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Hussein Mesmar

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**BEST CONSTANT AND MOUNTAIN-PASS SOLUTIONS FOR A
SUPERCRITICAL HARDY-SOBOLEV PROBLEM IN THE
PRESENCE OF SYMMETRIES**

HUSSEIN MESMAR

1. INTRODUCTION

Let (M, g) be a compact Riemannian manifold without boundary of dimension $n \geq 3$. Given $a, h \in C^0(M)$ and a weight $\omega > 0$ a.e., $2 < q < p$, we are interested in the existence of distributional solutions $u \in H_1^2(M)$ to the nonlinear problem

$$(1) \quad \begin{cases} \Delta_g u + a(x)u = \omega(x)u^{p-1} + h(x)u^{q-1} & \text{in } M, \\ u > 0 & \text{in } M \end{cases}$$

where $\Delta_g := -\operatorname{div}_g(\nabla)$ is the Laplacian with minus sign convention and $H_1^2(M)$ is the Sobolev space defined as the completion of $C^\infty(M)$ for the norm

$$u \mapsto \|u\|_{H_1^2} := \sqrt{\|u\|_2^2 + \|\nabla u\|_2^2}$$

where the norms $\|\cdot\|_q$ on the Lebesgue spaces $L^q(M)$ are taken with respect to the Riemannian element of volume dv_g . The classical Sobolev embedding yields

$$H_1^2(M) \hookrightarrow L^p(M) \text{ continuously iff } 1 \leq p \leq 2^* := \frac{2n}{n-2}, \text{ compactly iff } p < 2^*$$

and it is classical that, when $\omega \equiv 1$, problem (1) has a variational structure when $p \leq 2^*$. The difficulty is in the critical case $p = 2^*$ where many methods have been developed since the pioneer contributions of Aubin [2], Trudinger [19] and Schoen [17].

In the present paper, we add a nonconstant weight and we consider a *supercritical* case which breaks the variational structure. In order to get a supercritical exponent, following Moser [15] and Hebey-Vaugon [10], we impose the invariance of the problem under the action of isometries in order to lower in some sense the dimension of the manifold. Let G be a closed subgroup of the group of isometries $\operatorname{Isom}_g(M)$ of (M, g) and $k = \min_{x \in M} \dim Gx$, where Gx denotes the orbit of a point $x \in M$ under the action of G . We say that a function $\phi : M \rightarrow \mathbb{R}$ is G -invariant if $\phi \circ \sigma(x) = \phi(x)$ for any $x \in M$ and $\sigma \in G$. If G acts freely, then M/G has the structure of a manifold of dimension $(n - k)$: therefore the critical exponent on M/G is $\frac{2(n-k)}{n-k-2} > 2^*$, and, by going back on M , artificially, classical variational methods allows to get G -invariant solutions to (1) with a supercritical exponent.

In the present work, we consider the more intricate case where G does not act freely. We make the fundamental Assumption 8 on G inspired by the work of Saintier [16] (more details are in Section 2): note that under this assumption, we cannot consider problem (1) on the quotient M/G since it is not necessarily a manifold.

We fix $x_0 \in M$ and $s \in [0, 2)$. We define $2^*(k, s) = \frac{2(n-k-s)}{n-k-2}$, $2^*(k) = 2^*(k, 0)$ and we consider $q \in (2, 2^*(k))$. We investigate the existence of a solution $u \in H_1^2(M)$ to

$$(2) \quad \begin{cases} \Delta_g u + au = \frac{u^{2^*(k,s)-1}}{d_g(x, Gx_0)^s} + hu^{q-1} & \text{in } M, \\ u > 0 & \text{in } M \end{cases}$$

where d_g is the Riemannian distance on (M, g) . We refer to Djadli [6] for considerations on the case $s = 0$. Solutions to (2) are critical points of the functional

$$u \mapsto J_q(u) = \frac{1}{2} \int_M (|\nabla u|_g^2 + au^2) dv_g - \frac{1}{2^*(k,s)} \int_M \frac{|u|^{2^*(k,s)}}{d_g(x, Gx_0)^s} dv_g - \frac{1}{q} \int_M h|u|^q dv_g$$

which is defined for $u \in H_1^2(M) \cap L^{2^*(k,s)}(M, d(\cdot, Gx_0)^{-s})$. Our problem is *supercritical* from the viewpoint of Hardy-Sobolev embedding. Namely

$$H_1^2(M) \hookrightarrow L^p(M, d(\cdot, x_0)^{-s}) \text{ continuously if } 1 \leq p \leq 2^*(0, s) = \frac{2(n-s)}{n-2}.$$

Since $2^*(k, s) > 2^*(0, s)$ is supercritical as soon as $k > 0$, there is no embedding from $H_1^2(M)$ into $L^{2^*(k,s)}(M, d(\cdot, x_0)^{-s})$ and the problem does not have a variational formulation in the space $H_1^2(M)$. To overcome this difficulty, we work on

$$H_{1,G}^2(M) := \{u \in H_1^2(M) / u \circ \sigma(x) = u(x) \text{ for all } \sigma \in G \text{ a.e } x \in M\}.$$

In the sequel, $L_G^p(M, d_g(x, Gx_0)^{-s} dv_g)$ is the weighted space of G -invariant functions endowed with the norm $\|f\|_{p,s} = \| |f|^p d_g(\cdot, Gx_0)^{-s} \|_1$. Our first result is an improvement of the integrability:

Theorem 1 (Mesmar). *Let (M, g) be a compact Riemannian manifold without boundary of dimension $n \geq 3$ and G a subgroup of isometries of (M, g) . We suppose that Assumption 8 is verified, and we give ourselves k and Gx_0 as in these hypothesis. Assume that $k \geq 2$ and $n - k \geq 3$. Then, for any $p \in [1, 2^*(k, s)]$, $H_{1,G}^2(M)$ is continuously embedded in $L_G^p(M, \frac{dv_g}{d_g(x, Gx_0)^s})$. Moreover if $1 \leq p < 2^*(k, s)$ the embedding is compact.*

Note that the case $s = 0$ was handled by Faget [7] and the case $k = 0$ was handled by Jaber [12]. From this inclusion, we define the first best constant of the Hardy-Sobolev inequalities in the presence of isometries. It follows from Theorem 1 that there exists $L, B > 0$ such that for all $u \in H_{1,G}^2(M)$:

$$(3) \quad \|u\|_{2^*(k,s),s}^2 \leq L \|\nabla u\|_2^2 + B \|u\|_2^2$$

We define the first best constant to be

$$L_0 = L_0(M, g, n, k, s, x_0) = \inf\{L \geq 0 \text{ such that } \exists B \text{ s.t. (3) holds } \forall u \in H_{1,G}^2(M)\}$$

Definition 2. *For $m \geq 3$ we define the best constant of Euclidean Sobolev space*

$$(4) \quad \frac{1}{K_s(m)} = \inf_{\phi \in D_1^2(\mathbb{R}^m) \setminus \{0\}} \frac{\int_{\mathbb{R}^m} |\nabla \phi|^2 dX}{\left(\int_{\mathbb{R}^m} \frac{|\phi|^{2^*(k,s)}}{|X|^s} dX \right)^{\frac{2}{2^*(k,s)}}} > 0$$

where $D_1^2(\mathbb{R}^m)$ is the completion of $C_c^\infty(\mathbb{R}^m)$ for $\|\nabla \cdot\|_2$.

Aubin [2] and Talenti [18] have computed the best constant $K_0(m)$, and Lieb [13] has computed $K_s(m)$ explicitly for $s > 0$. We denote by $2^* = 2^*(0, 0)$ and $2^*(k) = 2^*(k, 0)$. We define

$$A := \min\{\text{Vol}_g(\mathcal{O}) / \mathcal{O} \text{ principal orbit of dimension } k\},$$

Faget [7] proved that the best constant for G -invariant Sobolev inequality is

$$(5) \quad L_0(M, g, n, k, 0, x_0) = \frac{K_0(n-k)}{A^{1-\frac{2}{2^*(k)}}}.$$

When $s > 0$, Jaber [12] showed that

$$L_0(M, g, n, 0, s, x_0) = K_s(n) = \frac{1}{(n-2)(n-s)} \left(\frac{1}{2-s} w^{n-1} \frac{\Gamma^2(\frac{n-2}{2-s})}{\Gamma(\frac{2(n-s)}{2-s})} \right)^{-\frac{2-s}{n-s}}$$

where Γ is the Euler function. In this article, we use these properties to prove that

Theorem 3 (Mesmar). *Let (M, g) a compact Riemannian manifold without boundary of dimension $n \geq 3$ and G a subgroup of isometries of (M, g) . We suppose that Assumption 8 is verified, and we give ourselves k and Gx_0 as in these hypothesis. For all $k \in [1, n-2)$, $s \in (0, 2)$ we have*

$$L_0(M, g, n, k, s, x_0) = \frac{K_s(n-k)}{A^{1-\frac{2}{2^*(k,s)}}}.$$

In particular, for all $\epsilon > 0$, there exists B_ϵ , such that for all $u \in H_{1,G}^2(M)$, we have that

$$(6) \quad \|u\|_{2^*(k,s),s}^2 \leq \left(\frac{K_s(n-k)}{A^{1-\frac{2}{2^*(k,s)}}} + \epsilon \right) \|\nabla u\|_2^2 + B_\epsilon \|u\|_2^2$$

In the sequel, for any $k \in \mathbb{N}$, $C_G^k(M)$ denotes the space of functions in $C^k(M)$ that are G -invariant. In the spirit of Aubin [2], we are then in position to get solutions to (2) when $h \equiv 0$:

Theorem 4 (Mesmar). *Let (M, g) be a compact Riemannian manifold without boundary of dimension $n \geq 3$ and G a closed subgroup of isometries of (M, g) . We suppose that Assumption 8 is verified, and we define $k := \min_{x \in M} \dim Gx$ and Gx_0 as in these hypothesis. We let $a \in C_G^0(M)$ be such that $\Delta_g + a$ is coercive and that*

$$\inf_{u \in H_{1,G}^2(M) \setminus \{0\}} \frac{\int_M (|\nabla u|^2 + au^2) dv_g}{\left(\int_M |u|^{2^*(k,s)} \frac{dv_g}{d_g(x, Gx_0)^s} \right)^{\frac{2}{2^*(k,s)}}} < \frac{K_s(n-k)^{-1}}{A^{1-\frac{2}{2^*(k,s)}}}.$$

Then the infimum is achieved by a positive function $u \in H_{1,G}^2(M) \cap C^{0,\alpha}(M) \cap C_{loc}^{1,\nu}(M \setminus \{Gx_0\})$ for all $\alpha \in (0, \min\{1, 2-s\})$, $\nu \in (0, 1)$. Up to multiplication by a constant, u satisfies

$$(7) \quad \begin{cases} \Delta_g u + au = \frac{u^{2^*(k,s)-1}}{d_g(x, Gx_0)^s} & \text{in } M, \\ u > 0 & \text{in } M \end{cases}$$

We are then left with proving the strict inequality above. Via test-function estimates, we are able to realize it to obtain the following:

Theorem 5 (Mesmar). *Let (M, g) be a compact Riemannian manifold without boundary of dimension $n \geq 3$ and G a closed subgroup of isometries of (M, g) . We suppose that Assumption 8 is verified, and we define $k := \min_{x \in M} \dim Gx$ and Gx_0 as in these hypothesis. We let $a \in C_G^0(M)$ be such that $\Delta_g + a$ is coercive. We assume that $n - k \geq 4$ and that*

$$\bar{a}(\bar{x}_0) < \frac{(n-k-2)(6-s)}{12(2n-2k-2-s)} \text{Scal}_{\bar{g}}(\bar{x}_0) - \frac{(n-k-2)(n-k-4)}{4(2n-2k-2-s)} \frac{\Delta_{\bar{g}} \bar{f}(\bar{x}_0)}{f(\bar{x}_0)}$$

Then there exists a positive G -invariant solution to (7).

We now consider the existence of a positive G -invariant solution for the perturbed problem (2). Such solutions are critical points for

(8)

$$v \mapsto J_q^+(v) = \frac{1}{2} \int_M (|\nabla v|_g^2 + av^2) dv_g - \frac{1}{2^*(k,s)} \int_M \frac{v_+^{2^*(k,s)}}{d_g(x, Gx_0)^s} dv_g - \frac{1}{q} \int_M hv_+^q dv_g$$

Using Ambrosetti-Rabinowitz's Mountain-Pass-Lemma [1], we get the following:

Theorem 6 (Mesmar). *Let (M, g) be a compact Riemannian manifold without boundary of dimension $n \geq 3$ and G a closed subgroup of isometries of (M, g) . We suppose that Assumption 8 is verified, and we define $k := \min_{x \in M} \dim Gx$ and Gx_0 as in these hypothesis. We let $a, h \in C_G^0(M)$ be such that $h \geq 0$ and $\Delta_g + a$ is coercive. If there exists $u_0 \in H_{1,G}^2(M)$, $u_0 \not\equiv 0$, $u \geq 0$ such that*

$$(9) \quad \sup_{t \geq 0} J_q^+(tu_0) < \frac{(2-s)}{2(n-k-s)} \frac{K_s(n-k)^{-\frac{n-k-s}{2-s}}}{A^{1-\frac{2}{2^*(k,s)}}$$

Then the perturbed Hardy-Sobolev equation admits a non-trivial solution in $C_G^0(M) \cap H_{1,G}^2(M)$. In addition, we have $u > 0$ in M and $u \in C^{0,\alpha} \cap C_{loc}^{1,\nu}(M \setminus \{Gx_0\})$, for all $\alpha \in (0, \min\{1, 2-s\})$, $\nu \in (0, 1)$.

We are left with going below a specific threshold. Here again, this is performed via test-functions estimates. We then obtain the following existence result:

Theorem 7 (Mesmar). *Let (M, g) be a compact Riemannian manifold without boundary of dimension $n \geq 3$ and G a closed subgroup of isometries of (M, g) . We suppose that Assumption 8 is verified, and we define $k := \min_{x \in M} \dim Gx$ and Gx_0 as in these hypothesis. We let $a, h \in C_G^0(M)$ be such that $h \geq 0$ and $\Delta_g + a$ is coercive. We suppose that*

- (1) $n - k \geq 4$;
- (2) $q > \frac{n-k}{n-k-2}$;
- (3) $h(x_0) > 0$.

Then the perturbed Hardy-Sobolev equation:

$$\Delta_g u + au = \frac{u^{2^*(k,s)-1}}{d_g(x, Gx_0)^s} + hu^{q-1}; \quad u > 0 \text{ in } M$$

admits a non-trivial solution in $C_G^0(M) \cap H_{1,G}^2(M)$. In addition $u \in C^{0,\alpha}(M) \cap C_{loc}^{1,\nu}(M \setminus \{Gx_0\})$, for all $\alpha \in (0, \min\{1, 2-s\})$, $\nu \in (0, 1)$.

