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# Approaching concepts of linear algebra with discrete mathematics: a situation from ongoing mathematical research for freshmen

Cécile Ouvrier-Bufferet

**Abstract.** Have you ever considered presenting concepts of linear algebra with objects and problems from discrete mathematics? This article provides such a situation. An epistemological analysis “à la Lakatos” of this situation is developed: it underscores how the discrete situation allows a mathematical experience for freshmen and contains interesting and even challenging material for all levels, including teachers, lecturers and research mathematicians. Productions of freshmen from scientific and non-scientific courses who face the situation are also analyzed through their defining processes.

**Keywords.** Discrete mathematics, generating, minimality, definition, defining process.

**Résumé.** Avez-vous déjà envisagé de présenter des concepts d’algèbre linéaire avec des objets et des problèmes issus des mathématiques discrètes ? Cet article présente une telle situation. Une analyse épistémologique “à la Lakatos” de cette situation est développée : elle souligne comment la situation discrète permet une expérience mathématique pour des étudiants de première année et contient du matériel intéressant et même stimulant pour tous les niveaux, y compris les enseignants, les chargés de cours et les mathématiciens. Les productions des étudiants de première année de filières scientifiques et non scientifiques sont également analysées à travers leurs processus de définition.

**Mots-clés.** Mathématiques discrètes, générateur, minimalité, définition, processus de définition.

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## 1. Introduction

The main feature of my mathematical and didactical research concerns the heuristics, and more precisely the “defining processes” which are used in mathematical research. Several studies about problem-solving have underscored the various interests of putting learners in a mathematical enquiry with open problems. These studies concern proof, modeling, changes of representations, use of new technologies etc. but little is found about “defining” even if the study of defining activities is a discreet but constant didactical topic of research in mathematics education since the 90s (e.g Mariotti & Fischbein, 1997; Zandieh & Rasmussen, 2010; Ouvrier-Bufferet, 2003a, 2013). This is one of the reasons that prompts me to initiate further discussion about “definitions” between mathematicians and mathematics educators, through this article: the definitions and their construction have definitely a place in mathematical research. I have the following guiding idea: I consider that definitions are concepts holders. During a research process, they are not finite products. Definitions evolve through several statements, each one giving new features of the involved connected concepts. In fact, to understand how concept formation works implies exploring the wide field of mathematical definitions considered as concepts holders. Studying the dialectic between the construction of definitions and the formation of concepts should be useful to design didactical situations where learners would have to build new concepts in order to solve a problem. That leads me to several major questions: how can one characterize defining processes in mathematics? Are learners able to define? Do they understand concepts better if they are involved in a defining process? I therefore had to work out a theoretical framework through epistemological, didactical and empirical research in order to characterize defining processes (Ouvrier-Bufferet, 2013, 2015ab). I have conducted several experiments<sup>1</sup> at the university level: the results of these experiments show that the students’ ability to define is real.

To begin with, I will present a part of my theoretical framework borrowed from epistemology, that of Lakatos (1961, 1976), focusing on definitional procedures. I will use it in order to highlight the potential defining process in an original problem - displacements on a regular grid - still partially open from a mathematical point of view. This problem should bring a wealth of interesting material for students, lecturers and researchers. I will then compare the processes of freshmen at university, from scientific and non-scientific courses, when they face this problem. The concluding section argues that this research opens new opportunities to teach abstract contents at university, involving

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<sup>1</sup> My experiments were conducted in discrete mathematics with the following concepts which are of different natures: trees (a known discrete concept, graspable in several ways), discrete straight lines (a concept which is still at work, for instance in the perspective of the design of a discrete geometry) and a wide study of properties of displacements on a regular grid (e.g. Ouvrier-Bufferet, 2003a, 2003b, 2006, 2011).

discrete mathematics but also defining processes, and engaging students in a mathematical exploration of a real ongoing problem, namely a research situation.

## 2. Framework

Exploring the wide question of “definitions” can lead us to deal with philosophical notions (such as those of Aristotle), logical constraints linked to the construction of a formal theory, epistemological tendencies (e.g. Lakatos 1976) and cognitive aspects (e.g. Vinner, 1991, with *concept image* and *concept definition*). Different kinds of definitions have been described in the literature (procedural and structural, nominalist and essentialist, lexical and stipulative, intuitive and formal, among many others). Nevertheless, I would like to stress that looking at definitions as specific statements does not give us any clues to characterize defining processes. From a cognitive point of view, Freudenthal (1973, p. 458) has studied two kinds of defining activities: the *descriptive (a posteriori) defining* and the *constructive (a priori) defining*. In both cases, the mathematical concept involved is already known by the learners; the learners do not build it in order to solve a problem, they do not encounter a quite new concept. Indeed, the descriptive defining “outlines a known object by singling out a few characteristics properties”, and the constructive defining “models new objects out of familiar ones” by examining changes to a given definition. I would like to go further in the direction of the constructive defining, but for concepts unknown by the learners, and from an epistemological point of view. Let me consider an aspect of Lakatos’s work (the concept-stretching), which in my opinion does not get all the attention it deserves: “A definitional procedure is a procedure of concept formation” (1961, p. 54). This kind of procedure focuses on knowledge expressed in skills and processes. Lakatos’s original contribution to the debate consisted in his attempt at a modeling of the mathematical discovery while integrating both the social<sup>2</sup> and the conceptual<sup>3</sup> aspects of the matter. He was strongly influenced by Pólya’s mathematical heuristics, Hegel’s dialectic, and Popper’s fallibilism. Lakatos has clearly influenced mathematics education, especially in the development of new kinds of interactions between teacher and students and in the implementation of the game of proofs and refutations in the classroom<sup>4</sup> (e.g. Larsen & Zandieh, 2008). However, Lakatos’s modeling of definitional procedures, more present in his PhD thesis (1961) than in his book ‘Proof and Refutations’ (1976), is little known, or rather little used. I will take this work about defining processes into account and test it on a research situation.

