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Approaching concepts involved in different fields with discrete mathematics: a situation from ongoing mathematical research

Cécile Ouvrier-Buffet

Abstract. This article develops a situation, coming from ongoing mathematical research, that can enable students to experience a mathematical activity involving the construction of definitions. An epistemological analysis "à la Lakatos" of this situation is developed: it highlights how the discrete situation allows a mathematical experience for undergraduate students and contains interesting and even challenging material for all levels, including teachers, lecturers and research mathematicians. The discrete situation opens perspectives for higher education, especially on defining processes and proofs, with concepts important across multiple areas of mathematics (generating set, minimality) in the background.

Keywords. Discrete mathematics, generating set, minimality, definition, defining process.

Résumé. Cet article propose une situation, issue de la recherche en mathématiques, pouvant permettre de faire vivre aux étudiants une activité mathématique impliquant la construction de définitions. Une analyse épistémologique "à la Lakatos" de cette situation est développée : elle souligne comment la situation discrète permet une expérience mathématique pour des étudiants à l'université et contient du matériel intéressant et même stimulant pour tous les niveaux, y compris les enseignants, les chargés de cours et les mathématiciens. La situation proposée apporte ainsi des perspectives, pour l'enseignement supérieur, en particulier sur les processus de définition, la preuve, avec, en arrière-plan, des concepts transversaux aux mathématiques (ensemble générateur, minimalité).

Mots-clés. Mathématiques discrètes, ensemble générateur, minimalité, définition, processus de définition.

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1. Introduction

The main feature of my mathematical and didactical research concerns the heuristics, and more precisely the "defining processes" which are used in mathematical research. Several studies about problem-solving have underscored the various interests of putting learners in a mathematical enquiry with open problems (e.g. Schoenfeld, 1994; Mousoulides & Sriraman, 2020; Balacheff, in press). These studies concern proof, modeling, changes of representations, use of new technologies etc. but little is found about "defining" even if the study of defining activities is a discreet but constant didactical topic of research in mathematics education since the 90s (e.g Mariotti & Fischbein, 1997; Zandieh & Rasmussen, 2010; see Ouvrier-Buffet 2013 for a synthesis). This is one of the reasons that prompts me to initiate further discussion about "definitions" between mathematicians and mathematics educators, through this article: the definitions and their construction have definitely a place in mathematical research. I have the following guiding idea: I consider that definitions are concepts holders. During a research process, they are not finite products. Definitions evolve through several statements, each one giving new features of the involved connected concepts. In fact, to understand how concept formation works implies exploring the wide field of mathematical definitions considered as concepts holders. Studying the dialectic between the construction of definitions and the formation of concepts should be useful to design didactical situations where learners would have to build new concepts in order to solve a problem. That leads me to several major questions: how can one characterize defining processes in mathematics? Are learners able to define? Do they understand concepts better if they are involved in a defining process? How are linked defining and proving processes? I therefore had to work out a theoretical framework through epistemological, didactical and empirical research in order to characterize defining processes (Ouvrier-Buffet, 2013, 2015ab). I have conducted several experiments[1] at the university level: the results of these experiments show that the students' ability to define is real. Moreover, I have interviewed mathematicians who think suitable to implement defining activities at the university level, but without conceptualizing the way it can be implemented with students (Ouvrier-Buffet 2013, 2015b). They are very interested in the

[1] My experiments were conducted in discrete mathematics with the following concepts which are of different natures: trees (a known discrete concept, graspable in several ways), discrete straight lines (a concept which is still at work in mathematics and computer science, for instance in the perspective of the design of a discrete geometry) and a wide study of properties of displacements on a regular grid (e.g. Ouvrier-Buffet, 2003a, 2003b, 2006, 2011).

didactical research that can lead to new situations for the university. The main objective of such an approach is to engage students in a real mathematical activity dealing with skills important in mathematics (defining, proving), while trying to overcome the well-known difficulties of students at the secondary-tertiary transition. Indeed, several syntheses (e.g. Gueudet & Vandebrouck 2022, Selden 2012) converge on students' difficulties and causes at this transition marked by a change in requirements and in the relationship to the objects and processes of proof, more complex and more formal, which requires an acculturation to the practices of mathematicians: abstract mathematical notions raising conceptualization difficulties, difference in institutional culture between secondary and higher education, lack of articulation between semantic, syntactic and pragmatic approaches etc. Early university students use inappropriate reasoning about mathematical concepts and their relationships and fail to construct proofs (e.g. Selden & Selden 2003). Besides, the students themselves feel that they have great difficulty with the logic and formalism.

In this context, activities involving defining processes and proofs seem promising as pointed out by researchers in various domains e.g. Larsen & Zandieh (2005) quoting Freudenthal (1973) or Edwards & Ward (2008). Indeed, Edwards & Ward (2008) insist on the fact that undergraduate students "(…) do not use definitions the way mathematicians do, even in the apparent absence of any other course of action." (p. 417) and ask for a promotion of an understanding of the role of definitions in mathematics. But how to characterize this role, the defining processes and defining activities? Does it mean transferring some of the research process done by mathematicians to the classroom? How to implement such a process with students in university? For what purposes?

To begin with, I will present a part of my theoretical framework borrowed from epistemology, that of Lakatos (1961, 1976), focusing on definitions and definitional procedures. I will use it in order to highlight the potential defining processes in an original problem - displacements on a regular grid - still partially open from a mathematical point of view. This problem should bring a wealth of interesting material for students, lecturers and researchers, mainly in order to propose supervised research projects to students. I will then give some elements of the processes of freshmen at university, from scientific and non-scientific courses, when they face this problem, in order to illustrate the features and the obstacles of defining processes. The concluding section argues that this research opens new opportunities to deal with abstract contents at university, involving discrete mathematics but also defining processes, and engaging students in a mathematical exploration of a real ongoing problem, namely a research situation (in the sense of Gravier & Ouvrier-Buffet, 2022).

2. Epistemological framework

Exploring the wide question of "definitions" can lead us to deal with philosophical notions (such as those of Aristotle, Leibniz or Popper), logical constraints linked to the construction of a formal theory, epistemological tendencies (e.g. Lakatos 1976) and cognitive aspects (e.g. Vinner, 1991, with *concept image* and *concept definition*). As pointed out by Alcock & Simpson (2009, p.3-17), "definitions themselves may be complicated and considerably harder to work with" (p. 13) for students. They use Vinner's tool to exemplify and underline students' difficulties with various concepts (functions, limits, convergence, groups…) when concept images (i.e. knowledge and experience associated with a concept) and spontaneous conceptions interfere with concept definition (i.e. the mathematical

definition). Regarding these students' difficulties with definitions, Alcock & Simpson (2009) also stress the need "to give students access to more authentic mathematical experience." (p. 16), for example in small-class situations.

I would like to underline that looking at definitions as specific statements does not give us any clues to characterize defining processes. From a cognitive point of view, Freudenthal (1973, p.134) emphasized that "Establishing a definition can be an essential feat, more essential than finding a proposition or a proof". He studied two kinds of definitions that the learners have acquainted in their mathematical experience: the *descriptive definition* and the *constructive and creative definition*. But, in both cases, the mathematical concept involved is already known by the learners; the learners do not build it in order to solve a problem, they do not encounter a quite new concept. Indeed, the descriptive defining "outlines a known object by singling out a few characteristics properties" (ibid., p. 457), and the constructive defining "models new objects out of familiar ones" by examining changes to a given definition. Freudenthal (1973) provides several examples about the constructive defining process, where the definition of a concept is modified by excluding, generalizing, specializing, replacing or adding properties to the definition. Here, we can say that a new concept is thus created. I would like to go further than Freudenthal in the direction of the constructive defining process, but for concepts unknown by the learners. So, my aim is then to design defining situations with unfamiliar concepts for students in order to avoid any pre-existing concept images or conceptions.

Let me consider an aspect of Lakatos's work (the concept-stretching), which in my opinion does not get all the attention it deserves: "A definitional procedure is a procedure of concept formation" (1961, p. 54). This kind of procedure focuses on knowledge expressed in skills and processes. Lakatos's original contribution to the debate consisted in his attempt at a modeling of the mathematical discovery while integrating both the social[2] and the conceptual[3] aspects of the matter. He was strongly influenced by Pólya's mathematical heuristics, Hegel's dialectic, and Popper's fallibilism. Lakatos has clearly influenced mathematics education, especially in the development of new kinds of interactions between teacher and students and in the implementation of the game of proofs and refutations in the classroom[4] (e.g. Larsen & Zandieh, 2008). However, Lakatos's modeling of definitional procedures, more present in his PhD thesis (1961) than in his book 'Proof and Refutations' (1976), is little known, or rather little used. I will take this work about defining processes into account and test it on a research situation.

We should be well-aware that the processes (the defining processes and the process of proof and refutations) described by Lakatos strongly depend on the starting situation which consists of: a situation of classification (which delimitates what is the class of polyedra), an initial conjecture (Euler formula), a first representation of the mathematical objects at stake[5] (polyedra), and a new proof (that of Cauchy, where a change of thinking is required because it implies a topological view). In particular,

---

[2] The debate context.

