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Robust optimization for lot-sizing problems under yield uncertainty

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Abstract

Production yield can be highly volatile and uncertain, especially in industries where exogenous and environmental factors such as the climate or raw material quality can impact the manufacturing process. Thus, for production planning, it is necessary to take into account the production yield uncertainty to obtain robust and efficient plans. In this paper, we consider lot-sizing problems under yield uncertainty. We propose a multi-period, single-item lot-sizing problem with backorder and yield uncertainty via a robust optimization methodology. First, we formulate a robust model under a budgeted uncertainty set, which is optimized under the worst case perspective to ensure the feasibility of the proposed plan for any realization of the yield described by the uncertainty set. Second, we analyze the structure of the optimal lot-sizing solution, and we derive the optimal robust policy for the special case of the inventory management problem under a box uncertainty. These results help us develop a dynamic program with polynomial complexity for the lot-sizing problem with stationary yield rate. Finally, extensive computational experiments show the robustness and effectiveness of the proposed model through an average and worst case analyses. The results demonstrate that the robust approach immunizes the system against uncertainty. Moreover, a comparison of the robust model with the nominal model, the deterministic model with safety stock, and the stochastic model shows that the robust model balances the costs better by reducing the backorders at the expense of more often producing a larger amount of goods.

Keywords: Lot-sizing, Production, Uncertainty, Robust Optimization

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1. Introduction

Many industries constantly face exogenous factors that can affect the quality of their products. In addition, the new products have become increasingly more complex with shorter life cycles, and product customization breaks the regularity of the production process and increases the number of failure sources (Duncan et al., 2021). It becomes particularly challenging to precisely estimate the production yields that are necessary in the production process. In this context, accounting for yield uncertainty is crucial in production planning because an underestimation of the production yield leads to excess inventory, whereas an overestimation creates significant stock-outs (Yano & Lee, 1995; Khouja, 1999).

The production yield incorporates quality factors into the lot-sizing model by measuring the expected quantity of non-defective items resulting from the release of a given production lot. Traditionally, this production yield is estimated based on historical data or machine specifications, but such estimations can be imprecise. In practice, the production yield is subject to multiple sources of uncertainty, such as deviations from standard operating procedures, environmental factors such as temperature and humidity, failures in the machinery of the system, a lack of a quality control system, material imperfections, process dysfunction, workforce inefficiency (Grosfeld-Nir & Gerchak, 2004). The yield uncertainty concerns the inability to predict precisely the output quantities associated with a lot size. This uncertainty occurs in many industries, such as electronics (e.g., Akella et al., 1992; Schemeleva et al., 2018), pharmaceuticals (Cho, 2010), food (Kazaz, 2004), agriculture (e.g., Jones et al., 2001; Anderson & Monjardino, 2019; Roell et al., 2020), steel and metallurgical industries (Lalpoor et al., 2011), as well as in remanufacturing processes (Panagiotidou et al., 2013).

In this paper, we will consider lot-sizing problems under yield uncertainty. Lot size decisions are a crucial step in production planning when aiming to meet customers' needs and minimize the overall costs (Dolgui & Proth, 2010). Although modelers often rely on deterministic variants of lot-sizing problems (LSPs) based on the hypothesis that all data is known or can be correctly predicted, in practice, many parameters are uncertain (Jans & Degraeve, 2008). The discrepancy between data estimation and their actual values can have a critical impact on the quality of the lot-sizing decision. There is a need to develop LSPs that take these uncertainties into account. There are static and adaptive strategies for production planning under uncertainty (Bookbinder & Tan, 1988). In a static strategy, the production decisions are fixed for the entire horizon. In an adaptive strategy, some decisions are updated to react to the realization of the uncertainty in the previous periods. This work provides the first in-depth investigation of a static strategy for robust

LSPs under yield uncertainty.

A static strategy has a practical relevance to reduce system nervousness at minimal computational effort (Blackburn et al., 1986). Sereshti et al. (2021) indicates that static lot-sizing plans do not experience nervousness, as the amount of assembly and production remains the same despite the realization of uncertainty. The authors also report that the static strategy combined with the receding horizon is a good approximation of the corresponding robust adaptive strategy. Thus, the static solution can be executed in a rolling horizon framework to tackle the dynamic strategy (Blackburn et al., 1986; Sereshti et al., 2021).

The yield uncertainty may have different impacts depending on the situation, such as an increase in the production costs, processing duration, or lead times, and it often results in a waste of materials and available resources. The consequences of these losses can be highly damaging to the system (Inderfurth & Vogelgesang, 2013). Most studies on LSPs with uncertain yield consider the single-item single-period problem (Yano & Lee, 1995; Khouja, 1999). In this simple setting, the optimal lot size can be derived through a mathematical analysis based on the newsboy inventory management model (Khouja, 1999). This technique, however, cannot be efficiently applied in a more general context of multi-period lot-sizing, because it can lead to poor solutions (Yano & Lee, 1995).

Another classic technique to increase the feasibility of a production plan over exogenous uncertainties (such as demand and supply delivery lead time) and endogenous uncertainties (such as production yield) is the use of safety stocks (Dolgui & Prodhon, 2007). Although the safety stock approach helps mitigate the impact of uncertain yield, safety stock calculations rely on separate calculations and strong assumptions. This results in sub-optimal solutions as shown in Thevenin et al. (2020). The authors indicate that the safety stock levels are computed either manually to meet a given service level in the master production schedule, or based on strict assumptions (e.g., static demand, base stock policy, lot-for-lot policy, critical stock policy). Since the lot sizes and the safety stocks are not defined simultaneously, this often results in sub-optimal decisions.

Safety stock can be integrated into the LSPs as a lower bound on the inventory level (De Bodt et al., 1982). Zhao et al. (2001) report different approaches to define the safety stock level at the master production schedule with the following parameters: safety coverage, service level and standard deviation of the forecast errors. The newsboy model, as presented in Khouja (1999), helps to determine safety stocks to balance inventory and backorder cost, rather than to satisfy a given service level. To improve the quality of non-deterministic decisions, formulating the problems as mixed-integer linear programs (MILP) has emerged as promising approach.

Stochastic programming (SP) and robust optimization (RO) are good methodologies to incorporate uncertainty into the decision-making process in a systematic way. They implicitly calculate a satisfactory inventory level to achieve greater cost savings and reduce nervousness (Thevenin et al., 2020). SP represents uncertain parameters with their probability distributions, and the aim here is to make decisions by minimizing the expected costs in a MILP formulation (Birge & Louveaux, 2011). Although SP is efficient for small-size instances with a limited number of scenarios, it does not scale up well for large instances or for a large number of scenarios, notably those within dynamic contexts (Brandimarte, 2006). Unlike the SP method, RO does not rely on a probabilistic distribution. RO tackles non-deterministic problems when distributional information on the unknown parameters is inaccurate or incomplete, and it optimizes for the worst case value of the uncertain parameter within a bounded and convex uncertainty set (Ben-Tal et al., 2009; Bertsimas et al., 2011). RO models often yield feasible and tractable easy-to-solve formulations, although modelers should be careful of possible conservative behavior in the decision-making process (Thiele, 2010).

In this paper we propose a methodology based on robust optimization for the non-stationary multi-period LSP under yield uncertainty, and we analyze it in terms of its applicability, optimality, and efficiency. To the best of our knowledge, we are the first to consider robust optimization for an LSP under yield uncertainty in a non-stationary production context, where the production parameters such as costs, demands, and production yield rates may change at each production period. The contribution of the current paper is fourfold. First, we derive an optimal policy for the stationary case of the nominal and maximum deviation values of the uncertain yield. This special case considers the box uncertainty set without setup, much like the case for the demand uncertainty presented by Bertsimas & Thiele (2006). Second, we propose a polynomial-time dynamic programming algorithm for the special case of the lot-sizing problem with stationary yield rate, non-stationary costs and demands, and box uncertainty set. Third, we propose a robust optimization formulation for a non-stationary multi-period LSP under budgeted uncertainty set and yield uncertainty. Finally, we perform an in-depth analysis of the resulting methods in terms of the quality of the solution, scalability, stability, robustness, and flexibility. In particular, we compare the production plans resulting from the robust, nominal, deterministic with safety stock in, and stochastic models. Although the robust models guarantee the robustness and feasibility of the proposed plan, the stochastic programs seek the production plan with the best expected costs. In addition, the deterministic problem with safety stock aims to reduce the risk of shortages. Thus, we intend to analyze when each technique is best suited to deal with production yield uncertainty.

This paper is organized as follows: Section 2 gives a review of previous work on non-deterministic lot-sizing problems, with a focus on uncertain production yield. Section 3 formally describes the considered problem and introduces the robust optimization methodology, more specifically, the mixed-integer linear formulation for the non-stationary case of the problem, the optimal robust properties and policies, and a dynamic programming formulation for the problem with stationary yield and non-stationary costs and demands. Section 4 presents the instances and simulation framework used in our experiments, as well as the experimental results. Finally, Section 5 concludes this work and provides some future research directions.

2. Literature Review

Because of their practical importance, LSPs have attracted a wide range of research from the manufacturing and operations research communities. Although most studies concern the deterministic LSP, there is a growing amount of research on non-deterministic lot-sizing. The bibliography on non-deterministic LSPs by Aloulou et al. (2014) confirms the prevalence of studies on demand uncertainty, which is the most natural source of uncertainty within a production planning context. From the literature on LSPs, it is evident that only a few studies consider other uncertain parameters such as lead time uncertainty, production yield, and cost uncertainty, even though these other parameters may similarly affect solution quality (Brahimi et al., 2017). Aloulou et al. (2014); Brahimi et al. (2017) also indicate that the most common approaches to solve non-deterministic LSPs rely on mathematical analytical methods (e.g., newsboy models or the use of safety stocks in the nominal problem), stochastic programming and robust optimization, among others. We refer the interested readers to Pochet & Wolsey (2006); Brahimi et al. (2017) for further information on the deterministic LSPs, and to Aloulou et al. (2014); Brahimi et al. (2017) for references on the non-deterministic problems. As our work concerns the application of robust optimization for lot-sizing under yield uncertainty, the rest of this section reviews the related publications on robust optimization for lot-sizing problems.

The robust LSP has gained interest from the research community over the last decades, and different approaches exist to solve robust problems (Gabrel et al., 2014). Robust models are often handled through reformulation per constraint and dualization, or adversarial approaches. Note that the adversarial techniques (Yanikoğlu et al., 2019) refer to the approaches which does not rely on a tractable reformulation of the robust counterpart but typically by iteratively solving a restricted robust model with a limited set of uncertainty realizations to determine a possible robust

solution, and then finding a scenario realization which makes this solution candidate infeasible. If such a scenario is found, this scenario realization is then added to the restricted robust model. The process terminates when a possible robust solution is guaranteed to be feasible. The reformulation per constraint and dualization leverages the optimality condition of the duality theory. In fact, the constraint-wise formulation, where the uncertain parameters appear only in the constraints, allows us to directly exploit the duality technique of Beck & Ben-Tal (2009), for which the best dual in robust models is proved to be equal to the worst primal. Thus, a robust counterpart model can be defined by reformulating all the constraints subject to uncertainty by a worst case perspective of the uncertain parameter under a predetermined uncertainty set.

Tractable robust models requires a convex optimization formulation which is are often handled by dualization (Bertsimas & de Ruiter, 2016), reformulation of the robust counterpart (e.g., See & Sim, 2010; Guillaume et al., 2017), Lagrangian duality (Rodrigues et al., 2021), dynamic programs (e.g., Bienstock & Özbay, 2008; Agra et al., 2016) and decomposition algorithms (e.g., Bienstock & Özbay, 2008; Attila et al., 2021). Although these techniques often yield feasible formulations that are easy to solve, they can lead to conservative solutions (Thiele, 2010). Gorissen & Den Hertog (2013) discuss different ways to alleviate the conservatism of the formulations. For further information on recent advances in robust optimization, we refer the interested readers to Gabrel et al. (2014).

