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# Exponential stability of periodic difference delay systems and 1-D hyperbolic PDEs of conservation laws

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**Abstract:** This paper applies a recent characterization, due to the authors, of exponential stability for linear periodic difference delay systems to the stability of one-dimensional hyperbolic systems of PDE, and elaborates on the case where the delays are rationally independent.

*Keywords:* linear periodic systems, difference delay systems, exponential stability, hyperbolic system, Henry-Hale theorem, rationally independent delays.

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## 1. INTRODUCTION

We deal with the stability of *time-periodic linear difference-delay systems*; i.e., dynamical systems of the form

$$y(t) = \sum_{j=1}^N D_j(t) y(t - \tau_j) \quad \text{for } t \geq s, \quad (1)$$

where  $s \in \mathbb{R}$  is the initial time,  $d$  and  $N$  are positive integers,  $D_j : \mathbb{R} \rightarrow \mathbb{C}^{d \times d}$  is a continuous  $T$ -periodic matrix-valued function:

$$D_j(t + T) = D_j(t), \quad 1 \leq j \leq N. \quad (2)$$

Without loss of generality, we order the delays and assume that  $T$  is strictly larger than all of them:

$$0 < \tau_1 < \dots < \tau_N < T \quad (3)$$

(otherwise we may replace  $T$  by  $kT$ , with  $k > 0$  the smallest integer such that  $kT > \tau_N$ ).

**Plan of the paper.** The present introduction reviews motivation from engineering for the study of periodic linear difference delay systems; it also introduces definitions and basic facts about System (1), as well as well-known results about time-invariant and periodic linear delay system, along with a recent necessary and sufficient condition for exponential stability obtained by the authors in Baratchart et al. (2022). The next three sections contain the contributions of the paper, all based on that result. Section 2 contains an analog of the Hale-Silkowski Theorem in the periodic case (see Theorem 3 below), when the delays are rationally independent. This yields a simple sufficient and necessary stability condition in that specific case. Section 3 gives a sufficient stability condition in terms of the norms of the matrices  $(D_j(\cdot))_{j \in \{1, \dots, N\}}$ , in the special case where the columns of System (1) are *disjoint*; see definition at the beginning of that section. Since delay systems are often used to model transport equations, Section 4 applies what precedes, in particular

Section 3, to stability criteria for this type of equations. Section 5 concludes with some open questions.

### 1.1 Motivation from electronics

Delay systems arise naturally in electronic engineering; see the pioneering works of Brayton (1967), for instance. In particular, System (1) recurs in that field because it models the behaviour at very high frequency of electronic components. Indeed, circuits contain linear, passive components as well as nonlinear, active ones, all of which can be described using finitely many state variables; but they also contain transmission lines, typically modelled by simple hyperbolic Partial Differential Equations (PDE) like the lossless telegrapher equations, that make the state space of the circuit infinite-dimensional. Integrating the telegrapher equations, one obtains a model comprised of delay-difference as well as differential equations. Using first order approximation, issues of stability for the circuit around a periodic trajectory reduce to investigating the exponential stability of the time-periodic linear system obtained by linearising around the periodic solution. This yields a network of delay-difference equations with boundary conditions coupled by differential equations, the stability of which is intimately connected with the stability of a periodic linear difference delay system of the form (1) *via* a compact perturbation argument; cf. Fueyo (2020); Hale and Verduyn Lunel (1993).

Due to the huge number of components in the circuit, it is known from electronic engineering textbooks that stability of the above-mentioned linearised system cannot be tested efficiently in the time domain. Instead, one considers a family of input-output systems, obtained by perturbing the linearised system by a small current  $i$  at some node of the circuit and observing the resulting perturbation of the voltage  $v$  between two nodes. Using Fourier expansions, stability is studied through the sin-

gularities of the harmonic transfer function (HTF) which is a doubly-infinite matrix (no initial nor final index for rows and columns) depending on a complex variable (see Suárez (2009); Möllerstedt (2000); Louarroudi (2014))). It is this electronic context that motivated our study; in fact, (Baratchart et al., 2022, Thm. 3) on which we base our analysis expresses that the HTF, when viewed as an operator-valued function, is holomorphic in some right half-plane with strictly negative abscissa. It has to be pointed out that rational approximation techniques used in electronic engineering, see for instance Jugo et al. (2001); Cooman et al. (2018), might be adapted to investigate numerically the stability of this kind of systems.