[3] The connection between concept and proof, on the one hand, and between concept formation and definitions, on the other hand.

[4] The limits of the use of the Lakatosian method have been pointed out by De Villiers (2002) and Hanna (2007), both of them quoting Conway. In particular, they said that Lakatos's method couldn't be applied generally. In defense of Lakatos, in his original thesis, he denies that the unique logic of mathematical discovery exists.

[5] The intended concept is then partially known *a priori*.

with this context of geometrical objects, it means that the concepts have a pre-existence, as well as there exists a pre-axiomatic theory. Very few situations following the whole Lakatosian model of refutations have been experimented in mathematics education with freshmen: all of them have the previously quoted aspects (e.g. Larsen & Zandieh 2005, 2008; Zandieh & Rasmussen 2010) and emphasize the didactical interests of revising a definition and its dialectic with proving processes. I will extend the use of (a part of) the Lakatosian modeling to a less restrictive situation (see section 3). Let me first present this modeling which is specific to definitions.

I point out that Lakatos leaves aside the first encounter with a mathematical object and also the axiomatization, the latter being, for him, the end of the creative process. His study of the mathematical process leads him to focus on refutational processes and definitions in connection with concept formation. Lakatos precisely identifies three kinds of definitions: the *naive definitions*, the *zero-definitions* and the *proof-generated definitions*, whose respective functions are to give a name, to communicate a result and to prove. A *naive definition* can be stated first, but it cannot evolve[6], contrary to a *zero-definition* that marks the beginning of the research process (hence the term "zero"). A zero-definition can be vague, outside of any formalism, it does not matter: it has to evolve into a "better" one. Hence a fundamental question arises: how can a zero-definition mutate? Lakatos relies on two cases (see also Larvor, 1998):

- The concept is altered by the presentation of a new kind of object. Here, the conjecture should be protected from such a counter-example (a monster), the zero-definition can be modified;

- Or the concept is altered by the presentation of a new proof: a work of translation is engaged, when one deals with a new proof in another mathematical framework. This is where a system of mathematical concepts can grow. Then, a zero-definition becomes a *proof-generated definition* with a validation by a proof. Note that it is every difficult to transpose such processes in education, because, usually, definitions come first, are not built by students and not revisited in proof.

A 'good' characteristic of a concept that lends itself to a defining activity is that there are several potential zero-definitions (for example, in different mathematical frameworks or with different views of the concept), as proved by Ouvrier-Buffet (2013) with a synthesis of literature. Indeed, working on zero-definitions makes possible the process of constructing a mathematical concept: the dialectic between concept construction and definition construction is visible in the evolution of zero-definitions.

I will show you in an example that this modeling of definition processes is helpful to analyze mathematical situations involving defining processes and also to characterize students' procedures. The chosen situation involves concepts which are unfamiliar. The mathematical results are also unfamiliar and may be surprising for students. They belong to discrete mathematics. This choice of discrete concepts is justified by the fact that discrete mathematics is an active branch of contemporary mathematics with accessible problems and concepts, several ways of reasoning, questioning and modeling. It also brings new ways of dealing with axiomatic construction and providing mathematical experiences (e.g. Ouvrier-Buffet 2020, Hart & Sandefur 2018).

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[6] Lakatos explains that we can do nothing with a naive concept, because we cannot extend it, unless you are able to fit it in a theory, then the considered concept will not be naive anymore!

3. A discrete problem: epistemological *a priori* analysis

3.A. The problem

I present here a very natural statement of the problem I consider. Take a regular squared grid[7] and some displacements which can be described with integers (or only positive integers) on this grid (Table 1). A starting point is given. An experimental fact is easy to grasp: starting from the starting point, a point of the grid is reachable or not, when one uses the given displacements. So "the" question: which points of the grid can one reach using positive integer combinations of these displacements?

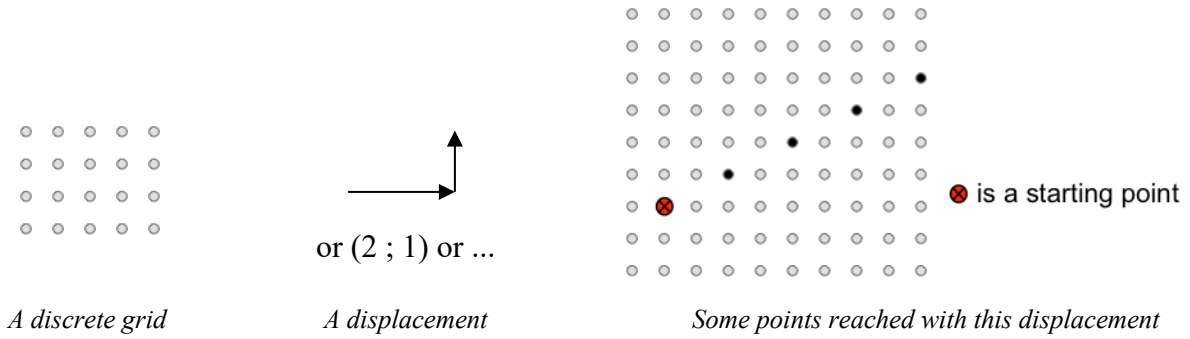

Table 1 – The objects at stake

There are a lot of underlying questions: is it possible to reach *all* the points of the grid? or *some sectors* of the grid? Can one characterize *generating sets* of displacements? Are they *minimal* ones? What about the *cardinality* of minimal generating sets? Etc.

This situation makes it possible to work on objects (grids, displacements and points reached) and implies mathematical questions on concepts to be defined such as *generator* and *minimality*. The terms *generator*, *generating set*[8] are used in mathematics (and also in physics) in various domains: linear algebra, commutative and abstract algebra (generating set of a group, a ring, an ideal, a module), in category theory, in topology, in differential equations (with tangent space), in graph theory, in computer science and combinatorics. The study of the "idea" of the concept of generator/generating set is usually done separately in each of these domains, but it crosses different fields in mathematics, as are the questions about minimality and minimum cardinality. Here, minimality is used when searching for a minimal generating set, i.e. a set of displacements that can be used to reach every point of the grid, but, if even one of those displacements is missing, this ability is lost. And the cardinality of minimal generating sets is not necessarily unique.

Let me develop the underlying defining process at stake in this problem.

[7] One can take another regular tessellation... not easy!

[8] But also linear combination (and its derivative concepts: linear independence and linear dependence). See the development of epistemological justification of Harel (2018).

3.B. An epistemological *a priori* analysis of the problem

I will propose one possible exploration[9] of the discrete problem through the Lakatosian kinds of definitions, while making room for heuristics as they can be involved in such a mathematical enquiry.

According to Pólya (1945, 1954), heuristic is a set of strategies for solving mathematical problems to learn, to teach and to reconstruct mathematics. Pólya emphasizes induction and analogy, but also several heuristics, such as generalizing (a result or a problem), specializing, modifying the problem, tabulating the observations, simplifying a problem, studying examples, studying extreme cases, studying concepts/problems/results from various standpoints, using what we call in mathematics education *generic examples*[10]. I consider that the following actions are heuristics: proving, conjecturing, refuting, creating, modeling, defining, extending but also transforming a questioning process, being able to mobilize non-linear reasoning, experimenting, decomposing-recomposing. In fact, fallibilism is central concern for Lakatos, with a specific focus on defining processes (not considered by Pólya) and their interactions with proof, reaching the growing of the mathematical knowledge. I will emphasize this dimension of the mathematical activity and discovery in the analysis of the mathematical problem. This kind of analysis has helped me to design a didactical situation for freshmen (see appendix and Ouvrier-Buffet 2003a, 2011).

The aim of the following epistemological analysis of the problem is then to highlight the potentialities of the discrete situation with regard to the defining process and to point out the concepts at stake, in the way Pólya and Lakatos did.

I will use letter G for the regular squared grid, A for a starting point, and the term *integer point* or *point* for lattice point. I have chosen to define a *displacement* on G with two positive integers and two directions (among up, down, left and right). For instance, "2 squares right and 3 squares down". A displacement can be represented with a vector. One can also consider displacements and their opposite displacements (and so integer coordinates), it is still a real issue. The order of the displacements does not interfere because displacements are commutative (property easy to prove).

3.B.a. First heuristic: exploring "small cases" to highlight phenomena and properties

There is an obvious set of four displacements which allows us to reach every point of G (Fig. 1). It is also obvious that two displacements are not enough to reach all the integer points: either all the points of a sector of G (i.e. a set of integer points bounded by two rays) are reached (Fig. 2), or only some points of a sector are reached (Fig. 3). I would like to underscore two major facts:

- The question of "reaching integer points" can be refined: one can study sets of displacements which allow to reach all the points of a given sector and/or sets of displacements which allow to reach

[9] I have solved the mathematical problem with a team of mathematicians whose research is in discrete mathematics. I encourage the reader to explore the discrete problem and to suggest other kinds of enquiries than the one proposed in this paper.