Bertsimas & Thiele (2004) proposed a first reformulation per constraint and dualization to solve the robust LSP and its variations or simplifications (e.g., the inventory management problem, which is a simplification of the lot-sizing problem where setup decisions are not taken into account). The authors consider the inventory management problem under a budgeted uncertainty set and uncertain demand, which results in a MILP formulation that can be solved by a commercial solver. The authors suggest carrying out a simulation to define a budget of uncertainty that best mitigates the uncertainties. The objective is to find a suitable compromise between cost and robustness (Thiele, 2010). In the same dualization vein, Zhang (2011) propose a formulation for the uncapacitated LSP with uncertain demand, where the inner maximization is reformulated as a longest path problem. The authors take the dual of the longest path problem to obtain the final robust model as an MILP, and this final model can also be solved by a commercial solver. Considering an example of adversarial approach, Bienstock & Özbay (2008) propose some decomposition techniques to iteratively constrain the space of realization of the uncertain parameter. The resulting approach provides plans that are sufficiently robust, and it scales well to solve large instances.

The production yield was first introduced by Bowman (1955) as the proportion of items that

were accepted, and that reached a high enough quality to be sent to costumers. We use the same notion to capture the proportion of good quality items in a lot. Studies on robust LSPs under yield uncertainty are scarce. Unlike the case of uncertain demand for which the quantity of quality goods obtained from the production is known, yield uncertainty affects the quantity of quality items obtained from a production lot. The dependence of the optimal amount of quality goods on the realization of the production yield increases the complexity of the adaptive formulation for the LSP (Ben-Tal et al., 2005). Since the production yield multiplies the decision variable, the adaptive robust model is intractable. Therefore, our goal is to derive solutions within a static decision strategy to understand the impact of the uncertain yield on the lot-sizing decision in a context with low or no nervousness, and to solve it with tractable models. We refer the reader to Yano & Lee (1995) for more on an LSP under yield uncertainty.

To the best of our knowledge, Kazemi Zanjani et al. (2010) and Quezada et al. (2020) are the only publications on stochastic programming for an LSP under yield uncertainty; and there is no existing study on robust optimization for lot-sizing under yield uncertainty. While Kazemi Zanjani et al. (2010) address the uncertain yield in terms of quality of raw materials, Quezada et al. (2020) formulate the production yield of refurbished items in a remanufacturing context. These articles show that the stochastic method is efficient within a static strategy because it minimizes the occurrence of the backorder. However, the stochastic models require the use of a sufficiently large scenario set to approximate properly the underlying distributions, and they may not scale well. Considering the robust optimization approach, Vayanos et al. (2012) tackle a stationary inventory management model with an uncertain production yield and fixed inventory and backorder costs, but they ignore the setup decisions and production costs. The authors propose a constraint sampling approximation to mitigate over conservative solutions. In a similar approach, Chu et al. (2019) propose a robust model for the procurement perspective of a stationary inventory management problem under the budgeted uncertainty set. They restrict the maximum value of the production yield to its nominal value, and they show that the problem can be formulated as a nominal problem with modified deterministic demand in terms of the accumulated deviation of both the uncertain demand and uncertain yield. The authors analyze the impact of the budget controlling the uncertainty and average and standard deviation of the uncertain parameters on the average performance of the robust models. Even though Chu et al. (2019) propose an insightful analysis of a inventory management problem with uncertain yield, they perform their studies in a procurement perspective, for which the production yield is set to its nominal value. Thus, Chu et al. (2019) do not allow obtaining an

amount of quality goods larger than the nominal ordered quantities to be obtained. In addition, Chu et al. (2019) do not compare the performance of the robust optimization approach with the stochastic programming and deterministic models.

This paper aims to fill the knowledge gap on the impact of the uncertain production yield on a multi-period lot-sizing problem within a robust perspective for a non-stationary case of production yield, costs and demands. Our work differs from the aforementioned literature in several aspects. First, to the best of our knowledge, the current paper is the first to formulate the non-stationary case of single-item and multi-period LSPs under yield uncertainty via robust optimization. Second, we derive the optimal robust policies for the single-period and multi-period LSPs with stationary nominal value and maximum deviation of the yield rate, with non-stationary inventory/backorder/production costs and demands, and with zero setup costs. Third, we propose a polynomial-time dynamic programming algorithm based on optimal robust policies to solve the considered problem for the special case where setup costs are considered. This helps us provide simple tools to compute a sufficiently robust plan, better suited for small decision settings or when an efficient commercial solver is not available. Fourth, we give an in-depth analysis of the impact of robust optimization for production planning based on numerical experiments. The results show that robust optimization is highly efficient and produces production plans that are more robust to different yield scenarios when compared to other decision-making methods (e.g., stochastic program, nominal problem, deterministic problem with safety stock) in terms of quality of the solution, cost savings, and robustness to changes or uncertainties in the system.

3. Problem statement and proposed robust approaches

This section introduces the considered model and some robust solution approaches. For the sake of clarity, we present the nominal lot-sizing problem prior to introducing the robust optimization models under an uncertainty set. First, we provide a robust optimization model for the LSP under yield uncertainty. Second, we derive a closed-form solution for the single-period model with a box uncertainty set. Next, we show how this closed-form solution can be extended to the multi-period inventory management problem, where we provide an optimal policy for the inventory management problem. Based on the optimal policy for the inventory management problem, we derive properties on the optimal solution of the lot-sizing problem. Finally, the aforementioned properties allow us to develop a dynamic programming algorithm to compute optimal solutions for the LSP with uncertain yield, with non-stationary costs and demands, and with stationary nominal value and maximum

deviation of the production yield.

3.1. Nominal problem

A single-item multi-period uncapacitated LSP with backorder and production yield determines the quantity to produce in each period of the finite planning horizon $T = \{1, \dots, |T|\}$. The objective is to meet demands efficiently and with quality goods while minimizing the overall costs. For each period $t \in T$, we are given the setup cost s_t , the unit production cost v_t , the inventory holding cost h_t , the backorder cost b_t , and the demand d_t . The model comprises the following decision variables: the lot size X_t to be produced, the inventory level I_t and the backorder level B_t at the end of the period, and the setup decision Y_t , such that $Y_t = 1$ if a setup occurs in t ($X_t > 0$) and $Y_t = 0$ otherwise. We define ρ_t as the strictly positive production yield in period t , such that $0 < \rho_t \leq 1$.

The formulation of the deterministic LSP with production yield is as follows:

$$\min \sum_{t \in T} s_t Y_t + v_t X_t + h_t I_t + b_t B_t \quad (1)$$

s.t. :

$$I_t - B_t = I_{t-1} - B_{t-1} + \rho_t X_t - d_t \quad \forall t \in T \quad (2)$$

$$X_t \leq M_t Y_t \quad \forall t \in T \quad (3)$$

$$X_t, I_t, B_t \geq 0 \quad \forall t \in T$$

$$Y_t \in \{0, 1\} \quad \forall t \in T$$

Without a loss of generality, we assume that there is no stock or backorder at the beginning of the planning horizon. The objective function (1) minimizes the total cost comprising the setup, unit production, inventory, and backorder costs. The inventory balance constraints (2) compute the level of backorder and inventory in period t from the demand, the produced goods at period t , and the inventory and backorder levels in period $t - 1$. The constraints (3) are setup-forcing constraints that relate the production quantities (X_t) to the setup decisions (Y_t), whereas $M_t = \frac{\sum_{t \in T} d_t}{\min_{\tau \leq t} \rho_\tau}$. These constraints set the setup variable Y_t to 1 if any production occurs in period t , and the setup remains inactive otherwise ($Y_t = 0$). In addition, constraints (3) can represent the capacity constraint by setting $M_t = \min\{C_t, \frac{\sum_{t \in T} d_t}{\min_{\tau \leq t} \rho_\tau}\}$, where C_t is the available capacity in period t .

3.2. Definition of the uncertainty set

The robust optimization considers that the uncertain parameter belongs to a bounded and convex uncertainty set. However, the tractability of the robust counterpart model depends on the

tractability of the uncertainty set (Bertsimas & Brown, 2009). Modelers often rely on some statistical consideration of historical data or previous knowledge of the studied system. The first robust optimization models used the *box uncertainty set*, introduced by Soyster (1973), which describes the uncertainty within an interval of possible values and which is bounded by its minimal and maximal acceptable realizations. To alleviate the over-conservatism of this approach, Bertsimas & Sim (2004) propose the *box polyhedral uncertainty set*, which is also known as the *budgeted uncertainty set*, where the uncertain parameter takes values within a range of values whose size is controlled by the decision-maker through a budget of uncertainty Γ . This budget of uncertainty reduces the size of the uncertainty set and reflects the degree of risk aversion of the decision-maker. The budget is a threshold for the number of uncertain parameters that can take their worst value (Thiele, 2010). This budget indicates a degree of acceptable variance of the uncertainty from its nominal value, where the larger the budget, the more averse to risk the decision-maker is. Note that the box uncertainty corresponds to the budgeted uncertainty set with the largest Γ value.

In the present work, we consider the widely adopted budgeted uncertainty set that is given by $\mathcal{U}_t = \{-1 \leq Z_\tau^t \leq 1 : \sum_{\tau=1}^t |Z_\tau^t| \leq \Gamma_t; t \in T; \tau \leq t\}$. This set is based on the nominal value and maximum deviation of the uncertain yield because these values are largely used in statistical quality control to bound the quality in terms of the key performance indicators (Godfrey, 1999). These values are easily obtained from historical data, and they reflect the basic characteristics of the uncertain parameter. We estimate the uncertain production yield $\tilde{\rho}$ through a natural parameterization $\tilde{\rho}(Z) = \bar{\rho} + Z\hat{\rho}$, with $Z \in [-1, 1]$. Here, the uncertain yield belongs to a range centered on its nominal value $\bar{\rho}$ and spread by its maximum deviation $\hat{\rho}$. The disturbance arising from the nominal value is given by the term Z . Therefore, we replace the production yield ρ_t in constraints (2) by the uncertain production yield $\tilde{\rho}_t$.

3.3. A robust counterpart formulation for the LSP with uncertain yield

In similar manner to Bertsimas & Thiele (2004), we rely on the reformulation per constraint approach and robust formulation under a budgeted uncertainty set to determine the production plan that minimizes the total costs for the LSP under yield uncertainty. The robust model is similar to the nominal model, but the inventory balance constraints are replaced by a pair of inequalities based on the convexity and the piecewise linearity of the inventory and backorder cost functions. Because the backorder corresponds to a negative stock level, these costs are complementary (Bertsimas & Thiele, 2004). We can then drop the inventory and backorder variables, so we directly compute the inventory and backorder costs according to the difference between the number of quality goods and demand. Thus, the constraints (5) and (6) replace the constraints (2), and H_t represents

either the inventory or backorder cost in period t , and the constraints (5) (resp. (6)) compute the worst case inventory (resp. backorder) costs. In addition, the inventory and backorder costs constraints are subject to yield uncertainty, and they are optimized under the worst case realization of the uncertain parameter in the uncertainty set. Since the reformulation per constraint and dualization approach is applied, H_t is independently defined for each period t as the highest cost between the worst inventory cost and the worst backorder cost under the uncertainty set \mathcal{U}_t . While the adversarial approach usually requires more computation time than the dualization method, it yields less conservative lot-sizing solutions than the reformulation per constraints and dualization approach. In the reformulation per constraint and dualization approach, the worst case is computed independently in each period (the computation of the inventory/backorder costs in each period may rely on different yield realizations), whereas the adversarial approach finds a unique worst case yield vector for all time periods. As a result, the total cost of the robust model includes the setup cost, the production costs, and the maximum periodic inventory and backorder costs for any production yield value in the uncertainty set \mathcal{U}_t for each period t . Therefore, we can model a robust counterpart for the LSP with uncertain yield as follows:

$$\min \sum_{t \in T} \left(s_t Y_t + v_t X_t + H_t \right) \quad (4)$$

s.t. :

$$H_t \geq \max_{\tilde{\rho} \in \mathcal{U}_t} \left[h_t \sum_{\tau=1}^t (\tilde{\rho}_\tau X_\tau - d_\tau) \right] \quad \forall t \in T \quad (5)$$

$$H_t \geq \max_{\tilde{\rho} \in \mathcal{U}_t} \left[-b_t \sum_{\tau=1}^t (\tilde{\rho}_\tau X_\tau - d_\tau) \right] \quad \forall t \in T \quad (6)$$

$$X_t \leq M_t Y_t \quad \forall t \in T$$

$$X_t, H_t \geq 0 \quad \forall t \in T$$

$$Y_t \in \{0, 1\} \quad \forall t \in T$$

where the upper bound on the production quantity M_t can be set based on the lowest possible value of the production yield, that is, $\min_{\tau \leq t} (\bar{\rho}_\tau - \hat{\rho}_\tau)$. Thus, for an LSP under yield uncertainty, M_t is set to $M_t = \frac{\sum_{t \in T} d_t}{\min_{\tau \leq t} (\bar{\rho}_\tau - \hat{\rho}_\tau)}$.

