State-Feedback and Static-Output-Feedback Parametric Controllers Based on a Discrete-Time LPV Grid Based Approach

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State-Feedback and Static-Output-Feedback Parametric Controllers Based on a Discrete-Time LPV Grid Based Approach ⋆

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Abstract: This work presents a two-step LMI-based approach for the synthesis of State-Feedback and Static-Output-Feedback controllers for discrete-time Linear Parameter Varying systems. The benefit of the proposed approach lies on the fact that the two-step LMI method allows to enforce a common parametric structure on the controller for all the parameter space. This results in a final controller implementation that does not requires any interpolation step and thus, is of easy implementation. An illustrative example is given for an autonomous vehicle steering control design case.

Keywords: Stability and stabilization, LPV $\mathcal{H}_\infty$ Control, Automotive Dynamics.

1. INTRODUCTION

In this work we propose a two-step LMI optimization problem to achieve the synthesis of parametric State-Feedback (SF) and Static-Output-Feedback (SOF) controllers for discrete-time Linear Parameter Varying (LPV) systems. Such an strategy was first introduced for the synthesis of LTI controllers in (Gahinet and Apkarian, 1994). In this work we translate those results to the SF and SOF cases and with the use of Parameter Dependent Lyapunov Functions (PDLF) and gridding approaches it is here extended for discrete-time LPV systems.

The first LMI optimization step allows to prove the feasibility of the SF or SOF design, respectively. Then, the second step is an LMI optimization problem for controller reconstruction, with the controller matrices as the only LMI variables to be found. This can then be exploited to impose a parameter dependent structure directly on the controller structure without requiring a change of variables to obtain LMI conditions.

For standard LPV feedback approaches, e.g. as in Polytopic LPV: (Apkarian et al. (1995)), Polyquadratic LPV: (Daafouz and Bernussou (2001); De Caigny et al. (2010)), Grid-Based LPV: (Wu et al. (1996)), ..., the online implementation of the controller requires an interpolation of point-wise controllers. For complex systems, with a large number of varying parameters, this interpolation can be quite cumbersome as the number of point-wise controllers increases exponentially with the number of parameters. The main contribution of the approach presented in this paper comes from the ease of implementation of the LPV controller. If the proposed parametric controller exists, its implementation is then self scheduled based on the imposed parameter dependent structure, without requiring an online interpolation and independent of the number of grid points considered.

The paper is organized as follows. Section 2 presents some well-known results from the literature. Section 3 describes the proposed discrete-time LPV grid based approach. Sections 4 and 5 deal with the LMI synthesis conditions and controller reconstruction steps respectively. Section 6 presents an illustrative example of an autonomous vehicle steering control design. Finally, in Section 7, conclusions are drawn and futures perspectives are suggested.

The paper notation is the following. $\|\cdot\|_2$ represents the L_2-norm. x^T represents the transpose of x. X^{-1} represents the inverse of X, matrix $X > 0$ represents that X is symmetric positive-definite, $\mathcal{X} = ker(X)$ represents that $\mathcal{X}$ is a base of the null space of X, for simplification $\text{He}(X) = X + X^T$ and $\star$ in an LMI represents a symmetric element transposed.

2. USEFUL LITERATURE RESULT

Consider the discrete-time system:

$$M : \begin{cases} x(k+1) = \mathcal{A}x(k) + \mathcal{B}w(k) \\ z(k) = \mathcal{C}x(k) + \mathcal{D}w(k) \end{cases} \tag{1}$$

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where $x \in \mathbb{R}^{n_x}$ is the state vector, $u \in \mathbb{R}^{n_u}$ are the control inputs, $w \in \mathbb{R}^{n_w}$ is the vector of exogenous inputs with bounded energy such that $w \in L_2$ and $z \in \mathbb{R}^{n_z}$ are the output control performances. The upper bound on the induced L_2-norm, generalization of the $\mathcal{H}_\infty$ norm, of the system M can be computed making use of the discrete-time Bounded-Real-Lemma (BRL), presented in the following.