## 1.2 Preliminaries

A *solution* to the difference delay system under examination is a map  $y(\cdot) : [s - \tau_N, +\infty) \rightarrow \mathbb{C}^d$  that satisfies (1). It is uniquely determined by its restriction to  $[s - \tau_N, s]$ ; *i.e.* by requiring that  $y(s + \theta) = \phi(\theta)$  for  $-\tau_N \leq \theta \leq 0$ , where the initial data  $\phi : [s - \tau_N, s] \rightarrow \mathbb{C}^d$  may belong to various functional spaces depending on the context. For example, if we put  $C^0([s, +\infty), \mathbb{C}^d)$  for the space of continuous functions defined on  $[s, +\infty)$  with values in  $\mathbb{C}^d$ , a necessary and sufficient condition for a continuous map  $\phi : [s - \tau_N, s] \rightarrow \mathbb{C}^d$  to be the initial condition of a continuous solution to (1) is that  $\phi \in C_s$  where

$$C_s := \{\phi \in C^0([-\tau_N, 0], \mathbb{C}^d) \mid \phi(0) = \sum_{j=1}^N D_j(s) \phi(-\tau_j)\} \quad (4)$$

Since the system is  $T$ -periodic, we have that  $C_{s+T} = C_s$ . For  $\phi \in C_s$ , an easy recursion shows that system (1) has a unique continuous solution  $y$  with initial data  $\phi$ . Thus, we may define, for all  $t, s, t \geq s$ , the solution operator:

$$U(t, s) : C_s \rightarrow C_t \\ \phi \mapsto U(t, s)\phi, \text{ the map } \theta \mapsto y(t + \theta). \quad (5)$$

For  $x \in \mathbb{C}^d$ , let  $\|x\|$  denote its Euclidean norm. We endow  $C_s$  with the familiar  $C^0$  (supremum) norm:

$$\|\phi\|_{C^0} = \sup_{\theta \in [-\tau_N, 0]} \|\phi(\theta)\|. \quad (6)$$

We may now formally define exponential stability:

*Definition 1.* System (1) is called  $C^0$ -exponentially stable if there exist  $\gamma, K > 0$  such that :

$$\|U(t, s)\phi\|_{C^0} \leq K e^{-\gamma(t-s)} \|\phi\|_{C^0}, \quad (7)$$

for all  $s \in \mathbb{R}$ , all  $t \geq s$  and all  $\phi \in C_s$ .

## 1.3 Stability results for time-invariant systems

We assume in this section that the maps  $t \mapsto D_j(t)$  are constant; *i.e.*,  $D_j(\cdot) \equiv D_j$  for  $j = 1, \dots, N$ , where  $D_j \in \mathbb{C}^{d \times d}$ . In this case, System (1) reduces to

$$y(t) = \sum_{j=1}^N D_j y(t - \tau_j) \quad \text{for all } t \geq s. \quad (8)$$

There are classically two ways of studying the exponential stability of such a system: one proceeds in the frequency domain *via* Laplace transforms or one devises Lyapunov-krasovskii functionals (see Gomez et al. (2019); Răsvan

and Niculescu (2002)). We focus here on the first one. Assumption (2) is true for any  $T > 0$  while all considerations and definitions above apply. The sets  $C_s$  do not depend on  $s$ , the linear operator  $U(t, s)$  depends only on  $t - s$ , and the system clearly defines a semigroup though we do not elaborate on this. Let us simply recall the following characterisation of exponential stability, first given in Cruz and Hale (1970) when there are finitely many commensurate delays, then generalised by Henry to the case of infinitely many, not necessarily commensurate delays Henry (1974), and subsequently carried over by Hale to distributed delay systems, see (Hale and Verduyn Lunel, 1993, Ch. 9).

*Theorem 2.* (Henry-Hale). Assume that the matrices  $D_j$  are constant,  $1 \leq j \leq N$ . A necessary and sufficient condition for System (8) to be  $C^0$ -exponentially stable is the existence of a real number  $\beta < 0$  such that, for all  $p$  in  $\{z \in \mathbb{C} \mid \Re(z) \geq \beta\}$ , one has:

$$I_d - \sum_{j=1}^N e^{-p\tau_j} D_j \text{ is invertible in } \mathbb{C}^{d \times d}. \quad (9)$$

Another classical result, stated below as Theorem 3 and due to Avellar and Hale (1980) (though originally introduced in the PhD dissertation of Silkowski (1976)), deals with the stability of System (8) when the delays are *rationally independent*; that is, if none of them can be written as a linear combination of the others with rational coefficients. In that case, there is a sufficient and necessary criterion for stability, often called the Hale-Silkowski criterion (cf Chitour et al. (2016)). It is simpler to check because it depends on parameters that vary in a compact set, whereas the condition given in the Henry-Hale theorem must be satisfied in a right half plane. Moreover, when the Hale-Silkowski criterion is verified then exponential stability holds whatever the delays (rationally independent or not). The two main arguments in the proof, which are not given here, are the Henry-Hale theorem and Kronecker's approximation theorem.

