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# LOGARITHMIC GROSS-PITAEVSKII EQUATION

RÉMI CARLES AND GUILLAUME FERRIERE

ABSTRACT. We consider the Schrödinger equation with a logarithmic nonlinearity and non-trivial boundary conditions at infinity. We prove that the Cauchy problem is globally well posed in the energy space, which turns out to correspond to the energy space for the standard Gross-Pitaevskii equation with a cubic nonlinearity, in small dimensions. We then characterize the solitary and travelling waves in the one dimensional case.

## 1. INTRODUCTION

We consider the Schrödinger equation with a logarithmic nonlinearity,

$$(logGP) \quad i\partial_t u + \Delta u = \lambda u \ln |u|^2, \quad u|_{t=0} = u_0,$$

where  $x \in \mathbb{R}^d$ ,  $d \geq 1$ ,  $\lambda > 0$ , and with the boundary condition at infinity

$$|u(t, x)| \rightarrow 1 \quad \text{as} \quad |x| \rightarrow \infty.$$

Such boundary condition is reminiscent of the standard Gross-Pitaevskii equation

$$(1.1) \quad i\partial_t u + \Delta u = (|u|^2 - 1)u, \quad u|_{t=0} = u_0,$$

whose Cauchy problem was studied in [8, 20, 21, 22, 43, 44]. The most complete result regarding this aspect is found in [21, 22], where, for  $d \leq 3$ , (1.1) is proved to be globally well posed in the energy space

$$E_{GP} := \{u \in H_{loc}^1(\mathbb{R}^d) \mid \mathcal{E}_{GP}(u) < \infty\},$$

where  $\mathcal{E}_{GP}$  is the Ginzburg-Landau energy

$$\mathcal{E}_{GP}(u) := \|\nabla u\|_{L^2(\mathbb{R}^d)}^2 + \frac{1}{2} \int_{\mathbb{R}^d} (|u|^2 - 1)^2 dx.$$

See also [33] for an analogous energy-critical problem. The logarithmic nonlinearity was introduced in the context of Schrödinger equations in [9], as it is the only nonlinearity satisfying the following tensorization property: if  $u_1(t, x_1), \dots, u_d(t, x_d)$  are solutions to the one-dimensional equation, then  $u(t, x) := u_1(t, x_1) \times \dots \times u_d(t, x_s)$  solves the  $d$ -dimensional equation. This model has regained interest in various domains of physics: quantum mechanics [42], quantum optics [9, 30, 35, 10], nuclear physics [32], Bohmian mechanics [38], effective quantum gravity [45], theory of superfluidity and Bose-Einstein condensation [1]. The logarithmic model may generalize the Gross-Pitaevskii equation, used in the case of two-body interaction, to the case of three-body interaction; see [45, 46].

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**1.1. Cauchy problem.** From the mathematical point of view, the Cauchy problem for (logGP) is more intricate than it may seem at first glance. The difficulty does not lie in the behavior of the logarithm at infinity, of course, but in its singularity at the origin. In the case of vanishing boundary condition at infinity,  $u_0 \in H^1(\mathbb{R}^d)$ , the Cauchy problem was solved in [14] (case  $\lambda < 0$ , see also [31]) and [12, 26] (for any  $\lambda \in \mathbb{R}$ ), by constructing solutions of a regularized equation converging to a solution to the exact equation, which turns out to be unique. Even when nontrivial boundary conditions at infinity are considered, like in the present paper, the singularity of the logarithm at the origin is the main difficulty, as the cancellation of the wave function is difficult (if not impossible) to rule out for *all*  $(t, x) \in \mathbb{R} \times \mathbb{R}^d$ . The nonlinearity fails to be locally Lipschitz continuous, so the fixed point argument employed in [21] in the case of (1.1) is hard to implement. Even if a solution of the form  $u = 1 + H^1$  is considered in the one dimensional case (where the  $H^1$ -norm control the  $L^\infty$  norm), it is not obvious to make a solution global in time, even if  $\|u_0 - 1\|_{H^1} \ll 1$ . Indeed, the conservation of the energy in the context of (logGP) reads  $d\mathcal{E}_{\log\text{GP}}/dt = 0$ , where

$$\mathcal{E}_{\log\text{GP}}(u) := \|\nabla u\|_{L^2(\mathbb{R}^d)}^2 + \lambda \int_{\mathbb{R}^d} (|u|^2 \ln |u|^2 - |u|^2 + 1) dx.$$

Consider  $F$  the antiderivative of the logarithm, such that  $F(1) = 0$ :  $F(y) = y \ln y - y + 1$  for  $y > 0$ , and  $F(0) = 1$ . The above potential energy is the integral in space of  $F(|u|^2)$ . We note that  $F \geq 0$  on  $[0, \infty)$ , and Taylor's formula yields

$$0 \leq F(y) = y^2 \int_0^1 \frac{1-s}{1+sy} ds \leq y^2,$$

and so  $0 \leq \mathcal{E}_{\log\text{GP}}(u) \leq \mathcal{E}_{\text{GP}}(u)$ . But this is not enough to solve (logGP) in the energy space, neither locally or globally in time. As a byproduct of our analysis, we will see that in low space dimensions,  $d \leq 4$ , the energy spaces for (logGP),

$$E_{\log\text{GP}} := \{v \in H_{\text{loc}}^1(\mathbb{R}^d) \mid \mathcal{E}_{\log\text{GP}}(v) < \infty\},$$

and (1.1) coincide,  $E_{\log\text{GP}} = E_{\text{GP}}$ . Note that the energy space  $E_{\text{GP}}$  was described very accurately in [21, 22]. We consider the notion of mild solution, based on Duhamel's formula:

$$u(t) = e^{it\Delta} u_{\text{in}} - i\lambda \int_0^t e^{i(t-s)\Delta} \left( u(s) \ln |u(s)|^2 \right) ds.$$

As we construct solutions to (logGP) as weak solutions, it is sensible to ask whether or not these solutions are also mild solutions, in the same fashion as in [41]. Our main result regarding the Cauchy problem for (logGP) is the following:

**Theorem 1.1.** *For any  $u_0 \in \mathcal{E}_{\log\text{GP}}$ , there exists a unique  $u \in L_{\text{loc}}^\infty(\mathbb{R}, E_{\log\text{GP}})$  solution to (logGP). Moreover, the flow of (logGP) enjoys the following properties:*

- *The energy is conserved,  $\mathcal{E}_{\log\text{GP}}(u(t)) = \mathcal{E}_{\log\text{GP}}(u_0)$  for all  $t \in \mathbb{R}$ .*
- *$u - u_0 \in C^0(\mathbb{R}, L^2)$ .*
- *$u$  is also a mild solution.*
- *If  $\Delta u_0 \in L^2$ , then  $u - u_0 \in W_{\text{loc}}^{1,\infty}(\mathbb{R}, L^2) \cap L_{\text{loc}}^\infty(\mathbb{R}, H^2)$ .*

**1.2. Solitary and traveling waves.** There are many references regarding the existence, description and the stability of solitary and traveling waves for (1.1), as can be seen for instance from the results evoked in [18]. In order to consider both solitary and traveling waves, we look for solutions of the form

$$(1.2) \quad u(t, x) = e^{i\omega t} \phi(x - ct),$$

where  $\omega \in \mathbb{R}$ ,  $c \in \mathbb{R}^d$  and  $\phi \in E_{\log\text{GP}}$ . In the case  $c = 0$ , we classically call the solution a solitary waves, and if  $c \neq 0$ , we call it traveling wave (stationary wave if  $\omega = c = 0$ ). From the mathematical point of view, the study of traveling waves for (1.1) (with  $\omega = 0$ ) goes back to [8], and has known many developments since; see e.g. [2, 3, 15, 16, 37] and references therein. In the one dimensional case, traveling waves for (1.1) are either constant, or such that  $0 \leq c < \sqrt{2}$ , and given explicitly by

$$\phi_c(x) = \sqrt{1 - \frac{c^2}{2}} \tanh\left(\sqrt{1 - \frac{c^2}{2}} \frac{x}{\sqrt{2}}\right) + i \frac{c}{\sqrt{2}},$$

up to space translation and a multiplicative constant of modulus 1; see e.g. [2]. This solution  $u(t, x) = \phi_c(x - ct)$  is usually referred to as *dark soliton*, and *black soliton* (or *kink solution*) in the case  $c = 0$  (the only case where  $\phi$  has a zero). In the case of (logGP), we do not have such an explicit formula, but somehow a very similar description. In the general case  $d \geq 1$ , we have:

**Theorem 1.2.** *If  $\phi \in E_{\log\text{GP}}$  and  $u$  is a traveling wave of the form (1.2), then  $\omega = 0$ .*

We now focus on the one-dimensional case. First, like for (1.1), if  $|c|$  is too large, then there is no non-constant traveling wave:

**Theorem 1.3.** *Let  $d = 1$  and  $c^2 \geq 2\lambda$ . Any solution to (logGP) of the form (1.2) is constant.*

We now come to the description of non-constant traveling waves:

**Theorem 1.4.** *Assume  $d = 1$  and  $c$  such that  $c^2 < 2\lambda$ . There exists a unique non-constant traveling wave in the following sense: there exists a traveling wave  $\phi_c$  such that any non-constant traveling wave  $u$  of the form (1.2) is such that*

$$\phi = e^{i\theta} \phi_c(\cdot - x_0),$$

for some constants  $\theta, x_0 \in \mathbb{R}$ . If  $c \neq 0$ ,  $\phi_c$  never vanishes. In the case  $c = 0$ ,  $\phi_0$  can be taken real-valued and increasing, and then

$$\lim_{x \rightarrow \pm\infty} \phi_0(x) = \pm 1.$$

We emphasize that qualitatively, we obtain the same properties as the black soliton, even though it is probably hopeless to get an explicit expression in the case of (logGP).

**1.3. Content and notations.** In Section 2, we reduce the study of (logGP) in the energy space, by analyzing the potential energy and characterizing the energy space  $E_{\log\text{GP}}$ . In Section 3, we prove that any solution in the energy space is unique, and that it is a mild solution. The heart of Theorem 1.1 is established in Section 4, where a weak solution is constructed by an approximation procedure. In Section 5, we show that  $H^2$  regularity is propagated by the flow, completing the proof of Theorem 1.1. We prove general results regarding solitary and traveling waves in Section 6, in particular Theorem 1.2. In Section 7, we analyze stationary waves in the one-dimensional case (Theorem 1.4 in the case  $c = 0$ ). One-dimensional traveling waves are studied in Section 8, where we complete the proof of Theorems 1.4 and 1.3. We conclude this paper by open questions in Section 9.

We define the  $L^2$ -bracket by

$$\langle f, g \rangle = \int_{\mathbb{R}^d} f \bar{g} \, dx.$$

For  $k$  an integer, we denote by  $\mathcal{C}^k$  the class of  $k$  times continuously differentiable functions, and if  $0 < s < 1$ , by  $\mathcal{C}^{k,s}$  the class of  $k$  times continuously differentiable functions whose derivative of order  $k$  is Hölder continuous with exponent  $s$ . If  $s = 1$ ,  $\mathcal{C}^{k,1}$  is the class of  $k$  times continuously differentiable functions whose derivative of order  $k$  is Lipschitzian.

## 2. PRELIMINARY RESULTS

**2.1. Generalities.** First, we recall some properties about the logarithm. The first property was discovered in [14], and is crucial for uniqueness issues related to (logGP).

**Lemma 2.1** ([14, Lemma 1.1.1]). *There holds*

$$\left| \operatorname{Im} \left( (z_2 \ln |z_2|^2 - z_1 \ln |z_1|^2)(\overline{z_2} - \overline{z_1}) \right) \right| \leq 2|z_2 - z_1|^2, \quad \forall z_1, z_2 \in \mathbb{C}.$$

The next lemma measures the continuity of the nonlinearity in (logGP):

**Lemma 2.2.** *For all  $\varepsilon \in (0, 1)$  and  $x, y \in \mathbb{C}$ ,*

$$\left| x \ln |x|^2 - y \ln |y|^2 \right| \leq C \left( |x|^\varepsilon |\ln |x|| + |y|^\varepsilon |\ln |y|| \right) |x - y|^{1-\varepsilon} + 2|x - y|.$$

*Proof.* Let  $x, y \in \mathbb{C}$  and assume without loss of generality that  $|x| \leq |y|$ . We also assume  $x \neq 0$ , otherwise the estimate is obvious. Then, we have

$$\left| x \ln |x|^2 - x \ln |y|^2 \right| = 2|x| |\ln |x| - \ln |y|| \leq 2|x| \frac{||x| - |y||}{|x|} \leq 2|x - y|.$$

On the other hand,

$$\left| x \ln |y|^2 - y \ln |y|^2 \right| \leq 2|x - y| |\ln |y|| \leq 2^{1+\varepsilon} |x - y|^{1-\varepsilon} |y|^\varepsilon |\ln |y||,$$

hence the conclusion.  $\square$

Then, we also recall a result about norms on finite dimensional subspace of  $L^2$ .

**Lemma 2.3.** *If  $\{f_k\}_{k \leq n}$  is a finite family of  $L^2$  linearly independent functions, then there exists  $C > 0$  such that, for all  $\lambda_k \in \mathbb{C}$ ,*

$$\max |\lambda_k| \leq C \left\| \sum_{k=0}^n \lambda_k f_k \right\|_{L^2}.$$

*Proof.* Since  $\{f_k\}$  is a basis of  $\operatorname{Vect} \{f_k\}$ ,  $(\lambda_k) \mapsto \sum_{k=0}^n \lambda_k f_k$  is an isomorphism from  $\mathbb{C}^n$  into  $\operatorname{Vect} \{f_k\}$ . Since we are in finite dimension, the conclusion easily follows.  $\square$

**2.2. Potential energy.** We start by a result on the potential energy  $\mathcal{E}_{\text{pot}}$ , given by

$$\mathcal{E}_{\text{pot}}(v) := \int \left( |v|^2 \ln |v|^2 - |v|^2 + 1 \right) dx.$$

Let

$$E_{\text{pot}}(v) := \int (|v| - 1)^2 \ln(2 + |v|) dx.$$

We know that both previous quantities are non-negative. The following result shows that they are actually equivalent.

**Lemma 2.4.** *There exists  $K_0 > 0$  such that for all  $v \in E_{\text{logGP}}$ ,*

$$\frac{1}{K_0} E_{\text{pot}}(v) \leq \mathcal{E}_{\text{pot}}(v) \leq K_0 E_{\text{pot}}(v).$$

*Proof.* Taylor formula yields, for  $y \geq 0$ ,

$$y^2 \ln y^2 - y^2 + 1 = 4(y-1)^2 \int_0^1 (1 + \ln(1-s+sy))(1-s) ds.$$

Distinguishing  $y < 1$  and  $y > 1$ , we get:

$$\ln(2+y) \lesssim \int_0^1 (1 + \ln(1-s+sy))(1-s) ds \lesssim \ln(2+y),$$

and the result follows, by considering  $y = |v|$ .  $\square$

This second functional  $E_{\text{pot}}$  appears to be more convenient than  $\mathcal{E}_{\text{pot}}$ . For instance, since  $\ln 2 \leq \ln(2+|x|) \leq \ln 2 + C_\delta \|x| - 1|^\delta$  for all  $\delta > 0$ , we can relate  $E_{\text{pot}}$  to the  $L^2$ -norm of  $|v| - 1$ , but also to any  $L^p$ -norm, for  $p > 2$  as follows:

**Lemma 2.5.** *Let  $p > 2$ . There exists  $C_p$  such that for all  $v \in E_{\text{logGP}}$ ,*

$$\ln(2) \| |v| - 1 \|_{L^2}^2 \leq E_{\text{pot}}(v) \leq \ln(2) \| |v| - 1 \|_{L^2}^2 + C_p \| |v| - 1 \|_{L^p}^p.$$

Last, we state a differential result about the potential energy, which follows from standard arguments, that may be found in [13] ([Proposition 1.5.2, proof of Theorem 3.3.5. Recall that the function  $F$  is given on  $(0, \infty)$  by  $F(y) = y \ln y - y + 1$ ).