Lemma 1. (Extended Discrete-Time BRL). [De Caigny et al. (2010); Hilhorst et al. (2014)] Given a discrete-time system M and a positive scalar value γ_∞, then the following three statements are equivalent:

1.
$$\sup_{w(k)\neq 0} \frac{\|z(k)\|_2}{\|w(k)\|_2} \leq \gamma_\infty \tag{2}$$

2.
$$\begin{bmatrix} G^T + G - X(k+1) & G^T\mathcal{A}^T & G^T\mathcal{C}^T & 0 \\ \star & X(k) & 0 & \mathcal{B} \\ \star & \star & \gamma_\infty I & \mathcal{D} \\ \star & \star & \star & \gamma_\infty I \end{bmatrix} > 0 \tag{3}$$

3.
$$\begin{bmatrix} G^T + G - X(k+1) & G^T\mathcal{A} & G^T\mathcal{B} & 0 \\ \star & X(k) & 0 & \mathcal{C}^T \\ \star & \star & \gamma_\infty I & \mathcal{D}^T \\ \star & \star & \star & \gamma_\infty I \end{bmatrix} > 0 \tag{4}$$

where $X(k) \in \mathbb{R}^{n_x \times n_x}$, $X(k) > 0$, and $G \in \mathbb{R}^{n_x \times n_x}$ is a slack variable.

3. DISCRETE-TIME PARAMETER DEPENDENT LYAPUNOV FUNCTIONS

Let us consider a discrete-time LPV system

$$x(k+1) = A(\rho)x(k), \tag{5}$$

where $\rho \in \mathbb{R}^m$ is a vector of m varying parameters. Let us consider parameter dependent Lyapunov matrices with the following structure:

$$X(\rho(k)) = \sum_{n=1}^{N} \theta_n(\rho(k)) X_n, \tag{6}$$

such that $\theta_n(\rho(k)) \in \mathbb{R}$ form the basis function, X_n are constant symmetric matrices and N determines the length of the basis function. Given the associated PDLF $V(x,\rho) \in \mathbb{R}$:

$$V(x,\rho) = x(k)^T X(\rho(k)) x(k), \tag{7}$$

the stability condition according to Lyapunov theory for (5) leads to:

$$A(\rho(k))^T X(\rho(k+1)) A(\rho(k)) - X(\rho(k)) < 0 \tag{8}$$

By considering a PDLF both $\rho(k)$ and $\rho(k+1)$ are present on the LMI condition. As it is standard practice in LPV theory, the value of $\rho(k)$ is assumed to be known, however, the exact future value $\rho(k+1)$ is usually unknown. Nevertheless, if the maximum parameter variation between two sample instances $\nu \in \mathbb{R}^m$ is known, then, it is possible to define a min-max polytope box with $j = 1, \ldots, 2^m$ vertexes such that it bounds $\rho(k+1)$ as:

$$\rho(k+1) \in [\rho(k) - \nu,\ \rho(k) + \nu] \tag{9}$$

Furthermore, condition (8) imposes an infinite number of LMI conditions to be checked due to the infinite possible values of the varying parameter vector ρ. To deal with this issue, let us consider a frozen grid on the parameter space with $i = 1, \ldots, Z$ grid points, such that for each grid point i the value of the frozen varying parameter vector is defined as ρ_i.

Let us combine this gridding approach on the parameter space of $\rho \in \mathbb{R}^m$ and bounds on $\rho(k+1)$ by the known maximum parameter variation $\nu \in \mathbb{R}^m$. Then, for each of the considered frozen grid points values ρ_i, the vertexes of the polytope (9) that bounds the parameter variation are defined as ρ_i^j, $j = 1, \ldots, 2^m$. Fig. 1 shows this grid and bound approach for the case of a grid with $m = 2$ varying parameters. Note that the order of the ρ_i^j vertexes is arbitrary.

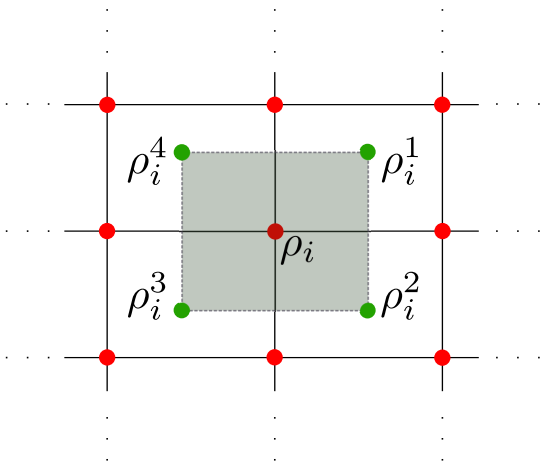

Fig. 1. Parameter variation bounds ρ_i^j for the frozen grid point ρ_i.

Using this discrete-time grid-based framework, (8) can be recast as a finite set of LMIs such that $\forall(\rho_i, \rho_i^j)$:

$$A_i^T X_i^j A_i - X_i < 0 \tag{10}$$

where for simplification of the notation $X_i = X(\rho_i)$, $X_i^j = X(\rho_i^j)$ and $A(\rho_i) = A_i$. This framework is employed in the following sections for the synthesis of discrete-time LPV controllers.