3.4. A robust mixed integer linear formulation for the LSP with uncertain yield reformulation

Since our robust counterpart is a constraint-wise formulation, for which the uncertain yield occurs on the right side of the inventory and backorder cost constraints, it can be addressed via the reformulation per constraint and dualization approach. In addition, we rely on the budgeted uncertainty set to alleviate the conservatism with a sufficiently good budget of uncertainty. We only

show the detailed steps of reformulation per constraint for the inventory inequalities (5), because the application for backorder inequalities is analogous. In fact, the inequalities differ only by the sign and the costs associated with the inventory level. Thus, the inventory level is negative in case of a backorder, and positive if the production exceeds the demand. As a result, within the worst case perspective of the robust approach, the optimal plan corresponds to a decision leading to a higher cost among these two groups of constraints.

The constraints (5) are nonlinear because of the maximization function. Therefore, the reformulation per constraint allows us to transform the non-linear constraints into linear ones. To avoid the inclusion of all possible quantifiers of the uncertainty in the uncertainty set and to linearize the constraints, the reformulation per constraint and dualization approach consists of three steps: 1) reformulation of the constraints subject to uncertainty as a worst case reformulation; 2) dualization of the reformulation problem; and 3) dropping the dualized reformulation into the initial formulation without the inner optimization term (Ben-Tal & Nemirovski, 2000).

The first step is the worst case reformulation of the constraints (5) that becomes:

$$H_t \geq h_t \left[\sum_{\tau=1}^t (\bar{\rho}_\tau X_\tau - d_\tau) + \max_{Z \in \mathcal{U}_t} \sum_{\tau=1}^t \hat{\rho}_\tau X_\tau Z_\tau^t \right] \quad \forall t \in T \quad (7)$$

For the inventory cost constraints, the worst case scenario occurs only when the deviation is positive, that is, when $Z_\tau^t \geq 0$ for $t \in T$; $\tau \leq t$. Then, we perform the second step to obtain a dual. Assuming that λ and μ are the dual variables, by following the dualization technique presented by Beck & Ben-Tal (2009), we obtain the follow primal-dual problem:

$$\begin{array}{ll} \max \sum_{\tau=1}^t \hat{\rho}_\tau X_\tau Z_\tau^t & \min \Gamma_t \lambda_t + \sum_{\tau=1}^t \mu_\tau^t \\ \sum_{\tau=1}^t Z_\tau^t \leq \Gamma_t & \xrightarrow{\text{dualized}} \lambda_t + \mu_\tau^t \geq \hat{\rho}_\tau X_\tau \quad \forall \tau \leq t \\ Z_\tau^t \leq 1 \quad \forall \tau \leq t & \lambda_t, \mu_\tau^t \geq 0 \quad \forall \tau \leq t \end{array}$$

We can finally perform the third step, and replace the worst case reformulation with its dual formulation. Therefore, the constraints (7) are reformulated as follows:

$$H_t \geq h_t \left[\sum_{\tau=1}^t (\bar{\rho}_\tau X_\tau - d_\tau) + \min_{\lambda_t + \mu_\tau^t \geq \hat{\rho}_\tau X_\tau} \left(\Gamma_t \lambda_t + \sum_{\tau=1}^t \mu_\tau^t \right) \right] \quad \forall t \in T$$

Similarly, the application of the reformulation per constraint and dualization in the constraints (6) leads to the following reformulation:

$$H_t \geq -b_t \left[\sum_{\tau=1}^t (\bar{\rho}_\tau X_\tau - d_\tau) - \min_{\lambda_t + \mu_\tau^t \geq \hat{\rho}_\tau X_\tau} \left(\Gamma_t \lambda_t + \sum_{\tau=1}^t \mu_\tau^t \right) \right] \quad \forall t \in T$$

Note that for each period t , the computation of the worst case inventory cost in the constraints (5) maximizes $\sum_{\tau \leq t} \hat{\rho}_\tau Z_\tau^t$, while the computation of the worst case backorder cost in the constraints

(6) seeks to minimize this quantity. The uncertainty set \mathcal{U}_t is symmetric in Z_τ^t . This symmetry is due the convexity and piecewise linearity exploited in the inventory balance constraints. As a result, the worst realization of $Z_\tau^t \in [-1, 1] \forall \tau \leq t$ for the constraints (6) are the opposite of the values obtained for the constraints (5), and no feasible production yield in the uncertainty set realizes both constraints at once. Thus, we can use the same dual variables (λ_t and μ_τ^t) for both cases, but they have a positive coefficient in the constraints (9) and negative in the constraints (10).

From the duality theory, since the primal worst equals dual best holds (Beck & Ben-Tal, 2009), it only takes at least one pair of feasible dual variables in the inner minimization to ensure that the solution holds for the outer minimization. Therefore, we can drop the minimization terms of the reformulated constraints. Hence, we obtain the final reformulation of the robust counterpart under budgeted uncertainty set, which is given as follows:

$$\min \sum_{t \in T} s_t Y_t + v_t X_t + H_t \quad (8)$$

s.t. :

$$H_t \geq h_t \left[\sum_{\tau=1}^t (\bar{\rho}_\tau X_\tau - d_\tau + \mu_\tau^t) + \lambda_t \Gamma_t \right] \quad \forall t \in T \quad (9)$$

$$H_t \geq -b_t \left[\sum_{\tau=1}^t (\bar{\rho}_\tau X_\tau - d_\tau - \mu_\tau^t) - \lambda_t \Gamma_t \right] \quad \forall t \in T \quad (10)$$

$$\lambda_t + \mu_\tau^t \geq \hat{\rho}_\tau X_\tau \quad \forall t \in T; \tau \leq t \quad (11)$$

$$X_t \leq M_t Y_t \quad \forall t \in T$$

$$X_t, H_t, \lambda_t \geq 0 \quad \forall t \in T$$

$$\mu_\tau^t \geq 0 \quad \forall t \in T; \tau \leq t$$

$$Y_t \in \{0, 1\} \quad \forall t \in T$$

Our final robust reformulation shares some similarities with the model for the inventory management problem with uncertain demand proposed by Bertsimas & Thiele (2004). However, in the model for uncertain demand, the deviation affects the uncertain parameter (demand) only. On the contrary, in our model, the deviation corresponds to the worst case impact of the uncertain parameter on the resulting production quantity.

3.5. Properties of an optimal robust policy for the inventory management problem

This section derives a closed-form solution and the optimal robust policy for special cases of the considered problem. The closed-form solution can be determined for the inventory management problem under the following assumptions. First, we consider the inventory management problem

where the setup and the unit production cost are equal to zero, while the inventory and backorder costs are non-stationary. Second, we consider a stationary case of the nominal value and maximum deviation of the yield. Third, our policy is derived from the inventory management problem under a box uncertainty set, where the box uncertainty set is given by $\mathcal{U}_t = \{-1 \leq Z_\tau^t \leq 1 : \sum_{\tau=1}^t |Z_\tau^t| \leq t, t \in T, \forall \tau \leq t\}$. Fourth, we assume that the unit inventory cost h_t is lower than the unit backorder cost b_t , and that demands are positive to obtain a positive production quantity X_t . The first two assumptions are common in the inventory management literature, and they are also often encountered in practice. Furthermore, although the worst case perspective from a model under a box uncertainty set is a conservative approach to handle the robust LSP with uncertain yield, it helps us derive the property of an optimal policy to compute a fully immunized plan.

This section is organized as follows. First, we prove that the optimal policy for the inventory management problem under yield uncertainty sets the production quantity such that the worst case inventory cost equals the worst case backorder cost. Second, we derive the optimal policy for the single period problem with uncertain yield. Third, we extend the single period policy for the multi-period inventory management problem. Fourth, we show that these policies remain valid for the LSP that accounts for unit production costs under mild assumptions. Finally, we indicate the need to anticipate the impact of uncertain yield on future costs in order to extend the policy to the budgeted-based model and the problem with non-stationary yield information.

Based on Proposition 3.1 we obtain the property for an optimal solution for the inventory management problem with uncertain yield, Proposition 3.2 gives the optimal policy for the single-period problem.

Proposition 3.1. *The optimal production quantity for the inventory management problem is at the point where the worst inventory cost is equal to the worst backorder cost.*

Proof. See Appendix A □

Proposition 3.2. *The optimal robust policy for the stationary case of a single-period inventory management problem under box uncertainty set and yield uncertainty is as follows:*

$$X = \frac{d}{\bar{\rho} + \hat{\rho} \left(\frac{h-b}{h+b} \right)}$$

Proof. See Appendix B □

We extend our analysis to the multi-period inventory management problem under box uncertainty set and yield uncertainty, with stationary nominal value and maximum deviation of the yield,

with non-stationary costs, and with no setup costs. It leads to Proposition 3.3.

Proposition 3.3. *The optimal robust policy for the multi-period inventory management problem under box uncertainty set and yield uncertainty, with no setup costs, with stationary nominal value and maximum deviation of the production yield, non-stationary inventory, backorder and unit production costs is given as follows:*

$$X_t = \frac{\bar{\rho}d_t + \hat{\rho} \left[\left(\frac{h_{t-1}-b_{t-1}}{h_{t-1}+b_{t-1}} \right) \bar{D}_t - \left(\frac{h_t-b_t}{h_t+b_t} \right) \bar{D}_{t-1} \right]}{\left(\bar{\rho} + \hat{\rho} \left(\frac{h_t-b_t}{h_t+b_t} \right) \right) \left(\bar{\rho} + \hat{\rho} \left(\frac{h_{t-1}-b_{t-1}}{h_{t-1}+b_{t-1}} \right) \right)}$$

if the following condition is respected: $v_t \leq (\bar{\rho} - \hat{\rho})b_t, \forall t \in T$.

Proof. See Appendix C □

Note that for the special case of the multi-period inventory management problem under box uncertainty set and yield uncertainty, with stationary production, inventory and backorder costs (i.e., $v_t = v_{t-1} = v$, $h_t = h_{t-1} = h$ and $b_t = b_{t-1} = b$) and without setup cost, the optimal robust policy can be simplified as follows:

$$X_t = \frac{d_t}{\bar{\rho} + \hat{\rho} \left(\frac{h-b}{h+b} \right)}$$

Note also that this formula is a direct extension from Proposition 3.2, if we replace d_t with d . In addition, if $v = 0$, then the condition $v \leq (\bar{\rho} - \hat{\rho})b$ is always valid.

Proposition 3.3 does not hold for the case with non-stationary nominal value and maximum deviation of the production yield. The policy described in Proposition 3.3 is myopic in the sense that the computation of X_t does not consider parameter values at period later than t (i.e., $v_\tau, h_\tau, b_\tau, d_\tau$, for $\tau > t$). We explain with an example that a myopic policy cannot be optimal if the nominal value and maximum deviation of the yield are not constant. Let us consider an extreme case for three periods, with $h = 1$, $b = 10$, $\bar{\rho} = (0.55, 1, 0.6)$, $\hat{\rho} = (0.45, 0, 0.4)$, $d = (15, 10, 25)$, $s = v = 0$. The robust model returns an optimal value equal to 175 with a lot size $X = (0, 50, 0)$, where the production for the entire production horizon is done in the second period for which the yield is maximum. A myopic policy cannot obtain the aforementioned optimal solution, because this solution requires not performing the setup in period 1 to favor production in period 2, where the deviation of the production yield from its nominal value is zero. The policy should anticipate the occurrence of better production yields in later periods for the definition of an optimal production quantity that respects the inventory balance constraints. However, it is not trivial to impose such condition to obtain an analytical solution.

