i}{x^{\nu_i}}$, then the Saito criterion yields $x^{\nu_1}\tilde{\omega}_1\wedge x^{\nu_2}\tilde{\omega}_2=\widetilde{u}x^{\nu}\tilde{f}x\mathrm{d}x\wedge\mathrm{d}y$. Since we have a good basis, that is, $\nu_1+\nu_2=\nu$, one has $\tilde{\omega}_1\wedge\tilde{\omega}_2=u\tilde{f}x\mathrm{d}x\wedge\mathrm{d}y$. Locally around $(0,-\epsilon)$ for $i=1,2$ we have

$$\tilde{\omega}_i=(A_i^{\nu_i}(1,y)+yB_i^{\nu_i}(1,y)+x(\cdots))\,\mathrm{d}x+x(B_i^{\nu_i}(1,y)+(\cdots))\,\mathrm{d}y$$

We notice that the form

$$\varpi_2=\frac{1}{x}\left(\tilde{\omega}_2-\frac{A_2^{\nu_2}(1,y)+yB_2^{\nu_2}(1,y)}{A_1^{\nu_1}(1,y)+yB_1^{\nu_1}(1,y)}\tilde{\omega}_1\right)$$

is holomorphic at $(0,-\epsilon)$ and $\{\tilde{\omega}_1,\varpi_2\}$ is a Saito basis for $\widetilde{S}:\{\tilde{f}=0\}$. A computation shows that the cofactor associated to $\tilde{\omega}_1$ is written $g_1^{'}=\tilde{g}_1+\nu\tilde{B}_1$. Moreover, one has $\tilde{\omega}_1=\left(\tilde{A}_1+y\tilde{B}_1\right)\mathrm{d}x+x\tilde{B}_1\mathrm{d}y=A'\mathrm{d}x+B'\mathrm{d}y$. Now,

$$(A_1^{\nu_1}(1,y)+yB_1^{\nu_1}(1,y)+x(\cdots))\frac{\partial\tilde{f}}{\partial y}-x\tilde{B}_1\frac{\partial\tilde{f}}{\partial x}=g_1'\tilde{f}.$$

If $x$ divides $g_1'$ then $\tilde{\omega}_1$ would be dicritical and this is not possible. Therefore,

$$I_{(0,-\epsilon)}\left(x, g_1'\tilde{f}\right) = I_{(0-\epsilon)}\left(A_1^{\nu_1}(1,y) + yB_1^{\nu_1}(1,y), x\right) + I_{(0,-\epsilon)}\left(x, \frac{\partial \tilde{f}}{\partial y}\right).$$

and, by Corollary 8, $I_{(0,-\epsilon)}(x, g_1') = i(\omega_1) - 1 = \mathfrak{i}(S) - 1$.
Notice that $\tilde{B}_1$ and $g_1'$ cannot have a common divisor, since it would be a common divisor of $\tilde{g}_1$ and $\tilde{B}_1$ that is impossible by Lemma 7. So,

$$I_{(0,-\epsilon)}\left(\tilde{B}_1, \tilde{g}_1\right) = I_{(0,-\epsilon)}\left(x\tilde{B}_1, g_1'\right) - \mathfrak{i}(S) + 1 = I_{(0,-\epsilon)}\left(B_1', g_1'\right) - \mathfrak{i}(S) + 1.$$

Moreover,

$$\begin{aligned} I_{(0,-\epsilon)}\left(\frac{\partial \tilde{f}}{\partial y}, \tilde{B}_1\right) &= I_{(0,-\epsilon)}\left(\frac{\partial \tilde{f}}{\partial y}, B_1'\right) - I_{(0,-\epsilon)}\left(\frac{\partial \tilde{f}}{\partial y}, x\right) \\ &= I_{(0,-\epsilon)}\left(\frac{\partial \tilde{f}}{\partial y}, B_1'\right) - I_{(0,-\epsilon)}\left(\tilde{f}, x\right) + 1. \end{aligned}$$