Remark 1. Note that for the case when there is no parameter variation, e.g. $\nu = 0$, then $X_i^j = X_i$, $\forall(\rho_i, \rho_i^j)$.

4. SF AND SOF CONTROLLER EXISTENCE CONDITIONS

The main contributions of this work are presented here and in the next section. First, two propositions are introduced that prove the existence of SF and SOF controllers respectively, for discrete-time LPV systems under induced L_2-norm performances.

4.1 $\mathcal{H}_\infty$ State-Feedback Controller Existence

Let us consider the following discrete-time LPV system:

$$N(\rho): \begin{cases} x(k+1) = A(\rho)x(k) + B_u(\rho)u(k) + B_w(\rho)w(k) \\ z(k) = C_z(\rho)x(k) + D_u(\rho)u(k) + D_w(\rho)w(k) \end{cases} \tag{11}$$

where $x \in \mathbb{R}^{n_x}$ is the state vector, $u \in \mathbb{R}^{n_u}$ are the control inputs, $w \in \mathbb{R}^{n_w}$ are the exogenous inputs with bounded

energy such that $w \in L_2$ and $z \in \mathbb{R}^{n_z}$ are the exogenous outputs. $\rho \in \mathbb{R}^m$ is the vector of varying parameters.

The following proposition provides sufficient conditions to prove the existence of a feasible SF controller $u(k) = K(\rho)x(k)$ such that $\frac{\|z\|_2}{\|w\|_2}$ is minimized.

Remark 2. The following proposition is actually given considering the way in which the LMI problem is formulated for computation. However, note that the proof is carried out as it should be theoretically, so for all ρ. This remark is also valid for Proposition 2.

Proposition 1. Given a positive scalar γ_∞ and a discrete-time LPV system $N(\rho)$, if there exist matrices $X(\rho) > 0$, with $X(\rho) \in \mathbb{R}^{n_x \times n_x}$, and $G \in \mathbb{R}^{n_x \times n_x}$ such that the following LMI problem is feasible $\forall(\rho_i, \rho_i^j)$:

$$X_i > 0,\ X_i^j > 0 \tag{12}$$

$$\mathcal{N}_{Mi}^T \begin{bmatrix} G^T + G - X_i^j & G^T A_i^T & G^T C_{zi}^T & 0 \\ \star & X_i & 0 & B_{wi} \\ \star & \star & \gamma_\infty I & D_{wi} \\ \star & \star & \star & \gamma_\infty I \end{bmatrix} \mathcal{N}_{Mi} > 0 \tag{13}$$

with $X_i = X(\rho_i)$ and $X_i^j = X(\rho_i^j)$ where $\rho_i, i = 1, ..., Z$ are the frozen grid points, with parameter variation at each grid point bounded as defined by (9) with vertexes $\rho_i^j, j = 1, ..., 2^m$, and $\mathcal{N}_{Mi} = ker(\begin{bmatrix} 0 & B_{ui}^T & D_{ui}^T & 0 \end{bmatrix})$.

Then, there exists a state-feedback control law $u(k) = K(\rho)x(k)$ such that $\frac{\|z\|_2}{\|w\|_2} \leq \gamma_\infty$. If the optimal γ_∞ is required, the LMI minimization problem for γ_∞ is still an LMI problem with variables γ_∞, X and G.

Proof: Considering a state-feedback control law $u(k) = K(\rho)x(k)$, system (11) can be rewritten as (1) with

$$\begin{aligned} \mathcal{A}(\rho) &= A(\rho) + B_u(\rho)K(\rho)\ ,\ \mathcal{B}(\rho) = B_w(\rho) \\ \mathcal{C}(\rho) &= C_z(\rho) + D_u(\rho)K(\rho)\ ,\ \mathcal{D}(\rho) = D_w(\rho) \end{aligned} \tag{14}$$

Substituting (14) in (3) the following condition is obtained:

$$\Psi(\rho) + \mathrm{He}\left(\begin{bmatrix} 0 \\ B_u(\rho) \\ D_u(\rho) \\ 0 \end{bmatrix} K(\rho) \begin{bmatrix} G & 0 & 0 & 0 \end{bmatrix} \right) > 0 \tag{15}$$

where

$$\Psi(\rho) = \begin{bmatrix} G^T + G - X(k+1) & G^T A(\rho)^T & G^T C_z(\rho)^T & 0 \\ \star & X(k) & 0 & B_w(\rho) \\ \star & \star & \gamma_\infty I & D_w(\rho) \\ \star & \star & \star & \gamma_\infty I \end{bmatrix} \tag{16}$$