We should be well-aware that the processes (the defining processes and the process of proof and refutations) described by Lakatos strongly depend on the starting situation which consists of: a situation of classification (which delimitates what is the class of polyedra), an initial conjecture (Euler formula), a first representation of the mathematical objects at stake<sup>5</sup> (polyedra), and a new proof (that of Cauchy, where a change of thinking is required because it implies a topological view).

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<sup>2</sup> The debate context.

<sup>3</sup> The connection between concept and proof, on the one hand, and between concept formation and definitions, on the other hand.

<sup>4</sup> The limits of the use of the Lakatosian method have been pointed out by De Villiers (2002) and Hanna (2007), both of them quoting Conway. In particular, they said that Lakatos’s method couldn’t be applied generally. In defense of Lakatos, in his original thesis, he denies that the unique logic of mathematical discovery exists.

<sup>5</sup> The intended concept is then partially known *a priori*.

In particular, with this context of geometrical objects, it means that the concepts have a pre-existence, as well as there exists a pre-axiomatic theory. Very few situations following the whole Lakatosian model of refutations have been experimented in mathematics education: all of them have the previously quoted aspects (e.g. Larsen & Zandieh, 2008). I will extend the use of (a part of) the Lakatosian modeling to a less restrictive situation (see section 3). Let me first present this modeling which is specific to definitions.

I point out that Lakatos leaves aside the first encounter with a mathematical object and also the axiomatization, the latter being, for him, the end of the creative process. His study of the mathematical process leads him to focus on refutational processes and definitions in connection with concept formation. Lakatos precisely identifies three kinds of definitions: the naive definitions, the zero-definitions and the proof-generated definitions, whose respective functions are to denominate, to communicate a result and to prove. A naive definition can be stated first, but it cannot evolve<sup>6</sup>, contrary to a zero-definition that marks the beginning of the research process (hence the term “zero”). A zero-definition can be vague, outside of any formalism, it does not matter: it has to evolve into a “better” one. Hence a fundamental question arises: how can a zero definition mutate? Lakatos relies on two cases (see also Larvor, 1998):

- The concept is altered by the presentation of a new kind of object. Here, the conjecture should be protected from such a counter-example (a monster), the zero definition can be modified;
- Or the concept is altered by the presentation of a new proof: a work of translation is engaged, when one deals with a new proof in another mathematical framework. This is where a system of mathematical concepts can grow. Then a zero definition becomes a proof-generated definition with a validation by a proof.

This modeling of defining processes can seem to be reduced, but I will show you in an example that it is helpful to analyze mathematical situations involving defining processes and also to characterize students' procedures. The chosen situation involves concepts which are unfamiliar. The mathematical results are also unfamiliar and may be surprising for students. They belong to discrete mathematics and are close to concepts from linear algebra whose teaching and learning are known to be problematic.

### 3. Discrete problem *versus* linear algebra

#### 3.A. Well-known obstacles in teaching and learning linear algebra

Let me summarize the main results of studies on the teaching of linear algebra in order to emphasize some recurrent difficulties and obstacles and then to justify my attempt to problematize certain concepts in a new perspective (see also Carlson et al. 1997, Kalman & Day, 2001). Dorier explains that students “are overwhelmed by the number of new definitions and feel like they are landing on a new planet” (Dorier, 2000, p. 185). He underscores the obstacle of formalism in linear algebra as well, characterizing this field of mathematics as a unifying and generalizing theory (Dorier, 2000; Dorier et al. 2000a; Dorier et al. 2000b). The lack of connections with previous

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<sup>6</sup> Lakatos explains that we can do nothing with a naive concept, because we cannot extend it, unless you are able to fit it in a theory, then the considered concept will not be naive anymore!

knowledge, the lack of practice in basic logic and set theory (misunderstanding of implicit quantifiers and of necessary and sufficient conditions) and the impossibility of using geometrical intuition also lead to difficulties for students in learning linear algebra. Besides, the multiple proofs and kinds of reasoning in this branch of mathematics make it complex for students (e.g. Sierpiska, 2000; Uhlig, 2002). This last point leads us to the problem of the meaning and of the control we can have on proofs when we construct them, especially in an abstract theory. Several propositions, different in nature, have been made to help overcome students' difficulties (Harel, 1998; Rogalski, 2000; Uhlig, 2002), using the historical roots of linear algebra (for instance, through the concept of 'rank' and the pivotal cluster of concepts 'linear combination', 'dependence' and 'independence'). Most of them are produced within long-term studies (didactical engineering in the meaning of Artigue, 2014) and all of them are within the strict field of algebra. Furthermore, the attempts to connect linear algebra to 2D and 3D geometry in order to give an image of some concepts have shown their limits. Moreover, such approaches convey other obstacles: the problem of these metaphorical approaches has been studied by Hillel (2000) and is also pointed out by Harel (1990, 1998). Indeed, geometry is limited to three dimensions and some concepts (such as linear (in)dependence or rank) have a very limited range of representation in the geometric context. And the loss of meaning through the transition to algebraic formalism actually turns out to be an issue in this type of geometric approach. I would also like to stress the complexity related to the understanding of the concepts in linear algebra with formal definitions, because "most of the models used in linear algebra for generalization and embodiment of concepts involve difficult mathematical ideas which students are not prepared to deal with." (Harel, 1990, p. 388). Besides, Harel's results underscore that, in linear algebra, "(...) students do not build effective concept images; rather, they place their full reliance on concept definitions, by memorizing them verbatim. (...) Once the concept definitions are forgotten, students are unable to retrieve or rebuild them on their own." (Harel, 1998, p. 499). The importance of involving students in a mathematical experience here appears crucial to me.

