Initial Trace of Positive Solutions of Some Diffusion Equations with Absorption

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1 Introduction

If we consider a nonnegative $C^{2,1}$ function $(x,t) \mapsto u(x,t)$ satisfying a diffusion equation

$$\partial_t u - A(u, \nabla u, D^2 u) + b(u, \nabla u) = 0 \tag{1.1}$$

in $Q_T := \mathbb{R}^N \times (0,T)$, a natural question is to understand what are the data on which the function depends, besides the structural assumptions on the functions A and B. One important approach is to associate to this function a general notion of initial value that we call the *initial trace problem*. The initial trace problem is two-fold:

1- Is it possible to define *in a suitable way* the limit value of $u(.,t)$ when $t \to 0$? This limit, whenever it exists, is called the *initial trace of* u, noted $tr\,(u)$. In most cases it is an outer regular nonnegative Borel measure in $\mathbb{R}^N$.
2- Given an outer regular nonnegative Borel measure ν in $\mathbb{R}^N$, is it possible to find a

nonnegative solution u of (1.1) in Q_T such that $tr\,(u) = \nu$? This step requires a sharp study of the possible initial traces.
3- Is the correspondence between the set of initial traces ν and the set of positive solutions of (1.1) in Q_T one-to-one ? More simply, is a positive solution defined in a unique way by its initial trace.

In this formulation A is a real valued Caratheodory function defined in $\mathbb{R} \times \mathbb{R}^N \times M^N(\mathbb{R})$ and B a real valued Caratheodry function defined in $\mathbb{R}^N \times \mathbb{R}$.

In this full generality the problem is hard to analyse except for the mere diffusion equation

$$\partial_t u - A(u, \nabla u, D^2 u) = 0, \tag{1.2}$$

where the two cases of the porous-media equation (with $A(u, \nabla u, D^2 u) = \Delta u^m$) and the p-Laplace diffusion equation (with $A(u, \nabla u, D^2 u) = div(|\nabla u|^{p-2}\nabla u)$) are fairly well understood. In these cases the initial trace is a nonnegative Radon measure with some growth at infinity. When the equation contains a reaction term, the situation is completely changed, even in the mere case where $A(u, \nabla u, D^2 u) = \Delta u$.

$$\partial_t u - \Delta u + b(u, \nabla u) = 0. \tag{1.3}$$

The sign of the reaction term plays an important role. Surprisingly the question of identifying the initial trace of a solution of (4.161) is much easier if $b(u, \nabla u)$ is nonpositive, e.g. $b(u, \nabla u) = -u^q$. In that case the function u is super-caloric and it always admits an initial trace in the class of nonnegative Radon measures in $\mathbb{R}^N$. The second question of reconstructing the solution from its initial trace is more involved, and the associated question of uniqueness is even deeper. In this paper we will concentrate on the case where the perturbation term is a nonlinear power term under the form

$$b(u, \nabla u) = \epsilon u^q \tag{1.4}$$

where $\epsilon = \pm 1$. The first easy to prove results dealing with the the case $\epsilon = -1$ and $q > 0$, or $\epsilon = 1$ and $0 < q \leq 1$ is the following.

Theorem 1 *Let $\epsilon = -1$ and $q > 0$ or $\epsilon = 1$ and $0 < q \leq 1$. If u is a nonnegative solution of (1.3) in Q_T there exists a nonnegative Radon measure μ in $\mathbb{R}^N$ such that*

$$\lim_{t \to 0} \int_{\mathbb{R}^N} u(x,t)\zeta(x)dx = \int_{\mathbb{R}^N} \zeta d\mu(x) \qquad \text{for all } \zeta \in C_c^\infty(\mathbb{R}^N). \tag{1.5}$$

The problems arising from the study of the model case

$$\partial_t u - \Delta u + |u|^{q-1}u = 0 \quad \text{in } Q_T, \tag{1.6}$$

is now fairly well understood after the initial work of Marcus and Véron [43] who put into light that the initial trace has to be understood in the sense of Borel measures and the exhaustive study of the supercritical case by Marcus and Véron [45] and Gkikas and

Véron [31], [32]. Note that this study followed the very complete analysis of the boundary trace of positive solutions of

$$-\Delta u + u^q = 0 \quad \text{in } \Omega, \tag{1.7}$$

which was carried on by Marcus and Véron [45], [46] and concluded by Marcus in the remarkable paper [41] to which the construction of Gkikas and Véron [32] that we will developed thoroughly in the sequel is much indebted.

Concerning (1.6), Marcus and Véron pointed out the key role of the critical exponent $q_c := 1 + \frac{2}{N}$ and showed that the analysis is very different according to the position of q with respect to q_c Their starting result concerning this equation is the following

Theorem 2 *Let $q > 1$ and u is a positive solution of (1.5) in Q_T. Then there exist a closed set $\mathcal{S} \subset \mathbb{R}^N$ and a nonnegative Radon measure μ in $\mathcal{R} := \mathbb{R}^N \setminus \mathcal{S}$ such that,*
(i) For any $\zeta \in C_c^\infty(\mathcal{R})$ there holds

$$\lim_{t\to 0} \int_{\mathbb{R}^N} u(x,t)\zeta(x)dx = \int_{\mathbb{R}^N} \zeta d\mu(x). \tag{1.8}$$

(ii) For any $y \in \mathcal{S}$ and any $\epsilon > 0$, there holds

$$\lim_{t\to 0} \int_{B_\epsilon(y)} u(x,t)dx = \infty. \tag{1.9}$$

The set $\mathcal{S} := \mathcal{S}ing(u)$ (resp. $\mu := \mu(u)$) is called the singular (resp. regular) part of the initial trace of u. Conversely we have an existence and uniqueness result in the subcritical case.

Theorem 3 *Let $1 < q < q_c$. Then for any couple $(\mathcal{S}, \mu)$ where $\mathcal{S}$ is a closed subset of $\mathbb{R}^N$ and μ a nonnegative Radon measure in $\mathcal{R} := \mathbb{R}^N \setminus \mathcal{S}$, there exists a unique positive solution u of (1.5) in $Q_\infty := \mathbb{R}^N \times (0,\infty)$ with initial trace $(\mathcal{S}, \mu)$.*

When $q \geq q_c$ not every measure is admissible for being the measure part of the initial trace of a positive solution of (1.5), neither every closed set can be the singular part. To answer this question it is necessary to introduce the Riesz $(\frac{2}{q}, q')$-capacity of a Borel set $E \subset \mathbb{R}^N$.

$$\dot{R}_{\frac{2}{q},q'}(E) = \inf\left\{ \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\zeta(x)-\zeta(y)|^{q'}}{|x-y|^{N-\frac{2}{q-1}}} dxdy : \zeta \in C_c^\infty(\mathbb{R}^N), 0 \leq \zeta \leq 1, \zeta \geq \mathbf{1}_E \right\}. \tag{1.10}$$

This capacity plays a fundamental role since Baras and Pierre proved in [9], [8] two fundamental results which assert that

A Borel set $S \subset \mathbb{R}^N$ is a removable singularity for any solution u of (1.5) in $Q_\infty := \mathbb{R}^N \times (0,\infty)$ which is continuous in $\overline{Q}_\infty \setminus (S \times \{0\})$ if and only if

$$\dot{R}_{\frac{2}{q},q'}(S) = 0. \tag{1.11}$$

Similarly there exists a solution u of (1.6) in Q_∞ such that $u(.,0)=\mu$, where μ is a bounded measure in $\mathbb{R}^N$ if and only if

$$\text{If } S\subset\mathbb{R}^N \text{ is a Borel set, } \dot{R}_{\frac{2}{q},q'}(S)=0\Longrightarrow|\mu(S)|=0. \tag{1.12}$$

For the sake of completeness, we sketch a proof in Theorem 3.12. Note that the boundedness assumption on μ is not necessary.
If $\mathcal{R}\subset\mathbb{R}^N$ is open, we denote by $\mathfrak{M}_q(\mathcal{R})$ the space of Radon measures μ in $\mathcal{R}$ satisfying

$$\text{If } S\subset\mathcal{R} \text{ is a Borel set, } \dot{R}_{\frac{2}{q},q'}(S)=0\Longrightarrow|\mu(S)|=0. \tag{1.13}$$

The positive cone of this space is denoted by $\mathfrak{M}_q^+(\mathcal{R})$.

If $\mathcal{S}\subset\mathbb{R}^N$ is closed and μ is a positive Radon measure in $\mathcal{S}^c$ we define

$$\partial_\mu\mathcal{S}:=\left\{y\in\mathcal{S}:\mu(B_\epsilon(y)\cap\mathcal{S}^c)=\infty,\,\forall\epsilon>0\right\}, \tag{1.14}$$

which is the set of blowing points of the measure, and

$$\mathcal{S}^*:=\left\{y\in\mathcal{S}:\dot{R}_{\frac{2}{q},q'}(B_\epsilon(y)\cap\mathcal{S})>0,\,\forall\epsilon>0\right\}, \tag{1.15}$$

which is the set of intrinsically non-removable points of $\mathcal{S}$.

Theorem 4 *Let $q\geq q_c$. A couple $(\mathcal{S},\mu)$ where $\mathcal{S}$ is a closed subset of $\mathbb{R}^N$ and $\mu\in\mathfrak{M}_q^+(\mathcal{R})$ where $\mathcal{R}:=\mathbb{R}^N\backslash\mathcal{S}$, is the initial trace of a positive solution u of (1.5) in $Q_\infty:=\mathbb{R}^N\times(0,\infty)$ if and only if $\mathcal{S}=\partial_\mu\mathcal{S}\cup\mathcal{S}^*$.*

A striking aspect of the super critical case, i.e. $q\geq q_c$, is that there exist infinitely many solutions when $\mathcal{S}$ is not empty and the solution constructed in Theorem 2 is actually the maximal solution with any initial trace $(\mathcal{S},0)$. This has resulted in a finer definition of the initial trace called the *precise trace*. The basic idea of this extension is to replace the Euclidean topology which served as a basic tool in the definition of the trace process by the thin $\mathfrak{T}_q$-topology associated to the $(\frac{2}{q},q')$-capacity. Note that this process was developed by Marcus and Véron [46] in a similar way for analysing the boundary trace of positive solutions of

$$-\Delta u+|u|^{q-1}u=0\quad\text{in }\Omega\subset\mathbb{R}^N. \tag{1.16}$$

When $q\geq q_c$ it is proved in [32] that any nonnegative solution u of (1.5) in Q_T admits a precise singular initial set $\mathcal{S}_q(u)$ which is the set of $\xi\in\mathbb{R}^N$ such that for any thin-neighbourhood U (for the $\mathfrak{T}_q$-topology) of ξ there holds

$$\int_0^T\int_{\mathbb{R}^N}u^q\left(\mathbb{H}[\mathbf{1}_U]\right)^{2q'}dxdt=\infty \tag{1.17}$$

where $\mathbb{H}[\mathbf{1}_U]$ is the heat potential in Q_∞ of the characteristic function of U. The set $\mathcal{R}_q(u):=\mathcal{S}_q^c(u)$ is the fine regular set of the initial trace. It is the set of the $\xi\in\mathbb{R}^N$ such that there exists a thin-neighbourhood U of ξ with the property that

$$\int_0^T\int_{\mathbb{R}^N}u^q\left(\mathbb{H}[\mathbf{1}_U]\right)^{2q'}dxdt<\infty. \tag{1.18}$$

Essentially the precise regular set of the initial trace carries a nonnegative Radon measure $\mu_{\mathcal{R}_q(u)}$, absolutely continuous with respect to the Riesz capacity $\dot{R}_{\frac{2}{q},q'}$, and such that for any bounded test function η belonging to the Besov space $B^{\frac{2}{q},q'}(\mathbb{R}^N)$ with "support" in $\mathcal{R}_q(u)$ (more precisely $\mathfrak{T}_q$-support in a sense which will be defined in the text), there holds

$$\lim_{t\to 0}\int_{\mathbb{R}^N} u(x,t)\eta_+^{2q'}dx = \int_{\mathbb{R}^N}\eta_+^{2q'}d\mu_{\mathcal{R}_q(u)}. \tag{1.19}$$

This allows to define a solution $v_{\mathcal{R}_q(u)}$ of (1.5) corresponding to this measure $\mu_{\mathcal{R}_q(u)}$. It is called the *regular component of* u. For defining the *singular component of* u we first denote by $U_{\mathcal{S}_q(u)}$ the maximal solution of (1.5) with an initial trace vanishing in $\mathcal{S}_q^c(u)$. The singular component is $[u]_{\mathcal{S}_q(u)}$ which is the maximal solution of (1.5) bounded from above by u and with initial trace vanishing in $\mathcal{S}_q^c(u)$. The couple $(\mu_{\mathcal{R}_q(u)},\mathcal{S}_q(u))$ is called *the precise initial trace.*

The main results in the supercritical case ($q \ge q_c$) are summarised by the following statement.

Theorem 5 *1- If u is a nonnegative solution of (1.5) in Q_T, then the function $v_{\mathcal{R}_q(u)}\oplus[u]_{\mathcal{S}_q(u)}$, which is the largest solution dominated by the super-solution $v_{\mathcal{R}_q(u)}+[u]_{\mathcal{S}_q(u)}$ admits for precise initial trace $(\mu_{\mathcal{R}_q(u)},\mathcal{S}_q(u))$.*
2- The solution $v_{\mathcal{R}_q(u)}\oplus[u]_{\mathcal{S}_q(u)}$ is σ-moderate in the sense that it is the increasing limit of solutions u_{μ_n} with initial data μ_n which are nonnegative bounded measures belonging to $B^{\frac{2}{q},q'}(\mathbb{R}^N)$. It is the unique σ-moderate solution with such a trace.
3- Any positive solution u of (1.5) is σ-moderate.

As a consequence there exists a one to one correspondence between the set of nonnegative solutions u of (1.5) and the set of couples $(\mu_{\mathcal{R}_q(u)},\mathcal{S}_q(u))$.

These notes are based upon the articles [43], [47], [46] written jointly with M. Marcus and [31], [32] written with K. Gkikas, in both cases in a very fruitful collaboration.

2 Equation with sublinear or reaction forcing term

In order to point out the differences between the absorption case which is the main object of this survey and the sublinear or forcing cases which are indeed much easier to treat, we we present some standard results in these later cases.

2.1 The heat equation

We first consider the basic approach of the trace problem for the heat equation. Let $u\in C^{2,1}(Q_T)$ be a positive solution of

$$\partial_t u-\Delta u=0 \quad \text{in } Q_T. \tag{2.1}$$

If $G\subset\mathbb{R}^N$ is any bounded domain, we denote by λ_G the first eigenvalue of $-\Delta$ in $H_0^1(G)$ and by ϕ_G the corresponding first positive eigenfunction normalized by $\max\phi_G=1$. Then

$$\frac{d}{dt}\int_G u(x,t)\phi_G^2(x)dx+2\lambda_G\int_G u(x,t)\phi_G^2(x)dx=2\int_G u(x,t)|\nabla\phi_G|^2(x)dx.$$

Therefore the function

$$t \mapsto e^{2\lambda_G t} \int_G u(x,t)\phi_G^2(x)dx$$

is nondecreasing. It admits a finite nonnegative limit $M_u(G)$ when $t \to 0$ and

$$e^{2\lambda_G \tau} \int_G u(x,\tau)\phi_G^2(x)dx - M_u(G) = \int\!\!\int_{Q_\tau^G} u(x,t)|\nabla \phi_G|^2(x)dxdt < \infty,$$

where $Q_T^G = G \times (0,T)$. This implies in particular that $u \in L^1(Q_\tau^G)$ for any $\tau < T$. Then, if $\zeta \in C_c^\infty(\mathbb{R}^N)$ there exists $\ell(\zeta)$ with the property that

$$\ell(\zeta) = \lim_{t\to 0} \int_{\mathbb{R}^N} u(x,t)\zeta(x) = \int_{\mathbb{R}^N} u(x,\tau)\zeta(x) - \int_0^\tau \int_{\mathbb{R}^N} u(x,s)\Delta\zeta(x)dxds$$

The mapping $\zeta \mapsto \ell(\zeta)$ is a positive linear functional, hence it extends as a unique Radon measure in $\mathbb{R}^N$ that we denote μ. The following characterisation of the measures μ is proved in [4], [5]

Let u be a nonnegative solution of (2.1) in Q_T and μ be the initial trace of u, then

$$\int_{\mathbb{R}^N} e^{-a|x|^2} d\mu(x) < \infty \quad \textit{for all } a < \frac{1}{4T}. \tag{2.2}$$

Conversely, if μ is a nonnegative Radon measure in $\mathbb{R}^N$ satisfying (2.2) the function u defined in Q_T by

$$u(t,x) = \frac{1}{(4\pi t)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}} d\mu(y) < \infty \tag{2.3}$$

is the unique positive solution of (2.1) with initial trace μ.

Definition 2.1 *If μ is a Radon measure in $\mathbb{R}^N$, we denote by $\mathbb{H}[\mu]$ the heat potential of μ, defined by*

$$\mathbb{H}[\mu](x,t) = \frac{1}{(4\pi t)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}} d\mu(y) = \int_{\mathbb{R}^N} H(x,y,t)d\mu(y), \tag{2.4}$$

provided this formula has a meaning, e.g. if μ is bounded. The function $H(x,y,t) := \frac{1}{(4\pi t)^{\frac{N}{2}}} e^{-\frac{|x-y|^2}{4t}}$ is called the heat kernel in Q_∞.

This result is the extension to higher dimension of Widder representation theorem proved in 1-D in [56].

2.2 Equations with sublinear or forcing reaction. Proof of Theorem 1

2.2.1 The sublinear case

We assume that $u \in C^{2,1}(Q_T)$ is a positive solution of (1.3) where b is continuous in $\mathbb{R} \times \mathbb{R}^N$ and satisfies

$$0 \le b(r,\xi) \le cr + d \quad \text{for all } r \ge 0 \text{ and } \xi \in \mathbb{R}^N, \tag{2.5}$$

for some $c > 0$ and $d \geq 0$. As in the case of heat equation we call ϕ_G the first normalised eigenfunction of a bounded smooth domain $G \subset \mathbb{R}^N$, with associated eigenfunction λ_G. Then

$$\frac{d}{dt}\int_G u(x,t)\phi_G^2(x)dx + (2\lambda_G + c)\int_G u(x,t)\phi_G^2(x)dx + d\int_G \phi_G^2(x)dx \geq 0.$$

This implies that the function

$$t \mapsto e^{(2\lambda_G+c)t}\left(\int_G u(x,t)\phi_G^2(x)dx + \frac{d}{2\lambda_G + c}\int_G \phi_G^2(x)dx\right)$$

is increasing. Therefore it admits a finite limit (positive) when $t \to 0$ and thus there exists

$$M_u(G) := \lim_{t\to 0}\int_G u(x,t)\phi_G^2(x)dx. \tag{2.6}$$

Since for any $0 < \tau < T$

$$\int_0^\tau\int_G b(u,\nabla u)(x,t)\phi_G^2(x)dxdt \leq c\int_0^\tau\int_G u(x,t)\phi_G^2(x)dxdt + d\int_0^\tau\int_G \phi_G^2(x)dxdt < \infty,$$

and G is arbitrary, it follows that $u + b(u,\nabla u) \in L^1(Q_G^\tau)$ where $Q_G^\tau := G \times (0,\tau)$. As in the case of heat equation we obtain that there exists a Radon measure μ in $\mathbb{R}^N$ such that for any $\zeta \in C_c^\infty(\mathbb{R}^N)$ there holds

$$\lim_{t\to 0}\int_{\mathbb{R}^N} u(x,t)\zeta(x)dx = \int_{\mathbb{R}^N}\zeta d\mu(x), \tag{2.7}$$

and also for any $t \in (0,T)$,

$$\int_{\mathbb{R}^N} u(x,t)\zeta(x) - \int_0^t\int_{\mathbb{R}^N} u(x,s)\Delta\zeta(x)dxds + \int_0^t\int_{\mathbb{R}^N} b(u,\nabla u)(x,t)\zeta(x)dxds = \int_{\mathbb{R}^N}\zeta d\mu(x). \tag{2.8}$$

This implies that u admits μ as an initial trace in the sense defined by (2.7), and it is a solution of

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta u + b(u,\nabla u) = 0 \quad &\text{in } Q_T \\ u(.,0) = \mu \quad &\text{in } \mathbb{R}^N. \end{aligned} \tag{2.9}$$

Finally, this problem can be solved for any nonnegative bounded measure μ, and more generally for any nonnegative Radon measure μ such that the heat potential $\mathbb{H}[\mu]$ can be defined.

2.2.2 The forcing reaction case

We assume that b satisfies

$$b(r,\xi) \leq 0 \quad \text{for all } r \geq 0 \text{ and } \xi \in \mathbb{R}^N. \tag{2.10}$$

As in the previous case the function

$$t \mapsto e^{2\lambda_G t} \int_G u(x,t)\phi_G^2(x)dx$$

is increasing, thus (2.6) holds. From (1.3) we have for any $0 < t < \tau < T$,

$$\int_G u(x,\tau)\phi_G^2(x)dx + 2\lambda_G \int_t^\tau \int_G u(x,s)\phi_G^2(x)dxds = 2\int_t^\tau \int_G u(x,s)|\nabla\phi_G|^2(x)dxds + \int_t^\tau \int_G b(u,\nabla u)(x,s)\phi_G^2(x)dxds + \int_G u(x,t)\phi_G^2(x)dx.$$

Because (1.3) holds $u\phi_G^2 \in L^1(Q_\tau^G)$, hence, by the monotone convergence theorem both $b(u,\nabla u)\phi_G^2$ and $u|\nabla\phi_G|^2$ are integrable in Q_τ^G. If $\zeta \in C_c^\infty(\mathbb{R}^N)$ we obtain that (2.8), (1.5) and (2.9) hold for some nonnegative Radon measure μ.

Remark. In the model case, the existence of solutions to

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta u - u^q = 0 \qquad & \text{in } Q_G^T \\ u(.,0) = \mu \qquad & \text{in } \mathbb{R}^N, \end{aligned} \tag{2.11}$$

with $q > 1$ depends on two factors: the total mass of the positive measure μ and its concentration. A necessary and sufficient condition is provided in [10, Théorème 3.2] under the form of a duality argument. In [55, Theorem 3] a sufficient condition for the existence of a global solution in Q_∞ is as follows,

$$(q-1)\int_0^\infty \|\mathbb{H}[\mu](s,.)\|_{L^\infty}^{q-1}\, ds \le 1. \tag{2.12}$$

3 The rough trace

This section is devoted to the construction and the study of the properties of the *rough initial trace* of positive solution of

$$\partial_t u - \Delta u + u^q = 0 \quad \text{in } Q_\infty, \tag{3.13}$$

when $q > 1$. The qualifier of *rough* will be justified later on in connection with surprising non-uniqueness results.

Proof of Theorem 2. Let u be a nonnegative solution of 3.13 in Q_∞ and $y \in \mathbb{R}^N$, then the following alternative holds:
(i) either there exists $\alpha > 0$ such that

$$\iint_{Q_1^{B_\alpha(y)}} u^q(x,t)dxdt < \infty, \tag{3.14}$$

(ii) or for any $\alpha > 0$

$$\iint_{Q_1^{B_\alpha(y)}} u^q(x,t)dxdt = \infty. \tag{3.15}$$

If (3.14) holds, then $u \in L^1(Q_T^B)$ for any open ball $B \subset \overline{B} \subset B_\alpha(y)$, and for any $\zeta \in C_c^\infty(B)$ there holds

$$\frac{d}{dt}\left(\int_B u(x,t)\zeta dx + \int_t^1 \int_B (u\Delta\zeta - u^q\zeta)\, dxd\tau\right) = 0. \tag{3.16}$$

Consequently there exists $\ell(\zeta)$ defined by

$$\ell(\zeta) := \lim_{t\to 0}\int_B u(x,t)\zeta dx = \int_B u(x,1)\zeta dx + \int_0^1 \int_B (u\Delta\zeta - u^q\zeta)\, dxd\tau. \tag{3.17}$$

The mapping $\zeta \mapsto \ell(\zeta)$ is a positive linear functional on $C_c^\infty(B)$, hence it is a Radon measure μ_B in B.

If (3.32) holds let B as above and ϕ_B be the first eigenfunction of $-\Delta$ in $H_0^1(B)$ with maximal value 1 and corresponding eigenfunction λ_B. Then

$$\begin{aligned}\frac{d}{dt}\int_B u(x,t)\phi_B^{2q'}dx + 2q'\lambda_B\int_B u(x,t)\phi_B^{2q'}dx \\ - 2q'(2q'-1)\int_B u(x,t)\phi_B^{2q'-2}\left|\nabla\phi_B\right|^2 dx + \int_B u^q(x,t)\phi_B^{2q'}dx = 0,\end{aligned}$$

where we have set $B = B_\alpha(y)$ and $q' = \frac{q}{q-1}$. Since

$$\int_B u(x,t)\phi_B^{2q'-2}\left|\nabla\phi_B\right|^2 dx \le \frac{\delta^q}{q}\int_B u^q(x,t)\phi_B^{2q'}dx + \frac{1}{\delta^{q'}q'}\int_B \left|\nabla\phi_B\right|^{2q'} dx,$$

for suitable $\delta > 0$ and $c > 0$, we have that

$$\frac{d}{dt}\left(e^{2q'\lambda_B t}\int_B u(x,t)\phi_B^{2q'}dx\right) + \frac{e^{2q'\lambda_B t}}{2}\int_B u^q(x,t)\phi_B^{2q'}dx \le ce^{2q'\lambda_B t}\int_B \left|\nabla\phi_B\right|^{2q'} dx.$$

Then

$$\begin{aligned}e^{2q'\lambda_B}\int_B u(x,1)\phi_B^{2q'}dx + \frac{1}{2}\int_t^1 e^{2q'\lambda_B\tau}\int_B u^q(x,\tau)\phi_B^{2q'}dxd\tau \\ \le e^{2q'\lambda_B t}\int_B u(x,t)\phi_B^{2q'}dx + c\int_t^1 e^{2q'\lambda_B\tau}\int_B \left|\nabla\phi_B\right|^{2q'} dxd\tau.\end{aligned} \tag{3.18}$$

Therefore

$$\lim_{t\to 0}\int_B u(x,t)\phi_B^{2q'}dx = \infty. \tag{3.19}$$

The set of points y such that (i) holds is clearly open and its union is called *the regular set* $\mathcal{R}$. By a partition of unity there exists a unique nonnegative Radon measure μ on $\mathcal{R}$ such that for any $\zeta \in C_c^\infty(\mathcal{R})$ there holds

$$\lim_{t\to 0}\int_{\mathbb{R}^N} u(x,t)\zeta dx = \int_{\mathbb{R}^N}\zeta d\mu(x). \tag{3.20}$$

For any $y \in \mathcal{S}$ and any $\alpha > 0$ we have (3.19), therefore we define a Borel measure ν such that for any Borel set $E \subset \mathbb{R}^N$

$$\nu(E) = \begin{cases} \int_E d\mu(x) & \text{if } E \subset \mathcal{R} \\ \infty & \text{if } E \cap \mathcal{S} \neq \emptyset, \end{cases} \tag{3.21}$$

and it is outer regular.

3.1 The a priori estimate

The function ϕ_∞ defined on $(0, \infty)$ by

$$\phi_\infty(t) = \left(\frac{1}{t(q-1)}\right)^{\frac{1}{q-1}} \tag{3.22}$$

is the maximal solution of the differential equation $u' + u^q = 0$ on $(0, \infty)$.
For any $R > 0$, let w_R be unique solution of

$$\begin{aligned} -\Delta w + w^q = 0 \quad &\text{in } B_R \\ \lim_{|x| \to R} w(x) = \infty. \end{aligned} \tag{3.23}$$

Existence follows from the universal Keller-Osserman upper construction and uniqueness from the fact that

$$w_R(x) = R^{-\frac{2}{q-1}} w_1(x/R). \tag{3.24}$$

For any $y \in \mathbb{R}^N$, $R > 0$ and $\epsilon > 0$ the function

$$u_{\epsilon,R,y}(x, t = \phi_\infty(t - \epsilon) + w_R(x - y) \tag{3.25}$$

is a super solution of (3.13) in $B_R(y) \times (\epsilon, \infty)$. Hence it dominates u therein. Letting $\epsilon \to 0$ and $R \to \infty$, yields

$$u(x, t) \leq \phi_\infty(t) \quad \text{for all } (x, t) \in \mathbb{R}^N \times (0, \infty). \tag{3.26}$$

This a priori estimate admits a localised version.

Proposition 3.1 *Let $q > 1$and $R > 0$.*
1- There exists a unique nonnegative solution $u := u_{\infty,R}$ of (3.13) in Q_∞ such that

$$\lim_{t \to 0} u(x, t) = \infty \quad \textit{uniformly in } B_R, \tag{3.27}$$

and

$$\lim_{t \to 0} u(x, t) = 0 \quad \textit{locally uniformly in } \overline{B}_R^c. \tag{3.28}$$

Furthermore

$$\lim_{t \to 0} t^{\frac{1}{q-1}} u_{\infty,R}(x, t) = \left(\frac{1}{q-1}\right)^{\frac{1}{q-1}} \qquad \textit{locally uniformly in } B_R, \tag{3.29}$$

and for any $\alpha > \frac{3-q}{q-1}$ *there exists* $C_\alpha > 0$ *such that*

$$u_{\infty,R}(x,t) \le C_\alpha t^{-\frac{1}{q-1}} \left(\frac{|x|-R}{\sqrt{t}}\right)^\alpha e^{-\frac{(|x|-R)^2}{4t}} \quad \text{for all } (x,t) \in Q_\infty \ \text{ s.t. } |x|-R \ge \sqrt{t}. \tag{3.30}$$

2- There exists a unique nonnegative solution $u := u_{\infty,R^c}$ *of (3.13) in* Q_∞ *such that*

$$\lim_{t\to 0} u(x,t) = 0 \quad \text{uniformly in } B_R, \tag{3.31}$$

and

$$\lim_{t\to 0} u(x,t)dx = \infty \quad \text{locally uniformly in } \overline{B}_R^c. \tag{3.32}$$

Furthermore

$$\lim_{t\to 0} t^{\frac{1}{q-1}} u_{\infty,R^c}(x,t) = \left(\frac{1}{q-1}\right)^{\frac{1}{q-1}} \qquad \text{uniformly in } B^c_{R+\epsilon} \tag{3.33}$$

for any $\epsilon > 0$. *Next, for any* $\theta \in (0,1)$ *and* $\alpha < \frac{3-q}{q-1}$, *there exists* $C_{\alpha,\theta} > 0$ *such that*

$$u_{\infty,R^c}(x,t) \le C_{\alpha,\theta} t^{-\frac{1}{q-1}} \left(\frac{\theta R-|x|}{\sqrt{t}}\right)^\alpha e^{-\frac{(\theta R-|x|)^2}{4t}} \quad \text{for all } (x,t) \in Q_\infty \ \text{ s.t. } |x| \le \theta R - \sqrt{t}. \tag{3.34}$$

Proof. Step 1-1- There exists a unique $C^\infty(0,\infty))$ *function* W *with positive value satisfying*

$$\begin{aligned} \partial_t W - \partial_{xx} W + W^q = 0 \qquad & \text{in } (0,\infty) \\ \lim_{x\to 0} W(x,t) = \infty \qquad & \text{for all } t > 0 \\ \lim_{t\to 0} W(x,t) = 0 \qquad & \text{for all } x > 0. \end{aligned} \tag{3.35}$$

This function is self-similar and endows the form

$$W(x,t) = t^{-\frac{1}{q-1}} \widetilde{W}\left(\tfrac{x}{\sqrt{t}}\right), \tag{3.36}$$

where $\widetilde{W}$ is the unique positive solution of

$$\begin{aligned} \widetilde{W}'' + \tfrac{\eta}{2}\widetilde{W}' + \tfrac{1}{q-1}\widetilde{W} - \widetilde{W}^q = 0 \qquad & \text{in } (0,\infty) \\ \lim_{\eta\to 0} \widetilde{W}(\eta) = \infty \\ \lim_{\eta\to\infty} \eta^{\frac{2}{q-1}} \widetilde{W}(\eta) = 0. \end{aligned} \tag{3.37}$$

The construction is as follows. Let $k > 1$ and $\zeta = \zeta_k$ be the solution of

$$\begin{aligned} \partial_t \zeta - \partial_{xx}\zeta + \zeta^q = 0 \qquad & \text{in } \mathbb{R} \\ \zeta(.,0) = \mathbf{1}_{[-k,0]} \qquad & \text{on } \mathbb{R}. \end{aligned} \tag{3.38}$$

The correspondence $k \mapsto \zeta_k$ is increasing. Since ζ_k is bounded from above by 1, ζ_k converges to ζ_∞ which is the unique solution of

$$\begin{array}{rl} \partial_t \zeta - \partial_{xx}\zeta + \zeta^q = 0 & \text{in } \mathbb{R} \times (0,\infty) \\ \zeta(.,0) = \mathbf{1}_{(-\infty,0]} & \text{on } \mathbb{R}. \end{array} \tag{3.39}$$

For $\ell > 0$ we denote by T_ℓ the scaling transformation which leaves (3.13) equivariant,

$$T_\ell[\phi](x,t)) = \ell^{\frac{2}{q-1}} \phi\left(\ell x, \ell^2 t\right). \tag{3.40}$$

Then $T_\ell[\zeta_\infty] := \zeta_{\infty,\ell}$ is the solution of (3.13) in Q_∞ with initial data $\ell^{\frac{2}{q-1}}\mathbf{1}_{(-\infty,0]}$. Again $\ell \mapsto \zeta_{\infty,\ell}$ is increasing. Since the function

$$x \mapsto \left(\frac{2(q+1)}{(q-1)^2}\right)^{\frac{1}{q-1}} x^{-\frac{2}{q-1}} := C_q x^{-\frac{2}{q-1}} \quad \text{for all } \ x > 0,$$

is a solution of

$$\begin{array}{rl} \partial_t v - \partial_{xx} v + v^q = 0 & \text{in } \ (0,\infty) \times (0,\infty) \\ v(0,t) = \infty & \text{in } \ (0,\infty), \end{array} \tag{3.41}$$

we have

$$\zeta_\infty(x,t) \le C_q x^{-\frac{2}{q-1}} \quad \text{in } \ (0,\infty) \times (0,\infty),$$

which implies that, for all $\ell > 1$,

$$\zeta_\infty(x,t) \le \zeta_{\infty,\ell}(x,t) \le \phi_\infty(t)\mathbf{1}_{(-\infty,0]}(x) + \min\left\{C_q|x|^{-\frac{2}{p-1}}, \phi_\infty(t)\right\}\mathbf{1}_{(0,\infty)}(x). \tag{3.42}$$

Thus $\zeta_{\infty,\ell}$ converges to some function W when $\ell \to \infty$, and W satisfies (3.13). Because there holds for any $\phi \in C_c^\infty((0,\infty))$,

$$\int_0^\infty \int_{\mathbb{R}} \left(-\zeta_{\infty,\ell}\left(\partial_t\phi + \Delta\phi\right) + \zeta_{\infty,\ell}^q \phi\right) dxdt = 0,$$

the function W satisfies the same upper bound (3.42) as $\zeta_{\infty,\ell}$ and it is a solution of

$$\begin{array}{rl} \partial_t \zeta - \partial_{xx}\zeta + \zeta^q = 0 & \text{in } \mathbb{R} \\ \zeta(.,0) = 0 & \text{on } (0,\infty) \\ \lim\limits_{t\to 0} \zeta(x,t) = \infty & \text{for all } x \le 0. \end{array} \tag{3.43}$$

Finally, for any $k > 0$, $T_k \circ T_\ell = T_{k\ell}$, hence $T_k[\zeta_{\infty,\ell}] = \zeta_{\infty,k\ell}$, which implies that $T_k[W] = W$ for all $k > 0$, hence W is self-similar. This implies that $W(x,t) = t^{-\frac{1}{q-1}}\widetilde{W}\left(\frac{x}{\sqrt{t}}\right)$ and W satisfies

$$\begin{array}{rl} \widetilde{W}'' + \dfrac{\eta}{2}\widetilde{W}' + \dfrac{1}{q-1}\widetilde{W} - \widetilde{W}^q = 0 & \text{in } (0,\infty) \\ \lim\limits_{\eta\to\infty} \eta^{\frac{2}{q-1}}\widetilde{W}(\eta) = 0 & \\ \lim\limits_{\eta\to 0} \widetilde{W}(\eta) = \infty. & \end{array} \tag{3.44}$$

The behaviour of W can be obtained by matching asymptotic expansion, if we consider the function $\eta \mapsto \widetilde{W}_\alpha := \eta^\alpha e^{-\frac{\eta^2}{4}}$ which is a supersolution (resp. subsolution) when $\eta \to \infty$ if $\alpha > \frac{3-q}{q-1}$ (resp. $\alpha < \frac{3-q}{q-1}$). Thus for any $\alpha > \frac{3-q}{q-1}$ there exists $C_\alpha > 0$ such that

$$\widetilde{W}(\eta) \le C_\alpha \eta^\alpha e^{-\frac{\eta^2}{4}} \quad \text{for all } \eta \text{ in } [1, \infty). \tag{3.45}$$

Inequality (3.34) follows from this estimate.

Step 1-2- We claim that there exists a unique positive function $u_{\infty,R}$ which satisfies (3.13)

$$\lim_{t\to 0} u_{\infty,R}(x,t) = \infty \qquad \textit{locally uniformly in } B_R, \tag{3.46}$$

and

$$\lim_{t\to 0} u_{\infty,R}(x,t) = 0 \qquad \textit{uniformly in } B^c_{R+\epsilon}. \tag{3.47}$$

for any $\epsilon > 0$.