So, as $\frac{\partial \tilde{f}}{\partial y} = \widetilde{\frac{\partial f}{\partial y}}$ and combining all the above relation yields

$$\begin{aligned} \mu - \tau = {} & I_{(0,-\epsilon)}\left(\frac{\partial \tilde{f}}{\partial y}, B_1'\right) - I_{(0,-\epsilon)}\left(\tilde{f}, x\right) + 1 - \left(I_{y=-\epsilon}\left(B_1', g_1'\right) - \mathfrak{i}(S) + 1\right) \\ & + \nu\left(\frac{\partial f}{\partial y}\right)\nu(B_1) - \nu(B_1)\nu(g_1) - \nu + 1 \\ = {} & I_{(0,-\epsilon)}(g_1', g_2') + (\nu - 1)\nu_1 - \nu_1\nu(g_1) - \nu + \mathfrak{i}(S). \end{aligned}$$

As $I_{(0,-\epsilon)}(g_1', g_2') = \widetilde{\mu} - \widetilde{\tau}$ and $\nu(g_1) = \nu_1 - 1$, we obtain finally

$$\mu - \tau = \widetilde{\mu} - \widetilde{\tau} + (\nu_1 - 1)(\nu_2 - 1) + \mathfrak{i}(S) - 1.$$

□

Let us analyze the examples previously considered.

**Example (1).** For $S_1$ we have a good basis with $\nu_1 = 1$, $\nu_2 = p - 1$ and $\mathfrak{i}(S_1) = 1$, then $\mu - \tau = 0$ as classically known.

**Example (2).** Notice that for $S_2$ we have $\mathfrak{i}(S_2) = i(\omega_1) = 1$, $\nu_1 = \nu_2 = 2$ and $\widetilde{S}_3$ is regular, so $\widetilde{\mu} - \widetilde{\tau} = 0$. In this way,

$$1 = I(g_1, g_2) = \mu - \tau = 0 + (2-1)(2-1) + 1 - 1.$$

So, the formula in the previous theorem holds although $S_2$ does not admit any good basis.

**Example (3).** For $S_3$ we get $\mathfrak{i}(S_3) = i(\omega_1) = 2$, $\nu_1 = 2, \nu_2 = 3$ and $\widetilde{S}_3$ is analytically equivalent to $S_2$, so $\widetilde{\mu} - \widetilde{\tau} = 1$. In this way,

$$4 = I(g_1, g_2) = \mu - \tau = 1 + (2-1)(3-1) + 2 - 1.$$

**Example (4).** As we presented above, $S_4$ does not have a good basis. We have $\mathfrak{i}(S_4) = i(\omega_1) = 1$, $\nu_1 = \nu_2 = 3$ and $\widetilde{\mu} - \widetilde{\tau} = 0$, but in this case,

$$5 = I(g_1, g_2) = \mu - \tau \neq 4 = 0 + (3-1)(3-1) + 1 - 1.$$

A more detailed analysis shows that Lemma 9 is not valid in this case because the intersection of the tangent cone of $g_1$ and $g_1$ is $x=0$ that is distinct to the tangent cone $y=0$ of $S_4$.

## 4. The minimal Tjurina number and the Dimca-Greuel question for plane branches.

Given a curve $S$, we denote by $L=L(S)$ its topological class. Although the Milnor number is constant in $L$, the same is not true for the Tjurina number $\tau(S)$. On the other hand, as $\tau(S)$ is upper semicontinuous, the minimum value $\tau_{\min}$ for curves in $L$ is achieved generically and it should be computed by the sole topological data (see Chapitre III, Appendice of [11] by Teissier).
For a topological class $L$ given by characteristic exponents $(\beta_0,\beta_1)$, Delorme in [3] presented a formula for the dimension of the generic component of the Moduli space that allow us to compute $\tau_{\min}$. For an arbitrary topological class, Peraire (see [8]) presented an algorithm to compute the $\tau_{\min}$ using the flag of the Jacobian ideal.
In this section, using the last theorem and results of [5], we give an alternative method to compute $\tau_{\min}$ in a fixed topological class $L$ and as a bonus we are able to answer a question of Dimca-Greuel for the irreducible plane curves.
If $S$ admits a good basis we can not insure that the same is valid for $\widetilde{S}$ (see Example (3)). However, this property is true generically.