[10] "induction suggest deduction: the particular case suggests the general proof" (Pólya, 1954, p.50). Such a definition leads to Balacheff's definition of generic examples: "involves making explicit the reasons for the truth of an assertion by means of operations or transformations on an object that is not there in its own right, but as a characteristic representative of its class" (Balacheff, 1988, p.219).

all the integer points. In both cases, one can talk about "*generating sets*" (for a sector or for G) and define it.

- If one tries to build generating sets made of three displacements – in order to do "better" from the optimization point of view than Fig. 1 – one can start from sets of two displacements for instance. Take Fig. 2: every point of the sector bounded by the two given displacements is reachable. I call this property "with Full Density" (FD). But, it is not always the case: Fig. 3 shows us that some points of the sector bounded by the displacements are not reachable. And now, I try to go "a little bit everywhere"[11] (ALBE) when adding at least one displacement. A geometrical argument (translation of the reached sector for instance) brings a third displacement (as represented on Fig. 4, taking quadrants into account, or as we can operate on Fig. 2 with a very acute angle). The property ALBE can be characterized in the following way: "for any point, there is a point A "near" P such that the distance[12] between P and A is bounded (independently of P)."

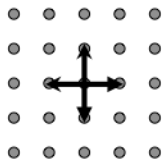

Fig. 1

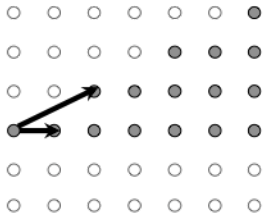

Fig. 2

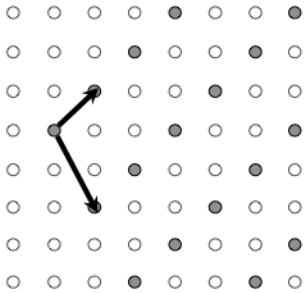

Fig. 3

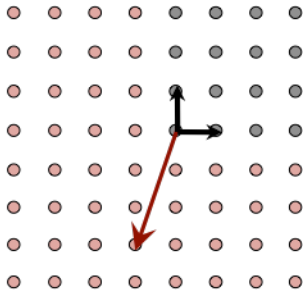

Fig. 4

Table 2 – Exploring "small cases"

The exploration of small cases leads us to the following results:

- I have emphasized two properties when trying to build generating sets: "with Full Density" (FD) and "a little bit everywhere" (ALBE).

- One can reach all the points of the grid when the FD and ALBE properties are satisfied simultaneously. Then, a zero-definition of a generating set can be stated (see 3.B.b).

- To reach the four cardinal points or to generate the four unit displacements implies to reach all the integer points. One then considers the grid through two directions (the horizontal and the vertical ones).

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[11] Allusion to "almost everywhere". One can also link this property to the discrete mathematical field of "covers" and raise the question of the minimal cover.
[12] Distance on the grid or an Euclidian distance, to define.

If I outline here a vertical development of this discrete problem (as done in Sally & Sally, 2007), I should point out the wider NP-Hard Frobenius Problem (Ramirez Alfonsin, 2006), also called the "coin problem": "given coins of denominations $a_1, a_2,..., a_n$ (with no common factor), what is the largest amount that cannot be changed?". Dealing with small cases of the Diophantine Frobenius problem can lead to the Pick Theorem, which has a lot of interesting applications (Sally & Sally 2007, Dissa 2020). Such a development would require another article.

3.B.b. Second heuristic: stating zero-definitions to progress in the solving process

In the sense of Lakatos (1961, 1976) and interviewed mathematicians (Ouvrier-Buffet 2013, 2015b), a zero-definition[13] marks a beginning of a research process. It can be modified in order to protect a primitive conjecture or because the concept is altered by the presentation of a proof: the proof is the master and the definitions evolve in order to continue the research. The formalization and the axiomatization come later. All such "local" definitions can have different functions: to give a name, to classify, to bring up several ways to grasp a concept, to circumscribe pre-formal ideas, to work on a proof, to delimit the range of use of an idea or of a conjecture or a proof, to communicate, to delimit a local theory.

After the exploration of small cases, a first natural zero-definition of a generating set can be stated in order to give a name to this concept, as follows: "a generating set is a set of displacements that allows access to all the points of G". Such a zero-definition is not operational to build such sets (except in the graphic register, with only two displacements) and is very "costly" if we want to check whether a set of displacements is a generating one. That is the reason why such a zero-definition should evolve in a problem where the number of displacements is superior to 2. It can evolve into an operational definition including two properties, namely, FD and ALBE, illustrated during the previous phase. Such zero-definitions can be locally validated by the exploration of small cases and by their use for proving that a set is a generating one.

One can give the following zero-definitions of FD and ALBE:

- $Zdef_{FD}$: a set of displacements has the FD property if and only if[14] all the points of a sector of the grid are reached;
- $Zdef_{ALBE}$: a set of displacements has the ALBE property if and only if there exists a positive number μ such that for any point X in the grid there exists a reachable point Y, "close to X", i.e. whose distance from X does not exceed μ (here, it is possible to avoid a metric altogether).

The minimality aspect is difficult and requires more than the study of small cases. A first zero-definition of a minimal generating set can be: "a minimal generating set of displacements is a set of three displacements allowing access to all the points of the grid" (Fig. 4). Solving a problem with four independent displacements (Fig.1 is a counterexample) allows the invalidation of this zero-definition. Then, a second zero-definition can be stated as follows: "a minimal generating set of

[13] that can also be called "working definition" or "conjectured definition".

[14] The use of the expression "if and only if" for a definition is sometimes questioned by some philosophers or teachers for whom a definition is only a denomination (see Ouvrier-Buffet, 2003a). I choose not to deal with such conceptions in this article.

displacements is a set of non-dependent displacements". This zero-definition has a geometrical root; it can mobilize knowledge on collineation of vectors in the plane and coplanarity in space.

As described by Ouvrier-Buffet (2011) with illustrations, "The status of these zero-definitions should evolve to that of proof-generated definitions through the following proof path: from proving the existence of minimal generating sets of displacements, to building such sets, to proving that a set is a generating one, and to proving that a set is a minimal generating one." (p. 173). It is obvious that the question of the existence is not easy.

3.B.c. Third heuristic: stating new questions and generalization, first proofs

After this prior exploration, more queries emerge:

1) **Existence**: do such generating sets exist?

2) **Building generating sets**: How can one build generating sets? Other generating sets? Do such sets have other mathematical properties? An integer k being given, can one build all the generating sets with k displacements? This last question makes sense when we deal with the notion of minimal generating sets.

3) **Removing a displacement**: Is it possible to remove a displacement without changing the set of reachable points? One can say that a set of displacements is *minimal* when removing any of its displacements modifies the set of reachable points. With this definition, how to characterize a *minimal generating set* of displacements?

4) **Cardinality**: Do the minimal generating sets of displacements always have the same number of elements? The answer is obviously no (see Fig. 1 and 4), and that is surprising, at least for students who are used to dealing with geometrical analysis and vector spaces. In fact, one can bring the discrete problem on the grid back to a problem on Z, and then mobilize knowledge from number theory. It is easy to build generating sets with different cardinalities. E = {1; -1} and F = {2; 3; -6} are two simple examples of minimal generating sets for Z (for E, if I remove 1, I loose the ALBE property. For F, if I remove 2 or 3, I loose the FD property and if I remove -6, I loose the ALBE property).

A generalization leads us to the following result:

Proposition 1: for any integer k, there exist, in Z, minimal generating sets of displacements with k elements.

Remark: the search of the proof of proposition 1 can lead the solver to the use of known arithmetical results, but it can also lead the solver to explore arithmetical properties (see for instance Duchet (1994) for an experiment with 14-year-old students).

And now, if one wants to build a minimal generating set on the squared grid, one can keep in mind the horizontal/vertical representation and choose displacements carefully, in order to keep the minimality. Therefore, one has the following proposition:

Proposition 2: for any integer k, there exist, on grid G, minimal generating sets of displacements with k elements.

Indications for the proof (in order to underscore its accessibility): one constructs a set of horizontal minimal generating displacements with (k-2) elements in order to generate Z (see above, k

being as big as one wants) and then adds two vertical displacements in order to go everywhere by translation. □

More questions to explore: can one build other kinds of minimal generating sets than those built in the previous proof? Do other proofs exist? I now directly address the readers: what kind of exploration of this discrete problem would you have conducted? With which major mathematical questions? What kinds of heuristics would you have mobilized? And what happens if we change the grid or the mathematical framework (e.g. the continuous case), following Lakatos' heuristic principle of translation?

3.B.d. Potential difficulties for students

The previous epistemological analysis emphasizes the mathematical concepts involved in the situation (mainly generating sets, minimality), with a focus on the defining process and the zero-definitions. With undergraduate students, we can expect that the defining process may be difficult if the students are not familiar with it. Moreover, the concepts involved in the situation are essential in linear algebra, a field taught in the early years of university and widely studied by didacticians. We can then, *a priori*, identify some potential difficulties for students facing the discrete situation in terms of these two dimensions.