**Lemma 2.6.** *Let  $I$  be an open interval. If  $u \in u_0 + C^0(I, L^2) \cap L_{\text{loc}}^\infty(I, H^1) \cap C^1(I, H^{-1})$  and  $\varepsilon > 0$ , then  $\int (1 - \varepsilon \Delta)^{-2} F(|u(t)|^2) dx$  is well defined for all  $t \in I$  and*

$$\frac{d}{dt} \int (1 - \varepsilon \Delta)^{-2} F(|u(t)|^2) dx = 2 \left\langle (1 - \varepsilon \Delta)^{-1} \left( u(s) \ln |u(t)|^2 \right), (1 - \varepsilon \Delta)^{-1} \partial_t u(t) \right\rangle.$$

Moreover, for all  $t \in I$ ,

$$\int (1 - \varepsilon \Delta)^{-2} F(|u(t)|^2) dx \xrightarrow{\varepsilon \rightarrow 0} \mathcal{E}_{\text{pot}}(u(t)).$$

**2.3. Energy space.** As for the energy space, Lemma 2.4 leads to

$$E_{\text{logGP}} = \{v \in H_{\text{loc}}^1(\mathbb{R}^d) \mid \nabla v \in L^2(\mathbb{R}^d) \text{ and } E_{\text{pot}}(v) < \infty\}.$$

We prove an even more explicit description of  $E_{\text{logGP}}$ :

**Lemma 2.7.** *The energy space is characterized by:*

$$\begin{aligned} E_{\text{logGP}} &= \{v \in H_{\text{loc}}^1(\mathbb{R}^d) \mid \nabla v \in L^2(\mathbb{R}^d) \text{ and } |v| - 1 \in L^2(\mathbb{R}^d)\} \\ &= \{v \in H_{\text{loc}}^1(\mathbb{R}^d) \mid \nabla v \in L^2(\mathbb{R}^d) \text{ and } |v| - 1 \in H^1(\mathbb{R}^d)\}. \end{aligned}$$

Moreover, there exists  $C > 0$  such that for all  $v \in E_{\text{logGP}}$ ,  $\| |v| - 1 \|_{H^1}^2 \leq C \mathcal{E}_{\text{logGP}}(v)$ .

*Proof.* First, we show that  $E_{\text{logGP}} \subset \{v \in H_{\text{loc}}^1 \mid \nabla v \in L^2 \text{ and } |v| - 1 \in L^2\}$ .

Let  $v \in E_{\text{logGP}}$ :  $\|\nabla v\|_{L^2} < \infty$ , and  $\mathcal{E}_{\text{pot}}(v) < \infty$ , which is equivalent to  $E_{\text{pot}}(v) < \infty$ . Lemma 2.5 yields  $|v| - 1 \in L^2$ .

Conversely, let  $v \in H_{\text{loc}}^1$  such that  $|v| - 1, \nabla v \in L^2$ . Then  $\nabla(|v| - 1) = \nabla v \cdot \frac{v}{|v|}$  and  $|\nabla(|v| - 1)| \leq |\nabla v|$ , which shows that  $f = |v| - 1 \in H^1$ . Since  $f \geq -1$ , we readily have

$$E_{\text{pot}}(v) = \int |f|^2 \ln(3+f) dx \leq \int |f|^2 \ln(3+|f|) dx \lesssim \int |f|^2 (1+|f|^\varepsilon) dx < \infty,$$

where  $\varepsilon > 0$  is arbitrarily small, and where we have used Sobolev embedding to conclude.

From the previous arguments, if  $v \in E_{\log\text{GP}}$ , the  $L^2$ -norm of  $|v| - 1$  can be estimated thanks to Lemma 2.5 and Lemma 2.4, whereas  $|\nabla(|v| - 1)| \leq |\nabla v|$ , hence the conclusion.  $\square$

**Corollary 2.8.** *For all  $\delta \geq 1$ ,  $\varepsilon > 0$ , and  $p$  such that*

$$(2.1) \quad p, p + \varepsilon \in \begin{cases} (2, \infty] & \text{if } d = 1, \\ (2, \infty) & \text{if } d = 2, \\ (2, 2^*] & \text{if } d \geq 3, \quad 2^* := \frac{2d}{d-2}, \end{cases}$$

there exists  $C_{p,\varepsilon,\delta}$  such that if  $v \in E_{\log\text{GP}}$ , then  $v|\ln|v|^2|^\delta \in L^p(\mathbb{R}^d)$  and

$$\left\| v|\ln|v|^2|^\delta \right\|_{L^p} \leq C_{p,\varepsilon,\delta} (\mathcal{E}_{\log\text{GP}}(v)^{1/2} + \mathcal{E}_{\log\text{GP}}(v)^{(1+\varepsilon)/2}).$$

*Proof.* We know that  $|v| - 1 \in H^1$  and, since  $\delta \geq 1$ ,

$$|v|\ln|v|^2|^\delta \leq C_\delta ||v| - 1| (\ln(2 + ||v| - 1|))^\delta \leq C_{\varepsilon,\delta} (||v| - 1| + ||v| - 1|^{1+\frac{\varepsilon}{p}}).$$

This inequality is easily established, for instance by considering separately the regions  $\{|v| < 1/2\}$ ,  $\{|v| > 2\}$ , and  $\{1/2 \leq |v| \leq 2\}$ . The result then follows from Sobolev embedding, Lemma 2.7 and Lemma 2.4.  $\square$

*Remark 2.9.* This corollary reveals an important difference with the case of vanishing boundary condition at infinity: in [26, 12], momenta in  $L^2$  ( $\| |x|^\alpha u \|_{L^2}$  for some  $\alpha \in (0, 1]$ ) are considered in order to control the nonlinearity in the region  $\{|u| < 1\}$ . The situation is obviously different in the case of nontrivial boundary condition at infinity, and the above corollary is crucial for the rest of this paper.

Lemma 2.7 shows that for  $d \leq 4$ ,  $E_{\log\text{GP}} = E_{\text{GP}}$ , in view of the identity

$$|u|^2 - 1 = (|u| - 1)(|u| - 1 + 2),$$

and the Sobolev embedding  $H^1(\mathbb{R}^d) \subset L^4(\mathbb{R}^d)$  for  $d \leq 4$ . In particular, Gérard [21] proved that  $E_{\text{GP}} + H^1 \subset E_{\text{GP}}$  for  $d \geq 1$ , and therefore the same holds for  $E_{\log\text{GP}}$  when  $d \leq 4$ . This is also true in all dimensions :

**Lemma 2.10.** *Let  $d \geq 1$ . For all  $v \in E_{\log\text{GP}}$  and for all  $f \in H^1$ , we have  $v + f \in E_{\log\text{GP}}$ . In addition, if  $p > 2$  satisfies (2.1), there exists  $C_p$  such that for every such  $v$  and  $f$ ,*

$$\mathcal{E}_{\log\text{GP}}(v + f) \leq C_p \mathcal{E}_{\log\text{GP}}(v) + C_p \mathcal{E}_{\log\text{GP}}(v)^p + C_p \|f\|_{H^1}^2 + C_p \|f\|_{H^1}^p.$$

*Proof.* Let  $v \in E_{\log\text{GP}}$  and  $f \in H^1$ . We know that  $\nabla v, \nabla f \in L^2$ , so  $\nabla(v + f) \in L^2$ . From Lemma 2.7, there only remains to show that  $|v + f| - 1 \in L^2$ . For this, we have for all  $x, y \in \mathbb{C}$

$$|x + y| = |x| + \int_0^1 \frac{x + ry}{|x + ry|} \cdot y \, dr.$$

Applying this equality to  $u$  and  $f$ , we get

$$|u + f| - 1 = |u| - 1 + \int_0^1 \frac{u + rf}{|u + rf|} \cdot f \, dr,$$

hence

$$||u + f| - 1| \leq ||u| - 1| + |f|,$$

and, for any  $p \in [1, \infty]$ ,

$$\| |u + f| - 1 \|_{L^p} \leq \| |u| - 1 \|_{L^p} + \| f \|_{L^p}$$

Moreover, Lemmas 2.4 and 2.5 yield, for any  $p > 2$  satisfying (2.1),

$$\begin{aligned} \mathcal{E}_{\log\text{GP}}(v + f) &\leq \| \nabla v + \nabla f \|^2 + C \| |v + f| - 1 \|_{L^2}^2 + C_p \| |v + f| - 1 \|_{L^p}^p \\ &\lesssim \| \nabla v \|_{L^2}^2 + \| \nabla f \|_{L^2}^2 + \| |u| - 1 \|_{L^2}^2 + \| f \|_{L^2}^2 + \| |u| - 1 \|_{L^p}^p + \| f \|_{L^p}^p \\ &\lesssim \mathcal{E}_{\log\text{GP}}(v) + \| |u| - 1 \|_{H^1}^p + \| f \|_{H^1}^2 + \| f \|_{H^1}^p \\ &\lesssim \mathcal{E}_{\log\text{GP}}(v) + \mathcal{E}_{\log\text{GP}}(v)^p + \| f \|_{H^1}^2 + \| f \|_{H^1}^p. \end{aligned} \quad \square$$

We conclude by adapting [21, Lemma 1]. Introduce the Zhidkov space

$$X^1(\mathbb{R}^d) = \{ u \in L^\infty(\mathbb{R}^d), \nabla u \in L^2(\mathbb{R}^d) \}.$$

**Lemma 2.11.** *Let  $d \geq 1$ . We have  $E_{\log\text{GP}} \subset X^1(\mathbb{R}^d) + H^1(\mathbb{R}^d)$ .*

*Proof.* Proceeding like in the proof of [21, Lemma 1], consider  $\chi \in C_0^\infty(\mathbb{C})$  a cutoff function such that  $0 \leq \chi \leq 1$ ,  $\chi(z) = 1$  for  $|z| \leq 2$  and  $\chi(z) = 0$  for  $|z| \geq 3$ . Decompose  $u \in E_{\log\text{GP}}$  as

$$u = u_1 + u_2, \quad u_1 = \chi(u)u, \quad u_2 = (1 - \chi(u))u.$$

Then  $\| u_1 \|_{L^\infty} \leq 3$ , and since  $|u| \geq 2$  on the support of  $u_2$ , for any  $\delta > 1$ , we may find  $C_\delta$  such that

$$|u_2| \leq C_\delta \| |u| - 1 \|^{\delta}.$$

Since  $|u| - 1 \in H^1(\mathbb{R}^d)$  from Lemma 2.7,  $|u| - 1 \in L^{2\delta}(\mathbb{R}^d)$  by Sobolev embedding (provided that  $\delta \leq \frac{d}{d-2}$  if  $d \geq 3$ ), hence  $u_2 \in L^2$ . The properties  $\nabla u_1, \nabla u_2 \in L^2$  are straightforward, and we refer to [21] for details.  $\square$

### 3. REGULARITY AND UNIQUENESS

We begin by proving the following point in Theorem 1.1:

**Lemma 3.1.** *Let  $I$  an interval. If  $u \in L_{\text{loc}}^\infty(I, E_{\log\text{GP}})$  (that is,  $t \mapsto \mathcal{E}_{\log\text{GP}}(u(t)) \in L_{\text{loc}}^\infty(I)$ ) satisfies (logGP) in  $\mathcal{D}'(I \times \mathbb{R}^d)$ , then  $u - u_0 \in C^0(I, L^2)$ .*

*Proof.* Since  $u \in L_{\text{loc}}^\infty(I, E_{\log\text{GP}})$ , we know from Corollary 2.8 that  $u \ln |u|^2 \in L_{\text{loc}}^\infty(I, L^2)$ . Moreover,  $\Delta u \in L_{\text{loc}}^\infty(I, H^{-1})$ . Therefore, (logGP) yields  $\partial_t u = \partial_t(u - u_0) \in L_{\text{loc}}^\infty(I, H^{-1})$ , with  $u(0) - u_0 = 0$ . This proves that  $u - u_0 \in C^0(I, H^{-1})$ . Since  $\nabla(u - u_0) \in L_{\text{loc}}^\infty(I, L^2)$ , we obtain the result by interpolation.  $\square$

This result allows us to infer the uniqueness of the solution of (logGP).

**Theorem 3.2.** *Let  $u_0 \in E_{\log\text{GP}}$  and  $I$  be a bounded interval containing 0. There exists at most one solution  $u \in L^\infty(I, E_{\log\text{GP}})$  to (logGP) in  $\mathcal{D}'(I \times \mathbb{R}^d)$ .*

*Proof.* Let  $u, v \in L^\infty(I, E_{\log\text{GP}})$  solve (logGP). Then  $u(t) - u_0$  and  $v(t) - u_0$  are continuous from  $I$  to  $L^2$  from Lemma 3.1. Therefore,  $w := u - v \in C^0(I; L^2)$ . This error satisfies (in  $\mathcal{D}'(I \times \mathbb{R}^d)$ )

$$i\partial_t w + \Delta w = \lambda(u \ln |u|^2 - v \ln |v|^2).$$



Since on the one hand  $\Delta w \in L^\infty(I, H^{-1})$ , and on the other hand  $u \ln |u|^2, v \ln |v|^2 \in L^\infty(I, L^2)$ , the previous equality is also satisfied in  $L^\infty(I, H^{-1})$ . As  $w \in L^\infty(I, H^1)$ , we can take the  $H^{-1} \times H^1$  bracket against  $w$  of the previous equality, which yields

$$(3.1) \quad i \langle \partial_t w, w \rangle_{H^{-1}, H^1} - \|\nabla w\|_{L^2}^2 = \lambda \int (u \ln |u|^2 - v \ln |v|^2) (\overline{u - v}) \, dx.$$

Thanks to Lemma 2.1, we have

$$\left| \operatorname{Im} \int (u \ln |u|^2 - v \ln |v|^2) (\overline{u - v}) \, dx \right| \leq 2 \|u - v\|_{L^2}^2 = 2 \|w\|_{L^2}^2.$$

Therefore, taking the imaginary part of (3.1) leads to

$$|\operatorname{Re} \langle \partial_t w, w \rangle_{H^{-1}, H^1}| = \frac{1}{2} \left| \frac{d}{dt} \|w\|_{L^2}^2 \right| \leq 2\lambda \|w\|_{L^2}^2.$$

Since  $w(0) = 0$ , Gronwall lemma concludes the proof.  $\square$

We end this section with a link between regularity and mild solution.

**Lemma 3.3.** *Let  $I$  be an open interval of  $\mathbb{R}$ ,  $u_0 \in E_{\log \text{GP}}$  and  $u \in L_{\text{loc}}^\infty(I, E_{\log \text{GP}})$  satisfying (logGP) in  $\mathcal{D}'(I \times \mathbb{R}^d)$ . Then  $u$  is a mild solution, and*

$$\int_0^t e^{-is\Delta} (u(s) \ln |u(s)|^2) \, ds \in H^1(\mathbb{R}^d) \quad \text{for all } t \in I.$$

*Proof.* From Corollary 2.8, we have  $u \ln |u|^2 \in L_{\text{loc}}^\infty(I, L^2)$ , thus we can define

$$v_{\text{NL}}(t) = -i\lambda \int_0^t e^{-is\Delta} u(s) \ln |u(s)|^2 \, ds \in \mathcal{C}^0(I, L^2) \cap W_{\text{loc}}^{1, \infty}(I, H^{-2}).$$

On the other hand, since  $\nabla u_0 \in L^2$  and  $u_0 \in L^\infty + L^2$  (from Lemma 2.11), we can define  $e^{it\Delta} u_0$  and we have  $e^{it\Delta} u_0 - u_0 \in \mathcal{C}^0(I, L^2) \cap \mathcal{C}^1(I, H^{-1})$  (see [21]). Therefore,

$$v := e^{it\Delta} u_0 + e^{it\Delta} v_{\text{NL}}(t) \in u_0 + \mathcal{C}^0(I, L^2) \cap W_{\text{loc}}^{1, \infty}(I, H^{-2}),$$

and we can compute

$$\begin{aligned} \partial_t v &= i\Delta e^{it\Delta} \left[ u_0 - i\lambda \int_0^t e^{-is\Delta} u(s) \ln |u(s)|^2 \, ds \right] \\ &\quad + e^{it\Delta} \partial_t \left[ u_0 - i\lambda \int_0^t e^{-is\Delta} u(s) \ln |u(s)|^2 \, ds \right] \\ &= i\Delta v + e^{it\Delta} \left[ -i\lambda e^{-it\Delta} u(t) \ln |u(t)|^2 \right] \\ &= i\Delta v - i\lambda u(t) \ln |u(t)|^2, \end{aligned}$$

where the equality is to be understood in  $L_{\text{loc}}^\infty(I, H^{-2})$ . Let  $w = u - v$ . Then,  $w \in \mathcal{C}^0(I, L^2)$  from Lemma 3.1 along with the previous arguments, and with the previous equality, we have, in  $L_{\text{loc}}^\infty(I, H^{-2})$ ,

$$\partial_t w - i\Delta w = 0.$$

Since  $w(0) = 0$  by construction, we get  $w = 0$ , which proves the mild formulation. Last, we know that  $\nabla(u(t) - u_0) \in L^2$  and  $\nabla(e^{it\Delta} u_0 - u_0) \in L^2$  for all  $t \in I$ , therefore

$$\nabla v_{\text{NL}}(t) \in L^2, \quad \forall t \in I.$$

Since we already know that  $v_{\text{NL}}(t) \in L^2$ , we get the conclusion.  $\square$

## 4. CONSTRUCTION OF A SOLUTION

In this section, we construct a global solution  $u \in L^\infty(\mathbb{R}, \mathcal{E}_{\log\text{NLS}})$  to (logGP). We adapt the method used by Ginibre and Velo [24] to construct global weak solutions to non-linear Schrödinger equations (NLS) by compactness. Here, the framework is different since the solution is not in  $L^2$ . However, we know that we should have  $u(t) - u_0 \in H^1$ . Therefore, we will approximate the solution on  $u_0 + X_m$  with  $X_m$  a sequence of finite dimensional linear subspaces approximating  $H^1$ .