Note that (15) is not an LMI due do the multiplication of the controller matrix $K(\rho)$ and the slack variable matrix G. However, thanks to the Projection Lemma, condition (15) is equivalent to the following conditions, which are independent of the controller matrix.

$$\mathcal{N}_M(\rho)^T \Psi(\rho) \mathcal{N}_M(\rho) > 0, \tag{17}$$

$$\mathcal{N}_N^T \Psi(\rho) \mathcal{N}_N > 0. \tag{18}$$

With $\mathcal{N}_N = ker(\begin{bmatrix} G & 0 & 0 & 0 \end{bmatrix})$, condition (18) leads to the trivial condition $X(k) > 0$ and can thus be discarded for controller design. Condition (17) is independent of the controller and is thus an LMI with variables $X(\rho)$ and G, and with $\mathcal{N}_M(\rho) = ker(\begin{bmatrix} 0 & B_u(\rho)^T & D_u(\rho)^T & 0 \end{bmatrix})$.

Condition (17) implies an infinite number of LMI to be checked due to the dependency on the varying parameter. Following the grid-based approach discussed in Sect. 3, (17) can be transformed into a finite number of LMIs to be considered where each grid point ρ_i is bounded by a polytope due to parameter variation and each vertex of this polytope is defined as ρ_i^j. Applying this discrete-time gridding approach, (17) is transformed into (13). This concludes the proof. ■

4.2 $\mathcal{H}_\infty$ Static-Output-Feedback Controller Existence

Let us consider now the following discrete-time LPV system:

$$N(\rho) : \begin{cases} x(k+1) = A(\rho)x(k) + B_u(\rho)u(k) + B_w(\rho)w(k) \\ z(k) = C_z(\rho)x(k) + D_u(\rho)u(k) + D_w(\rho)w(k) \\ y(k) = C_y(\rho)x(k) + F_w(\rho)w(k) \end{cases} \tag{19}$$

where $x \in \mathbb{R}^{n_x}$ is the state vector, $u \in \mathbb{R}^{n_u}$ are the control inputs, $w \in \mathbb{R}^{n_w}$ are the exogenous inputs with bounded energy such that $w \in L_2$, $z \in \mathbb{R}^{n_z}$ are the exogenous outputs and $y \in \mathbb{R}^{n_y}$ are the measured outputs. $\rho \in \mathbb{R}^m$ is the vector of varying parameters.

In this case, a SOF control law $u(k) = K(\rho)y(k)$ is synthesized to get induced L_2-norm performances in closed-loop, i.e. such that $\frac{\|z\|_2}{\|w\|_2}$ is minimized. The following proposition provides sufficient conditions to prove the existence of a feasible SOF controller $K(\rho)$.

Proposition 2. Given a positive scalar γ_∞ and a discrete-time LPV system $N(\rho)$, if there exist matrices $X(\rho) > 0$, with $X(\rho) \in \mathbb{R}^{n_x \times n_x}$, and $G \in \mathbb{R}^{n_x \times n_x}$ such that the following LMI problem is feasible $\forall(\rho_i, \rho_i^j)$:

$$X_i > 0,\ X_i^j > 0 \tag{20}$$

$$\mathcal{N}_{Mi}^T \begin{bmatrix} G^T + G - X_i^j & G^T A_i^T & G^T C_{zi}^T & 0 \\ \star & X_i & 0 & B_{wi} \\ \star & \star & \gamma_\infty I & D_{wi} \\ \star & \star & \star & \gamma_\infty I \end{bmatrix} \mathcal{N}_{Mi} > 0 \tag{21}$$

$$\mathcal{N}_{Ni}^T \begin{bmatrix} G^T + G - X_i^j & G^T A_i & G^T B_{wi} & 0 \\ \star & X_i & 0 & C_{zi}^T \\ \star & \star & \gamma_\infty I & D_{wi}^T \\ \star & \star & \star & \gamma_\infty I \end{bmatrix} \mathcal{N}_{Ni} > 0 \tag{22}$$

with $X_i = X(\rho_i)$ and $X_i^j = X(\rho_i^j)$ where $\rho_i, i = 1, ..., Z$ are the frozen grid points, with parameter variation at each grid point bounded as defined by (9) with vertexes $\rho_i^j, j = 1, ..., 2^m$, and $\mathcal{N}_{Mi} = ker(\begin{bmatrix} 0 & B_{ui}^T & D_{ui}^T & 0 \end{bmatrix})$, $\mathcal{N}_{Ni} = ker(\begin{bmatrix} 0 & C_{yi} & F_{wi} & 0 \end{bmatrix})$.

Then, there exists a SOF control law $u(k) = K(\rho)y(k)$ such that $\frac{\|z\|_2}{\|w\|_2} \leq \gamma_\infty$. If the optimal γ_∞ is required, the LMI minimization problem for γ_∞ is still an LMI problem with variables γ_∞, X and G.