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2. PRELIMINARIES ON MANIFOLDS INVARIANT BY A GROUP OF ISOMETRIES

Let (M, g) be a compact Riemannian manifold without boundary of dimension n , G a closed group of isometries of (M, g) and $k := \min_{x \in M} \dim Gx \geq 1$ where Gx is the orbit of point x under the action of G . Let $x_0 \in G$ be such that $\dim Gx_0 = k$.

Let us now recall some known geometric results and fix some notations. These results are in the book by Bredon [4]: we refer to Hebey-Vaugon [10], Faget [7] and Saintier [16] for the following exposition. Let G' be a closed subgroup of $Isom_g(M)$: then G' is a Lie group. On the other hand for all $x \in M$, we denote $\bar{x} := \Pi(x)$ where $\Pi : M \rightarrow M/G'$ the canonical surjection. We define $G'x := \{gx/g \in G'\}$ the orbit of x under G' and $S'x := \{g \in G'/g(x) = x\}$ the stabilizer of x under G' . Note that $G'x$ is a compact submanifold isomorphic to the quotient group $G'/S'x$, and we define a principal orbit $G'x$ if its stabilizer is minimal ($\forall y \in M$, $S'y$ contains a conjugate subgroup of $S'x$), in particular the principal orbits are of maximum dimension but the converse is not true.

Define $\Omega := \bigcup \mathcal{O}_P$ where the union is taken on all principal orbits. Then Ω is a dense open set of M and Ω/G' is a smooth manifold that can be endowed with a Riemannian distance \bar{g} such that the surjection $\Omega \rightarrow \Omega/G'$ is a Riemannian submersion. We define

$$(10) \quad \tilde{g} = \bar{v}^{\frac{2}{n-k-2}} \bar{g}$$

with $\bar{v}(\bar{x}) = \text{vol}(\Pi^{-1}(\bar{x})) = \text{vol}(G'x)$. We assume that $k = \min_{x \in M} \dim Gx \geq 1$. In the present paper, we make the following fundamental assumption:

Assumption 8. *For any Gx_0 , a G -orbit of minimal dimension k , we can find $\delta > 0$, and G' a closed subgroup of $Isom_g(M)$ such that:*

- (1) $G'x_0 = Gx_0$;
- (2) For all $x \in B_\delta(Gx_0) := \{y \in M/d_g(y; Gx_0) < \delta\}$, then $G'x$ is principal and $G'x \subset Gx$.

In particular $\dim G'x = \dim Gx_0 = k$, $\forall x \in B_\delta(Gx_0)$.

We now consider the quotient $N := B_\delta(Gx_0)/G'$ that turns to be a $(n - k)$ -Riemannian manifold. This manifold is then equipped with the metrics \bar{g} and \tilde{g} . We let $\eta \in C_c^\infty(\mathbb{R})$ be such that $0 \leq \eta \leq 1$, $\eta(x) = 1$ for $x \in B(0; 1)$ and $\eta(x) = 0$ for $x \in B(0; 2)^c$. For all $\bar{x}_1 \in N$ and $\delta' \in (0, \frac{i_{\tilde{g}}(N)}{2})$, we define $\eta_{\bar{x}_1, \delta'}(\bar{x}) = \eta(\delta'^{-1} d_{\tilde{g}}(\bar{x}_1, \bar{x}))$ for all $\bar{x} \in N$.

 3. PROPERTIES OF $L^p(M, d_g(x, Gx_0)^{-s} dv_g)$: PROOF OF THEOREM 1

We let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$, G a group of isometries of M , $x_0 \in M$ such that Gx_0 is an orbit of minimal dimension $k \in [1, n - 2)$, and $s \in (0, 2)$. We assume that Assumption 8 is verified.

We prove the continuous embedding of Theorem 1. By Hebey-Vaugon [10], the first Sobolev inclusion in the presence of symmetries (the case where $s = 0$), implies the existence of a positive constant C such that

$$(11) \quad \|u\|_{2^*(k)} \leq C \|u\|_{H_{1,G}^2(M)} \text{ for all } u \in H_{1,G}^2(M).$$

Hardy's inequality on \mathbb{R}^{n-k} writes

$$(12) \quad \int_{\mathbb{R}^{n-k}} \frac{\phi^2}{|X|^2} dX \leq \frac{4}{(n-k-2)^2} \int_{\mathbb{R}^{n-k}} |\nabla \phi|^2 dX, \quad \forall \phi \in C_c^\infty(\mathbb{R}^{n-k})$$

Let $u \in C_G^\infty(M)$ be such that there exists $r_0 > 0$, such that $\text{supp } u \subset B_{r_0}(Gx_0)$. We denote $\bar{u} = u \circ \Pi$, where Π is the projection of x on the $(n-k)$ -Riemannian manifold $N := B_{r_0}(Gx_0)/G'$. By (12), there exists $C_1 > 0$ such that

$$\int_N \frac{\bar{u}^2}{d_{\bar{g}}(\bar{x}, \bar{x}_0)^2} dv_{\bar{g}} \leq C_1 \|\nabla \bar{u}\|_2^2.$$

On the other hand, there exists $C_2, C_3 \geq 0$ such that

$$\int_M \frac{u^2}{d_g(x, Gx_0)^2} dv_g = \int_N \frac{\bar{u}^2}{d_{\bar{g}}(\bar{x}, \bar{x}_0)^2} \bar{u} dv_{\bar{g}} \leq C_2 \int_N \frac{\bar{u}^2}{d_{\bar{g}}(\bar{x}, \bar{x}_0)^2} dv_{\bar{g}}$$

And

$$\|\nabla u\|_2^2 \leq C_3 \|\nabla \bar{u}\|_2^2$$

We put together these three inequalities. Then, for any $u \in C_G^\infty(M)$ such that $\text{supp } u \subset B_{r_0}(Gx_0)$, there exists $C > 0$ such that

$$\int_M \frac{u^2}{d_g(x, Gx_0)^2} dv_g \leq C \|u\|_{H_{1,G}^2(M)}^2$$

Let us now set $p = p(s) > 1$ satisfying $\frac{1}{p} + \frac{s}{2} = 1$. For a function $u \in C_G^\infty(M)$ such that $\text{supp } u \subset B_{r_0}(Gx_0)$, by Hölder, we have

$$\int_M |u|^{2^*(k,s)-s} \left(\frac{u}{d_g(x, Gx_0)} \right)^s dv_g \leq \left(\int_M |u|^{\frac{2(2^*(k,s)-s)}{2-s}} dv_g \right)^{\frac{1}{p}} \left(\int_M \frac{u^2}{d_g(x, Gx_0)^2} dv_g \right)^{\frac{s}{2}}$$

Since $\frac{2(2^*(k,s)-s)}{2-s} = 2^*(k)$, raising the previous inequality to the power $\frac{2}{2^*(k,s)}$ we get

$$\left(\int_M |u|^{2^*(k,s)} \frac{dv_g}{d_g(x, Gx_0)^s} \right)^{\frac{2}{2^*(k,s)}} \leq \left(\int_M |u|^{\frac{2-s}{2^*(k,s)}} dv_g \right)^{\frac{2-s}{2^*(k,s)}} \left(\int_M \frac{u^2}{d_g(x, Gx_0)^2} dv_g \right)^{\frac{s}{2^*(k,s)}}$$

Grouping this inequality with (11) and using $\frac{2^*(k)(2-s)}{2 \cdot 2^*(k,s)} + \frac{s}{2^*(k,s)} = 1$, we get

$$(13) \quad \begin{aligned} \left(\int_M |u|^{2^*(k,s)} \frac{dv_g}{d_g(x, Gx_0)^s} \right)^{\frac{2}{2^*(k,s)}} &\leq C \left(\|u\|_{H_{1,G}^2(M)}^2 \right)^{\frac{2^*(k)(2-s)}{2 \cdot 2^*(k,s)}} C' \left(\|u\|_{H_{1,G}^2(M)}^2 \right)^{\frac{s}{2^*(k,s)}} \\ &\leq C'' \|u\|_{H_{1,G}^2(M)}^2 \end{aligned}$$

Where C, C', C'' are strictly positive constants only depends on M, k , and s . By density of $C_G^\infty(M)$, we get that (13) holds for all $u \in H_{1,G}^2(M)$, which proves the continuous embedding of Theorem 1.

We split the proof of the compactness embedding of Theorem 1 into two steps:

Step I: Let us first show that we have $H_{1,G}^2(M) \hookrightarrow L^q(M, d_g(x, Gx_0)^{-s} dv_g)$. We fix $u \in H_{1,G}^2(M)$, by Hölder's inequality we have that

$$\|u\|_{q,s} \leq \|u\|_{2^*(k,s)} \left(\int_M \frac{dv_g}{d_g(x, Gx_0)^s} \right)^{\frac{1}{q} - \frac{1}{2^*(k,s)}}$$

With (13), we get that

$$\|u\|_{q,s} \leq C \|u\|_{H_{1,G}^2(M)} \left(\int_M \frac{dv_g}{d_g(x, Gx_0)^s} \right)^{\frac{1}{q} - \frac{1}{2^*(k,s)}}$$

where C is independent of u . Hence the continuous injection follows.

Step II: Now let us prove the compactness of the embedding. Let $(u_i)_{i \in \mathbb{N}}$ be a bounded sequence in $H_{1,G}^2(M)$. The reflexivity of $H_{1,G}^2(M)$ and the Rellich-Kondrakov theorem yields the existence of $u \in H_{1,G}^2(M)$ such that, up to a subsequence also denoted by $(u_i)_i$, we have that

- (1) $u_i \rightharpoonup u$ weakly in $H_{1,G}^2(M)$
- (2) $u_i \rightarrow u$ strongly in $L^2(M)$
- (3) $u_i(x) \rightarrow u(x)$ for a.e $x \in M$.

Let us fix $\epsilon > 0$. For all $i \in \mathbb{N}$, $R > 0$, we define

$$\Omega_i(R) := \{x \in M / |u_i(x) - u(x)| < R\}$$

On the one hand, the continuous injection of the previous part and by the fact that (u_i) is bounded in $H_{1,G}^2(M)$ yield $C, C' > 0$ such as

$$\begin{aligned} \int_{M \setminus \Omega_i(R)} |u_i - u|^q \frac{dv_g}{d_g(x, Gx_0)^s} &\leq \frac{1}{R^{2^*(k,s)-q}} \int_{\Omega_i(A)} |u_i - u|^{2^*(k,s)} \frac{dv_g}{d_g(x, Gx_0)^s} \\ &\leq \frac{C}{R^{2^*(k,s)-q}} \|u_i - u\|_{H_{1,G}^2(M)}^{2^*(k,s)} \leq \frac{C'(n, k, s, u)}{R^{2^*(k,s)-q}} \end{aligned}$$

There exists $R_0 = R_0(\epsilon)$ such that for all $R > R_0$ we have

$$\frac{C''(n, k, s, u)}{R^{2^*(k,s)-q}} < \frac{\epsilon}{2}$$

On the other hand, let us set the sequence of functions $(f_i)_{i \in \mathbb{N}}$ defined on M by

$$f_i(x) = \frac{|u_i(x) - u(x)|^q}{d_g(x, Gx_0)^s} \mathbf{1}_{\Omega_i(R)}(x) \text{ for } i \in \mathbb{N} \text{ and a.e. } x \in M.$$

Then, by Lebesgue's theorem, we deduce that there exists $i_0 = i_0(\epsilon) > 0$, such that for all $i > i_0$ we have that

$$\int_M f_i dv_g < \frac{\epsilon}{2}.$$

Then, for all $R > R_0$, and $i > i_0$, we have that

$$\begin{aligned} \|u_i - u\|_{q,s}^q &= \int_M |u_i - u|^q \frac{dv_g}{d_g(x, Gx_0)^s} \\ &= \int_{M \setminus \Omega_i(R)} |u_i - u|^q \frac{dv_g}{d_g(x, Gx_0)^s} + \int_M f_i dv_g < \epsilon \end{aligned}$$

Hence $(u_i)_{i \in \mathbb{N}}$ converges u in the space $L^q(M, d_g(x, Gx_0)^{-s} dv_g)$. This proves the compactness of the embedding when $q < 2^*(k, s)$.

This ends the proof of Theorem 1.

4. BEST CONSTANTS ON RIEMANNIAN MANIFOLDS UP TO ϵ

This section is devoted to the proof of Theorem 3. We let (M, g) and G as in Assumption 8. Given $\epsilon > 0$, we prove (6), that is the existence of $B_\epsilon > 0$ such that

$$\|u\|_{L^{2^*(k,s),s}}^2 \leq \left(\frac{K_s(n-k)}{A^{1-\frac{2}{2^*(k,s)}}} + \epsilon \right) \|\nabla u\|_2^2 + B_\epsilon \|u\|_2^2 \text{ for all } u \in H_{1,G}^2(M)$$

We follow the work of Jaber [11] who proved this inequality for $k = 0$.

Step I: Construction of a finite covering of $N = B_{r_0}(Gx_0)/G'$ by exponential balls. We put the $(n-k)$ -Riemannian manifold (N, \bar{g}) . For δ the Euclidean metric on \mathbb{R}^{n-k} and by compactness criterion of M (more details in Jaber [11], Proposition 1.2.1 and Appendix I), and therefore on N , we have for all $\bar{x} \in N$, and all $\rho > 0$, there exists $r = r(x, \rho) \in (0, \frac{i_{\bar{g}}(N)}{2})$, such that $\lim_{\rho \rightarrow 0} r(x, \rho) = 0$ and the exponential map $(B_{2r}(\bar{x}), \exp_{\bar{x}}^{-1})$ satisfies the following properties:

$$(14) \quad \begin{cases} (1-\rho)\delta \leq \bar{g} \leq (1+\rho)\delta \\ (1-\rho)^{\frac{m}{2}} dx \leq dv_{\bar{g}} \leq (1+\rho)^{\frac{m}{2}} dx \\ D_\delta^{-1}|T|_\delta \leq |T|_{\bar{g}} \leq D_\delta|T|_\delta \text{ for all 1-covariant tensor } T \text{ on } N \end{cases}$$

With $\lim_{\rho \rightarrow +\infty} D_\delta = 1$.