*Remark 4.1.* In [12], for vanishing boundary condition at infinity, another approximation is considered, consisting in removing the singularity of the logarithm by saturating the nonlinearity, for  $\varepsilon > 0$ ,

$$(4.1) \quad i\partial_t u^\varepsilon + \Delta u^\varepsilon = \lambda u^\varepsilon \ln(|u^\varepsilon|^2 + \varepsilon), \quad u|_{t=0} = u_0,$$

and letting  $\varepsilon \rightarrow 0$ . This approximation has the advantage of working whichever the sign of  $\lambda$  is, while the approach introduced initially in [14] (see also [31]) seems to be bound to the nondispersive case  $\lambda < 0$ . We do not consider this approach here.

**4.1. Finite dimensional approximation.** Let  $\{w_j\}_{j \in \mathbb{N}}$  an Hilbertian basis of  $L^2$  with all  $w_j \in H^1$  and  $m \in \mathbb{N}$ . One may think for instance of Hermite functions, all the more since in Lemma 5.1 below, in order to prove the propagation of  $\dot{H}^2$  regularity, we further require  $w_j \in H^2$ . We take  $X_m := \text{Vect}(w_j)_{j \leq m}$  and look for an approximation  $u_m$  of the form

$$(4.2) \quad u_m(t, x) = u_0(x) + \sum_{k=0}^m g_{m,k}(t) w_k(x) =: u_0(x) + \varphi_m(t, x),$$

satisfying

$$(4.3) \quad \left\langle w_j, i\partial_t u_m + \Delta u_m - \lambda u_m \ln |u_m|^2 \right\rangle_{H^1, H^{-1}} = 0, \quad 0 \leq j \leq m,$$

with initial condition  $u_m(t) = u_0$ , which is equivalent to  $g_{m,k}(0) = 0$  for all  $0 \leq k \leq m$ . By substitution, (4.3) is equivalent to

$$(4.4) \quad \begin{aligned} i\dot{g}_{m,j} - \langle \nabla w_j, \nabla u_0 \rangle - \sum_{k=0}^m g_{m,k}(t) \langle \nabla w_j, \nabla w_k \rangle \\ = \lambda \left\langle w_j, f \left( u_0 + \sum_{k=0}^m g_{m,k}(t) w_k \right) \right\rangle, \end{aligned}$$

where  $f(x) = x \ln |x|^2$ . First, we study the last term.

**Lemma 4.2.** *The map  $\varphi \mapsto \langle w_j, f(\varphi) \rangle$  is well defined and continuous from  $u_0 + H^1$  into  $\mathbb{C}$  for any  $j \in \mathbb{N}$ . More precisely, it is  $C^{0,\varepsilon}(u_0 + H^1, \mathbb{C})$  for all  $\varepsilon \in (0, 1)$ .*

*Proof.* By Corollary 2.8 along with Lemma 2.10, we know that the right-hand side is well defined. Moreover, at fixed  $j$ , taking  $\varphi, \psi \in u_0 + H^1$ , we get

$$\begin{aligned} |\langle w_j, f(\varphi) \rangle - \langle w_j, f(\psi) \rangle| &= |\langle w_j, f(\varphi) - f(\psi) \rangle| \leq \int |w_j| |f(\varphi) - f(\psi)| dx \\ &\leq C \int |w_j| \left( |\varphi|^\varepsilon |\ln |\varphi|| + |\psi|^\varepsilon |\ln |\psi|| \right) |\varphi - \psi|^{1-\varepsilon} dx \\ &\quad + 2 \int |w_j| |\varphi - \psi| dx, \end{aligned}$$

for any  $\varepsilon \in (0, 1)$ , where we have used Lemma 2.2. The second term of the last inequality is obviously Lipschitzian with respect to the  $L^2$ -norm of the difference. As for the first term, by Hölder inequality with exponents  $2, \frac{2}{\varepsilon}$  and  $\frac{2}{1-\varepsilon}$ , we control it by

$$\|w_j\|_{L^2} \|\varphi - \psi\|_{L^2}^{1-\varepsilon} \left( \left\| |\varphi| \ln |\varphi| \right\|_{L^2}^{\frac{1}{\varepsilon}} \right)^{\varepsilon} + \left\| |\psi| \ln |\psi| \right\|_{L^2}^{\frac{1}{\varepsilon}} \right)^{\varepsilon}.$$

Now,  $\left\| |\varphi| \ln |\varphi| \right\|_{L^2}^{\frac{1}{\varepsilon}}$  and  $\left\| |\psi| \ln |\psi| \right\|_{L^2}^{\frac{1}{\varepsilon}}$  are locally bounded thanks to Corollary 2.8 and Lemma 2.10 once again. These estimates yield the conclusion.  $\square$

**Lemma 4.3.** *There exists a solution  $\{g_{m,k}\}_{k \leq m}$  to (4.4) on a maximal time interval  $(-T_m, T_m)$ , meaning that  $T_m < \infty$  if and only if*

$$\limsup_{t \rightarrow T_m} \sup_{j \leq m} |g_{m,j}(t)| = \infty.$$

(And similarly for  $T_m$ .) Moreover, the  $g_{m,k}$ 's are  $\mathcal{C}^{1,1-\varepsilon}(-T_m, T_m)$  for all  $\varepsilon \in (0, 1)$ .

*Proof.* The last term of (4.4) is continuous (and even  $\mathcal{C}^{0,\varepsilon}$  for all  $\varepsilon \in (0, 1)$ ) with respect to the  $g_{m,k}$ 's thanks to Lemma 4.2, and so are obviously the other terms. The conclusion comes from Peano Theorem and the fact that (4.4) is an autonomous ODE.  $\square$

We will then show that such a solution is global. First, we prove an intermediate result.

**Lemma 4.4.** *For all  $m \in \mathbb{N}$ ,  $(\nabla w_k)_{k \leq m}$  are linearly independent.*

*Proof.* Let  $\lambda_k \in \mathbb{C}$  such that  $\sum_{k=0}^m \lambda_k \nabla w_k = 0$ . Then  $\nabla \psi = 0$ , with  $\psi := \sum_{k=0}^m \lambda_k w_k \in H^1$ . Therefore,  $\psi = 0$ , and we conclude by the fact that  $\{w_k\}$  is an Hilbertian basis of  $L^2$ .  $\square$

**Lemma 4.5.** *The solution given by Lemma 4.3 is global. Moreover, it satisfies*

$$\mathcal{E}_{\log \text{GP}}(u_m(t)) = \mathcal{E}_{\log \text{GP}}(u_0), \quad \text{and} \quad \|\nabla \varphi_m(t)\|_{L^2} \leq 2\sqrt{\mathcal{E}_{\log \text{GP}}(u_0)} \quad \text{for all } t \in \mathbb{R}.$$

*Proof.* For this, we only have to prove that all the  $g_{m,k}$ s are (locally) bounded. Come back to (4.3): by multiplying by  $\dot{g}_{m,j}$  and summing over  $j \leq m$ , we get, in view of (4.2),

$$\left\langle \partial_t u_m, i \partial_t u_m + \Delta u_m - \lambda u_m \ln |u_m|^2 \right\rangle = 0.$$

By taking the real part of this equation, we obtain

$$\frac{d}{dt} \mathcal{E}_{\log \text{GP}}(u_m(t)) = 0.$$

Therefore,  $\mathcal{E}_{\log \text{GP}}(u_m(t)) = \mathcal{E}_{\log \text{GP}}(u_0)$  for all  $t \in (-T_m, T_m)$ . In particular,  $\|\nabla u_m(t)\|_{L^2}$  is uniformly bounded, and thus so is  $\|\nabla \varphi_m(t)\|_{L^2}$ : for  $t \in (-T_m, T_m)$ ,

$$\begin{aligned} \|\nabla \varphi_m(t)\|_{L^2} &\leq \|\nabla u_0\|_{L^2} + \|\nabla u_m(t)\|_{L^2} \leq \|\nabla u_0\|_{L^2} + \sqrt{\mathcal{E}_{\log \text{GP}}(u_m(t))} \\ &\leq 2\sqrt{\mathcal{E}_{\log \text{GP}}(u_0)}. \end{aligned}$$

We know that  $\nabla \varphi_m(t) = \sum_{k=0}^m g_{m,k}(t) \nabla w_k$ , thus Lemma 4.4 shows that we can apply Lemma 2.3, which gives

$$\max_{0 \leq k \leq m} |g_{m,k}(t)| \leq C_m \|\nabla \varphi_m(t)\|_{L^2} \leq 2C_m \sqrt{\mathcal{E}_{\log \text{GP}}(u_0)} < \infty, \quad \forall t \in (-T_m, T_m).$$

This proves that the  $g_{m,k}$ 's are actually globally bounded and therefore the solution is global.  $\square$

**4.2. Uniform estimates.** We have already shown the conservation of the energy and an estimate of the  $L^2$ -norm of  $\nabla\varphi_m$ , uniform both in  $t$  and  $m$ . Now, we proceed with its  $L^2$ -norm.

**Lemma 4.6.** *Let  $u_m = u_0 + \varphi_m$  the solution to (4.3) given by Lemma 4.3. Then  $\varphi_m$  is bounded in  $\mathcal{C}^{0, \frac{1}{2}}(I, L^2)$  uniformly in  $m$ , for every bounded interval  $I$ .*

*Proof.* Since all  $g_{m,k}$ 's are continuous, we already know that  $\varphi_m$  is continuous in time with values in  $L^2$ . By multiplying (4.3) by  $g_{m,j}(t)$  and summing over  $j$ , we obtain

$$\left\langle \varphi_m, i\partial_t\varphi_m + \Delta u_0 + \Delta\varphi_m - \lambda u_m \ln |u_m|^2 \right\rangle_{H^1, H^{-1}} = 0,$$

and the imaginary part gives

$$\frac{1}{2} \frac{d}{dt} \|\varphi_m\|_{L^2}^2 = \operatorname{Im} \langle \nabla\varphi_m, \nabla u_0 \rangle + \lambda \operatorname{Im} \langle \varphi_m, u_m \ln |u_m|^2 \rangle,$$

since  $\operatorname{Im} \langle \varphi_m, \Delta\varphi_m \rangle_{H^1, H^{-1}} = 0$ . Then, we can estimate the right-hand side :

$$|\langle \nabla\varphi_m, \nabla u_0 \rangle| \leq \|\nabla\varphi_m\|_{L^2} \|\nabla u_0\|_{L^2} \leq 2\mathcal{E}_{\log\text{GP}}(u_0),$$

thanks to Lemma 4.5, and

$$\begin{aligned} \left| \langle \varphi_m, u_m \ln |u_m|^2 \rangle \right| &\leq \|\varphi_m\|_{L^2} \left\| u_m \ln |u_m|^2 \right\|_{L^2} \\ &\lesssim \|\varphi_m\|_{L^2} \left( \mathcal{E}_{\log\text{GP}}(u_0)^{1/2} + \mathcal{E}_{\log\text{GP}}(u_0) \right), \end{aligned}$$

by Corollary 2.8. Since  $\varphi_m(0) = 0$ , Gronwall lemma gives the uniform boundedness of the  $L^2$  norm on every bounded interval  $I$ . Define

$$\xi_m(t) := -\Delta\varphi_m - \Delta u_0 + \lambda u_m \ln |u_m|^2.$$

Then  $\xi_m$  is bounded in  $L^\infty(I, H^{-1})$  uniformly in  $m$  for every bounded interval  $I$ . Moreover, for every  $t, s \in I$ ,

$$(4.5) \quad \|\varphi_m(t) - \varphi_m(s)\|_{L^2}^2 = 2 \int_s^t \langle \varphi_m(\tau) - \varphi_m(s), \partial_t\varphi_m(\tau) \rangle_{H^1, H^{-1}} d\tau.$$

By multiplying again (4.3) at time  $\tau$  by  $g_{m,j}(\tau) - g_{m,j}(s)$  and summing over  $j$ , there holds

$$\langle \varphi_m(\tau) - \varphi_m(s), \partial_t\varphi_m(\tau) \rangle_{H^1, H^{-1}} = \langle \varphi_m(\tau) - \varphi_m(s), \xi_m(\tau) \rangle_{H^1, H^{-1}}.$$

Since we know that the  $H^1$ -norms of  $\varphi_m(\tau)$  and  $\varphi_m(s)$  are bounded uniformly in  $m$  for  $\tau, s \in I$ , we get that  $\left| \langle \varphi_m(\tau) - \varphi_m(s), \xi_m(\tau) \rangle_{H^1, H^{-1}} \right|$  is bounded uniformly in  $m \in \mathbb{N}$ , for  $s, \tau \in I$ . Then, we get by (4.5)

$$\|\varphi_m(t) - \varphi_m(s)\|_{L^2}^2 \leq C_I |t - s|,$$

for all  $t, s \in I$ , with  $C_I$  depending on  $I$  but not on  $m$ , hence the conclusion.  $\square$

### 4.3. Convergence.

**Lemma 4.7.** *Let  $u_m = u_0 + \varphi_m$  the solution to (4.3) given by Lemma 4.3. There exists a subsequence of  $(\varphi_m)_m$  (still denoted by  $\varphi_m$ ) and*

$$\varphi \in \mathcal{C}^{0,1}(\mathbb{R}, H^{-1}) \cap \mathcal{C}^{0, \frac{1}{2}}(\mathbb{R}, L^2) \cap L_{\text{loc}}^\infty(\mathbb{R}, H^1)$$

such that  $\varphi_m$  converges to  $\varphi$  as  $m \rightarrow \infty$  in the following sense:

- $\varphi_m \xrightarrow{*} \varphi$  in  $L^\infty(I, H^1)$  for its weak-\* topology, for every bounded interval  $I$ ,
- $\varphi_m(t) \rightarrow \varphi(t)$  in  $L^2$  for its weak topology, for every  $t \in \mathbb{R}$ .

Moreover,  $\lambda u_m \ln |u_m|^2$  converges to  $i\partial_t\varphi + \Delta\varphi + \Delta u_0$  for the weak-\* topology of  $L^\infty(\mathbb{R}, L^2)$ .

*Proof.* We know by Lemmas 4.5 and 4.6 that  $\varphi_m$  is uniformly bounded in  $L^\infty(I, H^1)$ , which is the dual of  $L^1(I, H^{-1})$ , for every bounded interval  $I$ . Then, this sequence is relatively compact for the weak-\* topology of  $L^\infty(I, H^1)$ . By diagonal extraction, there is a subsequence (still denoted by  $\varphi_m$ ) which converges to some  $\varphi \in L^\infty_{\text{loc}}(\mathbb{R}, H^1)$  for the weak-\* topology of  $L^\infty(I, H^1)$ , for every bounded interval  $I$ .

Besides,  $u_m \ln |u_m|^2$  is also uniformly bounded in  $L^\infty(\mathbb{R}, L^2) = \left(L^1(\mathbb{R}, L^2)\right)'$  thanks to Corollary 2.8 and the conservation of the energy. Therefore, it is also relatively compact for the weak-\* topology of  $L^\infty(\mathbb{R}, L^2)$ .

Let  $\theta \in C_c^\infty(\mathbb{R})$  and  $j \leq m$ :

$$\begin{aligned} \lambda \int \theta(\tau) \langle w_j, u_m \ln |u_m|^2 \rangle d\tau &= \int \dot{\theta}(\tau) \langle w_j, i\varphi_m(\tau) \rangle d\tau - \int \theta(t) \langle \nabla w_j, \nabla \varphi_m(\tau) \rangle d\tau \\ &\quad - \int \theta(t) \langle \nabla w_j, \nabla u_0 \rangle d\tau \\ &\xrightarrow{m \rightarrow \infty} \int \dot{\theta}(\tau) \langle w_j, i\varphi(\tau) \rangle d\tau - \int \theta(t) \langle \nabla w_j, \nabla \varphi(\tau) \rangle d\tau \\ &\quad - \int \theta(t) \langle \nabla w_j, \nabla u_0 \rangle d\tau. \end{aligned}$$

Since  $(w_j)_j$  is an Hilbertian basis of  $L^2$ , this proves that  $\lambda u_m \ln |u_m|^2$  also converges to  $i\partial_t\varphi + \Delta\varphi + \Delta u_0$  in  $\mathcal{D}'(\mathbb{R}, L^2)$ . Therefore, the convergence is also in the weak-\* topology of  $L^\infty(\mathbb{R}, L^2)$ , thus we get  $i\partial_t\varphi + \Delta\varphi + \Delta u_0 \in L^\infty(\mathbb{R}, L^2)$ .