**Theorem 12.** *Let $L$ the topological class of plane branch given by the characteristic exponents $(\beta_0,\beta_1,\dots,\beta_s)$, $\tau_{\min}$ the minimal Tjurina number in $L$ and $\widetilde{\tau}_{\min}$ the minimal Tjurina number in $\widetilde{L}$. If $S$ is generic in $L$, then*

$$\mu-\tau_{\min}=\widetilde{\mu}-\widetilde{\tau}_{\min}+\left(\left[\frac{\beta_0}{2}\right]-1\right)\left(\beta_0-\left[\frac{\beta_0}{2}\right]-1\right)+\mathrm{i}(S)-1. \tag{4.1}$$

*Moreover, if $n=\left[\frac{\beta_1}{\beta_1-\beta_0}\right]$, then $\mathrm{i}(S)=\left[\frac{\beta_0}{2}\right]+1-p_1(S)$, where $p_1(S)$ can be computed in the following way:*

- *if $\beta_0$ is even then* $p_1(S)=\begin{cases} 1 & \text{if } n=2\\ 1 & \text{if } \beta_1 \text{ is even}\\ \dfrac{n-1}{2} & \text{if } \beta_1 \text{ is odd and } n \text{ odd}\\ \dfrac{n-2}{2} & \text{if } \beta_1 \text{ is odd and } n \text{ is even}\end{cases}$
- *if $\beta_0$ is odd then* $p_1(S)=\begin{cases} 0 & \text{if } n=2\\ 1 & \text{if } \beta_1 \text{ is odd}\\ \dfrac{n-3}{2} & \text{if } \beta_1 \text{ is even and } n \text{ odd}\\ \dfrac{n-2}{2} & \text{if } \beta_1 \text{ is even and } n \text{ is even.}\end{cases}$

*Proof.* Suppose that $\beta_0=\nu(S)$ is even. According to the Generic Basis Theorem, $S$ admits a good basis $\{\omega_1',\omega_2'\}$ with $\nu(\omega_1')=\nu(\omega_2')=\frac{\beta_0}{2}$. For generic $\alpha_1,\alpha_2\in\mathbb{C}$ $\{\omega_1=\omega_1'+\alpha_2\omega_2',\omega_2=\omega_2'+\alpha_1\omega_1'\}$ remain a good basis with $\nu_1=\nu_2=\frac{\beta_0}{2}$ and

$i(\omega_1) = i(\omega_2)$.
Now, according to [5] - using the notations of the mentioned paper, it refers to the case $\delta_1 = 0$ and $\delta_2 = 1$ - we obtain $\nu_1 + 1 = \frac{\beta_0}{2} + 1 = \sum_{q\in\mathbb{P}^1} \mathrm{Ind}(\widetilde{\mathcal{F}}, C, q) = i(\omega_1) + p_1(S)$, that is,

$$\mathfrak{i}(S) = i(\omega_1) = \nu_1 + 1 - p_1(S) = \frac{\beta_0}{2} + 1 - p_1(S) = \left[\frac{\beta_0}{2}\right] + 1 - p_1(S)$$

where $p_1(S)$ is described in [5].
Now, suppose $\beta_0$ is odd and let $\{\omega_1', \omega_2'\}$ be a Saito basis for $S \cup l$ with $l$ a generic line that without loss of generality can be considered $x = 0$. As $\nu(S \cup l)$ is even, by the same above argument, we can suppose that

$$\nu(\omega_1') = \nu(\omega_2') = \frac{\beta_0 + 1}{2} = \left[\frac{\beta_0}{2}\right] + 1 \quad \text{and} \quad \mathfrak{i}(S \cup l) = i(\omega_1') = i(\omega_2').$$

Denoting $\omega_i' = (a_i(y) + x(\cdots))\,\mathrm{d}x + x(\cdots)\,\mathrm{d}y$ and considering generic $\alpha_1, \alpha_2 \in \mathbb{C}$ we obtain a good Saito basis $\{\omega_1 = \omega_1' + \alpha_2\omega_2', \omega_2 = \omega_2' + \alpha_1\omega_1'\}$ such that $\nu(a_1(y) + \alpha_2 a_2(y)) = \nu(a_2(y) + \alpha_1 a_1(y))$,

$$i(\omega_1) = i(\omega_1') = i(\omega_2') = i(\omega_2) \text{ and } \nu(\omega_1) = \nu(\omega_1') = \nu(\omega_2') = \nu(\omega_2).$$