Step II: We prove that for all $\epsilon > 0$, there exists $r_0 > 0$ such that for all $u \in C_c^\infty(B_{r_0}(Gx_0))$ that is G -invariant, we have:

$$(15) \quad \|u\|_{L^{2^*(k,s),s}}^2 \leq \left(\frac{K_s(n-k)}{A^{1-\frac{2}{2^*(k,s)}}} + \frac{\epsilon}{2} \right) \|\nabla u\|_2^2$$

By the definition of the best Sobolev constant (4), we have for all $\Phi \in C_c^\infty(\mathbb{R}^{n-k})$

$$(16) \quad \left(\int_{\mathbb{R}^{n-k}} \frac{|\Phi|^{2^*(k,s)}}{|X|_\xi^s} dX \right)^{\frac{2}{2^*(k,s)}} \leq K_s(n-k) \int_{\mathbb{R}^{n-k}} |\nabla \Phi|_\xi^2 dX,$$

where ξ denotes the Euclidean metric on \mathbb{R}^{n-k} . We let $r_0 > 0$ that will be fixed later. Let $u \in C_c^\infty(B_{r_0}(Gx_0))$ be G -invariant. Define $N := B_{r_0}(Gx_0)/G \subset \mathbb{R}^{n-k}$ equipped with its metric \bar{g} . We note $\bar{u} = u \circ \Pi$ where Π is the canonical projection from (M, g) to (N, \bar{g}) (see Assumption 8). It follows from (14) applied on N and (16) that

$$\begin{aligned} \|\bar{u}\|_{L^{2^*(k,s)}(N, \frac{dv_{\bar{g}}}{d_{\bar{g}}(\bar{x}, \bar{x}_0)^s})}^2 &\leq (1+\rho)^{\frac{n-k}{2^*(k,s)}} K_s(n-k) \int_{\mathbb{R}^{n-k}} |\nabla(\bar{u} \circ \exp_{\bar{x}_0})|_\delta^2 dX \\ &\leq D_\rho^2 (1+\rho)^{\frac{n-k}{2^*(k,s)}} (1-\rho)^{-\frac{n-k}{2}} K_s(n-k) \int_N |\nabla \bar{u}|_{\bar{g}}^2 dv_{\bar{g}} \end{aligned}$$

Since $\lim_{\rho \rightarrow 0} D_\rho^2 (1+\rho)^{\frac{n-k}{2^*(k,s)}} (1-\rho)^{-\frac{n-k}{2}} = 1$, it follows that for all $\epsilon > 0$, there exists $\rho_0 > 0$ such that for all $0 < \rho < \rho_0$, we have

$$\left(\int_{B_{r_0}(\bar{x}_0)} |\bar{u}|^{2^*(k,s)} \frac{dv_{\bar{g}}}{d_{\bar{g}}(\bar{x}, \bar{x}_0)^s} \right)^{\frac{2}{2^*(k,s)}} \leq \left(K_s(n-k) + \frac{\epsilon}{2} \right) \int_{B_{r_0}(\bar{x}_0)} |\nabla \bar{u}|_{\bar{g}}^2 dv_{\bar{g}}$$

According to Faget [7] (formula (1.10)), for any function $f \in C_G^\infty(M)$, and for any $\epsilon > 0$, there exists U_{Gx_0} a neighborhood of Gx_0 , such that

$$\int_{U_{Gx_0}} f dv_g \leq \frac{1+\epsilon}{1-\epsilon} \text{Vol}(Gx_0) \int_{U_{Gx_0}/G} \bar{f} dv_{\bar{g}}$$

And also

$$\int_{U_{Gx_0}} |\nabla f|_g^2 dv_g \geq \frac{1-\epsilon}{(1+\epsilon)^2} \text{Vol}(Gx_0) \int_{U_{Gx_0}/G} |\nabla \bar{f}|_{\bar{g}}^2 dv_{\bar{g}}$$

Let us then set r_0 small enough such that $B_{r_0}(Gx_0) \subset U_{Gx_0}$, we will have

$$\begin{aligned} & \left(\int_{B_{r_0}(Gx_0)} |u|^{2^*(k,s)} \frac{dv_g}{d_g(x, Gx_0)^s} \right)^{\frac{2}{2^*(k,s)}} \\ & \leq \left(\frac{1+\epsilon}{1-\epsilon} \text{Vol}(Gx_0) \right)^{\frac{2}{2^*(k,s)}} \left(\int_{B_{r_0}(\bar{x}_0)} |\bar{u}|^{2^*(k,s)} \frac{dv_{\bar{g}}}{d_{\bar{g}}(\bar{x}, \bar{x}_0)^s} \right)^{\frac{2}{2^*(k,s)}} \\ & \leq \left(\frac{1+\epsilon}{1-\epsilon} \text{Vol}(Gx_0) \right)^{\frac{2}{2^*(k,s)}} \left(K_s(n-k) + \frac{\epsilon}{2} \right) \int_{B_{r_0}(\bar{x}_0)} |\nabla \bar{u}|_{\bar{g}}^2 dv_{\bar{g}} \\ & \leq \left(\frac{1+\epsilon}{1-\epsilon} \right)^{\frac{2}{2^*(k,s)}+2} \text{Vol}(Gx_0)^{\frac{2}{2^*(k,s)}-1} \left(K_s(n-k) + \frac{\epsilon}{2} \right) \int_M |\nabla u|_g^2 dv_g \\ & \leq \left(\frac{1+\epsilon}{1-\epsilon} \right)^{\frac{2}{2^*(k,s)}+2} A^{\frac{2}{2^*(k,s)}-1} \left(K_s(n-k) + \frac{\epsilon}{2} \right) \int_M |\nabla u|_g^2 dv_g \end{aligned}$$

And this is true for any $\epsilon > 0$. So for all $\epsilon > 0$ we have that (15) holds.

Step III: We prove that for all $\epsilon > 0$, $r_0 > 0$ and $u \in C_c^\infty(M \setminus B_{r_0}(Gx_0))$ G -invariant, there exists $B_{s,\epsilon} > 0$ such that:

$$(17) \quad \|u\|_{2^*(k,s),s}^2 \leq \left(\frac{K_s(n-k)}{A^{1-\frac{2}{2^*(k,s)}}} + \frac{\epsilon}{2} \right) \|\nabla u\|_2^2 + B_{s,\epsilon} \|u\|_2^2$$

For all $x_1 \in M$, such that $d_g(x_1, Gx_0) > r_0$, we set $u \in C_c^\infty(B_{r_1}(Gx_1))$ G -invariant. Then for all x in the support of u , we have that $d_g(x, Gx_0) \geq r_0 > 0$. Therefore,

$$\begin{aligned} \left(\int_M |u|^{2^*(k,s)} \frac{dv_g}{d_g(x, Gx_0)^s} \right)^{\frac{2}{2^*(k,s)}} &= \left(\int_{B_{r_1}(Gx_1)} |u|^{2^*(k,s)} \frac{dv_g}{d_g(x, Gx_0)^s} \right)^{\frac{2}{2^*(k,s)}} \\ &\leq \frac{1}{r_0^{\frac{2s}{2^*(k,s)}}} \left(\int_{B_{r_1}(Gx_1)} |u|^{2^*(k,s)} dv_g \right)^{\frac{2}{2^*(k,s)}} \end{aligned}$$

Since $2^*(k,s) < 2^*(k)$, by Faget [7], the embedding $H_{1,G}^2(M) \hookrightarrow L^{2^*(k,s)}$ is compact. Then for all $\beta > 0$, there exists $B_\beta > 0$ such that for all $u \in H_{1,G}^2(M)$, we have

$$\|u\|_{2^*(k,s)}^2 \leq \beta \|\nabla u\|_2^2 + B_\beta \|u\|_2^2,$$

which yields (17).

Step IV: We show (6) for all $u \in C_G^\infty(M)$. To do this, let $\phi \in C_c^\infty(\mathbb{R}^+)$ be such that $0 \leq \phi \leq 1$ and $\phi(x) = 1$ for all $x \leq 1$ and $\phi(x) = 0$ for all $x \geq 2$. For all

$\delta > 0$, we set $\eta := \phi\left(\frac{d_g(x, Gx_0)}{2\delta}\right) \in C_c^\infty(B_\delta(Gx_0))$. Note that η and $(1 - \eta)$ are G -invariant, and furthermore there exists $H > 0$ such that

$$(18) \quad |\nabla \eta^{\frac{1}{2}}|_g, |\nabla(1 - \eta)^{\frac{1}{2}}|_g \leq H$$

Let $\epsilon > 0$, and set $u \in C_G^\infty(M)$. Since $\frac{2^*(k,s)}{2^*(k)} > 1$, so according to Minkowski inequality we have:

$$\begin{aligned} & \left(\int_M |u|^{2^*(k,s)} \frac{dv_g}{d_g(x, Gx_0)^s} \right)^{\frac{2}{2^*(k,s)}} \leq \|\eta u^2 + (1 - \eta)u^2\|_{\frac{2^*(k,s)}{2}, s} \\ & \leq \|\eta u^2\|_{\frac{2^*(k,s)}{2}, s} + \|(1 - \eta)u^2\|_{\frac{2^*(k,s)}{2}, s} \\ & \leq \left(\int_M |\eta^{\frac{1}{2}} u|^{2^*(k,s)} \frac{dv_g}{d_g(x, Gx_0)^s} \right)^{\frac{2}{2^*(k,s)}} + \left(\int_M |(1 - \eta)^{\frac{1}{2}} u|^{2^*(k,s)} \frac{dv_g}{d_g(x, Gx_0)^s} \right)^{\frac{2}{2^*(k,s)}} \end{aligned}$$

By (15), and (17), for all $\epsilon > 0$, there exists $B_{s,\epsilon} > 0$ such that

$$\begin{aligned} & \left(\int_M |u|^{2^*(k,s)} \frac{dv_g}{d_g(x, Gx_0)^s} \right)^{\frac{2}{2^*(k,s)}} \\ & \leq \left(\frac{K_s(n-k)}{A^{1-\frac{2}{2^*(k,s)}}} + \frac{\epsilon}{2} \right) \int_M \left(|\nabla(\eta^{\frac{1}{2}} u)|_g^2 + |\nabla((1 - \eta)^{\frac{1}{2}} u)|_g \right) dv_g + B_{s,\epsilon} \|u\|_2^2 \\ & = \left(\frac{K_s(n-k)}{A^{1-\frac{2}{2^*(k,s)}}} + \frac{\epsilon}{2} \right) \int_M \left(\eta^{\frac{1}{2}} |\nabla u|_g + |u| |\nabla \eta^{\frac{1}{2}}|_g \right)^2 dv_g \\ & \quad + \left(\frac{K_s(n-k)}{A^{1-\frac{2}{2^*(k,s)}}} + \frac{\epsilon}{2} \right) \int_M \left((1 - \eta)^{\frac{1}{2}} |\nabla u|_g + |u| |\nabla(1 - \eta)^{\frac{1}{2}}|_g \right)^2 dv_g + B_{s,\epsilon} \|u\|_2^2 \\ & \leq \left(\frac{K_s(n-k)}{A^{1-\frac{2}{2^*(k,s)}}} + \frac{\epsilon}{2} \right) \left(\int_M \left(\eta |\nabla u|_g^2 + 2\eta^{\frac{1}{2}} |\nabla u|_g |u| |\nabla \eta^{\frac{1}{2}}|_g + |u|^2 |\nabla \eta^{\frac{1}{2}}|_g^2 \right) dv_g \right. \\ & \quad \left. + \int_M \left((1 - \eta) |\nabla u|_g^2 + 2\eta^{\frac{1}{2}} |\nabla u|_g |u| |\nabla(1 - \eta)^{\frac{1}{2}}|_g + |u|^2 |\nabla(1 - \eta)^{\frac{1}{2}}|_g^2 \right) dv_g \right) \\ & \quad + B_{s,\epsilon} \|u\|_2^2 \end{aligned}$$

With (18), we get

$$(19) \quad \begin{aligned} & \left(\int_M |u|^{2^*(k,s)} \frac{dv_g}{d_g(x, Gx_0)^s} \right)^{\frac{2}{2^*(k,s)}} \\ & \leq \left(\frac{K_s(n-k)}{A^{1-\frac{2}{2^*(k,s)}}} + \frac{\epsilon}{2} \right) \left(\|\nabla u\|_2^2 + 4H \|\nabla u\|_2 \|u\|_2 + 2H^2 \|u\|_2^2 \right) + B_{s,\epsilon} \|u\|_2^2 \end{aligned}$$

Let ϵ_0 be such that

$$\left(\frac{K_s(n-k)}{A^{1-\frac{2}{2^*(k,s)}}} + \frac{\epsilon}{2} \right) (1 + \epsilon_0) \leq \frac{K_s(n-k)}{A^{1-\frac{2}{2^*(k,s)}}} + \epsilon$$

And using the fact that

$$4H \|\nabla u\|_2 \|u\|_2 \leq \epsilon_0 \|\nabla u\|_2^2 + \frac{(2H)^2}{\epsilon_0} \|u\|_2^2$$

Using (19), we get

$$\begin{aligned} \left(\int_M |u|^{2^*(k,s)} \frac{dv_g}{d_g(x, Gx_0)^s} \right)^{\frac{2}{2^*(k,s)}} &\leq \left(\frac{K_s(n-k)}{A^{1-\frac{2}{2^*(k,s)}}} + \frac{\epsilon}{2} \right) (1 + \epsilon_0) \|\nabla u\|_2^2 + B_\epsilon \|u\|_2^2 \\ &\leq \left(\frac{K_s(n-k)}{A^{1-\frac{2}{2^*(k,s)}}} + \epsilon \right) \int_M |\nabla u|_g^2 dv_g + B_\epsilon \int_M |u|^2 dv_g \end{aligned}$$

with

$$B_\epsilon = \left(\frac{(2H)^2}{\epsilon_0} + 2H^2 \right) \left(\frac{K_s(n-k)}{A^{1-\frac{2}{2^*(k,s)}}} + \frac{\epsilon}{2} \right) + B_{s,\epsilon}$$

Which completes this step.

Part V: By density of $C_G^\infty(M)$ in $H_{1,G}^2(M)$, we get that (6) holds for all $u \in H_{1,G}^2(M)$, which proves (6).