Since the equation and the initial conditions are invariant under the transformation T_ℓ, we can assume that $R = 1$. If $\mathbf{e}$ is a unit vector we denote by $v_\mathbf{e}$ the function defined by

$$v_\mathbf{e}(x,t) = \begin{cases} W(\langle x - \mathbf{e}, \mathbf{e}\rangle, t) & \text{if } (\langle x - \mathbf{e}, \mathbf{e}\rangle > 0, t > 0 \\ \infty & \text{if } (\langle x - \mathbf{e}, \mathbf{e}\rangle \le 0, t > 0, \end{cases} \tag{3.48}$$

and by $H^+_\mathbf{e}$ (resp. $H^+_\mathbf{e}$) the half space $\{x : \langle x - \mathbf{e}, \mathbf{e}\rangle > 0\}$ (resp. $\{x : \langle x - \mathbf{e}, \mathbf{e}\rangle \le 0\}$). Then $v_\mathbf{e}$ satisfies (3.13) in Q_∞ with initial data $v_\mathbf{e}(x,0) = 0$ if $x \in H^+_\mathbf{e}$ and $v_\mathbf{e}(x,0) = \infty$ if $x \in H^-_\mathbf{e}$. Then

$$v_1 = \inf\{v_\mathbf{e} : |\mathbf{e}| = 1\} \tag{3.49}$$

is a supersolution of (3.13) in Q_∞ which satisfies $v_1(x,0) = 0$ if $|x| > 1$ and $v_1(x,0) = \infty$ if $|x| \le 1$. For $k > 0$ let u_k be the solution of (3.13) in Q_∞ with initial data $k\mathbf{I}_{B_1}$. Then $u_k \le v_1$. Since $k \mapsto u_k$ is increasing. Hence there exists a nonnegative function u which is a solution of of (3.13) in Q_∞ such that

$$\lim_{t\to 0} u(x,t) = 0 \quad \text{if } |x| > 1 \ \text{ and } \ \lim_{t\to 0} u(x,t) = \infty \quad \text{if } |x| \le 1. \tag{3.50}$$

By construction u is a minimal solution and by (3.45), and (3.30) holds.

Let $\widetilde{u}$ be another nonnegative function solution of (3.13) in Q_∞ satisfying (3.50). For $\ell < 1$ and $R > \ell^{-1}$, there exists $\epsilon_{\ell,R}$ such that

$$u(x,t) \le w_R(x) \quad \text{for all } (x,t) \in Q_\infty \ \text{ s.t. } \ell^{-1} \le |x| < R \text{ and } 0 < t \le \epsilon_{\ell,R},$$

where w_R is defined in (3.23). Therefore the supersolution $(x,t) \mapsto T_\ell[\widetilde{u}](x, t-\epsilon_{\ell,R}) + w_R(x)$ defined in $B_R \times (\epsilon_{\ell,R}, \infty)$, is larger than u on $\partial B_R \times (\epsilon_{\ell,R}, \infty)$ and for $t = \epsilon_{\ell,R}$. Hence

$$u(x,t) \le T_\ell[\widetilde{u}](x, t-\epsilon) + w_R(x) \quad \text{for all } \ (x,t) \in B_R \times (\epsilon_{\ell,R}, \infty).$$

When $R \to \infty$, $\epsilon_{\ell,R} \to 0$ and $w_R(x) \to 0$. This implies

$$u(x,t) \le T_\ell[\widetilde{u}](x,t) \quad \text{for all } \ (x,t) \in Q_\infty.$$

Letting $\ell \to 1$ yields $u \le \widetilde{u}$. Similarly $\widetilde{u} \le u$.

Step 1-3- The function $u_{\infty,R}$ satisfies

$$\lim_{t\to 0} t^{\frac{1}{q-1}} u_{\infty,R}(x,t) = \left(\frac{1}{q-1}\right)^{\frac{1}{q-1}} \qquad \text{locally uniformly in } B_R. \tag{3.51}$$

In order to prove this claim, for any $R' < R$ we construct a function $\psi \in C_c^2(\mathbb{R}^N)$ such that $0 \le \psi \le 1$, $\psi = 1$ in $B_{R'}$, $\psi = 0$ in $\overline{B}_R^c$ and $-\Delta\psi \le C\psi$ for some $C = CR, R') > 0$. For any $\delta > 0$, the function

$$(x,t) \mapsto X(x,t) := (1-\delta)\psi(x)\phi_\infty(t)$$

satisfies

$$\partial_t X - \Delta X + X^q \le (1-\delta)\phi_\infty\psi\left(\phi_\infty\left((1-\delta)^{q-1} - 1\right) - \frac{\Delta\psi}{\psi}\right).$$

Then there exists $\epsilon > 0$ such that the above expression is negative for $0 < t \le \epsilon$. Therefore $u_{\infty,R} \ge X$ in $\mathbb{R}^N \times (0,\epsilon]$. This implies

$$\liminf_{t\to 0} t^{\frac{1}{q-1}} \inf\{u_{\infty,R}(x,t) : x \in B_{R'}\} \ge (1-\delta)\left(\frac{1}{q-1}\right)^{\frac{1}{q-1}}. \tag{3.52}$$

Since δ is arbitrary, we obtained the claim from (3.26).

Step2-1- We claim that there exists a unique positive function u_{∞,R^c} which satisfies (3.13),

$$\lim_{t\to 0} u_{\infty,R^c}(x,t) = \infty \qquad \text{uniformly in } B_R^c, \tag{3.53}$$

and

$$\lim_{t\to 0} u_{\infty,R^c}(x,t) = 0 \qquad \text{locally uniformly in } B_R. \tag{3.54}$$

The proof uses the previous constructions. For any $k > 0$ we denote by v_k the solution of

$$\begin{aligned} \partial_t v - \Delta v + v^q &= 0 && \text{in } Q_\infty \\ v(.,0) &= k\mathbf{1}_{B_R^c} && \text{in } \mathbb{R}^N. \end{aligned} \tag{3.55}$$

The sequence $\{v_k\}$ is increasing and it satisfies

$$v_k(x,t) \le \phi_\infty(t)\mathbf{1}_{\overline{B}_R^c}(x) + \inf\{\phi_\infty(t), w_R(x)\}\mathbf{1}_{B_R}(x). \tag{3.56}$$

Then it converges to a positive solution of (3.13) in Q_∞ that we denote u_{∞,R^c}. Therefore u_{∞,R^c} satisfies

$$\iint_{Q_\infty}\left((-\partial_t\zeta - \Delta\zeta)u_{\infty,R^c} + \zeta u_{\infty,R^c}^p\right)dxdt = 0 \tag{3.57}$$

for all $\zeta \in C_c^\infty(B_R)$. Using (3.53) and (3.57 it implies that u_{∞,R^c} vanishes on B_R. Uniqueness of such a solution is obtained by the same scaling and shifting argument as in the Steps 1-2.

Step2-2- Improved estimates. As a subsolution of (3.13) in Q_∞ we take

$$v_2(x,t) = \sup\{v_{-\mathbf{e}}(-x,t) : \mathbf{e} \in \partial B_1\}. \tag{3.58}$$

With the same notations as in Step 1-2, v_2 is a subsolution, and $v_2(t,x) \to 0$ when $t \to 0$ if $x \in B_1$ and $v_2(t,x) \to \infty$ when $t \to 0$ and $x \in B_1^c$. This implies that (3.33) holds. The construction of the supersolution is more subtle: for $0 < \theta < 1$ there exists an integer n_θ such that

$$B_\theta \subset \bigcap_{1 \le j \le n_\theta} H_{j,\theta} \subset B_1,$$

where

$$H_{j,\theta} = \left\{x \in \mathbb{R}^N : \langle x - \theta \mathbf{e}_j, \mathbf{e}_j\rangle < 0\right\} \quad \text{with } \mathbf{e}_j \in \partial B_1.$$

Hence the function

$$v_{2,\theta}(x,t) = \sum_{1 \le j \le n_\theta} W(\langle -x + \theta \mathbf{e}_j, \mathbf{e}_j\rangle, t), \tag{3.59}$$

is a supersolution of (3.13) in Q_∞ which dominates u_{∞,R^c}. If $x \in B_\theta$, we have that $\mathrm{dist}\,(x, H_{j,\theta}^c) \le \theta - |x|$, therefore

$$u(x,t) \le n_\theta t^{-\frac{1}{q-1}} \widetilde{W}(\frac{\theta - |x|}{\sqrt{t}}), \tag{3.60}$$

which implies thanks to (3.30),

$$u(x,t) \le n_\theta C_\alpha t^{-\frac{1}{q-1}} \left(\frac{\theta - |x|}{\sqrt{t}}\right)^\alpha e^{-\frac{(\theta-|x|)^2}{4t}} \quad \text{for } (x,t) \in Q_\infty \text{ s.t. } |x| \le \theta - \sqrt{t}. \tag{3.61}$$

From this inequality estimate (3.34) follows by rescaling. □

3.2 The subcritical case

For a given $q > 1$ it not always possible to find a solution of (3.13) belonging to $C(\overline{Q}_\infty \setminus \{0\})$ vanishing on $\mathbb{R}^N \times \{0\} \setminus \{0\}$. Indeed Brezis and Friedman [18] proved the following results

Theorem 3.2 *Let*

$$q_c = 1 + \tfrac{2}{N}. \tag{3.62}$$

If $q \ge q_c$ any solution u of (3.13) belonging to $C(\overline{Q}_\infty \setminus \{0\})$ and vanishing on $\mathbb{R}^N \times \{0\} \setminus \{0\}$ is identically 0.
If $1 < q < q_c$, for any $c \in \mathbb{R}$ there exists a unique solution $u := u_{c\delta_0}$ of

$$\begin{array}{ll} \partial_t u - \Delta u + |u|^{q-1}u = 0 & \text{in } Q_\infty \\ u(.,0) = c\delta_0 & \text{in } \mathcal{D}'(\mathbb{R}^N) \end{array} \tag{3.63}$$

where δ_0 is the Dirac mass at 0. Furthermore if $\{\rho_n\}$ is a sequence of positive integrable functions which converges weakly to $c\delta_0$ in the sense of distributions in $\mathbb{R}^N$, then the sequence of functions $\{u_{\rho_n}\}$ which satisfy

$$\begin{array}{ll} \partial_t u - \Delta u + |u|^{q-1}u = 0 & \text{in } Q_\infty \\ u(.,0) = \rho_n & \text{in } \mathcal{D}'(\mathbb{R}^N) \end{array} \tag{3.64}$$

converges to $u_{c\delta_0}$ locally uniformly in Q_∞.

An important consequence of the previous result is the existence of *very singular solutions* which was first discovered by Brezis, Peletier and Terman in [19]. We give below an alternative proof of its construction and uniqueness.

Theorem 3.3 *Suppose $1 < q < q_c$. Then there exists a unique positive C^∞ function f defined on $[0,\infty)$ such that*

$$\begin{aligned} f'' + \left(\frac{N-1}{\eta} + \frac{\eta}{2}\right) f' + \frac{1}{q-1} f - f^p = 0 \quad \text{on } (0,\infty) \\ f'(0) = 0 \quad \text{and} \quad \lim_{\eta\to\infty} \eta^{\frac{2}{q-1}} f(\eta) = 0. \end{aligned} \tag{3.65}$$

Furthermore

$$f(\eta) = Ae^{-\frac{\eta^2}{4}} \eta^{\frac{2}{q-1}-N} \left(1 + \frac{2q}{q-1}\left(\frac{2}{q-1} - N\right)\eta^{-2} + o(\eta^{-2})\right) \quad \text{as } \eta \to \infty. \tag{3.66}$$

Proof. For any $\epsilon > 0$, u_c is bounded from above by the solution $u_{\infty,\epsilon}$ of (3.13) with initial data $u_{\infty,\epsilon}(x,0) = \infty \times \mathbf{1}_{B_\epsilon}(x)$ which is defined in the proof of Proposition 3.1-Step 1. When $c \to \infty$, u_c increases and converges to some solution u_∞ wich is a positive solution of (3.13) and is bounded from above by $u_{\infty,\epsilon}$. Because of uniqueness there holds $T_\ell[u_c] = u_{c\,\ell^{\frac{2}{q-1}-N}}$ where T_ℓ is defined in (3.40). Therefore

$$T_\ell[u_\infty] = u_\infty \quad \text{for any } \ell > 0.$$

Hence u_∞ is self-similar and radial because of uniqueness as u_c is, thus it endows the form

$$u_\infty(x,t) = t^{-\frac{2}{q-1}} f\left(\tfrac{|x|}{\sqrt{t}}\right), \tag{3.67}$$

and f satisfies the ODE (3.66). Because $u_\infty(x,t) \to 0$ for $x \neq 0$ when $t \to 0$, it implies that $\lim_{\eta\to\infty} \eta^{\frac{2}{q-1}} f(\eta) = 0$. The function f is a positive radial and bounded solution of

$$-\Delta_\eta f - \frac{1}{2}\eta.\nabla f - \frac{1}{q-1} f + f^p = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}.$$

Hence the singularity at $\eta = 0$ is removable. Thus f is C^∞ in $\mathbb{R}^N$ and $f'(0) = 0$. Similarly $T_\ell[u_{\infty,\epsilon}] = u_{\infty,\ell^{-1}\epsilon}$. Therefore $u_{\infty,\epsilon}$ decreases and converges when $\epsilon \to 0$ to the function $u_{\infty,0}$ which is a positive self-similar solution of (3.13), say

$$u_{\infty,0}(x,t) = t^{-\frac{2}{q-1}} \widetilde{f}\left(\tfrac{|x|}{\sqrt{t}}\right), \tag{3.68}$$

and $\widetilde{f}$ is a positive solution of (3.63). Since $u_\infty \le u_{\infty,0}$, one has $f \le \widetilde{f}$. Actually, f (resp. $\widetilde{f}$) is the minimal (resp. maximal) solution of (3.65). Estimate (3.66) is obtained by the classical method of matching asymptotic expansion.
For uniqueness, it follows from the fact that $0 < f(0) \le \widetilde{f}(0)$ combined to the expansion (3.66) that there exists $A > 1$ such that

$$f(\eta) \le \widetilde{f}(\eta) \le Af(\eta) \quad \text{for all } \eta \ge 0. \tag{3.69}$$

Actually, only the truncated expansion

$$Ae^{-\frac{\eta^2}{4}}\eta^{\frac{2}{q-1}-N}(1+o(1)) \quad \text{as } \eta\to\infty,$$

which is easily obtained as in Proposition 3.1-step 2 is needed. If $f\neq\widetilde{f}$, then $f<\widetilde{f}$ by the maximum principle. We set

$$W=f-\frac{1}{2A}(\widetilde{f}-f).$$

By convexity

$$\left(1+\frac{1}{2A}\right)f^p\leq\left(\left(1+\frac{1}{2A}\right)f-\frac{1}{2A}\widetilde{f}\right)^p+\frac{1}{2A}\widetilde{f}^p.$$

Hence W satisfies

$$-\Delta_\eta W-\frac{1}{2}\eta.\nabla W-\frac{1}{q-1}W+W^p\geq 0 \quad \text{in } \mathbb{R}^N.$$

Since $\widetilde{W}=\left(\frac{1}{2}+\frac{1}{2A}\right)f$ is smaller than W and satisfies

$$-\Delta_\eta W-\frac{1}{2}\eta.\nabla W-\frac{1}{q-1}W+W^p\leq 0 \quad \text{in } \mathbb{R}^N,$$

there exists a positive and radial function f^* satisfying

$$-\Delta_\eta f^*-\frac{1}{2}\eta.\nabla f^*-\frac{1}{q-1}f^*+f^{*p}=0 \quad \text{in } \mathbb{R}^N,$$

and such that $0<f^*<f$, which contradicts the minimality of f. □.

The following result is fundamental in the study of the singlar points of the initial trace of a solution u of (3.13) in the subcritical case.

Lemma 3.4 *Suppose $1<q<q_c$, u is a positive solution of (3.13) and $y\in\mathbb{R}^N$ is such that*

$$\limsup_{t\to 0}\int_{B_\epsilon(y)}u(x,t)dx=\infty \quad \textit{for all } \epsilon>0. \tag{3.70}$$

Then

$$u(x,t)\geq t^{-\frac{2}{q-1}}f\left(\frac{|x-y|}{\sqrt{t}}\right). \tag{3.71}$$

Proof. For any $\epsilon>0$, there exists a sequence $\{t_n\}$ decreasing to 0 such that

$$\int_{B_\epsilon(y)}u(x,t_n)dx=M(\epsilon,n)\to\infty \quad \text{as } n\to\infty.$$

Let $c>0$, then for $n\geq n_0=n_0(\epsilon,c)$, $M(\epsilon,n)>c$, hence there exist ϵ_n and $k_n>0$, both depending on c such that

$$\int_{B_{\epsilon_n}(y)}\min\{u(x,t_n),k_n\}dx=c.$$

Let u_n be the solution of (3.13) with initial data $u_n(x,0) = \min\{u(x,t_n), k_n\}\mathbf{1}_{B_{\epsilon_n}(y)}$. By the maximum principle

$$u(x,t+t_n) \geq u_n(x,t) \quad \text{for } (x,t) \in Q_\infty.$$

By Theorem 3.2 u_n converges to $u_{c\delta_y}$ when $n \to \infty$. Hence

$$u(x,t) \geq u_{c\delta_y}(x,t) \quad \text{for } (x,t) \in Q_\infty.$$

Since $c > 0$ is arbitrary, the claim follows from the fact that $\lim_{c\to\infty} u_{c\delta_y}(x,t) \to u_\infty(x-y,t)$ by Theorem 3.3. □

Proposition 3.5 *Suppose $1 < q < q_c$. Then for any closed set $\mathcal{S} \subset \mathbb{R}^N$ there exists a unique positive solution of (3.13) with initial trace $(\mathcal{S}, 0)$.*

Proof. Step 1- Construction of the minimal solution $\underline{u}_{\mathcal{S},0}$. Let $\{a_n\} \subset \mathcal{S}$ be a sequence of points dense in $\mathcal{S}$ and $\mu_n = n\sum_{j=1}^{n} \delta_{a_j}$. Then the sequence u_{μ_n} of solutions of (3.13) is increasing. By Lemma 3.4

$$u_{\mu_n} \geq \sup\{u_{n\delta_{a_j}} : 1 \leq j \leq n\}. \tag{3.72}$$

Furthermore, by (3.16), for any $y \in \mathcal{S}^c$ and $R = \operatorname{dist}(y, \mathcal{S})$ there holds

$$u_{\mu_n}(x,t) \leq Ct^{-\frac{1}{q-1}} \left(\frac{|x-y|}{\sqrt{t}}\right)^{\alpha} e^{-\frac{(R-[x-y])^2}{4t}} \qquad \text{for all } (x,t) \in B_R(y) \times (0,\infty), \tag{3.73}$$

where $C = C(\alpha, q) > 0$ and $\alpha > \frac{3-p}{p-1}$, and classicaly, $u_{\mu_n}(x,t) \leq \phi_\infty(t)$. Therefore the sequence $\{u_{\mu_n}\}$ increases and converges to some function denoted by $\underline{u}_{\mathcal{S},0}$ which is a positive solution of (3.13) and satisfies the same estimate from above (3.73) as u_{μ_n}. By 3.71) and 3.72) there holds

$$\underline{u}_{\mathcal{S},0}(x,t) \geq t^{-\frac{2}{q-1}} f\left(\frac{|x-a_n|}{\sqrt{t}}\right) \quad \text{for all } (x,t) \in B_R(y) \times (0,\infty) \text{ and } n \in \mathbb{N}. \tag{3.74}$$

Because $\{a_n\}$ is dense in $\mathcal{S}$, this last inequality implies that for any $y \in \mathcal{S}$ and $\epsilon > 0$,

$$\int_{B_\epsilon(y)} \underline{u}_{\mathcal{S},0}(x,t)dx \to \infty \quad \text{as } t \to 0. \tag{3.75}$$

Step 2- We claim that the function $\underline{u}_{\mathcal{S},0}$ is the minimal solution with initial trace $(\mathcal{S}, 0)$. Let u be such a solution. For $n \in \mathbb{N}^*$ we consider a double sequence of real numbers $\{\epsilon_{n,\ell}\}$ such that

$$\epsilon_{n,\ell} \leq \min\{|a_j - a_i| : 1 \leq i, j \leq n,\ i \neq j\} \quad \text{for all } \ell \in \mathbb{N}^*.$$

and since the set $\{a_j\}_{j\in\mathbb{N}}$ is dense in $\mathcal{S}$, for any ℓ there holds

$$\lim_{n\to\infty} \epsilon_{n,\ell} = 0.$$

We assume also

$$\lim_{\ell\to\infty}\epsilon_{n,\ell}=0\quad\text{for all } n\in\mathbb{N}^*.$$

For any $n\in\mathbb{N}^*$, $\ell\in\mathbb{N}^*$ and $j=1,...,n$, there holds

$$\int_{B_{\epsilon_{n,\ell}}(a_j)}u(x,t)dx\to\infty\quad\text{as } t\to 0.$$

Then for fixed $n\in\mathbb{N}^*$ there exists $t_{n,\ell}>0$ such that

$$\int_{B_{\epsilon_{n,\ell}}(a_j)}u(x,t_{n,\ell})dx\geq 2n\quad\text{for all } j=1,...,n.$$

Since $\epsilon_{n,\ell}\to 0$ when $\ell\to\infty$, it follows that $t_{n,\ell}\to 0$ under the same condition. Consequently there exist positive numbers $m_{j,n,\ell}$ for $j=1,...,n$ such that

$$\int_{B_{\epsilon_{n,\ell}}(a_j)}\min\{u(x,t_{n,\ell}),m_{j,n,\ell}\}dx=n.$$

We set

$$\rho_{n,\ell}(x)=\sum_{j=1}^{n}\min\{u(x,t_{n,\ell}),m_{j,n,\ell}\}\mathbf{1}_{B_{\epsilon_{n,\ell}}(a_j)}(x).\tag{3.76}$$

Then

$$\lim_{\ell\to\infty}\rho_{n,\ell}=\mu_n:=n\sum_{j=1}^{n}\delta_{a_j}\quad\text{in the sense of distributions in }\mathbb{R}^N.\tag{3.77}$$

Since $u(x,t_{n,\ell})\geq\rho_{n,\ell}(x)$ we have that $u(x,t+t_{n,\ell})\geq u_{n,\ell}(x,t)$ in Q_∞ where $u_{n,\ell}$ is the solution of (3.13) with initial data $\rho_{n,\ell}$. By Theorem 3.2 $u_{n,\ell}$ converges to u_{μ_n} defined in Step 1. Hence $u\geq u_{\mu_n}$. Letting $n\to\infty$ implies $u\geq\underline{u}_{\mathcal{S},0}$.

Step 3- Construction of the maximal solution $\overline{u}_{\mathcal{S},0}$. For $\epsilon>0$ we set

$$\mathcal{S}_\epsilon=\{x\in\mathbb{R}^N:\text{dist}\,(x,\mathcal{S})\leq\epsilon\}.$$

For $R>0$ we also define $\mathcal{S}_{\epsilon,R}=\mathcal{S}_\epsilon\cap\overline{B}_R$. Let $u=u_{\epsilon,R,n}$ be the solution of (3.13) with initial data $n\mathbf{1}_{\mathcal{S}_{\epsilon,R}}$. The mapping $(R,n)\mapsto u_{\epsilon,R,n}$ is increasing and bounded from above by ϕ_∞, hence there exists

$$u_\epsilon:=\lim_{\substack{n\to\infty\\R\to\infty}}u_{\epsilon,R,n}.$$

The mapping $R\mapsto u_{\epsilon,R}$ is increasing therefore there exists a limit u_ϵ when $R\to\infty$ which satisfies

$$\lim_{t\to 0}t^{\frac{1}{q-1}}u_\epsilon(x,t)=\left(\frac{1}{q-1}\right)^{\frac{1}{q-1}},\tag{3.78}$$

uniformly on any ball B_θ interior to $\mathcal{S}_\epsilon$ and

$$u_\epsilon(x,t)\leq Ct^{-\frac{1}{q-1}}\left(\frac{|x-y|}{\sqrt{t}}\right)^{\alpha}e^{-\frac{(R-[x-y[)^2}{4t}}\qquad\text{for all }(x,t)\in B_R(y)\times(0,\infty),\tag{3.79}$$

for all $y \in \mathcal{S}_\epsilon^c$ where $R = \operatorname{dist}(y, \mathcal{S}_\epsilon)$. This implies that the initial trace of u_ϵ is $(\mathcal{S}_\epsilon, 0)$. It is a consequence of the construction of u_ϵ as the limit of $u_{\epsilon,R,n}$ when $(n, R) \to (\infty, \infty)$ that the mapping $\epsilon \mapsto u_\epsilon$ is decreasing with limit $\overline{u}_{\mathcal{S}}$. Furthermore $\overline{u}_{\mathcal{S}} \geq \underline{u}_{\mathcal{S},0}$. Using (3.79) applied with $y \in \mathcal{S}^c$ and $R = \operatorname{dist}(y, \mathcal{S})$ we deduce that $\overline{u}_{\mathcal{S}}$ has initial trace $(\mathcal{S}, 0)$, and from now it is denoted $\overline{u}_{\mathcal{S},0}$.

Step 4- We claim that the function $\overline{u}_{\mathcal{S},0}$ is the maximal solution with initial trace $(\mathcal{S}, 0)$. Assume u is any positive solution of (3.13) with initial trace $(\mathcal{S}, 0)$ and for $R > 0$ let w_R be the solution of (3.23). For $\epsilon > 0$ the function $u_\epsilon + w_R$ is a supersolution of the equation in $B_R \times (0, \infty)$, thus for any $\delta > 0$ the function $(1+\delta)(u_\epsilon + w_R)$ is also a supersolution of the equation in $B_R \times (0, \infty)$. Since $u(x,t) \to 0$ when $t \to 0$ uniformly in $B_R \setminus \mathcal{S}_\epsilon$ we obtain that $u \leq (1+\delta)(u_\epsilon + w_R)$ in $B_R \times (0, \infty)$. Letting successively $\delta \to 0$, $R \to \infty$, here we use (3.24) and $\epsilon \to 0$, we infer that $u \leq \overline{u}_{\mathcal{S},0}$.

Step 5- We claim that there exists $K > 1$ such that $\overline{u}_{\mathcal{S},0} \leq K \underline{u}_{\mathcal{S},0}$. If $y \in \mathcal{S}$ there holds by (3.26) and (3.71) that

$$f(0) t^{-\frac{1}{q-1}} \leq \underline{u}_{\mathcal{S},0}(y,t) \leq \overline{u}_{\mathcal{S},0}(y,t) \leq \left(\frac{1}{q-1}\right)^{\frac{1}{q-1}} t^{-\frac{1}{q-1}}. \tag{3.80}$$

Thus the claim follows with $K = \left(\frac{1}{q-1}\right)^{\frac{1}{q-1}} (f(0))^{-1}$.

If $y \in \mathcal{S}^c$ let $z \in \mathcal{S}$ such that $|z - y| = \operatorname{dist}(y, \mathcal{S}) := d_y$. Then by (3.71) and (3.32),

$$f\left(\tfrac{d_y}{\sqrt{t}}\right) t^{-\frac{1}{q-1}} \leq \underline{u}_{\mathcal{S},0}(y,t) \leq \overline{u}_{\mathcal{S},0}(y,t) \leq C e^{-\frac{d_y^2}{4t}} t^{-\frac{1}{q-1}} \left(\frac{d_y}{\sqrt{t}}\right)^{\alpha}. \tag{3.81}$$

where $\alpha > \frac{3-q}{q-1}$. For $\sigma > 0$ we put

$$\mathcal{P}_\sigma = \left\{(y,t) : \frac{d_y}{\sqrt{t}} \leq \sigma\right\}.$$

If $(y,t) \in \mathcal{P}_\sigma$ there exists $K_\sigma > 0$ such that

$$\overline{u}_{\mathcal{S},0}(y,t) \leq K_\sigma \underline{u}_{\mathcal{S},0}(y,t). \tag{3.82}$$

Next we prove that for any $c > 1$ there exists σ_c such that for any $\sigma \geq \sigma_c$ there holds

$$\overline{u}_{\mathcal{S},0}(y,t) \leq K_\sigma \underline{u}_{\mathcal{S},0}(y, Ct) \quad \text{for all } (x,t) \in Q_\infty \setminus \mathcal{P}_\sigma. \tag{3.83}$$

It follows from expansion (3.66) that we have

$$e^{-\frac{d_y^2}{4t}} t^{-\frac{1}{q-1}} \left(\tfrac{d_y}{\sqrt{t}}\right)^{\alpha} \leq f\left(\tfrac{d_y}{\sqrt{ct}}\right) (ct)^{-\frac{1}{q-1}}, \tag{3.84}$$

which implies (3.83).

Next, for $\tau > 0$, let $u_{1,\tau}$ and $u_{2,\tau}$ be the solutions of (3.13) with respective initial data

$$\begin{aligned} u_{1,\tau} &= K_\sigma \mathbf{1}_{\mathcal{P}_\sigma}(x,\tau) \underline{u}_{\mathcal{S},0}(x,\tau) \\ u_{2,\tau} &= (1 - \mathbf{1}_{\mathcal{P}_\sigma}(x,\tau)) \underline{u}_{\mathcal{S},0}(x, C\tau). \end{aligned} \tag{3.85}$$

It is known and easy to prove that the solutions of (3.13) are uniquely determined by their initial data ([17]). The function $u_{1,\tau}+u_{2,\tau}$ is a supersolution and

$$\begin{aligned}(u_{1,\tau}+u_{2,\tau})(x,0) &= K_\sigma \mathbf{1}_{\mathcal{P}_\sigma}(x,\tau)\underline{u}_{\mathcal{S},0}(x,\tau)+(1-\mathbf{1}_{\mathcal{P}_\sigma}(x,\tau))\underline{u}_{\mathcal{S},0}(x,C\tau)\\ &\geq \overline{u}_{\mathcal{S},0}(x,\tau).\end{aligned}$$

Since $K_\sigma \underline{u}_{\mathcal{S},0}(x,\tau)\geq u_{1,\tau}(x,0)$ it follows that

$$K_\sigma \underline{u}_{\mathcal{S},0}(x,\tau+t)\geq u_{1,\tau}(x,t)\quad \text{for all } (x,t)\in Q_\infty.$$

Similarly $\underline{u}_{\mathcal{S},0}(x,C\tau)\geq u_{2,\tau}(x,0)$, therefore

$$\underline{u}_{\mathcal{S},0}(x,C\tau+t)\geq u_{2,\tau}(x,t)\quad \text{for all } (x,t)\in Q_\infty.$$

Combining these two inequalities we have that

$$\overline{u}_{\mathcal{S},0}(x,t+\tau)\leq u_{2,\tau}(x,t)+u_{1,\tau}(x,t)\leq K_\sigma \underline{u}_{\mathcal{S},0}(x,\tau+t)+\underline{u}_{\mathcal{S},0}(x,C\tau+t). \tag{3.86}$$

Letting $\tau\to 0$ yields

$$\overline{u}_{\mathcal{S},0}(x,t)\leq (1+K_\sigma)\underline{u}_{\mathcal{S},0}(x,t)\quad \text{for all } (x,t)\in Q_\infty. \tag{3.87}$$

Next we set $K=1+K_\sigma$ and

$$W=\underline{u}_{\mathcal{S},0}-\frac{1}{2K}\left(\overline{u}_{\mathcal{S},0}-\underline{u}_{\mathcal{S},0}\right).$$

If $\overline{u}_{\mathcal{S},0}\neq \underline{u}_{\mathcal{S},0}$, then $\overline{u}_{\mathcal{S},0}>\underline{u}_{\mathcal{S},0}$ and W is a supersolution of (3.13) by the same convexity argument used in the proof of Theorem 3.3. Note also that $\left(\frac{1}{2}+\frac{1}{2K}\right)\underline{u}_{\mathcal{S},0}$ is a subsolution of (3.13) smaller than W. Hence there exists a solution $\widetilde{u}$ of (3.13) satisfying

$$\left(\frac{1}{2}+\frac{1}{2K}\right)\underline{u}_{\mathcal{S},0}\leq \widetilde{u}\leq W. \tag{3.88}$$

This implies that the initial trace of $\widetilde{u}$ is also $(\mathcal{S},0)$. Since $W<\underline{u}_{\mathcal{S},0}$, we have a contradiction with the minimality of $\underline{u}_{\mathcal{S},0}$. □

The next result shows that the initial trace provides a one to one correspondence between the set of nonnegative solutions of (3.13) and the set of couples $(\mathcal{S},\mu)$ where $\mathcal{S}$ is a closed subset of $\mathbb{R}^N$ and μ a nonnegative Radon measure on $\mathcal{R}:=\mathbb{R}^N\setminus\mathcal{S}$.

Theorem 3.6 *Suppose $1<q<q_c$. Then for any closed set $\mathcal{S}\subset\mathbb{R}^N$ and any positive Radon measure μ on $\mathcal{R}:=\mathbb{R}^N\setminus\mathcal{S}$ there exists a unique positive solution of (3.13) with initial trace $(\mathcal{S},\mu)$.*

Proof. Step 1- Construction of the minimal solution. The principle is standard. We set $\mathcal{S}_\epsilon=\{x\in\mathbb{R}^N:\text{dist}\,(x,\mathcal{S})\leq\epsilon\}$. For $R>0$ we define

$$\mu_{\epsilon,R}=\mathbf{1}_{\mathcal{S}_\epsilon^c\cap B_R}\mu.$$

and denote by $u_{\mu_n+\mu_{\epsilon,R}}$ the solution of (3.13) with initial data $\mu_n+\mu_{\epsilon,R}$ where μ_n has been defined in the proof of Proposition 3.5-Step 1. Clearly $(\epsilon,R,n)\mapsto u_{\mu_n+\mu_{\epsilon,R}}$ is increasing in n and R and decreasing with respect to ϵ and we have

$$\max\{u_{\mu_n},u_{\mu_{\epsilon,R}}\}\le u_{\mu_n+\mu_{\epsilon,R}}\le u_{\mu_n}+u_{\mu_{\epsilon,R}}. \tag{3.89}$$

If we set

$$\underline{u}_{\mathcal{S},\mu}=\lim_{\epsilon\to0,n\to\infty,R\to\infty}u_{\mu_n+\mu_{\epsilon,R}}, \tag{3.90}$$

then

$$\max\{u_{(\mathcal{S},0)},u_\mu\}\le\underline{u}_{\mathcal{S},\mu}\le u_{(\mathcal{S},0)}+u_\mu. \tag{3.91}$$

Note that we have used Brezis'uniqueness result to obtain that u_μ is uniquely determined by μ. Inequality (3.91) implies that the initial trace of $\underline{u}_{\mathcal{S},\mu}$ is $(\mathcal{S},\mu)$.
Let u be any positive solution of (3.13) with initial trace $(\mathcal{S},\mu)$. If $k>R$ we denote by w_k the solution of (3.23) in B_k. Then $u+w_k$ is a supersolution of (3.13) in $Q_\infty^{B_k}$. Because $B_k=(B_k\cap\mathcal{S}_\epsilon)\cup(B_k\cap\mathcal{S}_\epsilon^c)$, there holds

$$\lim_{t\to0}\int_{B_k\cap\mathcal{S}_\epsilon}(u_{\mu_n+\mu_{\epsilon,R}}-u)_+dx=\lim_{t\to0}\int_{B_k\cap\mathcal{S}_\epsilon}(u_{\mu_n}-u)_+dx=0,$$

and

$$\lim_{t\to0}\int_{B_k\cap\mathcal{S}_\epsilon^c}(u_{\mu_n+\mu_{\epsilon,R}}-u)_+dx=\lim_{t\to0}\int_{B_k\cap\mathcal{S}_\epsilon^c}(u_{\mu_{\epsilon,R}}-u)_+dx=0.$$

Therefore the subsolution $(u_{\mu_n+\mu_{\epsilon,R}}-u-w_k)_+$ has zero initial data and it vanishes on $\partial_\ell(Q_\infty^{B_k}):=\partial B_k\times(0,\infty)$, thus it is identically zero. Therefore

$$u_{\mu_n+\mu_{\epsilon,R}}\le u-w_k\quad\text{in } Q_\infty^{B_k}. \tag{3.92}$$

Letting $k\to\infty$, $\epsilon\to0$ and $n\to\infty$ yields

$$\underline{u}_{\mathcal{S},\mu}\le u\quad\text{in } Q_\infty. \tag{3.93}$$

Step 2- Construction of the maximal solution. For $n,\epsilon>0$ we set $\mu_\epsilon=\mathbf{1}_{\mathcal{S}_\epsilon^c}\mu$ and $\mu_\epsilon^n=\mathbf{1}_{\mathcal{S}_\epsilon^c}\mu+n\mathbf{1}_{\mathcal{S}_\epsilon}dx$. Let $u_{n,\epsilon}=u_{\emptyset,\mu_\epsilon^n}$. When $n\to\infty$, the sequence $\{u_{n,\epsilon}\}$ increases and converges to a solution with initial trace $(\mathcal{S}_\epsilon,\mu_\epsilon)$ denoted by $u_{\mathcal{S}_\epsilon,\mu_\epsilon}$.
Let $k,\theta>0$, then $u_{\theta,\epsilon,k}:=(1+\theta)u_{\mathcal{S}_\epsilon,\mu_\epsilon}+w_k$ is a supersolution of (3.13) in Q_∞. By (3.46), the function $(u-u_{\theta,\epsilon,k})_+$ satisfies

$$\lim_{t\to0}\int_{\mathcal{S}_\epsilon\cap B_k}(u-u_{\theta,\epsilon,k})_+(x,t)dx=0.$$

In the cylinder $Q_\infty^{B_k}$ the function $u_{\theta,\epsilon,k}$ is a supersolution of (3.13) with initial data $(1+\theta)\mu$ and infinite boundary data. Hence it dominates u therein. Consequently

$$\lim_{t\to0}\int_{\mathcal{S}_\epsilon^c\cap B_k}(u-u_{\theta,\epsilon,k})_+(x,t)dx=0.$$

which yields

$$\lim_{t\to 0}\int_{B_k}(u-u_{\theta,\epsilon,k})_+(x,t)dx=0.$$

Because $u_{\theta,\epsilon,k}$ has infinite value on $\partial_\ell Q_\infty^{B_k}$ we deduce that $u_{\theta,\epsilon,k}\geq u$ in $Q_\infty^{B_k}$. Letting successively $k\to\infty$, $\delta\to 0$ and $\epsilon\to 0$ we obtain that

$$u_{\mathcal{S}_\epsilon,\mu_\epsilon}\geq u\quad \text{in } Q_\infty.$$

When $\epsilon\to 0$, $u_{\mathcal{S}_\epsilon,\mu_\epsilon}$ is decreasing and it converges to a solution $\overline{u}_{\mathcal{S},\mu}$ of (3.13) in Q_∞ with initial trace $(\mathcal{S},\mu)$ and is larger than any positive solution u with the same initial trace.