Faced with these obstacles, what kind of new approach can one propose? I have chosen to explore the following track: to engage students in a mathematical questioning close to that of linear algebra (but more "intuitive") in order to explore relations between concepts similar to those of linear algebra. Similar but not so interrelated. And I have found a nice discrete problem, outside formal algebra, to achieve that.

### 3.B. A discrete problem

I present here a very natural statement of the problem I consider. Take a regular squared grid<sup>7</sup> and some displacements which can be described with integers (or only positive integers) on this grid (Table 1). A starting point is given. An experimental fact is easy to grasp: starting from the starting point, a point of the grid is reachable or not, when one uses the given displacements. So "the" question: which points of the grid can one reach using positive integer combinations of these displacements?

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<sup>7</sup> One can take another regular tessellation... not easy!

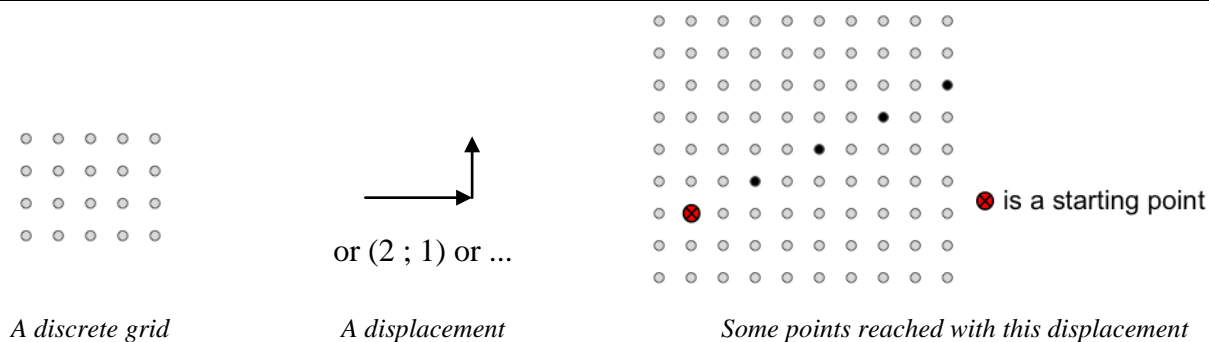


Table 1 – The objects at stake

There are a lot of underlying questions: is it possible to reach *all* the points of the grid? or *some sectors* of the grid? Can one characterize *generating sets* of displacements? Are they *minimal* ones? What about the *cardinality* of minimal generating sets? Etc.

### 3.C. An epistemological analysis of the problem

I will propose one possible exploration<sup>8</sup> of the discrete problem – the most natural and intuitive one – through the Lakatosian kinds of definitions, while making room for heuristics<sup>9</sup> as they can be involved in such a mathematical enquiry. The analysis of the mathematical problem has helped me to design a didactical situation for freshmen (see section 4 and Ouvrier-Buffet 2003a, 2011).

I will use letter  $G$  for the regular squared grid,  $A$  for a starting point, and the term *integer point* or *point* for lattice point. I have chosen to define a *displacement* on  $G$  with two positive integers and two directions (among up, down, left and right). For instance, “2 squares right and 3 squares down”. A displacement can be represented with a vector. One can also consider displacements and their opposite displacements (and so integer coordinates), it is still a real issue. The order of the displacements does not interfere because displacements are commutative (property easy to prove).

### 3.D. Exploring “small cases”<sup>10</sup>

There is an obvious set of four displacements which allows us to reach every point of  $G$  (Fig. 1). It is also obvious that two displacements are not enough to reach all the integer points: either all the points of a sector of  $G$  (i.e. a set of integer points bounded by two rays) are reached (Fig. 2), or only some points of a sector are reached (Fig. 3). I would like to underscore two major facts:

- The question of “reaching integer points” can be refined: one can study sets of displacements which allow to reach all the points of a given sector and/or sets of displacements which allow to reach all the integer points. In both cases, one can talk about “*generating sets*” (for a sector or for  $G$ ).

<sup>8</sup> I encourage the reader to explore the discrete problem and to suggest other kinds of enquiries than the one proposed in this paper.

<sup>9</sup> According to Pólya (1945, 1954), heuristic is a set of strategies for solving mathematical problems to learn, to teach and to reconstruct mathematics. For instance, I consider that the following actions are heuristics: proving, conjecturing, refuting, creating, modeling, defining, extending but also transforming a questioning process, being able to mobilize non-linear reasoning, experimenting, decomposing-recomposing.

<sup>10</sup> i.e. for two, three, and four displacements.

- If one tries to build generating sets made of three displacements – in order to do “better” from the optimization point of view than Fig. 1 – one can start from sets of two displacements for instance. Take Fig. 2: every point of the sector bounded by the two given displacements is reachable (I call this property “with Full Density” (FD)). It is not always the case: Fig. 3 shows us that some points of the sector bounded by the displacements are not reachable. And now, I try to go “a little bit everywhere”<sup>11</sup> (ALBE) when adding at least one displacement. A geometrical argument (translation of the reached sector for instance) brings a third displacement (as represented on Fig. 4, taking quadrants into account, or as we can operate on Fig. 2 with a very acute angle).