While Proposition 3.2 can be adapted to take into account the budgeted uncertainty set, the extension of Proposition 3.3 becomes more complicated. For the single period problem, the budgeted set restricts only the range of the maximum deviation, which leads to a worst case scenario that changes from $|Z| = 1$ to $|Z| = \min\{\Gamma, 1\}$. If the optimal policy for the budgeted uncertainty set in the single-period model can be obtained by replacing $\hat{\rho}$ by $\min\{\hat{\rho}; \hat{\rho}\Gamma\}$, that is not the case for Proposition 3.3. Returning to the example from the previous paragraph, we set the nominal value to $\bar{\rho} = 0.55$ and the maximum deviation to $\hat{\rho} = 0.05$. To transform the box-based formulation into a budgeted-based formulation, we replace $\hat{\rho}_t$ by $\hat{\rho}_t\Gamma_t$ in the policy given in Proposition 3.3. Assuming $\Gamma_t = 0.5t$, our policy gives a lot size $X = (28.33, 19.64, 51.16)$ which leads to a budgeted-based optimal value equal to 12, while the robust model returns a budgeted-based optimal value equal to 10 with a lot size $X = (28.33, 19.24, 47.97)$. The budget of uncertainty prevents setting all the values of Z_t to 1 or -1 . In each period, the value of Z_t changes such that it takes the highest value for the period with largest production quantity. However, the disturbance values are period independent. As the value of Z_t changes depending on the decided production quantity, the anticipation of the occurrence of better production yields in later periods is also needed to compute an optimal lot. Similarly to the case of non-stationary nominal value and maximum deviation of the yield rate, it is complex to define an analytical solution.

Our myopic policy gives an optimal solution for the case of stationary nominal value and maximum deviation under a box uncertainty set. In Section 3.6 we provide a dynamic program based on the optimal policy to solve the stationary LSP under yield uncertainty with non-stationary demands and costs (including the setup and production costs). As our myopic policy does not optimize within a global perspective of the planning horizon, it does not anticipate the occurrence of better production yields for later periods, and so it gives a sub-optimal solution for the case of non-stationary nominal value and maximum deviation of the yield rate, and also for the budgeted-based robust solution.

3.6. Dynamic programming for the uncapacitated robust LSP with uncertain yield and static costs

In this section, we propose a dynamic programming algorithm to address the lot-sizing problem with non-stationary costs and demands, and with a stationary nominal value and maximum deviation of the yield. The proposed dynamic programming algorithm extends the method of Zangwill (1969) that computes a solution from a succession of regeneration intervals. The approach of Zangwill (1969) is based on the property that between any two periods with production there is a regeneration period where the inventory level is 0. This property is not true in the robust LSP

under yield uncertainty, since we consider backorders and uncertain yield. However, Proposition 3.4 shows that between two periods with production there is a point where the worst inventory cost is equal to the worst backorder cost. We define such a period as a *regeneration period*. In addition, the approach of Zangwill (1969) must be adjusted because our extension for the case of uncertain yield allows the backorder at the end of the production horizon. This section successively introduces the main concepts that support our reasoning, the property of an optimal solution for the LSP with uncertain yield and the optimal policy in Section 3.6.1, and finally the dynamic programming algorithm in Section 3.6.2.

3.6.1. Structure of the optimal solution to the LSP under yield uncertainty

Let us define the cumulative demand as $\bar{D}_t = \sum_{\tau=1}^t d_\tau$ and the cumulative production quantity up to period t as $\bar{X}_t = \sum_{\tau=1}^t X_\tau$. The worst inventory cost ($\hat{I}_t(\bar{X}_t)$) and the worst backorder cost ($\hat{B}_t(\bar{X}_t)$) in period t can be written as follows:

We assume $\bar{D}_t = \sum_{\tau=1}^t d_\tau$ to be the cumulative demand, while $\bar{X}_t = \sum_{\tau=1}^t X_\tau$ is the cumulative production quantity in period t . Similarly to the proof of Proposition 3.1, $\hat{I}_t(\bar{X}_t)$ and $\hat{B}_t(\bar{X}_t)$ can be given as follows:

$$\begin{aligned}\hat{I}_t(\bar{X}_t) &= \max \{h_t [\bar{X}_t(\bar{\rho} + \hat{\rho}) - \bar{D}_t] ; 0\} \\ \hat{B}_t(\bar{X}_t) &= \max \{b_t [\bar{D}_t - \bar{X}_t(\bar{\rho} - \hat{\rho})] ; 0\}\end{aligned}$$

Definition 3.1 (Regeneration period). A regeneration period is a period r where the worst inventory cost ($\hat{I}_r(\bar{X}_r)$) equals to the worst backorder cost ($\hat{B}_r(\bar{X}_r)$). Therefore, the regeneration period r is such that: $\hat{I}_r(\bar{X}_r) = \hat{B}_r(\bar{X}_r)$.

Definition 3.2 (Regeneration interval). A regeneration interval $[m, n]$ is a set of consecutive periods such that $\hat{I}_m(\bar{X}_m) = \hat{B}_m(\bar{X}_m)$, $\hat{I}_n(\bar{X}_n) = \hat{B}_n(\bar{X}_n)$ and $\hat{I}_t(\bar{X}_t) \neq \hat{B}_t(\bar{X}_t)$ for all $t \in [m + 1, n - 1]$. In addition, the production is strictly positive in the setup period¹ β in the interval $[m, n]$, and it is 0 for all periods t , $t \neq \beta$ in the interval $[m + 1, n - 1]$.

Proposition 3.4. *The solution to the robust LSP under yield uncertainty with stationary nominal value and maximal deviation of the yield is a succession of regeneration intervals. For each regeneration interval, there is a setup period, beta, that minimizes the total cost over the interval.*

Proof. Given two consecutive periods with production β and γ (with no production in the interval $[\beta + 1, \gamma - 1]$), the worst case inventory and backorder cost H_τ in the interval $[\beta, \gamma]$ depends only on

¹A setup period can be a regeneration period for a regeneration interval of only one period.

the cumulative production \bar{X}_β in period β . Note that the costs after period γ will depend on the production quantity of period γ . Therefore, the production quantity in period β only impacts the costs in the interval $[\beta, \gamma]$. We redefine the worst case inventory or backorder cost in period τ in terms of the cumulative production in t by $H_\tau(\bar{X}_t)$ that is given as follows:

$$H_\tau(\bar{X}_t) = \max \left\{ \begin{array}{l} \hat{I}_\tau(\bar{X}_t) \\ \hat{B}_\tau(\bar{X}_t) \end{array} \right\} = \max \left\{ \begin{array}{l} h [\bar{X}_t(\bar{\rho} + \hat{\rho}) - \bar{D}_\tau] \\ b [\bar{D}_\tau - \bar{X}_t(\bar{\rho} - \hat{\rho})] \end{array} \right\}$$

where \bar{D}_t is the cumulative demand up to period t .

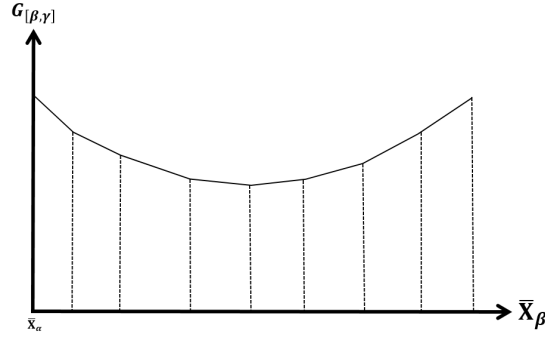


Figure 1: Piecewise inventory and backorder cost functions

Figure 1 illustrates the total cost $G_{[\beta, \gamma]}(\bar{X}_\beta)$ over the interval $[\beta, \gamma]$ as a function of the cumulative production \bar{X}_β in period β . \bar{X}_β must be chosen to minimize the sum of the inventory and backorder cost in the interval $[\beta, \gamma]$: $G_{[\beta, \gamma]}(\bar{X}_\beta) = \sum_{\tau \in [\beta, \gamma]} H_\tau(\bar{X}_\beta)$. If the production quantity is low in the period with production β (e.g., $\bar{X}_\beta = \bar{X}_\alpha$, where α is the period with production before β), the function $\sum_{\tau=\beta}^\gamma H_\tau(\bar{X}_\beta)$ corresponds to the sum of the backorder $\sum_{\tau=\beta}^\gamma \hat{B}_\tau(\bar{X}_\beta)$ in all periods, because the production is too low to cover the demand in any period of the interval $[\beta, \gamma]$, and the worst case cost corresponds to backorder over the entire interval $[\beta, \gamma]$. If the quantity \bar{X}_β increases to meet the demand in period β , the worst case cost corresponds to inventory in period β and to backorder afterwards $\sum_{\tau=\beta}^\gamma H_\tau(\bar{X}_\beta) = \hat{I}_\beta(\bar{X}_\beta) + \sum_{\tau=\beta+1}^\gamma \hat{B}_\tau(\bar{X}_\beta)$. In the last segment, where \bar{X}_β is large enough to meet the demand until period γ , the worst case cost corresponds to $\sum_{\tau=\beta}^\gamma \hat{I}_\tau(\bar{X}_\beta)$. Note that backorder costs are decreasing with the production quantity, whereas the inventory costs are increasing with \bar{X}_β . Therefore, $G_{[\beta, \gamma]}$ is a piecewise linear and convex function, such that the minimum is at a breakpoint. Each breakpoint in the function $G_{[\beta, \gamma]}$ corresponds to the case where the worst case backorder cost equals the worst case inventory cost in a period. As X_β is chosen to minimize $G_{[\beta, \gamma]}$, it is chosen such that the worst case backorder cost equals the worst case inventory cost in a period $n \in [\beta, \gamma]$. We call such a period a regeneration period, and a regeneration interval is a set of period $[m, n]$ where m and n are regeneration periods. As there is a regeneration period

between each successive period with setup, each regeneration interval contains one period with setup. In the case the last regeneration period r occurs before the end of the production horizon, that is $r < T$, then we cannot define a complete regeneration interval and the minimum cost in the interval $[r, T]$ corresponds to $G_{[r, T]} = \sum_{\tau=r+1}^T H_{\tau}(\bar{X}_r)$. In this case, all demands are backordered from the last regeneration period until the end of the production horizon. \square

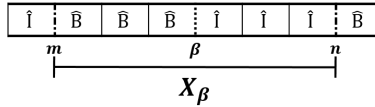


Figure 2: Representation of the regeneration period and regeneration interval concepts

Let m be the last regeneration period before starting production in the period β , such that n is the next regeneration period after β and $m \leq \beta \leq n$. Figure 2 illustrates the regeneration interval concept. Here, \hat{I} (resp. \hat{B}) indicates that the worst case cost at each period corresponds to the inventory (resp. backorder) costs. As previously mentioned, the production quantity in period β in the regeneration interval $[m, n]$ is chosen such that, at the regeneration periods m and n , the worst inventory cost is equal to the worst backorder cost. In addition, the worst cost corresponds to inventory costs (resp. backorder cost) from period β to $n - 1$ (resp. $m + 1$ to $\beta - 1$).

Proposition 3.4 indicates that the solution to LSP under yield uncertainty is a succession of regeneration periods, where each lot size covers the demand over the regeneration interval in which the production quantity is defined. Proposition 3.5 gives the optimal lot size to cover each regeneration interval.

Proposition 3.5. *The amount to produce in the setup period β to cover the regeneration interval $[m, n]$ (with $\beta \in [m, n]$ and $m < n$) is given by X_{β} , and it is calculated as follows:*

$$X_{\beta} = \bar{X}_n - \bar{X}_m$$

Proof. The previous proposition (Proposition 3.4) show that the optimal quantity to produce to fulfil demands over a regeneration interval $[m; n]$ is hold in period β where the cost is minimal. From Equation (C.1), the cumulative production quantity \bar{X}_m (resp. \bar{X}_n) at regeneration period m and n (respectively) is $\bar{X}_m = \frac{\sum_{\tau=1}^m d_{\tau}}{\bar{\rho} + \hat{\rho} \left(\frac{h_m - b_m}{h_m + b_m} \right)}$ (resp. $\bar{X}_n = \frac{\sum_{\tau=1}^n d_{\tau}}{\bar{\rho} + \hat{\rho} \left(\frac{h_n - b_n}{h_n + b_n} \right)}$). Note that Equation (C.1) applies here because the production in period β to cover a regeneration interval $[m; n]$ equals the difference between the cumulative production quantity in n (the end of the interval) and the cumulative production quantity in m (the beginning of the interval). Since there is no production

in interval $[\beta + 1, n]$, the cumulative production \bar{X}_β in period β is equal to \bar{X}_n . In addition, there is no production in interval $[m + 1, \beta - 1]$ and the cumulative production in this interval corresponds to \bar{X}_m . The production quantity in period β (with $m < n$) corresponds to the difference between the cumulative production at the end of the regeneration interval and the cumulative production at its beginning. Therefore, $X_\beta = \bar{X}_n - \bar{X}_m$. Note that if a setup period is a regeneration period, then the lot size in period β should cover only the demand from period β . For this special case, the lot size is directly computed by the policy $X_\beta = \frac{d_\beta}{\bar{\rho} + \hat{\rho} \left(\frac{h_\beta - b_\beta}{h_\beta + b_\beta} \right)}$ given in Proposition 3.3. \square

3.6.2. The dynamic program for the LSP with uncertain yield

Our dynamic program for the LSP under yield uncertainty iteratively defines successive regeneration intervals from the first production period until the end of the production horizon. For each interval, it defines the setup period that minimizes the costs over the interval. The dynamic programming recursion requires low computational effort because it works with the optimal cumulative policies that are easily computed, and it exploits forward recursion to avoid recalculating decision values that have already been defined.