Step 3- End of the proof. With the notations of Steps 1 and 2, we set

$$Z_{\epsilon,\mu_\epsilon}=u_{\mathcal{S}_\epsilon,\mu_\epsilon}-u_{\mu_n+\mu_\epsilon}\quad \textit{and}\ \ Z_{\epsilon,0}=u_{\mathcal{S}_\epsilon,0}-u_{\mu_n}.$$

Then

$$\partial_t(Z_{\epsilon,\mu_\epsilon}-Z_{\epsilon,0})-\Delta(Z_{\epsilon,\mu_\epsilon}-Z_{\epsilon,0})+u^q_{\mathcal{S}_\epsilon,\mu_\epsilon}-u^q_{\mu_n+\mu_\epsilon}-(u^q_{\mathcal{S}_\epsilon,0}-u^q_{\mu_n})=0.$$

Now

$$u^q_{\mathcal{S}_\epsilon,\mu_\epsilon}-u^q_{\mu_n+\mu_\epsilon}=\frac{u^q_{\mathcal{S}_\epsilon,\mu_\epsilon}-u^q_{\mu_n+\mu_\epsilon}}{u_{\mathcal{S}_\epsilon,\mu_\epsilon}-u_{\mu_n+\mu_\epsilon}}Z_{\epsilon,\mu_\epsilon},$$

and

$$u^q_{\mathcal{S}_\epsilon,0}-u^q_{\mu_n}=\frac{u^q_{\mathcal{S}_\epsilon,0}-u^q_{\mu_n}}{u_{\mathcal{S}_\epsilon,0}-u_{\mu_n}}Z_{\epsilon,0}.$$

Since

$$u_{\mathcal{S}_\epsilon,\mu_\epsilon}\geq\max\left\{u_{\mu_n+\mu_\epsilon},u_{\mathcal{S}_\epsilon,0}\right\}\quad \textit{and}\ \ u_{\mu_n}\leq\min\left\{u_{\mathcal{S}_\epsilon,0},u_{\mu_n+\mu_\epsilon}\right\},$$

the convexity of the function $r\mapsto r^q$ on $\mathbb{R}_+$ implies that

$$\frac{u^q_{\mathcal{S}_\epsilon,\mu_\epsilon}-u^q_{\mu_n+\mu_\epsilon}}{u_{\mathcal{S}_\epsilon,\mu_\epsilon}-u_{\mu_n+\mu_\epsilon}}\geq\frac{u^q_{\mathcal{S}_\epsilon,0}-u^q_{\mu_n}}{u_{\mathcal{S}_\epsilon,0}-u_{\mu_n}}\geq 0.$$

Therefore

$$\partial_t(Z_{\epsilon,\mu_\epsilon}-Z_{\epsilon,0})-\Delta(Z_{\epsilon,\mu_\epsilon}-Z_{\epsilon,0})+\frac{u^q_{\mathcal{S}_\epsilon,0}-u^q_{\mu_n}}{u_{\mathcal{S}_\epsilon,0}-u_{\mu_n}}(Z_{\epsilon,\mu_\epsilon}-Z_{\epsilon,0})\leq 0.$$

Since

$$\lim_{t\to\infty}\int_{\mathbb{R}^N}(Z_{\epsilon,\mu_\epsilon}-Z_{\epsilon,0})_+(x,t)=0,$$

it follows by the maximum principle that $Z_{\epsilon,\mu_\epsilon}\leq Z_{\epsilon,0}$. Letting $n\to\infty$ and $\epsilon\to 0$ implies

$$\overline{u}_{\mathcal{S},\mu}-\underline{u}_{\mathcal{S},\mu}\leq\overline{u}_{\mathcal{S},0}-\underline{u}_{\mathcal{S},0}.$$

Uniqueness follows by Proposition 3.5. □

Extensions and comments. The initial trace of positive solutions of (3.13) in the cylinder Q^Ω_∞ can be defined similarly. If u is such a solution, it admits an initial trace

in Ω which consists in a closed subset $\mathcal{S} \subset \Omega$ and a Radon measure μ defined in $\Omega \setminus \mathcal{S}$. Furthermore the value of u on the parabolic boundary $\partial_\ell Q_\infty^\Omega$ has to be taken into account in order to prove results of existence and uniqueness. This theory is developed in [43] in the following framework:
(i) $\Omega \subset \mathbb{R}^N$ is a smooth domain.
(ii) $u\lfloor_{\partial_\ell Q_\infty^\Omega} = f \in L^1(\partial_\ell Q_\infty^\Omega)$.
(iii) μ is a positive Radon measure in Ω which is bounded in a neighbourhood of $\partial\Omega$.
(iv) $1 < q < 1 + \frac{2}{N}$.

Under these conditions and the subcriticality assumption, the initial trace provides a one to one correspondence between the sets of positive solutions u of

$$\begin{aligned} \partial_t u - \Delta u + u^q = 0 \qquad & \text{in } Q_\infty^\Omega \\ u = f \qquad & \text{on } \partial_\ell Q_\infty^\Omega, \end{aligned} \tag{3.94}$$

and the set of couples $(\mathcal{S}, \mu)$ where $\mathcal{S}$ is a closed subset of Ω and μ a nonnegative Radon measure μ in $\mathcal{R} := \Omega \setminus \mathcal{S}$ which is bounded in a neighbourhood of $\partial\Omega$.

3.3 The supercritical case

The next lemma shows that a very singular solution of (3.13) cannot exist if $q \geq q_c$.

Lemma 3.7 *Let $q \geq q_c$, then problem (3.65) admits no positive solution.*

Proof. Let f be such a solution. Since $f(\eta) = o(\eta^{-\frac{2}{q-1}})$ as $\eta \to \infty$, by matching asymptotic expansion we obtain that for any $\alpha > \frac{2}{q-1} - N$ there exists $c_\alpha > 0$ such that

$$f(\eta) \leq c_\alpha \eta^\alpha e^{-\frac{\eta^2}{4}} \quad \text{for all } \eta \geq 1. \tag{3.95}$$

Then it follows from the equation that

$$f'(\eta) \leq c'_\alpha \eta^{\alpha+1} e^{-\frac{\eta^2}{4}} \quad \text{for all } \eta \geq 1. \tag{3.96}$$

Set $\phi_1(\eta) = e^{-\frac{\eta^2}{4}}$, then

$$\left(\eta^{N-1} e^{\frac{\eta^2}{4}} \phi_1'(\eta)\right)' = \frac{N}{2} \eta^{N-1} e^{\frac{\eta^2}{4}} \phi_1(\eta).$$

We write 3.65) under the form

$$\left(\eta^{N-1} e^{\frac{\eta^2}{4}} \phi_1'(\eta)\right)' + \eta^{N-1} e^{\frac{\eta^2}{4}} \left(\frac{1}{q-1} f - f^q\right) = 0.$$

Multiplying by ϕ_1 and integrating on $(0, \infty)$, which is justified by (3.95) and (3.96), we infer that

$$\int_0^\infty \left(\left(\frac{1}{q-1} - \frac{N}{2}\right) f - f^q\right) \eta^{N-1} e^{\frac{\eta^2}{4}} \phi_1 d\eta = 0.$$

This leads to a contradiction because $\frac{1}{q-1} - \frac{N}{2} \leq 0$. □

The following result proved by Brezis and Friedman [18] points out the role of the exponent q_c the study of singularities of solutions of (3.13).

Theorem 3.8 *Let $q \geq q_c$, $\Omega \subset \mathbb{R}^N$ be a domain containing 0 and $u \in C(\overline{Q_T^\Omega} \setminus \{(0,0)\})$ be a solution of (3.13) in Q_T^Ω vanishing at $t = 0$ except at $x = 0$. Then u can be extended as a continuous function in $C(\overline{Q_T^\Omega})$.*

Proof. We can assume that $\overline{B}_R \subset \Omega$ and we first assume that u_+ vanishes on $\partial B_R \times (0,T)$. Then for any $\epsilon > 0$ the function u_+ is bounded from above by the function $u_{\epsilon,\infty}$ which satisfies (3.13) in Q_∞ and has initial trace $(\overline{B}_\epsilon, 0)$. By scaling

$$T_\ell[u_{\epsilon,\infty}] = u_{\ell^{-1}\epsilon,\infty} \quad \text{for all } \ell > 0,$$

and since $\epsilon \mapsto u_{\epsilon,\infty}$ is increasing, there exists $u_{0,\infty} = \lim_{\epsilon \to 0} u_{\epsilon,\infty}$. Furthermore $u_{0,\infty}$ is selfsimilar and $u_+ \leq u_{0,\infty}$. By Lemma 3.7 $u_{0,\infty} = 0$, thus $u_+ = 0$.
In the general case we denote by ϕ the boundary value of u on $\partial_\ell Q_T^{B_R}$, and by ψ the solution of

$$\begin{aligned} \partial_t \psi - \Delta \psi = 0 \qquad & \text{in } Q_T^{B_R} \\ \psi = \phi_+ \qquad & \text{on } \partial_\ell Q_T^{B_R} \\ \psi(.,0) = 0 \qquad & \text{in } B_R. \end{aligned}$$

Then $(u-\psi)_+$ is a subsolution of (3.13) in $Q_T^{B_R}$. By the previous argument, $(u-\psi)_+ = 0$. Hence u_+ is bounded from above. Similarly u_- is bounded, this implies that u remains bounded in $Q_T^{B_R}$. Standard regularity results imply that u vanishes on $B_R \times \{0\}$ and the claim follows. □

When $q \geq q_c$ there exists no solution of (3.13) in Q_∞ with a Dirac measure as an initial data. This phenomenon is general and the next result proved in [9] shows that if $\mu \in \mathfrak{M}(\mathbb{R}^N)$ the problem with measure initial data

$$\begin{aligned} \partial_t u - \Delta u + |u|^{q-1}u = 0 \qquad & \text{in } Q_\infty \\ u(.,0) = \mu \qquad & \text{in } \mathbb{R}^N, \end{aligned} \tag{3.97}$$

can be solved only if the measure μ is not too concentrated.

Definition 3.9 *Let $\mu \in \mathfrak{M}(\mathbb{R}^N)$. A function $u \in L^q_{loc}(\overline{Q}_\infty) \cap C(Q_\infty)$ is a weak solution of (3.97) if for all for all $\zeta \in C_c^2(\overline{Q}_\infty)$ there holds*

$$\int_0^\infty \int_{\mathbb{R}^N} \left(-u\left(\partial_t \zeta + \Delta \zeta\right) + |u|^{q-1}u\zeta\right) dxdt = \int_{\mathbb{R}^N} \zeta(x,0) d\mu(x). \tag{3.98}$$

A measure μ for which (3.97) is solvable is called q-admissible.

For the sake of completeness we introduce the Bessel capacity $cap_{\frac{2}{q},q'}$ (relative to $\mathbb{R}^N$) defined by

$$\begin{aligned} cap_{\frac{2}{q},q'}(E) = \inf \Bigg\{ \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\zeta(x) - \zeta(y)|^{q'}}{|x-y|^{N - \frac{2}{q-1}}} dxdy + \|\zeta\|^{q'}_{L^{q'}(\mathbb{R}^N)} \\ : \zeta \in C_c^\infty(\mathbb{R}^N), 0 \leq \zeta \leq 1, \zeta \geq \mathbf{1}_E \Bigg\}. \end{aligned} \tag{3.99}$$

The properties of Bessel capacities $cap_{s,p}$ which are associated to the Besov space $B^{s,p}$ and the Bessel kernel G_s are fully developed in [1]. It is noticeable that when $q > q_c$ the zero-capacity Borel sets of the Riesz and the Bessel capacities coincide since there holds [1, 5.6.1]

$$\dot{R}_{\frac{2}{q},q'}(E) \leq cap_{\frac{2}{q},q'}(E) \leq A\left(\dot{R}_{\frac{2}{q},q'}(E) + \left(\dot{R}_{\frac{2}{q},q'}(E)\right)^{\frac{N}{N-\frac{2}{q-1}}}\right). \tag{3.100}$$

Theorem 3.10 *A measure $\mu \in \mathfrak{M}(\mathbb{R}^N)$ is q-admissible if and only if*

$$cap_{\frac{2}{q},q'}(F) = 0 \Longrightarrow |\mu|(F) = 0, \tag{3.101}$$

for all Borel set $F \subset \mathbb{R}^N$.

Before proving this result we give an equivalence of norms estimate which will be used in the sequel.

Lemma 3.11 *Assume $q \geq q_c$. Then for any $T > 0$ there exists $c = c(n, q, T) > 0$ such that for any bounded measure $\mu \in B^{-\frac{2}{q},q}(\mathbb{R}^N)$ there holds*

$$c^{-1} \left\|\mu\right\|_{B^{-\frac{2}{q},q}(\mathbb{R}^N)} \leq \left\|\mathbb{H}[\mu]\right\|_{L^q(Q_T)} \leq c \left\|\mu\right\|_{B^{-\frac{2}{q},q}(\mathbb{R}^N)}. \tag{3.102}$$

Furthermore, if $q > q_c$, there holds

$$c^{-1} \left\|\mu\right\|_{B^{-\frac{2}{q},q}(\mathbb{R}^N)} \leq \left\|\mathbb{H}[\mu]\right\|_{L^q(Q_\infty)} \leq c\left(\left\|\mu\right\|_{B^{-\frac{2}{q},q}(\mathbb{R}^N)} + \left\|\mu\right\|_{\mathfrak{M}(\mathbb{R}^N)}\right). \tag{3.103}$$

Proof. If $\mu \in B^{-2/q,q}(\mathbb{R}^N)$, there exists a unique $\omega \in B^{2-2/q,q}(\mathbb{R}^N)$ such that $\mu = (I-\Delta)\omega$, and $\|\mu\|_{B^{-2/q,q}} \approx \|\omega\|_{B^{2-2/q,q}}$. Applying standard interpolation methods to the analytic semi-group $e^{-t(I-\Delta)} = e^{-t}e^{t\Delta}$ (see e.g. [11], [54]) we obtain,

$$\begin{aligned}\left(\int\int_{Q_\infty} \left|t^{1/q}(I-\Delta)\mathbb{H}[\omega]\right|^q dx\frac{e^{-qt}dt}{t}\right)^{1/q} &= \left(\int\int_{Q_\infty} \left|t^{1/q}\mathbb{H}[\mu]\right|^q dx\frac{e^{-qt}dt}{t}\right)^{1/q} \\ &\approx \|\omega\|_{B^{2-2/q,q}} \\ &\approx \|\mu\|_{B^{-2/q,q}}.\end{aligned} \tag{3.104}$$

Clearly

$$e^{-qT}\int\int_{Q_T} \left|t^{1/q}\mathbb{H}[\mu]\right|^q dx\frac{dt}{t} \leq \int\int_{Q_\infty} \left|t^{1/q}\mathbb{H}[\mu]\right|^q dx\frac{e^{-qt}dt}{t},$$

and

$$\begin{aligned}\int\int_{Q_\infty} \left|t^{1/q}\mathbb{H}[\mu]\right|^q dx\frac{e^{-qt}dt}{t} &= \sum_{n=0}^{\infty}\int\int_{Q_{T+n+1}\setminus Q_{T+n}} \left|t^{1/q}\mathbb{H}[\mu]\right|^q dx\frac{e^{-qt}dt}{t} \\ &= \sum_{n=0}^{\infty}\int\int_{Q_T} |\mathbb{H}[\mu](s+n)|^q\, e^{-q(s+n)}ds \\ &\leq \left(\sum_{n=0}^{\infty} e^{-qn}\right)\int\int_{Q_T} \left|t^{1/q}\mathbb{H}[\mu]\right|^q \frac{dt}{t},\end{aligned}$$

and (3.102) follows. Notice that $\||\mathbb{H}[\mu](.,t)|\|_{L^q}^q \le ct^{-N(q-1)/2}\|\mu\|_{\mathfrak{M}}^q$, therefore $\mathbb{H}[\mu]$ belongs to $L^q(Q_\infty)$ if $q > q_c$ (but not if $q = q_c$). If $q > q_c$ (equivalently $N(q-1)/2 > 1$),

$$\begin{aligned}\int\int_{Q_\infty}\left|t^{1/q}\mathbb{H}[\mu]\right|^q dx\frac{dt}{t} &= \sum_{n=0}^{\infty}\int\int_{Q_{T+n+1}\setminus Q_{T+n}}\left|t^{1/q}\mathbb{H}[\mu]\right|^q dx\frac{dt}{t}\\ &= \int\int_{Q_T}\left|t^{1/q}\mathbb{H}[\mu]\right|^q dx\frac{dt}{t} + \int\int_{Q_T}\sum_{n=1}^{\infty}|\mathbb{H}[\mu](s+n)|^q\,dxds\\ &\le \int\int_{Q_T}\left|t^{1/q}\mathbb{H}[\mu]\right|^q dx\frac{dt}{t} + C\left(\sum_{n=1}^{\infty}n^{-N(q-1)/2}\right)\|\mu\|_{\mathfrak{M}}^q.\end{aligned}$$

Thus we obtain (3.103). □

Proof of Theorem 3.10 We present here an abridged proof slightly different from the original one due to Baras and Pierre [9]. We first notice that if $1 < q < q_c$ any function in $B^{\frac{2}{q},q'}(\mathbb{R}^N)$ coincides with a continuous function. Hence only the empty set has zero $cap_{\frac{2}{q},q'}$-capacity. Therefore any measure in $\mathbb{R}^N$ is q-admissible. From now on we assume that $q \ge q_c$. Let F be a compact set with zero $cap_{\frac{2}{q},q'}$-capacity and $\{\zeta_n\}$ a sequence as in the previous theorem. We take $\widetilde{\phi}_n = \widetilde{\phi}\mathbb{H}[\zeta_n]$ for test functions, where now $\widetilde{\phi} \in C_c^\infty(\mathbb{R}^N \times [0,\infty))$ is nonnegative, takes value in $[0,1]$ and is equal to 1 in a neighbourhood of F. Then (3.116) is replaced by

$$\int_0^\infty\int_{\mathbb{R}^N}\left(u^q\widetilde{\phi}_n - u\left(\partial_t\widetilde{\phi}_n + \Delta\widetilde{\phi}_n\right)\right)dxdt = \int_{\mathbb{R}^N}(\widetilde{\phi}\mathbb{H}[\zeta_n])(.,0)d\mu(x) \ge \mu(F). \tag{3.105}$$

Since $\zeta_n \to 0$ and $0 \le \zeta_n \le 1$, $\widetilde{\phi}_n \to 0$ a.e. and $\partial_t\widetilde{\phi}_n + \Delta\widetilde{\phi}_n \to 0$ in $L^{q'}_{loc}(\mathbb{R}^N \times [0,\infty)))$. Thus the left-hand side of 3.105) converges to 0, which implies $\mu(F) = 0$.

Conversely, if μ is a nonnegative measure which vanishes on Borel sets with zero $cap_{\frac{2}{q},q'}$-capacity, it can be proved by the Hahn-Banach theorem (see [30]) that there exists an increasing sequence $\{\mu_n\}$ of nonnegative bounded measures belonging to $B^{-\frac{2}{q},q}(\mathbb{R}^N)$ which converges to μ. We first prove that a nonnegative bounded measure μ belonging to $B^{-\frac{2}{q},q}(\mathbb{R}^N)$ is q-admissible. By the previous lemma, $\mathbb{H}[\mu]$ belongs to $L^q_{loc}(\mathbb{R}^N \times [0,\infty))$. Next, for $k > 0$, we set $g_k(r) = \text{sign}(u)\min\{|u|^p, k^p\}$ and we denote by u_k the solution of

$$\begin{aligned}\partial_t u - \Delta u + g_k(u) &= 0 \qquad &&\text{in } Q_\infty\\ u(.,0 &= \mu \qquad &&\text{in } \mathbb{R}^N.\end{aligned} \tag{3.106}$$

For $0 < k < \ell$ one has $0 < u_\ell < u_k < \mathbb{H}[\mu]$. We denote by u the limit of the u_k. Since for any $\zeta \in C_c^\infty(\mathbb{R}^N \times [0,\infty))$ there holds

$$\int_0^\infty\int_{\mathbb{R}^N}\left(g_k(u_k)\zeta - \left(\partial_t\zeta + \Delta\zeta\right)u_k\right)dxdt = \int_{\mathbb{R}^N}\zeta(.,0)d\mu, \tag{3.107}$$

and $g_k(u_k) \le (\mathbb{H}[\mu])^p \in L^1_{loc}(\mathbb{R}^N \times [0,\infty))$ we deduce by the Lebesgue dominated convergence theorem that

$$\int_0^\infty \int_{\mathbb{R}^N} \left(u^p \zeta - (\partial_t \zeta + \Delta \zeta)\, u\right) dxdt = \int_{\mathbb{R}^N} \zeta(.,0) d\mu. \tag{3.108}$$

This prove that u is a nonnegative solution of (3.97) and μ is q-admissible. Finally if μ is a nonnegative measure satisfying (3.101), there exists an increasing sequence of q-admissible measures $\{\mu_n\}$ converging to μ. For each n, let u_n be the solution of (3.97) with initial data μ_n. Then the sequence $\{u_n\}$ is nondecreasing. For any nonnegative $\zeta \in C_c^2(\overline{Q}_\infty)$ there holds

$$\int_0^\infty \int_{\mathbb{R}^N} \left(-u_n\, (\partial_t \zeta + \Delta \zeta) + u_n^q \zeta\right) dxdt = \int_{\mathbb{R}^N} \zeta(x,0) d\mu_n(x). \tag{3.109}$$

Let u be the limit of the increasing sequence $\{u_n\}$. By the Beppo-Levi convergence theorem one has (3.98). This implies in particular that $u \in L^q_{loc}(\overline{Q}_\infty)$. If ζ is nolonger nonnegative then

$$\lim_{n\to\infty} \int_0^\infty \int_{\mathbb{R}^N} u_n^q \zeta dxdt,$$

by the dominated convergence theorem. Since

$$\lim_{n\to\infty} \int_0^\infty \int_{\mathbb{R}^N} u_n\, (\partial_t \zeta + \Delta \zeta)\, dxdt = \int_0^\infty \int_{\mathbb{R}^N} u\, (\partial_t \zeta + \Delta \zeta)\, dxdt$$

it follows that u is a weak solution of (3.97) and therefore μ is q-admissible.
For a general measure μ satisfying (3.101), we write the Jordan decomposition $\mu = \mu_+ - \mu_-$ and the proof follows. □

Baras and Pierre proved in [9] a general removability result which involves the Bessel capacities of a set.

Theorem 3.12 *Let $q > 1$ and $F \subset \mathbb{R}^N$ a closed set. A function $u \in C(\overline{Q}_\infty \setminus F)$ which is a solution of (3.13) in Q_∞ can be extended continuously to a function in $C(\overline{Q}_\infty)$ if and only if*

$$cap_{\frac{2}{q},q'}(F) = 0 \quad \textit{where} \quad q' = \tfrac{q}{q-1}. \tag{3.110}$$

Proof. We give an abridged proof in order to point out the duality method introduced in [9]. We recall that the heat potential of a measure ω is

$$\mathbb{H}[\omega](x,t) = \frac{1}{(4\pi t)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}} d\omega(y) = \int_{\mathbb{R}^N} H(x,y,t) d\omega(y). \tag{3.111}$$

Without loss of generality, we can assume that F is a compact subset of B_R. Since $cap_{\frac{2}{q},q'}(F) = 0$, there exists a sequence $\{\zeta_n\} \subset C_c^\infty(\mathbb{R}^N)$ such that $\zeta_n = 1$ on F, $0 < \zeta_n \le 1$ and

$$\|\zeta_n\|_{B^{\frac{2}{q},q'}} \to 0 \quad \text{as } n \to \infty.$$

We can assume that the support of ζ_n is included into B_{R+1}. Let $\theta \in C_c^\infty(\mathbb{R}^N)$ such that $0 \le \theta \le 1$, $\theta = 1$ in B_{R+1} and $\theta = 0$ in B_{R+2}^c. We set $\eta_n := \theta \mathbb{H}[1-\zeta_n]$ and take η_n^α for test function where $\alpha > 0$.

By a straightforward computation based on Hölder's inequality we get

$$\int_0^1 \int_{\mathbb{R}^N} u^q \eta_n^\alpha dxdt \le c_\alpha \int_0^1 \int_{\mathbb{R}^N} \left(\eta_n^{\alpha - q'} \left(|\partial_t \eta_n|^{q'} + |\Delta \eta_n|^{q'} \right) + |\nabla \theta|^{q'} |\nabla \eta_n|^{q'} \right. \\ \left. + \eta_n^{\alpha - 2q'} |\nabla \eta_n|^{2q'} \right) dxdt + \int_{\mathbb{R}^N} (u \eta_n^\alpha)(.,1) dx. \tag{3.112}$$

We fix $\alpha = 2q'$. Replacing η_n by its value, we obtain,

$$\eta_n^{\alpha - q'} |\partial_t \eta_n|^{q'} \le \theta^{q'} |\partial_t \mathbb{H}[\zeta_n]|^{q'} \le |\partial_t \mathbb{H}[\zeta_n]|^{q'}.$$

At this point we use the interpolation results associated to the analytic semigroup in $L^{q'}(\mathbb{R}^N)$ generated by $-\Delta$, see e.g. [54, Section 1.14.5]. We get

$$\int_0^1 \int_{\mathbb{R}^N} \eta_n^{\alpha - q'} |\partial_t \mathbb{H}[\zeta_n]|^{q'} dt \le c \left\| \eta_n \right\|^{q'}_{B^{\frac{2}{q}, q'}}. \tag{3.113}$$

Similarly

$$\int_0^1 \int_{\mathbb{R}^N} \eta_n^{\alpha - q'} |\Delta \eta_n|^{q'} dt \le c \left\| \eta_n \right\|^{q'}_{B^{\frac{2}{q}, q'}}. \tag{3.114}$$

For the last term, we use Triebel's result combined with Gagliardo-Nirenberg inequality

$$\int_0^1 \int_{\mathbb{R}^N} \eta_n^{\alpha - 2q'} |\nabla \eta_n|^{2q'} dt \le c \left\| \eta_n \right\|^{q'}_{L^\infty} \left\| \eta_n \right\|^{q'}_{B^{\frac{2}{q}, 2q'}} \le c \left\| \eta_n \right\|^{q'}_{B^{\frac{2}{q}, q'}}. \tag{3.115}$$

Letting $n \to \infty$ and using the fact that $\eta_n \to 0$ in $B^{\frac{2}{q}, q'}$, we infer that $u \in L^q(\mathbb{R}^N \times (0,1))$. In order to prove that u is a solution, we take $\phi_n = \phi \mathbb{H}[1-\zeta_n]$ for test function where $\phi \in C_c^\infty(\mathbb{R}^N \times [0,\infty))$. Then

$$\int_{\mathbb{R}^N} (u \phi_n)(.,0) dx = \int_0^\infty \int_{\mathbb{R}^N} \left(u^q \phi_n - u \left(\partial_t \phi_n + \Delta \phi_n \right) \right) dxdt. \tag{3.116}$$

By computation,

$$(\partial_t \phi_n + \Delta \phi_n) = \mathbb{H}[1-\zeta_n] \partial_t \phi - \phi \partial_t \mathbb{H}[\zeta_n] + \mathbb{H}[1-\zeta_n] \Delta \phi - \phi \Delta \mathbb{H}[\zeta_n] - 2 \nabla \phi . \nabla \mathbb{H}[\zeta_n].$$

When $n \to \infty$, we have that

$$\mathbb{H}[1-\zeta_n] \left(\partial_t \phi + \Delta \phi \right) \to \partial_t \phi + \Delta \phi$$

in $L^\infty_{loc}(\mathbb{R}^N \times [0,\infty))$, and

$$\phi \partial_t \mathbb{H}[\zeta_n] + \phi \Delta \mathbb{H}[\zeta_n] + 2 \nabla \phi . \nabla \mathbb{H}[\zeta_n] \to 0$$

in $L^p(\mathbb{R}^N \times [0,\infty))$. Hence, we infer that

$$\int_{\mathbb{R}^N} (u\phi)(.,0)dx = \int_0^\infty \int_{\mathbb{R}^N} \left(u^q\phi - u\left(\partial_t\phi + \Delta\phi\right)\right) dxdt \tag{3.117}$$

from (3.116).
The converse is a consequence of the fact that any compact set with positive $cap_{\frac{2}{q},q'}$-capacity is the support of a nonnegative measure (*the capacitary measure*) μ belonging to the space $B^{-\frac{2}{q},q}(\mathbb{R}^N)$ see [1, Theorem 2.2.7]. By Theorem 3.10 any nonnegative bounded measure belonging to $B^{-\frac{2}{q},q}(\mathbb{R}^N)$ is q-admissible. The result follows. □

In order to prove some analogue of Theorem 3.6 in the case $q \geq q_c$, there are conditions both on the measure μ which has to satisfy a non-concentration condition such as (3.101) and on the singular set $\mathcal{S}$ which cannot be locally removable. Furthermore the singular set $\mathcal{S}$ can locally be created because the measure μ may be locally unbounded.

Definition 3.13 *Assume $q > 1$. Let $\mathcal{S} \subset \mathbb{R}^N$ be a closed set and μ a nonnegative Radon measure on $\mathcal{R} := \mathbb{R}^N \setminus \mathcal{S}$ satisfying (3.101) for all Borel sets $F \subset \mathcal{S}$. We denote*

$$\partial_\mu\mathcal{S} = \{x \in \mathcal{S} :\ \mu(B_\epsilon(x) \cap \mathcal{S}^c) = \infty \text{ for all } \epsilon > 0\}, \tag{3.118}$$

and

$$\mathcal{S}^* = \left\{x \in \mathcal{S} :\ cap_{\frac{2}{q},q'}(B_\epsilon(x) \cap \mathcal{S}) > 0 \text{ for all } \epsilon > 0\right\}. \tag{3.119}$$

The next result is proved in [43].

Theorem 3.14 *Let $q \geq q_c$. There exists a maximal positive solution u of (3.13) in Q_∞ with initial trace $(\mathcal{S}, \mu)$ if and only if μ satisfies (3.101) or all Borel set $F \subset \mathcal{S}$ and*

$$\mathcal{S} = \partial_\mu\mathcal{S} \cup \mathcal{S}^*. \tag{3.120}$$

Proof. Step 1: Construction of $\overline{u}_\mu$. Let $\{K_n\}$ be an increasing sequence of compact subsets of $\mathcal{R} := \mathbb{R}^N \setminus \mathcal{S}$ such that $\cup_n K_n = \mathcal{R}$, $\mu_n = \mathbf{1}_{K_n}\mu$. Since $\mu_n \leq \mu$, it follows from Theorem 3.10 that there exists a unique solution u_n to (3.97) with initial data $\mu = \mu_n$. The sequence $\{u_n\}$ is increasing and it converges to some nonnegative solution $\overline{u}_\mu$ of (3.13) in Q_∞. By Proposition 3.1-2, $\overline{u}_\mu(.,t)$ converges to 0 when $t \to 0$ locally uniformly in the interior of $\mathcal{S}$. For any $y \in \mathcal{R}$ and $R > 0$ such that $\overline{B}_R(y) \subset \mathcal{R}$, $\overline{u}_\mu$ is bounded from above in $Q_\infty^{B_R(y)}$ by $w_R(y - .) + u_{\mathbf{1}_{B_R(y)}\mu}$ where $u_{\mathbf{1}_{B_R(y)}\mu}$ is the solution of (3.13) in $Q_\infty^{B_R(y)}$ with initial data $\mathbf{1}_{B_R(y)}\mu$ and vanishing on $\partial_\ell Q_\infty^{B_R(y)}$, and w_R is defined in (3.23). Since $w_R(y - .) + u_{\mathbf{1}_{B_R(y)}\mu}$ is bounded in $L^q(Q_T^{B_{R'}(y)})$ for any $T > 0$ and $R' < R$ it follows that $\overline{u}_\mu$ satisfies (3.98) for any $\zeta \in C_c^\infty(Q_\infty^{B_R(y)})$.
Step 2: Characterization of $\partial_\mu\mathcal{S}$. For any $x \in \partial_\mu\mathcal{S}$, $\epsilon > 0$ and $n \in \mathbb{N}$, $\overline{u}_\mu$ is bounded from below by the solution $u_{n,\epsilon}$ of (3.13) in Q_∞ with initial data $\mathbf{1}_{K_n \cap B_\epsilon(y)}\mu$. Furthermore for any nonnegative $\zeta \in C_c^\infty(B_\epsilon(y))$,

$$\liminf_{t\to 0} \int_{B_\epsilon(y)} u(x,t)\zeta(x)dx \geq \lim_{t\to 0} \int_{B_\epsilon(y)} u_{n,\epsilon}(x,t)\zeta(x)dx = \int_{K_n \cap B_\epsilon(y)} \zeta(x)d\mu(x). \tag{3.121}$$

We can take ζ such that $\zeta = 1$ on $B_{\epsilon'}(y)$ for some $0 < \epsilon' < \epsilon$. When $n \to \infty$, we have

$$\liminf_{n\to\infty} \int_{K_n \cap B_\epsilon(y)} \zeta(x) d\mu(x) = \int_{B_\epsilon(y)\cap \mathcal{S}^c} \zeta(x) d\mu(x) = \infty,$$

hence y belongs the singular set of the initial trace of $\overline{u}_\mu$ that we denote by $\mathcal{S}ing(\overline{u}_\mu)$. Therefore

$$\partial_\mu \mathcal{S} \subset \mathcal{S}ing(\overline{u}_\mu).$$

Conversely, if $y \notin \partial_\mu \mathcal{S}$ there exists $\delta > 0$ such that $\mu(B_\delta(y) \cap S^c) = m_{\delta,y} < \infty$. Then for any nonnegative $\zeta \in C_c^\infty(B_\delta(y))$, one has

$$\lim_{t\to 0} \int_{\mathbb{R}^N} \overline{u}_\mu(x,t)\zeta(x) dx = \int_{\mathbb{R}^N} \zeta(x) d\mu(x) < \infty.$$

This implies that $y \notin \mathcal{S}ing(\overline{u}_\mu)$. Thus

$$\partial_\mu \mathcal{S} = \mathcal{S}ing(\overline{u}_\mu). \tag{3.122}$$

Step 3: Construction of $\overline{u}_\mathcal{S}$. By thickening $\mathcal{S}$ into $\mathcal{S}_\epsilon = \{x \in \mathbb{R}^N : \text{dist}\,(x, \mathcal{S}) \le \epsilon\}$ we construct an increasing sequence of solutions $\{u_{\mathcal{S}_\epsilon}\}$ with initial trace $(\mathcal{S}_\epsilon, 0)$. When $\epsilon \downarrow 0$, $\{u_{\mathcal{S}_\epsilon}\}$ decreases and converges to some nonnegative solution $\overline{u}_\mathcal{S}$ of (3.13) in Q_∞. Let $y \in \mathcal{S}^*$. Then for any $\epsilon > 0$ the set $B_\epsilon(y) \cap \mathcal{R}$ has positive $cap_{\frac{2}{q},q'}$-capacity. Hence there exists a positive measure $\mu_{\epsilon,y}$ in the dual space $B^{-\frac{2}{q},q}(\mathbb{R}^N)$ with support in $B_\epsilon(y) \cap \mathcal{R}$. For $n \in \mathbb{N}_*$ we denote by $u_{n\mu_{\epsilon,y}}$ the solution of (3.13) in $Q_\infty^{B_\epsilon(y)}$ with initial data $n\mu_{\epsilon,y}$ and which vanishes on $\partial_\ell Q_\infty^{B_\epsilon(y)}$. Then $\overline{u}_\mathcal{S} \ge u_{n\mu_{\epsilon,y}}$ in $Q_\infty^{B_\epsilon(y)}$. Hence

$$\liminf_{t\to 0} \int_{B_\epsilon(y)} \overline{u}_\mathcal{S}(x,t) dx \ge n \int_{B_\epsilon(y)} d\mu_{\epsilon,y}.$$

Since n is arbitrary this implies that

$$\lim_{t\to 0} \int_{B_\epsilon(y)} \overline{u}_\mathcal{S}(x,t) dx = \infty, \tag{3.123}$$

hence y belongs the singular set of the initial trace of $\overline{u}_\mathcal{S}$ that we denote by $\mathcal{S}ing(\overline{u}_\mathcal{S})$. Thus

$$\mathcal{S}^* \subset \mathcal{S}ing(\overline{u}_\mathcal{S}).$$

Conversely, if $y \in \mathcal{S} \setminus \mathcal{S}^*$, there exists $\delta > 0$ such that $cap_{\frac{2}{q},q'}(B_\delta(y) \cap \mathcal{S}) = 0$. For $0 < \epsilon < \delta' < \delta$ we denote by $u_{1,\epsilon}$ (resp. $u_{2,\epsilon}$) the solution of (3.13) in Q_∞ with initial trace $\mathcal{S}_\epsilon \cap \overline{B}_{\delta'}(y)$ (resp $\mathcal{S}_\epsilon \cap B^c_{\delta'}(y)$). Then

$$u_{\mathcal{S}_\epsilon} \le u_{1,\epsilon} + u_{2,\epsilon}.$$

When $\epsilon \to 0$, $u_{1,\epsilon} \downarrow u_{1,0}$ (resp. $u_{2,\epsilon} \downarrow u_{2,0}$) where $u_{1,0}$ is a solution of (3.13) in Q_∞ with $\mathcal{S}ing(u_{1,0}) \subset \mathcal{S} \cap \overline{B}_{\delta'}(y)$ (resp $\mathcal{S}ing(u_{2,0}) \subset \mathcal{S} \cap B^c_{\delta'}(y)$) and with no regular part. By Theorem 3.8 $u_{1,0} = 0$. Since

$$\overline{u}_\mathcal{S} \le u_{1,0} + u_{2,0} = u_{2,0},$$

we have for any $0 < \delta'' < \delta'$,

$$\lim_{t\to 0}\int_{B_{\delta''}(y)} \overline{u}_{\mathcal{S}}(x,t)dx \le \lim_{t\to 0}\int_{B_{\delta''}(y)} u_{2,0}(x,t)dx = 0,$$

hence $y \notin \mathcal{S}ing(\overline{u}_{\mathcal{S}})$. We conclude that

$$\mathcal{S}^* = \mathcal{S}ing(\overline{u}_{\mathcal{S}}). \tag{3.124}$$

Step 4: Construction of a solution u with initial trace $(\mathcal{S},\mu)$. Since $\max\{\overline{u}_{\mathcal{S}}, \overline{u}_\mu\}$ is a subsolution of (3.13) and $\overline{u}_{\mathcal{S}} + \overline{u}_\mu$ a supersolution, and since $\max\{\overline{u}_{\mathcal{S}}, \overline{u}_\mu\} \le \overline{u}_{\mathcal{S}} + \overline{u}_\mu$, there exists a solution u such that

$$\max\{\overline{u}_{\mathcal{S}}, \overline{u}_\mu\} \le u \le \overline{u}_{\mathcal{S}} + \overline{u}_\mu. \tag{3.125}$$

Then

$$\mathcal{S}ing(u) = \mathcal{S}ing(\overline{u}_{\mathcal{S}}) \cup \mathcal{S}ing(\overline{u}_\mu) = \mathcal{S}^* \cup \partial\mu\mathcal{S}. \tag{3.126}$$

Therefore $\mathcal{S}ing(u) = \mathcal{S}$ if and only if (3.120) holds.
As in the proof of Theorem 3.6-Step 2, the fonction constructed above is the maximal solution of (3.13) with initial trace $(\mathcal{S},\mu)$. □

We end this section with a non uniqueness result which asserts that in the supercritical case there could exist many positive solutions of (3.13) with the same initial trace with a non-empty singular set. This was proved first by Le Gall [39] in the framework of the *Brownian Snake*, with $q = 2$ and $N \ge 3$.