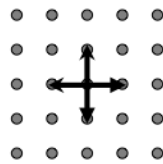


Fig. 1

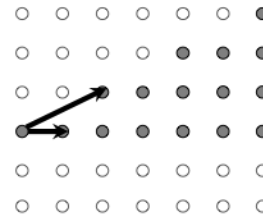


Fig. 2

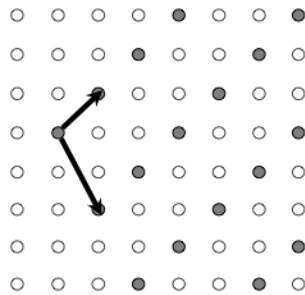


Fig. 3

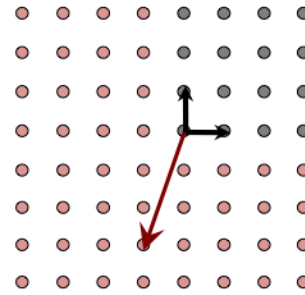


Fig. 4

Table 2 – Exploring “small cases”

The exploration of small cases leads us to these results:

- Two major properties emerge: one can give the following zero-definitions of “with Full Density” (FD) and “a little bit everywhere” (ALBE).

o  $Zdef_{FD}$ : a set of displacements has the FD property if and only if<sup>12</sup> all the points of a sector of the grid are reached;

o  $Zdef_{ALBE}$ : a set of displacements has the ALBE property if and only if there exists a positive number  $\mu$  such that for any point X in the grid there exists a reachable point Y, “close to X”, i.e. whose distance from X does not exceed  $\mu$  (here, it is possible to avoid a metric altogether).

<sup>11</sup> Allusion to “almost everywhere”. One can also link this property to the discrete mathematical field of “covers” and raise the question of the minimal cover.

<sup>12</sup> The use of the expression “if and only if” for a definition is sometimes questioned by some philosophers or teachers for whom a definition is only a denomination (see Ouvrier-Buffet, 2003a). I choose not to deal with such conceptions in this article.



- One can reach all the points of the grid when the FD and ALBE properties are satisfied simultaneously. Then, they imply a zero-definition of a generating set.

- To reach the four cardinal points or to generate the four unit displacements implies to reach all the integer points. One then considers the grid through two directions (the horizontal and the vertical ones). If I outline here a vertical development of this discrete problem (as done in Sally & Sally, 2007), I should point out the wider NP-Hard Frobenius Problem (Ramirez Alfonsin, 2006), also called the “coin problem”: “given coins of denominations  $a_1, a_2, \dots, a_n$  (with no common factor), what is the largest amount that cannot be changed?”. Dealing with small cases of the Diophantine Frobenius problem can lead to the Pick Theorem, which has a lot of interesting applications (Sally & Sally, 2007). Such a development would require another article.

After this prior exploration, more “natural” queries emerge:

1) **Existence**: do such generating sets exist?

2) **Building generating sets**: How can one build generating sets? Other generating sets? Do such sets have other mathematical properties? An integer  $k$  being given, can one build all the generating sets with  $k$  displacements? This last question makes sense when we deal with the notion of minimal generating sets.

3) **Removing a displacement**: Is it possible to remove a displacement without changing the set of reachable points? One can say that a set of displacements is *minimal* when removing any of its displacements modifies the set of reachable points. With this definition, how to characterize a *minimal generating set* of displacements?

4) **Cardinality**: Do the minimal generating sets of displacements always have the same number of elements? The answer is obviously no (see Figures 1 and 4), and that is surprising, at least for students who are used to dealing with geometrical analysis and “kind” vector spaces. In fact, one can bring the discrete problem on the grid back to a problem on  $\mathbb{Z}$ , and then mobilize knowledge from number theory. It is easy to build generating sets with different cardinalities.  $E = \{1; -1\}$  and  $F = \{2; 3; -6\}$  are two simple examples of minimal generating sets for  $\mathbb{Z}$  (for  $E$ , if I remove 1, I loose the ALBE property. For  $F$ , if I remove 2 or 3, I loose the FD property and if I remove -6, I loose the ALBE property).

A generalization leads us to the following result:

**Proposition 1**: for any integer  $k$ , there exist, in  $\mathbb{Z}$ , minimal generating sets of displacements with  $k$  elements.

Remark: the search of the proof of proposition 1 can lead the solver to the use of known arithmetical results, but it can also lead the solver to explore arithmetical properties (see for instance (Duchet, 1994) for an experiment with 14 year old students).

And now, if one wants to build a minimal generating set on the squared grid, one can keep in mind the horizontal/vertical representation and choose displacements carefully, in order to keep the minimality. Therefore, one has the following proposition:

**Proposition 2**: for any integer  $k$ , there exist, on grid  $G$ , minimal generating sets of displacements with  $k$  elements.

*Indications for the proof* (in order to underscore its accessibility): one constructs a set of horizontal minimal generating displacements with  $(k-2)$  elements in order to generate  $Z$  (see above,  $k$  being as big as one wants) and then adds two vertical displacements in order to go everywhere by translation.  $\square$

More questions to explore: can one build other kinds of minimal generating sets than those built in the previous proof? Do other proofs exist? I now directly address the readers: what kind of exploration of this discrete problem would you have conducted? With which major mathematical questions? What kinds of heuristics would you have mobilized?