It follows from (6) that

$$(20) \quad L_0(M, g, n, k, s, x_0) \leq \frac{K_s(n-k)}{A^{1-\frac{2}{2^*(k,s)}}}.$$

We prove the reverse inequality. Take $L_1 > 0$ such that there exists B_{L_1} such that

$$(21) \quad \|u\|_{L^{2^*(k,s)}(N, d_{\tilde{g}}(\bar{x}, \bar{x}_0)^s)}^2 \leq L_1 \|\nabla u\|_2^2 + B_{L_1} \|u\|_2^2$$

for all $u \in H_{1,G}^2(M)$. We take $\Phi \in C_c^\infty(\mathbb{R}^{n-k})$ and we let $(\Phi_\mu)_{\mu>0}$ be the family in $C_c^\infty(N)$ defined for all $\mu > 0$ small enough satisfying $\mu \leq \frac{\rho_0}{R}$ by:

$$\Phi_\mu(\bar{x}) = \Phi(\mu^{-1} \exp_{\bar{x}_0}^{-1}(\bar{x}))$$

where the exponential map is taken with respect to the metric \tilde{g} on $N := B_\delta(\bar{x}_0)/G$. Applying inequality (21) to our sequence Φ_μ , we write:

$$(22) \quad \|\Phi_\mu\|_{L^{2^*(k,s)}(N, d_{\tilde{g}}(\bar{x}, \bar{x}_0)^s)}^2 \leq L_1 \|\nabla \Phi_\mu\|_2^2 + B_{L_1} \|\Phi_\mu\|_2^2$$

Let $\epsilon > 0$. By compactness criterion of \bar{N} , there exists $R_\epsilon > 0$, such that for all $p \in B_{R_\epsilon}(0) \subset \mathbb{R}^{n-k}$ we have:

$$(1 - \epsilon)\xi \leq (\exp_{\bar{x}_0}^* \tilde{g})(p) \leq (1 + \epsilon)\xi$$

in the sense of bilinear forms. We fix $\mu_0 = \mu_0(\epsilon)$ such that for all $\mu < \mu_0$, we have $R\mu < R_\epsilon$. Then for all $\mu < \mu_0$, with a change of variables, we write

$$\begin{aligned} &\left(\int_{B_{\mu R}(\bar{x}_0)} |\Phi_\mu|^{2^*(k,s)} \frac{dv_{\tilde{g}}}{d_{\tilde{g}}(\bar{x}, \bar{x}_0)^s} \right)^{\frac{2}{2^*(k,s)}} \\ &\geq (1 - \epsilon)^{\frac{n-k}{2^*(k,s)}} \left(\int_{B_{\mu R}(\bar{x}_0)} \frac{|\Phi|^{2^*(k,s)}(\mu^{-1} \exp_{\bar{x}_0}^{-1}(x))}{d_{\tilde{g}}(\bar{x}, \bar{x}_0)^s} dv_{(\exp_{\bar{x}_0}^{-1})^* \xi} \right)^{\frac{2}{2^*(k,s)}} \\ (23) \quad &= \mu^{n-k-2} (1 - \epsilon)^{\frac{n-k}{2^*(k,s)}} \left(\int_{B_R(0)} \frac{|\Phi|^{2^*(k,s)}(X)}{|X|^s} dX \right)^{\frac{2}{2^*(k,s)}} \end{aligned}$$

And further

$$\begin{aligned}
\int_N |\nabla \Phi_\mu|_g^2 dv_{\tilde{g}} &= \int_{B_{\mu R}(\bar{x}_0)} |\nabla \Phi(\mu^{-1} \exp_{\bar{x}_0}^{-1}(x))|_g^2 dv_{\tilde{g}} \\
&\leq (1 + \epsilon)^{\frac{m}{2}} \int_{B_{\mu R}(\bar{x}_0)} |\nabla \Phi(\mu^{-1} \exp_{\bar{x}_0}^{-1}(x))|_{(\exp_{\bar{x}_0}^{-1})^* \xi}^2 dv_{(\exp_{\bar{x}_0}^{-1})^* \xi} \\
(24) \quad &= \frac{(1 + \epsilon)^{\frac{m}{2}}}{1 - \epsilon} \mu^{m-2} \int_{B_R(0)} |\nabla \Phi|_\delta^2 dX
\end{aligned}$$

With the same arguments, we show that

$$(25) \quad \int_N \Phi_\mu^2 dv_{\tilde{g}} \leq (1 + \epsilon)^{\frac{n-k}{2}} \mu^{n-k} \int_{B_R(0)} \Phi^2 dX$$

Combining (23), (24) and (25) and (22) we will have for all $\epsilon > 0$, $\mu < \mu_0$ than

$$\begin{aligned}
&\left(\int_{B_R(0)} \frac{|\Phi|^{2^*(k,s)}(X)}{|X|^s} dX \right)^{\frac{2}{2^*(k,s)}} \\
&\leq \frac{(1 + \epsilon)^{\frac{n-k}{2}}}{(1 - \epsilon)^{\frac{n-k}{2^*(k,s)}}} \left[\frac{L_1}{(1 - \epsilon)} \int_{B_R(0)} |\nabla \Phi|_\delta^2 dX + \mu^2 B_{L_1} \int_{B_R(0)} \Phi^2 dX \right]
\end{aligned}$$

Passing $\epsilon \rightarrow 0$ and $\mu \rightarrow 0$, we get

$$(26) \quad \left(\int_{B_R(0)} \frac{|\Phi|^{2^*(k,s)}(X)}{|X|^s} dX \right)^{\frac{2}{2^*(k,s)}} \leq L_1 \int_{B_R(0)} |\nabla \Phi|_\delta^2 dX$$

It then follows from the definition (4) that

$$(27) \quad L_1 \geq K_s(n - k)$$

Now, let $L \in \mathbb{R}$ be such that there exists B_L such that

$$\|u\|_{2^*(k,s),s}^2 \leq L \|\nabla u\|_2^2 + B_L \|u\|_2^2 \text{ for all } u \in H_{1,G}^2(M).$$

Define the family $x \mapsto \phi_\mu(x) = \Phi_\mu(\bar{x})$ in $C_G^\infty(M)$, where Φ_μ is as above. We have

$$\begin{aligned}
\|\phi_\mu\|_{2^*(k,s),s}^2 &= \left(\int_{B_{\mu R}(Gx_0)} |\phi_\mu|^{2^*(k,s)} \frac{dv_g}{d_g(x, x_0)^s} \right)^{\frac{2}{2^*(k,s)}} \\
&= \left(\int_{B_{\mu R}(\bar{x}_0)} \bar{\nu}^{\frac{-2}{n-k-2}} |\Phi_\mu|^{2^*(k,s)} \frac{dv_{\tilde{g}}}{d_{\tilde{g}}(\bar{x}, \bar{x}_0)^s} \right)^{\frac{2}{2^*(k,s)}} \\
(28) \quad &\leq (A + C_R \mu)^{-1 + \frac{2}{2^*(k,s)}} \left(\int_{B_{\mu R}(\bar{x}_0)} |\Phi_\mu|^{2^*(k,s)} \frac{dv_{\tilde{g}}}{d_{\tilde{g}}(\bar{x}, \bar{x}_0)^s} \right)^{\frac{2}{2^*(k,s)}}
\end{aligned}$$

And by definition of \tilde{g} , we have

$$\int_N |\nabla \Phi_\mu|_g^2 dv_{\tilde{g}} = \int_M |\nabla \phi_\mu|_g^2 dv_g$$

Using (26) in (28), we have

$$\|\phi_\mu\|_{2^*(k,s),s}^2 \leq (A + c'_R \mu)^{1 - \frac{2}{2^*(k,s)}} L \int_M |\nabla \phi_\mu|_g^2 dv_g$$

We then apply (21) on Φ_μ with $L_1 = (A + c'_R \mu)^{1 - \frac{2}{2^*(k,s)}} L$. By using (27), we get that $L \geq \frac{K_s(n-k)}{A^{1 - \frac{2}{2^*(k,s)}}}$, and then $L_0(M, g, n, k, s, x_0) \geq \frac{K_s(n-k)}{A^{1 - \frac{2}{2^*(k,s)}}}$. With (20), we get the result and Theorem 3 is proved.

Remarque 9. *In the case $s = 0$, this result still holds as proved by Faget [7].*

Remarque 10. *Theorem 3 does not allow to conclude if L_0 is achieved or not, in other words for the moment we do not know if (6) holds for $\epsilon = 0$.*

5. APPLICATIONS OF THE MOUNTAIN-PASS LEMMA: PROOF OF THEOREMS 6 AND 7

This section is devoted to the proof of the existence Theorems 6 and 7. We will need the following computational lemma:

Lemma 11. *For all $q \in (2, 2^*(k, s))$, there exists $C_1 = C_1(q) > 0$, $C_2 = C_2(q) > 0$, and $C_3 = C_3(q) > 0$ such that for all $x, y \in \mathbb{R}$*

$$\begin{aligned} ||x + y|^q - |x|^q - q|x|^{q-2}xy| &\leq C_1 (|x|^{q-2}|y|^2 + |y|^q) \\ |(x + y)|x + y|^q - x|x|^q| &\leq C_2 (|x|^q|y| + |y|^{q+1}) \\ |x|^q - |y|^q - |x - y|^q &\leq C_3 (|x|^{q-1}|x - y| + |x - y|^{q-1}|y|) \end{aligned}$$

Lemma 12. *Let (X, g_X) be a compact Riemannian manifold, $(f_k)_{k \in \mathbb{N}}$ a sequence of bounded functions in $L^p(X)$ with $p > 1$. If $(f_k)_{k \in \mathbb{N}}$ converges almost everywhere to a function f , then $f \in L^p(X)$ and $(f_k)_{k \in \mathbb{N}}$ converges weakly to f in $L^p(X)$*

Proof. A proof, inspired by Hebey, is in Jaber ([11], Lemma 6.5.1). □

We use the following version of the Mountain-Pass Lemma by Ambrosetti-Rabinowitz seen in Jaber [12]: Concerning terminology, we say that a sequence $(u_N)_{N \in \mathbb{N}} \in E$ is a Palais-Smale sequence (PS) for $J \in C^1(E)$ at the level $\beta \in \mathbb{R}$ if $J(u_N) \rightarrow \beta$ and $J'(u_N) \rightarrow 0$ strongly in the dual E'

Theorem 13 (Ambrosetti-Rabinowitz [1]). *Let $J \in C^1(E, \mathbb{R})$, where $(E, \|\cdot\|_E)$ is a Banach space. We assume that*

- (1) $J(0) = 0$
- (2) *There exists $\alpha, r > 0$ such that $J(v) \geq \alpha$ for all $v \in E$ verifying $\|v\|_E = r$*
- (3) *There exists $v_0 \in E$ such that $\limsup_{t \rightarrow +\infty} J(tv_0) < 0$*

We consider $t_0 > 0$ large enough such that $\|t_0 v_0\|_E > r$ and $J(t_0 v_0) < 0$. Then J admits a Palais-Smale sequence at level β_0 , where

$$\beta_0 = \min_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t))$$

and $\Gamma = \{\gamma \in C^0([0,1], E) / \gamma(0) = 0, \gamma(1) = t_0 v_0\}$. In particular, $0 < \beta_0 \leq \sup_{t \geq 0} J(tv_0)$.

Proposition 14. *For all $u_0 \in H_{1,G}^2(M)$, $u_0 \not\equiv 0$, $u_0 \geq 0$, there exists $\beta_0 = \beta_0(u_0) > 0$ and a sequence $(u_N)_{n \in \mathbb{N}} \in H_{1,G}^2(M)$ which is a Palais-Smale sequence for the functional J_q at level β_0 . Moreover $\beta_0 \leq \sup_{t \geq 0} J_q(tu_0)$.*

Proof. We take $E = H_{1,G}^2(M)$ and $J = J_q \in C^1(E, \mathbb{R})$ in Theorem 13. Since $J_q(0) = 0$, we have (1). Concerning (2), the coercivity of the operator $\Delta_g + a$ and the Sobolev and Cotsiolis-Ledoux inclusions [5] imply that there exist $C_i > 0$, $i = 1, 2, 3$ such that for all $v \in H_{1,G}^2(M)$, we have:

$$(29) \quad J_q(v) \geq C_1 \|v\|_{H_{1,G}^2(M)}^2 - C_2 \|v\|_{H_{1,G}^2(M)}^{2^*(k,s)} - C_3 \|v\|_{H_{1,G}^2(M)}^q$$

We consider the real function $f(r) = C_1 - C_2 r^{2^*(k,s)-2} - C_3 r^{q-2}$. As $q, 2^*(k,s) > 2$, then $f(r) \rightarrow C_1$ when $r \rightarrow 0$. In other words, there exists $r_0 > 0$, such that for all $r < r_0$, $f(r) > \frac{C_1}{2}$. So, for all $v \in H_{1,G}^2(M)$, satisfying $\|v\|_{H_{1,G}^2(M)} = \frac{r_0}{2}$, by (29) we have $J_q(v) \geq \frac{C_1 r_0^2}{8}$. Regarding (3), for $u_0 \in H_{1,G}^2(M) \setminus \{0\}$, we have that

$$\begin{aligned} J_q(tu_0) &= \frac{t^2}{2} \int_M (|\nabla u_0|^2 + au_0) dv_g - \frac{t^{2^*(k,s)}}{2^*(k,s)} \int_M |u_0|^{2^*(k,s)} \frac{dv_g}{d_g(x, Gx_0)^s} \\ &\quad - \frac{t^q}{q} \int_M h|u_0|^q \frac{dv_g}{d_g(x, Gx_0)^s} \\ &:= at^2 - bt^{2^*(k,s)} - ct^q \\ &= t^{2^*(k,s)} \left(at^{2-2^*(k,s)} - b - ct^{q-2^*(k,s)} \right) \leq t^{2^*(k,s)} \left(at^{2-2^*(k,s)} - b \right) \end{aligned}$$

Where $a, b > 0$ and $c \geq 0$. Then $J_q(tu_0) \rightarrow -\infty$ and (iii) holds.

Let $t_0 > 0$, $\beta_0 = \beta_0(u_0)$ and Γ as in the Mountain Pass lemma. Then for all $\gamma \in \Gamma$, we have that $\sup_{t \in [0,1]} J_q(\gamma(t)) \geq \lambda$, where $\lambda = C_1 \frac{r_0^2}{8}$, hence $\beta_0 \geq \lambda > 0$. So for $t_0 > 0$ large enough, our Proposition is proven. \square

Proposition 15. *Let $p, q > 0$ such that $p+q = 2^*(k,s)$, $u \in H_{1,G}^2(M)$, $f \in L^\infty(M)$, and $(w_n)_{N \in \mathbb{N}}$ a bounded sequence in $H_{1,G}^2(M)$ which almost converges everywhere to $w \in H_{1,G}^2(M)$. Then, up to a subsequence, we have*

$$\lim_{n \rightarrow \infty} \int_M f|u|^p w_n^q \frac{dv_g}{d_g(x, Gx_0)^s} = \int_M f|u|^p w^q \frac{dv_g}{d_g(x, Gx_0)^s}$$

Proof. Let $r = \frac{2^*(k,s)}{p} > 1$ and r' are conjugate. The inclusion of Theorem 1, implies that there exists $C = C(M, g, k, s) > 0$ such that

$$\|v\|_{L^{2^*(k,s)}(M, \frac{dv_g}{d_g(x, Gx_0)^s})} \leq C \|v\|_{H_{1,G}^2(M)} \text{ for all } v \in H_{1,G}^2(M).$$

Hence, $f|u|^p \in L^r(M, \frac{dv_g}{d_g(x, Gx_0)^s})$. On the other hand $\| |w_n|^q \|_{L^{r'}(M, \frac{dv_g}{d_g(x, Gx_0)^s})} = \|w_n\|_{L^{2^*(k,s)}(M, \frac{dv_g}{d_g(x, Gx_0)^s})}^q$, hence the sequence $(|w_n|^q)_{N \in \mathbb{N}}$ is bounded, and we apply

Lemma 12 and verify that $(|w_n|^q)_{N \in \mathbb{N}}$ converges weakly to $(|w|^q)_{N \in \mathbb{N}}$ in $L^{r'}(M, \frac{dv_g}{d_g(x, Gx_0)^s})$. This completes the proof of this proposition. \square

Proposition 16. *Let (M, g) and G be as in the Assumption 8. Let $a, h \in C_G^0(M)$ be such that $\Delta_g + a$ is coercive and $h \geq 0$. Let $(u_N)_{N \in \mathbb{N}} \in H_{1,G}^2(M)$ be a Palais-Smale sequence for J_q at level $\beta \in \mathbb{R}$. We assume that*

$$\beta < c_{n,k,s} \frac{K_s(n-k)}{A^{1-\frac{2}{2^*(k,s)}}}^{-\frac{n-k-s}{2-s}}, \text{ with } c_{n,k,s} = \frac{2-s}{2(n-k-s)}.$$

Then there exists $u \in H_{1,G}^2(M)$ such that $J_q(u) = \beta$ and up to a subsequence (u_N) converges strongly to u in $H_{1,G}^2(M)$. Moreover $J'_q(u) = 0$ and $J_q(u) = \beta$.