Theorem 3.15 *If $q \ge q_c$, there exist infinitely many solutions with initial trace $(\mathbb{R}^N, 0)$.*

Proof. Let $\{a_n\}$ be a dense sequence in $\mathbb{R}^N$, $\{\epsilon_n\}$ a sequence of positive numbers such that the series $\sum_n \epsilon_n$ is convergent and $\{u_n\}$ the sequence of maximal solutions of (3.13) in Q_∞ with initial trace $(\overline{B}_{\epsilon_n}(a_n), 0)$. We have

$$u_n(x,t) = \widetilde{u}_n(|x - a_n|, t).$$

The function $\widetilde{u}_n$ is radial and radially decreasing for fixed t. Furthermore $t \mapsto \widetilde{u}_n(z,t)$ is decreasing if $z \in B_{\epsilon_n}(a_n)$. We set

$$\eta_n = \sup\{\widetilde{u}_n(x,t) : (x,t) \in \mathbb{R}^N \times [1,\infty)\} = \widetilde{u}_n(0,1).$$

Since $q \ge q_c$, $\widetilde{u}_n \to 0$ uniformly on $\mathbb{R}^N \times [\epsilon,\infty)$ when $n \to \infty$, for any $\epsilon > 0$. For any $E > 0$ we can choose the ϵ_n such that the following inequality holds

$$\sum_{n\ge 0} \eta_n \le E.$$

Since $|u_n(x,t)|^{q-1} \le \frac{1}{t(q-1)}$, it follows by the parabolic Harnack inequality that the series $\sum_{n\ge 0} u_n$ converges normally on any compact subset of $\mathbb{R}^N \times (0,\infty)$, and we denote by $\overline{U}$ its sum. Since $(a+b)^q \ge a^q + b^q$ for any $a, b \ge 0$, $\overline{U}$ is a supersolution of (3.13). We set

$$\underline{U} = \sup\{u_n : n \in \mathbb{N}\}.$$

Then $\underline{U}$ is a subsolution of (3.13) and it is smaller than $\overline{U}$. Therefore there exists a positive solution U of (3.13) in Q_∞ such that

$$\underline{U} \le U \le \overline{U}. \tag{3.127}$$

For any $y \in \mathbb{R}^N$ and $\epsilon > 0$ there exist infinitely many a_n such that $B_{\epsilon_n}(a_n) \subset B_\epsilon(y)$, for such an n

$$\lim_{t\to 0} \int_{B_{\epsilon_n}(a_n)} u_n(x,t)dx = \infty.$$

Hence $y \in \mathcal{S}ing(u)$. Since the sequence $\{\epsilon_n\}$ can be chosen such that $\sum_{n\ge 0} \eta_n \le E$, we obtain

$$0 < U(0,1) \le E. \tag{3.128}$$

This ends the proof. □

This result can be improved in the following way, ([43, Proposition 4.14]).

Theorem 3.16 *If $q > q_c$, for any $\epsilon > 0$ there exists a positive solution u of (3.13) in Q_∞ with initial trace $(\mathbb{R}^N, 0)$ and a Borel set $F \subset \mathbb{R}^N$ shuch that*

$$\int_{\mathbb{R}^N} \mathbf{1}_E dx < \epsilon,$$

and

$$\lim_{t\to 0} u(x,t) = 0 \qquad \text{for all } x \in \mathbb{R}^N \setminus E.$$

Starting from this result it appeared clear that the definition of the initial trace performed by an averaging of the function $u(.,t)$ in an Euclidean neighborhood of a point y is not suitable to distinguish between the different solutions of (3.13). The idea of using the fine topolgy associated to the $cap_{\frac{2}{q},q'}$-topology is due to S. Kuznetsov. It was first used in [29] in the framework of the study of the boundary trace of positive solutions of

$$-\Delta u + u^q = 0 \tag{3.129}$$

in a domain $\Omega \subset \mathbb{R}^N$. In [45, 46], a sharper definition, suitable for all the supercritical exponents in the semilinear elliptic problem (1.16), was introduced and developed. This is this method, adapted to the parabolic case in [32] that we present in the next section. It will apply to all the exponents $q \ge q_c$.

4 The capacitary representation

4.1 σ-moderate solutions

We denote by $\mathfrak{M}^b_+(\mathbb{R}^N)$ the set of nonnegative bounded Radon measures in $\mathbb{R}^N$. If μ is a q-admissible measure, we denote by u_μ the solution of (3.97).

Definition 4.1 *A positive solution u of (3.13) in Q_∞ is called σ-moderate if there exists an increasing sequence $\{\mu_n\} \subset B^{-\frac{2}{q},q'}(\mathbb{R}^N) \cap \mathfrak{M}^b_+(\mathbb{R}^N)$ such that the corresponding solution $u := u_{\mu_n}$ of (3.97) converges to u locally uniformly in Q_∞.*

If F is a closed subset of $\mathbb{R}^N$ we set

$$\underline{u}_F = \max\{u_\mu : \mu \in B^{-\frac{2}{q},q'}(\mathbb{R}^N) \cap \mathfrak{M}^b_+(\mathbb{R}^N), \mu(F^c) = 0\}. \tag{4.1}$$

4.1.1 Besov and Bessel capacitary potentials

The main goal of this section is to prove that $\underline{u}_F$ coincides with the maximal solution $\overline{u}_F$ of (3.13) in Q_∞ with initial trace $(F, 0)$. We introduce several tools linked to Bessel and Besov capacities relative to a domain $\Omega \subset \mathbb{R}^N$. If K is a compact subset of the domain $\Omega \subset \mathbb{R}^N$, we set

$$\mathcal{T}_\Omega(K) = \{\eta \in C^\infty_c(\Omega), 0 \le \eta \le 1, \eta = 1 \text{ on } K\}. \tag{4.2}$$

Definition 4.2 *Let $s \in (0,1)$, $p > 1$ such that $sp \le N$ and $\Omega \subset \mathbb{R}^N$ be a domain. The Besov capacity $\dot{R}^\Omega_{s,p}$ of a compact set $K \subset \Omega$ is*

$$\dot{R}^\Omega_{s,p}(K) = \inf\left\{\|\phi\|^p_{\dot{B}_{s,p}} : \phi \in \mathcal{T}_\Omega(K)\right\}, \tag{4.3}$$

where $\dot{B}_{s,p}$ is the Aronszajn-Slobodeckij norm defined by

$$\|\phi\|_{\dot{B}_{s,p}} = \left(\int_\Omega\int_\Omega \frac{|\phi(x)-\phi(y)|^p}{|x-y|^{N+sp}}dxdy\right)^{\frac{1}{p}}. \tag{4.4}$$

If $\Omega = \mathbb{R}^N$ the Besov capacity coincides with the Riesz capacity defined in (1.10).
The Bessel capacity $cap^\Omega_{s,p}$ relative to Ω is defined by

$$cap^\Omega_{s,p}(K) = \inf\left\{\|\phi\|^p_{B_{s,p}} : \phi \in \mathcal{T}_\Omega(K)\right\}, \tag{4.5}$$

and $cap^{\mathbb{R}^N}_{s,p} = cap_{s,p}$.

In the sequel we will see that the capacity $\dot{R}^\Omega_{\frac{2}{q},q'}(K)$ is more suitable for the computations in our problem than the Bessel capacity $cap_{\frac{2}{q},q'}$.

Definition 4.3 *Let $q > 1$. If $F \subset \mathbb{R}^N$ is a closed set we denote for $n \in \mathbb{N}$ and $(x,t) \in Q_\infty$*

$$F_n := F_n(x,t) = \{y \in F : d_n \le |x-y| \le d_{n+1}\} \quad \text{where } d_n = \sqrt{nt},$$

$$\Gamma_n = B_{d_{n+1}} \setminus B_{d_n} = \left\{y \in \mathbb{R}^N : d_n \le |y| \le d_{n+1}\right\}.$$

The Bessel-capacitary potential of F is the function W_F defined by

$$W_F(x,t) = \frac{1}{t^{\frac{N}{2}}}\sum_{n=0}^\infty d_{n+1}^{N-\frac{2}{q-1}} e^{-\frac{n}{4}} cap_{\frac{2}{q},q'}\left(\frac{F_n}{d_{n+1}}\right) \quad \text{for all } (x,t) \in Q_\infty. \tag{4.6}$$

Similarly, the Besov capacitary potential of F is $\widetilde{W}_F$ defined by

$$\widetilde{W}_F(x,t) = \frac{1}{t^{\frac{N}{2}}} \sum_{n=0}^{\infty} d_{n+1}^{N-\frac{2}{q-1}} e^{-\frac{n}{4}} \dot{R}^{\Gamma_n}_{\frac{2}{q},q'} \left(\frac{F_n}{d_{n+1}} \right) \quad \text{for all } (x,t) \in Q_\infty. \tag{4.7}$$

The Besov capacitary potential of F is invariant by the scaling T_ℓ in the sense that for any $\ell > 0$,

$$\ell^{\frac{1}{q-1}} \widetilde{W}_F(\sqrt{\ell}x, \ell t) := T_\ell[\widetilde{W}_F](x,t) = \widetilde{W}_{\frac{F}{\sqrt{\ell}}}(x,t) \quad \text{for all } (x,t) \in Q_\infty. \tag{4.8}$$

The Besov capacity is linked to the Bessel capacity through the following directional Poincaré inequality [43].

Lemma 4.4 *Let $b > a > 0$ and Ω be a domain in $\mathbb{R}^N$ such that $\Omega \subset H_{a,b} := \{x = (x_1, x') : a < x_1 < b\}$. If $s \in (0,1)$ and $p > 1$ verify $sp \leq N$, there exists $\lambda = \lambda(N, s, p, \frac{b}{a}) > 0$ such that*

$$\int \int_{\Omega \times \Omega} \frac{|\eta(x) - \eta(y)|^p}{|x-y|^{N+sp}} dxdy \geq \lambda (b-a)^{-sp} \int_\Omega |\eta(x)|^p dx \quad \text{for all } \eta \in C_c^\infty(\Omega). \tag{4.9}$$

It is noticeable that the above domain Ω is not necessarily bounded, in which case the standard Poincaré inequality is easy to prove, but it is only contained in a strip of finite thickness. The Aronszajn-Slobodeckij norm in $C_c^\infty(\Omega)$ is smaller than the standard $B^{s,p}$-norm associated to the Bessel potential $G_s := \mathcal{F}[(1+|\xi|^2)^{-\frac{s}{2}}]$ (see [1]) and defined by

$$\|\phi\|_{B^{s,p}} = \|\phi\|_{\dot{B}_{s,p}} + \|\phi\|_{L^p} \quad \text{for all } \phi \in C_c^\infty(\Omega)$$

However, thanks to Lemma 4.4 there holds

Lemma 4.5 *Let $b > a > 0$ and Ω be a domain in $\mathbb{R}^N$ such that $\Omega \subset \Gamma_{a,b} = B_b \setminus \overline{B}_a$. If $s \in (0,1)$ and $p > 1$ verify $sp \leq N$, there exists $\lambda = \lambda(N, s, p, \frac{b}{a}) > 0$ such that*

$$\|\phi\|_{\dot{B}^{s,p}} \leq \|\phi\|_{B^{s,p}} \leq (1 + C(b-a)^s) \|\phi\|_{\dot{B}^{s,p}} \quad \text{for all } \eta \in C_c^\infty(\Omega). \tag{4.10}$$

The following properties of Bessel capacities $cap^\Omega_{s,p}$ relative to Ω and Besov capacities relative to $\mathbb{R}^N$ are classical and easy to establish.

Lemma 4.6 *For any $\tau > 0$ and any Borel set $K \subset \Omega$ there holds*

$$\dot{R}^\Omega_{s,p}(K) = \tau^{N-sp} \dot{R}^{\tau^{-1}\Omega}_{s,p}(\tau^{-1}K). \tag{4.11}$$

If $b > a$ and $\Omega \subset B_b \setminus \overline{B}_a$ there exists $c = c(b-a, \frac{b}{a}, N, s, p) > 0$ such that

$$\frac{1}{c} cap^\Omega_{s,p}(K) \leq \dot{R}^\Omega_{s,p}(K) \leq c cap^\Omega_{s,p}(K). \tag{4.12}$$

Finally if $K \subset \Omega' \subset \overline{\Omega}' \subset \Omega$, there exists $c = c(\mathrm{dist}\,(\Omega', \Omega^c), N, s, p) > 0$ such that

$$\frac{1}{c} cap^\Omega_{s,p}(K) \leq cap^\Omega_{s,p}(K) \leq c cap_{s,p}(K). \tag{4.13}$$

4.1.2 Heat potential and Besov space

If Ω is a bounded domain in $\mathbb{R}^N$, $p > 1$ and $s \in (0,1)$, we extend any $\eta \in C_c^\infty(\Omega)$ by zero in Ω^c and set

$$\|\eta\|_{\widetilde{B}^{s,p}} = \left(\int\int_{Q_\infty} |t^{1-\frac{s}{2}} \partial_t \mathbb{H}[\eta]|^p dx \frac{dt}{t} \right)^{\frac{1}{p}}. \tag{4.14}$$

It is proved in [11] that the following equivalence of norms holds for the Besov space $B^{s,p}(\Omega)$,

$$C^{-1} \|\eta\|_{B^{s,p}} := C^{-1} \left(\|\eta\|_{L^p} + \|\eta\|_{\dot{B}^{s,p}} \right) \le \|\eta\|_{L^p} + \|\eta\|_{\widetilde{B}^{s,p}} \le C \left(\|\eta\|_{L^p} + \|\eta\|_{\dot{B}^{s,p}} \right) \tag{4.15}$$

for all $\eta \in C_c^\infty(\Omega)$ for some $C = C(s,p,N) > 0$. Actually it is easy to see by scaling that the two norms $\|.\|_{\dot{B}^{s,p}}$ and $\|.\|_{\widetilde{B}^{s,p}}$ are universally equivalent in the sense that there exists $C = C(s,p,N) > 0$ such that for any domain Ω and any $\eta \in C_c^\infty(\Omega)$,

$$C^{-1} \|\eta\|_{\dot{B}^{s,p}} \le \|\eta\|_{\widetilde{B}^{s,p}} \le C \|\eta\|_{\dot{B}^{s,p}}. \tag{4.16}$$

If K is a compact subset of Ω and $\eta \in \mathcal{T}_\Omega(K)$ we set

$$R[\eta] = |\partial_t \mathbb{H}[\eta]| + |\nabla \mathbb{H}[\eta]|^2. \tag{4.17}$$

Lemma 4.7 *There exists $C = C(N,q) > 0$ such that for every $\eta \in \mathcal{T}_\Omega(K)$ there holds*

$$\|\eta\|^{q'}_{\widetilde{B}^{\frac{2}{q},q'}} \le \|R[\eta]\|^{q'}_{L^{q'}} := \int\int_{Q_\infty} (R[\eta])^{q'} dxdt \le C \|\eta\|^{q'}_{\widetilde{B}^{\frac{2}{q},q'}}. \tag{4.18}$$

Proof. Using the Gagliardo-Nirenberg inequality in $\mathbb{R}^N$, an elementary elliptic estimate and the fact that $0 \le \mathbb{H}[\eta] \le 1$,we see that

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla \mathbb{H}[\eta](.,t)|^{2q'} dx &\le C \left\| D^2 \mathbb{H}[\eta](.,t) \right\|^{q'}_{L^{q'}} \|\mathbb{H}[\eta](.,t)\|^{q'}_{L^\infty} \\ &\le C \|\Delta \mathbb{H}[\eta](.,t)\|^{q'}_{L^{q'}}, \end{aligned}$$

for all $t > 0$. Since $\partial_t \mathbb{H}[\eta] = \Delta \mathbb{H}[\eta]$, (4.18) follows. □

4.2 Estimate from above

The main result that we prove in this section is the following upper estimate

Theorem 4.8 *Let $q \ge q_c$. There exists a positive constant $c = c(N,q)$ such that for any closed subset $F \subset \mathbb{R}^N$ any nonnegative function $u \in C^{2,1}(Q_\infty) \cap C(\overline{Q}_\infty \setminus F \times \{0\})$ verifying*

$$\begin{aligned} \partial_t u - \Delta u + u^p = 0 \qquad & \text{in } Q_\infty \\ \lim_{t \to 0} u(x,t) = 0 \qquad & \text{locally uniformly in } F^c, \end{aligned} \tag{4.19}$$

satisfies

$$u(x,t) \le C W_F(x,t) \quad \text{for all } (x,t) \in Q_\infty, \tag{4.20}$$

where W_F is the capacitary potential defined in (4.6).

We will first consider the case where $F = K$ is a compact set.

4.2.1 Global L^q estimates

Let $K \subset B_r \subset \overline{B}_r \subset B_{r+\rho}$ where $r, \rho > 0$ be a compact set. We set

$$\mathcal{T}_{r,\rho}(K) = \{\eta \in C_c^\infty(B_{r+\rho}),\ 0 \le \eta \le 1,\ \eta = 1 \text{ on } K\}.$$

If $\eta \in \mathcal{T}_{r,\rho}(K$ we set

$$\eta^* = 1 - \eta \quad \text{and} \quad \zeta = (\mathbb{H}[\eta^*])^{2q'}.$$

Lemma 4.9 *If u is a positive function satisfying (4.19), there exists $C = C(N,q) > 0$ such that for every $T > 0$ and every compact set $K \subset B_r$,*

$$\int\int_{Q_T} u^q \zeta dx\, dt + \int_{\mathbb{R}^N} (u\zeta)(x,T)dx \le C\|R[\eta]\|_{L^{q'}}^{q'} \qquad \forall \eta \in \mathcal{T}_{r,\rho}(K). \tag{4.21}$$

Proof. By assumption η^* vanishes in an open neighbourhood $\mathcal{N}_1$ of K, for any open set $\mathcal{N}_2$ such that $K \subset \mathcal{N}_2 \subset \overline{\mathcal{N}}_2 \subset \mathcal{N}_1$ there exists $C_{\mathcal{N}_2}, c_{\mathcal{N}_2} > 0$ such that

$$\mathbb{H}[\eta^*](x,t) \le C_{\mathcal{N}_2} e^{-\frac{c_{\mathcal{N}_2}}{t}} \qquad \text{for all } (x,t) \in Q_\infty^{\mathcal{N}_2}.$$

By Proposition 3.1-2, this implies

$$\lim_{t\to 0} \int_{\mathbb{R}^N} (u\zeta)(x,t)dt = 0.$$

Taking ζ as a test function, we obtain

$$\int\int_{Q_T} u^q \zeta dxdt + \int_{\mathbb{R}^N} (u\zeta)(x,T)dx = \int\int_{Q_T} (\partial_t \zeta + \Delta\zeta)\, u dxdt. \tag{4.22}$$

Since

$$\partial_t\zeta + \Delta\zeta = 2q' H[\eta^*]^{2q'-1} (\partial_t H[\eta] + \Delta\mathbb{H}[\eta]) + 2q'(2q'-1)\mathbb{H}[\eta*]^{2q'-2} \left|\nabla \mathbb{H}[\eta^]\right|^2,$$

we deduce

$$\left|\int\int_{Q_T} (\partial_t\zeta + \Delta\zeta)\, u dxdt\right| \le c(q) \left(\int\int_{Q_T} u^q \zeta dxdt\right)^{\frac{1}{q}} \left(\int\int_{Q_T} R[\eta]^{q'} dxdt\right)^{\frac{1}{q'}},$$

where $R[\eta]$ is defined in (4.17). The proof follows from Lemma 4.7. □

Proposition 4.10 *Under the assumptions of Lemma 4.9, let $r, \rho > 0$, $T \ge (r+\rho)^2$,*

$$\mathcal{E}_{r+\rho} = \{(x,t) \in Q_\infty : |x|^2 + t \le (r+\rho)^2\},$$

and $Q_{r+\rho,T} = Q_T \setminus \mathcal{E}_{r+\rho}$. Then there exists $C = C(N,q,T) > 0$ such that

$$\int\int_{Q_{r+\rho,T}} u^q dxdt + \int_{\mathbb{R}^N} u(x,T)dx \le C\, \|R[\eta]\|_{L^{q'}}^{q'} \quad \text{for all } \eta \in \mathcal{T}_{r,\rho}. \tag{4.23}$$

Proof. In view of the previous lemma we have to show that under the above assumptions on T and η, there exists some $C = C(N, q, T) > 0$ such that

$$\zeta = \mathbb{H}[\eta^*]^{2q'} \geq C.$$

Since, by assumption $K \subset B_r$, $\eta^* = 1$ outside $B_{r+\rho}$ and $0 \leq \eta^* \leq 1$, we have

$$\begin{aligned}\mathbb{H}[\eta^*](x,t) \geq \mathbb{H}[1 - \mathbf{1}_{B_{r+\rho}}](x,t) &= \frac{1}{(4\pi t)^{\frac{N}{2}}} \int_{|y|>r+\rho} e^{\frac{|x-y|^2}{4t}} dy \\ &= 1 - \frac{1}{(4\pi t)^{\frac{N}{2}}} \int_{|y|\leq r+\rho} e^{\frac{|x-y|^2}{4t}} dy.\end{aligned}$$

If $(x,t) \in Q_{r+\rho,T}$, we write $x = (r+\rho)\xi$, $y = (r+\rho)\upsilon$ and $t = (r+\rho)^2\tau$. Then $(\xi,\tau) \in Q_{1,\frac{T}{(r+\rho)^2}}$ and

$$\frac{1}{(4\pi t)^{\frac{N}{2}}} \int_{|y|\leq r+\rho} e^{-\frac{|x-y|^2}{4t}} dy = \frac{1}{(4\pi\tau)^{\frac{N}{2}}} \int_{|\upsilon|\leq 1} e^{-\frac{|\xi-\upsilon|^2}{4\tau}} d\upsilon.$$

It is therefore easy to verify that

$$\max\left\{ \frac{1}{(4\pi\tau)^{\frac{N}{2}}} \int_{|\upsilon|\leq 1} e^{-\frac{|\xi-\upsilon|^2}{4\tau}} d\upsilon : (\xi,\tau) \in Q_{1,\frac{T}{(r+\rho)^2}} \right\} = \ell, \tag{4.24}$$

and $\ell = \ell(N, \frac{T}{(r+\rho)^2}) \in (0,1)$. Actually ℓ is independent of $\frac{T}{(r+\rho)^2}$ Êif this quantity is larger than 1. Putting $C = (1-\ell)^{-1}$ we deduce (4.23). □

4.2.2 Pointwise upper estimates

In this section the assumptions of Lemma 4.9 are fulfilled.

Lemma 4.11 *There exists a constant $C = C(N,q) > 0$ such that, for any $\eta \in \mathcal{T}_{r,\rho}(K)$*

$$u(x,(r+\rho)^2) \leq C \frac{\|R[\eta]\|_{L^{q'}}^{q'}}{(r(r+\rho))^{\frac{N}{2}}} \qquad \textit{for all } x \in \mathbb{R}^N. \tag{4.25}$$

Proof. Integrating the equation

$$\int_s^T \int_{\mathbb{R}^N} u^q dx d\tau + \int_{\mathbb{R}} u(x,T)dx = \int_{\mathbb{R}} u(x,s) \qquad \text{for all } T\ s > 0, \tag{4.26}$$

and by Proposition 4.10 we have that

$$\int_{\mathbb{R}^N} u(x,s)dx \leq C \int\int_{Q_T} (R[\eta])^{q'} dxdt \qquad \text{for all } T > s > (r+\rho)^2. \tag{4.27}$$

Since

$$u(x,s+\tau) \leq \mathbb{H}[u(s,.)](x,\tau) \leq \frac{1}{(4\pi\tau)^{\frac{N}{2}}} \int_{\mathbb{R}^N} u(y,s)dy,$$

we obtain (4.25) from (4.26) and (4.27) with $s = (r+\rho)^2$ and $\tau = (r+2\rho)^2 - (r+\rho)^2 \approx r(r+\rho)$ if $\rho = o(r)$. □

In the next result we show an integral estimate of u on the lateral boundary of $Q_t^{B_r}$.

Lemma 4.12 *Let $\gamma \geq r + 2\rho$ and $c > 0$, and either $N = 1, 2$ and $0 \leq t \leq c\gamma^2$, or $N \geq 3$ and $t > 0$. Then, for any $\eta \in \mathcal{T}_{r,\rho}(K)$, there holds*

$$\int_0^t \int_{\partial B_\gamma} u(x,\tau) dS d\tau \leq C \left\| R[\eta] \right\|_{L^{q'}}^{q'}, \tag{4.28}$$

where $C > 0$ depends on N, q and c if $N = 1, 2$ or N and q if $N \geq 3$.

Proof. Assume first that $N = 1, 2$ and set $G^\gamma := B_\gamma^c \times (-\infty, 0)$ and $\partial_\ell G^\gamma := \partial B_\gamma^c \times (-\infty, 0)$. Let h_γ be the function

$$h_\gamma(x) = 1 - \frac{\gamma}{|x|},$$

and ψ_γ the solution of

$$\begin{array}{rl} \partial_t \psi_\gamma + \Delta \psi_\gamma = 0 & \text{in } G^\gamma \\ \psi_\gamma = 0 & \text{in } \partial_\ell G^\gamma \\ \psi_\gamma(.,0) = h_\gamma & \text{in } B_\gamma^c. \end{array} \tag{4.29}$$

Then the function $\widetilde{\psi}(x,\tau) = \psi_\gamma(\gamma x, \gamma^2 \tau)$ satisfies

$$\begin{array}{rl} \partial_t \widetilde{\psi} + \Delta \widetilde{\psi} = 0 & \text{in } G^1 \\ \widetilde{\psi} = 0 & \text{in } \partial_\ell G^1 \\ \widetilde{\psi}(.,0) = h_1 & \text{in } B_1^c. \end{array} \tag{4.30}$$

By the maximum principle $\widetilde{\psi} \leq 1$ and by Hopf lemma

$$-\frac{\partial \widetilde{\psi}}{\partial \mathbf{n}} \lfloor_{\partial B_1 \times [-c,0]} \geq \theta > 0, \tag{4.31}$$

where $\theta = \theta(N, c)$. Thus

$$-\frac{\partial \psi_\gamma}{\partial \mathbf{n}} \lfloor_{\partial B_\gamma \times [-\gamma^2, 0]} \geq \frac{\theta}{\gamma}. \tag{4.32}$$

Multiplying the equation by $\psi_\gamma(x, \tau - t) = \psi_\gamma^*(x,\tau)$ and integrating on $B_\gamma^c \times (0,t)$ implies

$$\begin{array}{rl} \displaystyle\int_0^t \int_{B_\gamma^c} u^q \psi_r^* dx d\tau + \int_{B_\gamma^c} (u h_\gamma)(x,t) dx - \int_0^t \int_{\partial B_\gamma} \frac{\partial u}{\partial \mathbf{n}} \psi_\gamma^* dS d\tau & \\ \displaystyle = -\int_0^t \int_{\partial B_\gamma} \frac{\partial \psi_\gamma^*}{\partial \mathbf{n}} u d\sigma d\tau. & \end{array} \tag{4.33}$$

Since $0 \leq \psi_\gamma^* \leq 1$, we derive (4.28) from (4.32) and Proposition 4.10 since $B_\gamma^c \times (0,t) \subset \mathcal{E}_\gamma^c$, first by taking $t = T = \gamma^2 \geq (r+2\rho)^2$, and then for any $t \leq \gamma^2$t If $N \geq 3$, we proceed as above except that we introduce a new function

$$h_\gamma(x) = 1 - \left(\frac{\gamma}{|x|} \right)^{N-2}.$$

This function is harmonic, thus the solution ψ_γ of (4.29) coincides with h_γ, and $\theta = N-2$ is independent of the length of the time interval. This ends the proof. □

The following estimates concerning solution of the heat equation are easy to obtain from the Gaussian integral representation and left to the reader.

Lemma 4.13 *I- Let M, $a > 0$ and $\eta \in L^\infty(\mathbb{R}^N)$ such that*

$$0 \le \eta(x) \le Me^{-a|x|^2} \qquad a.e.\ in\ \mathbb{R}^N. \tag{4.34}$$

Then, for any $t > 0$,

$$0 \le \mathbb{H}[\eta](x,t) \le \frac{M}{(4at+1)^{\frac{N}{2}}} e^{-\frac{a|x|^2}{4at+1}} \qquad for\ all\ x \in \mathbb{R}^N. \tag{4.35}$$

II- Let M, a, $b > 0$ and $\eta \in L^\infty(\mathbb{R}^N)$ such that

$$0 \le \eta(x) \le Me^{-a(|x|-b)_+^2} \qquad a.e.\ in\ \mathbb{R}^N. \tag{4.36}$$

Then, for any $t > 0$,

$$0 \le \mathbb{H}[\eta](x,t) \le \frac{Me^{-\frac{a(|x|-b)_+^2}{4at+1}}}{(4at+1)^{\frac{N}{2}}} \qquad for\ all\ x \in \mathbb{R}^N. \tag{4.37}$$

Lemma 4.14 *There exists a constant $C = C(N,q) > 0$ such that, for any $\eta \in \mathcal{T}_{r,\rho}(K)$, there holds*

$$u(x,(r+2\rho)^2) \le C \max\left\{ \frac{r+\rho}{(|x|-r-2\rho)^{N+1}}, \frac{|x|-r-2\rho}{(r+\rho)^{N+1}} \right\} e^{-\frac{(|x|-(r+2\rho))^2}{4(r+2\rho)^2}} \|R[\eta]\|_{L^{q'}}^{q'}, \tag{4.38}$$

for any $x \in \mathbb{R}^N \setminus B_{r+3\rho}$.

Proof. The heat kernel in $B_1^c \times (0,\infty)$ with Dirichlet data on $\partial B_1^c \times (0,\infty)$ satisfies

$$H^{B_1^c}(x',x';t',s') \le C(t'-s')^{-\frac{N}{2}-1}(|x|-1)e^{-\frac{|x'-y'|^2}{4(t'-s')}} \quad \text{for } t' > s'. \tag{4.39}$$

If we denote $x = (r+2\rho)x'$ and $t = (r+2\rho)^2 t'$ for $(x,t) \in B_{r+2\rho}^c \times (0,T)$, then

$$u(x,t) \le (|x|-r-2\rho) \int_0^t \int_{\partial B_{r+\rho}} \frac{e^{-\frac{|x-y|^2}{4(t-s)}}}{(t-s)^{\frac{N}{2}+1}} dS(y)ds. \tag{4.40}$$

The right-hand side term in (4.40) is bounded from above by

$$\max\left\{ \frac{C(|x|-r-2\rho)}{(t-s)^{1+\frac{N}{2}}} e^{-\frac{(|x|-r-2\rho)^2}{4(t-s)}} : s \in (0,t) \right\} \int_0^t \int_{\partial B_{r+2\rho}} u(y,s)d\sigma(y)ds. \tag{4.41}$$

We fix $t=(r+2\rho)^2$ and $|x|\geq r+3\rho$. Since

$$\max\left\{\frac{e^{-\frac{(|x|-r-2\rho)^2}{4s}}}{s^{1+\frac{N}{2}}}: s\in(0,(r+2\rho)^2)\right\}$$
$$=(|x|-r-2\rho)^{-2-N}\max\left\{\frac{e^{-\frac{1}{4\sigma}}}{\sigma^{\frac{N}{2}+1}}: 0<\sigma<\left(\frac{r+2\rho}{(|x|-r-2\rho)}\right)^2\right\},$$

a direct technical computation shows that

$$\max\left\{\frac{e^{-\frac{(|x|-r-2\rho)^2}{4s}}}{s^{1+\frac{N}{2}}}: s\in(0,(r+2\rho)^2)\right\}\leq C(N)\rho^{-2-N}e^{\left(\frac{|x|-r-2\rho}{2r+4\rho}\right)^2}. \tag{4.42}$$

Combining this estimate with (4.40), (4.41) and Lemma 4.12, one gets (4.38). □

Remark. Since there exists $C>0$ such that

$$(|x|-r-2\rho)e^{\left(\frac{|x|-r-2\rho}{2r+4\rho}\right)^2}\leq C\frac{(r+\rho)^2}{\rho}e^{-\left(\frac{|x|-r-3\rho}{2r+4\rho}\right)^2}\quad\text{for all } x\in B^c_{r+3\rho}, \tag{4.43}$$

the following variant of (4.38) holds for all $x\in B^c_{r+3\rho}$,

$$u(x,(r+2\rho)^2)\leq C\max\left\{\frac{(r+\rho)^3}{\rho(|x|-r-2\rho)^{N+1}},\frac{1}{\rho(r+\rho)^{N+1}}\right\}e^{-\left(\frac{|x|-r-3\rho}{2r+4\rho}\right)^2}\|R[\eta]\|^{q'}_{L^{q'}}. \tag{4.44}$$

Next, we give a sharp pointwise upper bound of $u(x,t)$ when t is bounded from below.

Lemma 4.15 *There exists a constant $C=C(N,q)>0$ such that for any $\eta\in\mathcal{T}_{r+\rho}(K)$ the following estimate holds,*

$$u(x,t)\leq\frac{C\widetilde{M}e^{-\frac{(|x|-r-3\rho)_+^2}{4t}}}{t^{\frac{N}{2}}}\|R[\eta]\|^{q'}_{L^{q'}}\quad\textit{for all } (x,t)\in\mathbb{R}^N\times[(r+\rho)^2,\infty), \tag{4.45}$$

where

$$\widetilde{M}=\widetilde{M}(x,r,\rho)=\begin{cases}\left(1+\frac{r}{\rho}\right)^{\frac{N}{2}} & \textit{if } |x|<r+3\rho\\ \frac{(r+\rho)^{N+3}}{\rho(|x|-r-2\rho)^{N+2}} & \textit{if } r+3\rho\leq|x|\leq c_N^*(r+2\rho)\\ 1+\frac{r}{\rho} & \textit{if } |x|\geq c_N^*(r+2\rho),\end{cases} \tag{4.46}$$

with $c_N^*=1+\sqrt{4+2N}$.