Defining processes. In mathematics education, the experimental conditions of defining situations are always very specific. The experiments are carried out by the researchers themselves, with small numbers of students (from 2 to 15 or even 25), sometimes even with gifted students, in long-term situations (summer school for example) from primary school to university. The students' research work is always group work with supposed institutionalizations driven by the researcher (these institutionalizations are not described or questioned, but their existence is apparent in the articles). In these articles, when a student's defining activity is described, the levers used by the researcher and which allow an evolution of the students' processes are often not made explicit, except in research involving a didactical engineering (e.g. Ouvrier-Buffet 2006, 2011; Chorlay 2019; Lecorre 2016). Taking into account the literature in mathematics education (e.g. the research of Larsen & Zandieh, Zandieh & Rasmussen and Ouvrier-Buffet), the main difficulties students have with defining processes are the following: if the teacher does not ask for a definition or does not interfere, the students do not define (because it is unusual for them in their mathematical culture) or do not know where to stop when rewriting definitions. Playing with counter-examples (such as the Lakatosian one) should also be encouraged by the teacher. A positive point is that students do not have difficulties to produce different definitions (when asked by the teacher) and are able to question their equivalence.

Possible difficulties coming from linear algebra. Obstacles will emerge in the discrete situations if students have concept images coming from linear algebra and geometry. In their synthesis paper, Stewart and al. (2019) point out that there is "a scarce literature on proof in linear algebra" (p. 4) and emphasize the results that "students could not completely and correctly state the formal definition" (p. 12) and have "differences in attribution meanings to certain words and expressions" (p. 4) ; students also have a rare understanding of basis (p. 5) and struggle to grasp proofs. Moreover, Stewart & Thomas (2019, p. 1070) recall that Britton and Henderson (2009, p. 964) "claim that conceptual understanding is at the root of the problem" and that Inglis & Alcock (2012) emphasize that "students may be more concerned with processing algebraic manipulations and focus proportionately less on

words and logical relationships". I would also like to stress the complexity related to the understanding of the concepts in linear algebra with formal definitions and definitional reasoning (Harel, 1990 & 2017)[15] and to point out the students' difficulties with the form of mathematical statements (definitions and proofs), structures, the use of quantifiers and with linear combination, linear dependance, linear independence and its negation (Harel, 2017).

All of these results are in line with previous research on the teaching and learning of linear algebra which is still being quoted. Indeed, Dorier explains that students "are overwhelmed by the number of new definitions and feel like they are landing on a new planet" (Dorier, 2000, p. 185). He underscores the obstacle of formalism in linear algebra as well, characterizing this field of mathematics as a unifying and generalizing theory (Dorier, 2000; Dorier et al. 2000a; Dorier et al. 2000b). The lack of connections with previous knowledge, the lack of practice in basic logic and set theory (misunderstanding of implicit quantifiers and of necessary and sufficient conditions) and the impossibility of using geometrical intuition also lead to difficulties for students in learning linear algebra. Besides, the multiple proofs and kinds of reasoning in this branch of mathematics make it complex for students (e.g. Sierpinska, 2000; Uhlig, 2002). Several propositions, different in nature, have been made to help overcome students' difficulties (e.g. Harel, 1998; Rogalski, 2000; Uhlig, 2002; Stewart et al. 2018, 2019), using the historical roots of linear algebra (for instance, through the concept of 'rank' and the pivotal cluster of concepts 'linear combination', 'dependence' and 'independence'). Most of them are produced within long-term studies (didactical engineering in the meaning of Artigue, 2014) and all of them are within the strict field of algebra. Furthermore, the attempts to connect linear algebra to 2D and 3D geometry in order to give an image of some concepts have shown their limits (e.g. Gueudet, 2006). Moreover, such approaches convey other obstacles: the problem of these metaphorical approaches has been studied by Hillel (2000) and is also pointed out by Harel (2017). Indeed, geometry is limited to three dimensions and some concepts (such as linear (in)dependence or rank) have a very limited range of representation in the geometric context. And the loss of meaning through the transition to algebraic formalism actually turns out to be an issue in this type of geometric approach. We do not know if the students will use knowledge coming from their courses of linear algebra, but it may disturb the way they solve the problem: indeed, the link between the discrete case and the "continuous" case is not obvious.

The importance of involving students in a "new" mathematical experience here appears crucial.

3.C. Main features of the discrete problem compared with "classical" linear algebra

From an historical point of view, the fundamental importance of modules over a ring is confirmed, vector spaces over a field being as a subsidiary notion (as done in Mac Lane and Birkhoff's Algebra (1967))[16]. The contemporary works of researchers in didactics about the teaching and the learning of linear algebra take this branch of mathematics as a "self-sufficient" full-fledged system. It is obvious

[15] Besides, Harel's results underscore that, in linear algebra, "(...) students do not build effective concept images; rather, they place their full reliance on concept definitions, by memorizing them verbatim. (...) Once the concept definitions are forgotten, students are unable to retrieve or rebuild them on their own." (Harel, 1998, p. 499).

[16] See Moore (1995) for a wider historical study. In particular, Moore points out that "Birkhoff notwithstanding, the recent history of linear algebra has continued to confirm the fundamental importance of modules over a ring, with vector spaces over a field as a vital but subsidiary notion. This was how the subject was treated in Mac Lane and Birkhoff's Algebra (1967), where modules precede vector spaces-- presumably under the influence of Mac Lane." (Moore, 1995, p. 295).

that making connections with modules over a ring is not the easiest way to teach linear algebra at the beginning of university. The discrete situation I suggest can be a way to situate linear algebra's questionings in a wider context, but in an easier way than Mac Lane and Birkhoff's approach. It also gives a "natural" exploration of some results that are not "always true". Indeed, a minimal generating system, a maximal linearly independent system are not basis in the general case. For instance, for Z/(6), considered as a Z-module, there is no linearly independent system (non-empty) and {2; 3} is a minimal generating system. And in Q, considered as a Z-module, every system reduced to one element (non-zero) is a maximal linearly independent system and there exists no basis.

Let me go back to the discrete problem. The discrete problem is an open problem, which the students do not know. They can "do" mathematics, as a researcher does, without looking at a pre-existing course in a textbook. The discrete object "displacement" is different than in the continuous case: there is no + and – and there is no modification of a displacement by a scalar multiplication. In linear algebra, the scalar multiplication brings the FD property; and the use of + and – brings us the possibility to go everywhere but not necessarily with the ALBE property. Moreover, the question of "dimension" is typically a part of linear algebra whereas the discrete case deals with the question of "minimality": the concept of "dimension" would be problematized, provided that students have already encountered this concept in other contexts.

The queries are common to discrete and continuous problems, but it seems that they are more intuitive in the discrete case, even if the mathematics behind the displacements can be quite advanced. Furthermore, the concepts are highly interrelated in vector space whereas the discrete problem clearly separates the concepts (the "same" concepts as in vector space): it then especially allows work to be done on the links between the concepts involved in the problem and on the necessary and sufficient conditions. Besides, the links between concepts are more important than the concepts themselves, which are not an obstacle in studying the discrete case. The discrete queries – which can also be found in other fields of mathematics – and an exploration of links between definitions and proofs should be fruitful.

The situation proposed in discrete mathematics is then decontextualized in comparison with a classical introduction of concepts in linear algebra. The concepts of "generator", "minimality" but also "dependence", and later "basis", can be studied while avoiding excessive formalism. The shift from N and Z to R^+ and R has to be introduced by the teacher. All these features show the potentialities of this discrete situation, in order to highlight mathematical questionings, under certain didactical conditions. Such conditions should have the following characteristics: a specific research contract with the students, a long-term situation, and an institutionalization of enquiries which are common to the discrete problem, to linear algebra and to other domains in which the concepts being explored are used. If such conditions are not met in the university classroom, supervised research projects may be considered with twofold aims: to study the differences between discrete and continuous situations and to work on a mathematical experience involving searching, defining, and proving. It reaches Harel's propositions about definitional reasoning i.e. "the way of thinking by which one examines concepts and proves assertions in terms of *well-defined* statements—is likely to understand the concept of *dimension of a subspace* as intended—the number of vectors in a basis of the subspace—but he or she would also realize that such a definition is meaningless without answering the question

whether all bases of a subspace have the same number of vectors. Another student, for whom definitional reasoning has not yet reached full maturity, may have the same understanding without realizing the need to settle this question." (Harel, 2018, p. 5).

4. Design of a didactical situation: choices and challenges

4.A. A potential evolution of zero-definitions to proof-generated definitions?