## 4. Design of a didactical situation: choices and challenges

### 4.A. A potential evolution of zero-definitions to proof-generated definitions?

The shift from zero-definitions to proof-generated definitions is not easy to re-implement in the classroom when one designs a didactical situation. Indeed, Lakatos has based his historical reconstruction upon several mathematical backgrounds, using Cauchy's proof as a motivation to boost concept formation. And yet, this part of the defining process would be bringing learning potentials to teach proof and highlight the relations between proof and definitions, especially at university. In fact, no study in mathematics education has showed yet didactical situations involving such a shift from zero definitions to proof-generated definitions. If one wants to make the most of the Lakatosian view of definitions, one can start by focusing on the zero definitions and on their potential evolution to other "better" zero-definitions. It implies a view of the constructive and tentative nature of definitions which are socially constructed. The proof generation aspect comes later. I do not forget that some studies reveal that students can have difficulties in understanding the "very nature of mathematical definitions" (Edward & Ward, 2004, p. 411). Studying the zero definitions and their features can be a real opportunity for us to solve this understanding problem.

### 4.B. Zero-definitions at stake

I have designed a situation (made up by three problems) for freshmen who have not yet encountered linear algebra (or very partially) starting from the problem of displacements. I kept in mind the epistemological, logical and cognitive obstacles mentioned above, in particular the fact that students may have concept images coming from analytical geometry for instance (they can think that two displacements are enough to reach every integer points). I have tried to motivate the emergence of questionings and of zero definitions (generating sets, minimal generating sets, FD, ALBE). The validation of the constructed zero-definitions comes in their use when building new sets of displacements and testing new sets of displacements (generating and minimal aspects). Table 3 synthesizes the choices I have made for the design of the didactical situation (see also the appendix).

Problems	Exploration leads to ...	Validation / Potential evolution
Problem 1: Two displacements on the grid	- assimilate the rules of the displacements and the questionings.	Aim: to dismiss students' pre-existent concept image coming from vector space of dimension 2

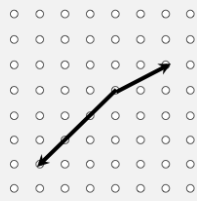


Fig. 5

- in order to generate all the points of the grid, more than two displacements are required”.

- a *conjecture*: “three displacements are enough to reach all the points of the grid.”

- a first natural  $Zdef_{generating\ set}$ : “a generating set is a set of displacements which allows access to all the points of  $G$ ”.

-  $Zdef_{ALBE}$  and  $Zdef_{FD}$  possibly.

where two vectors are enough to build a basis.

This  $Zdef$  is not operational to build generating sets (even if it is operational in the graphic register for two displacements).

The use of them to prove that a set is a generating one testifies to their validity.

Problem 2: Two or three displacements or four displacements where two of them are dependent

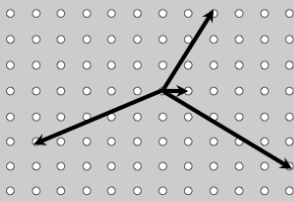


Fig. 6

-  $Zdef_{minimal\ generating\ set}$ : “a minimal generating set of displacements is a set of three displacements allowing access to all the points of the grid”.

-  $Zdef_{minimal\ generating\ set\ bis}$ : “a minimal generating set of displacements is a set of non-dependent displacements”.

- Towards the *proof* that “three displacements can be enough to generate all the points of the grid”. But one cannot conclude that all the generating sets are made up of three elements.

To explore a problem where four displacements are independent allows the invalidation of this  $Zdef$ .

Aims: To prove the existence of minimal generating sets, to generate such sets, to prove that a set is a generating one and to prove that a set is a minimal generating one.

Distinction between *minimal* and *minimum*.

Problem 3: A minimal generating set made up of 4 displacements (one can prove that this generating set is a minimal one)

To demonstrate that “three displacements are not always enough”. In fact, the set presented here features four displacements and is a minimal generating one.

The corollary of the exchange theorem (i.e. the existence of dimension, true in a vector space) is false in the discrete case<sup>13</sup>.

<sup>13</sup> Remark: the invariance of dimension makes sense when one shows in which instances it does not work.

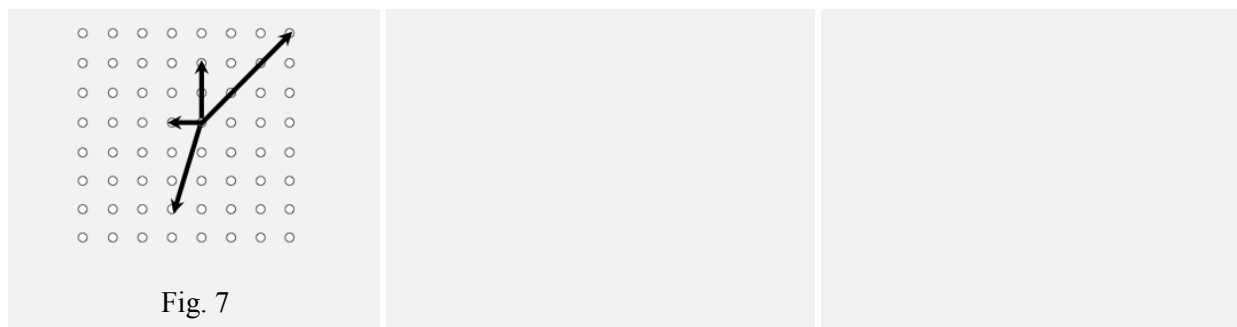


Fig. 7

Table 3 – Choices for the didactical situation

#### 4.C. Main features of the discrete problem compared with “classical” linear algebra

From an historical point of view, the fundamental importance of modules over a ring is confirmed, vector spaces over a field being as a subsidiary notion (as done in Mac Lane and Birkhoff’s Algebra (1967))<sup>14</sup>. The contemporary works of researchers in didactics about the teaching and the learning of linear algebra take this branch of mathematics as a “self-sufficient” full fledged system. It is obvious that making connections with modules over a ring is not the easiest way to teach linear algebra at the beginning of university. The discrete situation I suggest is a way to situate linear algebra’s questionings in a wider context, but in an easier way than Mac Lane and Birkhoff’s approach. It also gives a “natural” exploration of some results that are not “always true”. Indeed, a minimal generating system, a maximal linearly independent system are not basis in the general case. For instance, for  $Z/(6)$ , considered as a  $Z$ -module, there is no linearly independent system (non-empty) and  $\{2; 3\}$  is a minimal generating system. And in  $Q$ , considered as a  $Z$ -module, every system reduced to one element (non-zero) is a maximal linearly independent system and there exists no basis.