Proof. By the maximum principle

$$u(x,t)\leq\mathbb{H}[u(.,(r+2\rho)^2](x,t-(r+2\rho)^2)\quad\text{for any } t\geq(r+2\rho)^2.$$

By Lemma 4.11 and (4.44),

$$u(x,(r+2\rho)^2) \le C\widetilde{M} e^{-\frac{(|x|-r-3\rho)^2}{4(r+2\rho)^2}} \|R[\eta]\|_{L^{q'}}^{q'},$$

where

$$M^\dagger = M^\dagger(x,r,\rho) = \begin{cases} (r(r+\rho)^{-\frac{N}{2}} & \text{if } |x| < r+3\rho \\ \frac{(r+\rho)^{N+3}}{\rho(|x|-r-2\rho)^{N+2}} & \text{if } r+3\rho \le |x| \le c_N^*(r+2\rho) \\ 1+\frac{r}{\rho} & \text{if } |x| \ge c_N^*(r+2\rho). \end{cases}$$

Applying Lemma 4.13 with $a=(2r+4\rho)^{-2}$, $b=r+3\rho$ and t replaced by $t-(r+2r)^2$ implies

$$u(x,t) \le C\frac{M^\dagger(r+2\rho)^N}{t^{\frac{N}{2}}} e^{-\frac{(|x|-r-3\rho)^2}{4t}} \|R[\eta]\|_{L^{q'}}^{q'} \tag{4.47}$$

for $|x| \ge r+3\rho$ and $t \ge (r+2r)^2$, which implies (4.45). □

Finally we obtain an upper bound of $u(x,t)$ when t is not bounded from below.

Lemma 4.16 *There exists a constant $C=C(N,q)>0$ such that for any $\eta \in \mathcal{T}_{r+\rho}(K)$ the following estimate holds when $0<t\le(r+2\rho)^2$,*

$$u(x,t) \le C(r+\rho)\max\left\{\frac{1}{(|x|-r-2\rho)^{N+1}}, \frac{1}{\rho t^{\frac{N}{2}}}\right\} e^{-\frac{(|x|-r-3\rho)^2}{4t}} \|R[\eta]\|_{L^{q'}}^{q'}, \tag{4.48}$$

for any $(x,t) \in \left(\mathbb{R}^N \setminus B_{r+3\rho}\right) \times (0,(r+2\rho)^2]$.

Proof. From Lemma 4.12 we deduce by a simple modification of (4.38) that for any $|x| \ge r+2\rho$, there holds

$$u(x,t) \le C(|x|-r-2\rho)(r+2\rho)\max\left\{\frac{e^{-\frac{(|x|-r-2\rho)^2}{4s}}}{s^{1+\frac{N}{2}}} : 0<s\le t\right\} \|R[\eta]\|_{L^{q'}}^{q'}. \tag{4.49}$$

Next,

$$\max\left\{\frac{e^{-\frac{(|x|-r-2\rho)^2}{4s}}}{s^{1+\frac{N}{2}}} : 0<s\le t\right\} = \begin{cases} (2N+4)^{1+\frac{N}{2}}(|x|-r-2\rho)^{-N-2}e^{-\frac{N+2}{2}} & \text{if } 0<|x|\le r+2\rho+\sqrt{2t(N+2)} \\ \frac{e^{-\frac{(|x|-r-2\rho)^2}{4t}}}{t^{1+\frac{N}{2}}} & \text{if } |x| > r+2\rho+\sqrt{2t(N+2)}. \end{cases}$$

When $x \in B_{r+3\rho}^c$, we have that

$$(|x|-r-2\rho)e^{-\frac{(|x|-r-2\rho)^2}{4t}} \le e^{-\frac{(|x|-r-3\rho)^2}{4t}} \begin{cases} \rho e^{-\frac{\rho^2}{4t}} & \text{if } 2t<\rho^2 \\ \frac{2t}{\rho}e^{-1+\frac{\rho^2}{4t}} & \text{if } \rho^2 \le 2t \le 2(r+2\rho)^2. \end{cases}$$

However, since

$$\frac{\rho}{t}e^{-\frac{\rho^2}{4t}} \leq \frac{4}{\rho},$$

we derive

$$(|x|-r-2\rho)e^{-\frac{(|x|-r-2\rho)^2}{4t}} \leq \frac{Ct}{\rho}e^{-\frac{(|x|-r-3\rho)^2}{4t}},$$

and (4.48) follows. □

4.2.3 The upper Wiener test estimate

Definition 4.17 *We denote by δ_2 and δ_∞ the two parabolic distances*

$$\begin{array}{ll} (i) & \delta_2[(x,t),(y,s)] = \sqrt{(x-y)^2+|t-s|} \\ (ii) & \delta_\infty[(x,t),(y,s)] = \max\{|x-y|, \sqrt{|t-s|}\}. \end{array} \tag{4.50}$$

If $K \subset \mathbb{R}^N$ and $i = 2$ or ∞,

$$\delta_i[(x,t),K] = \inf\{\delta_i[(x,t),(y,0)] : y \in \hat{E}K\} = \begin{cases} \sqrt{\mathrm{dist}^{\,2}(x,K)+|t|} & \text{if } i=2 \\ \max\{\mathrm{dist}\,(x,K), \sqrt{t}\} & \text{if } i=\infty. \end{cases}$$

For $\beta > 0$ and $i = 2$ or ∞, we denote by $\mathcal{B}^i_\beta$ the parabolic ball with center $m = (x,t)$ and radius β in the metric δ_i.

If $K \subset \mathbb{R}^N$ is any compact we denote by $\overline{u}_K$ the maximal solution of (3.13) with initial trace $(K,0)$.

If $m = (x,t) \in Q_T$ we set $d_K = \mathrm{dist}\,(x,K)$, $D_K = \max\{|x-y| : y \in K\}$ and $\lambda = \sqrt{d_K^2+t} = \delta_2(m,K)$. We define the slicing of K by setting $d_n = d_n(K,t) := \sqrt{nt}$ $n \in \mathbb{N}_*$, $d_n^\pm = \left(\sqrt{nt} \pm \frac{\sqrt{t}}{\sqrt{n}}\right)_+$ and

$$T_n^* = \overline{B}_{d_{n+1}^+}(x) \setminus B_{d_n^-}(x)\,,\; T_n = \overline{B}_{d_{n+1}}(x) \setminus B_{d_n}(x) \quad \text{for all } \; n \in \mathbb{N},$$

thus $T_0^* = \overline{B}_{2\sqrt{t}}(x)$, $T_0 = \overline{B}_{\sqrt{t}}(x)$, and set

$$K_n := K_n(x,t) = K \cap T_n \quad \text{and} \quad \mathcal{Q}_n := \mathcal{Q}_n(x,t) = K \cap T_n.$$

The main result of this section is the following upper estimate

Theorem 4.18 *Assume $q \geq q_c$, then there exists a constant $C = C(N,q,T) > 0$ such that*

$$\overline{u}_K \leq \frac{C}{t^{\frac{N}{2}}} \sum_{n=0}^{a_j} d_{n+1}^{N-\frac{2}{q-1}} cap_{\frac{2}{q},q'}\left(\frac{K_n}{d_{n+1}}\right) \quad \textit{for all } (x,t) \in Q_T, \tag{4.51}$$

where a_j is the largest integer such that $K_j \neq \emptyset$.

We can assume that $x = 0$. Furthermore, in considering the scaling transformation T_ℓ with $\ell > 0$ we can assume $t = 1$. Thus the new compact singular set of the initial trace becomes $\frac{1}{\sqrt{\ell}}K$ that we still denote by K. For $n \in \mathbb{N}_*$ set $\delta_n = d_{n+1} - d_n$, then $\frac{1}{2\sqrt{n+1}} \leq \delta_n \leq \frac{1}{2\sqrt{n}}$. By convention $\delta_0 = 1$. It is possible to exhibit a collection Θ_n of points $a_{n,j}$ with center on the sphere $\Sigma_n = \{y \in \mathbb{R}^N : |y| = (d_{n+1} + d_n)/2\}$, such that

$$T_n \subset \bigcup_{a_{n,j} \in \Theta_n} B_{\delta_n}(a_{n,j}), \quad |a_{n,j} - a_{n,k}| \geq \delta_n \ \text{ and } \ \#\Theta_n \leq C n^{N-1},$$

for some constant $C = C(N)$. If $K_{n,j} = K_n \cap B_{\delta_n}(a_{n,j})$, there holds

$$K = \bigcup_{0 \leq n \leq a_K} \ \bigcup_{a_{n,j} \in \Theta_n} K_{n,j}.$$

The first intermediate step is based on the *quasi-additivity* property of capacities developed in [2].

Lemma 4.19 *Let $q \geq q_c$. There exists a constant $C = C(N, q)$ such that*

$$\sum_{a_{n,j} \in \Theta_n} \dot{R}^{B_{2\delta_n}(a_{n,j})}_{2/q,q'}(K_{n,j}) \leq C d_{n+1}^{N - \frac{2}{q-1}} cap_{\frac{2}{q},q'}\left(\frac{K_n}{d_{n+1}}\right) \qquad \forall n \in \mathbb{N}_*. \tag{4.52}$$

Proof. The following result is proved in [2, Th 3]: if the spheres $B_{\rho_j^\theta}(b_j)$, $\theta = 1 - \frac{2}{N(q-1)}$, are disjoint in $\mathbb{R}^N$ and G is a Borel (more generally an analytic) subset of $\bigcup_j B_{\rho_j}(b_j)$ where the ρ_j are positive numbers smaller than some $\rho^* > 0$, there holds

$$cap_{\frac{2}{q},q'}(G) \leq \sum_j cap_{\frac{2}{q},q'}(G \cap B_{\rho_j}(b_j)) \leq A cap_{\frac{2}{q},q'}(G), \tag{4.53}$$

for some A depending on N, q and ρ^*. This property is called *quasi-additivity*. We define for $n \in \mathbb{N}_*$,

$$\widetilde{T}_n = d_{n+1} T_n, \quad \widetilde{K}_n = d_{n+1} K_n \quad \text{and} \quad \widetilde{\mathcal{Q}}_n = d_{n+1} \mathcal{Q}_n.$$

Since $K_{n,j} \subset B_{\delta_n}(a_{n,j})$, it follows that

$$\widetilde{K}_{n,j} := d_{n+1} K_{n,j} \subset B_{d_{n+1}\delta_n}(\widetilde{a}_{n,j}).$$

Note that by Lemma 4.6

$$\begin{aligned} \dot{R}^{B_{2\delta_n}(a_{n,j})}_{2/q,q'}(K_{n,j}) &= d_{n+1}^{\frac{2}{q-1}-N} \dot{R}^{B_{2\delta_n d_{n+1}}(d_{n+1}a_{n,j})}_{2/q,q'}(\widetilde{K}_{n,j}) \\ &\approx d_{n+1}^{\frac{2}{q-1}-N} cap^{B_{2\delta_n d_{n+1}}(d_{n+1}a_{n,j})}_{2/q,q'}(\widetilde{K}_{n,j}) \\ &\approx d_{n+1}^{\frac{2}{q-1}-N} cap_{\frac{2}{q},q'}(\widetilde{K}_{n,j}), \end{aligned} \tag{4.54}$$

where $\widetilde{K}_{n,j} = d_{n+1} K_{n,j}$. For a fixed $n > 0$ and each repartition Λ of points $\widetilde{a}_{n,j} = d_{n+1}\, a_{n,j}$ such that the balls $B_{2^\theta}(\widetilde{a}_{n,j})$ are disjoint, the quasi-additivity property holds: if we set

$$K_{n,\Lambda} = \bigcup_{a_{n,j}\in\Lambda} K_{n,j}\ , \quad \widetilde{K}_{n,\Lambda} = d_{n+1}\, K_{n,\Lambda} = \bigcup_{a_{n,j}\in\Lambda} \widetilde{K}_{n,j} \quad \text{and} \quad \widetilde{K}_n = d_{n+1}\, K_n,$$

then

$$\sum_{a_{n,j}\in\Lambda} cap_{\frac{2}{q},q'}(\widetilde{K}_{n,j}) \approx cap_{\frac{2}{q},q'}(\widetilde{K}_{n,\Lambda}). \tag{4.55}$$

The maximal cardinal of any such repartition Λ is of the order of Cn^{N-1} for some positive constant $C = C(N)$, therefore the number of repartitions needed for a full covering of the set $\widetilde{T}_n$ is of finite order depending only on the dimension. Because $\widetilde{K}_n$ is the union of the $\widetilde{K}_{n,\Lambda}$,

$$\sum_{a_{n,j}\in\Theta_n} cap_{\frac{2}{q},q'}(\widetilde{K}_{n,j}) = \sum_{\Lambda}\sum_{a_{n,j}\in\Lambda} cap_{\frac{2}{q},q'}(\widetilde{K}_{n,j}) \approx cap_{\frac{2}{q},q'}(\widetilde{K}_n). \tag{4.56}$$

By Lemma 4.6,

$$cap_{\frac{2}{q},q'}(\widetilde{K}_n) \le cap^{B_{2d_{n+1}}}_{2/q,q'}(\widetilde{K}_n) \approx d_{n+1}^{N-\frac{1}{q-1}} cap^{B_2}_{2/q,q'}\left(\frac{K_n}{d_{n+1}}\right) \approx d_{n+1}^{N-\frac{1}{q-1}} cap_{\frac{2}{q},q'}\left(\frac{K_n}{d_{n+1}}\right),$$

we obtain (4.52) by combining this last inequality with (4.54) and (4.56). □

Proof of Theorem 4.18. Step 1. We first notice that

$$\overline{u}_K \le \sum_{0\le n\le a_K}\ \sum_{a_{n,j}\in\Theta_n} \overline{u}_{K_{n,j}}. \tag{4.57}$$

Actually, since $K = \bigcup_n \bigcup_{a_{n,j}} K_{n,j}$, there holds $\overline{K_{\epsilon'}} \subset \bigcup_n \bigcup_{a_{n,j}} K_{n,j\,\epsilon}$ for any $0 < \epsilon' < \epsilon$. Because a finite sum of positive solutions of (3.13) is a super solution,

$$\overline{u}_{K_{\epsilon'}} \le \sum_{0\le n\le a_K}\ \sum_{a_{n,j}\in\Theta_n} \overline{u}_{K_{n,j\,\epsilon}}. \tag{4.58}$$

Letting successively ϵ' and ϵ go to 0 implies (4.57).

Step 2. Let $n \in \mathbb{N}$. Since $K_{n,j} \subset B_{\delta_n}(a_{n,j})$ and $|x - a_{n,j}| = (d_n + d_{n+1})/2$, we can apply the previous lemmas with $r = \delta_n$ and $\rho = r$. For $n \ge n_N$, there holds $t = 1 \ge (r + 2\rho)^2 = 9/(n+1)$ and $|x - a_{n,j}| = (\sqrt{n+1} - \sqrt{n})/2 \ge (2 + C_N)(3/\sqrt{n+1})$ (notice that $n_N \ge 8$). Thus

$$u_{K_{n,j}}(0,1) \le Ce^{\left(\sqrt{n}-3/\sqrt{n+1}\right)^2/4}\, R^{B_{2\delta_n}(a_{n,j})}_{2/q,q'}(K_{n,j}) \le Ce^{3/2}e^{-\frac{n}{4}} R^{B_{2\delta_n}(a_{n,j})}_{2/q,q'}(K_{n,j}). \tag{4.59}$$

Using Lemma 4.19 we obtain, with $d_n = d_n(1) = \sqrt{n+1}$,

$$\sum_{n=n_N}^{a_K}\ \sum_{a_{n,j}\in\Theta_n} u_{K_{n,j}}(0,1) \le C \sum_{n=n_N}^{a_K} d_{n+1}^{N-\frac{2}{q-1}} e^{-\frac{n}{4}} cap_{\frac{2}{q},q'}\left(\frac{K_n}{d_{n+1}}\right). \tag{4.60}$$

Finally, we apply Lemma 4.11 if $1 \le n < n_{_N}$ and get

$$\begin{aligned}\sum_{1}^{n_{_N}-1} \sum_{a_{n,j}\in\Theta_n} u_{K_{n,j}}(0,1) &\le C \sum_{1}^{n_{_N}-1} cap_{\frac{2}{q},q'}\left(\frac{K_n}{d_{n+1}}\right)\\ &\le C' \sum_{1}^{n_{_N}-1} d_{n+1}^{N-\frac{2}{q-1}} e^{-\frac{n}{4}} cap_{\frac{2}{q},q'}\left(\frac{K_n}{d_{n+1}}\right).\end{aligned} \tag{4.61}$$

For $n = 0$, we proceed similarly, in splitting K_1 in a finite number of sets $K_{1,i}$, depending only on the dimension, such that $\operatorname{diam} K_{1,i} < 1/3$. Combining (4.60) and (4.61), we derive

$$\overline{u}_K(0,1) \le C \sum_{n=0}^{a_K} d_{n+1}^{N-\frac{2}{q-1}} e^{-\frac{n}{4}} cap_{\frac{2}{q},q'}\left(\frac{K_n}{d_{n+1}}\right). \tag{4.62}$$

In order to derive the same result for any $t > 0$, we notice that

$$\overline{u}_K(y,t) = t^{-\frac{1}{q-1}} \overline{u}_{K/\sqrt{t}}(y/\sqrt{t},1).$$

Going back to the definition of $d_n = d_n(K,t) = \sqrt{nt} = d_n(K\sqrt{t},1)$, we derive from (4.62) and the fact that $a_{_{K,t}} = a_{_{K\sqrt{t},1}}$

$$\overline{u}_K(0,t) \le C t^{-\frac{N}{2}} \sum_{n=0}^{a_K} d_{n+1}^{N-\frac{2}{q-1}} e^{-\frac{n}{4}} cap_{\frac{2}{q},q'}\left(\frac{K_n}{d_{n+1}}\right), \tag{4.63}$$

with $d_n = d_n(t) = \sqrt{t(n+1)}$. This is (4.51) with $x = 0$, and a space translation leads to the final result. □

Proof of Theorem 4.8. Let $m > 0$ and $F_m = F \cap \overline{B}_m$. We denote by $U_{B_m^c}$ the maximal solution of (3.13) in Q_∞ the initial trace of which vanishes on B_m. It is straightforward by scaling to verify that such a solution is actually the unique positive solution of (3.13) which satisfies

$$\lim_{t\to 0} u(x,t) = \infty$$

uniformly on $B_{m'}^c$, for any $m' > m$. Furthermore

$$\lim_{m\to\infty} U_{B_m^c}(y,t) = \lim_{m\to\infty} m^{-\frac{2}{q-1}} U_{B_1^c}(y/m, t/m^2) = 0,$$

uniformly on any compact subset of $\overline{Q}_\infty$. Since $\overline{u}_{F_m} + U_{B_m^c}$ is a super-solution, it is larger that $\overline{u}_F$ and therefore $\overline{u}_{F_m} \uparrow \overline{u}_F$. Because $W_{F_m}(x,t) \le W_F(x,t)$ and $\overline{u}_{F_m} \le C_1 W_{F_m}(x,t)$, the result follows. □

Remark. It is clear that Theorem 4.8 still holds if u is a positive subsolution of (3.13) satisfying the initial trace condition (4.19).

The Bessel capacitary potential admits an integral form. The next result is a variant of Theorem 4.8.

Theorem 4.20 *Assume $q \geq q_c$. Then there exists a positive constant $C_1^* = C_1^*(N, q, T)$ such for any closed subset $F \subset \mathbb{R}^N$ there holds for all $(x,t) \in Q_T$,*

$$\overline{u}_F(x,t) \leq \frac{C_1^*}{t^{1+\frac{N}{2}}} \int_{\sqrt{t}}^{\sqrt{t(a_t+2}} e^{-\frac{s^2}{4t}} s^{N-\frac{2}{q-1}} cap_{\frac{2}{q},q'}\left(\frac{1}{s}B \cap B_1(x)\right) sds, \tag{4.64}$$

where $a_t = \min\left\{n \in \mathbb{N} : F \subset B_{\sqrt{(n+1)t}}(x)\right\}$.

Proof. We use the inequality

$$cap_{\frac{2}{q},q'}\left(\frac{F_n}{d_{n+1}}\right) \leq cap_{\frac{2}{q},q'}\left(\frac{F}{d_{n+1}} \cap B_1\right),$$

and we set

$$\phi(s) = cap_{\frac{2}{q},q'}\left(\frac{F}{s} \cap B_1\right) \quad \text{for all } s > 0. \tag{4.65}$$

Step 1. By [1], [45], there exists $c = c(N,q) > 0$ such that

$$\frac{1}{c}\phi(\alpha s) \leq \phi(s) \leq c\phi(\beta s) \quad \text{for all } s > 0 \text{ and } \frac{1}{2} \leq \alpha \leq 1 \leq \beta \leq 2. \tag{4.66}$$

Actually, if $\beta \in [1,2]$,

$$\phi(\beta s) = cap_{\frac{2}{q},q'}\left(\frac{1}{\beta}\left(\frac{1}{s}F \cap B_\beta(x)\right)\right) \approx cap_{\frac{2}{q},q'}\left(\frac{1}{s}F \cap B_\beta(x)\right) \geq \frac{1}{c}\phi(s),$$

and if $\alpha \in [\frac{1}{2}, 1]$,

$$\phi(\alpha s) = cap_{\frac{2}{q},q'}\left(\frac{1}{\alpha}\left(\frac{1}{s}F \cap B_\alpha(x)\right)\right) \approx cap_{\frac{2}{q},q'}\left(\frac{1}{s}F \cap B_\alpha(x)\right) \leq c\phi(s).$$

Step 2. By (4.66)

$$cap_{\frac{2}{q},q'}\left(\frac{F}{d_{n+1}} \cap B_1(x)\right) \leq ccap_{\frac{2}{q},q'}\left(\frac{F}{s} \cap B_1(x)\right) \quad \text{for all } s \in [d_{n+1}, d_{n+2}],$$

and $n \leq a_t$. Then

$$\begin{aligned} c\int_{d_{n+1}}^{d_{n+2}} s^{N-\frac{2}{q-1}} e^{-\frac{s^2}{4t}} cap_{\frac{2}{q},q'}\left(\frac{F}{s} \cap B_1(x)\right) sds \\ \geq cap_{\frac{2}{q},q'}\left(\frac{F}{d_{n+1}} \cap B_1(x)\right) \int_{d_{n+1}}^{d_{n+2}} s^{N-\frac{2}{q-1}} e^{-\frac{s^2}{4t}} sds. \end{aligned}$$

Because $N - \frac{2}{q-1} \geq 0$ as $q \geq q_c$, we obtain

$$\int_{d_{n+1}}^{d_{n+2}} s^{N-\frac{2}{q-1}} e^{-\frac{s^2}{4t}} sds \geq e^{-\frac{n+2}{4}} d_{n+1}^{N-\frac{2}{q-1}} (d_{n+2} - d_{n+1}) \geq \frac{t d_{n+1}^{N-\frac{2}{q-1}} e^{-\frac{n}{2}}}{4e^2}, \tag{4.67}$$

which implies (4.64). □

4.3 Estimate from below

If μ is a bounded nonnegative q-admissible measure, we recall that u_μ is the solution of (3.97). The maximal σ-moderate solution of (3.13) with an initial trace vanishing outside a closed set $F \subset \mathbb{R}^N$ is denoted by $\underline{u}_F$ and defined by

$$\underline{u}_F = \sup\{u_\mu : \mu \in \mathfrak{M}^b_+, \text{ and } q\text{-admissible s.t. } \mu(F^c) = 0\}. \tag{4.68}$$

The main result of this section is

Theorem 4.21 *Let $q \geq q_c$ and $T > 0$. Then there exists a positive constant $C = C(N, q, T)$ such that for any closed set $F \subset \mathbb{R}^N$,*

$$\underline{u}_F(x,t) \geq CW_F(x,t) \quad \textit{for all } (x,t) \in Q_\infty. \tag{4.69}$$

We first assume that F is compact and we denote it by K. If μ is q- admissible and nonnegative, then $u_\mu \leq \mathbb{H}[\mu]$. Since

$$u_\mu = \mathbb{H}[\mu] - \mathbb{G}[u^q],$$

where $\mathbb{G}$ is the Green parabolic heat potential, defined by

$$\mathbb{G}[f](x,t) = \int_0^t \mathbb{H}[f(.,s)](x,t-s)ds = \int_0^t \int_{\mathbb{R}^N} \frac{1}{(4\pi(t-s))^{\frac{N}{2}}} e^{-\frac{|x-y|^2}{4(t-s)}} f(y,s)dyds \tag{4.70}$$

for all $(x,t) \in Q_\infty$, there holds

$$\begin{aligned} u_\mu(x,t) &\geq \mathbb{H}[\mu](x,t) - \mathbb{G}[(\mathbb{H}[\mu])^q](x,t) \\ &\geq \left(\frac{1}{4\pi t}\right)^{\frac{N}{2}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}} d\mu(y) \\ &- \int_0^t \int_{\mathbb{R}^N} \frac{1}{(4\pi(t-s))^{\frac{N}{2}}} e^{-\frac{|x-y|^2}{4(t-s)}} \left(\frac{1}{(4\pi s)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{-\frac{|y-z|^2}{4s}} d\mu(z)\right)^q dyds \end{aligned} \tag{4.71}$$

for all $(x,t) \in Q_\infty$. The main idea of the proof is as follows: for any $(x,t) \in Q_T$ we construct a q-admissible bounded measure $\mu = \mu_{x,t}$ such that

$$\mathbb{H}[\mu_{x,t}](x,t) \geq CW_K(x,t), \tag{4.72}$$

and

$$\mathbb{G}[(\mathbb{H}[\mu_{x,t}])^q] \leq C\mathbb{H}[\mu_{x,t}] \quad \text{in } Q_\infty, \tag{4.73}$$

with constants C depending only on N, q and T. From this first estimate we replace $\mu_{x,t}$ by $\epsilon\mu_{x,t}$ with $\epsilon = (2C)^{-\frac{1}{q-1}}$ in order to obtain

$$u_{\epsilon\mu_{x,t}}(x,t) \geq \frac{1}{2}\mathbb{H}[\epsilon\mu_{x,t}](x,t) \geq \frac{C}{2}W_K(x,t).$$

If such an estimate holds for any $(x,t) \in Q_\infty$, it will follow by the definition of $\underline{u}_K$ that

$$\underline{u}_K \geq \frac{C}{2}W_K. \tag{4.74}$$

4.3.1 Estimate from below of the solution of the heat equation

The slicing of $\mathbb{R}^N$ used in the previous section is the intersection with $\mathbb{R}^N \times \{0\}$ of an extended slicing of Q_T that we construct as follows: if K is a compact subset of $\mathbb{R}^N$, $m = (x,t)$, we define d_K, λ, d_n as in Definition 4.17 and a_t as in Theorem 4.20.
Let $\alpha \in (0,1)$ to be fixed later on. For $n \in \mathbb{Z}$ we set

$$\mathcal{T}_n = \begin{cases} \mathcal{B}^2_{\sqrt{t(n+1)}}(m) \setminus \mathcal{B}^2_{\sqrt{tn}}(m) & \text{if } n \geq 1 \\ \mathcal{B}^2_{\alpha^{-n}\sqrt{t}}(m) \setminus \mathcal{B}^2_{\alpha^{1-n}\sqrt{t}}(m) & \text{if } n \leq 0, \end{cases}$$

and

$$\mathcal{T}_n^* = \mathcal{T}_n \cap Q_t \quad \text{if } n \in \mathbb{Z}.$$

For any $n \in \mathbb{N}_*$ and $m = (x,t) \in Q_T$, we recall that

$$\mathcal{Q}_n = K \cap \mathcal{B}^2_{\sqrt{t(n+1)}}(m) = K \cap B_{d_{n+1}}(x),$$

and

$$K_n = K \cap \mathcal{T}_{n+1} = K \cap \left(B_{d_{n+1}}(x) \setminus B_{d_n}(x)\right).$$

Let $\nu_n \in \mathfrak{M}^b_+(\mathbb{R}^N) \cap B^{-\frac{2}{q},q'}(\mathbb{R}^N)$ be the capacitary measure of the set $d_{n+1}^{-1}K_n$ (see [1, Section 2.2]). Then ν_n vanishes outside $d_{n+1}^{-1}K_n$ and satisfies

$$\nu_n(d_{n+1}^{-1}K_n) = cap_{\frac{2}{q},q'}(d_{n+1}^{-1}K_n) \quad \text{and} \quad \|\nu_n\|_{B^{-\frac{2}{q},q'}(\mathbb{R}^N)} = \left(cap_{\frac{2}{q},q'}(d_{n+1}^{-1}K_n)\right)^{\frac{1}{q}}. \tag{4.75}$$

Let μ_n be defined on any Borel set $A \subset K_n$ by

$$\mu_n(A) = d_{n+1}^{N-\frac{2}{q-1}} \nu_n(d_{n+1}^{-1}A). \tag{4.76}$$

We set

$$\mu_{t,K} = \sum_{n=0}^{a_t} \mu_n,$$

and

$$\mathbb{H}[\mu_{t,K}] = \sum_{n=0}^{a_t} \mathbb{H}[\mu_n].$$

Proposition 4.22 *Let $q \geq q_c$, then there holds*

$$\mu_{t,K} \geq \frac{1}{(4\pi t)^{\frac{N}{2}}} \sum_{n=0}^{a_t} e^{-\frac{n+1}{4}} d_{n+1}^{N-\frac{2}{q-1}} cap_{\frac{2}{q},q'}\left(\frac{d_{n+1}}{K_n}\right) \qquad \textit{for all } (x,t) \in Q_T. \tag{4.77}$$

Proof. We have

$$\mathbb{H}[\mu_n](x,t) = \frac{1}{(4\pi t)^{\frac{N}{2}}} \int_{K_n} e^{-\frac{|x-y|^2}{4t}} d\mu_n(y).$$

Furthermore

$$\int_{K_n} e^{-\frac{|x-y|^2}{4t}} d\mu_n(y) \le \left(\max\{e^{-\frac{|x-y|^2}{4t}} : y \in K_n\}\right) \mu_n(K_n)$$
$$\le \left(\max\{e^{-\frac{|x-y|^2}{4t}} : y \in K_n\}\right) d_{n+1}^{N-\frac{2}{q-1}} cap_{\frac{2}{q},q'}(d_{n+1}^{-1}K_n),$$

by (4.75) and (4.76). Thus

$$\int_{K_n} e^{-\frac{|x-y|^2}{4t}} d\mu_n(y) \le e^{-\frac{n+1}{4}} d_{n+1}^{N-\frac{2}{q-1}} cap_{\frac{2}{q},q'}(d_{n+1}^{-1}K_n),$$

by the definition of K_n and d_n, and (4.77) follows by the definition of $\mu_{t,K}$. □

4.3.2 Estimate from above of the nonlinear term

We write (4.71) under the form

$$u_\mu(x,t) \ge \sum_{n\in\mathbb{Z}} \mathbb{H}[\mu_n](x,t) - \int_0^t \int_{\mathbb{R}^N} H(x,y,t-s) \left(\sum_{n\in A_K} \mathbb{H}[\mu_n](y,s)\right)^q dyds \qquad (4.78)$$
$$= I_1 - I_2.$$

We recall that $\mu_n = 0$ if $n \notin A_K = \mathbb{N} \cap [1, a_t]$. Then

$$I_2 = \frac{1}{(4\pi)^{\frac{N}{2}}} \int_0^t \int_{\mathbb{R}^N} (t-s)^{\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left(\sum_{n\in A_K} \mathbb{H}[\mu_n](y,s)\right)^q dyds \qquad (4.79)$$
$$\le \frac{2^{q-1}}{(4\pi)^{\frac{N}{2}}} (J_\ell + J'_\ell),$$

where

$$J_\ell = \sum_{p\in\mathbb{Z}} \int\int_{\mathcal{T}_p^*} (t-s)^{\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left(\sum_{n\le p+\ell} \mathbb{H}[\mu_n](y,s)\right)^q dyds,$$

and

$$J'_\ell = \sum_{p\in\mathbb{Z}} \int\int_{\mathcal{T}_p^*} (t-s)^{\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left(\sum_{n> p+\ell} \mathbb{H}[\mu_n](y,s)\right)^q dyds.$$

In these expressions $\ell \in \mathbb{N}$ will be fixed later on.

Lemma 4.23 *Let $0 < a < b$ and $t > 0$, then*

$$\max\left\{\sigma^{-\frac{N}{2}} e^{-\frac{\rho^2}{4\sigma}} : 0 \le \sigma \le t,\ at \le \rho^2 + \sigma \le bt\right\} = \begin{cases} t^{-\frac{N}{2}} e^{-\frac{a}{4}} & \text{if } \frac{a}{2N} > 1 \\ \left(\frac{2N}{at}\right)^{\frac{N}{2}} e^{-\frac{N}{2}} & \text{if } \frac{a}{2N} \le 1. \end{cases} \qquad (4.80)$$

Proof. Set

$$\mathcal{J}(\rho,\sigma)=\sigma^{-\frac{N}{2}}e^{-\frac{\rho^2}{4\sigma}},$$

and

$$\mathcal{K}_{a,b,t}=\{(\rho,\sigma)\in[0,\infty)\times(0t]:at\le\rho^2+\sigma\le bt\}.$$

We notice that, for fixed σ, the maximum of $\mathcal{J}(.,\sigma)$ is achieved when ρ is minimal. If $\sigma\in[at,bt]$, the minimal value of ρ is zero, while if $\sigma\in(0,at)$, the minimal value is $\sqrt{at-s}$.
- Assume first $a\ge 1$, then $\mathcal{J}(at-\sigma,\sigma)=e^{\frac{1}{4}}\sigma^{-\frac{N}{4}}e^{-\frac{at}{4\sigma}}$. Thus if $1\le\frac{a}{2N}$, the minimal value of $\mathcal{J}(\sqrt{at-\sigma},\sigma)$ is $e^{\frac{1-2N}{4}}\left(\frac{2N}{at}\right)^{\frac{N}{2}}$, while if $\frac{a}{2N}<1$, this minimum is $e^{\frac{1}{4}}t^{-\frac{N}{2}}e^{-\frac{a}{4}}$.
- Assume now $a<1$, then

$$\begin{aligned}\max\{\mathcal{J}(\rho,\sigma):(\rho,\sigma)\in\mathcal{K}_{a,b,t}\}&=\max\left\{\max_{\sigma\in(at,t}\mathcal{J}(0,\sigma),\max_{\sigma\in(0,at}\mathcal{J}(\sqrt{at-\sigma},\sigma)\right\}\\&=\max\left\{(at)^{-\frac{N}{2}},e^{-\frac{1-2N}{4}}\left(\frac{2N}{at}\right)^{\frac{N}{2}}\right\}\\&=e^{-\frac{1-2N}{4}}\left(\frac{2N}{at}\right)^{\frac{N}{2}}.\end{aligned}$$

From these two estimates, (4.80) follows. □

Remark. The following variant of Lemma 4.23 will be useful in the sequel: *For any $\theta>\frac{1}{2N}$, there holds*

$$\max\{\mathcal{J}(\rho,\sigma):(\rho,\sigma)\in\mathcal{K}_{a,b,t}\}\le e^{\frac{1}{4}}\left(\frac{2N\theta}{t}\right)^{\frac{N}{2}}e^{\frac{a}{4}}\quad\text{if }\theta a\ge 1.\tag{4.81}$$

Lemma 4.24 *There exists a positive constant $C=C(N,q,\ell,T)$ such that*

$$J_\ell\le Ct^{-\frac{N}{2}}\sum_{n=1}^{a_t}d_{n+1}^{N-\frac{2}{q-1}}e^{-\frac{1+(n-\ell)_+}{4}}cap_{\frac{2}{q},q'}\left(\frac{K_n}{d_{n+1}}\right).\tag{4.82}$$

Proof. The set of the indices p for the summation in J_ℓ is reduced to $\mathbb{Z}\cap[-\ell+2,\infty)$, thus there holds $J_\ell=J_{1,\ell}+J_{2,\ell}$ where

$$J_{1,\ell}=\sum_{p=2-\ell}^{0}\int\!\!\int_{\mathcal{T}_p^*}(t-s)^{-\frac{N}{2}}e^{-\frac{|x-y|^2}{4(t-s)}}\left(\sum_{n<p+\ell}\mathbb{H}[\mu_n](y,s)\right)^q dyds$$

and

$$J_{2,\ell}=\sum_{p=1}^{\infty}\int\!\!\int_{\mathcal{T}_p^*}(t-s)^{-\frac{N}{2}}e^{-\frac{|x-y|^2}{4(t-s)}}\left(\sum_{n<p+\ell}\mathbb{H}[\mu_n](y,s)\right)^q dyds.$$

If $p = 2-p, ..., 0$,

$$(y,s) \in \mathcal{T}_p^* \Longrightarrow t\alpha^{2-2p} \leq |x-y|^2 + t - s \leq t\alpha^{-2p},$$

while if $p \geq 1$,

$$(y,s) \in \mathcal{T}_p^* \Longrightarrow pt \leq |x-y|^2 + t - s \leq (p+1)t.$$

By Lemma 4.23 and (4.81), there exists a positive constant $C = C(N, \ell, \alpha)$ such that

$$\max\left\{(t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} : (y,s) \in \mathcal{T}_p^*\right\} \leq Ct^{-\frac{N}{2}} e^{-\frac{\alpha^{2-p}}{4}}, \tag{4.83}$$

whenever $p = 2-\ell, ..., 0$, and

$$\max\left\{(t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} : (y,s) \in \mathcal{T}_p^*\right\} \leq Ct^{-\frac{N}{2}} e^{-\frac{p}{4}}, \tag{4.84}$$

when $p \geq 1$.
When $p = 2-\ell, ..., 0$

$$\left(\sum_{n=1}^{p-\ell-1} \mathbb{H}[\mu_n](y,s)\right)^q \leq C \sum_{n=1}^{p-\ell-1} \left(\mathbb{H}[\mu_n]\right)^q (y,s), \tag{4.85}$$

where $C = C(q,\ell) > 0$, therefore

$$\begin{aligned} J_{1,\ell} &\leq Ct^{-\frac{N}{2}} \sum_{p=2-\ell}^{0} e^{-\frac{\alpha^{2-p}}{4}} \sum_{n=1}^{p-\ell-1} \|\mathbb{H}[\mu_n]\|^q_{L^q(Q_T)} \\ &\leq Ct^{-\frac{N}{2}} \sum_{n=1}^{\ell-1} \|\mathbb{H}[\mu_n]\|^q_{L^q(Q_T)} \sum_{p=n-\ell+1}^{0} e^{-\frac{\alpha^{2-p}}{4}} \\ &\leq Ct^{-\frac{N}{2}} e^{-\frac{\alpha^{2\ell-2}}{4}} \sum_{n=1}^{\ell-1} \|\mathbb{H}[\mu_n]\|^q_{L^q(Q_T)}. \end{aligned} \tag{4.86}$$

When the set of indices p is not upper bounded, we introduce some extra parameter to be made precise later on. Then

$$\left(\sum_{n=1}^{p-\ell-1} \mathbb{H}[\mu_n](y,s)\right)^q \leq \left(\sum_{n=1}^{p-\ell-1} e^{\frac{q'\delta n}{4}}\right)^{\frac{q}{q'}} \sum_{n=1}^{p-\ell-1} e^{-\frac{q\delta n}{4}} \left(\mathbb{H}[\mu_n]\right)^q (y,s). \tag{4.87}$$

Remembering thet $\mu_n = 0$ if $n \geq a_t$, we obtain that there exists $C > 0$ depending also on δ such that

$$\begin{aligned} J_{2,\ell} &\leq Ct^{-\frac{N}{2}} \sum_{p=1}^{\infty} e^{\frac{\delta(p+\ell-1)q-p}{4}} \sum_{n=1}^{p-\ell-1} e^{-\frac{q\delta n}{4}} \|\mathbb{H}[\mu_n]\|^q_{L^q(Q_T)} \\ &\leq Ct^{-\frac{N}{2}} \sum_{n=1}^{\infty} e^{-\frac{q\delta n}{4}} \|\mathbb{H}[\mu_n]\|^q_{L^q(Q_T)} \sum_{p=n-\ell+1\vee 1}^{\infty} e^{\frac{\delta(p+\ell-1)q-p}{4}} \\ &\leq Ct^{-\frac{N}{2}} \sum_{n=1}^{\infty} e^{-\frac{1+(n-\ell)_+}{4}} \|\mathbb{H}[\mu_n]\|^q_{L^q(Q_T)}. \end{aligned} \tag{4.88}$$

We chose δ such that $\delta\ell q < 1$. Combining (4.86) and (4.88) and using Lemma 3.11 and (4.75) and (4.76) we obtain (4.82). □

The set of indices p such that the term μ_n is not zero in the summation J'_ℓ is $\mathbb{Z} \cap (-\infty, a_t - \ell]$. We write

$$J'_\ell = J'_{1,\ell} + J'_{2,\ell}$$

with

$$J'_{1,\ell} = \sum_{-\infty}^{p=0} \int\int_{\mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left(\sum_{n=1\vee p+\ell}^{\infty} \mathbb{H}[\mu_n](y,s) \right)^q$$

and

$$J'_{2,\ell} = \sum_{p=1}^{a_t-\ell} \int\int_{\mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left(\sum_{n=p+\ell}^{\infty} \mathbb{H}[\mu_n](y,s) \right)^q .$$

Lemma 4.25 *There exists a positive constant $C = C(N, q, \ell)$ such that*

$$J'_{1,\ell} \le C t^{1-\frac{N}{q}2} \sum_{n=0}^{a_t} e^{-\frac{(1+\beta_0)(n-h)_+}{4}} d_{n+1}^{Nq-2q'} cap_{2q,q'}^q \left(\frac{K_n}{d_{n+1}} \right), \tag{4.89}$$

where $\beta_0 = \frac{q-1}{4}$ and $h = \frac{2q(q+1)}{(q-1)^2}$.