*Theorem 3.* (Hale-Silkowski). If the delays are rationally independent then System (8) is exponentially stable if and only if, for all  $\theta_j \in [0, 2\pi]$  and  $j = 1, \dots, N$ , the following operator has spectrum compactly included in the open unit disk :

$$\sum_{j=1}^N e^{i\theta_j} D_j. \quad (10)$$

Furthermore, when this condition is satisfied, exponential stability of System (8) holds for any set of positive delays.

The first part of Theorem 3 follows from the implication (i)  $\Rightarrow$  (ii) in (Hale and Verduyn Lunel, 1993, Theorem 6.1, chapter 9). The second part is item (iii) of the same reference.

## 1.4 Stability in the periodic case

Generally speaking, the stability of periodic systems is studied through the monodromy operator; *i.e.*, the operator  $U(T, 0)$ , with  $U$  defined in (5). In fact, System (1) is  $C^0$ -exponentially stable if and only if the spectrum of the monodromy operator is strictly included in the unit

disk. Because the spectrum of  $U(T, 0)$  needs not consist of eigenvalues, the study of stability becomes complicated, even though explicit representations of the monodromy operator can be obtained in special cases when the period and the delays of the system are rationally dependent. Recently, a generalisation of the Henry-Hale theorem to the periodic case was proven in Baratchart et al. (2022), when the maps  $t \mapsto D_j(t)$  are  $T$ -periodic with Hölder-continuous derivative. In particular, this generalisation requires no assumption on the ratio between the delays and the period. As an extra piece of notation, let us put  $\check{D}_j(k)$ , for  $k \in \mathbb{Z}$ , to denote the  $k$ -th Fourier coefficient of  $D_j$ :

$$\check{D}_j(k) := \frac{1}{T} \int_0^T D_j(t) e^{-\frac{i2\pi kt}{T}} dt. \quad (11)$$

We let  $L_{D_j}$  be the (doubly infinite) block Laurent matrix associated with  $D_j(t)$ , whose entries are

$$(L_{D_j})_{k,n} := \check{D}_j(n - k), \quad n, k \in \mathbb{Z}. \quad (12)$$

Since each  $D_j(t)$  is a bounded function, such Laurent matrices act on the Hilbert space :

$$l_d^2(\mathbb{Z}) := \{z = (z_j)_{j \in \mathbb{Z}} \mid z_j \in \mathbb{C}^d, \sum_{j=-\infty}^{+\infty} \|z_j\|^2 < +\infty\}, \quad (13)$$

equipped with the norm  $\|\cdot\|_2$  given by

$$\|z\|_2 := \left( \sum_{j=-\infty}^{+\infty} \|z_j\|^2 \right)^{1/2}. \quad (14)$$

Note that  $\|L_{D_j} z\|_2 = \|D_j Z\|_{L^2(\mathbb{T})}$  where  $\mathbb{T} = \mathbb{R}/T\mathbb{Z}$  is the circle, while  $Z \in L^2(\mathbb{T})$  is the function whose Fourier coefficients are the  $z_k$ .

For  $A : l_d^2(\mathbb{Z}) \rightarrow l_d^2(\mathbb{Z})$  a linear operator, we denote by  $\|A\|_2$  its operator norm:

$$\|A\|_2 := \sup_{z \neq 0} \frac{\|Az\|_2}{\|z\|_2}. \quad (15)$$

Irrespectively of  $m \in \mathbb{N}$ , we put  $I_\infty$  for the identity operator on  $l_m^2(\mathbb{Z})$  and, for  $1 \leq j \leq N$ , we define the (doubly infinite) block diagonal matrix  $\tilde{D}_{\tau_j}$  by

$$\tilde{D}_{\tau_j} := \text{diag}\{\dots, e^{-4i\pi\tau_j/T} I_d, e^{-2i\pi\tau_j/T} I_d, I_d, e^{+2i\pi\tau_j/T} I_d, e^{+4i\pi\tau_j/T} I_d, \dots\}, \quad (16)$$

where  $I_d$  is the identity matrix of size  $d \times d$ . We then define a function  $R$  of the complex variable  $p$  with values in the bounded operators  $l_d^2(\mathbb{Z}) \rightarrow l_d^2(\mathbb{Z})$  (identified with their doubly infinite matrix in the Fourier basis):

$$R(p) := I_\infty - \sum_{j=1}^N e^{-p\tau_j} L_{D_j} \tilde{D}_{\tau_j}. \quad (17)$$

*Theorem 4.* (Baratchart et al. (2022), Theorem 3).