Let me go back to the discrete problem. The discrete problem is an open problem, which the students do not know. They can “do” mathematics, as a researcher does, without looking at a pre-existing course in a textbook. The discrete object “displacement” is different than in the continuous case: there is no  $+$  and  $-$  and there is no modification of a displacement by a scalar multiplication. In linear algebra, the scalar multiplication brings the FD property; and the use of  $+$  and  $-$  brings us the possibility to go everywhere but not necessarily ALBE. Moreover, the question of “dimension” is typically a part of linear algebra whereas the discrete case deals with the question of “minimality” (the concept of “dimension” would be problematized, provided that students have already encountered this concept in other contexts.)

The queries are common to discrete and continuous problems but they are more intuitive in the discrete case than in linear algebra, even if the mathematics behind the displacements can be quite advanced. Furthermore, the concepts are highly interrelated in vector space whereas the discrete problem clearly separates the concepts (the “same” concepts as in vector space): it then especially allows work to be done on the links between the concepts involved in the problem and on the necessary and sufficient conditions. Besides, the links between concepts are more important than

<sup>14</sup> See G.H. Moore, The Axiomatization of linear algebra: 1875-1940, *Historia Mathematica* 22 (1995), 262-303 for a wider historical study.

the concepts themselves, which are not an obstacle in studying the discrete case. The discrete queries – which can also be found in other fields of mathematics – and an exploration of links between definitions and proofs should be fruitful.

The situation proposed in discrete mathematics is then decontextualized in comparison with a classical introduction of concepts in linear algebra. The concepts of “generator”, “minimality” but also “dependence”, and later “basis”, can be studied while avoiding excessive formalism. The shift from  $\mathbb{N}$  and  $\mathbb{Z}$  to  $\mathbb{R}^+$  and  $\mathbb{R}$  has to be introduced by the teacher. All these features show the potentialities of this discrete situation, which can be an introduction to a linear algebra course, in order to highlight mathematical questionings, under certain didactical conditions. Such conditions should follow these features: a specific research contract in the classroom, a long-term situation, and an institutionalization of enquiries which are common to the discrete problem and to linear algebra.

## 5. Students faced with displacements on a regular grid

I have experimented the situation of displacements with freshmen from scientific and non-scientific courses at university. The first time with freshmen in science training for my didactical research, and the second time with future economists for the needs of my teaching, before a course about linear algebra. I will not go into the details of the methodology, it is not my purpose here. Students from scientific courses are used to working in groups (3 or 4 students per group, around 30 students) and the intervention (audio-taped) lasted 3 consecutive hours. The future economists (around 30 students) were not recorded (except one group), I will report notes I have taken for this group of students. I will give the general tracks followed by the students from scientific courses and focus on the main differences between these two groups of freshmen (see the appendix for excerpts).

### 5.A. With freshmen from scientific courses<sup>15</sup>

I must point out that the freshmen who took part in the intervention had a small academic knowledge in linear algebra: they knew the notions of “linear independency”, “basis” and “vector space” (they knew their definitions in a vague manner). The situation of displacements allows a work on mathematical objects graspable through a basic representation close to that of vectors. The objects do not represent an obstacle in themselves. The main difficulty lies in the fact that properties have to be characterized (generating sets, minimality, among others) and to be denominated. These specificities of the situation of displacements partially explain why the students did not engage in characterizing mathematical properties. Indeed, only some statements were produced but they were not recognized as important mathematical statements and they did not evolve into institutionalized definitions. Moreover, the persistence of concept images coming from analytical geometry was real (they said for instance: “We have four displacements. We need only two vectors as a basis.”). Nevertheless, I have identified a natural definition of “generating” (“to reach all the points of the grid”<sup>16</sup>). It has been transformed into an operational property (“to generate the four cardinal points

<sup>15</sup> For a more descriptive study of the students’ productions, see Ouvrier-Buffet (2003a).

<sup>16</sup> The sentences in quotes come from the students’ dialogues. I have formalized them a little bit, because students did not institutionalize them, they only used these properties or definitions in the action.

or unit displacements”). It has not been connected to the ALBE and FD properties, even if these properties were implicit in the students’ dialogues (the ALBE property is predominant). Furthermore, I have identified two *definitions-in-action*<sup>17</sup> (i.e. statements that enable students to pick up the most adequate selection of information according to the situation, but without any explicit definitions): one for “minimal generating set” (“when all the displacements are used during the search of four unit displacements”) and one for “minimal” (“a set of displacements is not minimal when one of them is an integer combination of the others”). Students can work efficiently with these two definitions-in-action, which are both connected to the same proposition-in-action (“to have the four directions represented is a necessary condition in order to have a generating set of displacements”). Such a definition-in-action can become powerful. But in this case, the definition-in-action of “minimal” was imported by the students from their previous knowledge in linear algebra. In fact, in this example, “minimal” is a translation of “independent”. This kind of definition-in-action (an imported one) blocks the students’ conceptualization. Students stayed in the action, in the proposed configurations. Their process did not move to a generalization which would have allowed an evolution of their definitions-in-action. For them, there is no explicit need for formalization (Harel, 1998). Moreover, the management conducted by the teacher-observer is not definition-oriented, it is neutral. A plausible hypothesis is that this distance between manipulation and formalization (formalization merely being a first step, not a complete theorization) is too rarely approached in the teaching process. It goes along the lines of previous epistemological and didactical results which conclude that formalism is a crucial obstacle in the teaching of linear algebra. It underlines that putting students into a mathematical experience is a necessary condition in order to help them in their learning process at university. Besides, the short time format (three hours) probably did not encourage the evolution of definitions, a wider exploration of defining processes requiring more time.