Proof. Since

$$(y,s) \in \mathcal{T}_p^* \quad \text{and} \quad (z,0) \in K_n \Longrightarrow |y-z| \ge (\sqrt{n} - \alpha^{-p})\sqrt{t}, \tag{4.90}$$

there holds by Lemma 4.23,

$$\mathbb{H}[\mu_n](y,s) \le \frac{1}{(4\pi s)^{\frac{N}{2}}} e^{-\frac{(\sqrt{n}-\alpha^{-p})^2 t}{4s}} \mu_n(K_n) \le C t^{-\frac{N}{2}} e^{-\frac{(\sqrt{n}-\alpha^{-p})^2}{4}} \mu_n(K_n).$$

Let $\{\epsilon_n\}$ be a sequence of positive numbers such that

$$A_\epsilon := \sum_{n=0}^{\infty} \epsilon_n < \infty,$$

then

$$\begin{aligned}
J'_{1,\ell} &\le C A_\epsilon^{\frac{q'}{q}} t^{-\frac{Nq}{2}} \sum_{p=-\infty}^{0} \int\int_{\mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \sum_{n=1\vee p+\ell}^{\infty} \epsilon_n^{-q} e^{-\frac{q(\sqrt{n}-\alpha^{-p})^2}{4}} (\mu_n(K_n))^q \, dyds \\
&\le C A_\epsilon^{\frac{q'}{q}} t^{-\frac{Nq}{2}} \sum_{n=1}^{\infty} \epsilon_n^{-q} (\mu_n(K_n))^q \sum_{p=-\infty}^{0\wedge n-\ell} e^{-\frac{q(\sqrt{n}-\alpha^{-p})^2}{4}} \int\int_{\mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} dyds \\
&\le C A_\epsilon^{\frac{q'}{q}} t^{-\frac{Nq}{2}} \sum_{n=1}^{\infty} \epsilon_n^{-q} (\mu_n(K_n))^q e^{-\frac{q(\sqrt{n}-1)^2}{4}} \int\int_{\cup_{p\le 0}\mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} dyds \\
&\le C A_\epsilon^{\frac{q'}{q}} t^{1-\frac{Nq}{2}} \sum_{n=1}^{\infty} \epsilon_n^{-q} e^{-\frac{q(\sqrt{n}-1)^2}{4}} (\mu_n(K_n))^q .
\end{aligned} \tag{4.91}$$

Set $h = \frac{2q(q+1)}{(q-1)^2}$ and $Q = \frac{q+1}{2}$, then $q(\sqrt{n}-1)^2 \geq Q(n-h)_+$ for $n \geq 1$. Then

$$\epsilon_n = e^{\frac{(q-1)(n-h)_+}{16q}} \Longrightarrow \epsilon_n^{-q} e^{-\frac{q(\sqrt{n}-1)^2}{4}} \leq e^{-\frac{(q+3)(n-h)_+}{16}}.$$

Therefore

$$J'_{1,\ell} \leq Ct^{1-\frac{N}{2}} \sum_{n=1}^{\infty} e^{-\frac{(1+\beta_0)(n-h)_+}{4}} \left(\mu_n(K_n)\right)^q.$$

This implies (4.89) from the properties of μ_n. □

The estimate of the term $J'_{2,\ell}$ is more involved. In order to help the reader to follow the idea, we first give a proof in dimension 1.

Lemma 4.26 *Assume $N = 1$, $q \geq 3$ and ℓ is an integer larger than 1. Then there exists a positive constant $C = C(q,\ell)$ such that*

$$J'_{2,\ell} \leq Ct^{-\frac{1}{2}} \sum_{n=\ell}^{a_t} e^{-\frac{n}{4}} d_{n+1}^{\frac{q-3}{q-1}} cap_{\frac{2}{q},q'}\left(\frac{K_n}{d_{n+1}}\right). \tag{4.92}$$

Proof. If $p \geq 1$, $n \geq p = \ell$ and if $y, s) \in \mathcal{T}_p^*$ and $z \in K_n$ there holds $|x-y| \geq \sqrt{t}\sqrt{p}$ and $|y-z| \geq \sqrt{t}\left(\sqrt{n}-\sqrt{p+1}\right)$. Therefore

$$J'_{2,\ell} \leq C\sqrt{t} \sum_{p=1}^{a_t-\ell} \frac{1}{\sqrt{p}} \int_0^t e^{-\frac{pt}{4(t-s)}} \left(\sum_{n=p+\ell}^{a_t} \frac{1}{\sqrt{s}} e^{-\frac{(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} \mu_n(K_n) \right)^q ds.$$

Let $\epsilon \in (0,q)$ be some parameter to be made more precise later on, then

$$\left(\sum_{n=p+\ell}^{a_t} \frac{1}{\sqrt{s}} e^{-\frac{(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} \mu_n(K_n) \right)^q$$
$$\leq \left(\sum_{n=p+\ell}^{a_t} e^{-\frac{\epsilon q'(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} \right)^{\frac{q'}{q}} \sum_{n=p+\ell}^{a_t} s^{-\frac{q}{2}} e^{-(q-\epsilon)\frac{(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} \left(\mu_n(K_n)\right)^q.$$

By comparison between series and integrals we have

$$\begin{aligned}
\sum_{n=p+\ell}^{a_t} e^{-\frac{\epsilon q'(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} &\leq \int_{p+\ell}^{\infty} e^{-\frac{\epsilon q'(\sqrt{x}-\sqrt{p+1})^2 t}{4s}} dx \\
&\leq 2\int_{\sqrt{p+\ell}-\sqrt{p+1}}^{\infty} e^{-\frac{\epsilon q' x^2 t}{4s}} (x+\sqrt{p+1}) dx \\
&\leq \tfrac{4s}{\epsilon q' t} e^{-\epsilon q' \frac{(\sqrt{p+\ell}-\sqrt{p+1})^2 t}{4s}} + 2\sqrt{p+1}\int_{\sqrt{p+\ell}-\sqrt{p+1}}^{\infty} e^{-\frac{\epsilon q' x^2 t}{4s}} dx \\
&\leq C\sqrt{\tfrac{(p+1)s}{t}} e^{-\epsilon q' \frac{(\sqrt{p+\ell}-\sqrt{p+1})^2 t}{2s}} \\
&\leq C\sqrt{\tfrac{(p+1)s}{t}}.
\end{aligned}$$

Set $q_\epsilon = q - \epsilon$, then

$$J'_{2,\ell} \leq C\epsilon^{-\frac{q'}{q}} t^{1-\frac{q}{2}} \sum_{n=\ell+1}^{\infty} (\mu_n(K_n))^q \sum_{p=1}^{n-\ell} \int_0^t \sqrt{s(t-s)} e^{-\frac{-pt}{4(t-s)}} e^{-q_\epsilon \frac{(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} ds,$$

where $C = C(\epsilon, q) > 0$. Since

$$\int_0^t \sqrt{s(t-s)} e^{-\frac{-pt}{4(t-s)}} e^{-q_\epsilon \frac{(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} ds = \int_0^1 \sqrt{s(1-s)} e^{-\frac{-p}{4(t-s)}} e^{-q_\epsilon \frac{(\sqrt{n}-\sqrt{p+1})^2}{4s}} ds,$$

we can apply Lemma 4.34 with $a = \frac{1}{2}$, $b = \frac{1}{2}$, $A = \sqrt{p}$, $B = \sqrt{q_\epsilon}(\sqrt{n} - \sqrt{p+1})$. For such a choice,

$$B \geq \sqrt{q_\epsilon}(\sqrt{p+\ell} - \sqrt{p+1}) \geq \frac{(\ell-1)\sqrt{q_\epsilon}}{\sqrt{p}} \Longrightarrow \kappa = (\ell-1)\sqrt{q_\epsilon},$$

and

$$\sqrt{\frac{A}{A+B}}\sqrt{\frac{B}{A+B}} \leq \frac{\sqrt[4]{p}\sqrt{\sqrt{n}-\sqrt{p}}}{\sqrt{n}}.$$

Therefore

$$\int_0^t \frac{e^{-\frac{pt}{4(t-s)}} e - q\frac{(\sqrt{n}-\sqrt{p+1})^2 t}{4t}}{\sqrt{s^q(t-s)}} ds \leq C \frac{\sqrt[4]{p}\sqrt{\sqrt{n}-\sqrt{p}}}{\sqrt{n}} e^{-\frac{(\sqrt{p}+\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1}))^2}{4}}.$$

This implies

$$J'_{2,\ell} \leq C t^{1-\frac{q}{2}} \sum_{n=\ell+1}^{a_t} \frac{(\mu_n(K_n))^q}{\sqrt{n}} \sum_{p=1}^{n-\ell} p^{\frac{2q-3}{4}} \sqrt{\sqrt{n}-\sqrt{p}}\, e^{-\frac{(\sqrt{p}+\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1}))^2}{4}}, \tag{4.93}$$

where $C = C(\epsilon, q, \ell) > 0$. Then, by Lemma 4.35,

$$J'_{2,\ell} \leq C t^{1-\frac{q}{2}} \sum_{n=\ell+1}^{a_t} n^{\frac{q-3}{4}} e^{-\frac{n}{4}} (\mu_n(K_n))^q. \tag{4.94}$$

Replacing $\mu_n(K_n)$ by its value $d_{n+1}^{\frac{q-3}{q-1}} cap_{\frac{2}{q}q'}\left(\frac{K_n}{d_{n+1}}\right)$, the expression when $N = 1$, and since $diam\left(\frac{K_n}{d_{n+1}}\right) \leq \frac{1}{n}$, we obtain

$$(\mu_n(K_n))^q \leq C\left(\frac{t}{n}\right)^{\frac{q-3}{2}} \mu_n(K_n) = C\left(\frac{t}{n}\right)^{\frac{q-3}{2}} d_{n+1}^{\frac{q-3}{q-1}} cap_{\frac{2}{q}q'}\left(\frac{K_n}{d_{n+1}}\right), \tag{4.95}$$

and the proof follows. □

Next we give the proof for $N \geq 2$. For this task we will use again the quasi-additivity property.

Lemma 4.27 *Assume $N \geq 2$ and ℓ is an integer larger than 1. There exists a positive constant $C_1 = C_1(N, q, \ell)$ such that*

$$J'_{2,\ell} \leq C_1 t^{-\frac{N}{2}} \sum_{n=\ell}^{a_t} e^{-\frac{n}{4}} d_{n+1}^{N-\frac{2}{q-1}} cap_{\frac{2}{q},q'}\left(\frac{K_n}{d_{n+1}}\right). \tag{4.96}$$

Proof. As in the prof Theorem 4.18 there exists a finite number J depending only on the dimension N of separated sub-partitions $\{\Theta^h_{t,n}\}_{h=1}^J$ of the rescaled sets $\widetilde{T}_n = \sqrt{\frac{n+1}{t}} T_n$ by the N-dimensional balls $B_2(\widetilde{a}_{n,j})$ where $\widetilde{a}_{n,j} = \sqrt{\frac{n+1}{t}} a_{n,j}$, $|a_{n,j}| = \frac{1}{2}(d_n + d_{n+1})$ and $|a_{n,j} - a_{n,k}| \geq \sqrt{\frac{4t}{n+1}}$. Furthermore $\#\Theta^h_{t,n} \leq C n^{N-1}$. We denote $K_{n,j} = K_n \cap B_{\sqrt{\frac{t}{n+1}}}(a_{n,j})$. We can write $\mu_n = \sum_{h=1}^J \mu_n^h$ and accordingly $J'_{2,\ell} \sum_{h=1}^J \mu_n^h J'^h_{2,\ell}$ where $\mu_n^h = \sum_{j \in \Theta^h_{t,n}} \mu_{n,j}$ and the $\mu_{n,j}$ are the capacitary measures of $K_{n,j}$ relative to $B_{n,j} := B_{\frac{6t}{5}}(a_{n,j}$, which means

$$\nu_{n,j}(K_{n,j}) = cap^{B_{n,j}}_{\frac{2}{q},q'}(K_{n,j} \quad \text{and} \quad \|\nu_{n,j}\|_{B^{-\frac{2}{q},q}(B_{n,j})} = \left(cap^{B_{n,j}}_{\frac{2}{q},q'}(K_{n,j}\right)^{\frac{1}{q}}. \tag{4.97}$$

Thus

$$J'_{2,\ell} = \sum_{p=1}^{a_t-\ell} \int\int_{\mathcal{T}_p^*} (-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left(\sum_{n=p+\ell}^{\infty} \sum_{h=1}^{J} \sum_{j \in Gth^h_{t,n}} \mathbb{H}[\mu_{n,j}(y,s)]\right)^q dyds. \tag{4.98}$$

We denote

$$J'^h_{2,\ell} = \sum_{p=1}^{a_t-\ell} \int\int_{\mathcal{T}_p^*} (-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left(\sum_{n=p+\ell}^{\infty} \sum_{j \in Gth^h_{t,n}} \mathbb{H}[\mu_{n,j}(y,s)]\right)^q dyds.$$

Since J depends only on N and q,

$$J'_{2,\ell} \leq C \sum_{h=1}^J J'^h_{2,\ell}.$$

If n and p are such that $n \geq \ell + 1$, we set

$$\lambda_{n,j,y} = \inf\left\{|y-z| : z \in B_{\sqrt{t}n+1}(a_{n,j}\right\} = |y - a_{n,j}| - \sqrt{t}n + 1.$$

Then

$$\sum_{n=p+\ell}^{a_t}\int_{K_n} e^{-\frac{|y-z|^2}{4t}}d\mu_n^h(z) = \sum_{n=p+\ell}^{a_t}\sum_{j\in Gth_{t,n}^h}\mathbb{H}[\mu_{n,j}(z)]$$
$$\leq \left(\sum_{n=p+\ell}^{a_t}\sum_{j\in Gth_{t,n}^h} e^{-\epsilon q'\frac{\lambda_{n,j,y}^2}{4s}}\right)^{\frac{1}{q'}}$$
$$\times\left(\sum_{n=p+\ell}^{a_t}\sum_{j\in Gth_{t,n}^h} e^{-q\lambda_{n,j,y}^2\frac{1-\epsilon}{4s}}\left(\mu_{n,j}(K_{n,j})\right)^q\right)^{\frac{1}{q}}$$

where $\epsilon>0$ will be made precise later on.

Step 1. We claim that

$$\sum_{n=p+\ell}^{a_t}\sum_{j\in Gth_{t,n}^h} e^{-\epsilon q'\frac{\lambda_{n,j,y}^2}{4s}} \leq C\sqrt{\frac{ps}{t}}, \tag{4.99}$$

where $C=C(\epsilon,q,N)>0$. If $y\in T_p$, let $z_y\in T_n$ such that $|y-z|=\mathrm{dist}\,(y,T_n)$ hence

$$\sqrt{t}(\sqrt{n}-\sqrt{p+1})\leq |y-z|\leq \sqrt{t}(\sqrt{n}-\sqrt{p+1}).$$

Let $Y=\frac{\sqrt{t(p+1)}}{|y|}y$, $\mathbf{e}=\frac{Y}{|Y|}$ and, for integers $k\in[-n,n]$, $b_k=\frac{k\sqrt{t}}{\sqrt{n}}\mathbf{e}$. We denote by $H_{n,k}$ the domain in $\mathbb{R}^N$ limited by the hyperplanes orthogonal to $\mathbf{e}$ going through the points $\frac{(k+1)\sqrt{t}}{\sqrt{n}}\mathbf{e}$ and $\frac{(k-1)\sqrt{t}}{\sqrt{n}}\mathbf{e}$, and by $G_{n,k}$ the spherical shell obtained by intersecting the spherical shell T_n with $H_{n,k}$. The number of points $a_{n,j}$ belonging to $G_{n,k}$ is smaller than $C(n+1-|k|)^{N-2}$ where $C=C(N)>0$. Let $\Lambda_{n,k}$ be the set of indices $j\in\Theta_{t,n}$ such that $a_{n,j}\in G_{n,k}$. Note that in $a_{n,j}\in G_{n,k}$, it is a consequence of Pythagora's theorem that $\lambda_{n,j,y}^2$ is larger than $t(n+p+1-2k\frac{p+1}{n})$. Therefore

$$\sum_{n=p+\ell}^{a_t}\sum_{j\in\Theta_{t,n}} e^{-\epsilon q'\frac{\lambda_{n,j,y}^2}{4s}} \leq C\sum_{n=p+\ell}^{a_t}\sum_{k=-n}^{n}(n+1-|k|)^{N-2}e^{\frac{\epsilon q'(n+p+1-2k\sqrt{p+1})t}{4s\sqrt{n}}}. \tag{4.100}$$

Case $N=2$. Summing a geometric series and using the inequality $\frac{e^u}{e^u-1}\leq 1+\frac{1}{u}$ on $(0,\infty)$, we obtain

$$\sum_{k=-n}^{n} e^{\frac{\epsilon q' k\sqrt{p+1}t}{2s\sqrt{n}}} \leq e^{\frac{\epsilon q'\sqrt{p+1}t}{2s}}\frac{e^{\frac{\epsilon q'\sqrt{p+1}t}{2s\sqrt{n}}}}{e^{\frac{\epsilon q'\sqrt{p+1}t}{2s\sqrt{n}}}-1} \leq e^{\frac{\epsilon q'\sqrt{p+1}t}{2s}}\left(1+\frac{2s\sqrt{n}}{\epsilon q' t\sqrt{p+1}}\right). \tag{4.101}$$

Therefore using comparison between series and integrals,

$$\sum_{n=p+\ell}^{a_t}\sum_{j\in Gth_{t,n}^h} e^{-\epsilon q'\frac{\lambda_{n,j,y}^2}{4s}} \leq C\sum_{n=p+\ell}^{a_t}\left(1+\frac{s\sqrt{n}}{t\sqrt{p}}\right)e^{-\epsilon q'\frac{(\sqrt{n}-\sqrt{p+1})^2 t}{4s}}$$
$$\leq C\int_{p+1}^{\infty} e^{-\epsilon q'\frac{(\sqrt{x}-\sqrt{p+1})^2 t}{4s}}dx + \frac{Cs}{t\sqrt{n}}\int_{p+1}^{\infty}\sqrt{x}e^{-\epsilon q'\frac{(\sqrt{x}-\sqrt{p+1})^2 t}{4s}}dx. \tag{4.102}$$

Next

$$\begin{aligned}\int_{p+1}^{\infty} e^{-\epsilon q' \frac{(\sqrt{x}-\sqrt{p+1})^2 t}{4s}} dx &= 2\int_{\sqrt{p+1}}^{\infty} e^{-\epsilon q' \frac{(y-\sqrt{p+1})^2 t}{4s}} y dy \\ &= 2\int_0^{\infty} e^{-\epsilon q' \frac{y^2 t}{4s}} y dy + 2\sqrt{p+1}\int_0^{\infty} e^{-\epsilon q' \frac{y^2 t}{4s}} dy \\ &= \frac{2s}{t}\int_0^{\infty} e^{-\epsilon q' \frac{y^2}{4}} z dz + \sqrt{\frac{(p+1)s}{t}}\int_0^{\infty} e^{-\epsilon q' \frac{z^2}{4}} dz,\end{aligned} \tag{4.103}$$

and

$$\begin{aligned}\int_{p+1}^{\infty} e^{-\epsilon q' \frac{(\sqrt{x}-\sqrt{p+1})^2 t}{4s}} \sqrt{x} dx &= 2\int_{\sqrt{p+1}}^{\infty} e^{-\epsilon q' \frac{(y-\sqrt{p+1})^2 t}{4s}} y^2 dy \\ &= 2\int_0^{\infty} e^{-\epsilon q' \frac{y^2 t}{4s}} (y+\sqrt{p+1})^2 dy \\ &\le 4\int_0^{\infty} e^{-\epsilon q' \frac{y^2 t}{4s}} y^2 dy + 4(p+1)\int_0^{\infty} e^{-\epsilon q' \frac{y^2 t}{4s}} dy \\ &\le 4\left(\frac{s}{t}\right)^{\frac{3}{2}}\int_0^{\infty} e^{-\epsilon q' \frac{z^2}{4}} z^2 dz + 4(p+1)\sqrt{\frac{s}{t}}\int_0^{\infty} e^{-\epsilon q' \frac{z^2}{4}} dz.\end{aligned} \tag{4.104}$$

Combined with (4.102), this inequalities imply

$$\sum_{n=p+\ell}^{a_t} \sum_{j\in\Theta_{t,n}} e^{\frac{\epsilon q' \lambda_{n,j,y}^2}{4}} \le C\sqrt{\frac{ps}{t}}. \tag{4.105}$$

Case $N > 2$. The value of the right-hand side of (4.100) is clearly an increasing function of N, hence it is sufficient to prove (4.100) when $N = 2 + 2d$ with $d \in \mathbb{N}_*$. There holds

$$\sum_{k=-n}^{n} (n+1-|k|)^d e^{\frac{\epsilon q' k t\sqrt{p+1}}{2s\sqrt{n}}} \le 2\sum_{k=0}^{n} (n+1-k)^d e^{\frac{\epsilon q' k t\sqrt{p+1}}{2s\sqrt{n}}}. \tag{4.106}$$

We set

$$\alpha = \epsilon q' \frac{t\sqrt{p+1}}{2s\sqrt{n}} \quad \textit{and } I_d = \sum_{k=0}^{n} (n+1-k)^d e^{k\alpha}.$$

Since

$$e^{k\alpha} = \frac{e^{(k+1)\alpha} - e^{k\alpha}}{e^{\alpha} - 1},$$

we use the Abel's transform and obtain

$$\begin{aligned}I_d &= \frac{1}{e^{\alpha}-1}\left(e^{(n+1)\alpha} - (n+1)^d + \sum_{k=1}^{n}\Big((n+2-k)^d - (n+1-k)d\Big)e^{k\alpha}\right) \\ &\le \frac{1}{e^{\alpha}-1}\left((1-d)e^{(n+1)\alpha} - (n+1)^d + de^{\alpha}\sum_{k=1}^{n}(n+1-k))e^{k\alpha}\right).\end{aligned}$$

Therefore the following induction relation holds

$$I_d \leq \frac{de^\alpha}{e^\alpha - 1} I_{d-1}. \tag{4.107}$$

We use again the fact that

$$\frac{de^\alpha}{e^\alpha - 1} \leq C\left(1 + \frac{s\sqrt{n}}{t\sqrt{p}}\right)$$

as in (4.101), and

$$I_d \leq C\left(1 + C\left(\frac{s\sqrt{n}}{t\sqrt{p}}\right)^{d+1}\right) I_0.$$

Therefore (4.102) is replaced by

$$\begin{aligned}\sum_{n=p+\ell}^{a_t} \sum_{j \in G t h_{t,n}^h} e^{-\epsilon q' \frac{\lambda_{n,j,y}^2}{4s}} &\leq C \sum_{n=p+\ell}^{a_t} \left(1 + \frac{s\sqrt{n}}{t\sqrt{p}}\right)^{d+1} e^{-\epsilon q' \frac{(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} \\ &\leq C \int_{p+1}^{\infty} e^{-\epsilon q' \frac{(\sqrt{x}-\sqrt{p+1})^2 t}{4s}} dx + \left(\frac{Cs}{t\sqrt{p}}\right)^{d+1} \int_{p+1}^{\infty} x^{\frac{d+1}{2}} e^{-\epsilon q' \frac{(\sqrt{x}-\sqrt{p+1})^2 t}{4s}} dx.\end{aligned} \tag{4.108}$$

Using the estimate of the first integral of the right-hand side of (4.108) that we have obtained in (4.103), we can concentrate on the second integral,

$$\begin{aligned}\int_{p+1}^{\infty} x^{\frac{d+1}{2}} e^{-\epsilon q' \frac{(\sqrt{x}-\sqrt{p+1})^2 t}{4s}} dx &= \int_0^{\infty} (y + \sqrt{p+1})^{d+2} e^{-\epsilon q' \frac{y^2 t}{4s}} dy \\ &\leq C \int_0^{\infty} y^{d+2} e^{-\epsilon q' \frac{y^2 t}{4s}} dy + C p^{1+\frac{d}{2}} \int_0^{\infty} e^{-\epsilon q' \frac{y^2 t}{4s}} dy \\ &\leq C \left(\frac{s}{t}\right)^{2+\frac{d}{2}} \int_0^{\infty} e^{-\epsilon q' \frac{z^2}{4}} dz \\ &\qquad + C \left(\frac{s}{t}\right)^{\frac{3}{2}} p^{1+\frac{d}{2}} \int_0^{\infty} e^{-\epsilon q' \frac{z^2}{4}} dz.\end{aligned} \tag{4.109}$$

We obtain (4.99) from (4.104), (4.108) and (4.109).

Step 2. Since $\mathcal{T}_p^* \subset \Gamma_p \times [0,t]$, where we recall it $\Gamma_p = B_{d+1}(x) \setminus B_{d_n}(x)$, the fact that $(y,s) \in \mathcal{T}_p^*$ implies $|x-y|^2 \geq (p-1)t$. Therefore $J'^h_{2,\ell}$ satisfies

$$\begin{aligned}J'^h_{2,\ell} &\leq C t^{\frac{1-q}{2}} \sum_{p=1}^{\infty} p^{\frac{q-1}{2}} \int_0^t \int_{\Gamma_p} (t-s)^{-\frac{N}{2}} s^{\frac{-q(N-1)+1}{2}} e^{-4\frac{|x-y|^2}{4(t-s}} \\ &\qquad \times \sum_{n=p+\ell}^{a_t} \sum_{j \in \Theta_{t,n}^h} e^{-\frac{q(1-\epsilon)\lambda_{n,j,y}^2}{4s}} (\mu_{n,j}(K_{n,j}))^q \, dyds \\ &\leq C t^{\frac{1-q}{2}} \sum_{n=p+\ell}^{a_t} \sum_{j \in \Theta_{t,n}^h} (\mu_{n,j}(K_{n,j}))^q \\ &\qquad \times \sum_{p=1}^{n-\ell} \int_0^t \int_{\Gamma_p} (t-s)^{-\frac{N}{2}} s^{\frac{-q(N-1)+1}{2}} e^{-4\frac{|x-y|^2}{4(t-s}} e^{-\frac{q(1-\epsilon)\lambda_{n,j,y}^2}{4s}} dyds,\end{aligned} \tag{4.110}$$

where $C = C(N, q, \epsilon) > 0$. Next we set $q_\epsilon = (1-\epsilon)q$. If we write

$$\begin{aligned}|y - a_{n,j}|^2 &= |x-y|^2 + |x - a_{n,j}|^2 - 2\langle y - x, a_{n,j} - x\rangle \\ &\geq pt + |x - a_{n,j}|^2 - 2\langle y - x, a_{n,j} - x\rangle,\end{aligned}$$

we deduce

$$\int_{\Gamma_p} e^{-\frac{q_\epsilon |y-a_{n,j}|^2}{4s}} dy = e^{-\frac{|x-a_{n,j}|^2}{4s}} \int_{\sqrt{tp}}^{\sqrt{t(p+1}} e^{-\frac{q_\epsilon r^2}{4s}} \int_{\partial B_r(x)} e^{\frac{\langle y-x, a_{n,j}-x\rangle}{2s}} dS_r(y) dr.$$

Since the value of the spherical integral is invariant by rotations in $\mathbb{R}^N$, we can assume that $a_{n,j} - x = (0, 0, 0, ..., |a_{n,j} - x|$. We then use the spherical coordinates in $\mathbb{R}^N$ with center x and the representation of $S^{N-1} = \{(\sin\phi.\sigma, \phi) : \sigma \in S^{N-2}, \phi \in [0, \pi]\}$. With this representation $\langle y - x, a_{n,j} - x\rangle = |y - x||a_{n,j} - x| \cos\phi$. This yields

$$\int_{\partial B_r(x)} e^{\frac{\langle y-x, a_{n,j}-x\rangle}{2s}} dS_r(y) = r^{N-1}|S^{N-2} \int_0^\pi e^{q_\epsilon \frac{|a_{n,j}-x| r\cos\phi}{2s}} \sin^{N-2}\phi d\phi.$$

By Lemma 4.36

$$\begin{aligned}\int_{\partial B_r(x)} e^{-q_\epsilon \frac{\langle y-x, a_{n,j}-x\rangle}{2s}} dS_r(y) &\leq C \frac{r^{N-1} e^{q_\epsilon r \frac{r|a_{n,j}-x|}{2s}}}{\left(1 + \frac{r|a_{n,j}-x|}{s}\right)^{\frac{N-1}{2}}} \\ &\leq C s^{\frac{N-1}{2}} \left(\frac{r}{|a_{n,j} - x|}\right)^{\frac{N-1}{2}}.\end{aligned} \tag{4.111}$$

Therfore

$$\int_{\Gamma_p} e^{-\frac{q_\epsilon |y-a_{n,j}|^2}{4s}} dy \leq C t^{\frac{N-1}{4}} p^{\frac{N-3}{4}} \frac{s^{\frac{N-1}{2}} e^{-q_\epsilon \frac{(|a_{n,j}-x| - \sqrt{t(p+1)})^2}{4s}}}{|a_{n,j} - x|^{\frac{N-1}{2}}}. \tag{4.112}$$

Since $|a_{n,j} - x| \geq \sqrt{tn}$ we obtain

$$\begin{aligned}&\int_0^t \int_{\Gamma_p} (t-s)^{-\frac{N}{2}} s^{-\frac{q(N-1)+1}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} e^{-q_\epsilon \frac{\lambda_{n,j,y}^2}{4s}} dy ds \\ &\qquad \leq C \frac{\sqrt{t} p^{\frac{N-3}{4}}}{n^{\frac{N-1}{4}}} \int_0^t (t-s)^{-\frac{N}{2}} s^{-\frac{q(N-1)+1}{2}} e^{-\frac{p}{4(t-s)}} e^{-q_\epsilon \frac{(\sqrt{tn} - \sqrt{(p+1)t})^2}{4t}} ds \\ &\qquad \leq C \frac{t^{\frac{1-q(N-2)}{2}}}{n^{\frac{N-1}{4}}} \int_0^1 (1-\tau)^{-\frac{N}{2}} \tau^{-\frac{q(N-1)+1}{2}} e^{-\frac{p}{4(1-\tau)}} e^{-q_\epsilon \frac{(\sqrt{n} - \sqrt{p+1})^2}{4\tau}} d\tau.\end{aligned} \tag{4.113}$$

We apply Lemma 4.34 with $A = \sqrt{p}$, $B = \sqrt{q_\epsilon}(\sqrt{n} - \sqrt{p+1})$, $b = \frac{(q-1)(N-1)+1}{2}$, $a = \frac{N}{2}$ and $\kappa = \frac{\sqrt{q_\epsilon}(\ell-1)}{8}$, as in the case $N = 1$. For these specific values

$$\begin{aligned}A^{1-a} B^{1-b} (A+B)^{a+b-2} &= p^{\frac{2-N}{4}} \left(\sqrt{q_\epsilon}\left(\sqrt{n} - \sqrt{p+1}\right)\right)^{\frac{1-(q-1)(N-1)}{2}} \\ &\quad \times \left(\sqrt{p} + \sqrt{q_\epsilon}\left(\sqrt{n} - \sqrt{p+1}\right)\right)^{\frac{(q-1)(N-1)+N-3}{2}} \\ &\leq C \left(\frac{n}{p}\right)^{\frac{N}{4} - \frac{1}{2}} \left(\frac{\sqrt{n} - \sqrt{p}}{\sqrt{n}}\right)^{\frac{1-(q-1)(N-1)}{2}},\end{aligned}$$

where $C = C(N, q, \kappa) > 0$. Hence

$$\begin{aligned}
&\int_0^t \int_{\Gamma_p} (t-s)^{-\frac{N}{2}} s^{-\frac{N}{2}} e^{-\frac{q_\epsilon |x-y|^2}{4s}} dyds \\
&\quad \le C \frac{t^{\frac{1-q(N-1)}{2}} p^{\frac{N-3}{4}}}{n^{\frac{N-1}{4}}} \left(\frac{n}{p}\right)^{\frac{N-2}{4}} \left(\frac{\sqrt{n}-\sqrt{p}}{\sqrt{n}}\right)^{\frac{1-(q-1)(N-1)}{2}} e^{-\frac{(\sqrt{p}+\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1}))^2}{4}} \qquad (4.114) \\
&\quad \le C t^{\frac{1-q(N-1)}{2}} p^{-\frac{1}{4}} n^{\frac{(q-1)(N-1)-2}{4}} \left(\sqrt{n}-\sqrt{p}\right)^{\frac{1-(q-1(N-1))}{2}} e^{-\frac{(\sqrt{p}+\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1}))^2}{4}}.
\end{aligned}$$

Then we deduce from (4.110), (4.114)

$$\begin{aligned}
J_{2,\ell}^h \le C t^{1-\frac{Nq}{2}} \sum_{n=\ell+1}^{a_t} \sum_{j \in \Theta_{t,n}^h} n^{\frac{(q-1)(N-1)-2}{4}} \left(\mu_{n,j}(K_{n,j})\right)^q \\
\times \sum_{p=1}^{n-\ell} p^{\frac{2q-3}{4}} \left(\sqrt{n}-\sqrt{p}\right)^{\frac{1-(q-1)(N-1)}{2}} e^{-\frac{(\sqrt{p}+\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1}))^2}{4}}.
\end{aligned} \qquad (4.115)$$

By Lemma 4.35 with $\alpha = \frac{2q-3}{4}$, $\beta = \frac{1-(q-1(N-1)}{2}$, $\delta = \frac{1}{4}$ and $\gamma = q_\epsilon$, we obtain

$$\sum_{p=1}^{n-\ell} p^{\frac{2q-3}{4}} \left(\sqrt{n}-\sqrt{p}\right)^{\frac{1-(q-1)(N-1)}{2}} e^{-\frac{(\sqrt{p}+\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1}))^2}{4}} \le C n^{\frac{N(q-1)+q-3}{4}} e^{-\frac{n}{4}}, \qquad (4.116)$$

thus

$$J_{2,\ell}'^h \le C t^{1-\frac{Nq}{2}} \sum_{n=\ell+1}^{a_t} n^{\frac{N(q-1)}{2}-1} e^{-\frac{n}{4}} \sum_{j \in \Theta_{t,n}^h} \left(\mu_{n,j}(K_{n,j})\right)^q. \qquad (4.117)$$

Because

$$\mu_{n,j}(K_{n,j}) = cap_{\frac{2}{q},q'}^{B_{n,j}}(K_{n,j}),$$

we use the rescaling procedure of Lemma 4.19 except that the scaling factor is $\sqrt{(N+1)t}$ instead of $\sqrt{N+1}$, so that the sets $\widetilde{T}_n$, $\widetilde{K}_n$, $\widetilde{K}_{n,j}$ and $\widetilde{\mathcal{Q}}_n$ remain unchanged. Using the quasi-additivity and the fact that $J_{2,\ell}' = \sum_{h=1}^{J} J_{2,\ell}'^h$, we deduce

$$J_{2,\ell}' \le C t^{-\frac{N}{2}} \sum_{n=\ell+1} a_t d_{n+1}^{N-\frac{2}{q-1}} cap_{\frac{2}{q},q'}\left(\frac{K_n}{d_{n+1}}\right), \qquad (4.118)$$

which implies (4.96). □

The proof of Theorem 4.21 follows from the previous estimates on J_1 and J_2.