Assume that the  $D_j : \mathbb{R} \rightarrow \mathbb{C}^{d \times d}$  are periodic and differentiable with Hölder continuous derivative for  $1 \leq j \leq N$ . A necessary and sufficient condition for System (1) to be exponentially stable is the existence of a real number  $\beta < 0$  such that :

- (i)  $R(p)$  is invertible  $l_d^2(\mathbb{Z}) \rightarrow l_d^2(\mathbb{Z})$  for all  $p$  in  $\{z \in \mathbb{C} \mid \Re(z) \geq \beta\}$ ,
- (ii) there is a positive number  $M$  such that  $\|R(p)^{-1}\|_2 \leq M$  for all  $p$  in  $\{z \in \mathbb{C} \mid \Re(z) \geq \beta\}$ .

Properties of almost periodic function easily imply that Theorem 4 is indeed a generalisation of the Henry-Hale Theorem 2, meaning that in the time-invariant case Theorem 4 reduces to Theorem 2.

We surmise that condition (ii) cannot be dispensed with. As the results of the paper are based on Theorem 4, we assume throughout that *the matrices  $D_j(\cdot)$  are differentiable with Hölder continuous derivative*

## 2. RATIONALLY INDEPENDENT DELAYS

As shown in Theorem 6, a simplification of Theorem 4 analogous to the Hale-Silkowski theorem of the time-invariant case can be produced when the delays are rationally independent.

Let us first recall some classical definition and properties. Let  $\mathcal{A} : \mathcal{E} \rightarrow \mathcal{E}$  a linear operator, where  $\mathcal{E}$  is a Banach endowed with a norm  $\|\cdot\|_{\mathcal{E}}$ . The operator  $\mathcal{A}$  is said to be bounded below if there exists  $c > 0$  such that:

$$\|\mathcal{A}x\|_{\mathcal{E}} \geq c\|x\|_{\mathcal{E}}. \quad (18)$$

Equivalently,  $\mathcal{A}$  is not bounded below if and only if there exists a sequence  $(x_k)_{k \in \mathbb{N}} \in \mathcal{E}^{\mathbb{N}}$  with  $\|x_k\|_{\mathcal{E}} = 1$  such that:

$$\mathcal{A}x_k \xrightarrow[k \rightarrow +\infty]{} 0. \quad (19)$$

We are now in position to prove a generalisation of the Hale-Silkowski theorem to the periodic case. For this, we introduce the following map:

$$R_{\text{HS}}(\rho, \theta_1, \dots, \theta_N) = I_\infty - \sum_{j=1}^N e^{-\rho\tau_j} L_{D_j} \tilde{D}_{\tau_j} e^{i\theta_j}, \quad (20)$$

for  $\rho \in \mathbb{R}$ ,  $\theta_j \in [0, 2\pi]$  and  $j = 1, \dots, N$ . We need a lemma.

*Lemma 5.* When the delays are rationally independent, assertions (A) and (B) below are equivalent:

- (A) there exists a real number  $\beta < 0$  such that :
  - (i)  $R(p)$  is invertible  $l_d^2(\mathbb{Z}) \rightarrow l_d^2(\mathbb{Z})$  for all  $p$  in  $\{z \in \mathbb{C} \mid \Re(z) \geq \beta\}$ ,
  - (ii) there exists a real number  $M$  such that  $\|R(p)^{-1}\|_2 \leq M$  for all  $p$  in  $\{z \in \mathbb{C} \mid \Re(z) \geq \beta\}$ .
- (B)  $R_{\text{HS}}(\rho, \theta_1, \dots, \theta_N)$  is invertible  $l_d^2(\mathbb{Z}) \rightarrow l_d^2(\mathbb{Z})$  for  $\rho \geq 0$  and  $\theta_j \in [0, 2\pi]$ ,  $j = 1, \dots, N$ .