From this and the fact that  $\Delta\varphi \in L^\infty_{\text{loc}}(\mathbb{R}, H^{-1})$ , we get  $\partial_t\varphi \in L^\infty_{\text{loc}}(\mathbb{R}, H^{-1})$ . Therefore,  $\varphi \in W_{\text{loc}}^{1,\infty}(\mathbb{R}, H^{-1})$  and thus  $\varphi \in C_{\text{loc}}^{0,1}(\mathbb{R}, H^{-1}) \cap L^\infty_{\text{loc}}(\mathbb{R}, H^1)$ . By interpolation, this leads to  $\varphi \in C_{\text{loc}}^{0,\frac{1}{2}}(\mathbb{R}, L^2)$ .

As for the convergence in the second item, let  $t \in \mathbb{R}$ . Since  $\varphi_m(t)$  is bounded in  $L^2$  uniformly in  $m$ , the sequence is relatively compact for the weak topology of  $L^2$ . Let  $\chi$  be the limit of a subsequence (still denoted by  $\varphi_m$ ). It is then enough to prove that  $\chi = \varphi(t)$ . Let  $\gamma \in (0, 1)$  Then,

$$\begin{aligned} \|\varphi(t) - \chi\|_{L^2}^2 &= \langle \varphi(t) - \chi, \varphi_m(t) - \chi \rangle + \frac{1}{2\gamma} \int_{t-\gamma}^{t+\gamma} \langle \varphi(t) - \chi, \varphi(t) - \varphi_m(t) \rangle d\tau \\ &= \langle \varphi(t) - \chi, \varphi_m(t) - \chi \rangle \\ &\quad + \frac{1}{2\gamma} \int_{t-\gamma}^{t+\gamma} \langle \varphi(t) - \chi, \varphi(t) - \varphi(\tau) + \varphi(\tau) - \varphi_m(\tau) + \varphi_m(\tau) - \varphi_m(t) \rangle d\tau. \end{aligned}$$

Since  $\varphi, \varphi_m \in C^{0,\frac{1}{2}}([t-1, t+1], L^2)$  uniformly in  $m$ , we have for all  $\tau \in [t-1, t+1]$

$$\begin{aligned} |\langle \varphi(t) - \chi, \varphi(t) - \varphi(\tau) \rangle| &\leq C_t |t - \tau|^{\frac{1}{2}} \|\varphi(t) - \chi\|_{L^2}, \\ |\langle \varphi(t) - \chi, \varphi_m(t) - \varphi_m(\tau) \rangle| &\leq C_t |t - \tau|^{\frac{1}{2}} \|\varphi(t) - \chi\|_{L^2}, \end{aligned}$$

hence

$$\begin{aligned}
\|\varphi(t) - \chi\|_{L^2}^2 &\leq \langle \varphi(t) - \chi, \varphi_m(t) - \chi \rangle + \frac{C_t}{2\gamma} \|\varphi(t) - \chi\|_{L^2} \int_{t-\gamma}^{t+\gamma} |t - \tau|^{\frac{1}{2}} d\tau \\
&\quad + \frac{1}{2\gamma} \int_{t-\gamma}^{t+\gamma} \langle \varphi(t) - \chi, \varphi(\tau) - \varphi_m(\tau) \rangle d\tau \\
(4.6) \quad &\leq \langle \varphi(t) - \chi, \varphi_m(t) - \chi \rangle + C_t \gamma^{\frac{1}{2}} \|\varphi(t) - \chi\|_{L^2} \\
&\quad + \frac{1}{2\gamma} \int_{t-\gamma}^{t+\gamma} \langle \varphi(t) - \chi, \varphi(\tau) - \varphi_m(\tau) \rangle d\tau.
\end{aligned}$$

Moreover, we know that

- $\langle \varphi(t) - \chi, \varphi_m(t) - \chi \rangle \xrightarrow{m \rightarrow \infty} 0$  by weak convergence in  $L^2$  of  $\varphi_m(t)$  to  $\chi$ ,
- $\int_{t-\gamma}^{t+\gamma} \langle \varphi(t) - \chi, \varphi(\tau) - \varphi_m(\tau) \rangle d\tau \xrightarrow{m \rightarrow \infty} 0$  by the weak-\* convergence in  $L^\infty((t-1, t+1), L^2)$  of  $\varphi_m$  to  $\varphi$ .

Therefore, taking the limit  $m \rightarrow \infty$  in (4.6), we get

$$\|\varphi(t) - \chi\|_{L^2}^2 \leq C_t \gamma^{\frac{1}{2}} \|\varphi(t) - \chi\|_{L^2}.$$

Since this is true for all  $\gamma \in (0, 1)$ , we get  $\|\varphi(t) - \chi\|_{L^2} = 0$ , hence the conclusion.  $\square$

**4.4. Equation and initial data for the limit.** We are now able to prove that  $u = u_0 + \varphi$  is indeed a solution to (logGP).

**Lemma 4.8.** *Let  $\varphi \in C^{0,1}(\mathbb{R}, H^{-1}) \cap C^{0,\frac{1}{2}}(\mathbb{R}, L^2) \cap L_{\text{loc}}^\infty(\mathbb{R}, H^1)$  defined in Lemma 4.7. Then  $u := u_0 + \varphi$  satisfies (logGP).*

*Proof.* First, since  $\varphi_m(0) \rightharpoonup \varphi(0)$  and  $\varphi_m(0) = 0$  by construction, we get  $\varphi(0) = 0$  and thus  $u|_{t=0} = u_0$ .

We also know from Lemma 4.7 that  $\lambda u_m \ln |u_m|^2$  converges to  $i\partial_t \varphi + \Delta \varphi + \Delta u_0$  for the weak-\* topology of  $L^\infty(\mathbb{R}, L^2)$ , and thus also for the weak topology of  $L^2(I \times \Omega)$  for every bounded interval  $I$  and every bounded open subset of  $\mathbb{R}^d$ .

We also know that  $\varphi_m$  is  $\frac{1}{2}$ -Hölder continuous in time with values in  $L^2(\mathbb{R}^d)$  on  $I$ , uniformly in  $m$  from Lemma 4.6, and thus also on  $L^2(\Omega)$ . Last,  $\varphi_m(t) \in H^1$  for all  $t \in I$  along with a uniform bound in  $m$  by Lemmas 4.5 and 4.6. Thus, Arzela–Ascoli Theorem for  $\varphi_m|_\Omega$  yields its relative compactness in  $\mathcal{C}(I, L^2(\Omega))$ , which shows that  $\varphi_m|_\Omega$  converges strongly to  $\varphi|_\Omega$  in  $\mathcal{C}(I, L^2(\Omega))$ , and thus also in  $L^2(I \times \Omega)$ . Up to a further subsequence,  $\varphi_m$  converges a.e. to  $\varphi$  in  $I \times \Omega$ , and so does  $u_m$  to  $u$ . By continuity of  $z \mapsto z \ln |z|^2$  on  $\mathbb{C}$ , we obtain the convergence of  $u_m(t, x) \ln |u_m(t, x)|^2$  to  $u(t, x) \ln |u(t, x)|^2$  a.e. in  $I \times \Omega$ . Besides, we also know that  $u_m \ln |u_m|^2$  is bounded in  $L^\infty(I, L^2)$  uniformly in  $m$ , and thus also in  $L^2(I \times \Omega)$ . Therefore, along this further subsequence,  $u_m \ln |u_m|^2$  converges weakly to  $u \ln |u|^2$  in  $L^2(I \times \Omega)$ .

Therefore, by uniqueness of the limit, we get  $\lambda u \ln |u|^2 = i\partial_t \varphi + \Delta \varphi + \Delta u_0$ , which gives the conclusion.  $\square$

**4.5. Conservation law.** We now prove that the energy of the solution that we have just constructed is independent of time. Note that this is necessarily *the* solution to (logGP), in view of Theorem 3.2.

**Lemma 4.9.** *Let  $\varphi \in C^{0,1}(\mathbb{R}, H^{-1}) \cap C^{0,\frac{1}{2}}(\mathbb{R}, L^2) \cap L_{\text{loc}}^\infty(\mathbb{R}, H^1)$  defined in Lemma 4.7 and  $u := u_0 + \varphi$ . Then  $u$  satisfies  $\mathcal{E}_{\text{log}}(u(t)) = \mathcal{E}_{\text{log}}(u_0)$  for all  $t \in \mathbb{R}$ .*

*Proof.* This proof is an adaptation of the techniques developed by Ozawa [39] to prove conservation laws from the mild formulation, and resumes the regularization procedure presented in e.g. [13]. We know that  $\varphi \in L_{\text{loc}}^\infty(\mathbb{R}, H^1)$ , thus Lemma 2.10 shows that  $u \in L_{\text{loc}}^\infty(\mathbb{R}, E_{\text{logGP}})$ . From Lemma 3.3,  $u$  is a mild solution. Moreover, we also know that  $\nabla u(t) \in L^2$  for all  $t \in \mathbb{R}$  (and not only a.e.). Therefore, for all  $\varepsilon > 0$ ,  $(1 - \varepsilon\Delta)^{-1}\nabla u(t)$  is well defined in  $L^2$  and

$$(1 - \varepsilon\Delta)^{-1}\nabla u(t) \xrightarrow{\varepsilon \rightarrow 0} \nabla u(t) \quad \text{in } L^2.$$

On the other hand, we also have

$$\begin{aligned} \|(1 - \varepsilon\Delta)^{-1}\nabla u(t)\|_{L^2}^2 &= \|e^{-it\Delta}(1 - \varepsilon\Delta)^{-1}\nabla u(t)\|_{L^2}^2 \\ &= \|(1 - \varepsilon\Delta)^{-1}e^{-it\Delta}\nabla u(t)\|_{L^2}^2 \\ &= \|(1 - \varepsilon\Delta)^{-1}e^{-it\Delta}\nabla u_0 + (1 - \varepsilon\Delta)^{-1}e^{-it\Delta}\nabla(u(t) - u_0)\|_{L^2}^2. \end{aligned}$$

The last line is well defined since  $\nabla u_0 \in L^2$ . Moreover, since  $u(t) - u_0 \in H^1$ , we have  $e^{-it\Delta}\nabla(u(t) - u_0) = \nabla e^{-it\Delta}(u(t) - u_0)$ , i.e.

$$\|(1 - \varepsilon\Delta)^{-1}\nabla u(t)\|_{L^2}^2 = \|(1 - \varepsilon\Delta)^{-1}e^{-it\Delta}\nabla u_0 + (1 - \varepsilon\Delta)^{-1}\nabla e^{-it\Delta}(u(t) - u_0)\|_{L^2}^2$$

Since  $(1 - \varepsilon\Delta)^{-1}\nabla$  is bounded on  $L^2$  (e.g. as a Fourier multiplier), we can substitute  $u(t)$  by the mild formulation :

$$\begin{aligned} &(1 - \varepsilon\Delta)^{-1}\nabla e^{-it\Delta}(u(t) - u_0) \\ &= (1 - \varepsilon\Delta)^{-1}\nabla \left( u_0 - i\lambda \int_0^t e^{-is\Delta} \left( u(s) \ln |u(s)|^2 \right) ds - e^{-it\Delta}u_0 \right). \end{aligned}$$

Moreover, since the Schrödinger semigroup  $e^{it\Delta}$  maps  $u_0 + H^1$  into itself, we have

$$e^{-it\Delta}\nabla u_0 = \nabla e^{-it\Delta}u_0,$$

which leads to

$$\|(1 - \varepsilon\Delta)^{-1}\nabla u(t)\|_{L^2}^2 = \left\| (1 - \varepsilon\Delta)^{-1}\nabla \left( u_0 - i\lambda \int_0^t e^{-is\Delta} \left( u(s) \ln |u(s)|^2 \right) ds \right) \right\|_{L^2}^2.$$

Now, we know thanks to Lemma 3.3 that  $\int_0^t e^{-is\Delta} \left( u(s) \ln |u(s)|^2 \right) ds \in H^1$ . Therefore,

$$\begin{aligned} &\|(1 - \varepsilon\Delta)^{-1}\nabla u(t)\|_{L^2}^2 \\ &= \left\| (1 - \varepsilon\Delta)^{-1}\nabla u_0 - i\lambda(1 - \varepsilon\Delta)^{-1}\nabla \int_0^t e^{-is\Delta} \left( u(s) \ln |u(s)|^2 \right) ds \right\|_{L^2}^2 \\ &= \|(1 - \varepsilon\Delta)^{-1}\nabla u_0\|_{L^2}^2 \\ &\quad - 2\lambda \operatorname{Re} \left\langle (1 - \varepsilon\Delta)^{-1}\nabla u_0, i(1 - \varepsilon\Delta)^{-1}\nabla \int_0^t e^{-is\Delta} \left( u(s) \ln |u(s)|^2 \right) ds \right\rangle \\ &\quad + \lambda^2 \left\| (1 - \varepsilon\Delta)^{-1}\nabla \int_0^t e^{-is\Delta} \left( u(s) \ln |u(s)|^2 \right) ds \right\|_{L^2}^2. \end{aligned}$$

Then, we compute the last two terms of the right-hand side: for the last term,

$$\begin{aligned}
& \left\| (1 - \varepsilon\Delta)^{-1} \nabla \int_0^t e^{-is\Delta} \left( u(s) \ln |u(s)|^2 \right) ds \right\|_{L^2}^2 \\
&= \left\| \int_0^t e^{-is\Delta} (1 - \varepsilon\Delta)^{-1} \nabla \left( u(s) \ln |u(s)|^2 \right) ds \right\|_{L^2}^2 \\
&= \operatorname{Re} \left\langle \int_0^t e^{-is\Delta} (1 - \varepsilon\Delta)^{-1} \nabla \left( u(s) \ln |u(s)|^2 \right) ds, \int_0^t e^{-is'\Delta} (1 - \varepsilon\Delta)^{-1} \nabla \left( u(s') \ln |u(s')|^2 \right) ds' \right\rangle \\
&= \operatorname{Re} \int_0^t \int_0^t \left\langle e^{-is\Delta} (1 - \varepsilon\Delta)^{-1} \nabla \left( u(s) \ln |u(s)|^2 \right), e^{-is'\Delta} (1 - \varepsilon\Delta)^{-1} \nabla \left( u(s') \ln |u(s')|^2 \right) \right\rangle ds ds' \\
&= 2 \operatorname{Re} \int \int_{0 \leq s' \leq s \leq t} \left\langle e^{-is\Delta} (1 - \varepsilon\Delta)^{-1} \nabla \left( u(s) \ln |u(s)|^2 \right), e^{-is'\Delta} (1 - \varepsilon\Delta)^{-1} \nabla \left( u(s') \ln |u(s')|^2 \right) \right\rangle ds ds',
\end{aligned}$$

by symmetry  $s \leftrightarrow s'$ . Therefore,

$$\begin{aligned}
& \left\| (1 - \varepsilon\Delta)^{-1} \nabla \int_0^t e^{-is\Delta} \left( u(s) \ln |u(s)|^2 \right) ds \right\|_{L^2}^2 \\
&= 2 \operatorname{Re} \int \int_{0 \leq s' \leq s \leq t} \left\langle (1 - \varepsilon\Delta)^{-1} \nabla \left( u(s) \ln |u(s)|^2 \right), e^{i(s-s')\Delta} (1 - \varepsilon\Delta)^{-1} \nabla \left( u(s') \ln |u(s')|^2 \right) \right\rangle ds ds' \\
&= 2 \operatorname{Re} \int_0^t \left\langle (1 - \varepsilon\Delta)^{-1} \nabla \left( u(s) \ln |u(s)|^2 \right), \int_0^s e^{i(s-s')\Delta} (1 - \varepsilon\Delta)^{-1} \nabla \left( u(s') \ln |u(s')|^2 \right) ds' \right\rangle ds \\
&= \frac{2}{\lambda} \operatorname{Im} \int_0^t \left\langle (1 - \varepsilon\Delta)^{-1} \nabla \left( u(s) \ln |u(s)|^2 \right), -i\lambda (1 - \varepsilon\Delta)^{-1} \nabla \int_0^s e^{i(s-s')\Delta} \left( u(s') \ln |u(s')|^2 \right) ds' \right\rangle ds \\
&= \frac{2}{\lambda} \operatorname{Im} \int_0^t \left\langle (1 - \varepsilon\Delta)^{-1} \nabla \left( u(s) \ln |u(s)|^2 \right), (1 - \varepsilon\Delta)^{-1} \nabla \left[ u(s) - e^{is\Delta} u_0 \right] \right\rangle ds.
\end{aligned}$$

For the other term,

$$\begin{aligned}
& \operatorname{Re} \left\langle (1 - \varepsilon\Delta)^{-1} \nabla u_0, i(1 - \varepsilon\Delta)^{-1} \nabla \int_0^t e^{-is\Delta} \left( u(s) \ln |u(s)|^2 \right) ds \right\rangle \\
&= - \operatorname{Im} \int_0^t \left\langle (1 - \varepsilon\Delta)^{-1} \nabla u_0, e^{-is\Delta} (1 - \varepsilon\Delta)^{-1} \nabla \left( u(s) \ln |u(s)|^2 \right) \right\rangle ds \\
&= - \operatorname{Im} \int_0^t \left\langle e^{is\Delta} (1 - \varepsilon\Delta)^{-1} \nabla u_0, (1 - \varepsilon\Delta)^{-1} \nabla \left( u(s) \ln |u(s)|^2 \right) \right\rangle ds \\
&= \int_0^t \operatorname{Im} \left\langle (1 - \varepsilon\Delta)^{-1} \nabla \left( u(s) \ln |u(s)|^2 \right), (1 - \varepsilon\Delta)^{-1} \nabla e^{is\Delta} u_0 \right\rangle ds.
\end{aligned}$$