In the same way as for Theorem 4.8, the estimate in Theorem 4.21 admits an integral form. Fortunately it yields the same form as for Theorem 4.20

Theorem 4.28 *Assume $q \geq q_c$. Then there exists a positive constant $C_2^* = C_2^*(N, q, T$ such for any closed subset $F \subset \mathbb{R}^N$ there holds for all $(x,t) \in Q_T$,*

$$\underline{u}_F(x,t) \geq \frac{C_2^*}{t^{1+\frac{N}{2}}} \int_{\sqrt{t}}^{\sqrt{t(a_t+2}} e^{-\frac{s^2}{4t}} s^{N-\frac{2}{q-1}} cap_{\frac{2}{q},q'}\left(\frac{1}{s}B \cap B_1(x)\right) sds, \tag{4.119}$$

where a_t is the smallest integer j such that $F \subset B_{\sqrt{jt}}(x)$.

Proof. We distinguish according $q = q_c$, or $q > q_c$, and for simplicity we denote $B_r = B_r(x)$ for the various values of r.

Case 1: $q = q_c \Longleftrightarrow N - \frac{2}{q-1} = 0$. Because $F_n = F \cap (B_{d_{n+1}} \setminus B_{d_n})$ there holds

$$cap_{\frac{2}{q},q'}\left(\frac{F_n}{d_{n+1}}\right) \geq cap_{\frac{2}{q},q'}\left(\frac{F}{d_{n+1}} \cap B_1\right) - cap_{\frac{2}{q},q'}\left(\frac{F \cap B_{d_n}}{d_{n+1}}\right),$$

Furthermore, since $d_{n+1} \geq d_n$,

$$cap_{\frac{2}{q},q'}\left(\frac{F \cap B_{d_n}}{d_{n+1}}\right) = cap_{\frac{2}{q},q'}\left(\frac{d_n}{d_{n+1}}\frac{F \cap B_{d_n}}{d_n}\right) \leq cap_{\frac{2}{q},q'}\left(\frac{F}{d_n} \cap B_1\right),$$

thus

$$cap_{\frac{2}{q},q'}\left(\frac{F_n}{d_{n+1}}\right) \geq cap_{\frac{2}{q},q'}\left(\frac{F}{d_{n+1}} \cap B_1\right) - cap_{\frac{2}{q},q'}\left(\frac{F}{d_n} \cap B_1\right),$$

it follows

$$\begin{aligned}
\sum_{n=1}^{a_t} e^{-\frac{n}{4}} cap_{\frac{2}{q},q'}\left(\frac{F_n}{d_{n+1}}\right) &\geq \sum_{n=1}^{a_t} e^{-\frac{n}{4}} cap_{\frac{2}{q},q'}\left(\frac{F}{d_{n+1}} \cap B_1\right) - \sum_{n=1}^{a_t} e^{-\frac{n}{4}} cap_{\frac{2}{q},q'}\left(\frac{F}{d_n} \cap B_1\right) \\
&\geq \sum_{n=1}^{a_t} e^{-\frac{n}{4}} cap_{\frac{2}{q},q'}\left(\frac{F}{d_{n+1}} \cap B_1\right) - e^{-\frac{1}{4}} \sum_{n=0}^{a_t-1} e^{-\frac{n}{4}} cap_{\frac{2}{q},q'}\left(\frac{F}{d_{n+1}} \cap B_1\right) \\
&\geq (1 - e^{-\frac{1}{4}}) \sum_{n=1}^{a_t-1} e^{-\frac{n}{4}} cap_{\frac{2}{q},q'}\left(\frac{F}{d_{n+1}} \cap B_1\right) - e^{-\frac{1}{4}} cap_{\frac{2}{q},q'}\left(\frac{F}{\sqrt{t}} \cap B_1\right).
\end{aligned}$$

Since, by (4.66),

$$cap_{\frac{2}{q},q'}\left(\frac{F}{s'} \cap B_1\right) \geq cap_{\frac{2}{q},q'}\left(\frac{F}{d_{n+1}} \cap B_1\right) \geq cap_{\frac{2}{q},q'}\left(\frac{F}{s} \cap B_1\right),$$

for any $s' \in [d_{n+1}, d_{n+2}]$ and $s \in [d_n, d_{n+1}]$, there holds

$$\begin{aligned}
te^{-\frac{n}{4}} cap_{\frac{2}{q},q'}\left(\frac{F}{d_{n+1}} \cap B_1\right) &\geq cap_{\frac{2}{q},q'}\left(\frac{F}{d_{n+1}} \cap B_1\right) \int_{d_n}^{d_{n+1}} e^{-s^2/4t} s\, ds \\
&\geq \int_{d_n}^{d_{n+1}} e^{-s^2/4t} cap_{\frac{2}{q},q'}\left(\frac{F}{s} \cap B_1\right) s\, ds.
\end{aligned}$$

This implies

$$W_F(x,t) \geq (1 - e^{-\frac{1}{4}}) t^{-(1+\frac{N}{2})} \int_0^{\sqrt{ta_t}} e^{-s^2/4t} cap_{\frac{2}{q},q'}\left(\frac{F}{s} \cap B_1\right) s\, ds.$$

Case 2: $q > q_c \Longleftrightarrow N - \frac{2}{q-1} > 0$. In that case it follows from Lemma 4.6 that

$$cap_{\frac{2}{q},q'}\left(\frac{F_n}{d_{n+1}}\right) \approx d_{n+1}^{\frac{2}{q-1}-N} cap_{\frac{2}{q},q'}\left(F_n\right).$$

Thus

$$W_F(x,t) \approx t^{-1-\frac{N}{2}} \sum_{n=0}^{a_t} e^{-\frac{n}{4}} cap_{\frac{2}{q},q'}\left(F_n\right).$$

Since

$$cap_{\frac{2}{q},q'}\left(F_n\right) \geq cap_{\frac{2}{q},q'}\left(F \cap B_{d_{n+1}}\right) - cap_{\frac{2}{q},q'}\left(F \cap B_{d_n}\right),$$

we obtain, using again Abel's transform,

$$\begin{aligned} t^{-\frac{N}{2}} \sum_{n=0}^{a_t} e^{-\frac{n}{4}} cap_{\frac{2}{q},q'}\left(F_n\right) &\geq (1-e^{-\frac{1}{4}}) t^{-\frac{N}{2}} \sum_{n=0}^{a_t-1} e^{-\frac{n}{4}} cap_{\frac{2}{q},q'}\left(F \cap B_{d_{n+1}}\right) \\ &\geq (1-e^{-\frac{1}{4}}) t^{-(1+\frac{N}{2})} \int_0^{\sqrt{ta_t}} e^{-\frac{s^2}{4t}} cap_{\frac{2}{q},q'}\left(F \cap B_s\right) s\, ds. \end{aligned}$$

Because $cap_{\frac{2}{q},q'}\left(F \cap B_s\right) \approx s^{N-\frac{2}{q-1}} cap_{\frac{2}{q},q'}\left(s^{-1}F \cap B_1\right)$, (4.119) follows. □

4.4 Applications

The main result of this section is the following,

Theorem 4.29 *Let $N \geq 1$, $q > 1$ and F be a closed subset of $\mathbb{R}^N$. Then $\overline{u}_F = \underline{u}_F$.*

Proof. When $1 < q < q_c$ this is proved in Proposition 3.5. The principle of the proof uses convexity and the integral forms of Theorem 4.8 and Theorem 4.21. The technique is an adaptation that we recall for the sake of completeness of the proof in the subcritical case. By Theorem 4.20 and Theorem 4.28 there exists a positive constant C, depending on N, q and T such that

$$\underline{u}_F \leq \overline{u}_F \leq C\underline{u}_F \quad \text{in } Q_T. \tag{4.120}$$

Let us assume that $\overline{u}_F \neq \underline{u}_F$. By the strong maximum principle $\overline{u}_F > \underline{u}_F$. By convexity $\widetilde{u} = \overline{u}_F - \frac{1}{2C}(\overline{u}_F - \underline{u}_F)$ is a super-solution, which is smaller than $\underline{u}_F$. If we set $\theta := \left(\frac{1}{2} + \frac{1}{2C}\right)$, then $0 < \theta < 1$ and $\theta\underline{u}_F$ is a subsolution smaller than $\underline{u}_F$. There exists a solution u^* of (3.13) which satisfies

$$\theta\underline{u}_F \leq u^* \leq \widetilde{u} < \underline{u}_F \quad \text{in } Q_T.$$

Hence u^* is a solution of (4.28). If μ is an admissible measure vanishing outside F, then $u_{\theta\mu}$ is the smallest solution above the subsolution θu_μ. Thus $u_{\theta\mu} \leq u^* < \underline{u}_F$. Since μ is arbitrary, we deduce $\underline{u}_F \leq u^* < \underline{u}_F$, which is a contradiction. □

Another consequence of the uniqueness result is the following equivalence of the discrete and integral capacitary potentials.

Proposition 4.30 *Assume $q \geq q_c$. Then there exist two positive constants $C_1^\dagger$, $C_2^\dagger$, depending only on N, q and T such that*

$$\begin{aligned} C_2^\dagger t^{-(1+\frac{N}{2})} \int_0^{\sqrt{ta_t}} s^{N-\frac{2}{q-1}} e^{-\frac{s^2}{4t}} cap_{\frac{2}{q},q'}\left(\frac{F}{s} \cap B_1(x)\right) s\, ds \leq W_F(x,t) \\ \leq C_1^\dagger t^{-(1+\frac{N}{2})} \int_{\sqrt{t}}^{\sqrt{t(a_t+2)}} s^{N-\frac{2}{q-1}} e^{-\frac{s^2}{4t}} cap_{\frac{2}{q},q'}\left(\frac{F}{s} \cap B_1(x)\right) s\, ds \end{aligned} \tag{4.121}$$

for any $(x,t) \in Q_T$.

Definition 4.31 *If F is a closed subset of $\mathbb{R}^N$, we define the $(\frac{2}{q}, q')$-integral parabolic capacitary potential $\mathcal{W}_F$ by*

$$\mathcal{W}_F(x,t) = t^{-1-\frac{N}{2}} \int_0^{D_F(x)} s^{N-\frac{2}{q-1}} e^{-\frac{s^2}{4t}} cap_{\frac{2}{q},q'}\left(\frac{F}{s} \cap B_1(x)\right) s\, ds \quad \forall (x,t) \in Q_\infty, \tag{4.122}$$

where $D_F(x) = \max\{|x-y| : y \in F\}$.

By an easy computation we obtain that

$$\begin{aligned} 0 \leq \mathcal{W}_F(x,t) - t^{-(1+\frac{N}{2})} \int_0^{\sqrt{ta_t}} s^{N-\frac{2}{q-1}} e^{-\frac{s^2}{4t}} cap_{\frac{2}{q},q'}\left(\frac{F}{s} \cap B_1(x)\right) s\, ds \\ \leq C \frac{t^{(q-3)/2(q-1)}}{D_F(x)} e^{-\frac{D_F^2(x)}{4t}}, \end{aligned} \tag{4.123}$$

and

$$\begin{aligned} 0 \leq t^{-(1+\frac{N}{2})} \int_0^{\sqrt{t(a_t+2)}} s^{N-\frac{2}{q-1}} e^{-\frac{s^2}{4t}} cap_{\frac{2}{q},q'}\left(\frac{F}{s} \cap B_1(x)\right) s\, ds - \mathcal{W}_F(x,t) \\ \leq C \frac{t^{(q-3)/2(q-1)}}{D_F(x)} e^{-\frac{D_F^2(x)}{4t}}, \end{aligned} \tag{4.124}$$

for some $C = C(N,q) > 0$. Furthermore

$$\mathcal{W}_F(x,t) = t^{-\frac{1}{q-1}} \int_0^{D_F(x)/\sqrt{t}} s^{N-\frac{2}{q-1}} e^{-\frac{s^2}{4t}} cap_{\frac{2}{q},q'}\left(\frac{F}{s\sqrt{t}} \cap B_1(x)\right) s\, ds. \tag{4.125}$$

The following result gives a sufficient condition in order that $\overline{u}_F$ has a strong blow-up (i.e. of the maximal order $t^{-1/(q-1)}$) at a point x.

Proposition 4.32 *Assume $q \geq q_c$ and F is a closed subset of $\mathbb{R}^N$. If there exists $\gamma \in [0,\infty)$ such that*

$$\lim_{\tau \to 0} cap_{\frac{2}{q},q'}\left(\frac{F}{\tau} \cap B_1(x)\right) = \gamma, \tag{4.126}$$

then

$$\lim_{t \to 0} t^{\frac{1}{q-1}} \overline{u}_F(x,t) = C\gamma, \tag{4.127}$$

for some $C = C(N,q) > 0$.

Proof. Clearly, condition (4.126) implies

$$\lim_{t\to 0} cap_{\frac{2}{q},q'}\left(\frac{F}{\sqrt{t}}\cap B_1(x)\right)=\gamma$$

for any $s>0$. Then (4.127) follows by Lebesgue's theorem. Notice also that the set of γ is bounded from above by a constant depending on N and q. □

In the next result we give a condition in order that the solution remains bounded at a point x. The proof is similar to the previous one.

Proposition 4.33 *Assume $q\ge q_c$ and F is a closed subset of $\mathbb{R}^N$. If*

$$\limsup_{\tau\to 0}\tau^{-\frac{2}{q-1}} cap_{\frac{2}{q},q'}\left(\frac{F}{\tau}\cap B_1(x)\right)<\infty, \tag{4.128}$$

then $\overline{u}_F(x,t)$ remains bounded when $t\to 0$.

Remark. If we assume that f is a convex function on $\mathbb{R}^+$ satisfying

$$c_2 r^q\le f(r)\le c_1 r^q \qquad \forall r\ge 0, \tag{4.129}$$

for some $0<c_2\le c_1$ we can construct in the same way as for (3.13) the solutions $\underline{u}_F$ and $\overline{u}_F$ for equation

$$\partial_t u-\Delta u+f(u)=0 \qquad \text{in } Q_T. \tag{4.130}$$

The bilateral estimate estimate (4.120) is still valid (up to change of the C_i). Since only convexity of f is used in the proof of Theorem 4.29, there still holds $\underline{u}_F=\overline{u}_F$. Similar extensions of Proposition 4.32 and Proposition 4.33 are also clear.

4.5 Appendix

We present here some highly technical computations which are not of particularly interest for the trace theory but are usefull in the proof of the results.

4.5.1 Generalized beta integrals

Lemma 4.34 *Let a and b be two real numbers, $a>0$ and $\kappa>0$. Then there exists a constant $C=C(a,b,\kappa)>0$ such that for any $A>0$, $B>\kappa/A$ there holds*

$$\int_0^1 (1-x)^{-a}x^{-b}e^{-A^2/4(1-x)}e^{-B^2/4x}dx\le Ce^{-(A+B)^2/4}A^{1-a}B^{1-b}(A+B)^{a+b-2}. \tag{4.131}$$

Proof. We first notice that

$$\max\left\{e^{-A^2/4(1-x)}e^{-B^2/4x}:0\le x\le 1\right\}=e^{-(A+B)^2/4}, \tag{4.132}$$

and it is achieved for $x_0=B/(A+B)$. Set $\phi(x)=(1-x)^{-a}x^{-b}e^{-A^2/4(1-x)}e^{-B^2/4x}$, thus

$$\int_0^1\phi(x)dx=\int_0^{x_0}\phi(x)dx+\int_{x_0}^1\phi(x)dx=I_{a,b}+J_{a,b}.$$

Put

$$u = \frac{A^2}{4(1-x)} + \frac{B^2}{4x}, \tag{4.133}$$

then

$$4ux^2 - (4u + B^2 - A^2)x + B^2 = 0. \tag{4.134}$$

If $0 < x < x_0$ this equation admits the solution

$$x = x(u) = \frac{1}{8u}\left(4u + B^2 - A^2 - \sqrt{16u^2 - 8u(A^2+B^2) + (A^2-B^2)^2}\right)$$

$$\int_0^{x_0} (1-x)^{-a} x^{-b} e^{-A^2/4(1-x) - B^2/4x} dx = -\int_{(A+B)^2/4}^{\infty} (1-x(u))^{-a} x(u)^{-b} e^{-u} x'(u) du$$

Putting $x' = x'(u)$ and differentiating (4.134),

$$4x^2 + 8uxx' - (4u + B^2 - A^2)x' - 4x = 0 \Longrightarrow -x' = \frac{4x(1-x)}{4u + B^2 - A^2 - 8ux}.$$

Thus

$$\int_0^{x_0} \phi(x) dx = 4\int_{(A+B)^2/4}^{\infty} \frac{(1-x(u))^{-a+1} x(u)^{-b+1} e^{-u} du}{4u + B^2 - A^2 - 8ux(u)}. \tag{4.135}$$

Using the explicit value of the root $x(u)$, we finally get

$$\int_0^{x_0} \phi(x) dx = 4\int_{(A+B)^2/4}^{\infty} \frac{(1-x(u))^{-a+1} x(u)^{-b+1} e^{-u} du}{\sqrt{16u^2 - 8u(A^2+B^2) + (A^2-B^2)^2}}, \tag{4.136}$$

and the factorization below holds

$$16u^2 - 8u(A^2+B^2) + (A^2-B^2)^2 = 16(u - (A+B)^2/4)(u - (A-B)^2/4).$$

We set $u = v + (A+B)^2/4$ and obtain

$$x(u) = \frac{v + (AB + B^2)/2 - \sqrt{v(v+AB)}}{2\left(v + (A+B)^2/4\right)},$$

and

$$1 - x(u) = \frac{v + (A^2 + AB)/2 + \sqrt{v(v+AB)}}{2\left(v + (A+B)^2/4\right)}.$$

We introduce the relation $\approx$ linking two positive quantities depending on A and B. It means that the two sided-inequalities up to multiplicative constants independent of A and B. Therefore

$$\int_0^{x_0} \phi(x) dx = 2^{a-b-4} e^{-(A+B)^2/4} \int_0^{\infty} \widetilde{\phi}(v) dv \quad \text{where}$$

$$\widetilde{\phi}(v) = \frac{\left(v + (AB+B^2)/2 - \sqrt{v(v+AB)}\right)^{1-b} \left(v + (A^2+AB)/2 + \sqrt{v(v+AB)}\right)^{1-a}}{e^v \left(v + (A+B)^2/4\right)^{2-a-b} \sqrt{v(v+AB)}}. \tag{4.137}$$

Case 1: $a \geq 1$, $b \geq 1$. First

$$\frac{\big(v+(A+B)^2/4\big)^{a+b-2}}{\sqrt{v(v+AB)}} \leq \frac{\big(v+(A+B)^2/4\big)^{a+b-2}}{\sqrt{v(v+\kappa)}} \approx \frac{\big(v+(A+B)^2\big)^{a+b-2}}{\sqrt{v(v+\kappa)}} \tag{4.138}$$

since $a+b-2 \geq 0$ and $AB \geq \kappa$. Next

$$\Big(v+(A^2+AB)/2+\sqrt{v(v+AB)}\Big)^{1-a} \approx (v+A(A+B))^{1-a}\,. \tag{4.139}$$

Furthermore

$$\begin{aligned} v+(AB+B^2)/2-\sqrt{v(v+AB)} &= B^2\frac{v+(A+B)^2/4}{v+B(A+B)/2+\sqrt{v(v+AB)}} \\ &\approx B^2\frac{v+(A+B)^2}{v+B(A+B)}. \end{aligned} \tag{4.140}$$

Then

$$\Big(v+(AB+B^2)/2-\sqrt{v(v+AB)}\Big)^{1-b} \approx B^{2-2b}\left(\frac{v+B(A+B)}{v+(A+B)^2}\right)^{b-1}. \tag{4.141}$$

It follows

$$\begin{aligned} \widetilde{\phi}(v) &\leq CB^{2-2b}\left(\frac{v+(A+B)^2}{v+A(A+B)}\right)^{a-1}\frac{(v+B(A+B))^{b-1}}{\sqrt{v(v+\kappa)}} \\ &\leq CB^{2-2b}\left(\frac{v+(A+B)^2}{v+A(A+B)}\right)^{a-1}\frac{v^{b-1}+(B^2+AB)^{b-1}}{\sqrt{v(v+\kappa)}}, \end{aligned} \tag{4.142}$$

where C depends on a, b and κ. The function $v \mapsto (v+(A+B)^2)/(v+A(A+B))$ is decreasing on $(0,\infty)$. If we set

$$C_1 = \int_0^\infty \frac{v^{b-1}e^{-v}dv}{\sqrt{v(v+\kappa)}} \quad \text{and} \quad C_2 = \int_0^\infty \frac{e^{-v}dv}{\sqrt{v(v+\kappa)}},$$

then

$$C_1 \leq K(B^2+AB)^{b-1}C_2$$

with $K = C_1\kappa^{1-b}/C_2$. Therefore

$$\int_0^{x_0}\phi(x)dx \leq Ce^{-(A+B)^2/4}B^{1-b}A^{1-a}(A+B)^{a+b-2}. \tag{4.143}$$

The estimate of $J_{a,b}$ is obtained by exchanging (A,a) with (B,b) and replacing x by $1-x$. *Mutadis mutandis*, this leads directely to the same expression as in 4.143 and finally

$$\int_0^{1}\phi(x)dx \leq Ce^{-(A+B)^2/4}A^{1-a}B^{1-b}(A+B)^{a+b-2}. \tag{4.144}$$

Case 2: $a \geq 1$, $b < 1$. Estimates (4.137), (4.138), (4.139), (4.140) and (4.141) are valid. Because $v \mapsto (v + B(A+B))^{b-1}$ is decreasing, (4.142) has to be replaced by

$$\widetilde{\phi}(v) \leq CB^{2-2b} \left(\frac{v + (A+B)^2}{v + A(A+B)} \right)^{a-1} \frac{\left(AB + B^2\right)^{b-1}}{\sqrt{v(v+\kappa)}}. \tag{4.145}$$

This implies (4.143) directly. The estimate of $J_{a,b}$ is performed by the change of variable $x \mapsto 1 - x$. If $x_1 = 1 - x_0$, there holds

$$J_{a,b} = \int_0^{x_1} x^{-a}(1-x)^{-b} e^{-A^2/4x} e^{-B^2/4(1-x)} dx = \int_0^{x_1} \Psi(x) dx.$$

Then

$$\int_0^{x_1} \Psi(x) dx = 2^{b-a-4} e^{-(A+B)^2/4} \int_0^{x_1} \widetilde{\Psi}(v) dv \quad \text{where}$$

$$\widetilde{\Psi}(v) = \frac{\left(v + (AB + A^2)/2 - \sqrt{v(v+AB)}\right)^{1-a} \left(v + (B^2 + AB)/2 + \sqrt{v(v+AB)}\right)^{1-b}}{e^v \left(v + (A+B)^2/4\right)^{2-a-b} \sqrt{v(v+AB)}}. \tag{4.146}$$

Equivalence (4.138) is unchanged; (4.139) is replaced by

$$\left(v + (B^2 + AB)/2 + \sqrt{v(v+AB)}\right)^{1-b} \approx (v + B(A+B))^{1-b}, \tag{4.147}$$

(4.140) by

$$v + (AB + A^2)/2 - \sqrt{v(v+AB)} \approx A^2 \frac{v + (A+B)^2}{v + A(A+B)}, \tag{4.148}$$

and (4.141) by

$$\left(v + (AB + A^2)/2 - \sqrt{v(v+AB)}\right)^{1-a} \approx A^{2-2a} \left(\frac{v + A(A+B)}{v + (A+B)^2} \right)^{a-1}. \tag{4.149}$$

Because $a > 1$, (4.142) turns into

$$\begin{aligned} \widetilde{\Psi}(v) &\leq CA^{2-2b}(v + (A+B)^2)^{b-1} \frac{(v + A^2 + AB)^{a-1}(v + B^2 + AB)^{1-b}}{\sqrt{v(v+\kappa)}} \\ &\leq Ce^{-(A+B)^2/4} A^{2-2b}(A+B)^{2b-2} \\ &\qquad \times \frac{v^{a-b} + (A^2 + AB)^{a-1} v^{1-b} + (B^2 + AB)^{1-b} v^{a-1} + A^{a-1} B^{1-b} (A+B)^{a-b}}{\sqrt{v(v+\kappa)}}. \end{aligned} \tag{4.150}$$

Because $AB \geq \kappa$, there exists a positive constant C, depending on κ, such that

$$\begin{aligned} \int_0^\infty \frac{v^{a-b} + (A^2 + AB)^{a-1} v^{1-b} + (B^2 + AB)^{1-b} v^{a-1}}{\sqrt{v(v+\kappa)}} e^{-v} dv \\ \leq CA^{a-1} B^{1-b} (A+B)^{a-b} \int_0^\infty \frac{e^{-v} dv}{\sqrt{v(v+\kappa)}}. \end{aligned} \tag{4.151}$$

Combining (4.150) and (4.151) leads to

$$\int_0^{x_1} \Psi(x)dx \leq Ce^{-(A+B)^2/4} A^{1-a} B^{1-b} (A+B)^{a+b-2}. \tag{4.152}$$

This, again, implies that (4.131) holds.

Case 3: $\max\{a,b\} < 1$. Inequalities (4.137)-(4.141) hold, but (4.142) has to be replaced by

$$\begin{aligned} \widetilde{\phi}(v) &\leq CB^{2-2b} \left(\frac{v + (A+B)^2}{v + A(A+B)} \right)^{a-1} \frac{\left(v + B^2 + AB\right)^{b-1}}{\sqrt{v(v+\kappa)}} \\ &\leq CB^{1-b}(A+B)^{2a+b-3} \frac{v^{1-a} + \left(A^2 + AB\right)^{1-a}}{\sqrt{v(v+\kappa)}} \end{aligned} \tag{4.153}$$

Noticing that

$$\int_0^\infty \frac{v^{1-a} e^{-v} dv}{\sqrt{v(v+\kappa)}} \leq C \left(A^2 + AB\right)^{1-a} \int_0^\infty \frac{e^{-v} dv}{\sqrt{v(v+\kappa)}},$$

it follows that (4.143) holds. Finally (4.144) holds by exchanging (A,a) and (B,b). □

4.5.2 Discrete generalized beta series

Lemma 4.35 . *Let α, β, γ, δ be real numbers and ℓ an integer. We assume $\gamma > 1$, $\delta > 0$ and $\ell \geq 2$. Then there exists a positive constant C such that, for any integer $n > \ell$*

$$\sum_{p=1}^{n-\ell} p^\alpha (\sqrt{n} - \sqrt{p}\,)^\beta e^{-\delta(\sqrt{p} + \sqrt{\gamma}(\sqrt{n} - \sqrt{p+1}))^2} \leq Cn^{\alpha - \beta/2} e^{-\delta n}. \tag{4.154}$$

Proof. The function $x \mapsto (\sqrt{x} + \sqrt{\gamma}(\sqrt{n} - \sqrt{x+1}))^2$ is decreasing on $[(\gamma-1)^{-1}, \infty)$. Furthermore there exists $C > 0$ depending on ℓ, α and β such that $p^\alpha(\sqrt{n} - \sqrt{p}\,)^\beta \leq Cx^\alpha(\sqrt{n} - \sqrt{x+1}\,)^\beta$ for $x \in [p, p+1]$. If we denote by p_0 the smallest integer larger than $(\gamma-1)^{-1}$, we derive

$$\begin{aligned} S &= \sum_{p=1}^{n-\ell} p^\alpha (\sqrt{n} - \sqrt{p}\,)^\beta e^{-(\sqrt{p} + \sqrt{\gamma}(\sqrt{n} - \sqrt{p+1}))^2/4} \\ &= \sum_{p=1}^{p_0-1} + \sum_{p_0}^{n-\ell} p^\alpha (\sqrt{n} - \sqrt{p}\,)^\beta e^{-\delta(\sqrt{p} + \sqrt{\gamma}(\sqrt{n} - \sqrt{p+1}))^2} \\ &\leq \sum_{p=1}^{p_0-1} p^\alpha (\sqrt{n} - \sqrt{p}\,)^\beta e^{-\delta(\sqrt{p} + \sqrt{\gamma}(\sqrt{n} - \sqrt{p+1}))^2} \\ &\qquad + C \int_{p_0}^{n+1-\ell} x^\alpha (\sqrt{n} - \sqrt{x}\,)^\beta e^{-\delta(\sqrt{x} + \sqrt{\gamma}(\sqrt{n} - \sqrt{x+1}))^2} dx, \end{aligned}$$

(notice that $\sqrt{n} - \sqrt{x} \approx \sqrt{n} - \sqrt{x+1}$ for $x \leq n - \ell$). Clearly

$$\sum_{p=1}^{p_0-1} p^\alpha (\sqrt{n} - \sqrt{p}\,)^\beta e^{-\delta(\sqrt{p} + \sqrt{\gamma}(\sqrt{n} - \sqrt{p+1}))^2} \leq C_0 n^\alpha (\sqrt{n} - \sqrt{n-\ell}\,)^\beta e^{-\delta n}, \tag{4.155}$$

for some C_0 independent of n. We set $y = y(x) = \sqrt{x+1} - \sqrt{x}/\sqrt{\gamma}$. Obviously

$$y'(x) = \frac{1}{2}\left(\frac{1}{\sqrt{x+1}} - \frac{1}{\sqrt{\gamma}\sqrt{x}}\right) \qquad \forall x \geq p_0.$$

Thus their exists $\epsilon = \epsilon(\delta, \gamma) > 0$ such that $\sqrt{2}\sqrt{x} \geq y(x) \geq \epsilon\sqrt{x}$ and $y'(x) \geq \epsilon/\sqrt{x}$. Moreover

$$\begin{aligned}
\sqrt{x} &= \frac{\sqrt{\gamma}\left(y + \sqrt{\gamma y^2 + 1 - \gamma}\right)}{\gamma - 1}, \\
\sqrt{n} - \sqrt{x} &= \frac{\sqrt{n}(\gamma-1) - \sqrt{\gamma}y - \sqrt{\gamma}\sqrt{\gamma y^2 + 1 - \gamma}}{\gamma - 1} \\
&= \frac{n(\gamma-1) + \gamma - 2y\sqrt{\gamma n} - \gamma y^2}{\sqrt{n}(\gamma-1) - \sqrt{\gamma}y + \sqrt{\gamma}\sqrt{\gamma y^2 + 1 - \gamma}} \\
&\approx \frac{n(\gamma-1) + \gamma - 2y\sqrt{\gamma n} - \gamma y^2}{\sqrt{n}}
\end{aligned}$$

since $y(x) \leq \sqrt{n}$. Furthermore

$$\begin{aligned}
n(\gamma-1) + \gamma - 2y\sqrt{\gamma n} - \gamma y^2 &= \gamma(\sqrt{n+1} + \sqrt{n}/\sqrt{\gamma} + y)(\sqrt{n+1} - \sqrt{n}/\sqrt{\gamma} - y) \\
&\approx \sqrt{n}(\sqrt{n+1} - \sqrt{n}/\sqrt{\gamma} - y),
\end{aligned}$$

because y ranges between $\sqrt{n+2-\ell} - \sqrt{n+1-\ell}\sqrt{\gamma} \approx \sqrt{n}$ and $\sqrt{p_0+1} - \sqrt{p_0}\sqrt{\gamma}$. Thus

$$(\sqrt{n} - \sqrt{x}\,)^\beta \approx \left(\sqrt{n+1} - \sqrt{n}/\sqrt{\gamma} - y\right)^\beta.$$

This implies

$$\begin{aligned}
&\int_{p_0}^{n+1-\ell} x^\alpha (\sqrt{n} - \sqrt{x}\,)^\beta e^{-\delta(\sqrt{x} + \gamma(\sqrt{n} - \sqrt{x+1}))^2} dx \\
&\qquad \leq C \int_{y(p_0)}^{y(n+1-\ell)} y^{2\alpha+1} \left(\sqrt{n+1} - \sqrt{n}/\sqrt{\gamma} - y\right)^\beta e^{-\gamma\delta(\sqrt{n}-y)^2} dy \\
&\qquad \leq C n^{\alpha+\beta/2+1} \int_{1-y(n+1-\ell)/\sqrt{n}}^{1-y(p_0)/\sqrt{n}} (1-z)^{2\alpha+1} (z + \sqrt{1+1/n} - 1 - 1/\sqrt{\gamma})^\beta e^{-\gamma\delta n z^2} dz.
\end{aligned} \tag{4.156}$$

Moreover

$$\begin{aligned}
1 - \frac{y(p_0)}{\sqrt{n}} &= 1 - \frac{1}{\sqrt{n}}\left(\sqrt{p_0+1} - \frac{\sqrt{p_0}}{\sqrt{\gamma}}\right), \\
1 - \frac{y(n-\ell+1)}{\sqrt{n}} &= 1 - \frac{\sqrt{n-\ell+2}}{\sqrt{n}} + \frac{\sqrt{n-\ell+1}}{\sqrt{n\gamma}} \\
&= \frac{1}{\sqrt{\gamma}}\left(1 + \frac{\sqrt{\gamma}\,(\ell-2) - \ell + 1}{2n} + \frac{\sqrt{\gamma}\,(\ell-2)^2 - (\ell-1)^2}{8n^2}\right) + O(n^{-3}).
\end{aligned} \tag{4.157}$$

Let θ fixed such that $1-\dfrac{y(n-\ell+1)}{\sqrt{n}}<\theta<1-\dfrac{y(p_0)}{\sqrt{n}}$ for any $n>p_0$. Then

$$\begin{aligned}\int_\theta^{1-y(p_0)/\sqrt{n}}(1-z)^{2\alpha+1}(z+\sqrt{1+1/n}-1-1/\sqrt{\gamma})^\beta e^{-\gamma\delta n z^2}dz&\\ \le C_\theta\int_\theta^{1-y(p_0)/\sqrt{n}}(1-z)^{2\alpha+1}e^{-\gamma\delta n z^2}dz&\\ \le C_\theta\, e^{-\gamma\delta n\theta^2}\int_\theta^{1-y(p_0)/\sqrt{n}}(1-z)^{2\alpha+1}dz&\\ \le C\, e^{-\gamma\delta n\theta^2}\max\{1,n^{-\alpha-1/2}\}.&\end{aligned}$$

Because $\gamma\theta^2>1$ we derive

$$\int_\theta^{1-y(p_0)/\sqrt{n}}(1-z)^{2\alpha+1}(z+\sqrt{1+1/n}-1-1/\sqrt{\gamma})^\beta e^{-\gamma\delta n z^2}dz\le Cn^{-\beta}e^{-\delta n},\qquad(4.158)$$

for some constant $C>0$. On the other hand

$$\begin{aligned}\int_{1-y(n+1-\ell)/\sqrt{n}}^{\theta}(1-z)^{2\alpha+1}(z+\sqrt{1+1/n}-1-1/\sqrt{\gamma})^\beta e^{-\gamma\delta n z^2}dz&\\ \le C'_\theta\int_{1-y(n+1-\ell)/\sqrt{n}}^{\theta}(z+\sqrt{1+1/n}-1-1/\sqrt{\gamma})^\beta e^{-\gamma\delta n z^2}dz.&\end{aligned}$$

The minimum of $z\mapsto(z+\sqrt{1+1/n}-1-1/\sqrt{\gamma})^\beta$ is achieved at $1-y(n+1-\ell)$ with value

$$\frac{\sqrt{\gamma}(\ell+1)+1-\ell}{2n\sqrt{\gamma}}+O(n^{-2}),$$

and the maximum of the exponential term is achieved at the same point with value

$$e^{-n\delta+((\ell-2)\sqrt{\gamma}+1-\ell)/2}(1+\circ(1))=C_\gamma e^{-n\delta}(1+\circ(1)).$$