Proof. We let $(u_N)_{N \in \mathbb{N}} \in H_{1,G}^2(M)$ be as in the statement, that is

$$(30) \quad J_q(u_N) \rightarrow \beta \text{ and } J'_q(u_N) \rightarrow 0 \text{ strongly in } H_{1,G}^2(M)',$$

where β is as above. Note that, for all $\psi \in H_{1,G}^2(M)$, we have

$$(31) \quad \begin{aligned} \langle J'_q(u_N), \psi \rangle &= \int_M ((\nabla u_N, \nabla \psi)_g + a u_N \psi) dv_g - \int_M \psi u_N |u_N|^{2^*(k,s)-2} \frac{1}{d_g(x, Gx_0)^s} dv_g \\ &\quad - \int_M h \psi u_N |u_N|^{q-2} dv_g \end{aligned}$$

We proceed here in 4 steps:

Step I: We prove that the sequence $(u_N)_{N \in \mathbb{N}}$ is bounded in $H_{1,G}^2(M)$. Indeed, since the operator $\Delta_g + a$ is coercive, there exists $C > 0$ such that

$$(32) \quad \begin{aligned} \|u_N\|_{H_{1,G}^2(M)}^2 &\leq C \int_M (|\nabla u_N|_g^2 + a u_N^2) dv_g \\ &\leq C \left[2J_q(u_N) + \frac{2}{2^*(k,s)} \int_M |u_N|^{2^*(k,s)} \frac{dv_g}{d_g(x, Gx_0)^s} + \frac{2}{q} \int_M h |u_N|^q dv_g \right] \end{aligned}$$

On the other hand, from (31) we get that

$$\begin{aligned} o(\|u_N\|_{H_{1,G}^2(M)}) &= \langle J'_q(u_N), u_N \rangle \\ &= \int_M (|\nabla u_N|_g^2 + a u_N^2) dv_g - \int_M |u_N|^{2^*(k,s)} \frac{dv_g}{d_g(x, Gx_0)^s} - \int_M h |u_N|^q dv_g \end{aligned}$$

Putting the previous equality in J_q , we have

$$(33) \quad \begin{aligned} 2J_q(u_N) &= \left(1 - \frac{2}{2^*(k,s)}\right) \int_M |u_N|^{2^*(k,s)} \frac{dv_g}{d_g(x, Gx_0)^s} \\ &\quad + \left(1 - \frac{2}{q}\right) \int_M h |u_N|^q dv_g + o(\|u_N\|_{H_{1,G}^2(M)}) \end{aligned}$$

By combining (32) and (33), we get

$$(34) \quad \begin{aligned} \|u_N\|_{H_{1,G}^2(M)}^2 &\leq C \left[\int_M |u_N|^{2^*(k,s)} \frac{dv_g}{d_g(x, Gx_0)^s} + \int_M h |u_N|^q dv_g \right] \\ &\quad + o(\|u_N\|_{H_{1,G}^2(M)}) \end{aligned}$$

Using (33), (30) and the fact that $h \geq 0$

$$\begin{aligned} &\left(1 - \frac{2}{2^*(k,s)}\right) \int_M |u_N|^{2^*(k,s)} \frac{dv_g}{d_g(x, Gx_0)^s} \\ &= 2\beta - \left(1 - \frac{2}{q}\right) \int_M h |u_N|^q dv_g + o(\|u_N\|_{H_{1,G}^2(M)}) \\ &\leq 2\beta + o(\|u_N\|_{H_{1,G}^2(M)}) \end{aligned}$$

This implies that

$$(35) \quad \int_M |u_N|^{2^*(k,s)} \frac{dv_g}{d_g(x, Gx_0)^s} = O(1) + o(\|u_N\|_{H_{1,G}^2(M)})$$

and

$$(36) \quad \begin{aligned} \left(1 - \frac{2}{q}\right) \int_M h |u_N|^q dv_g &= 2\beta - \frac{2}{2^*(k,s)} \int_M |u_N|^{2^*(k,s)} \frac{dv_g}{d_g(x, Gx_0)^s} + o(\|u_N\|_{H_{1,G}^2(M)}) \\ &= O(1) + o(\|u_N\|_{H_{1,G}^2(M)}) \end{aligned}$$

Grouping (34), (35) and (36) we get that

$$\|u_N\|_{H_{1,G}^2(M)}^2 = O(1) + o(\|u_N\|_{H_{1,G}^2(M)}).$$

And then $\|u_N\|_{H_{1,G}^2(M)}$ is bounded. This completes the proof of this step.

Step II: We claim that, up to extraction, there exists $u \in H_{1,G}^2(M)$ such that

- (1) $u_N \rightharpoonup u$ weakly in $H_{1,G}^2(M)$
- (2) $u_N \rightarrow u$ strongly in $L^{p_1}(M)$ for all $p_1 \in [2, 2^*]$,
- (3) $u_N \rightarrow u$ strongly in $L^{p_2}(M, \frac{dv_g}{d_g(x, Gx_0)^s})$ for all $p_2 \in [2, 2^*(k,s)]$,
- (4) $u_N(x) \rightarrow u(x)$ for a.e $x \in M$.

Moreover, $J'_q(u) = 0$.

The existence of u is a consequence of the boundedness of (u_N) and the compactness of the subcritical embeddings. Passing to the limit in $J'_q(u_N) = o(1)$ yields $J'_q(u) = 0$: we refer to Jaber [12] and Mesmar [14] for details.

Step III: We prove that for the subsequence of $(u_N)_{N \in \mathbb{N}}$ in Step II, we have that

$$(37) \quad \|\nabla(u_N - u)\|_2^2 = \int_M \frac{|u_N - u|^{2^*(k,s)}}{d_g(x, Gx_0)^s} dv_g + o(1), \text{ and}$$

$$(38) \quad c_{n,k,s} \|\nabla(u_N - u)\|_2^2 \leq \beta + o(1), \text{ with } c_{n,k,s} = \frac{2-s}{2(n-k-s)},$$

Indeed, we set $w_N = u_N - u$. Lemma 11 yields the existence of $C > 0$ such that

$$\left| |u_N|^{2^*(k,s)} - |u|^{2^*(k,s)} - |w_N|^{2^*(k,s)} \right| \leq C \left(|u|^{2^*(k,s)-1} |w_N| + |u| |w_N|^{2^*(k,s)-1} \right)$$

Using this inequality, and applying Proposition 15 to the sequence (w_N) , we get

$$(39) \quad \int_M \left(|u_N|^{2^*(k,s)} - |u_N - u|^{2^*(k,s)} \right) \frac{dv_g}{d_g(x, Gx_0)^s} = \int_M |u|^{2^*(k,s)} \frac{dv_g}{d_g(x, Gx_0)^s} + o(1).$$

Let us test $J'_q(u_N) - J'_q(u)$ on $u_N - u$

$$(40) \quad \begin{aligned} o(1) &= \langle J'_q(u_N) - J'_q(u), u_N - u \rangle \\ &= \int_M (|\nabla(u_N - u)|_g^2 + a(u_N - u)^2) dv_g + \int_M h(u_N - u) [u_N |u_N|^{q-2} - u |u|^{q-1}] dv_g \\ &\quad - \int_M (u_N - u) \left[u_N |u_N|^{2^*(k,s)-2} - u |u|^{2^*(k,s)-2} \right] \frac{dv_g}{d_g(x, Gx_0)^s} \\ &= \int_M (|\nabla(u_N - u)|_g^2 + a(u_N - u)^2) dv_g \\ &\quad - \int_M \left(|u_N|^{2^*(k,s)} - u u_N |u_N|^{2^*(k,s)-2} + |u|^{2^*(k,s)} - u u_N |u|^{2^*(k,s)-2} \right) \frac{dv_g}{d_g(x, Gx_0)^s} \\ &\quad - \int_M h (|u_N|^q + |u|^q - u u_N |u_N|^{q-2} - u u_N |u|^{q-2}) dv_g \end{aligned}$$

On the other hand, Lemma 12 yields

$$\begin{aligned} \lim_{n \rightarrow +\infty} \int_M u u_N |u_N|^{2^*(k,s)-2} \frac{dv_g}{d_g(x, Gx_0)^s} &= \int_M |u|^{2^*(k,s)} \frac{dv_g}{d_g(x, Gx_0)^s} \\ &= \lim_{n \rightarrow +\infty} \int_M u u_N |u|^{q-2} \frac{dv_g}{d_g(x, Gx_0)^s} \end{aligned}$$

and

$$\begin{aligned} \lim_{n \rightarrow +\infty} \int_M h u u_N |u_N|^{q-2} \frac{dv_g}{d_g(x, Gx_0)^s} &= \int_M h |u|^q \frac{dv_g}{d_g(x, Gx_0)^s} \\ &= \lim_{n \rightarrow +\infty} \int_M h u u_N |u|^{q-2} \frac{dv_g}{d_g(x, Gx_0)^s} \end{aligned}$$

Also, the weak convergence of (u_N) to u in $H_{1,G}^2(M)$, the last equalities, as well as (39) and (40) imply (37). The weak and strong convergence of Step II imply

$$\begin{aligned} J_q(u_N) - J_q(u) &= \frac{1}{2} \int_M (|\nabla u_N|_g^2 - |\nabla u|_g^2) dv_g + o(1) \\ &\quad - \frac{1}{2^*(k,s)} \int_M (|u_N|^{2^*(k,s)} - |u|^{2^*(k,s)}) \frac{dv_g}{d_g(x, Gx_0)^s} \\ (41) \quad &= \left(\frac{1}{2} - \frac{1}{2^*(k,s)} \right) \int_M (|\nabla u_N|_g^2 - |\nabla u|_g^2) dv_g + o(1) \end{aligned}$$

Since $\langle J'_q(u), u \rangle = 0$, we have that

$$J_q(u) = \left(\frac{1}{2} - \frac{1}{2^*(k,s)} \right) \int_M u^{2^*(k,s)} \frac{1}{d_g(x, Gx_0)^s} dv_g + \left(\frac{1}{2} - \frac{1}{q} \right) \int_M h u^q dv_g,$$

and then $J_q(u) \geq 0$ since $h \geq 0$. Moreover $\lim_{n \rightarrow +\infty} J_q(u_N) = \beta$. Then (41) yields

$$(42) \quad c_{n,k,s} \int_M (|\nabla u_N|_g^2 - |\nabla u|_g^2) dv_g \leq \beta + o(1)$$

By weak convergence of u_N , we have that

$$\begin{aligned} \int_M |\nabla(u_N - u)|_g^2 dv_g &= \int_M |\nabla u_N|_g^2 dv_g + \int_M |\nabla u|_g^2 dv_g - 2 \int_M (\nabla u, \nabla \phi)_g dv_g \\ (43) \quad &= \int_M (|\nabla u_N|_g^2 - |\nabla u|_g^2) dv_g + o(1) \end{aligned}$$

Combining (42) and (43) yields (38).

Step IV: It remains to show that

$$\lim_{N \rightarrow \infty} \int_M |\nabla u_N|_g^2 dv_g = \int_M |\nabla u|_g^2 dv_g$$

It follows from Theorem 3 that for all $\epsilon > 0$ there exists $B_\epsilon > 0$ such that

$$\left(\int_M \frac{|v|^{2^*(k,s)}}{d_g(x, Gx_0)^s} dv_g \right)^{\frac{2}{2^*(k,s)}} \leq \left(\frac{K_s(n-k)}{A^{1-\frac{2}{2^*(k,s)}}} + \epsilon \right) \|\nabla v\|_2^2 + B_\epsilon \|v\|_2^2.$$

for all $u \in H_{1,G}^2(M)$. Let us test this inequality for $v = u_N - u$, using the fact that u_N converges to u strongly in $L^2(M)$

$$\|u_N - u\|_{L^{2^*(k,s)}(M, \frac{dv_g}{d_g(x, Gx_0)^s})}^{2^*(k,s)} \leq \left(\frac{K_s(n-k)}{A^{1-\frac{2}{2^*(k,s)}}} + \epsilon \right)^{\frac{2^*(k,s)}{2}} \|\nabla(u_N - u)\|_2^{2^*(k,s)} + o(1)$$

Let us use this result with (37)

$$\begin{aligned}
o(1) &= \|\nabla(u_N - u)\|_2^2 - \int_M |u_N - u|^{2^*(k,s)} \frac{dv_g}{d_g(x, Gx_0)^s} \\
&= \|\nabla(u_N - u)\|_2^2 \left[1 - \left(\int_M |u_N - u|^{2^*(k,s)} \frac{dv_g}{d_g(x, Gx_0)^s} \right) \|\nabla(u_N - u)\|_2^{-2} \right] \\
&\geq \|\nabla(u_N - u)\|_2^2 \left[1 - \left(\frac{K_s(n-k)}{A^{1-\frac{2}{2^*(k,s)}}} + \epsilon \right)^{\frac{2^*(k,s)}{2}} \|\nabla(u_N - u)\|_2^{2^*(k,s)-2} + o(1) \right]
\end{aligned}$$

this inequality and (38) imply

$$o(1) \geq \|\nabla(u_k - u)\|_2^2 \left[1 - \left(\frac{K_s(n-k)}{A^{1-\frac{2}{2^*(k,s)}}} + \epsilon \right)^{\frac{2^*(k,s)}{2}} \left(\frac{\beta}{c_{n,k,s}} \right)^{\frac{2^*(k,s)-2}{2}} + o(1) \right]$$

Like $\beta < c_{n,k,s} \frac{K_s(n-k)}{A^{1-\frac{2}{2^*(k,s)}}}^{-\frac{n-k-s}{2-s}}$, then it follows from the previous inequality that for ϵ small enough, there exists $C > 0$ such that $\|\nabla(u_N - u)\|_2^2 \leq o(1)$. Hence the result. This proves Proposition 16. \square

We are now in position to prove Theorem 6.

Proof of Theorem 6: We do the proof in steps:

Step I: Suppose there is $u_0 \in H_{1,G}^2(M)$, $u_0 \neq 0$, $u_0 \geq 0$ such that (9) holds. It follows from Propositions 14 and 16 that there exists $u \in H_{1,G}^2(M) \setminus \{0\}$ such that $J_q(u) = \beta > 0$, hence $u \neq 0$, and $J'_q(u) = 0$, which satisfies (2) in the weak sense.

Step II: (Regularity) Arguing as in Ghoussoub-Robert [8] or Jaber [11], we get that $u \in C^{0,\alpha} \cap C_{loc}^{1,\nu}(M \setminus \{Gx_0\})$, for all $\alpha \in (0, \min\{1, 2-s\})$, $\nu \in (0, 1)$. We refer to Mesmar [14] for a detailed proof.