Therefore, there holds

$$\begin{aligned}
& \|(1 - \varepsilon\Delta)^{-1}\nabla u(t)\|_{L^2}^2 - \|(1 - \varepsilon\Delta)^{-1}\nabla u_0\|_{L^2}^2 \\
&= -2\lambda \operatorname{Im} \int_0^t \left\langle (1 - \varepsilon\Delta)^{-1}\nabla \left( u(s) \ln |u(s)|^2 \right), (1 - \varepsilon\Delta)^{-1}\nabla u(s) \right\rangle ds \\
&= 2\lambda \operatorname{Im} \int_0^t \left\langle (1 - \varepsilon\Delta)^{-1} \left( u(s) \ln |u(s)|^2 \right), (1 - \varepsilon\Delta)^{-1}\Delta u(s) \right\rangle ds \\
&= 2\lambda \operatorname{Im} \int_0^t \left\langle (1 - \varepsilon\Delta)^{-1} \left( u(s) \ln |u(s)|^2 \right), (1 - \varepsilon\Delta)^{-1} \left[ \Delta u(s) - \lambda u(s) \ln |u(s)|^2 \right] \right\rangle ds \\
&= -2\lambda \operatorname{Im} \int_0^t \left\langle (1 - \varepsilon\Delta)^{-1} \left( u(s) \ln |u(s)|^2 \right), (1 - \varepsilon\Delta)^{-1} i\partial_t u(s) \right\rangle ds \\
&= -2\lambda \operatorname{Re} \int_0^t \left\langle (1 - \varepsilon\Delta)^{-1} \left( u(s) \ln |u(s)|^2 \right), (1 - \varepsilon\Delta)^{-1} \partial_t u(s) \right\rangle ds.
\end{aligned}$$

Thanks to Lemma 2.6, we get

$$\begin{aligned}
& \|(1 - \varepsilon\Delta)^{-1}\nabla u(t)\|_{L^2}^2 - \|(1 - \varepsilon\Delta)^{-1}\nabla u_0\|_{L^2}^2 \\
&= -\lambda \int_0^t \frac{d}{ds} \left[ \int (1 - \varepsilon\Delta)^{-2} F(|u(s)|^2) dx \right] ds \\
&= \lambda \int (1 - \varepsilon\Delta)^{-2} F(|u_0|^2) dx - \lambda \int (1 - \varepsilon\Delta)^{-2} F(|u(t)|^2) dx,
\end{aligned}$$

which gives  $\mathcal{E}_{\log\text{GP}}(u(t)) = \mathcal{E}_{\log\text{GP}}(u_0)$  after taking the limit  $\varepsilon \rightarrow 0$ .  $\square$

## 5. HIGHER REGULARITY

In this section, we prove the propagation of the  $\dot{H}^2$  regularity by the flow of (logGP). We would like to proceed like in [12] (see also [14, 13]) directly on the solution  $u$  constructed, i.e. differentiating (logGP) with respect to time and get an  $L^2$ -energy estimate for  $\partial_t \varphi$ . A direct formal computation would lead to the local boundedness of this  $L^2$ -norm with the additional argument that  $\partial_t u(0) \in L^2$  as soon as  $\Delta u_0 \in L^2$ . However, this computation cannot directly be made rigorous since  $u \ln |u|^2$  cannot be differentiated due to the lack of regularity of the logarithm.

To overcome this difficulty, we shall work on the approximate solutions  $u_m$ . In [12], the authors worked on the approximate solutions to prove the propagation of the  $H^2$  regularity, since the above mentioned energy estimate is licit in the case of (4.1) (and provides bounds which are uniform in  $\varepsilon$ ). However, we cannot reproduce the same proof here: our approximate solutions do not satisfy an equation with a regularized nonlinearity. Indeed, the equations (4.3) can be put under the form

$$i\partial_t \varphi_m + \mathbb{P}_m \left( \Delta u_m - \lambda u_m \ln |u_m|^2 \right) = 0,$$

where  $\mathbb{P}_m$  is the orthogonal projector from  $L^2$  onto  $X_m$  (we can assume  $w_j \in H^2$  to make this rigorous).

Thus, the nonlinearity is still not smooth enough to make the computation rigorous. However, by assuming  $w_j \in H^2$  (for all  $j$ ), we automatically have  $\varphi_m(t) \in H^2$  for all  $t \in \mathbb{R}$  and  $m \in \mathbb{N}$ . Even more, we have  $\varphi_m \in \mathcal{C}^1(\mathbb{R}, H^2)$  since the  $g_{m,k}$ 's belong to  $\mathcal{C}^1(\mathbb{R})$  (see Lemma 4.3). Those crude bounds are *a priori* not uniform in  $m$ , but we show that we can improve them.

**Lemma 5.1.** *Let  $u_m$  given by Lemma 4.3 and assume  $w_j \in H^2$  for all  $j \in \mathbb{N}$ . Then, for all bounded interval  $I$ ,  $\partial_t u_m$  is bounded in  $C^0(I, L^2)$  uniformly in  $m$ .*

*Proof.* From Lemma 4.3, we know that the  $g_{m,j}$ 's are  $C^{1,\varepsilon}$  for all  $\varepsilon \in (0, 1)$ , therefore  $\partial_t u_m \in C^{0,\varepsilon}(\mathbb{R}, H^2)$ . Let  $\tau > 0$  and

$$\psi_{m,\tau}(t) := \frac{\|\partial_t \varphi_m(t+\tau)\|_{L^2}^2 - \|\partial_t \varphi_m(t)\|_{L^2}^2}{\tau},$$

which is well defined for all  $t \in \mathbb{R}$  and any  $\tau > 0$ . Our goal is to prove a bound on  $\psi_{m,\tau}(t)$  independent of  $\tau \leq 1$ . For this, we first rewrite

$$\psi_{m,\tau}(t) = \frac{1}{\tau} \operatorname{Re} \langle \partial_t \varphi_m(t+\tau) + \partial_t \varphi_m(t), \partial_t \varphi_m(t+\tau) - \partial_t \varphi_m(t) \rangle.$$

Since  $\partial_t \varphi_m(t+\tau)$  and  $\partial_t \varphi_m(t)$  are in  $\operatorname{Vect}(w_j)_{j \leq m}$ , we can use (4.3) and get

$$\begin{aligned} \tau \psi_{m,\tau}(t) &= -\operatorname{Im} \langle \partial_t \varphi_m(t+\tau) + \partial_t \varphi_m(t), \Delta \varphi_m(t+\tau) - \Delta \varphi_m(t) \rangle \\ &\quad + \lambda \operatorname{Im} \langle \partial_t \varphi_m(t+\tau) + \partial_t \varphi_m(t), u_m(t+\tau) \ln |u_m(t+\tau)|^2 - u_m(t) \ln |u_m(t)|^2 \rangle. \end{aligned}$$

For the first term, we get

$$\begin{aligned} \langle \partial_t \varphi_m(t), \Delta \varphi_m(t+\tau) - \Delta \varphi_m(t) \rangle &= \langle \partial_t \varphi_m(t), \Delta \varphi_m(t+\tau) - \Delta \varphi_m(t) - \tau \partial_t \Delta \varphi_m(t) \rangle \\ &\quad + \tau \langle \partial_t \varphi_m(t), \partial_t \Delta \varphi_m(t) \rangle. \end{aligned}$$

Since  $\langle \partial_t \varphi_m(t), \partial_t \Delta \varphi_m(t) \rangle = -\|\partial_t \nabla \varphi_m\|_{L^2}^2$ , the last term vanishes when we take the imaginary part. On the other hand, we have shown that  $\Delta \varphi_m \in C^{1,\varepsilon}(\mathbb{R}, L^2)$ . Hence, for all  $\tau \leq 1$ ,

$$\|\Delta \varphi_m(t+\tau) - \Delta \varphi_m(t) - \tau \partial_t \Delta \varphi_m(t)\|_{L^2} \leq C_{m,\varepsilon,t} \tau^{1+\varepsilon}.$$

Therefore, for all  $t \in \mathbb{R}$ ,  $\tau, \varepsilon \in (0, 1)$ ,

$$|\operatorname{Im} \langle \partial_t \varphi_m(t), \Delta \varphi_m(t+\tau) - \Delta \varphi_m(t) \rangle| \leq C_{m,\varepsilon,t} \tau^{1+\varepsilon},$$

and similarly for  $\operatorname{Im} \langle \partial_t \varphi_m(t+\tau), \Delta \varphi_m(t+\tau) - \Delta \varphi_m(t) \rangle$ .

Now, we also have

$$\begin{aligned} &\langle \partial_t \varphi_m(t), u_m(t+\tau) \ln |u_m(t+\tau)|^2 - u_m(t) \ln |u_m(t)|^2 \rangle \\ &= \frac{1}{\tau} \langle \varphi_m(t+\tau) - \varphi_m(t), u_m(t+\tau) \ln |u_m(t+\tau)|^2 - u_m(t) \ln |u_m(t+\tau)|^2 \rangle \\ &\quad - \frac{1}{\tau} \langle \varphi_m(t+\tau) - \varphi_m(t) - \tau \partial_t \varphi_m(t), u_m(t+\tau) \ln |u_m(t+\tau)|^2 - u_m(t) \ln |u_m(t+\tau)|^2 \rangle. \end{aligned}$$

For the first term, we can use Lemma 2.1 since  $\varphi_m(t+\tau) - \varphi_m(t) = u_m(t+\tau) - u_m(t)$ , so that

$$\begin{aligned} &\left| \operatorname{Im} \langle \varphi_m(t+\tau) - \varphi_m(t), u_m(t+\tau) \ln |u_m(t+\tau)|^2 - u_m(t) \ln |u_m(t+\tau)|^2 \rangle \right| \\ &\leq 2 \|u_m(t+\tau) - u_m(t)\|_{L^2}^2 = 2 \|\varphi_m(t+\tau) - \varphi_m(t)\|_{L^2}^2. \end{aligned}$$

For the second term, we can use Lemma 2.2 so that

$$\begin{aligned} &\left| \langle \varphi_m(t+\tau) - \varphi_m(t) - \tau \partial_t \varphi_m(t), u_m(t+\tau) \ln |u_m(t+\tau)|^2 - u_m(t) \ln |u_m(t+\tau)|^2 \rangle \right| \\ &\leq \|\varphi_m(t+\tau) - \varphi_m(t) - \tau \partial_t \varphi_m(t)\|_{L^2} \left\| u_m(t+\tau) \ln |u_m(t+\tau)|^2 - u_m(t) \ln |u_m(t+\tau)|^2 \right\|_{L^2} \end{aligned}$$

The first factor can be estimated like previously with the fact that  $\varphi_m \in C^{1,\varepsilon}(\mathbb{R}; L^2)$ :

$$\|\varphi_m(t+\tau) - \varphi_m(t) - \tau \partial_t \varphi_m(t)\|_{L^2} \leq C_{m,t,\varepsilon} \tau^{1+\varepsilon}.$$

As for the other factor, we use Lemma 2.2 and then Corollary 2.8 to obtain

$$\begin{aligned}
& \left\| u_m(t+\tau) \ln |u_m(t+\tau)|^2 - u_m(t) \ln |u_m(t+\tau)|^2 \right\|_{L^2} \\
& \lesssim \left\| \left( |u_m(t+\tau)|^{\frac{\varepsilon}{2}} |\ln |u_m(t+\tau)|| + |u_m(t)|^{\frac{\varepsilon}{2}} |\ln |u_m(t)|| \right) |u_m(t+\tau) - u_m(t)|^{1-\frac{\varepsilon}{2}} \right\|_{L^2} \\
& \quad + \|u_m(t+\tau) - u_m(t)\|_{L^2} \\
& \lesssim \left\| |u_m(t+\tau)|^{\frac{\varepsilon}{2}} |\ln |u_m(t+\tau)|| + |u_m(t)|^{\frac{\varepsilon}{2}} |\ln |u_m(t)|| \right\|_{L^{\frac{4}{\varepsilon}}} \left\| |u_m(t+\tau) - u_m(t)|^{1-\varepsilon} \right\|_{L^{\frac{2}{1-\frac{\varepsilon}{2}}}} \\
& \quad + \|u_m(t+\tau) - u_m(t)\|_{L^2} \\
& \lesssim \left( \left\| |u_m(t+\tau)| |\ln |u_m(t+\tau)|| \right\|_{L^2}^{\frac{2}{\varepsilon}} + \left\| |u_m(t)| |\ln |u_m(t)|| \right\|_{L^2}^{\frac{2}{\varepsilon}} \right) \|u_m(t+\tau) - u_m(t)\|_{L^2}^{1-\frac{\varepsilon}{2}} \\
& \quad + \|u_m(t+\tau) - u_m(t)\|_{L^2} \\
& \lesssim \left( \mathcal{E}_{\log\text{GP}}(u_0)^{\varepsilon/2} + \mathcal{E}_{\log\text{GP}}(u_0)^{(\varepsilon+\varepsilon^2)/2} \right) \|u_m(t+\tau) - u_m(t)\|_{L^2}^{1-\frac{\varepsilon}{2}} + \|u_m(t+\tau) - u_m(t)\|_{L^2}.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
& \left| \langle \varphi_m(t+\tau) - \varphi_m(t) - \tau \partial_t \varphi_m(t), u_m(t+\tau) \ln |u_m(t+\tau)|^2 - u_m(t) \ln |u_m(t+\tau)|^2 \rangle \right| \\
& \leq C_{m,t,\varepsilon} \left[ \tau^{2+\frac{\varepsilon}{2}} \left( \mathcal{E}_{\log\text{GP}}(u_0)^{\varepsilon/2} + \mathcal{E}_{\log\text{GP}}(u_0)^{\varepsilon} \right) \|\delta_\tau \varphi_m\|_{L^2}^{1-\frac{\varepsilon}{2}} + \tau^{2+\varepsilon} \|\delta_\tau \varphi_m\|_{L^2} \right],
\end{aligned}$$

where  $\delta_\tau \varphi_m := \frac{1}{\tau}(\varphi_m(t+\tau) - \varphi_m(t))$ . Thus, we get

$$\begin{aligned}
& \left| \text{Im} \langle \partial_t \varphi_m(t), u_m(t+\tau) \ln |u_m(t+\tau)|^2 - u_m(t) \ln |u_m(t)|^2 \rangle \right| \\
& \leq (2\tau + C_{m,t,\varepsilon} \tau^{1+\varepsilon}) \|\delta_\tau \varphi_m\|_{L^2}^2 \\
& \quad + C_{m,t,\varepsilon} \tau^{1+\frac{\varepsilon}{2}} \left( \mathcal{E}_{\log\text{GP}}(u_0)^{\varepsilon/2} + \mathcal{E}_{\log\text{GP}}(u_0)^{\varepsilon} \right) \|\delta_\tau \varphi_m\|_{L^2}^{1-\frac{\varepsilon}{2}}.
\end{aligned}$$

Similar computations can be done for  $\text{Im} \langle \partial_t \varphi_m(t+\tau), u_m(t+\tau) \ln |u_m(t+\tau)|^2 - u_m(t) \ln |u_m(t)|^2 \rangle$ , so we get

$$\begin{aligned}
(5.1) \quad |\psi_{m,\tau}(t)| & \leq C_{m,t,\varepsilon} \tau^\varepsilon + (4\lambda + C_{m,t,\varepsilon} \tau^\varepsilon) \|\delta_\tau \varphi_m\|_{L^2}^2 \\
& \quad + C_{m,t,\varepsilon} \tau^{1+\frac{\varepsilon}{2}} \left( \mathcal{E}_{\log\text{GP}}(u_0)^{\varepsilon/2} + \mathcal{E}_{\log\text{GP}}(u_0)^{\varepsilon} \right) \|\delta_\tau \varphi_m\|_{L^2}^{1-\frac{\varepsilon}{2}}.
\end{aligned}$$

At  $m$  fixed, since  $\varphi_m \in \mathcal{C}^{1,\varepsilon}(\mathbb{R}; L^2)$ , we know that as  $\tau \rightarrow 0$ ,  $\delta_\tau \varphi_m(t)$  converges strongly to  $\partial_t \varphi_m(t)$  in  $L^2$  for any  $t \in \mathbb{R}$ , but also that, for  $t \in \mathbb{R}$  fixed,  $\|\delta_\tau \varphi_m\|_{L^2}$  is uniformly bounded in  $\tau \leq 1$ . The previous estimate shows that  $\psi_{m,\tau}(t)$  is uniformly bounded in  $\tau \leq 1$  at  $m \in \mathbb{N}$  and  $t \in \mathbb{R}$  fixed, which means that

$$t \mapsto \|\partial_t \varphi_m(t)\|_{L^2}^2 \in W_{\text{loc}}^{1,\infty}(\mathbb{R}, \mathbb{R}).$$