Now the family

$$\left\{\omega_1, \frac{1}{x}\left(\omega_2 - \frac{a_2(y) + \alpha_1 a_1(y)}{a_1(y) + \alpha_2 a_2(y)}\omega_1\right)\right\}$$

is a good Saito basis for $S$. Finally, since $i\left(\frac{1}{x}\left(\omega_2 - \frac{a_2(y)+\alpha_1 a_1(y)}{a_1(y)+\alpha_2 a_2(y)}\omega_1\right)\right) \geq i(\omega_1)$, one has $\mathfrak{i}(S) = i(\omega_1)$. By the description of $p_1(S \cup l)$ given in [5] - using the notations of the article, it refers to the case $\delta_1 = 1$ and $\delta_2 = 1$ - we get

$$\mathfrak{i}(S) = i(\omega_1) = \frac{\nu(S) + 1}{2} - p_1(S) = \left[\frac{\beta_0}{2}\right] + 1 - p_1(S).$$

Thus, the proof of the formula is a consequence of Theorem 11 noticing that by the Generic Basis Theorem we have $\nu_1 = \left[\frac{\beta_0}{2}\right]$ and $\nu_2 = \beta_0 - \left[\frac{\beta_0}{2}\right]$. □

**Example (5).** In [8], Peraire computed the minimum Tjurina number for the topological class whose characteritic exponents are $(9, 12, 17)$. After five blowing-ups, we obtain a curve with multiplicity 2. The corresponding characteristics exponents of the sequence of blown-up curves are $(3, 14)$, $(3, 11)$, $(3, 8)$, $(3, 5)$, $(2, 3)$. Applying inductively the formula (4.1), one accumulates contribution to the difference $\mu - \tau_{\min}$. Actually, it can be seen that the respective contributions are 15, 1, 1, 1, 0, 0. Thus $\tau_{\min} = \mu - 18 = 98 - 18 = 80$ which coincides with the computation of Peraire.

The last theorem allow us obtain a formula to compute the minimum Tjurina number in a topological class using the multiplicity sequence.

**Corollary 13.** *Let $L$ a topological class of a singular plane branch determined by the multiplicity sequence $\nu_{(1)}, \nu_{(2)}, \ldots, \nu_{(N)}, \nu_{(N+1)} = 1, \ldots$. The minimal Tjurina number achieved in $L$ is*

$$\tau_{\min} = \sum_{i=1}^{N}\left(\nu_{(i)}^2 + \left[\frac{\nu_{(i)}}{2}\right]\left(\left[\frac{\nu_{(i)}}{2}\right] - \nu_{(i)} - 1\right) - 1 + p_1(S_{(i)})\right)$$

*where $S_{(i)}$ denote the curve with multiplicity $\nu_{(i)}$ in the canonical resolution process for a generic curve in $L$.*

*Proof.* Applying inductively the formula presented in the last theorem and using that $\mathfrak{i}\left(S_{(i)}\right) = \left[\frac{\nu_{(i)}}{2}\right] + 1 - p_1(S_{(i)})$ yields

$$\tau_{\min} = \mu - \sum_{i=1}^{N} \left( \left( \left[\frac{\nu_{(i)}}{2}\right] - 1 \right) \left( \nu_{(i)} - \left[\frac{\nu_{(i)}}{2}\right] - 1 \right) - \left( \mathfrak{i}\left(S_{(i)}\right) - 1 \right) \right)$$

$$= \mu + \sum_{i=1}^{N} \left( \left[\frac{\nu_{(i)}}{2}\right] \left( \left[\frac{\nu_{(i)}}{2}\right] - \nu_{(i)} - 1 \right) + \nu_{(i)} - 1 + p_1(S_{(i)}) \right). \tag{4.2}$$

As $\mu = \sum_{i=1}^{N} \nu_{(i)} \left(\nu_{(i)} - 1\right)$, we get the proof. □

In [4], Dimca and Greuel present an interesting question about the Tjurina number for curves in a given topological class $L$. More specifically, they ask if $4\tau(S) > 3\mu(S)$ for any curve in $L$.
As the Tjurina number is semicontinuous in $L$ and we have obtained a formula for the $\tau_{\min}$, we are able to given a lower bound for the Tjurina number in terms of the Milnor number and it answered positively the previous question for the irreducible case.