**Proof.** Note that (ii) in condition (A) amounts to require that  $R(p)$  is bounded below on the set  $\{p \in \mathbb{C} \mid \Re(p) \geq \beta\}$ , uniformly with respect to  $p$ ; i.e., there is  $\tilde{M} > 0$  such that:

$$\tilde{M}\|x\|_2 \leq \|R(p)x\|_2, \quad x \in l_d^2(\mathbb{Z}). \quad (21)$$

If (A) does not hold, there is a sequence  $(p_n) \in \mathbb{C}^{\mathbb{N}}$  with  $\lim_{n \rightarrow +\infty} \Re(p_n) = 0$  or  $\Re(p_n) \geq 0$  for all  $n \in \mathbb{N}$  verifying one of the two assertions:

- (a)  $R(p_n)$  is not invertible for all  $n \in \mathbb{N}$ .

(b)  $R(p_n)$  is invertible for all  $n$  and  $R(p_n)x_n \xrightarrow{n \rightarrow +\infty} 0$  for some sequence  $(x_n) \in (\mathbb{C}^d)^\mathbb{N}$  with  $\|x_n\| = 1$ .

Since  $R(p)$  is invertible for  $\Re(p)$  large enough by a Neumann series argument, the sequence  $(p_n)_{n \in \mathbb{N}}$  has bounded real part. So, by compactness, there exists a subsequence (still denoted with  $(p_n)$ ),  $\rho \geq 0$ , and  $\theta_j \in [0, 2\pi]$  for  $j = 1, \dots, N$  such that

$$\lim_{n \rightarrow +\infty} R(p_n) = R_{\text{HS}}(\rho, \theta_1, \dots, \theta_N), \quad (22)$$

where the convergence is understood in operator norm and the  $\theta_j$  are suitable accumulation points of the  $-\Im(p_n \tau_j)$  modulo  $2\pi$ . Using that invertible operators form an open set (in case (a)) or picking a limit point  $0 \neq x \in \mathbb{C}^d$  and observing that  $R_{\text{HS}}(\rho, \theta_1, \dots, \theta_N)x = 0$  (in case (b)), we deduce from (22) that condition (B) cannot hold.

Conversely, if (B) is not verified then there exist  $\rho \geq 0$  and  $\theta_j \in [0, 2\pi]$ ,  $j = 1, \dots, N$ , such that  $R_{\text{HS}}(\rho, \theta_1, \dots, \theta_N)$  is not invertible. By Kronecker's approximation theorem, there is a real sequence  $(\mu^{\{n\}})$  comprised of integral multiples of  $2\pi$  such that

$$e^{i(\theta_j - \mu^{\{n\}} \tau_j)} \xrightarrow{n \rightarrow +\infty} 1, \quad j = 1, \dots, N,$$

ensuing that

$$R(\rho + i\mu^{\{n\}}) \xrightarrow{n \rightarrow +\infty} R_{\text{HS}}(\rho, \theta_1, \dots, \theta_N). \quad (23)$$

Now, if (A) did hold,  $R(\rho + i\mu^{\{n\}})$  would be invertible with  $\|R^{-1}(\rho + i\mu^{\{n\}})\|_2 \leq M$  for all  $n \in \mathbb{N}$ , because  $\rho \geq 0$ . For  $\phi : l_d^2(\mathbb{Z}) \rightarrow l_d^2(\mathbb{Z})$  a bounded operator and  $p \in \mathbb{C}$  such that  $R^{-1}(p)$  exists, we have:

$$\phi = R^{-1}(p) (I_d - (I_d - R(p)\phi)). \quad (24)$$

Hence, by a Neumann series argument, the ball of center  $R^{-1}(\rho + i\mu^{\{n\}})$  and radius  $1/M$  in the space of bounded operators on  $l_d^2(\mathbb{Z})$  would consist entirely of invertible elements, contradicting (23). This proves the lemma.

We are now in position to state a generalisation of the Hale-Silkowski Theorem in the periodic case:

**Theorem 6.** If the delays are rationally independent, then a necessary and sufficient condition for System (1) to be exponentially stable is:

$$(1) \ I_\infty - \sum_{j=1}^N e^{-\rho \tau_j} L_{D_j} \tilde{D}_{\tau_j} e^{i\theta_j} \text{ is invertible } l_d^2(\mathbb{Z}) \rightarrow l_d^2(\mathbb{Z})$$

for  $\rho \geq 0$  and  $\theta_j \in [0, 2\pi]$  for  $j = 1, \dots, N$ .