The limit  $\tau \rightarrow 0$  in (5.1) then yields

$$\left| \frac{d}{dt} \|\partial_t \varphi_m(t)\|_{L^2}^2 \right| \leq 4\lambda \|\partial_t \varphi_m(t)\|_{L^2}^2.$$

On the other hand, we also have  $i\partial_t \varphi_m(0) = \mathbb{P}_m [\lambda u_0 \ln |u_0|^2 - \Delta u_0]$ , and since  $\mathbb{P}_m$  is an orthogonal projector, we get

$$\begin{aligned}
\|\partial_t \varphi_m(0)\|_{L^2} & \leq \lambda \left\| u_0 \ln |u_0|^2 \right\|_{L^2} + \|\Delta u_0\|_{L^2} \\
& \leq C_\varepsilon \lambda (\mathcal{E}_{\log\text{GP}}(u_0)^{\varepsilon/2} + \mathcal{E}_{\log\text{GP}}(u_0)^{\varepsilon}) + \|\Delta u_0\|_{L^2}.
\end{aligned}$$

Thanks to Gronwall Lemma, we get a bound which is independent of  $m$ : for all  $m \in \mathbb{N}$  and  $t \in \mathbb{R}$ ,

$$\|\partial_t \varphi_m(t)\|_{L^2}^2 \leq \left[ C_\varepsilon \lambda (\mathcal{E}_{\log\text{GP}}(u_0))^{\varepsilon/2} + \mathcal{E}_{\log\text{GP}}(u_0)^\varepsilon + \|\Delta u_0\|_{L^2} \right]^2 e^{4\lambda t}. \quad \square$$

**Corollary 5.2.** *Let  $u$  given by Lemma 4.7. Then, for all bounded interval  $I$ ,  $\partial_t u$  is bounded in  $L^\infty(I; L^2)$ , and so is  $\Delta u$ .*

*Proof.* We want to take the limit  $m \rightarrow \infty$  in Lemma 5.1. For this, we use the fact that  $\varphi_m \xrightarrow{*} \varphi$  in  $L^\infty(I; H^1)$  for its weak-\* topology for any bounded interval  $I$  by Lemma 4.7. Therefore,  $\partial_t \varphi_m \rightharpoonup \partial_t \varphi$  in  $\mathcal{D}'(I \times \mathbb{R}^d)$ . As the  $L^\infty(I; L^2)$  norm of  $\partial_t \varphi_m$  is uniformly bounded in  $m$  and since  $\mathcal{C}_c(I \times \mathbb{R}^d)$  is dense in  $L^\infty(I; L^2) = \left(L^1(I; L^2)\right)'$ , this limit shows that  $\partial_t \varphi = \partial_t u$  belongs to  $L^\infty(I; L^2)$ .

To conclude, we show that  $u \ln |u|^2 \in L^\infty(I; L^2)$ : in view of Corollary 2.8, for  $t \in I$ ,

$$\left\| u(t) \ln |u(t)|^2 \right\|_{L^2} \lesssim \mathcal{E}_{\log\text{GP}}(u_0)^{1/2} + \mathcal{E}_{\log\text{GP}}(u_0).$$

Hence, using (logGP),  $\Delta u \in L^\infty(I; L^2)$ . □

## 6. ON STATIONARY AND TRAVELING WAVES

In this section, we prove some rather general results regarding solitary and traveling waves. First, plugging (1.2) into (logGP), we get the equation for  $\phi$ :

$$(6.1) \quad -\omega \phi - ic \cdot \nabla \phi + \Delta \phi = \lambda \phi \ln |\phi|^2.$$

### 6.1. The only possible value for $\omega$ .

*Proof of Theorem 1.2.*  $u$  satisfies (logGP) with initial data  $\phi$  and  $u \in L^\infty(\mathbb{R}; E_{\log\text{GP}})$ . From Lemma 3.1, we thus know that  $u - \phi \in C^0(\mathbb{R}; L^2)$ . On the other hand, we know that, for any  $t \in \mathbb{R}$ ,  $x \in \mathbb{R}^d$  and any  $f$  smooth enough,

$$f(x - tc) - f(x) = -t \int_0^1 \nabla f(x - \tau c) \cdot c \, d\tau,$$

which leads to

$$\begin{aligned} \|f(x - tc) - f(x)\|_{L^2} &\leq |t| \int_0^1 \|\nabla f(x - \tau c) \cdot c\|_{L^2} \, d\tau \leq |t| \int_0^1 \|\nabla f(x - \tau c)\|_{L^2} |c| \, d\tau \\ &\leq |t| |c| \int_0^1 \|\nabla f\|_{L^2} \, d\tau \leq |t| |c| \|\nabla f\|_{L^2}. \end{aligned}$$

Since  $\nabla \phi \in L^2$ , we can also apply this inequality to  $\phi$ , which proves that for all  $t \in \mathbb{R}$ ,  $\phi(\cdot - ct) - \phi \in L^2$ .

The above two claims imply that  $u - \phi(\cdot - ct) \in L^2$  for all  $t \in \mathbb{R}$ . In view of (1.2), this means that  $(e^{i\omega t} - 1)\phi \in L^2$  for all  $t \in \mathbb{R}$ . Suppose that  $\omega \neq 0$ : considering  $t = \frac{\pi}{\omega}$ , we get  $-2\phi \in L^2$ , which is in contradiction with the fact that  $|\phi| - 1 \in L^2$  from Lemma 2.7. Therefore,  $\omega = 0$ . □

**6.2. Regularity and limits of a traveling wave in dimension  $d = 1$ .** In this section, we prove a result about the regularity and limits at infinity of traveling waves of the form (1.2) solution to (logGP) in dimension  $d = 1$ . In view of Theorem 1.2, such traveling waves satisfy

$$(6.2) \quad -ic\phi' + \phi'' = \lambda\phi \ln |\phi|^2.$$

**Lemma 6.1.** *A traveling wave  $\phi$  solution to (6.2) satisfies*

- $\phi \in C_b^2$  and  $\lim_{|x| \rightarrow \infty} |\phi(x)| = 1$ ,
- $\phi' \in H^1$  and  $\lim_{|x| \rightarrow \infty} \phi'(x) = 0$ .

*Proof.* Since  $\phi \in E_{\log GP}$ , we have  $\phi \in H_{\text{loc}}^1$  and in particular  $\phi \in C^{0, \frac{1}{2}}(\mathbb{R})$  by Sobolev embedding in dimension  $d = 1$ . Actually,  $\phi$  satisfies (6.2) with  $\omega = 0$  and  $\phi \ln |\phi|^2 \in L^2 \cap C^0$  from Corollary 2.8. Moreover,  $\phi' \in L^2$  since  $\phi \in E_{\log GP}$ . Therefore, we get  $\phi'' \in L^2$ , which leads to  $\phi' \in H^1$ . By Sobolev embedding, this leads in particular to  $\phi' \in C_b^0$  and, coming back to (6.2) once again, we get  $\phi'' \in C^0$ . This leads to  $\phi \in H_{\text{loc}}^2 \cap C^2$ .  $\phi' \in H^1$  also gives  $\lim_{|x| \rightarrow \infty} \phi'(x) = 0$ . Furthermore, with Lemma 2.7,  $|\phi| - 1 \in H^1$ , which means that  $\lim_{|x| \rightarrow \infty} |\phi(x)| = 1$  and that  $\phi$  is bounded, which shows that  $\phi''$  is also bounded by (6.2).  $\square$

## 7. SOLITARY WAVES IN THE ONE-DIMENSIONAL CASE

In this section, we prove Theorem 1.4 in the case  $c = 0$ . By Theorem 1.2, we know that  $\omega = 0$ . Thus, the equation we address in this section is:

$$(7.1) \quad \phi'' = \lambda\phi \ln |\phi|^2.$$

**7.1. Properties of a solitary wave in dimension  $d = 1$ .** We first assume that there exists a non-constant traveling wave  $u$  of the form (1.2) with  $c = 0$ , solution to (logGP) in the energy space, and we gather some of its properties.

First, we show that  $\phi$  satisfies an interesting energy equality.

**Lemma 7.1.** *If  $\phi \in E_{\log GP}$ , then*

$$(7.2) \quad |\phi'|^2 = \lambda (|\phi|^2 \ln |\phi|^2 - |\phi|^2 + 1).$$

*Proof.* Multiplying (7.1) by  $2\bar{\phi}'(x)$ , and taking the real value, we get

$$\frac{d}{dx} |\phi'|^2 = \lambda \frac{d}{dx} (|\phi|^2 \ln |\phi|^2 - |\phi|^2 + 1).$$

By the facts that  $\phi \in E_{\log GP}$  and  $|\phi|^2 \ln |\phi|^2 - |\phi|^2 + 1 \geq 0$ , we get  $|\phi'|^2 \in L^1$  and  $|\phi|^2 \ln |\phi|^2 - |\phi|^2 + 1 \in L^1$ , so we can integrate the above equation, and no additional constant of integration appears.  $\square$

**Corollary 7.2.** *If  $\phi \in E_{\log GP}$  is real-valued on some unbounded interval  $I$ , then either  $\phi'$  never vanishes and does not change sign on  $I$ , or  $\phi$  is constant on  $I$  (equals to  $\pm 1$ ).*

*Proof.* If there is some  $y \in I$  such that  $\phi'(y) = 0$ , then Lemma 7.1 shows that  $\phi(y) = \pm 1$  as  $x \mapsto x \ln x - x + 1$  vanishes only for  $x = 1$  on  $\mathbb{R}_+$ , and thus the uniqueness part of the Cauchy theorem applied to (7.1) gives (as long as it does not vanish)  $\phi \equiv \pm 1$  on  $I$ . On the other hand, if there is no such  $y$ ,  $\phi'$  cannot change sign on  $I$  since it is continuous and never vanishes.  $\square$

Then, we show that  $\phi$  can be taken real-valued up to a gauge on some neighborhood of  $+\infty$ .

**Lemma 7.3.** *There exists  $x^- \in \mathbb{R} \cup \{-\infty\}$  and  $\theta \in [0, 2\pi)$  such that  $\phi(x)e^{-i\theta}$  is real-valued and positive for all  $x \geq x^-$ , and, either  $x^- = -\infty$ , or  $\lim_{x \rightarrow x^-} \phi(x) = 0$ .*

*Proof.* We can write (7.1) as a first order differential system,

$$(7.3) \quad \begin{pmatrix} \phi \\ \psi \end{pmatrix}' = \begin{pmatrix} \psi \\ \lambda \phi \ln |\phi|^2 \end{pmatrix} =: F \left( \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right).$$

$F$  is a continuous function on  $\mathbb{C}^2$  and is of class  $\mathcal{C}^1$  on  $(\mathbb{C} \setminus \{0\}) \times \mathbb{C}$ . From Lemma 6.1, there exists  $x_0 \in \mathbb{R}$  such that  $\frac{3}{2} \geq |\phi(x)| \geq \frac{1}{2}$  and  $|\phi'(x)| \leq \frac{1}{2}$  for all  $x \geq x_0$ . Therefore, by Cauchy-Lipschitz Theorem, there exists a maximal interval  $I_+ = (x^-, x^+)$  such that  $x_0 \in I_+$  and  $(\phi, \phi')$  is the unique solution of (7.3) on  $I_+$  with initial data  $\phi(x_0)$  and  $\psi(x_0) = \phi'(x_0)$ , with values in  $\mathbb{C}^2 \setminus (\{0\} \times \mathbb{C})$ . Moreover, if  $|x^+| < \infty$ , then either  $\lim_{x \rightarrow x^+} \phi(x) = 0$  or  $\lim_{x \rightarrow x^+} (|\phi(x)| + |\psi(x)|) = \infty$ , and similarly for the limit  $x \rightarrow x^-$ .

However, since we already know that  $\frac{3}{2} \geq |\phi(x)| \geq \frac{1}{2}$  and  $|\phi'(x)| \leq \frac{1}{2}$  for all  $x \geq x_0$ , we can already conclude that  $x^+ = \infty$ . On the other hand, since  $\phi \in \mathcal{C}^2(\mathbb{R})$  by Lemma 6.1, we know that  $\lim_{x \rightarrow x^-} (|\phi(x)| + |\psi(x)|) = \infty$  cannot be true if  $x^- > -\infty$ .

On  $I_+$ ,  $\phi$  does not vanish. Hence, we can use a polar decomposition:  $\phi = \rho e^{i\theta}$ , with  $\rho$  and  $\theta$  real-valued and defined as

$$(7.4) \quad \rho = |\phi|, \quad \theta' = \frac{\text{Im}(\phi' \bar{\phi})}{\rho^2}, \quad \theta(x_0) \in [0, 2\pi) \text{ such that } e^{i\theta(x_0)} = \frac{\phi(x_0)}{\rho(x_0)}.$$

The regularity of  $\phi$  and the fact that it does not vanish on  $I_+$  ensure that  $\rho$  and  $\theta$  are well defined and  $\mathcal{C}^2$  on  $I_+$ . Thus, we can compute  $\phi''$  in terms of  $\rho, \theta$ , and their derivatives:

$$\phi'' = (\rho'' - \rho(\theta')^2 + 2i\rho'\theta') e^{i\theta}.$$

Substituting into (7.1), we get (after simplification by  $e^{i\theta}$ )

$$(7.5) \quad \rho'' - \rho(\theta')^2 + 2i\rho'\theta' + i\rho\theta'' = \lambda\rho \ln \rho^2.$$

By taking the imaginary part, we get  $\rho\theta'' + 2\rho'\theta' = 0$ . Since  $\rho$  does not vanish on  $I_+$ , we get for all  $x \in I_+$ :

$$\theta'(x) = \frac{c_0}{\rho(x)^2},$$

where  $c_0$  is a constant. Then, we can compute for all  $x \in I_+$

$$\phi'(x) = (\theta'(x) + i\rho(x)\theta'(x)) e^{i\theta(x)} = \left( \theta'(x) + i \frac{c_0}{\rho(x)} \right) e^{i\theta(x)},$$

which leads to  $\frac{c_0}{\rho} \in L^2(I_+)$  and at the same time  $\lim_{x \rightarrow \infty} \frac{c_0}{\rho} = c_0$  since we know that  $\lim_{x \rightarrow +\infty} \rho(x) = 1$ . Combining these two arguments yields  $c_0 = 0$ .

Hence,  $\theta$  is constant on  $I_+$ , which gives the conclusion.  $\square$

From now on, without loss of generality, we can assume that  $\theta = 0$  in Lemma 7.3, up to changing  $\phi$  to  $e^{-i\theta}\phi$ .

**Lemma 7.4.** *Let  $x^- \in \mathbb{R} \cup \{-\infty\}$  from Lemma 7.3. Then  $x^- > -\infty$ . Moreover,  $\phi(x) \in (0, 1)$  and  $\phi'(x) > 0$  for all  $x > x^-$ . Last,  $\phi(x^-) = 0$  and  $\phi'(x^-) = 1$ .*

*Proof.* From Corollary 7.2, we know that  $\phi'$  never vanishes and does not change sign on  $(x^-, \infty)$ . By contradiction, if  $\phi(x_2) > 1$  for some  $x_2 > x^-$ , then

- Either  $\phi' > 0$  on  $(x^-, \infty)$ , and thus  $\phi(x) \geq \phi(x_2)$  for all  $x \geq x_2$ , which is in contradiction with the fact that  $\lim_{x \rightarrow \infty} |\phi| = 1$ .

- Or  $\phi' < 0$  on  $(x^-, \infty)$ , which means that  $\phi(x) \geq \phi(x_2) > 1$  for all  $x^- < x \leq x_2$ . Therefore,  $x^- = -\infty$ , but then we have the same contradiction as in the previous case, for  $-\infty$ .