Step III: (Positivity) For all $u \in H_{1,G}^2(M)$ we set $u_+(x) := \max\{u(x), 0\}$ and $u_-(x) := \max\{-u(x), 0\}$ for a.e. $x \in M$. Doing the same analysis as for J_q , we show that $(J_q^+)'(u) \equiv 0$, where for all $v \in H_{1,G}^2(M)$ we set

$$\begin{aligned}
J_q^+(v) &= \frac{1}{2} \int_M (|\nabla v|_g^2 + av^2) dv_g \\
&\quad - \frac{1}{2^*(k,s)} \int_M \frac{|v_+|^{2^*(k,s)-1} v_+}{d_g(x, Gx_0)^s} dv_g - \frac{1}{q} \int_M h |v_+|^{q-1} v_+ dv_g
\end{aligned}$$

In particular, $\langle (J_q^+)'(u), u_- \rangle = 0$, hence

$$0 = - \int_M (|\nabla u_-|_g^2 + au_-^2) dv_g + \int_M |u_+|^{2^*(k,s)-1} \frac{u_-}{d_g(x, Gx_0)^s} dv_g + \int_M h |v_+|^{q-1} u_- dv_g$$

Therefore $\int_M (|\nabla u_-|_g^2 + au_-^2) dv_g = 0$. Using that $\Delta_g + a$ is coercive, we get that $\|u_-\|_{H_{1,G}^2(M)} = 0$, so $u \geq 0$. By the maximum principle for elliptic operators [3], we get that $u > 0$ on $M \setminus Gx_0$. This proves Theorem 6. \square

6. PROOF OF THEOREMS 7, 4 AND 5

6.1. **Test-function estimates.** To go below the threshold (9), we construct test-functions modeled on the extremal for the Hardy-Sobolev inequality (5) given by

$$(44) \quad \phi : X \in \mathbb{R}^m \mapsto (1 + |X|^{2-s})^{-\frac{m-2}{2-s}}, \quad m := n - k.$$

Given (M, g) and G which satisfy the Assumption 8 of Section 2. We define the test functions $(\bar{u}_\epsilon)_{\epsilon > 0}$ for all $\epsilon > 0$, $\bar{x} \in N := B_\delta(Gx_0)/G'$ by

$$(45) \quad \bar{u}_\epsilon(\bar{x}) := \left(\frac{\epsilon^{\frac{2-s}{2}}}{\epsilon^{2-s} + d_{\tilde{g}}(\bar{x}, \bar{x}_0)^{2-s}} \right)^{\frac{m-2}{2-s}} \quad \text{where } m := n - k \geq 3$$

where \tilde{g} is defined in (10).

Lemma 17. *Let (X, g) be a compact Riemannian manifold of dimension $m \geq 3$, and $(B_{\rho_0}(x_0), \exp_{x_0}^{-1})$ an exponential ball centered at x_0 and of radius $\rho \in (0, i_g(X))$. So setting $x = \exp_{x_0}(r\theta)$, for all $x \in B_{\rho_0}(x_0)$, we have that*

$$(46) \quad \int_{S^{m-1}} f \circ \exp_{x_0}(x) \sqrt{\det(g)}(x) d\theta = w_{m-1} \left[f(x_0) - \left(\frac{\Delta_g f(x_0)}{2m} + \frac{f(x_0) \text{Scal}_g(x_0)}{6m} \right) \right] + O(r^3)$$

For all $f \in C^0(X)$, where $r = d_g(x, x_0)$ and $d\theta$ and w_{m-1} are respectively the area element and the volume of the unit sphere S^{m-1} of \mathbb{R}^m .

Proof. Using the Cartan expansion of the metric g to order 2 (for the proof see Hebey [9]) for any point $x \in B_{\rho_0}(x_0)$, that is

$$g_{ij}(x) = \delta_{ij} + \frac{1}{3} R_{i\alpha\beta j} x^\alpha x^\beta + O(r^3).$$

With this expansion and a Taylor expansion of f , straightforward computations yield the result. \square

Remarque 18. *In the exponential map $(N, \exp_{\bar{x}_0}^{-1})$, we pass to the polar coordinates (r, θ) such that for all $\bar{x} \in N$, we have $r = d_{\tilde{g}}(\bar{x}, \bar{x}_0)$ and $\theta \in S^{m-1}$. So we have for all $\bar{x} \in B_{\rho_0}(\bar{x}_0)$*

$$(47) \quad dv_{\tilde{g}}(\bar{x}) = \sqrt{\det(\tilde{g})}(\bar{x}) d\bar{x} = \sqrt{\det(\tilde{g})}(\bar{x}) r^{m-1} d\theta$$

with $d\bar{x} = (\exp_{\bar{x}_0}^{-1})^*(dX)$, and with Gauss's Lemma, we have that $\tilde{g}_{1i}(\bar{x}) = \delta_{1i}$.

Lemma 19. *For ϕ as in (44), we have that*

$$\|\nabla \bar{u}_\epsilon\|_{L^2(N)}^2 = \begin{cases} \int_{\mathbb{R}^m} |\nabla \phi|^2 dX - \frac{\int_{\mathbb{R}^m} |X|^2 |\nabla \phi|^2 dX}{6m} \text{Scal}_{\tilde{g}}(\bar{x}_0) \epsilon^2 + o(\epsilon^2) & \text{if } m \geq 5 \\ \int_{\mathbb{R}^m} |\nabla \phi|^2 dX - \frac{w_3}{6} \text{Scal}_{\tilde{g}}(\bar{x}_0) \epsilon^2 \ln\left(\frac{1}{\epsilon}\right) + O(\epsilon^2) & \text{if } m = 4 \end{cases}$$

Proof. Since \bar{u}_ϵ is a radial function, so by the Gauss Lemma above, for all $\bar{x} \in N$

$$\begin{aligned} |\nabla \bar{u}_\epsilon|_{\tilde{g}}^2 &= \tilde{g}^{ij} \nabla_i \bar{u}_\epsilon \nabla_j \bar{u}_\epsilon = \tilde{g}^{rr} (\nabla_r \bar{u}_\epsilon)^2 \\ &= (m-2)^2 \epsilon^{m-2} \frac{r^{2(1-s)}}{(\epsilon^{2-s} + r^{2-s})^{\frac{2(m-s)}{2-s}}} \end{aligned}$$

We integrate $|\nabla \bar{u}_\epsilon|_{\bar{g}}^2$ over N using (47) and Cartan expansion (46) and by changing the variable $r = \epsilon\rho$, we write

$$\begin{aligned}
\int_N |\nabla \bar{u}_\epsilon|_{\bar{g}}^2 dv_{\bar{g}} &= (m-2)^2 \epsilon^{m-2} w_{n-k-1} \int_0^{\rho_0} r^{m+1} \frac{\left(1 - \frac{\text{Scal}_{\bar{g}}(\bar{x}_0)}{6m} r^2 + O(r^3)\right)}{r^{2s} (\epsilon^{2-s} + r^{2-s})^{\frac{2(m-s)}{2-s}}} dr \\
&= (m-2)^2 w_{m-1} \int_0^{\frac{\rho_0}{\epsilon}} \rho^{m+1} \frac{\left(1 - \frac{\text{Scal}_{\bar{g}}(\bar{x}_0)}{6m} (\epsilon\rho)^2 + O((\epsilon\rho)^3)\right)}{\rho^{2s} (1 + \rho^{2-s})^{\frac{2(m-s)}{2-s}}} d\rho \\
(48) \quad &:= A(\epsilon) - \frac{(m-2)^2}{6m} \text{Scal}_{\bar{g}}(\bar{x}_0) w_{m-1} B(\epsilon) + C(\epsilon)
\end{aligned}$$

where

$$\begin{aligned}
A(\epsilon) &:= (m-2)^2 w_{m-1} \int_0^{\frac{\rho_0}{\epsilon}} \frac{\rho^{m+1} d\rho}{\rho^{2s} (1 + \rho^{2-s})^{\frac{2(m-s)}{2-s}}} \\
B(\epsilon) &= \epsilon^2 \int_0^{\frac{\rho_0}{\epsilon}} \frac{\rho^{m+3} d\rho}{\rho^{2s} (1 + \rho^{2-s})^{\frac{2(m-s)}{2-s}}} \text{ and } C(\epsilon) = O\left(\int_0^{\frac{\rho_0}{\epsilon}} \frac{\rho^{m+1} (\epsilon\rho)^3 d\rho}{\rho^{2s} (1 + \rho^{2-s})^{\frac{2(m-s)}{2-s}}}\right)
\end{aligned}$$

Step I: We first compute $A(\epsilon)$:

$$\begin{aligned}
(49) \quad A(\epsilon) &= (m-2)^2 w_{m-1} \left[\int_0^{+\infty} \frac{\rho^{m+1} d\rho}{\rho^{2s} (1 + \rho^{2-s})^{\frac{2(m-s)}{2-s}}} - \int_{\frac{\rho_0}{\epsilon}}^{+\infty} \dots \right] \\
&= \int_{\mathbb{R}^m} |\nabla \phi|^2 dX - (m-2)^2 w_{m-1} \int_{\frac{\rho_0}{\epsilon}}^{+\infty} \frac{\rho^{m+1} d\rho}{\rho^{2s} (1 + \rho^{2-s})^{\frac{2(m-s)}{2-s}}} \\
&= \int_{\mathbb{R}^m} |\nabla \phi|^2 dX + O(\epsilon^{m-2})
\end{aligned}$$

Step II: On the other hand, $\rho \mapsto \frac{\rho^{m+3}}{\rho^{2s} (1 + \rho^{2-s})^{\frac{2(m-s)}{2-s}}} \in L^1(\mathbb{R}^m)$ as soon as $m \geq 5$.

Therefore for $m \geq 5$, we have that

$$\begin{aligned}
B(\epsilon) &= \epsilon^2 \int_0^{\frac{\rho_0}{\epsilon}} \frac{\rho^{m+3} d\rho}{\rho^{2s} (1 + \rho^{2-s})^{\frac{2(m-s)}{2-s}}} - \epsilon^2 \int_{\frac{\rho_0}{\epsilon}}^{+\infty} \frac{\rho^{m+3} d\rho}{\rho^{2s} (1 + \rho^{2-s})^{\frac{2(m-s)}{2-s}}} \\
&= \epsilon^2 (m-2)^{-2} w_{m-1}^{-1} \int_{\mathbb{R}^m} |X|^2 |\nabla \phi|^2 dX + O(\epsilon^{m-2})
\end{aligned}$$

When $m = 4$:

$$\begin{aligned}
B(\epsilon) &= \epsilon^2 \int_0^1 \frac{\rho^7 d\rho}{\rho^{2s} (1 + \rho^{2-s})^{\frac{2(4-s)}{2-s}}} + \epsilon^2 \int_1^{\frac{\rho_0}{\epsilon}} \frac{\rho^7 d\rho}{\rho^{2s} (1 + \rho^{2-s})^{\frac{2(4-s)}{2-s}}} \\
&= O(\epsilon^2) + \epsilon^2 \left(\int_1^{\frac{\rho_0}{\epsilon}} \frac{d\rho}{\rho} + \int_1^{\frac{\rho_0}{\epsilon}} \frac{\rho^7}{\rho^{2s}} \left[\frac{1}{(1 + \rho^{2-s})^{\frac{2(4-s)}{2-s}}} - \frac{1}{(\rho^{2-s})^{\frac{2(4-s)}{2-s}}} \right] d\rho \right) \\
&= O(\epsilon^2) + \epsilon^2 \ln\left(\frac{\rho_0}{\epsilon}\right) + \epsilon^2 \int_1^{\frac{\rho_0}{\epsilon}} \frac{\rho^{7-2s} O\left(\frac{1}{\rho^{2-s}}\right)}{(1 + \rho^{2-s})^{\frac{2(4-s)}{2-s}}} d\rho
\end{aligned}$$

And then

$$(50) \quad B(\epsilon) = \begin{cases} \epsilon^2 (m-2)^2 w_{m-1}^{-1} \int_{\mathbb{R}^m} |X|^2 |\nabla \phi|^2 dX + O(\epsilon^{m-2}) & \text{if } m \geq 5 \\ O(\epsilon^2) + \epsilon^2 \ln\left(\frac{1}{\epsilon}\right) & \text{if } m = 4 \end{cases}$$

Step III: Finally straightforward computations yield

$$(51) \quad C(\epsilon) = \begin{cases} o(\epsilon^2) & \text{if } m \geq 5 \\ O(\epsilon^2) & \text{if } m = 4 \end{cases}$$

Grouping (49), (50) and (51) in (48), we get the expected estimate. \square

Lemma 20. *Let $f \in C^2(N)$ be such that $\partial_i f(\bar{x}_0) = 0$ for $i = 1, \dots, m$, $m := n - k \geq 4$, then*

$$\begin{aligned} \int_N \frac{f|\bar{u}_\epsilon|^{2^*(k,s)}}{d_{\bar{g}}(\bar{x}, \bar{x}_0)^s} dv_{\bar{g}} &= f(x_0) \int_{\mathbb{R}^{n-k}} \frac{|\phi|^{2^*(k,s)}}{|X|^s} dX \\ &\quad - \epsilon^2 \left(\frac{\Delta_{\bar{g}} f(x_0)}{2m} + \frac{f(x_0)}{6m} \text{Scal}_{\bar{g}} \right) \int_{\mathbb{R}^{n-k}} |X|^{2-s} |\phi|^{2^*(k,s)} dX \\ &\quad + o(\epsilon^2) \end{aligned}$$

Proof. We compute $\|\bar{u}_\epsilon\|_{L^{2^*(k,s)}(N, d_{\bar{g}}(\bar{x}, \bar{x}_0)^{-s} dv_{\bar{g}})}^{2^*(k,s)}$ using (47) and the Cartan expansion (46) and performing a change of variable $r = \epsilon\rho$

$$(52) \quad \begin{aligned} \int_N f|\bar{u}_\epsilon|^{2^*(k,s)} \frac{dv_{\bar{g}}}{d_{\bar{g}}(\bar{x}, \bar{x}_0)^s} &= w_{m-1} \int_0^{\frac{\rho_0}{\epsilon}} \frac{\rho^{m-1} (f(\bar{x}_0) - D_{g,f}(\epsilon\rho)^2 + O((\epsilon\rho)^3))}{\rho^s (1 + \rho^{2-s})^{\frac{2(m-s)}{2-s}}} d\rho \\ &= f(\bar{x}_0)A'(\epsilon) - D_{g,f}w_{m-1}B'(\epsilon) + O(C'(\epsilon)) \end{aligned}$$

where

$$\begin{aligned} D_{g,f} &:= \frac{\Delta_{\bar{g}} f(x_0)}{2m} + \frac{f(x_0)}{6m} \text{Scal}_{\bar{g}}; \quad A'(\epsilon) := w_{m-1} \int_0^{\frac{\rho_0}{\epsilon}} \frac{\rho^{m-1} d\rho}{\rho^s (1 + \rho^{2-s})^{\frac{2(m-s)}{2-s}}}; \\ B'(\epsilon) &:= \epsilon^2 \int_0^{\frac{\rho_0}{\epsilon}} \frac{\rho^{m+3} d\rho}{\rho^s (1 + \rho^{2-s})^{\frac{2(m-s)}{2-s}}}; \quad C'(\epsilon) := \int_0^{\frac{\rho_0}{\epsilon}} \frac{\rho^{m-1} (\epsilon\rho)^3 d\rho}{\rho^s (1 + \rho^{2-s})^{\frac{2(m-s)}{2-s}}} \end{aligned}$$