**Proof.** This follows from Lemma 5 and Theorem 4.

Theorem 6 allows us to relax condition (ii) in Theorem 4, but we still incur a dependence on the real number  $\rho$ . In the Hale-Silkowski theorem for the time-invariant case, we can further eliminate  $\rho$  because, when the delays are rationally independent, the system is exponentially stable for all delays if it is for some. However, we surmise this is not true for periodic difference delay system, and believe that Theorem 6 cannot reduce to the classical Hale-Silkowski Theorem 3. Still, Theorem 6 gives a criterion for exponential stability of System (1) that needs only be verified on a compact set, thanks to a Neumann series argument.

### 3. THE CASE OF DISJOINT COLUMNS

In a special case, one can give a sufficient condition for exponential stability of System (1) in terms of the  $D_j(\cdot)$  themselves, and not the Laurent matrices associated to the Fourier series of the  $D_j(\cdot)$  as is the case in Theorem 4. Let us say that *the columns of system (1) are disjoint* if whenever  $k \in \{1, \dots, d\}$  is such that the  $k^{\text{th}}$  column of  $D_j(t)$  is nonzero, then the  $k^{\text{th}}$  column of  $D_i(t)$  is zero for every  $i \neq j$ . For  $A$  a  $d \times d$  matrix, we denote by  $\|A\|$  the operator norm induced by the norm  $\|\cdot\|$  on  $\mathbb{C}^d$ . The disjoint column property is restrictive, and few difference delay systems will enjoy it. However, this is the case for difference delay systems coming from 1-D hyperbolic difference delay equations; see Section 4.

**Proposition 7.** If the column of the periodic delay system (1) are disjoint and there exists an invertible constant diagonal matrix  $D$  such that

$$\left\| D \sum_{i=1}^N D_i(t) D^{-1} \right\| < 1 \quad (25)$$

for all  $t \in \mathbb{R}$ , then System (1) is exponentially stable.

**Proof.** Writing  $L_D$  for the Laurent matrix of  $D$ , we get from properties of Laurent's matrices (see Böttcher and Silbermann (1999)) and since the column are disjoint that

$$\begin{aligned} & \left\| L_D \sum_{i=1}^N e^{-p\tau_i} L_{D_i} \tilde{D}_i L_D^{-1} \right\|_2 \\ & \leq \sup_{t \in [0, T]} \left\| D \sum_{i=1}^N e^{-p\tau_i} D_i(t) D^{-1} \right\| < 1. \end{aligned} \quad (26)$$

for some  $\gamma < 0$  and  $p \in \{\tilde{p} \in \mathbb{C} | \Re(\tilde{p}) \geq \gamma\}$ . Hence,  $I_\infty - \sum_{j=1}^N e^{-p\tau_j} L_{D_j} \tilde{D}_{\tau_j}$  is invertible in  $l_d^2(\mathbb{Z})$  to  $l_d^2(\mathbb{Z})$  and uniformly bounded for such  $p$ , by a Neumann series argument. We now get the result from Theorem 4.  $\square$

### 4. 1-D HYPERBOLIC SYSTEMS

Classically, delay systems are instrumental to give stability conditions for 1-D hyperbolic PDEs, see for instance the pioneering works Brayton (1967); Abolinia and Myshkis (1960); Cooke and Krumme (1968) (see also Răsvan (2014) for a more recent study) considering difference-differential equations coming from such systems. The use of difference delay systems arising from 1-D hyperbolic PDEs with conservation laws is more recent (see Coron and Nguyen (2015); Chitour et al. (2016); Baratchart et al. (2021)). Below we prove that such systems are equivalent to a periodic difference delay system with the disjoint column property. We recast Theorem 4 and Theorem 6 in this framework, and applying Proposition 7 will allow us to recover some known stability results regarding linear time-varying 1-D hyperbolic PDE (see Baratchart et al. (2021)). More precisely, following the notation of (Bastin and Coron, 2016, Chapter 3), we consider the hyperbolic system

$$\partial_t R(t, x) + \Lambda \partial_x R(t, x) = 0, \quad (t, x) \in \Omega. \quad (27)$$