Therefore,  $\phi(x) \in (0, 1)$  for all  $x > x^-$ , and this leads to  $\phi' > 0$  on  $(x^-, \infty)$ . Now, remark that  $f(x) = x^2 \ln x^2 - x^2 + 1$  is decreasing on  $[0, 1]$ . Thus,  $\lambda f(\phi) = (\phi')^2$  is decreasing, which shows that, taking  $x_0 \in (x^-, \infty)$ , we have  $\phi'(x)^2 \geq \phi'(x_0)^2$  for all  $x^- < x \leq x_0$ , and thus  $\phi'(x) \geq \phi'(x_0)$ . By integration, we get for all  $x \in (x^-, x_0]$ ,

$$\phi(x) \leq \phi(x_0) + (x - x_0)\phi'(x_0).$$

Since the right hand side goes to  $-\infty$  as  $x \rightarrow -\infty$ , and since  $\phi > 0$  on  $(x^-, \infty)$ , this yields  $x^- > -\infty$ . Thus, we are in the second case of Lemma 7.3, and since  $\phi$  is continuous, we get  $\phi(x^-) = 0$ . Last, (7.2) holds for  $x = x^-$ , and we know by continuity that  $\phi'(x^-) \geq 0$ , which gives  $\phi'(x^-) = 1$ .  $\square$

**Lemma 7.5.** *Let  $x^- \in \mathbb{R}$  from Lemma 7.3. Then, for all  $x < x^-$ ,  $\phi(x)$  is real-valued, negative and increasing.*

*Proof.* From Lemma 7.4, we know that  $\phi(x^-) = 0$  and  $\phi'(x^-) = 1$ . Therefore, there exists  $\delta > 0$  such that  $\phi \neq 0$  on  $[x^- - \delta, x^-)$ . Define

$$x_0^- := \inf \{y < x^- \mid \forall x \in (y, x^-), \phi(x) \neq 0\} < x^- - \delta.$$

On  $I_2 := (x_0^-, x^-)$ , we can use a polar factorization in the same way as in (7.4):  $\phi = \rho e^{i\theta}$ , where  $\rho$  and  $\theta$  also satisfy (7.5) on  $I_2$ . Taking again the imaginary part and integrating the ODE, we have, like before,

$$\theta'(x) = \frac{c_0}{\rho(x)^2}, \quad \forall x \in I_2.$$

We still have then

$$\phi'(x) = \left( \theta'(x) + i \frac{c_0}{\rho(x)} \right) e^{i\theta(x)},$$

and once again we must have  $\frac{c_0}{\rho(x)} \in L^2(I_2)$  since  $\phi' \in L^2$ . However, we know that  $\phi(x^-) = 0$  and  $\phi'(x^-) = 1$ , which means that  $\phi(x) \sim x - x^-$  as  $x \rightarrow x^-$ . In terms of  $\rho$ , this implies  $\rho(x) \sim |x - x^-|$  as  $x \rightarrow x^-$ , and  $\frac{c_0}{\rho(x)} \in L^2(I_2)$  if and only if  $c_0 = 0$ . Once again,  $\theta$  is constant on  $I_2$ , and therefore so is  $\frac{\phi}{\rho}$ . With the previous asymptotics, we know that it tends to  $-1$  at  $x^-$ . Thus, we get

$$\phi = -\rho.$$

This shows that  $\phi$  is real-valued and negative on  $I_2$ . Then, we can apply Lemma 7.1 on  $(x_0^-, \infty)$ , which shows that  $\phi'$  does not vanish on this interval, and thus  $\phi'$  does not change sign once again, which proves that  $\phi' > 0$  on  $I_2$ . Therefore,  $\phi(x) \leq \phi(x^- - \delta) < 0$  for all  $x \in (x_0^-, x^- - \delta)$ , which proves that  $x_0^- = -\infty$ .  $\square$

It only remains to analyze the limit of  $\phi$  at  $-\infty$ .

**Lemma 7.6.** *We have*

$$\lim_{x \rightarrow -\infty} \phi(x) = -1$$

*Proof.* We have seen that  $\phi$  is increasing, thus it has a limit at  $-\infty$ . From Lemma 7.5, we know it is negative. Moreover,  $|\phi| - 1 \in L^2$ , which means that  $\phi + 1 \in L^2((-\infty, 0))$ . The conclusion easily follows.  $\square$

The conclusion of all the previous lemmas is the following:

**Corollary 7.7.** *If  $u$  is a non-constant traveling wave solution to (logGP) of the form (1.2) with  $c = 0$ , then there exists  $\theta \in \mathbb{R}$  such that  $e^{-i\theta}\phi$  is a real-valued, increasing,  $\mathcal{C}^2$  function with values in  $(-1, 1)$  which vanishes at a unique point  $x_0$ . Moreover,  $\phi_0(x) = e^{-i\theta}\phi(x + x_0)$  also satisfies (logGP),*

$$\lim_{x \rightarrow \pm\infty} \phi_0(x) = \pm 1.$$

and

$$(7.6) \quad (\phi_0')^2 = \lambda \left( \phi_0^2 \ln \phi_0^2 - \phi_0^2 + 1 \right) \quad \text{on } \mathbb{R}.$$

**7.2. Analysis of the new ODE and proof of Theorem 1.4.** From Corollary 7.7,  $\phi_0$  vanishes at  $x = 0$ , satisfies (7.6) and is strictly increasing, i.e.  $\phi_0' > 0$ . Thus,  $\phi_0$  satisfies the ODE

$$(7.7) \quad \phi_0' = \sqrt{\lambda} \sqrt{\phi_0^2 \ln \phi_0^2 - \phi_0^2 + 1} \quad \text{on } \mathbb{R}.$$

The uniqueness of this function is the topic of the following lemma.

**Lemma 7.8.** *There exists a unique function  $\phi_0$  satisfying (7.7) with the initial data  $\phi_0(0) = 0$ . It is defined on  $\mathbb{R}$  and satisfies*

$$\lim_{x \rightarrow \pm\infty} \phi_0(x) = \pm 1.$$

Moreover,  $\phi_0 \in E_{\log\text{GP}}$ .

*Proof.* We already know that  $f(x) = x \ln x - x + 1 \geq 0$  for all  $x \geq 0$ , and  $f$  vanishes only at  $x = 1$ . Moreover, by simple computations, we check that  $g(x) = f(x^2)$  is  $\mathcal{C}^1$  on  $\mathbb{R}$ , and  $\mathcal{C}^2$  on  $\mathbb{R} \setminus \{0\}$ . We also compute  $f'(1) = 0$  and  $f''(1) = 1$ , hence

$$g(x) \underset{x \rightarrow \pm 1}{\sim} \frac{1}{2} (x^2 - 1)^2.$$

From these facts, we deduce that  $h = \sqrt{g}$  is a  $\mathcal{C}^0(\mathbb{R}) \cap \mathcal{C}^1(\mathbb{R} \setminus \{-1, 1\})$  function such that  $h(x) \underset{x \rightarrow \pm 1}{\sim} |x^2 - 1|/\sqrt{2}$ . Therefore, it is  $\mathcal{C}^{0,1}$  locally on  $\mathbb{R}$ , and we may invoke Cauchy-Lipschitz Theorem: there exists a unique  $\phi_0$  satisfying  $\phi_0' = \sqrt{\lambda} h(\phi_0)$  with the initial condition  $\phi_0(0) = 0$ , on a maximal interval  $I \ni 0$  of existence.

Then,  $\phi_0$  cannot reach 1 or  $-1$ , because the constant function 1 and  $-1$  are both solutions to this ODE, and it would contradict the uniqueness in the Cauchy-Lipschitz Theorem. Since  $\phi_0$  is continuous, we infer that  $\phi_0(x) \in (-1, 1)$  for all  $x \in \mathbb{R}$ . Moreover, we can easily prove that there exists  $c_1 > 0$  such that

$$\begin{aligned} h(y) &\geq c_1(y + 1), \quad \forall y \in [-1, 0], \\ h(y) &\geq c_1(1 - y), \quad \forall y \in [0, 1]. \end{aligned}$$

Since we know that  $\phi_0(x) \in (-1, 0]$  for all  $x \leq 0$ , we can estimate

$$\phi_0'(x) \geq c_1 \sqrt{\lambda} (\phi_0(x) + 1), \quad \forall x \leq 0.$$

By integrating backward, we get for all  $x \leq 0$ ,

$$\begin{aligned} -1 &< \phi_0(x) \leq -1 + e^{\sqrt{\lambda} c_1 x}, \\ 0 &< \phi_0'(x) \leq c_1 \sqrt{\lambda} e^{\sqrt{\lambda} c_1 x}. \end{aligned}$$



Similarly, we also have for all  $x \geq 0$

$$\begin{aligned} 1 - e^{-\sqrt{\lambda}c_1x} &\leq \phi_0(x) < -1, \\ 0 < \phi_0'(x) &\leq c_1\sqrt{\lambda}e^{\sqrt{\lambda}c_1x}. \end{aligned}$$

Those estimates prove the expected limits at  $\pm\infty$ , and also show that  $\phi_0' \in L^2$  together with  $|\phi_0| - 1 \in L^2$ . We conclude that  $\phi_0 \in E_{\log\text{GP}}$  by Lemma 2.7.  $\square$

The previous result obviously proves that we have a set of solitary waves for (logGP).

**Corollary 7.9.** *For any  $\theta \in \mathbb{R}$  and any  $x_0 \in \mathbb{R}$ ,  $e^{i\theta}\phi_0(\cdot - x_0) \in E_{\log\text{GP}}$  is a solitary (stationary) wave of (logGP).*

It remains to show that they are the only possible solitary waves in order to prove Theorem 1.4 in the case  $c = 0$ .

*Proof of Theorem 1.4: case  $c = 0$ .* Let  $\phi \in E_{\log\text{GP}}$  and  $u$  be a traveling wave of the form (1.2) with  $c = 0$ . From Corollary 7.7, we can find  $\theta \in \mathbb{R}$  and  $x_0 \in \mathbb{R}$  such that  $e^{-i\theta}\phi(\cdot + x_0)$  is a real-valued, increasing,  $C^2$  function with values in  $(-1, 1)$ , which vanishes at 0, and satisfies (7.6), i.e. (7.7) by positivity of its derivative. It is therefore  $\phi_0$ , defined in Lemma 7.8 by uniqueness given by the same lemma:  $e^{-i\theta}\phi(\cdot + x_0) = \phi_0$ , i.e.  $\phi = e^{i\theta}\phi_0(\cdot - x_0)$ .  $\square$

## 8. TRAVELING WAVES IN THE ONE-DIMENSIONAL CASE

**8.1. Admissible velocities.** The goal of this section is to prove Theorem 1.3, and characterize velocities of nontrivial (that is, nonconstant) traveling waves. We assume that there exists a traveling wave of the form (1.2) solution to (logGP), i.e.  $\phi \in E_{\log\text{GP}}$  solution to (6.2).

**Lemma 8.1.** *The function  $\eta = 1 - |\phi|^2$  satisfies, on  $\mathbb{R}$ ,*

$$\frac{1}{2}(\eta')^2 = h_c(\eta),$$

where

$$h_c(y) := \lambda \left( (1-y)^2 \ln(1-y)^2 - (1-y)^2 + 1 \right) - \frac{2\lambda + c^2}{2} y^2.$$

*Proof.* We follow the same lines as in [2]. Writing  $\phi = \phi_1 + i\phi_2$ , (6.2) becomes

$$(8.1) \quad \begin{cases} \phi_1'' + c\phi_2' = \lambda\phi_1 \ln(\phi_1^2 + \phi_2^2), \\ \phi_2'' - c\phi_1' = \lambda\phi_2 \ln(\phi_1^2 + \phi_2^2). \end{cases}$$

We multiply the first equation by  $\phi_2$  and the second one by  $\phi_1$ , and we subtract in order to get:

$$(\phi_1'\phi_2 - \phi_1\phi_2')' = -c(\phi_2\phi_2' + \phi_1\phi_1') = \frac{c}{2}\eta', \quad \eta := 1 - |\phi|^2.$$

Moreover, we know that  $\phi$  is bounded by Lemma 6.1, thus  $\phi_1'\phi_2, \phi_1\phi_2' \in L^2$ , and  $\eta \in L^2$  by using Lemma 2.7. Thus, integrating the above identity yields

$$(8.2) \quad \phi_1'\phi_2 - \phi_1\phi_2' = \frac{c}{2}\eta.$$

Now, we multiply the first equation of (8.1) by  $\phi_1'$ , the second by  $\phi_2'$  and we sum in order to get:

$$\phi_1'\phi_1'' + \phi_2'\phi_2'' = \lambda(\phi_1\phi_1' + \phi_2\phi_2') \ln(\phi_1^2 + \phi_2^2),$$

which can be written as

$$\left(|\phi'|^2\right)' = \lambda \left(|\phi|^2 \ln |\phi|^2 - |\phi|^2 + 1\right)'.$$

We know that both  $|\phi'|^2$  and  $|\phi|^2 \ln |\phi|^2 - |\phi|^2 + 1$  (which is positive and whose integral over  $\mathbb{R}$  is bounded by  $\mathcal{E}_{\log\text{GP}}(\phi)$ ) are in  $L^1$ , hence:

$$|\phi'|^2 = \lambda(|\phi|^2 \ln |\phi|^2 - |\phi|^2 + 1) \quad \text{on } \mathbb{R}.$$

We deduce then

$$\begin{aligned} \eta'' &= -2|\phi'|^2 - 2(\phi_1\phi_1'' + \phi_2\phi_2'') \\ &= -2|\phi'|^2 - 2\phi_1 \left(-c\phi_2' + \lambda\phi_1 \ln |\phi|^2\right) - 2\phi_2 \left(c\phi_1' + \lambda\phi_2 \ln |\phi|^2\right) \\ &= -4\lambda(1-\eta) \ln(1-\eta) - 2\lambda\eta - c^2\eta = h_c'(\eta). \end{aligned}$$

Multiplying by  $\eta'$ , we obtain

$$\frac{1}{2} ((\eta')^2)' = (h_c(\eta))'.$$

Since  $\lim_{\pm\infty} \eta' = \lim_{\pm\infty} \eta = 0$  from Lemma 6.1, we conclude by integrating.  $\square$

*Remark 8.2.* This lemma may be extended to the case of any non-linearity  $F(u)$  with  $F$  a continuous function on  $\mathbb{C}$  of the form  $F(u) = u f(|u|^2)$  with  $f(1) = 0$ .

**Corollary 8.3.** *If  $c \neq 0$ , then  $\phi$  never vanishes.*

*Proof.* It comes from the fact that  $h_c(1) = -\frac{c^2}{2} < 0$  if  $c \neq 0$ , while  $h_c(\eta) = \frac{1}{2}(\eta')^2 \geq 0$ . Therefore,  $\eta(x) \neq 1$ , that is  $\phi(x) \neq 0$ , for all  $x \in \mathbb{R}$ .  $\square$

**Corollary 8.4.** *If  $c^2 \geq 2\lambda$ , then  $\eta \equiv 0$ .*

*Proof.* The limiting case  $c^2 = 2\lambda$  is treated like in [2]: if  $\eta$  is not identically zero, then in view of Lemma 6.1, by translation invariance, we may assume

$$|\eta(0)| = \max\{|\eta(x)|, x \in \mathbb{R}\} > 0, \quad \eta'(0) = 0.$$

In view of Lemma 8.1,  $h_c(\eta(0)) = 0$ . On the other hand, Corollary 8.3 implies that  $1 - \eta(x) > 0$  for all  $x \in \mathbb{R}$ , and direct computation yields

$$h_c''(y) = 4\lambda \ln(1-y), \quad 0 \leq y < 1.$$

Therefore,  $h_c''(y)$  is positive for  $y < 0$ , negative for  $0 < y < 1$ . As  $h_c'(0) = 0$ ,  $h_c'$  is negative on  $(-\infty, 0) \cup (0, 1)$ :  $h_c$  is decreasing on  $(-\infty, 1)$ . Since  $h_c(0) = 0$ , the only zero of  $h_c$  on  $[0, 1)$  is the origin, and so  $\eta(0) = 0$ , hence a contradiction:  $\eta$  is identically zero.

Suppose now  $c^2 > 2\lambda$ . Since  $h_c(0) = h_c'(0) = 0$ , and  $h_c''(0) = 2\lambda - c^2$ , there exists  $\varepsilon > 0$  such that  $h_c(y) < 0$  for all  $y \in [-\varepsilon, \varepsilon] \setminus \{0\}$ . On the other hand,  $h_c(\eta)$  is continuous, nonnegative (from Lemma 8.1) and tends to 0 at  $\pm\infty$ . The conclusion follows easily.  $\square$

**Lemma 8.5.** *If  $c^2 \geq 2\lambda$ , then  $\phi$  is constant.*

*Proof.* By Corollary 8.4, we know that  $|\phi(x)| = 1$  for all  $x \in \mathbb{R}$ . Since  $\phi$  is a  $\mathcal{C}_b^2(\mathbb{R})$  function, there exists a real-valued function  $\theta \in \mathcal{C}^2$  such that  $\phi = e^{i\theta}$  (defined like in (7.4) for instance). By substitution in (6.2),  $\theta$  satisfies then

$$c\theta' + i\theta'' - (\theta')^2 = 0.$$

By taking the imaginary part, we get  $\theta'' = 0$ . Since we must have  $\phi' = i\theta' e^{i\theta} \in L^2$ , we get  $\theta' \in L^2$  and thus  $\theta' = 0$ .  $\square$

**8.2. Nontrivial traveling waves.** We now consider the case  $0 < c^2 < 2\lambda$ . Define, for all  $y > 0$ ,

$$f_c(y) := \frac{c^2}{4} \left( \frac{1}{y^3} - y \right) + \lambda y \ln y^2,$$

$$g_c(y) := -\frac{c^2}{4} \frac{(1-y^2)^2}{y^2} + \lambda(y^2 \ln y^2 - y^2 + 1).$$

The following lemma is established by direct calculations:

**Lemma 8.6.** *Let  $c$  such that  $0 < c^2 < 2\lambda$ . There exists  $0 < y_0 < y_1 < 1$  such that the following holds:*

- $f_c$  has exactly two zeroes on  $(0, +\infty)$ :  $y_1$  and 1.
- $f_c$  is positive on  $(0, y_1) \cup (1, +\infty)$ , negative on  $(y_1, 1)$ .
- $g_c$  has exactly two zeroes on  $(0, +\infty)$ :  $y_0$  and 1.
- $g_c$  is negative on  $(0, y_0)$ , positive on  $(y_0, +\infty) \setminus \{0\}$ .
- There exists  $C_c > 0$  such that  $\frac{1}{C_c}(1-y)^2 \leq g_c(y) \leq C_c(1-y)^2$  for all  $y \in (y_1, 1)$ .