Arguing as in the proof of Lemma 19, we get that

$$A'(\epsilon) = \int_{\mathbb{R}^{n-k}} \frac{|\phi|^{2^*(k,s)}}{|X|^s} dX + O(\epsilon^{m-s})$$

$$B'(\epsilon) = \epsilon^2 w_{m-1}^{-1} \int_{\mathbb{R}^m} |X|^{2-s} |\phi|^{2^*(k,s)} dX + O(\epsilon^{m-s})$$

and

$$C'(\epsilon) = \begin{cases} O(\epsilon^3) & \text{if } m \geq 5; m = 4 \text{ and } s < 1 \\ O(\epsilon^3 \ln(\frac{1}{\epsilon})) & \text{if } m = 4 \text{ and } s = 1 \\ O(\epsilon^{m-s}) & \text{if } m = 4 \text{ and } s > 1 \end{cases}$$

By gathering these identities in (52), we will have the announced result. \square

Arguing as in the above proof of Lemma 19, we get that:

Lemma 21. *For all $g \in C^0(N)$, and $q \in (0, 2^*(k))$ we have*

$$\int_N g|\bar{u}_\epsilon|^q dv_{\bar{g}} = \begin{cases} \epsilon^{m - \frac{q(m-2)}{2}} g(\bar{x}_0) \int_{\mathbb{R}^m} |\phi|^q dX + o(\epsilon^{m - \frac{q(m-2)}{2}}) & \text{if } q > \frac{2^*(k)}{2} \\ \epsilon^{\frac{m}{2}} \ln(\frac{1}{\epsilon}) g(\bar{x}_0) w_{m-1} + O(\epsilon^{\frac{m}{2}} \ln \frac{1}{\epsilon}) & \text{if } q = \frac{2^*(k)}{2} \\ O\left(\epsilon^{\frac{q(m-2)}{2}(m-2)}\right) & \text{if } q < \frac{2^*(k)}{2} \end{cases}$$

In particular, when $q = 2$, we get that

$$\int_N g \bar{u}_\epsilon^2 dv_{\bar{g}} = \begin{cases} \epsilon^2 g(\bar{x}_0) \int_{\mathbb{R}^{n-k}} |\phi|^2 dX + o(\epsilon^2) & \text{si } m \geq 5 \\ g(\bar{x}_0) w_3 \epsilon^2 \ln\left(\frac{1}{\epsilon}\right) + O(\epsilon^2) & \text{if } m = 4 \end{cases}$$

Lemma 22. For $m = n - k \geq 5$, and ϕ as in (44), we define

$$\begin{aligned} k_1 &:= \frac{\int_{\mathbb{R}^m} |\phi|^2 dX}{\int_{\mathbb{R}^m} |\nabla \phi|^2 dX} \\ k_2 &:= -\frac{1}{2^*(k, s) 3m} \frac{\int_{\mathbb{R}^m} |X|^{2-s} |\phi|^{2^*(k, s)} dX}{\int_{\mathbb{R}^m} \frac{|\phi|^{2^*(k, s)}}{|X|^s} dX} + \frac{1}{6m} \frac{\int_{\mathbb{R}^m} |X|^2 |\nabla \phi|^2 dX}{\int_{\mathbb{R}^m} |\nabla \phi|^2 dX} \\ k_3 &:= \frac{1}{2^*(k, s) m} \frac{\int_{\mathbb{R}^m} |X|^{2-s} |\phi|^{2^*(k, s)} dX}{\int_{\mathbb{R}^m} \frac{|\phi|^{2^*(k, s)}}{|X|^s} dX} \end{aligned}$$

We have that

$$(53) \quad \frac{k_2}{k_1} = c'_{n, k, s} = \frac{(m-2)(6-s)}{12(2m-2-s)} \text{ and } \frac{k_3}{k_1} = c''_{n, k, s} = \frac{(m-2)(m-4)}{4(2m-2-s)}.$$

Proof. Let $p, q > 0$ such that $p - q > 1$. We consider $I_p^q = \int_0^{+\infty} \frac{t^q dt}{(1+t)^p}$, an integration by parts shows that $I_p^q = \frac{p}{q+1} I_{p+1}^{q+1}$. Moreover, one can easily show that $I_p^q = I_{p+1}^q + I_{p+1}^{q+1}$. These two relations allow us to find the following results which will be useful for the calculation of our main ratios:

$$(54) \quad I_{p+1}^q = \frac{p-q-1}{p} I_p^q \text{ and } I_{p+1}^{q+1} = \frac{q+1}{p-q-1} I_{p+1}^q$$

Now, we will calculate the ratio $\frac{k_2}{k_1}$ using (54). Passing in polar coordinates (r, θ) , and setting $t = r^{2-s}$ such that $dr = \frac{1}{2-s} t^{-\frac{1-s}{2-s}} dt$, we will have

$$(55) \quad \int_{\mathbb{R}^m} |\phi|^2 dX = w_{m-1} \int_0^{+\infty} \frac{r^{m-1} dr}{(1+r^{2-s})^{\frac{2(m-2)}{2-s}}} = \frac{w_{m-1}}{2-s} \int_0^{+\infty} \frac{t^{\frac{m}{2-s}-1}}{(1+t)^{\frac{2(m-2)}{2-s}}} dt$$

Similarly, we can prove that

$$(56) \quad \int_{\mathbb{R}^m} |X|^2 |\nabla \phi|^2 dX = \frac{(m-2)^2 w_{m-1}}{2-s} \int_0^{+\infty} \frac{t^{\frac{m}{2-s}+1}}{(1+t)^{\frac{2(m-2)}{2-s}+2}} dt$$

Using (55) and (56) and (54), for $q = \frac{m}{2-s} - 1$, $p = \frac{2(m-2)}{2-s}$, we will have

$$(57) \quad \frac{\int_{\mathbb{R}^m} |X|^2 |\nabla \phi|^2 dX}{\int_{\mathbb{R}^m} |\phi|^2 dX} = \frac{\frac{(m-2)^2}{2-s} w_{m-1} I_{p+2}^{q+2}}{\frac{w_{m-1}}{2-s} I_p^q} = \frac{m(m-2)(m+2-s)}{2(2m-2-s)}$$

In a similar way, we will have for $p = \frac{2(m-2)}{2-s}$, $q = \frac{m}{2-s} - 1$

$$(58) \quad \frac{\int_{\mathbb{R}^m} |X|^{2-s} |\phi|^{2^*(k, s)} dX}{\int_{\mathbb{R}^m} |\phi|^2 dX} = \frac{I_{p+2}^{q+1}}{I_p^q} = \frac{m(m-4)}{2(m-2)(2m-2-s)}$$

and for $p = \frac{2(m-s)}{2-s}$, $q = \frac{m-s}{2-s} - 1$

$$(59) \quad \frac{\int_{\mathbb{R}^m} |\nabla \phi|^2 dX}{\int_{\mathbb{R}^m} \frac{|\phi|^{2^*(k, s)}}{|X|^s} dX} = (m-2)^2 \frac{q+1}{p-1-q-1} = (m-2)(m-s)$$

By combining (57), (58), and (59), we get (53). \square

6.2. Proof of Theorem 7. We now go back to the initial manifold (M, g) . For $\rho > 0$, we let $N := B_{Gx_0}(\rho_0)/G' = B_{\bar{x}_0}(\rho_0)$ that is a Riemmanian manifold of dimension $m \geq 4$. We take the family $(\bar{u}_\epsilon)_{\epsilon > 0}$ defined as on (45), and we define

$$u_\epsilon(x) = \begin{cases} \left(\frac{\epsilon^{\frac{2-s}{2}}}{\epsilon^{2-s} + d_{\bar{g}}(\bar{x}, \bar{x}_0)^{2-s}} \right)^{\frac{m-2}{2-s}} & \text{if } x \in B_{\rho_0}(Gx_0) \\ \left(\frac{\epsilon^{\frac{2-s}{2}}}{\epsilon^{2-s} + \rho_0^{2-s}} \right)^{\frac{m-2}{2-s}} & \text{if } x \in M \setminus B_{\rho_0}(Gx_0) \end{cases}$$

Note that $u_\epsilon \in H_{1,G}^2(M)$. Let J_q be as in (8). For all $\epsilon > 0$, and $t \geq 0$, we get

$$\begin{aligned} J_q(tu_\epsilon) &= \frac{t^2}{2} \int_M (|\nabla u_\epsilon|_g^2 + au_\epsilon^2) dv_g - \frac{t^{2^*(k,s)}}{2^*(k,s)} \int_M \frac{u_\epsilon^{2^*(k,s)}}{d_g(x, Gx_0)^s} dv_g - \frac{t^q}{q} \int_M hu_\epsilon^q dv_g \\ (60) \quad &= \frac{t^2}{2} A_\epsilon - \frac{t^{2^*(k,s)}}{2^*(k,s)} B_\epsilon - \frac{t^q}{q} C_\epsilon \end{aligned}$$

Where

$$\begin{aligned} A_\epsilon &:= \int_N (|\nabla \bar{u}_\epsilon|_{\bar{g}}^2 + \bar{a}\bar{u}_\epsilon^2) dv_{\bar{g}} + \int_{M \setminus B_{\rho_0}(Gx_0)} (|\nabla u_\epsilon|_g^2 + au_\epsilon^2) dv_g \\ B_\epsilon &:= \int_N \bar{f} |\bar{u}_\epsilon|^{2^*(k,s)} \frac{dv_{\bar{g}}}{d_{\bar{g}}(\bar{x}, \bar{x}_0)^s} + \int_{M \setminus B_{\rho_0}(Gx_0)} u_\epsilon^{2^*(k,s)} \frac{dv_g}{d_g(x, Gx_0)^s} \\ C_\epsilon &:= \int_N \bar{h} |\bar{u}_\epsilon|^q dv_{\bar{g}} + \int_{M \setminus B_{\rho_0}(Gx_0)} hu_\epsilon^q dv_g \end{aligned}$$

with $\bar{h}(\bar{x}) = \bar{v}^{-\frac{2}{n-k-2}} h(\Pi(x))$, $\bar{a}(\bar{x}) = \bar{v}^{-\frac{2}{n-k-2}} a(\Pi(x))$, and $\bar{f}(\bar{x}) = \bar{v}^{-\frac{2-s}{n-k-2}} (\Pi(x))$. Outside the ball centered in Gx_0 , we have that

$$\begin{aligned} \int_{M \setminus B_{\rho_0}(Gx_0)} (|\nabla u_\epsilon|_g^2 + au_\epsilon^2) dv_g &= O(\epsilon^{m-2}) \\ \int_{M \setminus B_{\rho_0}(Gx_0)} u_\epsilon^{2^*(k,s)} \frac{dv_g}{d_g(x, Gx_0)^s} &= O(\epsilon^{m-s}) \\ \int_{M \setminus B_{\rho_0}(Gx_0)} hu_\epsilon^q dv_g &= O(\epsilon^{\frac{q(m-2)}{2}}) \end{aligned}$$

By using the equalities above, and Lemmas 19 and 20 we get

$$A_\epsilon = \begin{cases} \int_{\mathbb{R}^m} |\nabla \phi|^2 dX - \frac{\int_{\mathbb{R}^m} |X|^2 |\nabla \phi|^2 dX}{6m} \text{Scal}_{\bar{g}}(\bar{x}_0) \epsilon^2 + \epsilon^2 \bar{a}(\bar{x}_0) \int_{\mathbb{R}^m} |\phi|^2 dX + o(\epsilon^2) & \text{if } m \geq 5 \\ \int_{\mathbb{R}^4} |\nabla \phi|^2 dX - \frac{w_3}{6} \text{Scal}_{\bar{g}}(\bar{x}_0) \epsilon^2 \ln\left(\frac{1}{\epsilon}\right) + \bar{a}(\bar{x}_0) w_3 \epsilon^2 \ln\left(\frac{1}{\epsilon}\right) + o(\epsilon^2 \ln\left(\frac{1}{\epsilon}\right)) & \text{if } m = 4 \end{cases}$$

$$B_\epsilon = \bar{f}(\bar{x}_0) \int_{\mathbb{R}^m} \frac{|\phi|^{2^*(k,s)}}{|X|^s} dX - \epsilon^2 \left(\frac{\Delta_{\bar{g}} \bar{f}(\bar{x}_0)}{2m} + \frac{\bar{f}(\bar{x}_0)}{6m} \text{Scal}_{\bar{g}} \right) \int_{\mathbb{R}^m} |X|^{2-s} |\phi|^{2^*(k,s)} dX + o(\epsilon^2)$$

$$C_\epsilon = \begin{cases} \epsilon^{m - \frac{q(m-2)}{2}} \bar{h}(\bar{x}_0) \int_{\mathbb{R}^m} |\phi|^q dX + o\left(\epsilon^{m - \frac{q(m-2)}{2}}\right) & \text{si } q > \frac{2^*(k)}{2} \\ \epsilon^{\frac{m}{2}} \ln\left(\frac{1}{\epsilon}\right) \bar{h}(\bar{x}_0) w_{m-1} + o(\epsilon^2 \ln\left(\frac{1}{\epsilon}\right)) & \text{si } q = \frac{2^*(k)}{2} \\ O\left(\epsilon^{\frac{q(m-2)}{2}}\right) & \text{si } q < \frac{2^*(k)}{2} \end{cases}$$

Hence, when $\epsilon \rightarrow 0$

$$A_\epsilon \rightarrow A_0 := \int_{\mathbb{R}^m} |\nabla \phi|^2 dX, \quad B_\epsilon \rightarrow B_0 := f(x_0) \int_{\mathbb{R}^m} \frac{|\phi|^{2^*(k,s)}}{|X|^s} dX, \quad \text{et } C_\epsilon \rightarrow 0$$

Step I: We prove that for all $\epsilon > 0$, there exists a unique $t_\epsilon > 0$ such that $\sup_{t \geq 0} J_q(tu_\epsilon) = J_q(t_\epsilon u_\epsilon)$, and that

$$(61) \quad t_\epsilon = T_\epsilon(1 - \alpha_0 C_\epsilon + o(C_\epsilon)),$$

where $T_\epsilon = (A_\epsilon B_\epsilon^{-1})^{\frac{1}{2^*(k,s)-2}}$, $\alpha_0 > 0$, and the sequence t_ϵ converges and its limit t_0 verifies $t_0^{2^*(k,s)} B_0 = A_0$.