Proof: Considering a static-output-feedback control law $u(k) = K(\rho)y(k)$, system (19) can be rewritten as (1) with

$$\begin{aligned} \mathcal{A}(\rho) &= A(\rho) + B_u(\rho)K(\rho)C_y(\rho) \\ \mathcal{B}(\rho) &= B_w(\rho) + B_u(\rho)K(\rho)F_w(\rho) \\ \mathcal{C}(\rho) &= C_z(\rho) + D_u(\rho)K(\rho)C_y(\rho) \\ \mathcal{D}(\rho) &= D_w(\rho) + D_u(\rho)K(\rho)F_w(\rho) \end{aligned} \tag{23}$$

Substituting (23) in (3) the following condition is obtained:

$$\Psi(\rho) + \mathrm{He}\left(\begin{bmatrix} 0 \\ B_u(\rho) \\ D_u(\rho) \\ 0 \end{bmatrix} K(\rho) \begin{bmatrix} C_y(\rho)G & 0 & 0 & F_w(\rho) \end{bmatrix}\right) > 0 \tag{24}$$

where

$$\Psi(\rho) = \begin{bmatrix} G^T + G - X(k+1) & G^T A(\rho)^T & G^T C_z(\rho)^T & 0 \\ \star & X(k) & 0 & B_w(\rho) \\ \star & \star & \gamma_\infty I & D_w(\rho) \\ \star & \star & \star & \gamma_\infty I \end{bmatrix} \tag{25}$$

Additionally, substituting (23) in (4) the following condition is obtained:

$$\Phi(\rho) + \mathrm{He}\left(\begin{bmatrix} G^T B_u(\rho) \\ 0 \\ 0 \\ D_u(\rho) \end{bmatrix} K(\rho) \begin{bmatrix} 0 & C_y(\rho) & F_w(\rho) & 0 \end{bmatrix}\right) > 0 \tag{26}$$

where

$$\Phi(\rho) = \begin{bmatrix} G^T + G - X(k+1) & G^T A(\rho) & G^T B_w(\rho) & 0 \\ \star & X(k) & 0 & C_z(\rho)^T \\ \star & \star & \gamma_\infty I & D_w(\rho)^T \\ \star & \star & \star & \gamma_\infty I \end{bmatrix} \tag{27}$$

Making use of the Projection Lemma, conditions (24) and (26) are true if and only if the following four conditions are also true

$$\mathcal{N}_M(\rho)^T \Psi(\rho) \mathcal{N}_M(\rho) > 0, \tag{28}$$

$$\widetilde{\mathcal{N}_N}(\rho)^T \Psi(\rho) \widetilde{\mathcal{N}_N}(\rho) > 0, \tag{29}$$

$$\widetilde{\mathcal{N}_M}(\rho)^T \Phi(\rho) \widetilde{\mathcal{N}_M}(\rho) > 0, \tag{30}$$

$$\mathcal{N}_N(\rho)^T \Phi(\rho) \mathcal{N}_N(\rho) > 0, \tag{31}$$

where

$$\mathcal{N}_M(\rho) = ker(\begin{bmatrix} 0 & B_u(\rho)^T & D_u(\rho)^T & 0 \end{bmatrix}), \tag{32}$$

$$\widetilde{\mathcal{N}_N}(\rho) = ker(\begin{bmatrix} C_y(\rho)G & 0 & 0 & F_w(\rho) \end{bmatrix}), \tag{33}$$

$$\widetilde{\mathcal{N}_M}(\rho) = ker(\begin{bmatrix} B_u^T(\rho)G & 0 & 0 & D_u^T(\rho) \end{bmatrix}), \tag{34}$$

$$\mathcal{N}_N(\rho) = ker(\begin{bmatrix} 0 & C_y(\rho) & F_w(\rho) & 0 \end{bmatrix}). \tag{35}$$

However, due to the duality property between (3) and (4), conditions (24) and (26) are likewise dual equivalent conditions. As such, (28) and (30) as well as (31) and (29) are respectively equivalent conditions. This means that the SOF controller $K(\rho)$ in (24) and (26) exists if and only if both (28) and (31) hold true, while being able to ignore inequalities (29) and (30).