### **5.B. Main differences between freshmen from scientific and non-scientific backgrounds**

The mathematical exploration of the discrete problem is more natural and more intuitive for freshmen from non-scientific courses (NS group) than for freshmen from scientific courses (S group). It is not so surprising: the S group tries to mobilize their previous knowledge and that limits the students in their study of the mathematical problem. Moreover, studying cases where the cardinality of the minimal generating sets is not an invariant feature is not a common practice at the beginning of university. The NS group were more accurate on their characterization of sectors, focusing on the ALBE and the FD properties, but (again) without defining them (these students stayed in the action too). In fact, it is as though having (partial) knowledge in linear algebra blocks students in their exploration of the discrete problem. That is the reason why the place in a course of the discrete situation should be before a course of linear algebra in order to make room for the mathematical exploration of the discrete case. Then, an institutionalization of the questionings (common to the discrete and the continuous cases) should be done. Besides, a study of the relations

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<sup>17</sup> For this concept of definitions-in-action, I was inspired by Vergnaud (1996) and his characterization of *concepts-in-action*.

between the concepts involved in the discrete situation opens new opportunities to engage students in a work on implication (necessary and sufficient conditions).

I am studying the way one can use the discrete case during the first university year to introduce to concepts and questionings of linear algebra in the most beneficial manner. I pay attention to the distance between manipulation and formalization and also to the difficulties that students encounter with formulation and validation (in Brousseau's 1997 meaning): the articulation between defining and proving should be built in students' practices as well as a work on mathematical statements in order to deepen the work on proof. Such a research is linked to the study of defining and proving processes of mathematicians (e.g Ouvrier-Bufferet, 2013, 2015ab). This work is still in process.

## 6. Opening

“Ideas concerning the mathematical understanding of students at all levels are germane to the study of the mathematical process.” (Vinner, 1991, p. 120)

“Students should have experiences that focus on the use of mathematical definitions and experiences in the process of defining.” (Edward & Ward, 2004, p. 422)

### 6.A. An efficient tool in epistemology and didactic

The Lakatosian kinds of definitions (especially zero-definitions) allow an epistemological analysis useful from a didactical point of view, bringing a (partial) modeling of defining processes and elements in order to make the construction of definitions more dynamical. With the notion of definitions-in-action, students' processes can be described, as well as the lacks in students' defining processes. I also keep in mind Harel's necessity principle<sup>18</sup>, but it remains difficult to design didactical situations involving this principle for the defining processes. Not only because students have not yet encountered such processes, but also because “all” the features of the defining processes are not yet well known, even if some researchers have characterized the heuristics and behaviors of mathematicians (e.g. Burton, 2004; Schoenfeld, 1994; Ouvrier-Bufferet, 2013).

### 6.B. Discrete mathematics: a way to consider mathematical objects

Of primary interest in the situation of displacements on a grid is the use of discrete structures to grasp transversal concepts (such as generating sets, the question of minimality, etc.) and transversal questionings which belong to several fields of mathematics. Moreover, the discrete situation allows an access to the reasoning behind the construction of concepts and then contributes to the development of proof abilities through an active mathematical exploration of the problem (construction of relations between properties and proofs of implications) (see for other examples Gravier & Ouvrier-Bufferet, 2022; Ouvrier-Bufferet, 2020).

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<sup>18</sup> Harel (1998, p. 502) explains the Necessity Principle translating as three steps:

- 1) Recognize what constitutes an intellectual need for a particular population of students, relative to the concept to be learned.
- 2) Present the students with a problem that corresponds to their intellectual need, and from whose solution the concept can be elicited.
- 3) Help students elicit the concept from the problem solution.

If one tries to problematize the same concepts in other mathematical situations involving material objects (such as displacements, for the reason of the devolution of the problem), one can foresee the following tracks: to cover the plane with fractals, to generate some kinds of graphs. One also has to question the invariants in the discrete mathematics which are common to the continuous cases: if the discrete problems are sometimes (and even often) easier to grasp than the continuous ones, the mathematics behind can be quite advanced. Such a discussion leads us to further studies. There is room to explore new questionings about the teaching of continuous structures, with the help of the discrete ones. This comparatively young branch of mathematics arouses interest because it brings new potentialities for education: Goldin (2004, p. 58) emphasizes “how experiences in discrete mathematics may provide a basis for developing powerful heuristic processes and powerful affect” (cf. a special issue of ZDM about Discrete Mathematics and Proof in the High School, 2004). Exploring this avenue is worthwhile for the investigation of mathematical practice.

## Appendix

### 1) The problems which make up the situation

On a regular squared grid, for each problem, use a set of displacements and choose a starting point (called A), any one. The questions are:

- 1- Starting from point A, which points of the grid can we reach?
- 2- What are the consequences if we remove one or more displacements?