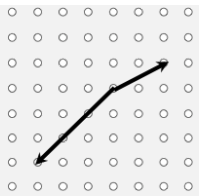
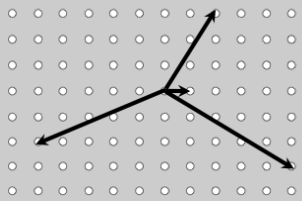
The shift from zero-definitions to proof-generated definitions is not easy to re-implement in the classroom when one designs a didactical situation. Indeed, Lakatos has based his historical reconstruction upon several mathematical backgrounds, using Cauchy's proof as a motivation to boost concept formation. And yet, this part of the defining process would be bringing learning potentials to teach proof and highlight the relations between proof and definitions, especially at university. In fact, no study in mathematics education has showed yet didactical situations involving such a shift from zero-definitions to proof-generated definitions. If one wants to make the most of the Lakatosian view of definitions, one can start by focusing on the zero-definitions and on their potential evolution to other "better" zero-definitions. It implies a view of the constructive and tentative nature of definitions which are socially constructed. The proof generation aspect comes later. I do not forget that some studies reveal that students can have difficulties in understanding the "very nature of mathematical definitions" (Edward & Ward, 2004, p. 411). Studying the zero-definitions and their features can be a real opportunity for us to solve this understanding problem.

4.B. Zero-definitions at stake

Starting from the problem of displacements, I have designed a situation (made up by three problems) for freshmen who have not yet encountered linear algebra (or only very partially), nor other mathematical domains involving generating sets. I kept in mind the epistemological, logical and cognitive obstacles mentioned above, in particular the fact that students may have concept images coming from analytical geometry for instance: they can think that two displacements are enough to reach every integer points.

I have tried to motivate the emergence of "natural" questionings and of zero-definitions (generating sets, minimal generating sets, FD, ALBE) with the exploration of "small" cases. The validation of the constructed zero-definitions will come in their use when building new sets of displacements and testing new sets of displacements (generating and minimal aspects). Table 3 synthesizes the choices I have made for the design of the didactical situation (see also the appendix).

Problems	Exploration leads to ...	Validation / Potential evolution
Problem 1: Two displacements on the grid	- assimilate the rules of the displacements and the questionings. - a fact: "in order to generate all the points of the grid, more than two displacements are required".	Aim: to dismiss students' pre-existent concept image coming from vector space of dimension 2 where two vectors are enough to build a basis.

 Fig. 5	- a *conjecture*: "three displacements are enough to reach all the points of the grid." - a first natural $Zdef_{generating\ set}$: "a generating set is a set of displacements which allows access to all the points of G". - $Zdef_{ALBE}$ and $Zdef_{FD}$ possibly.	This *Zdef* is not operational to build generating sets (even if it is operational in the graphic register for two displacements). The use of them to prove that a set is a generating one testifies to their validity.
Problem 2: Two or three displacements or four displacements where two of them are dependent  Fig. 6	- $Zdef_{\ minimal\ generating\ set}$: "a minimal generating set of displacements is a set of three displacements allowing access to all the points of the grid". - $Zdef_{\ minimal\ generating\ set\ bis}$: "a minimal generating set of displacements is a set of non-dependent displacements". - Towards the *proof* that "three displacements can be enough to generate all the points of the grid". But one cannot conclude that all the generating sets are made up of three elements.	To explore a problem where four displacements are independent allows the invalidation of this *Zdef*. Aims: To prove the existence of minimal generating sets, to generate such sets, to prove that a set is a generating one and to prove that a set is a minimal generating one. Distinction between *minimal* and *minimum*.
Problem 3: A minimal generating set made up of 4 displacements (one can prove that this generating set is a minimal one)	To demonstrate that "three displacements are not always enough". In fact, the set presented here features four displacements and is a minimal generating one.	The corollary of the exchange theorem (i.e. the existence of dimension, true in a vector space) is false in the discrete case[17].

[17] Remark: the invariance of dimension makes sense when one shows in which instances it does not work.

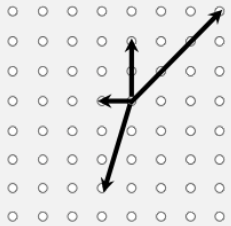
 Fig. 7		

Table 3 – Choices for the didactical situation

5. Undergraduate students faced with displacements on a regular grid

The situation was designed, implemented at the university and validated in the framework of a didactical engineering in (Ouvrier-Buffet, 2003). Then the choices made in Table 3. I have experimented it since several times in different contexts (research-action, doctoral training, pre-service teacher training).

In particular, I have experimented the situation of displacements with freshmen from scientific and non-scientific courses at university, the same year. The first time with freshmen in science training for my didactical research, and the second time with future economists for the needs of my teaching, before a course about linear algebra. I will not go into the details of the methodology, it is not my purpose here. Students from scientific courses are used to working in groups (3 or 4 students per group, around 30 students) and the intervention (audio-taped) lasted 3 consecutive hours. The future economists (around 30 students) were not recorded (except one group), I will report notes I have taken for this group of students. I will give the general tracks followed by the students from scientific courses and focus on the main differences between these two groups of freshmen (see the appendix for excerpts) and on their obstacles. My aim is to highlight the potentialities of the situation for the teaching at university level in terms of skills and definitional reasoning (in Harel's 2018 meaning above described). This experiment provided an isolated occasion for freshmen to experience situation of ongoing mathematics which is typical of mathematics, with an emphasis on defining processes.

5.A. With freshmen from scientific and non-scientific courses[18]

I must point out that the freshmen who took part in the intervention had a small academic knowledge in linear algebra: they knew the notions of "linear independency", "basis" and "vector space" (they knew their definitions in a vague manner). The situation of displacements allows a work on mathematical objects graspable through a basic representation close to that of vectors. The objects do not represent an obstacle in themselves. The main difficulty lies in the fact that properties have to be characterized (generating sets, minimality, among others) and to be named. These specificities of the situation of displacements partially explain why the students did not engage in characterizing mathematical properties. Indeed, only some statements were produced, but they were not recognized as important mathematical statements and they did not evolve into institutionalized zero-definitions. Moreover, the persistence of concept images coming from analytical geometry was real (they said for

[18] For a more descriptive study of the students' productions from scientific courses, see Ouvrier-Buffet (2003a).

instance: "We have four displacements. We need only two vectors as a basis."). Nevertheless, in students' works, I have identified a natural definition of "generating" ("to reach all the points of the grid"[19]). It has been transformed into an operational property ("to generate the four cardinal points or unit displacements"). It has not been connected to the ALBE and FD properties, even if these properties were implicit in the students' dialogues: the ALBE property is predominant. Furthermore, I have identified two statements that enable students to pick up the most adequate selection of information according to the situation, but without any explicit definitions (I call them *definitions-in-action*[20]):

- one for "minimal generating set": "when all the displacements are used during the search of four unit displacements",
- and one for "minimal": "a set of displacements is not minimal when one of them is an integer combination of the others".

Students can work efficiently with these two definitions-in-action, which are both connected to the same proposition-in-action ("to have the four directions represented is a necessary condition in order to have a generating set of displacements"). Such a definition-in-action can become powerful. But in this case, the definition-in-action of "minimal" was imported by the students from their previous knowledge in linear algebra. In fact, in this example, "minimal" is a translation of "independent". This kind of imported definition-in-action obviously blocks the students' conceptualization. Students stayed in the action, in the proposed configurations. Their process did not move to a generalization which would have allowed an evolution of their definitions-in-action. For them, there is clearly no explicit need for formalization (Harel, 1998). Moreover, the management conducted by the teacher-observer is not definition-oriented, it is neutral. A plausible hypothesis is that this distance between manipulation and formalization (formalization merely being a first step, not a complete theorization) is too rarely approached in the teaching process. It goes along the lines of previous epistemological and didactical results which conclude that formalism is a crucial obstacle in the teaching of linear algebra (e.g. Dorier, 2000; Harel, 2017). Besides, the short time format (three hours) probably did not encourage the evolution of definitions, a wider exploration of defining processes requiring more time. It underlines that putting students into a mathematical experience is a necessary condition in order to help them in their learning process at university.

The mathematical exploration of the discrete problem is more natural and more intuitive for freshmen from non-scientific courses (NS group) than for freshmen from scientific courses (S group). It is not so surprising: the S group tries to mobilize their previous knowledge and that limits the students in their study of the mathematical problem. Moreover, studying cases where the cardinality of the minimal generating sets is not an invariant feature is not a common practice at the beginning of university. The NS group were more accurate on their characterization of sectors, focusing on the ALBE and the FD properties, but (again) without defining them: these students stayed in the action too. In fact, it is as though having (partial) knowledge in linear algebra blocks students in their exploration of the discrete problem.

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[19] The sentences in quotes come from the students' dialogues. I have formalized them a little bit, because students did not institutionalize them, they only used these properties or definitions in the action.

[20] For definitions-in-action, I was inspired by Vergnaud (1996) and his characterization of *concepts-in-action.*

These students' productions show that the discrete situation engages them in a solving process and underscore the importance of the necessary teacher' guidance about defining processes and the statement of mathematical properties. Some excerpts are given in Appendix 2 to show how students represent the mathematical problem and formulate the properties (mainly in-action).