As (28) and (30) are independent from the controller block, they are LMI with variables $X(\rho)$ and G, however, due to parameter dependency theses LMI implies an infinite number of LMI to be checked. As discussed in Section 3, considering a grid of frozen values ρ_i across the considered space for the varying parameter vector ρ, (28) and (30) can be transformed into a finite number of LMIs to be considered where each grid point ρ_i is bounded by a polytope due to parameter variation and each vertex of this polytope is defined as ρ_i^j. Applying this discrete-time gridding approach, (28) and (30) are transformed into (21) and (22) respectively. This concludes the proof. ■

5. CONTROLLER RECONSTRUCTION

The two propositions introduced in Sect. 4 give conditions to prove the existence of either SF or SOF controllers respectively. However, it does not provide any mean to compute the controller itself. A second step is thus required. Note that if the LMI problem presented in Proposition 1 or Proposition 2 is feasible, one will find suitable matrices $X(\rho)^*$ and G^* that prove the existence of the controller, with subscript $*$ indicating that the numerical values of these matrices are known. Knowing $X(\rho)^*$ and G^*, notice that condition (15) as well as (24) are now LMI conditions with the controller matrix $K(\rho)$ as the only unknown variable.

Exploiting this fact, the following corollaries formalize the steps to follow to achieve the controller reconstruction after successful results obtained from the application of Proposition 1 and Proposition 2, respectively.

Remark 3. In the following corollaries it is assumed the controller $K(\rho)$ has a similar parameter dependent structure as the Lyapunov matrix in (6), that is:

$$K(\rho(k)) = \sum_{n=1}^{N} \theta_n(\rho(k)) K_n \tag{36}$$

However, note that it is not required that the parameter dependent controller $K(\rho)$ and the parameter dependent Lyapunov matrix $X(\rho)$ share identical basis functions $\theta_n(\rho)$.

Corollary 1. (SF Controller Reconstruction). Given a positive scalar γ_∞ and matrices $X(\rho)^*$ and G^*, substituting (14) in (3) leads to the following LMI problem with $K(\rho)$ as the only unknown variable.

$$\Psi_i^* + \mathrm{He}\left(\begin{bmatrix} 0 \\ B_{ui} \\ D_{ui} \\ 0 \end{bmatrix} K_i \begin{bmatrix} G^* & 0 & 0 & 0 \end{bmatrix}\right) > 0, \quad \forall(\rho_i, \rho_i^j) \tag{37}$$

If this LMI problem is feasible, the solution are the matrices K_n of the SF controller $K(\rho)$.

Corollary 2. (SOF Controller Reconstruction). Given a positive scalar γ_∞ and matrices $X(\rho)^*$ and G^*, substituting (23) in (3) leads to the following LMI problem with $K(\rho)$ as the only unknown variable.

$$\Psi_i^* + \mathrm{He}\left(\begin{bmatrix} 0 \\ B_{ui} \\ D_{ui} \\ 0 \end{bmatrix} K_i \begin{bmatrix} C_{yi}G^* & 0 & 0 & F_{wi} \end{bmatrix}\right) > 0, \quad \forall(\rho_i, \rho_i^j) \tag{38}$$

If this LMI problem is feasible, the solution are the matrices K_n of the SOF controller $K(\rho)$.

6. ILLUSTRATIVE EXAMPLE

In this section, the design of SF and SOF controllers is carried out for lateral steering control of an autonomous vehicle. To show the performance of the computed controllers a high-speed path tracking simulation example is given.

The vehicle used as reference in this illustrative example is a Renault Megane full car model, and the parameters

used in this section have been introduced in (Poussot-Vassal et al. (2011); Fergani (2014)). Additionally, (Fergani (2014)) also details the nonlinear dynamics used for vehicle simulation.

6.1 LPV SF Controller Design

The control objective is to track an externally provided yaw rate reference $\dot{\psi}_{ref}$ while minimizing the control effort and attenuating the effect of noise disturbances n on the $\dot{\psi}$ measurement, the generation of said reference is outside the scope of this work. The considered general plant interconnection for the induced L_2-norm synthesis problem is shown in Fig. 2.

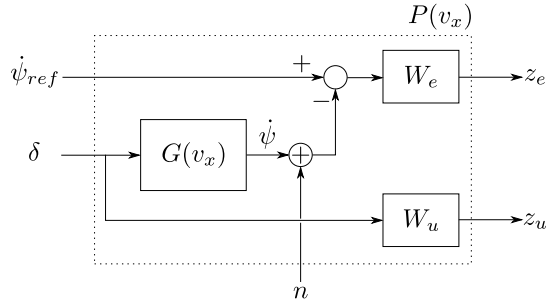

Fig. 2. LPV/$\mathcal{H}_\infty$ scheme used for control design.