The diagonal matrix  $\Lambda$  is defined as

$$\Lambda := \begin{pmatrix} \Lambda^+ & 0 \\ 0 & -\Lambda^- \end{pmatrix} \quad (28)$$

with  $\Lambda^+ = \text{diag}\{\lambda_1, \dots, \lambda_m\}$ ,  $\Lambda^- = \text{diag}\{\lambda_{m+1}, \dots, \lambda_n\}$  and  $\lambda_i > 0$ ,  $i = 1, \dots, n$ . Let us put

$$\Omega = \{(t, x) \in \mathbb{R}^2, 0 < x < 1 \text{ and } 0 < t < +\infty\}. \quad (29)$$

and write  $R := \begin{pmatrix} R^+ \\ R^- \end{pmatrix}$ , where

$$R^+ = \begin{pmatrix} R_1 \\ \vdots \\ R_m \end{pmatrix} \text{ and } R^- = \begin{pmatrix} R_{m+1} \\ \vdots \\ R_n \end{pmatrix}. \quad (30)$$

The (periodic time-varying) boundary conditions are :

$$\begin{pmatrix} R^+(t, 0) \\ R^-(t, 1) \end{pmatrix} = K(t) \begin{pmatrix} R^+(t, 1) \\ R^-(t, 0) \end{pmatrix}, \quad (31)$$

where  $K : \mathbb{R} \rightarrow \mathbb{C}^{n \times n}$  is assumed *differentiable with Hölder continuous derivative* and  $T$ -periodic for  $T$  a strictly positive real. First, let us make precise the meaning of (27) and (31). Indeed, (27) *a priori* requires some differentiability, while (31) requires that  $R^+$  and  $R^-$  extend to functions on the boundary of  $\Omega$  (this boundary is  $\{0\} \times [0, 1] \cup [0, +\infty) \times \{0\} \cup [0, +\infty) \times \{1\}$ ). We denote by  $\bar{\Omega}$  the union of  $\Omega$  and its boundary.

Concerning (27), it is understood in the sense of distributions; *i.e.*  $R(t, x) \in C^0(\Omega, \mathbb{R})$ , is said to be a solution of (27) if one has

$$\iint_{\Omega} (\partial_t \varphi^* + \partial_x \varphi^* \Lambda) R dt dx = 0 \quad (32)$$

for any smooth functions  $\varphi : \Omega \rightarrow \mathbb{R}^n$  with compact support contained in  $\Omega$ . By the characteristic method, it is well known that if  $R$  is a continuous function meeting equation (32), then there exist  $n$  functions  $x_1, \dots, x_n$  such that

$$R^+(t, x) = \begin{pmatrix} x_1(x - \lambda_1 t) \\ \vdots \\ x_m(x - \lambda_m t) \end{pmatrix} \quad (33)$$

and

$$R^-(t, x) = \begin{pmatrix} x_{m+1}(x + \lambda_{m+1} t) \\ \vdots \\ x_n(x + \lambda_n t) \end{pmatrix}, \quad (34)$$

where  $x_k$  is continuous in  $(-\infty, 1)$  for  $k = 1, \dots, m$  and  $x_k$  is continuous in  $(0, +\infty)$  for  $k = m+1, \dots, n$ . Considering their continuation in 1 and 0 respectively and letting  $y_k(t) = x_k(-\lambda_k t)$  for  $k = 1, \dots, m$ , as well as  $y_k(t) = x_k(1 + \lambda_k t)$  for  $k = m+1, \dots, n$  and  $\tau_i = \frac{1}{\lambda_i}$  for  $i = 1, \dots, n$ , equation (31) becomes:

$$\begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix} = K(t) \begin{pmatrix} y_1(t - \tau_1) \\ \vdots \\ y_n(t - \tau_n) \end{pmatrix}, \quad (35)$$

which is a periodic delay system of the form (1), having the nice property of being column disjoint as defined in

Section 3. We assume without loss of generality that the period  $T$  of the matrix  $K(\cdot)$  is greater than the largest delay  $\tau_{\max} = \sup_{i \in [1, n]} (\tau_i)$ .

**Theorem 8.** There is a unique map  $(t, x) \mapsto R(t, x)$ , continuous  $\bar{\Omega} \rightarrow \mathbb{R}^n$ , which is a solution of (31)-(32) satisfying  $R^+(0, x) = f_0(x)$  and  $R^-(0, x) = g_0(x)$  for  $0 \leq x \leq 1$ , where  $f_0 \in (C^0([0, 1]))^m$  and  $g_0 \in (C^0([0, 1]))^{n-m}$  are such that

$$\begin{pmatrix} f_0(0) \\ g_0(1) \end{pmatrix} = K(0) \begin{pmatrix} f_0(1) \\ g_0(0) \end{pmatrix}. \quad (36)$$