5.B. Conclusions: discrete displacements on a grid, what for?

In fact, the difficulty of the task is in solving the problem itself: the concepts one is working on do not need to be explicitly defined at the beginning of the solving process. These characteristics of the situation partly explain why there was no real activity to define the properties of the displacements. This shows that the students were not able to distance themselves sufficiently from the objects they were handling. Moreover, the use of some words and expressions (even if poorly used) coming from the students' courses of linear algebra may have short-circuited the definitional reasoning. Nevertheless, the "natural" definition of generator (to reach all the points of the grid) moved into an operational property (to build 4 points of elementary displacements); and I have identified two definitions-in-action (for minimal generating set and minimal, see 5.A above).

It seems that the place in a course of the discrete situation should be before a course of linear algebra in order to make room for the mathematical exploration of the discrete case. An institutionalization of the questionings (common to the discrete and the continuous cases) should be done: it should lead to identify the interest of the continuous case, where one has the theorem of the dimension, which is not true in the discrete case. Besides, a study of the relations between the concepts involved in the discrete situation opens new opportunities to engage students in a work on implication (necessary and sufficient conditions).

I am studying the way one can use the discrete case with undergraduate students to introduce to concepts and mathematical questionings in the most beneficial manner. I pay attention to the distance between manipulation and formalization and also to the difficulties that students encounter with formulation and validation (in Brousseau's 1997 meaning): the articulation between defining and proving should be built in students' practices as well as a work on mathematical statements in order to deepen the work on proof. Such a research is linked to the study of defining and proving processes of mathematicians (e.g Ouvrier-Buffet, 2013, 2015ab). This work is still in process.

6. Opening

> "Ideas concerning the mathematical understanding of students at all levels are germane to the study of the mathematical process." (Vinner, 1991, p. 120)

> "Students should have experiences that focus on the use of mathematical definitions and experiences in the process of defining." (Edward & Ward, 2004, p. 422)

6.A. An efficient tool in epistemology and didactic to study defining processes

The Lakatosian kinds of definitions (especially zero-definitions) allow an epistemological analysis useful from a didactical point of view, bringing a (partial) modeling of defining processes and elements in order to make the construction of definitions more dynamical. Adding the notion of

definitions-in-action, students' processes can be described, as well as the lacks in students' defining processes. It is a way to grasp definitional reasoning. I also keep in mind Harel's necessity principle[21] and the fact that "Intellectual need and epistemological justification are two sides of the same coin —they are different but inextricably related constructs." (Harel, 2018, p.11). It remains difficult to design didactical situations involving these constructs for the defining processes. Not only because students have not yet encountered such processes, but also because "all" the features of the defining processes are not yet well known, even if some researchers have characterized the heuristics and behaviors of mathematicians (e.g. Burton, 2004; Schoenfeld, 1994; Ouvrier-Buffet, 2013).

Bearing in mind the above described obstacles in the teaching and the learning of linear algebra, especially the fact that many students struggle with the understanding the basic concepts and questionings in linear algebra and with proof (e.g. Stewart & Thomas, 2019), emphasizing the definitional reasoning can be useful to address this issue. The long-term discrete situation opens a door, bringing in a frame where statements are not equivalent as in the continuous case and problematizing mathematical (difficult) questionings about generator, minimality and existence. I agree with Stewart & Thomas (2019, p. 1070) quoting Harel (1997)[22]: "While Harel acknowledges that this model of teaching requires many hours of class time, he believes there is no other alternative."

6.B. Discrete mathematics: a way to consider mathematical objects

Of primary interest in the situation of displacements on a grid is the use of discrete structures to grasp concepts (such as generating sets, minimality, etc.) and questionings which belong to several fields of mathematics. Moreover, the discrete situation allows an access to the reasoning behind the construction of concepts and then contributes to the development of proof abilities through an active mathematical exploration of the problem (construction of relations between properties and proofs of implications) (see for other examples Gravier & Ouvrier-Buffet, 2022; Ouvrier-Buffet, 2020).

If one tries to problematize the same concepts in other mathematical situations involving material objects (such as displacements, for the reason of the devolution of the problem), one can foresee the following tracks: to cover the plane with fractals, to generate some kinds of graphs. One also has to question the invariants in the discrete mathematics which are common to the continuous cases: if the discrete problems are sometimes (and even often) easier to grasp than the continuous ones, the mathematics behind can be quite advanced. Such a discussion leads us to further studies. There is room to explore new questionings about the teaching of continuous structures, with the help of the discrete ones. This comparatively young branch of mathematics arouses interest because it brings

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[21] Harel (1998, p. 502) explains the Necessity Principle translating as three steps:
1) Recognize what constitutes an intellectual need for a particular population of students, relative to the concept to be learned.
2) Present the students with a problem that corresponds to their intellectual need, and from whose solution the concept can be elicited.
3) Help students elicit the concept from the problem solution.

[22] Harel (1997, pp. 119–122) has offered four recommendations for achieving this goal.
1. Students should take an active part in the construction of relations between ideas and in the production of their justifications; 2. Students should be helped to build proofs on their intuitions; 3. Students should be encouraged to read proofs; 4. Students should learn that understanding a proof is more than understanding each of the proof's steps.

new potentialities for education: Goldin (2004, p. 58) emphasizes "how experiences in discrete mathematics may provide a basis for developing powerful heuristic processes and powerful affect" (cf. a special issue of ZDM about Discrete Mathematics and Proof in the High School, 2004). Exploring this avenue is worthwhile for the investigation of mathematical practice.

Appendix

1) The problems which make up the situation

On a regular squared grid, for each problem, use a set of displacements and choose a starting point (called A), any one. The questions are:

1- Starting from point A, which points of the grid can we reach?

2- What are the consequences if we remove one or more displacements?

Problems	Sets of displacements
Problem 1	d_1: 2 squares to the right and 1 square up d_2: 3 squares to the left and 3 squares down
Problem 2	d_1: 2 squares to the right and 3 squares up d_2: 5 squares to the left and 2 squares down d_3: 5 squares to the right and 3 squares down d_4: 1 square to the right
Problem 3	d_1: 3 squares to the right and 3 squares up d_2: 2 squares up d_3: 1 square to the left d_4: 1 square to the left and 3 squares down

2) Students' dialogues – Some excerpts to illustrate Section 5

Group from non-scientific background (NS group)

During the following excerpts, Samy and Yohan try to build a generating set of displacements (O is the Observer).

- *Concept image "two displacements are enough"*

444 S: But why couldn't it work with two displacements?

- *Definition-in-action of ALBE. Horizontal/vertical representation and use of number theory*

458 S: I can cover all the squares upright, because I can go down each time for one square. And with this... I can go to the right and to the left. Then, this set of displacements works for me. (...)

505 O: Why did you take 2 squares to the left?

506 S: Because I had 3 squares to the right.

507 Y: You want to come back from 1 (...)

510 S: I think that with a combination with 3 and 2, one reaches all the numbers.

- *Four cardinal points and four unit displacements* (see Fig. 10)

564 S: In fact, when I use these displacements several times, I reach these points (*he shows the four cardinal points*). They would symbolize a unit-displacement up, a unit-displacement to the right, a unit-displacement down and a unit-displacement to the left (...) they are commutative (...) the four elementary points ... I say "elementary" because they allow the production...

565 Y: ... of all the points.

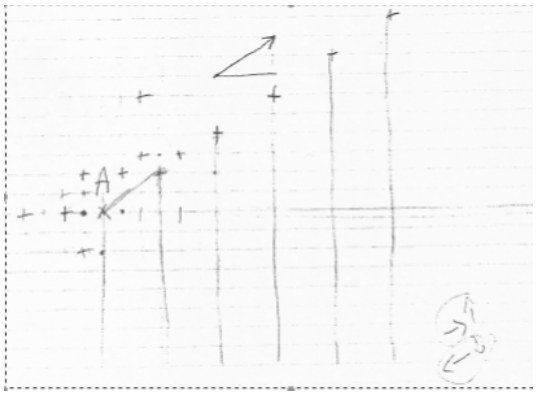


Fig. 8

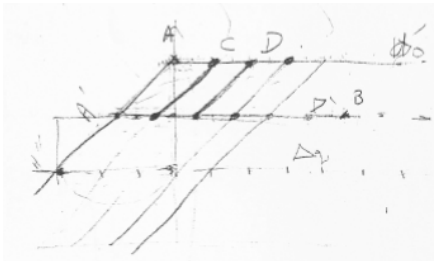


Fig. 9

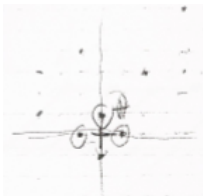


Fig. 10

Representations used by the NS group

Groups from scientific courses (excerpts of four groups)

- *Concept image "two displacements are enough"*

138 P: We have four displacements. We only need two vectors as a basis. (Yellow group – problem 2)

- *Knowledge from linear algebra*

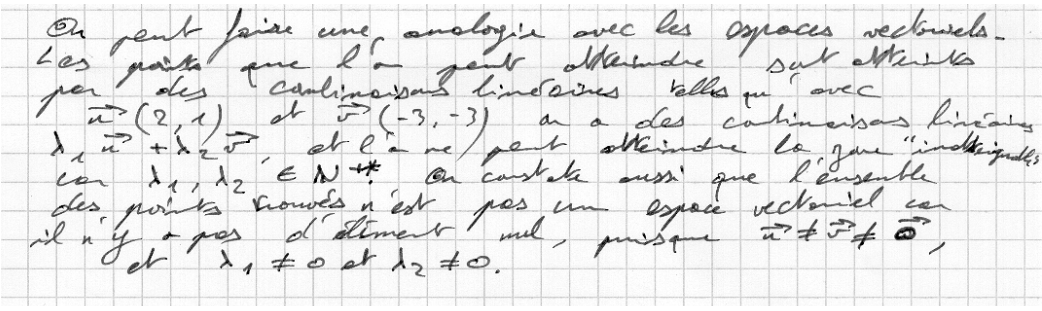
350 S: The thing is ... no, I don't think that it is the same thing (*as linear algebra*), because here, we can't reach this point.