*Proof.* We compute

$$f'_c(y) = \frac{c^2}{4} \left( -\frac{3}{y^4} - 1 \right) + 2\lambda \ln y + 2\lambda, \quad f''_c(y) = \frac{3c^2}{y^5} + \frac{2\lambda}{y} > 0.$$

As

$$\lim_{y \rightarrow 0} f'_c(y) = -\infty, \quad \lim_{y \rightarrow +\infty} f'_c(y) = +\infty,$$

the derivative  $f'_c$  has a unique zero on  $(0, +\infty)$ , which we denote by  $y_2 > 0$ . Since  $f'_c(1) = 2\lambda - c^2 > 0$ , we know that  $0 < y_2 < 1$ :  $f_c(1) = 0$ , hence  $f_c(y_2) < 0$ , and there exists a unique  $0 < y_1 < y_2$  such that  $f_c(y_1) = 0$ .

We note that  $f_c = \frac{1}{2}g'_c$ , and in view of the above pieces of information, we can draw:

$y$	0	$y_1$	$y_2$	1	$+\infty$
$f_c(y)$	$+\infty$	0	-	0	$+\infty$
$g_c(y)$	$-\infty$	+	0	$+\infty$	$+\infty$

In particular,  $g_c$  is increasing on  $(0, y_1)$ , from  $-\infty$  to a positive value, and there exists a unique  $y_0 \in (0, y_1)$  such that  $g_c(y_0) = 0$ , and the lemma follows easily.  $\square$

**Lemma 8.7.** *If  $c$  is such that  $0 < c^2 < 2\lambda$ , then  $\rho = |\phi|$  satisfies*

$$(8.3) \quad \rho'' = f_c(\rho),$$

$$(8.4) \quad (\rho')^2 = g_c(\rho).$$

Moreover, there exists  $x_0 \in \mathbb{R}$  such that  $\rho(x_0) = y_0$ , where  $y_0$  is defined in Lemma 8.6, and  $\theta \in \mathcal{C}^2$ , defined so that  $\phi = \rho e^{i\theta}$ , satisfies

$$(8.5) \quad \theta' = \frac{c}{2} \left( 1 - \frac{1}{\rho^2} \right).$$

*Proof.* By Corollary 8.3,  $\phi$  never vanishes. It is also a  $\mathcal{C}_b^2$  function. Therefore, we can define  $\rho$  and  $\theta$  as in (7.4) so that  $\phi = \rho e^{i\theta}$ . They satisfy

$$(8.6) \quad -ic\rho' + c\rho\theta' + \rho'' - \rho(\theta')^2 + 2i\rho'\theta' + i\rho\theta'' = \lambda\rho \ln \rho^2.$$

By taking the imaginary part, we get

$$(8.7) \quad -c\rho' + \rho\theta'' + 2\rho'\theta' = 0.$$

As  $\rho$  never vanishes, the solution of this ODE on  $\theta'$  takes the form

$$\theta'(x) = \frac{c}{2} + \frac{c_0}{\rho(x)^2},$$

where  $c_0$  is a real constant. Then, there holds

$$\phi' = \left( \rho' + i \left( \frac{c}{2}\rho + \frac{c_0}{\rho} \right) \right) e^{i\theta} \in L^2.$$

On the other hand, we already know that  $\lim_{\pm\infty} \rho = 1$ , thus we must have  $c_0 = -\frac{c}{2}$ , so that

$$\theta'(x) = \frac{c}{2} \left( 1 - \frac{1}{\rho(x)^2} \right).$$

Now, (8.6) reads

$$\begin{aligned} \rho'' &= -\frac{c^2}{2} \left( \rho - \frac{1}{\rho} \right) + \frac{c^2}{4} \rho \left( 1 - \frac{1}{\rho^2} \right)^2 + \lambda\rho \ln \rho^2 \\ &= -\frac{c^2}{2} \left( \rho - \frac{1}{\rho} \right) + \frac{c^2}{4} \left( \rho - \frac{2}{\rho} + \frac{1}{\rho^3} \right) + \lambda\rho \ln \rho^2 \\ &= \frac{c^2}{4} \left( \frac{1}{\rho^3} - \rho \right) + \lambda\rho \ln \rho^2. \end{aligned}$$

After multiplication by  $\rho'$  and integration, we find:

$$(\rho')^2 = -\frac{c^2}{4} \frac{(1 - \rho^2)^2}{\rho^2} + \lambda(\rho^2 \ln \rho^2 - \rho^2 + 1).$$

We also know that if  $\rho \equiv 1$ , then  $\theta' \equiv 0$  and thus  $\phi$  is constant. On the contrary, if  $\phi$  is not constant, then  $\rho$  is not either, and thus has an extremum since  $\lim_{\pm\infty} \rho = 1$ . Let  $x_0$  be extremal. Then  $\rho'(x_0) = 0$ , and therefore  $0 = g_c(\rho(x_0))$ , with  $\rho(x_0) \neq 1$ . Lemma 8.6 implies  $\rho(x_0) = y_0$ .  $\square$

**Lemma 8.8.** *There exists a unique  $\rho_c \in \mathcal{C}^2$  satisfying (8.3),  $\rho_c(0) = y_0$  and  $\rho'_c(0) = 0$ . Moreover,  $\rho_c \geq y_0 > 0$ ,  $\lim_{\pm\infty} \rho_c = 1$  and  $\rho_c - 1 \in H^1$ .*

*Proof.* Since  $f_c$  is  $\mathcal{C}^\infty$  on  $(0, \infty)$ , we can apply Cauchy-Lipschitz Theorem to get a local solution. By multiplying (8.3) by  $\rho'_c$  and integrating, this solution satisfies also (8.4). Lemma 8.6 then yields  $\rho_c \geq y_0$ , which proves that this solution is global.

Then, by (8.3) and Lemma 8.6 again, we get  $\rho''_c(0) > 0$ . Hence there exists  $\varepsilon > 0$  such that  $\rho'_c < 0$  on  $(-\varepsilon, 0)$  and  $\rho'_c > 0$  on  $(0, \varepsilon)$ , and  $\rho_c > y_0$  on  $(-\varepsilon, \varepsilon) \setminus \{0\}$ . Let

$$x^+ := \sup \{x > 0 \mid \rho'_c(y) > 0 \text{ for all } y \in (0, x)\},$$

$$x^- := \inf \{x < 0 \mid \rho'_c(y) < 0 \text{ for all } y \in (x, 0)\}.$$

From (8.4) and Lemma 8.6 along with the fact that  $\rho_c$  and  $\rho'_c$  are continuous, we know that either  $x^+ = +\infty$  or  $\rho_c(x^+) = 1$ , and similarly either  $x^- = -\infty$  or  $\rho_c(x^-) = 1$ . However, if  $\rho(x_1) = 1$  at some point  $x_1 \in \mathbb{R}$ , then (8.4) gives  $\rho'_c(x_1) = 0$  and the uniqueness in Cauchy-Lipschitz Theorem for (8.3) leads to  $\rho_c \equiv 1$ , hence a contradiction. Therefore,  $x^+ = +\infty$  and  $x^- = -\infty$ , which implies that  $\rho_c$  is decreasing on  $(-\infty, 0)$ , increasing on

$(0, \infty)$  and  $\rho_c(x) \in [y_0, 1)$  for all  $x \in \mathbb{R}$ . Thus, it has limits  $\ell_{\pm} > y_0$  at  $\pm\infty$  and must also satisfy  $\lim_{\pm\infty} \rho'_c = 0$  in view of (8.4). Therefore, the limit must satisfy  $g_c(\ell_{\pm}) = 0$ , which leads to  $\ell_{\pm} = 1$ .

Last, since  $\rho'_c > 0$  on  $(0, \infty)$ , (8.4) reads on this interval

$$\rho'_c = \sqrt{g_c(\rho_c)}.$$

By Lemma 8.6 and since  $\rho_c(x) \geq y_1$  for  $x$  large enough, we get

$$\rho'_c \geq \sqrt{C_c}(1 - \rho_c).$$

Thus  $\rho_c$  converges exponentially fast to 1 at  $+\infty$  by Gronwall lemma, and therefore  $\rho'_c$  converges exponentially fast to 0. The same holds as  $x \rightarrow -\infty$ , hence the conclusion  $\rho_c - 1 \in H^1$ .  $\square$

**Lemma 8.9.** *For  $c$  such that  $0 < c^2 < 2\lambda$  and  $\theta_0 \in \mathbb{R}$ , define  $\Theta_c$  by*

$$\Theta'_c = \frac{c}{2} \left(1 - \frac{1}{\rho_c^2}\right), \quad \Theta_c(0) = \theta_0,$$

where  $\rho_c$  is given by Lemma 8.8. Then  $\phi_c := \rho_c e^{i\Theta_c} \in E_{\log\text{GP}}$  and  $u$  defined by (1.2) is a traveling wave for (logGP).

*Proof.* Direct computations show that  $\Theta_c$  satisfies (8.7), and thus  $(\rho_c, \Theta_c)$  satisfies (8.6), which is equivalent to  $\phi_c$  satisfying (6.2) or  $u$  satisfying (logGP). Therefore, we only have to prove that  $\phi_c \in E_{\log\text{GP}}$ .

By Lemma 8.8, we already know that  $|\phi_c| - 1 = \rho_c - 1 \in H^1$ . Moreover,

$$\phi'_c = (\rho'_c + i\rho_c\Theta'_c) e^{i\Theta_c} = \left(\rho'_c + i\frac{c}{2} \frac{\rho_c^2 - 1}{\rho_c}\right) e^{i\Theta_c}.$$

By Lemma 8.8 again,  $\rho'_c \in L^2$  and  $\rho_c$  is bounded, and bounded away from 0, so that  $\frac{\rho_c^2 - 1}{\rho_c} \in L^2$ . We conclude  $\phi_c \in E_{\log\text{GP}}$  thanks to Lemma 2.7.  $\square$

*End of the proof of Theorem 1.4.* Let  $\phi \in E_{\log\text{GP}}$  and  $u$  a traveling wave of the form (1.2). By applying Lemma 8.7, we get  $x_0 \in \mathbb{R}$  such that  $\rho(x_0) = y_0$  where  $\rho = |\phi|$ , which yields  $\rho'(x_0) = 0$  through (8.4). Since  $\rho(\cdot + x_0)$  also satisfies (8.3), this function satisfies the assumptions of Lemma 8.8. Thus  $\rho(\cdot + x_0) = \rho_c$ , and since  $\theta$  satisfies (8.5),  $\phi$  is of the form  $\phi_c$  defined in Lemma 8.9 for some  $\theta_0$ .  $\square$

## 9. SOME OPEN QUESTIONS

**Multidimensional solitary and traveling waves.** We have classified solitary and traveling waves in the one-dimensional setting, using ODE techniques. The picture is of course rather different for  $d \geq 2$ , see typically the case of traveling waves for the Gross-Pitaevskii equation (1.1), in [8, 3], where the role of hydrodynamical formulation (via Madelung transform) is decisive. See also [2] and references therein.

We note that contrary to the case with vanishing boundary condition at infinity, one cannot easily construct multidimensional waves from one-dimensional ones by using the tensorization property evoked in the introduction. Recall that if  $u_1(t, x_1), \dots, u_d(t, x_d)$  are solutions to the one-dimensional logarithmic Schrödinger equation, then  $u(t, x) := u_1(t, x_1) \times \dots \times u_d(t, x_d)$  solves the  $d$ -dimensional logarithmic Schrödinger equation. In the case of vanishing boundary condition at infinity,  $u_j(t, \cdot) \in L^2(\mathbb{R})$ , hence  $u(t, \cdot) \in$

$L^2(\mathbb{R}^d)$ ; see e.g. [11] for some applications. On the other hand, if  $u_j(t, \cdot) \in E_{\log\text{GP}} = E_{\text{GP}}$  (the sets are the same in the one-dimensional case), then even for  $d = 2$ ,

$$(x_1, x_2) \mapsto u_1(t, x_1)u_2(t, x_2)$$

does not belong to the energy space  $E_{\log\text{GP}} = E_{\text{GP}}$  (case of  $\mathbb{R}^2$ ), as the spatial derivatives are not in  $L^2(\mathbb{R}^2)$ .

**Orbital stability.** In the case of the one-dimensional Gross-Pitaevskii equation, the orbital stability of traveling waves was proven in [23] by using the fact that this equation is integrable, in the sense that it admits a Lax pair, and solutions are analyzed thanks to Zakharov-Shabat's inverse scattering method. It seems unlikely that even for  $d = 1$ , (logGP) is completely integrable, so the stability of traveling waves will require a different approach.

In [36, 4], the orbital stability of the black soliton is studied without using the integrable structure, but variational arguments. In [7] (dark soliton) and [25] (black soliton), the orbital stability is improved to asymptotic stability (the modulation factors, which consist of a translation parameter and a phase shift, are analyzed). These approaches may be more suited to (logGP), but new difficulties due to the special form of the nonlinearity arise, starting with the control of the presence of vacuum,  $\{u = 0\}$ .

**Scattering.** Another way to study stability properties consists in establishing a scattering theory, considering the plane wave solution,  $u(t, x) = e^{ik \cdot x - i|k|^2 t}$ ,  $k \in \mathbb{R}^d$ , as a reference solution. By Galilean invariance, the analysis is reduced to the case  $k = 0$ , and the question is the behavior of the solution as  $t \rightarrow +\infty$  when the initial datum  $u_0 \equiv 1$  is perturbed. In such a framework, a crucial object is the linearized Schrödinger operator about the constant 1. Using in addition normal form techniques, a scattering theory for (1.1) was developed in [27, 28, 29] for  $d \geq 2$ , and resumed in [33, 34] in the case of the three-dimensional cubic-quintic Schrödinger equation (the quintic term introduces the new difficulty of an energy-critical factor).

In the case of (logGP), the new difficulty is due to the fact that the logarithmic nonlinearity is not multilinear in  $(u, \bar{u})$ , reminding of the difficulty of giving sense to linearization in such a context (see e.g. [19, 40]).

**Convergence toward other models.** It is well-established that in the long wave régime, the Gross-Pitaevskii equation converges to the KdV equation in the one-dimensional case, and to the KP-I equation in the multidimensional case; see e.g. [5, 6, 17]. More precisely, we may resume the general presentation from [17]: considering the nonlinear Schrödinger equation

$$i\partial_t u + \Delta u = f(|u|^2)u,$$

where the nonlinearity  $f$  is such that  $f(1) = 0$  and  $f'(1) > 0$ , the KdV and KP-I equations appear in the limit  $\varepsilon \rightarrow 0$ , after the rescaling

$$t = c\varepsilon^3 \tau, \quad X_1 = \varepsilon(x_1 - ct), \quad X_j = \varepsilon^2 x_j, \quad j \in \{2, \dots, d\}.$$

Writing

$$u(t, X) = (1 + \varepsilon^2 A^\varepsilon(t, X)) e^{i\varepsilon\varphi^\varepsilon(t, X)},$$

and plugging this expression into the equation solved by  $u$ , the formal limit  $\varepsilon \rightarrow 0$  in the obtained system provides a limit  $A$  solution to the KdV equation if  $d = 1$ , and to the KP-I equation if  $d \geq 2$ . We emphasize that in the usual Gross-Pitaevskii equation (1.1),  $f(y) = y - 1$ , in the logarithmic case (logGP) with  $\lambda = 1$ ,  $f(y) = \ln y$ , so we always have  $f(1) = 0$  and  $f'(1) > 0$ . The general case considered in [17] is analyzed under the

assumption  $f \in C^\infty(\mathbb{R}, \mathbb{R})$ , which is not satisfied in the logarithmic case. This is probably not the sharp assumption in order for the arguments presented in [17] to remain valid, but it is most likely that the proof uses a  $C^0$  regularity on  $[0, +\infty)$ , like in [15, 16, 37] (these papers are devoted to the existence of traveling waves in the same framework;  $C^1$  regularity of  $f$  is required only near 1). Again, the main difficulty in the case of the logarithmic nonlinearity lies in the control of vacuum, as  $f$  fails to be continuous at the origin.

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