**Proof.** System (31)-(32) leads to the difference delay system (35). Since the latter admits a unique continuous solution with initial conditions from  $C_s$ , we get the result *via* equations (33)-(34).  $\square$

**Definition 9.** System (31)-(32) is said to be exponentially stable if there exists  $C > 0$  and  $\gamma > 0$  such that

$$\left\| \begin{pmatrix} R^+(t, \cdot) \\ R^-(t, \cdot) \end{pmatrix} \right\|_{\infty} \leq C \left\| \begin{pmatrix} R^+(0, \cdot) \\ R^-(0, \cdot) \end{pmatrix} \right\|_{\infty} e^{-\gamma t}, \quad (37)$$

where:

$$\left\| \begin{pmatrix} R^+(t, \cdot) \\ R^-(t, \cdot) \end{pmatrix} \right\|_{\infty} = \sup_{x \in \Omega} (\|R^+(t, x)\| + \|R^-(t, x)\|), \quad (38)$$

for all  $t \geq 0$ .

For  $k = 1, \dots, n$ , we denote by  $K_k(\cdot)$  the  $n \times n$  matrix having the same  $k$ -th column as  $K(\cdot)$  while other columns are zero. For  $p \in \mathbb{C}$  let us introduce the operator  $R_{\text{hyp}}(p) : l_n^2(\mathbb{Z}) \rightarrow l_n^2(\mathbb{Z})$  as

$$R_{\text{hyp}}(p) := I_{\infty} - \sum_{j=1}^n e^{-p \tau_j} L_{k_j} \tilde{D}_{\tau_j}. \quad (39)$$

Theorem 4 leads to the following necessary and sufficient condition for exponential stability of 1-D hyperbolic PDE with periodic boundary conditions.

**Theorem 10.** A necessary and sufficient condition for System (31)-(32) to be exponentially stable is the existence of a real number  $\beta < 0$  such that :

- (i)  $R_{\text{hyp}}(p)$  is invertible  $l_n^2(\mathbb{Z}) \rightarrow l_n^2(\mathbb{Z})$  for all  $p$  in  $\{z \in \mathbb{C} | \Re(z) \geq \beta\}$ ,
- (ii) there is a positive number  $M$  such that  $\|R_{\text{hyp}}(p)^{-1}\|_2 \leq M$  for all  $p$  in  $\{z \in \mathbb{C} | \Re(z) \geq \beta\}$ .

**Proof.** A straightforward application of Theorem 4 gives that conditions (i)-(ii) amount to exponential stability of the system (35). Then, we obtain the exponential stability of System (31)-(32) via the equations (33)-(34).  $\square$

**Theorem 11.** If the delays are rationally independent, then a necessary and sufficient condition for System (31)-(32) to be exponentially stable is that

- (1)  $I_{\infty} - \sum_{j=1}^N e^{-\rho \tau_j} L_{D_j} \tilde{D}_{\tau_j} e^{i\theta_j}$  is invertible  $l_n^2(\mathbb{Z}) \rightarrow l_n^2(\mathbb{Z})$  for  $\rho \geq 0$  and  $\theta_j \in [0, 2\pi]$ .

**Proof.** Applying Theorem 6 gives that condition (1) is sufficient and necessary for exponential stability of

System (35). We deduce the result via the equivalence of System (31)-(32) and the equations (33)-(34).  $\square$

We finally mention a known result that applies to non-necessarily Hölder-smooth coefficients, see Baratchart et al. (2021)

**Theorem 12.** If there exists an invertible diagonal matrix  $D$  such that  $\|DK(t)D^{-1}\|_2 < 1$  and  $K(t)$  is a differentiable function with Hölder derivative, then exponential stability of System (31)-(32) holds.

**Proof.** Since system (35) enjoys column-disjointness we can apply Proposition 7 which gives us the exponential stability. Again, we deduce the exponential stability of System (31)-(32) from the equations (33)-(34).  $\square$

Theorem 12 is a generalisation of Theorem 3.7 from the book Bastin and Coron (2016), in the case of smooth linear periodic boundaries for hyperbolic systems.

## 5. CONCLUSION

The paper gave stability criteria for periodic linear difference delay systems when the column are disjoint or when the delay are rationally independent. Furthermore, it stated sufficient and necessary stability conditions for 1-D hyperbolic PDEs with linear periodic boundary conditions. Many interesting open questions are left here for further study. For instance, is exponential stability a continuous property in the delays and the period of the system, at least locally? Or else, can we extend our necessary and sufficient stability conditions to more general 1-D hyperbolic systems?

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