351 P: Each time, you try to reuse your courses, you realize that it is useless!

352 S: No, but here, it is obvious. But a linear combination, it is a 'lambda *v*' thing, where *v* is any vector...

353 P: I don't know what it is! (...)

373 S: Because it is precisely not a vector space, I mean that it looks like a vector space, but it is not a vector space. First, there is no zero element, and besides the 'lambda' that we can use can't be negative. How would you say that? ... in fact, you only have two kinds of displacements, but for instance, when you use the (*displacement*) (3; 3) like this, you can't use the (*displacement*) (3; 3) in the other direction. And the same thing happens with the (2; 1), you can't use the (-2; -1). (Pink group – problems 1 & 2)


On peut faire une analogie avec les espaces vectoriels. Les points que l'on peut atteindre sont atteints par des combinaisons linéaires telles qu'avec $\vec{u}(2,1)$ et $\vec{v}(-3,-3)$ on a des combinaisons linéaires $\lambda_1 \vec{u} + \lambda_2 \vec{v}$, et l'on ne peut atteindre la zone "inatteignable" car $\lambda_1, \lambda_2 \in \mathbb{N}^*$. On constate aussi que l'ensemble des points trouvés n'est pas un espace vectoriel car il n'y a pas d'élément nul, puisque $\vec{u} \neq \vec{v} \neq \vec{0}$, et $\lambda_1 \neq 0$ et $\lambda_2 \neq 0$.

(Translation) "We can make an analogy with vector spaces. The reachable points are reached with linear combinations such as with and we have linear combinations , and we can't reach the "unreachable" zone because . One also notices that the set of the found points is not a vector space because there is no zero element, because , and ". (Purple group – an excerpt of their notes)

- *FD property*

1430 R: We noticed that if we remove d4, it is not like removing d2 where a whole sector disappears. If we remove d4, a lot of points are removed, but we still have points in a whole zone. (Pink group – problem 2)

- *Four cardinal points and algebraic systems*

178 B: In order to go everywhere, you try, and if you find (1; 1), it means that you can reach this point. After that you go on with this system to see if you have solutions. You process with (1; -1) in order to see if you can reach this point and with (-1; 1) and with (-1; -1).

179 A: Yes! You can see if you can reach the four points around!

180 B: Therefore, it means that you can go everywhere. If we can prove that, we have no more questions. And if we can't prove it ... it means that there are other conditions, it is complicated, or there is a technique…

181 A: It is not these four points. We have to take those.

182 B: Ah yes! We have to take (1; 0) (-1; 0) (0; 1) and (0; -1). We have a lot of systems to solve! *(they solve systems)* There are many solutions, but proving that there is one is enough. And this solution should work for the others too...

199 A: With that, we have proven that we can go everywhere. (Blue group – problem 2)

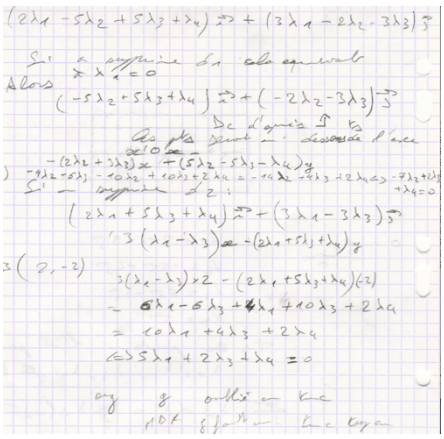


Fig. 11

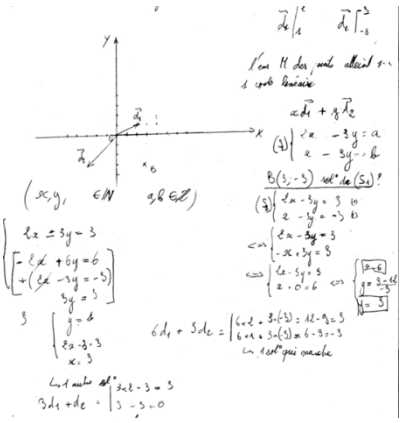


Fig. 12

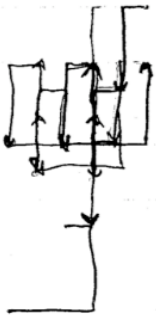


Fig. 13

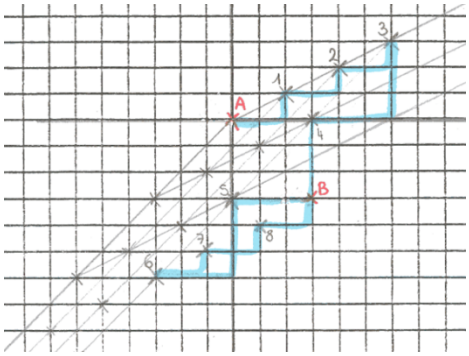


Fig. 14

Representations used by the S groups

Références

Alcock, L., & Simpson, A. (2009). *Ideas from mathematics education. An introduction for mathematicians.* The Higher Education Academy.

Artigue, M. (2014). Didactic engineering in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education*. Springer.

Balacheff, N. (1988). Aspects of proof in pupils' practice of school mathematics. In D. Pimm (Ed.), *Mathematics, teachers and children* (p. 216-235). Hodder & Stoughton.

Balacheff, N. (in press). Situations pour l'apprentissage de la preuve en mathématiques – Etat de la recherche et questions ouvertes. In Ouvrier-Buffet, C., Vandebrouck, F. & Vivier, L. (Eds), *Synthèses et perspectives en didactique des mathématiques – Preuve, modélisation et technologies numériques*. Episciences.

Britton, S., & Henderson, J. (2009). Linear algebra revisited: An attempt to understand students' conceptual difficulties. *International Journal of Mathematical Education in Science and Technology, 40*(7), 963–974.

Brousseau, G. (1997). *Theory of the Didactical Situations in Mathematics*. Kluwer Academic Publishers.

Burton, L. (2004). *Mathematicians as enquirers: learning about learning mathematics*. Kluwer Academic Publishers.

Corfield, D. (2002). Argumentation and the Mathematical Process. In G. Kampis, L. Kvasz & M. Stöltzner (Eds), *Appraising Lakatos: mathematics, methodology, and the man* (pp. 115-138). Kluwer Academic Publishers.

Chorlay, R. (2019). A pathway to a student-worded definition of limits at the secondary-tertiary transition. *International Journal of Research in Undergraduate Mathematics Education, 5*(3), 267-314.

De Villiers, M. (2002). A Fibonacci Generalization: a Lakatosian Example. *Mathematics in College*, 10-29.

Dissa, S. (2020). *Entre arithmétique et géométrie discrète - Une étude épistémologique et didactique autour du théorème de Bézout et du théorème de Pick*. PhD Thesis, Grenoble Alps University.

Dorier, J.-L. (Ed) (2000). *On the Teaching of Linear Algebra.* Kluwer Academic Publishers.

Dorier, J.-L., Robert, A., Robinet, J., Rogalski, M. (2000a). The Obstacle of Formalism in Linear Algebra. In Dorier J.-L. (Ed.), *On the Teaching of Linear Algebra* (pp. 85-94). Kluwer Academic Publishers.

Dorier, J.-L., Robert, A., Robinet, J., Rogalski, M. (2000b). The Meta Lever. In Dorier J.-L. (Ed.), *On the Teaching of Linear Algebra* (pp. 151-176). Kluwer Academic Publishers.

Duchet, P. (1994). Communication sur une grille, *Actes MATh.en.JEANS (1994)*, 45-50. Ed. MATh.en.JEANS, Paris, Web available: http://mathenjeans.free.fr/amej/edition/actes/actespdf/94045050.pdf

Edward, B.S. & Ward, M. B. (2004). Student (Mis)use of Mathematical Definitions. *The American Mathematical Monthly, 111*(5), 411-424.