The system $G(v_x)$ represents the lateral dynamics of the vehicle, given by the well-known bicycle model

$$\begin{bmatrix} \dot{y} \\ \ddot{\psi} \end{bmatrix} = \begin{bmatrix} -\frac{C_{\alpha f}+C_{\alpha r}}{mv_x} & -v_x - \frac{C_{\alpha f}l_f - C_{\alpha r}l_r}{mv_x} \\ -\frac{C_{\alpha f}l_f - C_{\alpha r}l_r}{I_z v_x} & -\frac{C_{\alpha f}l_f^2 + C_{\alpha r}l_r^2}{I_z v_x} \end{bmatrix} \begin{bmatrix} \dot{y} \\ \dot{\psi} \end{bmatrix} + \begin{bmatrix} \frac{C_{\alpha f}}{m} \\ \frac{C_{\alpha f}l_f}{I_z} \end{bmatrix} \delta \tag{39}$$

Notice that $G(v_x)$, and thus $P(v_x)$, are LPV systems with the longitudinal velocity v_x as a varying parameter.

To impose tracking performances, the weighting function W_e is given by a first-order LTI system with:

- *Inputs:* yaw rate error $e_{\dot{\psi}} \in \mathbb{R}$.
- *States:* $x_e \in \mathbb{R}$
- *Output:* tracking performance output $z_e \in \mathbb{R}$

To limit actuator bandwidth and the actuator gain, the weight W_u is given by a first-order LTI system with:

- *Inputs:* steering action δ.
- *States:* $x_u \in \mathbb{R}$
- *Output:* actuator performance output $z_u \in \mathbb{R}$

Note that $G(v_x)$ as given in (39) is a continuous-time model, in order to obtain a discrete-time grid-based representation of $P(v_x)$, multiple frozen values of v_x are considered. At each frozen grid point v_{x_i}, $G(v_{x_i})$ is then discretized with a sampling time $T_s = 0.01s$. For the discretization of $G(v_{x_i})$, the exact zero-order-hold discretization method is used, while the weighting functions are discretized employing the Tustin transform. After all systems have been discretized at each grid point, the point-wise generalized plant interconnection $P(v_{x_i})$ can be obtained.

The frozen values for the varying parameter v_x are in the range $15m/s$ to $35m/s$ and the considered bound on parameter variation $\nu = 5m/s^2$, with a coarse gridding by considering an interval between grid points of only $0.1m/s$. This amounts to a total of 201 grid points.

Finally, the considered Lyapunov matrix $X(v_x)$ in the PDLF is:

$$X(v_x) = X_1 + v_x X_2 + \frac{1}{v_x} X_3 \tag{40}$$

For the SF controller design, Proposition 1 is proven to be feasible, with an optimal upper bound on the closed-loop induced L_2-norm of the system of $\gamma_\infty = 0.667$.

The reconstruction of the SF control law, given by

$$\delta(k) = K_x(v_x)x(k) \tag{41}$$

with x being the full-state vector of the generalized plant $P(v_x)$:

$$x = \begin{bmatrix} \dot{y} & \dot{\psi} & x_e & x_u \end{bmatrix}^T, \tag{42}$$

is obtained following Corollary 1, imposing the following structure on the controller:

$$K_x(v_x) = K_{x_1} + v_x K_{x_2} + \frac{1}{v_x} K_{x_3} \tag{43}$$

Notice that despite considering 201 grid points, the second LMI optimization problem has only three decision variables, one for each of the K_{x_n} controller matrices in (43). Moreover, for implementation, with the varying parameter v_x known at each sampling instance, the controller can be computed directly according to the imposed parametrization on $K_x(v_x)$.

6.2 LPV SOF Controller Design

The control design for the SOF controller is similar to the one seen for the SF control design case, with the control performances enforced employing the generalized plant $P(v_x)$ given in Fig. 2. For the SOF controller case, the lateral velocity $\dot{y}$ is not available for feedback, so the measured outputs are given as:

$$\begin{aligned} y(k) = C_y x(k) + F_w(k) &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} w(k) \\ &= \begin{bmatrix} \dot{\psi} + n & x_e & x_u \end{bmatrix}^T \end{aligned} \tag{44}$$

Proposition 2 is proven to be feasible, with an optimal upper bound on the closed-loop induced L_2-norm of the system of $\gamma_\infty = 0.835$.

With the aim of computing a control law given by

$$\delta(k) = K_y(v_x)y(k), \tag{45}$$

the controller reconstruction is then carried according to Corollary 2, imposing the following structure on the controller:

$$K_y(v_x) = K_{y_1} + v_x K_{y_2} + \frac{1}{v_x} K_{y_3} \tag{46}$$

6.3 Time Domain Simulation

To assess the performance of the designed controllers in the time domain, a path tracking simulation is performed.