Problems	Sets of displacements
<b>Problem 1</b>	$d_1$ : 2 squares to the right and 1 square up $d_2$ : 3 squares to the left and 3 squares down
<b>Problem 2</b>	$d_1$ : 2 squares to the right and 3 squares up $d_2$ : 5 squares to the left and 2 squares down $d_3$ : 5 squares to the right and 3 squares down $d_4$ : 1 square to the right
<b>Problem 3</b>	$d_1$ : 3 squares to the right and 3 squares up $d_2$ : 2 squares up $d_3$ : 1 square to the left $d_4$ : 1 square to the left and 3 squares down

### 2) Students' dialogues – Some excerpts

#### Group from non-scientific background (NS group)

During the following excerpts, Samy and Yohan try to build a generating set of displacements (O is the Observer).



- *Concept image “two displacements are enough”*

444 S: But why couldn't it work with two displacements?

- *Definition-in-action of ALBE. Horizontal/vertical representation and use of number theory*

458 S: I can cover all the squares upright, because I can go down each time for one square. And with this... I can go to the right and to the left. Then, this set of displacements works for me. (...)

505 O: Why did you take 2 squares to the left?

506 S: Because I had 3 squares to the right.

507 Y: You want to come back from 1 (...)

510 S: I think that with a combination with 3 and 2, one reaches all the numbers.

- *Four cardinal points and four unit displacements (see Fig. 10)*

564 S: In fact, when I use these displacements several times, I reach these points (*he shows the four cardinal points*). They would symbolize a unit-displacement up, a unit-displacement to the right, a unit-displacement down and a unit-displacement to the left (...) they are commutative (...) the four elementary points ... I say “elementary” because they allow the production...

565 Y: ... of all the points.

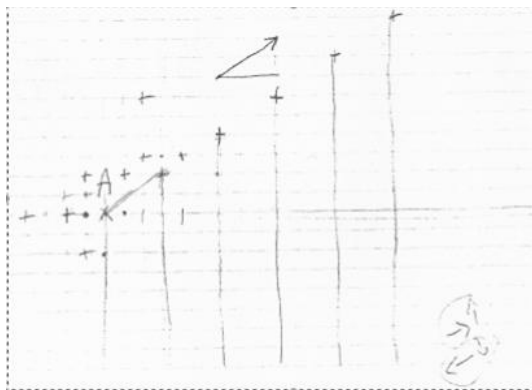


Fig. 8

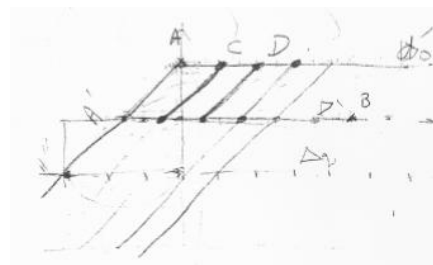


Fig. 9



Fig. 10

Representations used by the NS group

### Groups from scientific courses (excerpts of four groups)

- *Concept image “two displacements are enough”*

138 P: We have four displacements. We only need two vectors as a basis. (Yellow group – problem 2)

- *Knowledge from linear algebra*

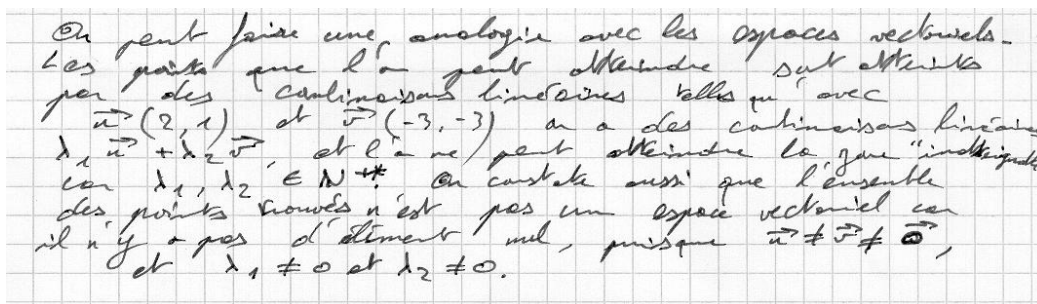
350 S: The thing is ... no, I don't think that it is the same thing (*as linear algebra*), because here, we can't reach this point.

351 P: Each time, you try to reuse your courses, you realize that it is useless!

352 S: No, but here, it is obvious. But a linear combination, it is a 'lambda  $v$ ' thing, where  $v$  is any vector...

353 P: I don't know what it is! (...)

373 S: Because it is precisely not a vector space, I mean that it looks like a vector space, but it is not a vector space. First, there is no zero element, and besides the 'lambda' that we can use can't be negative. How would you say that? ... in fact, you only have two kinds of displacements, but for instance, when you use the (*displacement*) (3; 3) like this, you can't use the (*displacement*) (3; 3) in the other direction. And the same thing happens with the (2; 1), you can't use the (-2; -1). (Pink group – problems 1 & 2)



(Translation) "We can make an analogy with vector spaces. The reachable points are reached with linear combinations such as with  and  we have linear combinations , and we can't reach the "unreachable" zone because . One also notices that the set of the found points is not a vector space because there is no zero element, because , and ". (Purple group – an excerpt of their notes)

#### - FD property

1430 R: We noticed that if we remove d4, it is not like removing d2 where a whole sector disappears. If we remove d4, a lot of points are removed, but we still have points in a whole zone. (Pink group – problem 2)

#### - Four cardinal points and algebraic systems

178 B: In order to go everywhere, you try, and if you find (1; 1), it means that you can reach this point. After that you go on with this system to see if you have solutions. You process with (1; -1) in order to see if you can reach this point and with (-1; 1) and with (-1; -1).

179 A: Yes! You can see if you can reach the four points around!

180 B: Therefore, it means that you can go everywhere. If we can prove that, we have no more questions. And if we can't prove it ... it means that there are other conditions, it is complicated, or there is a technique...



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