The function $MCI(m, \beta, n)$ gives the minimal cost over the regeneration interval $[m, n]$ for a production setup in period $\beta \in [m, n]$. $MCI(m, \beta, n)$ includes a setup cost, the production cost associated with the optimal production quantity for the regeneration interval $[m, n]$ (computed according to Proposition 3.5) and the inventory balance costs over the regeneration interval. From Proposition 3.4, we know that the cumulative production in n covers all the demands up to n and that the cumulative production in m covers all demands up to m . Thus, for the $MCI(m, \beta, n)$'s inventory management cost calculation is given by $\sum_{\tau=m+1}^{\beta-1} H_\tau(\bar{X}_m)$ in $[m + 1, \beta - 1]$ and by $\sum_{\tau=\beta}^n H_\tau(\bar{X}_n)$ in $[\beta, n]$. Thus, $MCI(m, \beta, n)$ is then given by $MCI(m, \beta, n) = \left\{ s_\beta + v_\beta X_\beta + \sum_{\tau=m+1}^{\beta-1} H_\tau(\bar{X}_m) + \sum_{\tau=\beta}^n H_\tau(\bar{X}_n) \right\}$, where $X_\beta = \bar{X}_n - \bar{X}_m$ if $m < n$, and $X_\beta = \frac{d_\beta}{\bar{\rho} + \hat{\rho} \left(\frac{h_\beta - b_\beta}{h_\beta + b_\beta} \right)}$ otherwise. Contrary to the deterministic case, it is possible to obtain a plan for which the backorder level at the end of the production horizon is not zero. Thus, we denote by $MCR(r, T)$ the minimal cost from the last regeneration period r to T , the end of the planning horizon, when only the backorder levels are considered. $MCR(r, T) = \sum_{\tau=r+1}^T H_\tau(\bar{X}_r)$.

Finally, we define the dynamic program recursive function by $F(t)$, which gives the optimal cost from period 0 to t . The costs up to period t correspond to the cost of the last regeneration period m plus the cost up to period t (that is in the interval $[m, t]$). Therefore, $F(t)$ can be computed recursively. Given the optimal cost $F(m)$ up to period $m < t$, the computation of $F(t)$ enumerates

all possible values for the last regeneration period m and the period with setup $\beta \in [m, t]$. Thus, $F(t)$ is given by

$$F(t) = \min_{m \leq \beta \leq t} \{F(m) + MCI(m, \beta, t)\} \quad \forall t \leq T - 1$$

Since the last regeneration period can occur before period T , the end of the production horizon, we should modify $F(T)$ to take into account the possibility of backordering all unmet demands from the last regeneration period r until T . Therefore, the optimal cost at the period T is given by

$$F(T) = \min \left\{ \min_{m \leq \beta \leq T} \{F(m) + MCI(m, \beta, T)\} ; \min_{m \leq T} \{F(m) + MCR(m, T)\} \right\}$$

Proposition 3.6. *The dynamic program for solving the LSP under yield uncertainty runs in $O(T^3)$.*

Proof. See Appendix Appendix D □

Proposition 3.6 indicates that our dynamic program performs in $O(T^3)$. The problem becomes NP-hard when extended to the capacitated context, and this can be verified with a reduction from the deterministic single-period capacitated LSP (Florian & Klein, 1971; Bitran & Yanasse, 1982).

4. Results and discussions

This section presents the experimental study, and its objective is threefold: 1) to demonstrate the robustness of the presented approaches in coping with a non-deterministic LSP; 2) to report an in-depth investigation on the robust LSP with uncertain yield, its performance, the quality of the solutions, and the computational efficiency, and 3) to evaluate and compare the performance of the different optimization approaches in terms of the average and worst case quality of the solution.

The experiments consider the following solution approaches: DET_{SS} , the deterministic model with safety stock, as presented in Appendix F; DET , the nominal model with $\rho_t = \bar{\rho}_t$, as presented in Section 3.1; SP , the stochastic program, as presented in Appendix G; RO_{Γ} , the LSP under yield uncertainty, with uncertain yield and budget Γ , as presented in Section 3.3; and finally DP , the dynamic program for computing an optimal robust plan for the stationary LSP under box uncertainty set and with uncertain yield, as presented in Section 3.6.

Note that DET and SP are natural benchmarks to compare solution approaches that cope with uncertainties. DET_{SS} corresponds to the approach commonly used in practice, where a safety stock is computed separately from the lot sizes. This section is organized as follows: Section 4.1 presents the instance generation method. Section 4.2 introduces the simulation framework used to compare the methods. Finally, Section 4.3 presents an analysis of the developed models. We discuss the

performance of the dynamic program in Section 4.3.1, the investigation of the price of robustness in terms of the budget of uncertainty in Section 4.3.2, and the performance and quality of the plans resulting from different optimization approaches in Section 4.3.3. We also highlight the advantages of using robust optimization to hedge against uncertainties in a highly uncertain context.

4.1. Instance generation

We generate each parameter of the instances using uniform distributions. We set the value of the support of these uniform distributions in a similar manner to Brandimarte (2006) to generate setup and inventory costs. As Brandimarte (2006) considers demand uncertainty, we generate the demand similarly to Alem et al. (2018)). The production costs, the inventory costs, the demands, the nominal values of the production yield rate, and the maximum deviations of the production yield were randomly generated using an uniform distribution within the following intervals: $v_t \in U(10, 20)$, $h_t \in U(1, 10)$, $d_t \in U(140, 480)$, $\bar{\rho}_t \in U(0.5, 0.7)$, and $\hat{\rho}_t \in U(0.1, 0.3)$, respectively. The setup costs are computed with the time between orders formula: $s_t = \frac{\bar{D}_t \cdot TBO^2 \cdot h_t}{2}$, where \bar{D}_t represents the average demand up to period t . The capacity can be computed as $C_t = \Phi \frac{\sum_{\tau \leq t} d_\tau}{\min_{\tau \leq t} (\bar{\rho}_\tau - \hat{\rho}_\tau)}$, where Φ is a factor to control the tightness of the capacity. To adapt these instances for the LSP under yield uncertainty, we generate the nominal and maximum deviation of the yield with uniform distributions with expectations in the range (0.5;0.7) and (0.1;0.3), respectively.

For instance, Chu et al. (2019) consider the stationary case, where they set the mean and standard deviation of the production yield to 0,95 and 0,05, respectively. As we aim to study the impact of yield uncertainty in the lot-sizing model, we consider here the non-stationary uncertain production quality, where the quantity of quality goods varies significantly between periods. This setting relates to complex production systems in which factors affecting production yield are difficult to measure or estimate precisely due to lack of data. We consider instances with 4, 12, and 24 periods, a time between orders of 1, 2, or 4, and a backorder cost that equals 2, 5, or 10 times the holding cost for each period t . In addition, we consider a capacity factor Φ of 25%, 50%, and 75% for the capacitated model, and we assume a capacity factor of 100% for the non-capacitated model. We generate the instances with a full factorial design for these four parameters, which leads to 108 instances. We set the inventory and backorder levels to zero at the beginning of the horizon. Because the optimal policies are valid only for the uncapacitated models and the special case with non-stationary costs and demands but with constant nominal value and maximum deviation of the production yield, we generate 81 additional instances resulting from the factorial design of the following parameters: 6, 12, 24, 30, 36, 48, 96, 192, 384 periods, time between orders of 1, 2 or 4,

and backorder costs equaling $2h_t$, $5h_t$, or $10h_t$. We increase the instance size compared to the other instances to measure in terms of computational time the performance of the dynamic program for larger size problems.

4.2. Simulation

We analyze the quality of the production plans through a simulation with $|\Omega| = 5000$ scenarios generated with Monte Carlo sampling, where each scenario represents a possible realization of the production yield over the horizon. We simulate the yield with a uniform distribution with support $[\bar{\rho}_t - \hat{\rho}_t; \bar{\rho}_t + \hat{\rho}_t]$ in period t . We note $EVPI$, the expected value of perfect information, the average cost of the perfect information solutions, where $EVPI_\omega$ is the cost of the solution computed with the deterministic model for scenario ω . To evaluate each optimization method, we fix X_t and Y_t obtained from the optimization step in the deterministic model for each scenario ω .

4.3. Experimental results

This section presents an average cost analysis for the uncapacitated and capacitated problems. The algorithms were implemented in Python 3.6, and the mathematical models were solved with CPLEX version 12.10. The experiments were run on Intel(R) Xeon Broadwell 2683/2.1GHz processors with 125GB of RAM. All the models for all the instances were solved until optimality. We compare the methods based on the objective value provided by each optimization approach (e.g., the objective function given in (4) for RO_Γ), the average computational time (in seconds), the expected value (Exp. Cost) of each solution approach evaluated in the simulation, along with the worst case cost in the simulation, and the 95th and 99th percentile cost (p.c.). We also report the relative difference between the expected value of perfect information $EVPI$ and the simulated expected value of each method given by $GAP_{EVPI} = 100 \times \frac{\text{Exp. Cost} - EVPI}{EVPI}$. In addition, we define by $GAP_{OPT} = 100 \times \frac{\text{Exp. Cost} - \text{Obj. Value}}{\text{Obj. Value}}$ the relative difference between the objective value of a solution approach and its simulated expected cost. Finally, we report the coefficient of variation CV , an index that indicates a high variability of the costs in the simulation. Thus, CV is the ratio of the standard deviation to the mean, such that the higher CV is, the more widely dispersed the values are from the mean.

4.3.1. Performance of the dynamic program

First, we analyze the performance of the dynamic program to solve the multi-period LSP under box uncertainty set and yield uncertainty, with constant nominal value and maximum deviation of

the production yield, and with non-stationary costs and demands. Table 1 reports the computational times (in seconds) for *DP* and *RO* on the 81 instances generated for the special case. Each column corresponds to 9 instances with the same planning horizon, and we indicate the average time to solve them.

Table 1: Computational time of the dynamic programming approach and the reformulated MILP

Approach	Planning horizon size								
	6	12	24	30	36	48	96	192	384
DP	0.002s	0.01s	0.15s	0.28s	0.54s	1.65s	22.05s	320.63s	5245.16s
RO	0.183s	0.21s	0.37s	0.35s	0.30s	0.48s	2.30s	26.41s	2182.94s

Table 1 shows that the robust dynamic program is computationally less demanding than the robust model for instances with at most 30 periods, while *DP* becomes more computationally demanding than *RO* for a production horizon greater than 30 periods. However, when we consider the instances with production horizon larger than 30 periods, *DP* requires around 10.35 minutes to determine a solution for instances with a planning horizon up to 384 periods, while *RO* takes only 4.05 minutes on average to compute the same solution. These results show that the *DP* approach is better suited for small instances or when an efficient commercial solver is not available. However, for medium and large sized instances, the computational effort required by *DP* becomes too expensive compared to the solution computed with the *MILP* robust model. Moreover, *RO* can deal with more general LSPs, which makes its performance more competitive with a problem-specific approach. Therefore, the MILP model can be extended to include different practical constraints while it can still remain efficient in terms of speed in computing an optimal solution.

4.3.2. Price of robustness

To analyze the impact of the budget Γ , we consider different budgets to represent the decision-maker's risk aversion. For each instance, the budget varies from low aversion ($\Gamma = 0.1T$) to extreme aversion $\Gamma = T$, going through a progression with the size of the production planning. The budget represents the maximum number of uncertain parameters that can take the worst case value (Thiele, 2010). Since Γ indicates the number of periods where the production yield can take its worst case realization, it is convenient to express Γ as a proportion of T . Table 2 indicates the impact of the budget of uncertainty on the costs for all the considered instances. While the Obj Value column gives the average of the objective function values computed in the optimization step, the remaining

columns report the average value for the features obtained in the simulation step.