The reference path is based on the first sector from the racing circuit of Montmeló in Barcelona. A white noise have been injected in the yaw rate $\dot{\psi}$ measurement signal, simulating a sensor noise of zero mean and 1.48 $[deg]$ standard deviation. Two simulation scenarios are compared, where the only difference is the considered steering controller, referenced in the following as SF case or SOF case, respectively. In both scenarios, the vehicle shares the same longitudinal speed v_x profile, which is available for the controller at each sampling instance as the varying parameter, shown in Fig. 3.

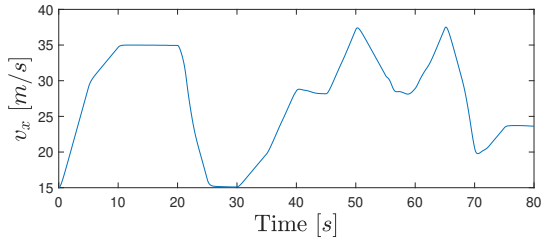

Fig. 3. Varying parameter v_x profile.

Figure 4 shows the desired reference trajectory (in blue) and for the SF (in red) and SOF (in yellow) cases the actual achieved trajectory. From these results, we can see that in terms of trajectory tracking both controllers perform fairly similar. As observed, the vehicle manages to achieve an almost perfect tracking of the reference track in both scenarios.

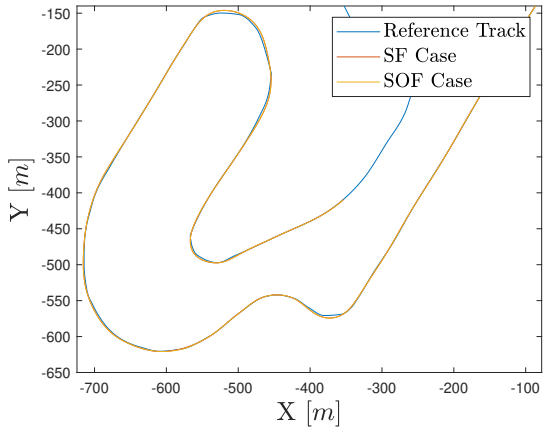

Fig. 4. Reference and followed paths by the vehicle in SF and SOF scenarios.

Fig. 5 presents the controller command δ for the SF (top) and for the SOF (bottom) cases, respectively. In both cases, the controller steering action is very similar for the most part. The most noticeable difference is that the SOF controller ensures a reduction of the effect of sensor noises on the controller command. This improvement can be attributed to the fact that the SOF control synthesis method based on Proposition 2 allows to account for disturbances in the feedback signal $y(k)$. This translates to an improvement of the controller in practice, even though the theoretical induced L_2-norm performance is degraded with respect to the state-feedback case.

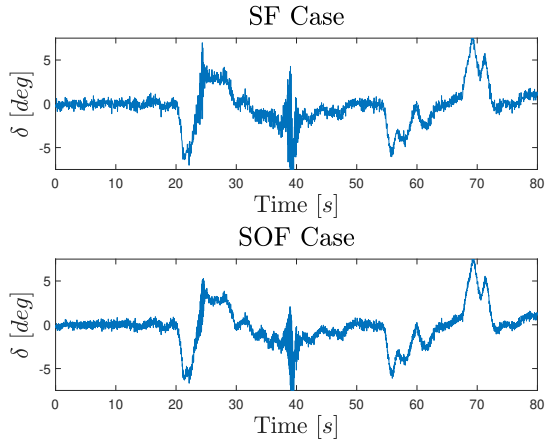

Fig. 5. Controller command δ fot the SF and SOF case.

7. CONCLUSION

In this work, a synthesis method has been proposed for the computation of parametric State-Feedback and Static-Output-Feedback controllers for discrete-time Linear Parameter Varying systems under induced L_2 performances. Arguably this method could be more conservative than state of the art LPV synthesis results based on point-wise controller variables. However, the major benefit of this approach is the ease of implementation effort for the obtained parametric controller as the real-time applied control law does not requires any interpolation step. This was showcased on an illustrative example for a lateral steering control problem based on a realistic vehicle simulation. Despite the fact that the presented design examples were based on coarse gridding approach, with 201 frozen grid values on the considered varying parameter range, the controller implementation still remained simple, while at the same time both parametric SF and SOF controllers achieved great performance.

In future works it is expected to validate the proposed parametric control synthesis method in a real platform and offer a comparison with recent LPV controllers synthesis results, which require for real-time implementation the interpolation of point-wise controllers. Additionally it should be studied the effect of the number of considered frozen grid-points on the obtained controller. Whereas an interesting extension is the design of Dynamic-Output-Feedback controllers.

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