**Corollary 14.** *Let $S$ be a singular irreducible plane curve. Then*

$$\tau\left(S\right) \geq \frac{3}{4}\mu\left(S\right) + \frac{\sqrt{1+4\mu(S)}-1}{8}.$$

*In particular, $4\tau(S) > 3\mu(S)$.*

*Proof.* We denote $\mu = \mu(S)$. It is sufficient to show the inequality for the $\tau_{\min}$. By (4.2), the relation below holds

$$4\tau_{\min} - 3\mu = \mu + 4\sum_{i=1}^{N} \left( \left[\frac{\nu_{(i)}}{2}\right] \left( \left[\frac{\nu_{(i)}}{2}\right] - \nu_{(i)} - 1 \right) + \nu_{(i)} - 1 + p_1(\nu_{(i)}) \right).$$

Now, using that $\mu = \sum_{i=1}^{N} \nu_{(i)} \left(\nu_{(i)} - 1\right)$ and $4\left[\frac{\nu_{(i)}}{2}\right] \left( \left[\frac{\nu_{(i)}}{2}\right] - \nu_{(i)} - 1 \right) = -\nu_{(i)}^2 - 2\nu_{(i)} + \delta_i$ with $\delta_i = 0$ if $\nu_{(i)}$ is even and $\delta_i = 3$ if $\nu_{(i)}$ is odd, we obtain

$$4\tau_{\min} - 3\mu = \sum_{i=1}^{N} \left( \nu_{(i)} + \delta_i + 4(p_1(S_{(i)}) - 1) \right).$$

Now, by Theorem 12 we have that:

- if $\nu_{(i)}$ is even, then $p_1(S) \geq 1$ and $\nu_{(i)} + 0 + 4(p_1(\nu_{(i)}) - 1) \geq \nu_{(i)}$,
- if $\nu_{(i)}$ is odd, then $p_1(S) \geq 0$ and $\nu_{(i)} + 3 + 4(p_1(\nu_{(i)}) - 1) \geq \nu_{(i)} - 1$.

So, the following inequality follows

$$4\tau_{\min} - 3\mu \geq \sum_{i=1}^{N}(\nu_{(i)} - 1). \tag{4.3}$$

As $\mu = \sum_{i=1}^{N}(\nu_{(i)} - 1)^2 + \sum_{i=1}^{N}\left(\nu_{(i)} - 1\right)$ we get $4\tau_{\min} - 3\mu \geq \mu - \sum_{i=1}^{N}(\nu_{(i)} - 1)^2$. Using (4.3), that is, $-(4\tau_{\min} - 3\mu)^2 \leq -\left(\sum_{i=1}^{N}(\nu_{(i)} - 1)\right)^2 \leq -\sum_{i=1}^{N}(\nu_{(i)} - 1)^2$, we obtain $4\tau_{\min} - 3\mu \geq \mu - (4\tau_{\min} - 3\mu)^2$ and consequently

$$\tau\left(S\right) \geq \tau_{\min} \geq \frac{3}{4}\mu + \left(\frac{-1 + \sqrt{1 + 4\mu}}{8}\right).$$

□

**Example (6).** Let us consider the topological class $L$ determined by the characteristic exponents $(141, 142)$. The Milnor number for any curve in $L$ is $\mu = (141 - 1)(142 - 1) = 19740$. Using the lower bound presented in the last result we obtain $\tau_{\min} \geq 14840$. For this topological class it follows by the Delorme result (*cf.* [3]) that $\tau_{\min} = 14910$.

While we submit the first version of this paper to Arxiv, we discover that, at the same time, a positive answer for the Dimca-Greuel question was obtained by Alberich-Carramiñana *et al.* and published in Arxiv [1] a few days before. Although the methods are a bit different, the key ingredient is still the formula for the generic dimension of the moduli space obtained in [5].

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