Table 2: Impact of the budget of uncertainty on the robust lot-sizing decision

Γ	Uncapacitated						Capacitated					
	Exp.	95 th p.c.	99 th p.c.	Worst	Obj	CV	Exp.	95 th p.c.	99 th p.c.	Worst	Obj	CV
	Cost			Cost	Value		Cost			Cost	Value	
0.1T	124,759	154,848	185,884	243,203	132,004	10.8%	132,974	163,341	195,299	246,471	143,413	10.9%
0.2T	137,315	154,685	165,599	195,557	169,567	7.2%	145,625	164,536	175,202	201,651	184,442	7.5%
0.3T	146,202	163,330	170,991	188,427	192,411	6.6%	156,249	174,874	182,625	200,529	210,890	6.7%
0.4T	155,134	174,025	181,590	191,107	218,111	7.0%	166,748	187,231	195,255	208,007	237,618	6.9%
0.5T	157,734	178,326	186,106	195,401	227,836	7.4%	169,338	190,815	199,027	210,768	247,834	7.1%
0.6T	159,507	181,895	190,214	197,804	236,085	7.9%	172,066	194,805	203,491	213,806	256,740	7.7%
0.7T	159,353	182,381	190,537	197,941	237,278	8.3%	171,205	194,614	203,547	213,448	258,356	7.9%
0.8T	158,777	182,053	190,164	197,690	237,573	8.4%	170,770	194,318	203,299	213,112	258,566	7.9%
0.9T	158,883	182,202	190,322	197,836	237,732	8.4%	170,782	194,377	203,370	213,232	258,805	8.0%
1.0T	158,883	182,202	190,322	197,836	237,732	8.4%	170,782	194,377	203,370	213,232	258,805	8.0%

As reported in Table 2, for both the uncapacitated and capacitated model, the best solutions in terms of objective value and expected cost are obtained for a very tiny budget (Γ equals $0.1T$ and $0.2T$), so the decision-maker should be willing to accept a high degree of risk resulting from uncertainties. Considering the 95th percentile costs, the lowest costs for the uncapacitated (resp. capacitated) model are obtained for Γ between $0.1T$ and $0.3T$, while the lowest 99th percentile costs are given by a budget equaling $0.2T$ and $0.3T$ for both variants. In addition, the lowest worst case costs are obtained for Γ between $0.2T$ and $0.5T$ for both uncapacitated and capacitated model, and the lowest coefficient of variation CV can be achieved for a budget that equals $0.3T$, $0.4T$ (resp. $0.3T$, $0.4T$ or $0.5T$). However, for Γ lower than $0.1T$, the robust objective value fails to cover the 95th percentile cost (and, consequently, the worst case cost). Therefore, with such a low Γ value, the decision-maker has no guarantee that the plan will be sufficiently immunized from uncertainties. On the other hand, although the solutions obtained for Γ greater than $0.4T$ achieve a robustness level that covers the worst case scenarios, they are overly conservative. In fact, for Γ between $0.5T$ and T , the objective value is much higher than the worst case cost. Thus, even if for the largest value of Γ the solutions are robust, they do not result in the best option in terms of the expected costs and stability. Therefore, the budget of uncertainty in the interval from $0.2T$ to $0.4T$ offers better production plans, since the decisions are sufficiently stable because of a low CV , the expected cost is relatively low, and the robust objective value covers at least the 99th percentile costs (and even

the worst case costs for when Γ equals $0.3T$ or $0.4T$).

4.3.3. Comparison of the LSP with uncertain yield resolution approaches

We now compare the production plans resulting from *RO*, *SP*, *DET* and *DET_{SS}*. We consider *RO* with a budget that is equal to $0.2T$, $0.3T$, and $0.4T$ because we have previously shown that these values result in better trade-offs in average costs, conservatism and robustness. We also consider the extreme case of the robust approach with a more conservative solution for $\Gamma = T$. All results reported in these sections are an average over all the considered instances. In addition, all values are computed during the simulation step except for the optimal values and computational time which are obtained from the optimization. Note that from the non-deterministic literature, nominal models ignore the occurrence of uncertainties, and they tend to have higher costs compared to other optimization approaches that take uncertainties into account. We also emphasize that the safety stock is a classic complementary approach to control uncertainties in a non-deterministic context, and such a safety stock can be incorporated into the deterministic model. The safety stock calculation, which relies on different assumptions, can be done separately as preprocessing step.

Table 3 shows the simulation results for the uncapacitated LSP. Here, the computation times required by *DET* (about 0.01s) and *DET_{SS}* (about 0.03s) are extremely low. Next, *RO* takes on average 0.38s to compute a solution, followed by *SP*, the most demanding approach, which takes about 19.6s to propose a production plan. *DET* is the less robust method because its GAP_{OPT} and its CV are the highest among all the methods tested, which indicates a costly solution highly impacted by yield disruptions. *DET* is followed by *DET_{SS}*, *SP* and *RO* solutions. Considering the quality of the solution in terms of GAP_{EVPI} , *SP* proposes the best expected cost followed respectively by the *DET_{SS}*, *DET* and *RO* models. However, *RO* is less impacted by disruptions of the production yield than *SP*, since the robust CV is about 7% on average, whereas *SP* gives a production plan whose CV is 15% on average. In addition, although *DET_{SS}* and *DET* give better expected costs than *RO*, these models lead to large costs for more adverse production yield values (since they have larger worst case, 99th and 95th percentile costs). Moreover, *DET_{SS}* and *DET* are much more impacted by yield uncertainty than *RO*. While, on average, *DET_{SS}* has a CV of around 19%, and *DET* has a CV of 25%.

Since *DET* and *DET_{SS}* do not offer good solutions in terms of robustness, we focus on the comparison between *RO* and *SP*. Although *SP* provides lower expected costs (115,624) than *RO* (149,384, on average), the robust model leads to a lower worst case cost (about 193,232 on average

Table 3: Performance of the uncapacitated models in terms of the average cost and worst case simulated costs

Model	Exp. Cost	95 th p.c.	99 th p.c.	Worst Cost	Comp. Time	CV	GAP _{EVPI}	GAP _{OPT}
<i>EVPI</i>	89,454	98,728	102,679	107,073		6%		
<i>DET_{SS}</i>	129,241	191,368	241,716	301,517	0.03	19%	36%	2%
<i>DET</i>	131,284	222,968	287,950	350,106	0.01	25%	27%	26%
<i>SP</i>	115,624	152,899	194,764	258,646	19.60	15%	20%	-4%
<i>RO_{Γ=0.2T}</i>	137,315	154,685	165,599	195,557	0.63	7%	44%	-18%
<i>RO_{Γ=0.3T}</i>	146,202	163,330	170,991	188,427	0.52	7%	53%	-23%
<i>RO_{Γ=0.4T}</i>	155,134	174,025	181,590	191,107	0.34	7%	60%	-27%
<i>RO_{Γ=T}</i>	158,883	182,202	190,322	197,836	0.04	8%	62%	-32%

over the different budgets) than the worst cost obtained with the stochastic plan (258,646). In addition, the robust models have a coefficient of variation of 7% on average, while *SP* has a *CV* equal to 15%. This value confirms that *SP* is more impacted by disturbances on the production yield than the robust plans. Nevertheless, we stress that the *SP* and *RO* methodologies have fundamentally different objectives. Although the stochastic program seeks the minimum expected costs, the robust optimization method aims the minimum objective value that covers the costs incurred even for the worst realization of the uncertain yield. In the same vein, the relative difference between *EVPI* and *RO_Γ* is greater than the *GAP_{EVPI}* between *EVPI* and *SP* because the robust models propose a production plan that remains cost effective, even for the worst realization of the uncertain yield for a well-chosen budget of uncertainty, while *SP* may be ineffective in case of adverse events. The robust strategy leads to more conservative solutions than the production plan proposed by the stochastic program, for which the strategy is defined regarding the probability of the realization of the uncertainty. While *SP* is known to be prone to changes in the underlying uncertainty (e.g., if the distribution changes), the *RO* remains stable and robust for different and unrelated uncertainty realizations. This can be verified in the column 99th percentile and worst case average costs, for which robust models are much less impacted by the uncertain parameter, leading to lower costs.

Table 4 presents the results for the capacitated version of the LSP. Much like the uncapacitated model, *DET* has the lowest computational time of 0.01s, followed by *DET_{SS}* (about 0.03s), *RO* (about 0.24s on average), and finally *SP* (around 15.64s). The capacitated *DET* is not robust enough in comparison with other approaches, since it has the highest *CV* and it gives the highest costs in terms of 95th and 99th percentile, and worst case costs. In addition, although the capacitated *DET_{SS}* model has a *GAP_{OPT}* as good as *SP*, its 95th and 99th percentile costs, as well as its

Table 4: Performance of the capacitated models in terms of the average cost and worst case simulated costs

Model	Exp. Cost	95 th p.c.	99 th p.c.	Worst Cost	Comp. Time	CV	GAP _{EVPI}	GAP _{OPT}
<i>EVPI</i>	99,655	112,824	118,132	124,212		8%		
<i>DET_{SS}</i>	136,003	188,261	232,546	289,263	0.03	17%	30%	-4%
<i>DET</i>	137,722	222,418	281,401	346,124	0.01	24%	22%	23%
<i>SP</i>	124,029	162,394	204,674	263,150	15.64	16%	17%	-4%
<i>RO_{Γ=0.2T}</i>	145,625	164,536	175,202	201,651	0.31	7%	38%	-20%
<i>RO_{Γ=0.3T}</i>	156,249	174,874	182,625	200,529	0.37	7%	48%	-25%
<i>RO_{Γ=0.4T}</i>	166,748	187,231	195,255	208,007	0.24	7%	56%	-28%
<i>RO_{Γ=T}</i>	170,782	194,377	203,370	213,232	0.03	8%	59%	-32%

worst case cost, are higher than the respective *SP* costs. *DET* is also less robust since its *CV* is around 24%. Therefore, *DET* and *DET_{SS}* are not competitive in terms of performance to mitigate uncertainties when compared to *SP* and *RO* models. While the lowest expected cost is given by *SP*, the lowest 99th percentile and worst case costs are given by the robust models. *RO* not only gives the lowest costs, especially for Γ equals 0.3*T* or 0.4*T*, but also fully immunizes the problem from uncertainties with a production plan whose objective value covers any realization of the uncertain yield. The *CV* of *RO* models remains the same in comparison with the uncapacitated version of the problem, and the values are still lower than the *CV* for the *SP* solutions (that decreases to 24%). This indicates that the robust plans are more stable than the stochastic ones for different realizations of the uncertain yield. In addition, the relative difference between the optimization methods and *EVPI* becomes lower, and the optimality gap for *RO* also decreases. The results show that the robust approach is efficient when mitigating uncertainties because it offers a good, stable and robust production plan. Although the expected costs from the robust models (159,851 on average) are higher than the respective *SP* costs (about 124,029), the robust 99th percentile cost (on average 189,113) is lower than the stochastic one (about 204,674). Similarly, the robust worst case cost (about 205,855 on average) is much lower than the stochastic one (around 263,150). In addition, for a budget of uncertainty greater than or equal to 0.3*T*, *RO* worst case cost is covered by the objective value. As a result, the *SP* model is less efficient when uncertainty information is relatively limited or if we want to limit a downside risk due to the realization of uncertainties.

The *RO* framework provides a systematic approach to determine a robust production plan that mitigates uncertainties with a conservatism partially controlled by the budget of uncertainty. Similarly to Hnaien & Afsar (2017), where the authors conclude that a robust capacitated LSP is

easier to solve than the uncapacitated version, we also observe the same pattern. Our capacitated *RO* model is generally easier to solve, which is not usually the case for the deterministic model where the capacitated model is more difficult to solve (e.g., Brahim et al. (2006)). Hnaien & Afsar (2017) indicate that the capacitated model has a bound on the lot size lower than the natural bound, and as the linear relaxation of the capacitated version may be less fractional, it may lead to faster calculations. However, there may be other explanations. There are some capacitated problems in the literature where the instances are designed to be the most challenging, and so time consuming. As a result, the computational effort to solve the capacitated problem may change depending on the instance configuration in size, structure (e.g., multi level), costs, setup features (e.g., setup times, setup carryover, high setup costs), resource availability (e.g., tight capacity), and resolution approach (Buschkühl et al., 2010). On the other hand, robust models search for solutions that are immunized from uncertainties, which leads to a large number of feasible solutions. As a consequence, capacitated *RO* models have a reduced amount of feasible solutions in the worst case perspective, since limited resources lead to more restrictions defining the solution space (Zhanga et al., 2020). Hence, the robust models with capacity constraints potentially require less computing effort. This has also been observed in other literature (e.g., Hnaien & Afsar (2017); Zhanga et al. (2020)).