Indeed, $\frac{dJ_q(tu_\epsilon)}{dt}$ vanishes for $t = 0$ or else for $t_\epsilon > 0$ such that $A_\epsilon = f_\epsilon(t_\epsilon)$, where $f_\epsilon(t) := t^{2^*(k,s)-2} B_\epsilon + t^{q-2} C_\epsilon$. Since $A_\epsilon > 0$, $B_\epsilon, C_\epsilon \geq 0$, $C_\epsilon > 0$, a straightforward analysis of f_ϵ yields $t_\epsilon \rightarrow t_0 := (A_0 B_0^{-1})^{\frac{1}{2^*(k,s)-2}}$ when $\epsilon \rightarrow 0$. Furthermore, we have that

$$\begin{aligned} t_\epsilon &= (A_\epsilon B_\epsilon^{-1} - t_\epsilon^{q-2} C_\epsilon B_\epsilon^{-1})^{\frac{1}{2^*(k,s)-2}} \\ &= T_\epsilon (1 - t_\epsilon^{q-2} A_\epsilon^{-1} C_\epsilon)^{\frac{1}{2^*(k,s)-2}} = T_\epsilon (1 - \alpha_0 C_\epsilon + o(C_\epsilon)) \end{aligned}$$

With $\alpha_0 = \frac{t_0^{q-2} A_0^{-1}}{2^*(k,s)-2}$. This completes our step.

Step II: We show that with subsequence near $(u_\epsilon)_{\epsilon > 0}$, we have

$$(62) \quad J_q(t_\epsilon u_\epsilon) = c_{n,k,s} (I(u_\epsilon))^{\frac{2^*(k,s)}{2^*(k,s)-2}} - \frac{T_0^q}{q} C_\epsilon + o(C_\epsilon)$$

With $c_{n,k,s} = \frac{2-s}{2(n-k-s)}$ and

$$I(u_\epsilon) = \frac{\int_M (|\nabla u_\epsilon|^2 + a u_\epsilon) dv_g}{\left(\int_M |u_\epsilon|^{2^*(k,s)} \frac{dv_g}{d_g(x, Gx_0)^s} \right)^{\frac{2}{2^*(k,s)}}}$$

Indeed, the expansions (60) and (61) of $J_q(tu_\epsilon)$ and t_ϵ imply

$$\begin{aligned} J_q(t_\epsilon u_\epsilon) &= \frac{t_\epsilon^2}{2} A_\epsilon - \frac{t_\epsilon^{2^*(k,s)}}{2^*(k,s)} B_\epsilon - \frac{t_\epsilon^q}{q} C_\epsilon \\ &= \frac{T_\epsilon^2}{2} A_\epsilon - \frac{T_\epsilon^{2^*(k,s)}}{2^*(k,s)} B_\epsilon + \alpha_0 (-T_\epsilon^2 A_\epsilon + T_\epsilon^{2^*(k,s)} B_\epsilon + T_\epsilon^q C_\epsilon) C_\epsilon - \frac{T_\epsilon^q}{q} C_\epsilon + o(C_\epsilon) \end{aligned}$$

The explicit definition of T_ϵ then yields (62).

Step III: We prove that

$$\begin{aligned} \frac{K_s(n-k)}{A^{1-\frac{2}{2^*(k,s)}}} I(u_\epsilon) &= 1 + \\ &\begin{cases} \left(k_1 \bar{a}(\bar{x}_0) - k_2 \text{Scal}_{\bar{g}}(\bar{x}_0) + k_3 \frac{\Delta_{\bar{g}} \bar{f}(\bar{x}_0)}{\bar{f}(\bar{x}_0)} \right) \epsilon^2 + o(\epsilon^2) & \text{si } m \geq 5, \\ w_3 \left(\int_{\mathbb{R}^4} |\nabla \phi|^2 dX \right)^{-1} \left(\bar{a}(\bar{x}_0) - \frac{1}{6} \text{Scal}_{\bar{g}} \right) \epsilon^2 \ln\left(\frac{1}{\epsilon}\right) + o\left(\epsilon^2 \ln\left(\frac{1}{\epsilon}\right)\right) & \text{if } m = 4, \end{cases} \end{aligned}$$

Where k_1, k_2, k_3 are constants as in Lemma 22. We distinguish 2 cases:

Case III.a: $m = 4$. In this case we have

$$I(u_\epsilon) = \frac{C_1 + C_2 w_3 \epsilon^2 \ln\left(\frac{1}{\epsilon}\right) + o\left(\epsilon^2 \ln\left(\frac{1}{\epsilon}\right)\right)}{(C_3 + C_4 \epsilon^2 + o(\epsilon^2))^{\frac{2}{2^*(k,s)}}}$$

with

$$C_1 = \int_{\mathbb{R}^4} |\nabla \phi|^2 dX; \quad C_2 = \bar{a}(\bar{x}_0) - \frac{1}{6} \text{Scal}_{\bar{g}}(\bar{x}_0); \quad C_3 = \bar{f}(\bar{x}_0) \int_{\mathbb{R}^4} \frac{|\phi|^{2^*(k,s)}}{|X|^s} dX;$$

$$C_4 = - \left(\frac{\bar{f}(\bar{x}_0)}{24} \text{Scal}_{\bar{g}}(\bar{x}_0) + \frac{\Delta_{\bar{g}} \bar{f}(\bar{x}_0)}{8} \right) \int_{\mathbb{R}^{n-k}} |X|^{2-s} |\phi|^{2^*(k,s)} dX.$$

It then follows that:

$$I(u_\epsilon) = \frac{1}{C_3^{\frac{2}{2^*(k,s)}}} \left[\frac{C_1 + C_2 w_3 \epsilon^2 \ln\left(\frac{1}{\epsilon}\right) 2 + o(\epsilon^2 \ln\left(\frac{1}{\epsilon}\right))}{\left(1 + \frac{C_4}{C_3} \epsilon^2 + o(\epsilon^2)\right)^{\frac{2}{2^*(k,s)}}} \right]$$

$$= \left(\frac{K_s(n-k)}{A^{1-\frac{2}{2^*(k,s)}}} \right)^{-1} \left(1 + \frac{C_2}{C_1} w_3 \epsilon^2 \ln\left(\frac{1}{\epsilon}\right) + o\left(\epsilon^2 \ln\left(\frac{1}{\epsilon}\right)\right) \right)$$

The proposition is proved in this case.

Case III.b: $m \geq 5$. In this case we have

$$I(u_\epsilon) = \frac{C_1 + C_2 \epsilon^2 + o(\epsilon^2)}{(C_3 + C_4 \epsilon^2 + o(\epsilon^2))^{\frac{2}{2^*(k,s)}}}$$

with

$$C_1 = \int_{\mathbb{R}^m} |\nabla \phi|^2 dX; \quad C_2 = \bar{a}(\bar{x}_0) \int_{\mathbb{R}^m} |\phi|^2 dX - \frac{1}{6m} \text{Scal}_{\bar{g}}(\bar{x}_0) \int_{\mathbb{R}^m} |X|^2 |\nabla \phi|^2 dX;$$

$$C_3 = \bar{f}(\bar{x}_0) \int_{\mathbb{R}^m} \frac{|\phi|^{2^*(k,s)}}{|X|^s} dX; \quad C_4 = - \left(\frac{\bar{f}(\bar{x}_0)}{6m} \text{Scal}_{\bar{g}}(\bar{x}_0) + \frac{\Delta_{\bar{g}} \bar{f}(\bar{x}_0)}{2m} \right) \int_{\mathbb{R}^m} |X|^{2-s} |\phi|^{2^*(k,s)} dX.$$

It follows that:

$$I(u_\epsilon) = \frac{1}{C_3^{\frac{2}{2^*(k,s)}}} \left[\frac{C_1 + C_2 \epsilon^2 + o(\epsilon^2)}{\left(1 + \frac{C_4}{C_3} \epsilon^2 + o(\epsilon^2)\right)^{\frac{2}{2^*(k,s)}}} \right]$$

$$= \frac{C_1}{C_3^{\frac{2}{2^*(k,s)}}} + \frac{C_1}{C_3^{\frac{2}{2^*(k,s)}}} \left(\frac{C_2}{C_1} - \frac{2C_4}{2^*(k,s)C_3} \right) \epsilon^2 + o(\epsilon^2)$$

Since ϕ is an extremal for the Hardy-Sobolev inequality (4), we get that

$$\frac{C_2}{C_1} - \frac{2C_4}{2^*(k,s)C_3} = k_1 \bar{a}(\bar{x}_0) - k_2 \text{Scal}_{\bar{g}}(\bar{x}_0) + k_3 \frac{\Delta_{\bar{g}} \bar{f}(\bar{x}_0)}{\bar{f}(\bar{x}_0)} \quad \text{and} \quad \frac{C_1}{C_3^{\frac{2}{2^*(k,s)}}} = \frac{K_s(n-k)^{-1}}{A^{1-\frac{2}{2^*(k,s)}}}$$

Lemma 22 completes the proof of this case.

Step IV: We show that there exists a constant $c > 0$ such that

$$J_q(t_\epsilon u_\epsilon) \leq c_{n,k,s} \left(\frac{K_s(n-k)}{A^{1-\frac{2}{2^*(k,s)}}} \right)^{\frac{2^*(k,s)}{2^*(k,s)-2}} - c h(x_0) \epsilon^{m-\frac{q(m-2)}{2}} + o(\epsilon^{m-\frac{q(m-2)}{2}})$$

Indeed, for $m \geq 5$

(63)

$$I(u_\epsilon) \leq \frac{K_s(n-k)^{-1}}{A^{1-\frac{2}{2^*(k,s)}}} + k_5 \left(\bar{a}(\bar{x}_0) - c'_{n,k,s} \text{Scal}_{\bar{g}}(\bar{x}_0) + c''_{n,k,s} \frac{\Delta_{\bar{g}} \bar{f}(\bar{x}_0)}{\bar{f}(\bar{x}_0)} \right) \epsilon^2 + o(\epsilon^2)$$

with, as in (53),

$$c'_{n,k,s} = \frac{(m-2)(6-s)}{12(2m-2-s)} \quad \text{and} \quad c''_{n,k,s} = \frac{(m-2)(m-4)}{4(2m-2-s)}$$

Consequently, there exists $k_5 := k_5(m, s) > 0$ such that

$$\begin{aligned}
& J_q(t_\epsilon u_\epsilon) \\
& \leq c_{n,k,s} \left(\frac{K_s(n-k)}{A^{1-\frac{2}{2^*(k,s)}}} \right)^{\frac{2^*(k,s)}{2^*(k,s)-2}} + k_6 \left(\bar{a}(\bar{x}_0) - c'_{n,k,s} \text{Scal}_{\bar{g}}(\bar{x}_0) + c''_{n,k,s} \frac{\Delta_{\bar{g}} \bar{f}(\bar{x}_0)}{\bar{f}(\bar{x}_0)} \right) \epsilon^2 \\
(64) \quad & + o(\epsilon^2) - \frac{T_0^q}{q} C_\epsilon + o(C_\epsilon)
\end{aligned}$$

with

$$C_\epsilon = \epsilon^{m-\frac{q(m-2)}{2}} \bar{h}(\bar{x}_0) \int_{\mathbb{R}^{n-k}} |\phi|^q dX + O\left(\epsilon^{\frac{q(m-2)}{2}}\right) + o(\epsilon^{m-\frac{q(m-2)}{2}})$$

Now $m - \frac{q(m-2)}{2} < 2$, it follows that $\epsilon^2 = o(\epsilon^{m-\frac{q(m-2)}{2}})$ which proves our equality. For $m = 4$, by the previous steps there exists $k_6 := k_6(m, s) > 0$ such that

$$I(u_\epsilon) \leq \frac{K_s(n-k)^{-1}}{A^{1-\frac{2}{2^*(k,s)}}} + k_5 (\bar{a}(\bar{x}_0) - c'_{n,k,s} \text{Scal}_{\bar{g}}(\bar{x}_0)) \epsilon^2 \ln\left(\frac{1}{\epsilon}\right) + o\left(\epsilon^2 \ln\left(\frac{1}{\epsilon}\right)\right)$$

This implies that

$$\begin{aligned}
& J_q(t_\epsilon u_\epsilon) \\
& \leq c_{n,k,s} \left(\frac{K_s(n-k)}{A^{1-\frac{2}{2^*(k,s)}}} \right)^{\frac{2^*(k,s)}{2^*(k,s)-2}} + k_6 (\bar{a}(\bar{x}_0) - c'_{n,k,s} \text{Scal}_{\bar{g}}(\bar{x}_0)) \epsilon^2 \ln\left(\frac{1}{\epsilon}\right) \\
(65) \quad & + o\left(\epsilon^2 \ln\left(\frac{1}{\epsilon}\right)\right) - \frac{T_0^q}{q} C_\epsilon + o(C_\epsilon)
\end{aligned}$$

In this case

$$C_\epsilon = \epsilon^{4-q} \bar{h}(\bar{x}_0) \int_{\mathbb{R}^{n-k}} |\phi|^q dX + o(\epsilon^{4-q})$$

Then $\epsilon^2 \ln(\frac{1}{\epsilon}) = o(C_\epsilon)$ which proves the rest of our step. We can now conclude. Since $h(x_0) > 0$ in (63), we get the conditions of Theorem 6, and this implies the conclusion of Theorem 7.

6.3. Proof of Theorems 4 and 5. We note that if $h \equiv 0$, then (62) becomes

$$J_q(t_\epsilon u_\epsilon) = c_{n,k,s} (I(u_\epsilon))^{\frac{2^*(k,s)}{2^*(k,s)-2}}$$

Therefore, if

$$\inf_{u \in H_1^2 \setminus \{0\}} \frac{\int_M (|\nabla u_\epsilon|^2 + a u_\epsilon) dv_g}{\left(\int_M |u_\epsilon|^{2^*(k,s)} \frac{dv_g}{d_g(x, Gx_0)^s} \right)^{\frac{2}{2^*(k,s)}}} < \frac{K_s(n-k)^{-1}}{A^{1-\frac{2}{2^*(k,s)}}},$$

by following the same argument in Step II of the proof of Theorem 7, we get that the conditions of Theorem 6 are verified, and this implies the conclusion of Theorem 4. Moreover, if

$$\bar{a}(\bar{x}_0) < \frac{(n-k-2)(6-s)}{12(2n-2k-2-s)} \text{Scal}_{\bar{g}}(\bar{x}_0) - \frac{(n-k-2)(n-k-4)}{4(2n-2k-2-s)} \frac{\Delta_{\bar{g}} \bar{f}(\bar{x}_0)}{\bar{f}(\bar{x}_0)}$$

Then we can deduce from (64) and (65) the proof of Theorem 5.

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HUSSEIN MESMAR, INSTITUT ÉLIE CARTAN, UMR 7502, UNIVERSITÉ DE LORRAINE, BP 70239, F-54506 VANDŒUVRE-LÈS-NANCY, FRANCE

Email address: hussein.mesmar@univ-lorraine.fr