Edward, B.S. & Ward, M. B. (2008). The role of mathematical definitions in mathematics and in undergraduate mathematics courses. In M.P. Carlson & C. Rasmussen (Eds). *Making the connection: research and teaching in undergraduate mathematics* (Vol. 73, pp. 223-232). MAA Notes, Mathematical Association of America.

Freundenthal, H. (1973). *Mathematics as an educational task.* Dordrecht, the Netherlands: Reidel.

Gueudet G. & Vandebrouck F. (2022). Transition secondaire-supérieur : Ce que nous apprend la recherche en didactique des mathématiques. *EpiDEMES, 1.*

Goldin, G.A. (2004). Problem Solving Heuristics, Affect, and Discrete Mathematics. *Zentralblatt für Didaktik der Mathematik, 36*(2), 56-60.

Gravier, S. & Ouvrier-Buffet, C. (2022). The mathematical background of proving processes in discrete optimization – exemplification with Research-Situations for the Classroom. *ZDM – Mathematics Education, 54*, 925-940.

Gueudet., G. (2006). Using Geometry to teach and learn Linear Algebra. *Research in Collegiate Mathematics Education, VI*, 171-195. ⟨hal-00926689⟩.

Hanna, G. (2007). The ongoing value of proof. In Boero, P. (Ed), *Theorems in school: from history, epistemology and cognition to classroom practice* (pp. 3-16). Sense Publishers.

Harel, G. (1990). Using geometric models and vector arithmetic to teach high-school students basic notions in linear algebra. *International Journal of Mathematics Education in Science and Technology, 21*, 387-392.

Harel, G. (1997). Moving beyond concept definition. In D. Carlson, C. R. Johnson, D. C. Lay, A. D. Porter, A. Watkins, & W. Watkins (Eds.), *Resources for teaching linear algebra, MAA Notes* (Vol. 42, pp. 107–126). Washington: MAA.

Harel, G. (1998). Two Dual Assertions: the First on Learning and the Second on Teaching (or Vice Versa). *American Mathematical Monthly, 105*(6), 497-507.

Harel, G. (2017). The learning and teaching of linear algebra: Observations and generalizations. *The Journal of Mathematical Behavior, 46*, 69-95.

Harel, G. (2018). The learning and teaching of linear algebra through the lenses of intellectual need and epistemological justification and their constituents. In S. Stewart, C. Andrews-Larson, A. Berman, M. Zandieh (eds), *Challenges and strategies in teaching linear algebra*. Springer.

Hart, E. & Sandefur, J. (eds) (2018). *Teaching and learning discrete mathematics worldwide: curriculum and research*. ICME13 monograph. Springer, Cham.

Hillel, J. (2000). Modes of Description and the Problem of Representation in Linear Algebra. In Dorier J.-L. (Ed.), *On the Teaching of Linear Algebra* (pp. 191-208). Kluwer Academic Publishers.

Inglis, M., & Alcock, L. (2012). Expert and novice approaches to reading mathematical proofs. *Journal for Research in Mathematics Education, 43*, 358–390.

Lakatos, I. (1961). *Essays in the Logic of Mathematical Discovery*. Thesis, Cambridge University Library.

Lakatos, I. (1976). *Proofs and refutations*. Cambridge: CUP.

Larvor, B. (1998). *Lakatos: An Introduction*. Routledge.

Larsen, S. & Zandieh, M. (2005). Conjecturing and Proving as Part of the Process of Defining. In Lloyd, G. M., Wilson, M., Wilkins, J. L. M., & Behm, S. L. (Eds.). *Proceedings of the 27th PME-NA*, pp. 797-804.

Larsen, S. & Zandieh, M. (2008). Proofs and Refutations in the Undergraduate Mathematics Classroom. *Educational Studies in Mathematics, 67*, 205-216.

Lecorre, T. (2016). *Des conditions de conception d'une ingénierie relative à la définition de limite – Elaboration d'un cadre basé sur un modèle de rationalité.* PhD Thesis, University of Grenoble-Alpes (France).

Mariotti, M.A. & Fischbein, E. (1997). Defining in classroom activities. *Educational Studies in Mathematics, 34*, 219–248.

Moore, G.H. (1995). The Axiomatization of linear algebra: 1875-1940. *Historia Mathematica, 22*, 262-303.

Mousoulides, N., & Sriraman, B. (2020). Heuristics in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education*. Springer. https://doi.org/10.1007/978-3-030-15789-0_172

Ouvrier-Buffet, C. (2003a). *Construction de définitions / construction de concept : vers une situation fondamentale pour la construction de définitions en mathématiques*. PhD Thesis, Université Joseph Fourier Grenoble, France, Web Available http://tel.archives-ouvertes.fr/tel-00005515/en/

Ouvrier-Buffet, C. (2003b). Can the Aristotelian and Lakatosian Conceptions constitute a tool for the analysis of a definition construction process*? Mediterranean Journal for Research in Mathematics Education, 2*, 19-36.

Ouvrier-Buffet, C. (2006). Exploring Mathematical Definition Construction Processes. *Educational Studies in Mathematics, 63*(3), 259-282.

Ouvrier-Buffet, C. (2011). A mathematical experience involving defining processes: in-action definitions and zero-definitions. *Educational Studies in Mathematics, 76*(2), 165-182.

Ouvrier-Buffet, C. (2013). *Modélisation de l'activité de définition en mathématiques et de sa dialectique avec la preuve – Étude épistémologique et enjeux didactiques.* Note de synthèse HDR. Université Paris Diderot.

Ouvrier-Buffet, C. (2015a). Modéliser l'activité de définition : vers de nouvelles perspectives en didactique. *Recherches en didactique des mathématiques, 35*(3), 313-356.

Ouvrier-Buffet, C. (2015b). A Model of mathematicians' approach to the defining processes. In K. Krainer & N. Vondrová (Eds.), *Proceedings of the ninth congress of the European Society For Research In Mathematics Education (CERME9)* (pp.2214-2220). Prague, Czech Republic: Charles University In Prague, Faculty of Education and ERME.

Ouvrier-Buffet, C. (2020). Discrete Mathematics Teaching and Learning. In S. Lerman (Ed.), *Encyclopedia of Mathematics Education*. Springer, Cham.

Pólya, G. (1954). *Mathematics and plausible reasoning, 2 vols*. Princeton: Princeton University Press, 1954.

Pólya, G. (1945). *How to solve it.* Princeton University Press, Princeton, 1973, First published 1945.

Ramirez Alfonsin, JL. (2006). *The Diophantine Frobenius Problem.* Oxford University Press.

Rogalski, M. (2000). The Teaching experimented in Lille. In Dorier J.-L. (Ed.), *On the Teaching of Linear Algebra* (pp. 133-149). Kluwer Academic Publishers.

Sally, J. D. & Sally, P. (2007). *Roots to research: a vertical development of mathematical problems.* AMS Bookstore.

Schoenfeld, A.H. (1994). *Mathematical Thinking and Problem-Solving.* Lawrence Erlbaum Associates.

Selden A. (2012). Transitions and proof and proving at tertiary level. In G. Hanna G. & M. De Villiers (Eds.), *Proof and proving in mathematics education* (p. 391-420). Springer.

Selden A. & Selden J. (2003). Validations of proofs considered as texts: Can undergraduates tell whether an argument proves a theorem? *JRME, 34*, 4-36.

Sierpinska, A. (2000). On Some Aspects of Students' Thinking in Linear Algebra. In Dorier J.-L. (Ed.), *On the Teaching of Linear Algebra* (pp. 209-246). Kluwer Academic Publishers.

Stewart, S., Andrews-Larson, C. & Zandieh, M. (2019). Linear algebra teaching and learning: themes from recent research and evolving research priorities. *ZDM Mathematics Education, 51,* 1017–1030.

Stewart, S., Thomas, M.O.J. (2019). Student perspectives on proof in linear algebra. *ZDM Mathematics Education, 51*, 1069–1082.

Swinyard, C. (2011). Reinventing the formal definition of limit: The case of Amy and Mike. *Journal of Mathematical Behavior, 30*, 93-114.

Uhlig, F. (2002). The Role of Proof in Comprehending and Teaching Elementary Linear Algebra. *Educational Studies in Mathematics, 50*(3), 335-346.

Vergnaud, G. (1996). The theory of conceptual fields. In L. Steffe, P.Nesher, P. Cobb, G. Goldin, & B. Greer (Eds.), *Theories of mathematical learning* (pp. 219-239). Mahwah, NJ: Erlbaum.

Vinner, S. (1991). The role of definitions in teaching and learning of mathematics. In D. Tall (Ed.), *Advanced Mathematical Thinking.* Kluwer.

Discrete Mathematics and Proof in the High School (2004*). Zentralblatt für Didaktik der Mathematik*, vol. 36 (2), 44-84 and vol. 36 (3), 82-116.

Zandieh, M. & Rasmussen, C. (2010). Defining as a mathematical activity: A framework for characterizing progress from informal to more formal ways of reasoning. *Journal of Mathematical Behavior, 29,* 57–75.

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