To analyze the cost components incurred in the simulated production system, Figure 3 indicates the setup frequency for the uncapacitated and capacitated models, Figure 4 reports the proportion of the average expected costs imputable to the setup, production, inventory, and backorder costs for the uncapacitated and capacitated models. Finally, Figure 5 reports the average lot size $\|\mathbf{X}\|$, inventory $\|\mathbf{I}\|$, and backorders $\|\mathbf{B}\|$ levels accumulated over the entire production horizon, and the inventory and backorder levels at the end of the production planning for the uncapacitated and capacitated models.

Figure 3 shows the setup frequency for different capacity levels. As expected, the frequency of setup increases when the capacity decreases. For the uncapacitated models, *RO* has the highest setup frequency (with setup frequency of 62% on average). It is followed by *SP* with a setup frequency of 46%, *DET_{SS}* with a setup frequency of 44%, and finally *DET* with a setup frequency of 43%. The *RO* models adopt a strategy with more frequent production setups to reduce the total costs and to avoid backorder costs when the problem faces low-yield values. Figure 3 also reports that the robust production plans for Γ from $0.2T$ to $0.4T$ not only mitigate uncertainties better, they also provide a configuration setup that is still efficient even if resource availability is disrupted.

As shown, for the different capacity factors, the setup frequency for these robust models does not suffer major disturbances.

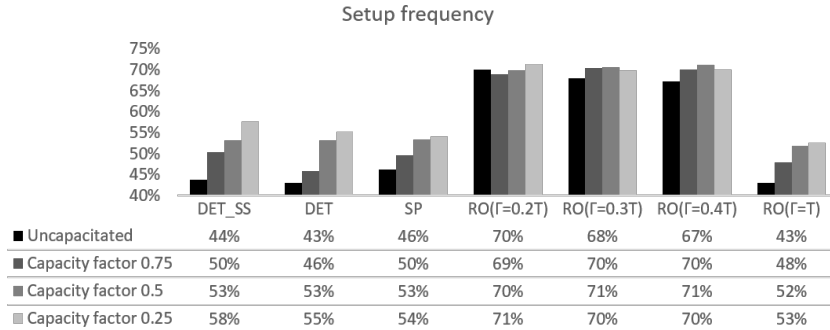
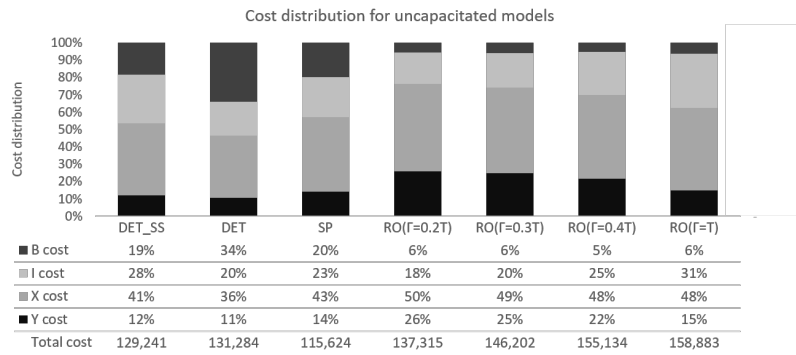


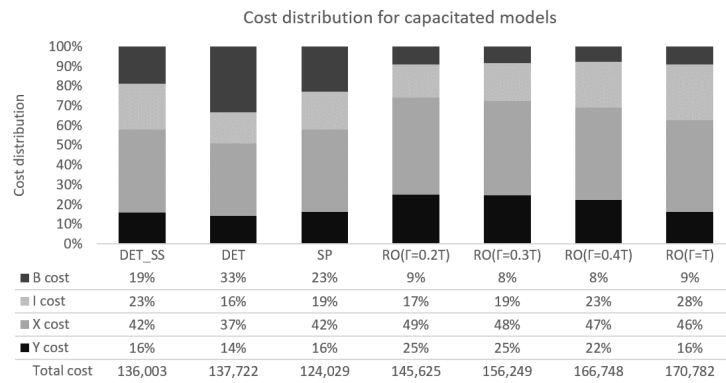
Figure 3: Characteristics of the solutions in terms of setup frequency

Figure 4 gives an overview of the cost distributions for all considered approaches. *RO* leads to fewer backorders (6% of the cost for *RO* when $\Gamma = 0.3T$ versus 20% for *SP*) at the expense of higher setup, production and inventory costs. *RO* tends to exceed the costs from *DET*, *DET_{SS}* and *SP* for all components except the backorder costs. However, the robust backorder costs are much lower than the respective *DET*, *DET_{SS}* and *SP* costs, which compensate for the highest setup, production and inventory costs. Therefore, the robust plan offers more possibilities for the decision-maker to take advantage of the available resources, while also reducing the impact of the uncertain events on the production plan. For instance, when the availability of the resources is more restricted, backordering becomes more frequent. *RO* manages to control (and even reduce) the backorder cost by increasing the lot size and the frequency of production, yet lot size and inventory levels remain acceptable. As a result, the robust model favors a large production level to meet demands, *DET_{SS}* relies on large stock level, and *DET* and *SP* take the risk of backordering goods.

Figure 5 shows that all approaches keep as many goods in stock over the entire production planning as possible. Although this strategy reduces the backorder levels (with little increase for the capacitated models), it requires larger lot sizes to keep enough quality goods to meet demands. The production quantity becomes even higher for the *DET_{SS}* than for *DET* to ensure that the safety stock is respected. *DET* and *SP* have quite similar production plans, with a more important difference in terms of the inventory level. For the uncapacitated problem, *RO* produces an amount of goods relatively close to that proposed by *SP* (6,411 units on average when considering all Γ values versus 6,033 units produced on average with *SP* and 6,229 units produced with *DET*), yet



(a) Uncapacitated problem



(b) Capacitated problem

Figure 4: Characteristics of the solutions in terms of the cost distribution

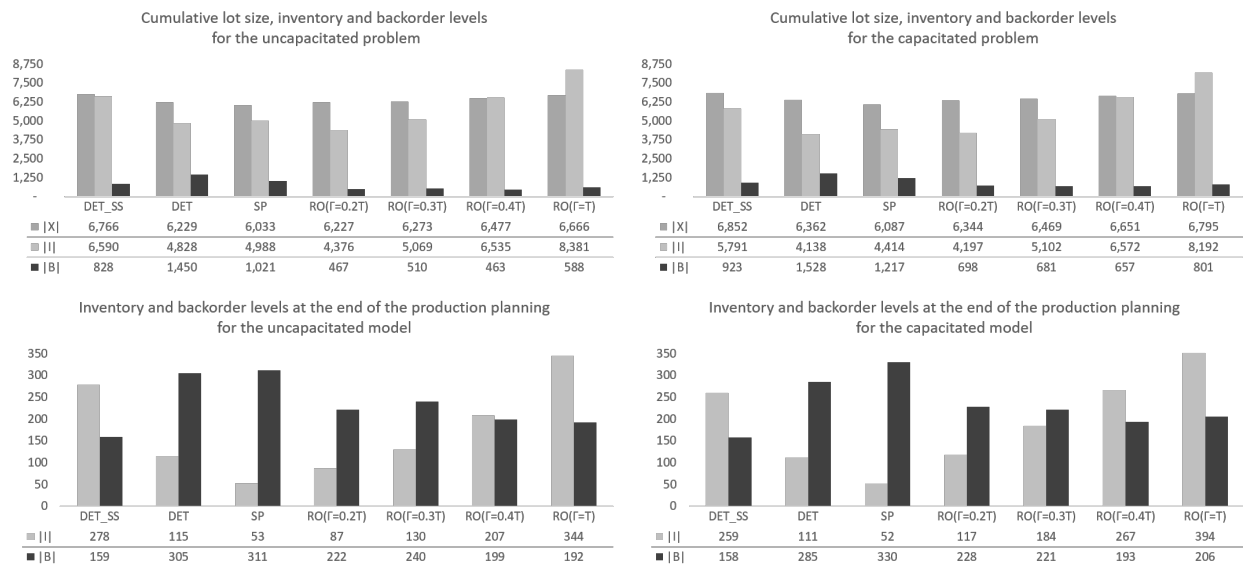


Figure 5: Characteristics of the solutions in terms of production, inventory and backorder levels

it leads to a low backorder level (an average of 507 units backordered with the robust plan versus 1,021 units backordered units with SP and 1,450 units with DET). Besides, the robust models have a larger inventory level on average of 192 units (resp. 241 units), versus 53 units (resp. 52 units) for SP and 115 units (resp. 111 units) for DET .

To conclude, the RO provides effective support for decision-makers. Contrary to DET_{SS} , DET and SP , RO provides an objective value that is larger than the expected simulated costs, and this can reassure the decision-maker. In addition, unlike other approaches, the robust plan covers even the most pessimistic scenario. When we investigate the stability and robustness of the proposed plans, the robust approaches provide the production plan that copes better with uncertainties because it tends to offer greater cost savings with a low impact of yield disturbance on the production plan. In addition, contrarily to SP , DET and DET_{SS} which adopt a strategy that places backorders more often to reduce the inventory and production costs, RO relies on a sufficiently low stock level that satisfies demands and which is supported by sparse production setups that minimize costs with sporadic production backorders. Therefore, the robust models mitigate the impact of the realization of unknown and pessimistic scenarios on the production plan better.

5. Conclusion

In this paper, we have introduced a robust formulation for lot-sizing under yield uncertainty. We show that the multi-period problem under box uncertainty set, where the average and standard

deviation of the production yield are constant over the planning horizon while costs and demands are not, can be solved in polynomial time with a dynamic programming approach. This work also proposes a mixed-integer linear program for the non-stationary LSP with uncertain yield, and it provides insights into robust production plans. Our results show that with a proper budget of uncertainty, the robust model mitigates uncertainties with a balance between production quantities, setup costs, and inventory management costs. In addition, the robust optimization method requires less computational effort than stochastic programming, and its solutions are less conservative, yet more robust, compared to the classical approaches to dealing with uncertainties on LSPs (represented here by the nominal problem and the deterministic problem with safety stock). Other major advantages of robust optimization over the other proposed approaches are that the robust approach requires little information about the uncertainty factors and no strong assumption on the uncertain parameter characteristic. Since the static solution is an upper bound of the adaptive solution, our contributions will serve as a basis for future work that address the adaptive strategy. Further investigation is still needed to propose an adaptive framework to cope with uncertainties within a static-dynamic, or even a dynamic, decision framework. The present work could also be extended to deal with multi-echelon systems.

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Appendices

Appendix A. Proof of Proposition 3.1

Proof. In the robust single-period inventory management problem under box uncertainty set, the worst inventory (resp. backorder) cost corresponds to the largest (resp. lowest) production yield, and this situation corresponds to $Z = 1$ (resp. $Z = -1$). For this special case, we denote by $\widehat{I}(X)$ the worst inventory cost and by $\widehat{B}(X)$ the worst backorder cost. These costs depend only on the lot size X , and they are given as follows:

$$\widehat{I}(X) = \max \{h[X(\bar{\rho} + \hat{\rho}) - d], 0\} \qquad \widehat{B}(X) = \max \{b[d - X(\bar{\rho} - \hat{\rho})], 0\}$$

The total cost $TC = \widehat{I}(X) + \widehat{B}(X)$ is piecewise linear convex in the lot size X , and it reaches its minimum when $\widehat{I}(X) = \widehat{B}(X)$. □

Appendix B. Proof of Proposition 3.2

Proof. Since the optimal policy achieves its minimum cost when the worst inventory cost is equal to the worst backorder cost, from Proposition 3.1, we can directly derive the optimal policy as the production quantity X for which $\widehat{I}(X) = \widehat{B}(X)$. Thus, with a little algebra, we can isolate X in the equality $h[X(\bar{\rho} + \hat{\rho}) - d] = b[d - X(\bar{\rho} - \hat{\rho})]$, that results in our optimal robust policy $X = \frac{d}{\bar{\rho} + \hat{\rho} \left(\frac{h-b}{h+b} \right)}$. □

Appendix C. Proof of Proposition 3.3

Proof. As in the proof for the single-period problem, we define the inventory and backorder costs in terms of the production quantities $X_{[t]}$ and disturbance from the mean $Z_{[t]}$ up to period t as follows:

$$I_t(X_{[t]}, Z_{[t]}) = \max \left\{ h_t \sum_{\tau=1}^t [X_\tau(\bar{\rho} + \hat{\rho}Z_\tau^t) - d_\tau] ; 0 \right\}$$

$$B_t(X_{[t]}, Z_{[t]}) = \max \left\{ b_t \sum_{\tau=1}^t [d_\tau - X_\tau(\bar{\rho} + \hat{\rho}Z_\tau^t)] ; 0 \right\}$$

where $X_{[t]}$ (resp. $Z_{[t]}$) is the vector of X_τ (resp. Z_τ^q) $\forall \tau \leq t; q \leq \tau$. We assume that the values of Z_τ^t are chosen independently for each period t . Therefore, the worst case perspective in a period t sets Z_τ^t to 1 (for inventory) or -1 (for backorder) for all $\tau \leq t$. Let us define the cumulative demand as $\bar{D}_t = \sum_{\tau=1}^t d_\tau$ and the cumulative production quantity up to period t as $\bar{X}_t = \sum_{\tau=1}^t X_\tau$. The worst inventory cost ($\hat{I}_t(\bar{X}_t)$) and the worst backorder cost ($\hat{B}_t(\bar{X}_t)$) in period t can be written as follows:

$$\hat{I}_t(\bar{X}_t) = \max \{ h_t [\bar{X}_t(\bar{\rho} + \hat{\rho}) - \bar{D}_t] ; 0 \}$$

$$\hat{B}_t(\bar{X}_t) = \max \{ b_t [\bar{D}_t - \bar{X}_t(\bar{\rho} - \hat{\rho})] ; 0 \}$$

The total cost TC_t for period t is given by $TC_t = v_t X_t + \hat{I}_t(\bar{X}_t) + \hat{B}_t(\bar{X}_t)$. TC_t is a piecewise linear convex function in \bar{X}_t . From Proposition 3.1, TC_t reaches its minimum when $\hat{I}_t(\bar{X}_t) = \hat{B}_t(\bar{X}_t)$. When $v_t \geq 0$, the function defining the total cost at period t only has an upward shift equivalent to the total production costs in period t where $\hat{I}_t(\bar{X}_t) = \hat{B}_t(\bar{X}_t)$. The period where the minimum is reached does not change. Therefore, the condition $v_t \leq (\bar{\rho} - \hat{\rho})b_t$ indicates that it is profitable to produce while the production cost is lower than the backorder cost for the worst realization of the production yield, that is, for the lowest production yield rate. Note that if $v_t > (\bar{\rho} - \hat{\rho})b_t$, the production cost becomes larger than the backorder cost, and TC_t is a strictly increasing function. Thus, TC_t reaches its minimum when $\hat{I}_t(\bar{X}_t) = \hat{B}_t(\bar{X}_t)$, if condition $v_t \leq (\bar{\rho} - \hat{\rho})b_t$ is respected.

Since the cumulative demand increases with t , we seek for the cumulative production quantities that optimize the total cost for each period and lead to a minimization of the sum maximum cost over the horizon up to t . With a little algebra, from the equality $\hat{I}_t(\bar{X}_t) = \hat{B}_t(\bar{X}_t)$, we define the optimal robust cumulative policy for each period t as follows:

$$\bar{X}_t = \frac{\bar{D}_t}{\bar{\rho} + \hat{\rho} \left(\frac{h_t - b_t}{h_t + b_t} \right)} = \frac{\sum_{\tau=1}^t d_\tau}{\bar{\rho} + \hat{\rho} \left(\frac{h_t - b_t}{h_t + b_t} \right)} \quad (\text{C.1})$$

Since the production in period t is equivalent to the difference between the cumulative production up to t given by \bar{X}_t and the cumulative production up to period $t - 1$ given by \bar{X}_{t-1} , with a little

algebra we define the optimal robust policy for the multi-period inventory management problem under box uncertainty set and yield uncertainty, and with non-stationary inventory and backorder costs as follows:

$$X_t = \bar{X}_t - \bar{X}_{t-1} = \frac{\bar{D}_t}{\bar{\rho} + \hat{\rho} \left(\frac{h_t - b_t}{h_t + b_t} \right)} - \frac{\bar{D}_{t-1}}{\bar{\rho} + \hat{\rho} \left(\frac{h_{t-1} - b_{t-1}}{h_{t-1} + b_{t-1}} \right)} = \frac{\bar{\rho} d_t + \hat{\rho} \left[\left(\frac{h_{t-1} - b_{t-1}}{h_{t-1} + b_{t-1}} \right) \bar{D}_t - \left(\frac{h_t - b_t}{h_t + b_t} \right) \bar{D}_{t-1} \right]}{\left(\bar{\rho} + \hat{\rho} \left(\frac{h_t - b_t}{h_t + b_t} \right) \right) \left(\bar{\rho} + \hat{\rho} \left(\frac{h_{t-1} - b_{t-1}}{h_{t-1} + b_{t-1}} \right) \right)}$$

□

Appendix D. Proof of Proposition 3.6

Proof. Since our dynamic program relies on Proposition 3.4 and Proposition 3.5, it addresses the uncapacitated LSPs under a box uncertainty set, with uncertain yield, with non-stationary costs and demands, and with stationary nominal value and maximum deviation of the production yield. Clearly, the execution time is dominated by the search for successive regeneration intervals and their respective setup periods for each t in function $F(t)$. The search for the best production setup β and regeneration interval $[m, n]$ that gives the minimal $MCI(m, \beta, n)$ cost takes at most $\mathcal{O}(T^2)$. One should also consider the function $MCR(r, T)$ that minimizes the costs from the last regeneration period to the end of the planning horizon in $\mathcal{O}(T)$ run time. However, $\mathcal{O}(T)$ is dominated by $\mathcal{O}(T^2)$. Since dominated complexities can be ignored, then we only consider $\mathcal{O}(T^2)$. With the forward approach, we can access previously computed values in $\mathcal{O}(1)$. Thus, the recursive function $F(t)$ takes only $\mathcal{O}(1)$ to access the values for $F(\tau)$, $\tau < t$. Running the dynamic program calls T times the recursive function $F(t)$. Consequently, the dynamic program takes at most $\mathcal{O}(T^3)$ run time to propose a solution over the entire planning horizon. □

Appendix E. The newsboy-based LSP under yield uncertainty

Khouja (1999) present an excellent review on the single period newsboy problem. An introduction to the problem under yield uncertainty is presented by Noori & Keller (1986). Some considerations about the lot sizing as a single-stage, single-item, multi-period newsboy problem with deterministic demand random yields was proposed in Yano (1989). The authors present the dynamic program for the newsboy-based formulation of the LSP with uncertain yield. Inspired by the newsboy-based LSP formulation proposed by Yano (1989), for each period $t \in T$, we can implement a newsboy-based problem (NB) to deal with a lot-sizing problem with uncertain yield with a lot-for-lot policy that is given as follows:

$$NB : \min E(Z_t) = v_t Q_t + b_t \int_0^{\beta_t} (d_t - \rho_t Q_t) f(\rho) d\rho + h_t \int_{\beta_t}^1 (\rho_t X_t - d_t) f(\rho) d\rho \quad (\text{E.1})$$

where $\beta_t = \frac{d_t}{Q_t}$ is the expected production yield rate, and Q_t the production quantity in period t .

The conditions for minimization optimality are $Q_t^* \Leftrightarrow \frac{\partial E(Z_t)}{\partial Q_t} = 0$; $\frac{\partial^2 E(Z_t)}{\partial Q_t^2} \geq 0$. We have $\frac{\partial E(Z_t)}{\partial Q_t} = v_t + h_t \bar{\rho}_t + (b_t + h_t)E(\beta)$ as a first condition for optimality. Therefore, production happens if and only if the first condition of optimality is met. Thus, we may produce when:

$$\frac{\partial E(Z_t^*)}{\partial Q_t^*} = v_t + h_t \bar{\rho}_t E\left(\frac{d_t}{d_t} = 1\right) + (b_t + h_t)E(\beta_t^*) = 0 \quad \therefore \quad E(\beta_t^*) = \frac{v_t + h_t \bar{\rho}_t}{b_t + h_t}$$

where $\beta_t^* = \frac{d_t}{X_t^*}$. The second condition for optimality is $\frac{\partial^2 E(Z_t)}{\partial Q_t^2} = \frac{(b_t + h_t)d_t^2}{Q_t^3}$. The optimality conditions say that given the expected production yield we may produce at least $Q_t = d_t$ otherwise, we may backorder the demands. Then, if any production is made, we have $\frac{\partial^2 E(Z_t)}{\partial Q_t^2} \geq 0$.

For the capacitated LSP, we check if the quantity to be produced respects the resource availability. Any quantity not supported by available capacity must be backordered. We repeat this process over the production planning horizon to compute the optimal newsboy-based LSP solution.

Appendix F. The deterministic LSP with safety stock

Inspired by the work of Absi & Kedad-Sidhoum (2009), we consider DET_{SS} , the deterministic LSP with production yield ρ_t and safety stock SS_t , which is given as follows:

$$\begin{aligned} & \min \sum_{t \in T} s_t Y_t + v_t X_t + h_t I_t + b_t B_t \\ & \text{s.t. :} \\ & I_t - B_t = I_{t-1} - B_{t-1} + \rho_t X_t - d_t & \forall t \in T \\ & X_t \leq M_t Y_t & \forall t \in T \\ & I_t \geq SS_t & \forall t \in T \\ & X_t, I_t, B_t \geq 0 & \forall t \in T \\ & Y_t \in \{0, 1\} & \forall t \in T \end{aligned}$$

We compute the safety stock with the newsboy-based solution, such that $SS_t = Q_t - d_t$, where Q_t is the production quantity defined with NB . This leads to a situation where backorders and inventory can occur in the same period, so that safety stock that is not respected is penalized through backorder costs.

Appendix G. The two-stage stochastic programming LSP with uncertain production yield

SP handles uncertainty through a mathematical program whose objective is to minimize the expected cost (Birge & Louveaux, 2011). The uncertain parameter is described by a probability distribution and some statistical indicators (e.g., mean and standard deviation) that are usually gathered by processing and analysis of data from historical data and other available data about the decision system. The stochastic programs are a natural benchmark to compare the production plans proposed by other methodologies and verify their performance and quality.

To evaluate the performance of our robust model, we propose a scenario-based stochastic program to represent the lot-sizing problem under yield uncertainty, based on the case study presented by Kazemi Zanjani et al. (2010). While the authors present a multi-stage formulation which deals with a dynamic decision problem, our work addresses a static decision strategy. A two stage approach corresponds to a static decision in which the plan for the entire production horizon is defined before the realization of the uncertainty. Since our work investigates the static version of the LSP with uncertain yield, a two stage model is a more appropriate benchmark to measure the quality of the static robust model. We consider a set Ω of possible yield scenarios, where each scenario ω has a probability p_ω of realization. ρ_t^ω is the realization of the uncertain yield for the period t of in scenario ω . The two-stage stochastic program for the LSP with uncertain yield is given as follows:

$$\begin{aligned}
 & \min \sum_{\omega \in \Omega} p_\omega \sum_{t \in T} s_t Y_t + v_t X_t + h_t I_t^\omega + b_t B_t^\omega \\
 & \text{s.t. :} \\
 & I_t^\omega - B_t^\omega = I_{t-1}^\omega - B_{t-1}^\omega + \rho_t^\omega X_t - d_t & \forall t \in T; \omega \in \Omega \\
 & X_t \leq M_t \cdot Y_t & \forall t \in T \\
 & X_t \geq 0 & \forall t \in T \\
 & I_t^\omega, B_t^\omega \geq 0 & \forall t \in T; \omega \in \Omega \\
 & Y_t \in \{0, 1\} & \forall t \in T
 \end{aligned}$$

Although SP is largely applied within optimization under uncertainties, this approach often suffers from scalability issues, being computationally prohibitive, and requiring advanced techniques to generate possible scenarios. For this, as many scenarios as possible are generated in order to reflect the uncertainty distribution, even though the number of scenarios may be limited to restrict the computational efforts. For our numerical experiments, we used this stochastic problem with 500 scenarios, where the yield realizations are randomly drawn from a uniform distribution with support $[\bar{\rho}_t - \hat{\rho}_t; \bar{\rho}_t + \hat{\rho}_t]$ for each period t .