We denote

$$z_{\gamma,n}=1+1/\sqrt{\gamma}-\sqrt{1+1/n}\quad\text{and}\quad I_\beta=\int_{1-y(n+1-\ell)/\sqrt{n}}^{\theta}(z-z_{\gamma,n})^\beta e^{-\gamma\delta n z^2}dz.$$

Since $1-y(n+1-\ell)\ge 1/\sqrt{2\gamma}$ for n large enough,

$$\begin{aligned}I_\beta&\le\sqrt{2\gamma}\int_{1-y(n+1-\ell)/\sqrt{n}}^{\theta}(z-z_{\gamma,n})^\beta z e^{-\gamma\delta n z^2}dz\\ &\le\frac{-\sqrt{2\gamma}}{2n\gamma\delta}\left[(z-z_{\gamma,n})^\beta e^{-\gamma\delta n z^2}\right]_{1-y(n+1-\ell)/\sqrt{n}}^{\theta}\\ &\qquad+\frac{\beta\sqrt{2\gamma}}{2n\gamma\delta}\int_{1-y(n+1-\ell)/\sqrt{n}}^{\theta}(z-z_{\gamma,n})^{\beta-1}z e^{-\gamma\delta n z^2}dz.\end{aligned}$$

But $1 - y(n+1-\ell)/\sqrt{n} - z_{\gamma,n} = (\ell-1)(1-1/\sqrt{\gamma})/2n$, therefore

$$I_\beta \le C_1 n^{-\beta-1} e^{-\delta n} + \beta C_1' n^{-1} I_{\beta-1}. \tag{4.159}$$

If $\beta \le 0$, we derive

$$I_\beta \le C_1 n^{-\beta-1} e^{-\delta n},$$

which inequality, combined with (4.156) and (4.158), yields to (4.154). If $\beta > 0$, we iterate and get

$$I_\beta \le C_1 n^{-\beta-1} e^{-\delta n} + C_1' n^{-1}(C_1 n^{-\beta} e^{-\delta n} + (\beta-1) C_1' n^{-1} I_{\beta-2}).$$

If $\beta - 1 \le 0$ we derive

$$I_\beta \le C_1 n^{-\beta-1} e^{-\delta n} + C_1 C_1' n^{-1-\beta} e^{-\delta n} = C_2 n^{-\beta-1} e^{-\delta n},$$

which again yields to (4.154). If $\beta - 1 > 0$, we continue up we find a positive integer k such that $\beta - k \le 0$, which again leads to

$$I_\beta \le C_k n^{-\beta-1} e^{-\delta n},$$

and finally to (4.154). □

4.5.3 Generalised Wallis integrals

Lemma 4.36 *For any integer $N \ge 2$ there exists a constant $c_N > 0$ such that*

$$\int_0^\pi e^{m\cos\theta} \sin^{N-2}\theta \, d\theta \le c_N \frac{e^m}{(1+m)^{(N-1)/2}} \qquad \forall m > 0. \tag{4.160}$$

Proof. Put $\mathcal{I}_N(m) = \displaystyle\int_0^\pi e^{m\cos\theta} \sin^{N-2}\theta \, d\theta$. Then $\mathcal{I}_2'(m) = \displaystyle\int_0^\pi e^{m\cos\theta}\cos\theta \, d\theta$ and

$$\begin{aligned}
\mathcal{I}_2''(m) &= \int_0^\pi e^{m\cos\theta}\cos^2\theta \, d\theta = \mathcal{I}_2(m) - \int_0^\pi e^{m\cos\theta}\sin^2\theta \, d\theta \\
&= \mathcal{I}_2(m) - \frac{1}{m}\int_0^\pi e^{m\cos\theta}\cos\theta \, d\theta \\
&= \mathcal{I}_2(m) - \frac{1}{m}\mathcal{I}_2'(m).
\end{aligned}$$

Thus $\mathcal{I}_2$ satisfies a Bessel equation of order 0. Since $\mathcal{I}_2(0) = \pi$ and $\mathcal{I}_2'(0) = 0$, $\pi^{-1}\mathcal{I}_2$ is the modified Bessel function of index 0 (usually denoted by I_0) the asymptotic behaviour of which is well known, thus (4.160) holds. If $N = 3$

$$\mathcal{I}_3(m) = \int_0^\pi e^{m\cos\theta}\sin\theta \, d\theta = \left[\frac{-e^{m\cos\theta}}{m}\right]_0^\pi = \frac{2\sinh m}{m}.$$

For $N > 3$ arbitrary

$$\mathcal{I}_N(m) = \int_0^\pi \frac{-1}{m}\frac{d}{d\theta}(e^{m\cos\theta})\sin^{N-3}\theta \, d\theta = \frac{N-3}{m}\int_0^\pi e^{m\cos\theta}\cos\theta\sin^{N-4}\theta \, d\theta. \tag{4.161}$$

Therefore,

$$\mathcal{I}_4(m) = \frac{1}{m}\int_0^\pi e^{m\cos\theta}\cos\theta\, d\theta = \mathcal{I}_2'(m),$$

and, again (4.160) holds since $I_0'(m)$ has the same behaviour as $I_0(m)$ at infinity. For $N \geq 5$

$$\mathcal{I}_N(m) = \frac{3-N}{m^2}\left[e^{m\cos\theta}\cos\theta\sin^{N-5}\theta\right]_0^\pi + \frac{N-3}{m^2}\int_0^\pi e^{m\cos\theta}\frac{d}{d\theta}\left(\cos\theta\sin^{N-5}\theta\right)d\theta.$$

Differentiating $\cos\theta\sin^{N-5}\theta$ and using (4.161), we obtain

$$\mathcal{I}_5(m) = \frac{4\sinh m}{m^2} - \frac{4\sinh m}{m^3},$$

while

$$\mathcal{I}_N(m) = \frac{(N-3)(N-5)}{m^2}\left(\mathcal{I}_{N-4}(m) - \mathcal{I}_{N-2}(m)\right), \tag{4.162}$$

for $N \geq 6$. Since the estimate (4.160) for $\mathcal{I}_2$, $\mathcal{I}_3$, $\mathcal{I}_4$ and $\mathcal{I}_5$ has already been obtained, a straigthforward induction implies the general result. □

Remark. Although it does not has any importance for our use, it must be noticed that $\mathcal{I}_N$ can be expressed either with hyperbolic functions if N is odd, or with Bessel functions if N is even.

5 The precise trace

In the supercritical case $q \geq q_c$, Theorem 3.15 has pointed out the necessity to introduce a finer definition of the initial trace which could distinguish among solutions of (3.13) which have the same initial trace in the sense defined previously.

5.1 Lattice structure of the set of positive solutions of (3.13)

The idea of analysing the algebraic structure of the set of positive solutions of the semilinear elliptic equation (1.16) is due to Dynkin [26]. It was intensively used by Marcus and Véron [46] in the construction of the precise boundary trace for such equations.

Definition 5.1 *We denote by $\mathcal{U}_+(Q_T)$ the set of nonnegative solutions of (3.13). All the elements of $\mathcal{U}_+(Q_T)$ belong to $C^{2,1}(Q_T)$.*

By a subsolution (resp. supersolution) of (3.13) in Q_T we mean a function $u \in L^q_{loc}$ satisfying

$$\iint_{Q_T}\left(-(\partial_t\zeta + \Delta\zeta)u + |u|^{q-1}u\zeta\right)dxdt \leq 0 \quad (\text{resp. } \geq 0) \text{ for all } \zeta \in C_c^\infty(Q_T)\,,\ \zeta \geq 0. \tag{5.1}$$

Lemma 5.2 *Let u be a subsolution of (3.13) in Q_T, then*

$$|u(x,t)| \le \left(\frac{1}{t(q-1)}\right)^{\frac{1}{q-1}} \quad \text{for almost all } (x,t) \in Q_T. \tag{5.2}$$

Proof. Because of Kato's inequality, the function $|u|$ is a subsolution of (3.13). Hence we can assume that u is nonnegative. Let $\{\rho_{\epsilon_n}\}$ ($\epsilon_n > 0$) be a sequence of $C_c^\infty(\mathbb{R}^{N+1})$ nonnegative functions with support in B_{ϵ_n} and total mass equal to 1. We assume that $\epsilon_n \to 0$, hence $\rho_{\epsilon_n} \to \delta_0$ is the sense of distributrions. Such a sequence is called a sequence of mollifiers. If $\epsilon_n < \epsilon$ the distribution $u_n := u * \rho_{\epsilon_n}$ is well defined and is C^∞ in $\mathbb{R}^N \times (\epsilon, T)$ where, by convexity, it satisfies

$$\partial_t u_n - \Delta u_n + u_n^q \le 0.$$

As in the proof of (3.22), for any $y \in \mathbb{R}^N$, the function $(x,t) \mapsto \phi_\infty(t-\epsilon) + w_R(x-y)$ where w_R is defined in (3.23) is a supersolution of (3.13) in $\mathbb{R}^N \times (e,T)$ which dominates u_n at $t = \epsilon$ and for $|x-y| \to R$. Hence it is larger than u_n in this domain. Letting $R \to \infty$ and $\epsilon \to 0$ yields

$$u_n(x,t) \le \phi_\infty(t) \quad \text{for all } (x,t) \in Q_T.$$

When $\epsilon_n \to 0$, u_n converges to u a.e. in Q_T and in $L^q_{loc}(Q_T)$. This implies (5.2). □

Proposition 5.3 *Let $T > 0$ and $u \in L^q_{loc}(Q_T)$ be nonnegative.*
(i) If u is a subsolution of (3.13) there exists a minimal solution v above u, that if U is any solution larger than u, then $u \le v \le U$.
(ii) If u is a continuous supersolution of (3.13) there exists a maximal solution w dominated by u, that is if U is any solution smaller than u, then $U \le w \le u$.
All the above inequalities hold both almost everywhere and in the sense of distributions.

Proof. (i) We use again the subsolutions $u_n := u * \rho_{\epsilon_n}$, and for $\epsilon, R > 0$ we denote by $v_n := v_{\epsilon_n,\epsilon,R}$ be the solution of

$$\begin{aligned} \partial_t v_n - \Delta v_n + v_n^q &= 0 && \text{in } B_R \times (\epsilon, T) \\ v_n &= u_n && \text{on } \partial B_R \times (\epsilon, T) \\ v_n(.,s) &= u_n(.,s) && \text{in } B_R. \end{aligned} \tag{5.3}$$

Then $v_n \ge u_n$ by the comparison principle. Furthermore v_n satisfies

$$v_n(x,t) \le \phi_\infty(t-\epsilon) + w_n(x),$$

where w_R is the large solution in B_R defined in (3.23). Hence it is locally bounded in $B_R \times (\epsilon, T)$ for any $\epsilon > 0$ and $R > 0$. Therefore, up to a subsequence $\{R_j\}$ such that $R_j \to \infty$, the sequence $\{v_{\epsilon_n,\epsilon,R_j}\}$ converges locally in $C^{2,1}(\mathbb{R}^N \times (\epsilon, T))$ to a nonnegative solution $v = v_{\epsilon_n,\epsilon}$ of (3.13) in $\mathbb{R}^N \times (\epsilon, T)$. Furthermore

$$v_{\epsilon_n,\epsilon}(x,t) \ge u_n(x,t) \text{ for all } (x,t) \in \mathbb{R}^N \times (\epsilon, T).$$

Since $v_{\epsilon_n,\epsilon}$ satisfies the uniform parabolic a priori estimates and the associated compactness properties, we infer that, up to a subsequence $v_{\epsilon_n,\epsilon} \to v_\epsilon$ locally in $C^{2,1}(\mathbb{R}^N \times (\epsilon, T))$ when

$\epsilon_n \to 0$. As for u_n it converges to u a.e. and in $L^p_{loc}(\mathbb{R}^N \times (\epsilon, T))$ for any $p < \infty$. Furthermore

$$v_\epsilon(x,t) \geq u(x,t) \text{ for all } (x,t) \in \mathbb{R}^N \times (\epsilon, T).$$

By letting $\epsilon \to 0$ using again the local compactness of $\{v_\epsilon\}$ in $C^{2,1}(Q_T)$, we obtain that up to a subsequence, v_ϵ converges locally to a nonnegative solution v of (3.13) in Q_T which dominates u therein. By construction v is smaller than any element of $\mathcal{U}_+(Q_T)$ which dominates u.

(ii) For $\epsilon, R > 0$ we denote by $w := w_{\epsilon,R}$ the solution of

$$\begin{array}{rl} \partial_t w - \Delta w - w^q = 0 & \text{in } B_R \times (\epsilon, T) \\ w = u & \text{on } \partial B_R \times (\epsilon, T) \\ w(.,\epsilon) = u(.,\epsilon) & \text{in } B_R. \end{array} \tag{5.4}$$

Note that the boundary values of w are well defined since u is continuous. By the comparison principle

$$0 \leq w_{\epsilon,R} \leq u \text{ in } B_R \times (\epsilon, T).$$

Furthermore $w_{\epsilon,R}$ dominates in $B_R \times (\epsilon, T)$ any nonnegative solution U smaller than u. Since u is continuous in Q_T, it is locally bounded therein. As in (i) the set of functions $\{w_{\epsilon,R}\}$ is eventually locally compact in $C^{2,1}(\mathbb{R}^N \times (0,T))$. We conclude as in (i). □

The following result has already been proved but we mention it for the sake of completeness.

Proposition 5.4 *Let u and v be nonnegative, locally bounded functions in Q_T.*
(i) If u and v are subsolutions (resp. supersolutions) then $\max\{u,v\}$ *(resp.* $\min\{u,v\}$*) is a subsolution (resp. a supersolution).*
(ii) If u and v are supersolutions then $u+v$ is a supersolution.
(iii) If u is a subsolution and v is a supersolution then $(u-v)_+$ is a subsolution.

The following notations have been introduced by Dynkin [26].

Notations Let u and v be nonnegative, locally bounded functions in Q_T.
(i) If u is a subsolution, $[u]_\dagger$ denotes the smallest solution dominating u.
(ii) If u is a continuous supersolution, $[u]^\dagger$ denotes the largest solution dominated by u.
(iii) If u and v are subsolutions then $u \vee v := [\max\{u,v\}]_\dagger$.
(iv) If u and v are continuous supersolutions, then $u \wedge v := [\min\{u,v\}]^\dagger$ and $u \oplus v = [u+v]^\dagger$.
(v) If u is a subsolution and v is a supersolution then $u \ominus v := [u-v]_\dagger$.

Proposition 5.5 *The following properties hold*
(i) $(u \vee v) \vee w = u \vee (v \vee w) = [\max\{u,v,w\}]_\dagger$,
(ii) $(u \wedge v) \wedge w = u \wedge (v \wedge w) = [\min\{u,v,w\}]^\dagger$.

Proposition 5.6 *(i) Let $\{u_k\}$ be a sequence of positive, continuous subsolutions of (3.13). Then $U := \sup u_k$ is a subsolution. The statement remains valid if subsolution is replaced by supersolution and* sup *by* inf.

(ii) Let $\mathcal{T}$ be a family of positive solutions of (3.13). Suppose that, for every u_1 and u_2 belonging to $\mathcal{T}$ there exists $v \in \mathcal{T}$ such that

$$\max\{u_1, u_2\} \leq v \qquad (\textit{resp.}\ \min\{u_1, u_2\} \geq v).$$

Then there exists a monotone sequence $\{u_n\} \subset \mathcal{T}$ such that

$$u_n \uparrow \sup_{\mathcal{T}} u, \qquad (\textit{resp.}\ \ u_n \downarrow \inf_{\mathcal{T}} u).$$

Therefore $\sup_{\mathcal{T}} u$ (resp. $\inf_{\mathcal{T}} u$) is a solution.

Proof. (i) We set

$$v_j = \max\{u_1, u_2, ...u_j\}.$$

By induction on j it is clear that v_j is a subsolution and the sequence $\{v_j\}$ is non-decreasing. Because of the universal upper bound (3.26) v_j converges to some function $\bar{v}$ when $j \to \infty$, and $\bar{v}$ is a subsolution which coincides with U. The proof for the min assertion is similar.

(ii) is already proved in [25] and we recall the construction. For every $x \in Q_T$, we set $\ell(x,t) = \sup\{u(x,t) : u \in \mathcal{T}\}$. Let $A = \{(x_n, t_n)\}$ be a countable dense subset of Q_T. For every n there exists a sequence $\{u_{m,n}\} \subset \mathcal{T}$ such that

$$\sup_m\{u_{m,n}(x_n,t_n)\} = \lim_{m\to\infty} u_{m,n}(x_n,t_n) = \ell(x_n,t_n).$$

We set $u_{m,1} = u_{m_1}$. Since $\mathcal{T}$ is closed with respect to the relation $\vee$, $u_{m_2} := u_{m,1} \vee u_{m,2}$ belongs to $\mathcal{T}$ and the sequence $\{u_{m_2}\}$ is increasing and it satisfies

$$\lim_{m_j\to\infty} u_{m_j}(x_j,t_j) = \ell(x_j,t_j) \quad \text{for } j = 1, 2.$$

By induction we construct an increasing subsequence $\{u_{m_n}\}$ of $\mathcal{T}$ such that

$$\lim_{m_n\to\infty} u_{m_n}(x_n,t_n) = \ell(x_n,t_n) \qquad \text{for all } n \in \mathbb{N}_*.$$

Let us denote by $\mathcal{T}_0$ the countable subset of $\mathcal{T}$ of functions $\{u_{m_n}\}$ and set $v = \sup \mathcal{T}_0$. Then $v(x_n,t_n) = \ell(x_n,t_n)$. Using estimate (2.1) and regularity results for parabolic equations we infer that the set $\mathcal{T}_0$ is relatively compact in the $C^{2,1}_{loc}(Q_T)$-topology. Hence, there exists a subsequence sequence of $\mathcal{T}_0$ still denoted by $\{u_{m_n,}\}$ which converges in this topology to a function w which is a nonnegative solution of (3.13) and such that $u_{m_n}(x_n,t_n) \to w(x_n,t_n)$ as $m_n \to \infty$. hence $w(x_n,t_n) = \ell(x_n,t_n)$. We claim now that $w = \sup_{\mathcal{T}} u$. Indeed, if $u \in \mathcal{T}$, $w(x_n,t_n) \geq \ell(x_n,t_n) \geq u(x_n,t_n)$. By continuity, $w(x,t) \geq u(x,t)$ for all $(x,t) \in Q_T$. Thus w is an upper bound of $\mathcal{T}$. It is clearly the least upper bound because any other upper bound $\widetilde{u} \in \mathcal{U}_+(Q_T)$ is larger than u_{m_n} on A, hence larger than w on A, and thus larger than w by density and continuity.

The proof concerning the existence of the greatest lower bound is similar if $\mathcal{T}$ is stable under $\wedge$.

□

The set $\mathcal{U}_+(Q_T)$ is partially ordered for the relation $\leq$. Since for any $u, v \in \mathcal{U}_+(Q_T)$, $u \wedge v$ and $u \vee v$ belong to $\mathcal{U}_+(Q_T)$, it is a *lattice*. Since, by Proposition 3.4, any nonempty subset $\mathcal{T}$ of $\mathcal{U}_+(Q_T)$ admits both a least upper bound (the supremum) and a greatest lower bound (the infimum), it is a *complete lattice*. In the case of semilinear elliptic equations, the similar result is to be found in [25, Theorem 5.1].

Corollary 5.7 *The set $\mathcal{U}_+(Q_T)$ is a complete lattice stable for the laws $\oplus$ and $\ominus$.*

5.2 Fine topology and Besov spaces

5.2.1 The $\mathfrak{T}_q$-fine topology

It is classical in potential theory that there exists a topology which is naturaly adapted to the study of subharmonic functions. This topology was initially introduced by Henri Cartan and its definition is expressed in terms of the Newtonian capacity $cap_{1,2}$. In the study of the initial trace the fine topology is the one associated to the $cap_{\frac{2}{q},q'}$ capacity. In this section we assume $q \geq q_c$ and we note $q' = \frac{q}{q-1}$.

Definition 5.8 *A set $F \subset \mathbb{R}^N$ is $(\frac{2}{q}, q')$-thin at $a \in \mathbb{R}^N$ if*

$$\int_0^1 \left(\frac{cap_{\frac{2}{q},q'}(F \cap B_s(a))}{s^{N-\frac{2}{q-1}}} \right)^{q-1} \frac{ds}{s} < \infty. \tag{5.5}$$

If the above integral is infinite, the set F is $(\frac{2}{q}, q')$-thick at a.
A set F is a $(\frac{2}{q}, q')$- fine neighbourhood of one of its points a if F^c is thin at a.
A set F is $(\frac{2}{q}, q')$-finely open, if F^c is thin at any point $a \in F$. It is $(\frac{2}{q}, q')$-finely closed if its complement F^c is $(\frac{2}{q}, q')$-finely open.

Notations and vocabulary For simplicity we will denote by $\mathfrak{T}_q$ the $cap_{\frac{2}{q},q'}$-fine topology associated to these notions (see [1, Chapter 6] for a detailled study of these notions).
Let $A, B \subset \mathbb{R}^N$
a) A is $\mathfrak{T}_q$-essentially contained in B, denoted by $A \subset^q B$, if

$$cap_{\frac{2}{q},q'}(A \cap B^c) = 0.$$

b) The sets A and B are $\mathfrak{T}_q$-equivalent, denoted by $A \sim^q B$ if

$$cap_{\frac{2}{q},q'}(A \Delta B) \quad \text{where } A \Delta B := (A \cap B^c) \cup (B \cap A^c).$$

c) The closure of a set A in the $\mathfrak{T}_q$-topology is called the $\mathfrak{T}_q$-closure and denoted by $\widetilde{A}$. The $\mathfrak{T}_q$-interior of A is denoted by $A^{\Diamond}$.
d) If $\epsilon > 0$, we denote by A_ϵ the ϵ-neighbourhood of A in the standard Euclidean topology associated to the distance function.
e) The set of all $\mathfrak{T}_q$-thick points of A is denoted by $b_q(A)$. It is the set of points a of A such that A is $(\frac{2}{q}, q')$-thick at a. The set of all $\mathfrak{T}_q$-thin points of A, is denoted by $e_q(A)$. The next result is essentially due to Kellog ([1, Corollary 6. 3.17]).

Proposition 5.9 *There holds*

$$A \text{ is } \mathfrak{T}_q\text{-open} \iff A \subset e_q(A^c), \quad B \text{ is } \mathfrak{T}_q\text{-closed} \iff b_q(B) \subset B.$$

Therefore

$$\widetilde{A} = A \cup b_q(A) \qquad A^{\Diamond} = A \cap e_q(A^c).$$

Furthermore the capacity $cap_{\frac{2}{q},q'}$ *possesses the Kellog property*

$$cap_{\frac{2}{q},q'}(A \cap e_q(A)) = cap_{\frac{2}{q},q'}(A \setminus b_q^c(A)) = 0. \tag{5.6}$$

Proposition 5.10 *(i) If* $Q \subset \mathbb{R}$ *is* $\mathfrak{T}_q$*-open, then* $e_q(Q^c)$ *is the largest* $\mathfrak{T}_q$*-open set which is equivalent to* Q*.*
(ii) If $F \subset \mathbb{R}$ *is* $\mathfrak{T}_q$*-closed, then* $b_q(F)$ *is the smallest* $\mathfrak{T}_q$*-closed set which is equivalent to* F*.*

It is often easier to use the related notions of quasi open or quasi closed sets although these notions are not equivalent. All details to be found in [1, Chapter 6].

Definition 5.11 *A set* $F \subset \mathbb{R}^N$ *is* $\mathfrak{T}_q$*-quasi open if for any* $\epsilon > 0$ *there exists an open set* $G \subset \mathbb{R}^N$ *verifying* $cap_{\frac{2}{q},q'}(G) < \epsilon$ *such that* $F \cap G^c$ *is open in the relative topology of* G^c*.*
A set F *is* $\mathfrak{T}_q$*-quasi closed if* F^c *is* $(\frac{2}{q}, q')$*-quasi open.*
A property $\mathcal{P}$ *holds* $\mathfrak{T}_q$*-quasi everywhere in an open set* $\Omega \subset \mathbb{R}^N$ *if it holds in* Ω *except on a set with zero* $cap_{\frac{2}{q},q'}$*-capacity. Abridged notation:* $\mathfrak{T}_q$*-q.e.*
A function f *defined* $\mathfrak{T}_q$*-q.e. in an open set* $\Omega \subset \mathbb{R}^N$ *is* $\mathfrak{T}_q$*-quasi continuous if for every* $\epsilon > 0$ *there exists an open set* $G \subset \Omega$ *such that* $cap_{\frac{2}{q},q'}(G) = 0$ *with the property that* $f\lfloor_{G^c}$ *is continuous in* G^c *for the induced topology.*

Proposition 5.12 *Any function* f *in* $B^{\frac{2}{q},q'}(\Omega)$ *is* $\mathfrak{T}_q$*-quasi continuous. Thus every element of* $B^{\frac{2}{q},q'}(\Omega)$ *admits a* $\mathfrak{T}_q$*-quasi continuous representative. Let* f_1 *and* f_2 *be two* $\mathfrak{T}_q$*-quasi continuous functions which coincide a.e. in* Ω*, then they coincide* $\mathfrak{T}_q$*-q.e.*

Remark. The notion of $(\frac{2}{q}, q')$-quasi openedness defines a quasi-topology. It is not a topology because an arbitrary union of quasi open sets may not be quasi open. However a *countable* union of quasi open sets is quasi open.

The next result is proved in [46, Proposition 2.1]. We list below a series of results concerning the $\mathfrak{T}_q$-topology and $\mathfrak{T}_q$-quasi topology which are used throughout this section. Their proofs can be found in [1, Proposition 6.4.13] for assertion (i), in [1, Proposition 6.4.12] for assertion (ii), in [1, Proposition 6.4.9] for assertion (iii) and in [1, Proposition 6.4.11] for assertion (iv). Assertions (v)-(viii) are classical in the theory of capacities as exposed in the same book.

Proposition 5.13 *Assume* $q \geq q_c$*.*
(i) Every $\mathfrak{T}_q$*-closed set is* $\mathfrak{T}_q$*-quasi closed.*

(ii) If F is $\mathfrak{T}_q$-quasi closed set, then $E \sim^q \widetilde{E}$.
(iii) A set F is $\mathfrak{T}_q$-quasi closed if and only if there exists a sequence of closed sets $\{F_n\}$ such that $cap_{\frac{2}{q},q'}(F \cap F_n^c) \to 0$.
(iv) There exists a positive constant $\widetilde{C}$ such that for every set F,

$$cap_{\frac{2}{q},q'}(\widetilde{F}) \leq \widetilde{C} cap_{\frac{2}{q},q'}(F).$$

(v) If E is $\mathfrak{T}_q$-quasi closed and $F \sim^q E$, then F is $\mathfrak{T}_q$-quasi closed.
vi) If $\{E_n\}$ is an increasing sequence of Borel sets of $\mathbb{R}^N$, then

$$cap_{\frac{2}{q},q'}\left(\bigcup_n E_n\right) = \lim_{n\to\infty} cap_{\frac{2}{q},q'}(E_n).$$

(vii) If $\{K_n\}$ is a decreasing sequence of compacts sets of $\mathbb{R}^N$, then

$$cap_{\frac{2}{q},q'}\left(\bigcap_n K_n\right) = \lim_{n\to\infty} cap_{\frac{2}{q},q'}(K_n).$$

(viii) For every Borel set $F \subset \mathbb{R}^N$ (and more generaly for every Suslin set), there holds

$$cap_{\frac{2}{q},q'}(F) = \inf\left\{cap_{\frac{2}{q},q'}(G), F \subset G,\ G \text{ open}\right\} = sup\left\{cap_{\frac{2}{q},q'}(K), K \subset F,\ K \text{ compact}\right\}.$$

As a consequence of (iii) there holds:

Corollary 5.14 *A set F is $\mathfrak{T}_q$-quasi closed if and only if there exists a sequence $\{F_n\}$ of $\mathfrak{T}_q$-quasi closed subsets of F such that $cap_{\frac{2}{q},q'}(F \cap F_n^c) \to 0$ as $n \to \infty$.*

Definition 5.15 *Let F be a $\mathfrak{T}_q$-quasi closed set.*
(i) An increasing sequence $\{F_n\}$ of closed subsets of F is called a $\mathfrak{T}_q$-stratification of F if $cap_{\frac{2}{q},q'}(F \cap F_n^c) \to 0$ as $n \to \infty$.
(ii) A $\mathfrak{T}_q$-stratification $\{F_n\}$ is called a proper $\mathfrak{T}_q$-stratification if $cap_{\frac{2}{q},q'}(F \cap F_n^c) \leq 2^{-n-1}$. The sets F_n can be chosen to be compact.
(iii) A $\mathfrak{T}_q$-open set V verifying $cap_{\frac{2}{q},q'}(F \cap V^c) = 0$ is called a $\mathfrak{T}_q$-quasi neighbourhood of F.

The next separation result is valid in any locally compact Hausdorff space.

Proposition 5.16 *Let X be a locally compact Hausdorff space, $K \subset X$ be a compact set contained in an open set A. Then there exists an open set G such that*

$$K \subset G \subset \overline{G} \subset A.$$

Although the fine topology is not locally compact (even if it is Hausdorff) it admits some separation results which are the counterpart of Proposition 5.16.

Lemma 5.17 *Let $F \subset \mathbb{R}^N$ be $\mathfrak{T}_q$-closed. Then:*
(i) If D is an open set such that $cap_{\frac{2}{q},q'}(F \cap D^c) = 0$, then there exists an open set $\mathcal{O}$ such that

$$F \subset^q \mathcal{O} \subset \widetilde{\mathcal{O}} \subset^q D. \tag{5.7}$$

(ii) If D is a $\mathfrak{T}_q$-open set that verifies $F \subset^q D$, there exists a $\mathfrak{T}_q$-open set $\mathcal{O}$ such that (5.7) holds.

Proof. Since $F \cap D \sim^q F$, $F \cap D$ is $\mathfrak{T}_q$-quasi closed and there exists a proper $\mathfrak{T}_q$-stratification $\{F_n\}$ of $F \cap D$ by compact sets such that $F \sim^q F' := \cup_{n=1}^{\infty} F_n$.
If E' is closed, the result follows by Proposition 5.16. If it is not the case, we can assume that $F_{n+1} \setminus F_n \neq \emptyset$ for all integer n. We apply Proposition 5.16 with $K = F_n$ and $G = F_n'$ is the open set containing F_n such that its closure $\overline{F}_n'$ is contained in D: because

$$cap_{\frac{2}{q},q'}(F_n \setminus F_{n-1}) \leq cap_{\frac{2}{q},q'}((E \cap D) \setminus F_n) \leq 2^{-n-1},$$

there exists an open set D_n containing $F_n \setminus F_{n-1}$ such that $cap_{\frac{2}{q},q'}(D_n) < 2^{-n}$. We have also,

$$D_n \cap F_n \subset \widetilde{D_n \cap F_n} \subset \widetilde{D}_n \subset D \quad \text{for all } n \in \mathbb{N}.$$

Since $F' = F_1 \cup_{n=2}^{\infty} (F_n \setminus F_{n-1})$ we have that

$$F' = \bigcup_{n=1}^{\infty} D_n \cap F_n' \subset \bigcup_{n=1}^{\infty} \widetilde{D_n \cap F_n'} \subset D.$$

It is therefore sufficient to prove that $\bigcup_{n=1}^{\infty} \widetilde{D_n \cap F_n'}$ is $\mathfrak{T}_q$-closed. Actually, for any $n \in \mathbb{N}$ we have

$$\begin{aligned} cap_{\frac{2}{q},q'}\left(\bigcup_{n=1}^{\infty} \widetilde{D_n \cap F_n'} \setminus \bigcup_{n=1}^{m} \widetilde{D_n \cap F_n'}\right) &\leq cap_{\frac{2}{q},q'}\left(\bigcup_{n=m+1}^{\infty} \widetilde{D_n \cap F_n'}\right) \leq \sum_{n=m+1}^{\infty} cap_{\frac{2}{q},q'}\left(\widetilde{D}_n\right) \\ &\leq c \sum_{n=m+1}^{\infty} cap_{\frac{2}{q},q'}(D_n) \leq c \sum_{n=m+1}^{\infty} 2^{-m} = c2^{-m}. \end{aligned}$$

Because $\bigcup_{n=1}^{m} \widetilde{D_n \cap F_n'}$ is $\mathfrak{T}_q$-quasi closed the result follows by Corollary 5.14. □

Lemma 5.18 *I- Let F be a $\mathfrak{T}_q$ closed set and $\{F_n\}$ a proper $\mathfrak{T}_q$-stratification of F. Then there exists a decreasing sequence of open sets $\{Q_j\}$ such that $\cup F_n := F' \subset Q_j$ for every $j \in \mathbb{N}$ and*
(i) $\cap_j Q_j = F'$, $\widetilde{Q}_{j+1} \subset Q_j$,
(ii) $\lim_{j\to\infty} cap_{\frac{2}{q},q'}(Q_j) = cap_{\frac{2}{q},q'}(E)$.
II- If A is a $\mathfrak{T}_q$ open set, there exists a decreasing sequence of open sets $\{A_n\}$ such that

$$A \subset \bigcap_n A_n := A' \ , \ cap_{\frac{2}{q},q'}(A_n \setminus A') \to 0 \ \text{as } n \to \infty \ , \ A \sim^q A'.$$

Furthermore there exists an increasing sequence of closed sets $\{E_j\}$ *such that* $E_j \subset A'$ *and*
(i) $\cup_j E_j = A'$, $E_j \subset^q E_{j+1}^{\Diamond}$,
(ii) $cap_{\frac{2}{q},q'}(E_j) \to cap_{\frac{2}{q},q'}(A')$ *when* $j \to \infty$.

Proof. Let $\{D_j\}$ be a decreasing sequence of open sets containing F such that

$$\lim_{j\to\infty} cap_{\frac{2}{q},q'}(D_j) = cap_{\frac{2}{q},q'}(F') = cap_{\frac{2}{q},q'}(F).$$

Case 1: F is closed. We can assume that $F_n = F$ for all n and we set $K_n = B_n(x) \cap F$ for some $x \in F$. By Proposition 5.16 there exists a decreasing sequence $\{\epsilon_{1,n}\}$ converging to 0 such that

$$F \subset Q_1 := \bigcup_{n=1}^{\infty} K_n^{\frac{\epsilon_{1,n}}{2}} \subset \overline{Q}_1 \subset D_1,$$

where $K_n = B_n(x) \cap F$ and, we recall it, $K_n^{\frac{\epsilon_{1,n}}{2}} := \{y \in \mathbb{R}^N : \operatorname{dist}(y, K_n) \leq \frac{\epsilon_{1,n}}{2}\}$. By Proposition 5.16 there exists a decreasing sequence $\{\epsilon_{2,n}\}$ converging to 0, such that $\epsilon_{2,n} \leq \epsilon_{1,n}$ for all n and

$$F \subset Q_2 := \bigcup_{n=1}^{\infty} K_n^{\frac{\epsilon_{1,n}}{4}} \subset \overline{Q}_2 \subset D_2.$$

Note that

$$\overline{Q}_2 \subset \bigcup_{n=1}^{\infty} \overline{K_n^{\frac{\epsilon_{1,n}}{4}}} \subset \bigcup_{n=1}^{\infty} K_n^{\frac{\epsilon_{1,n}}{2}}.$$

Since $\overline{K_n^{\frac{\epsilon_{1,n}}{4}}}$ is closed, we have $Q_2 \subset \overline{Q}_2 \subset Q_1$. By induction we construct a double sequence $\{\epsilon_{j,n}\}$ decreasing in n and converging to 0, non-increasing in j for any fixed n such that

$$F \subset Q_j := \bigcup_{n=1}^{\infty} K_n^{\frac{\epsilon_{j,n}}{2}} \subset \overline{Q}_j \subset D_j,$$

and

$$Q_{j+1} \subset \overline{Q}_{j+1} \subset Q_j \quad \text{for all } j \geq 1.$$

Noting that $F \subset Q_j \subset F^{2-j}$ we deduce that $F = \cap_j Q_j$. Finally,

$$cap_{\frac{2}{q},q'}(F) \leq \lim_{j\to\infty} cap_{\frac{2}{q},q'}(Q_j) \leq \lim_{j\to\infty} cap_{\frac{2}{q},q'}(D_j) = cap_{\frac{2}{q},q'}(F).$$

This yields the result in that case.

Case 2: F is only $\mathfrak{T}_q$ closed. There exists a proper $\mathfrak{T}_q$ stratification $\{F_n\}$ of F such that $F \sim^q F' := \cup_{n=1}^{\infty} F_n$. We can also assume that $F_{n+1} \cap F_n^c \neq \emptyset$ for all integer n.
As in Case 1, for each n we construct the sets Q_j^n relative to F_n that were denoted Q_j and were related to F. Because $cap_{\frac{2}{q},q'}(\widetilde{F_n \setminus F_{n-1}}) \leq ccap_{\frac{2}{q},q'}(F_n \setminus F_{n-1})$, we can choose an open set D_n^1 such that $cap_{\frac{2}{q},q'}(D_n^1) \leq c2^{-n}$. In view of Lemma 5.17 the set

$$Q_1 := \bigcup_{n=1}^{\infty} (D_n^1 \cap Q_n^1)$$

is open and

$$F' \subset Q_1 \subset \widetilde{Q}_1 \subset D_1.$$

Furthermore the set

$$\bigcup_{n=1}^{\infty} \widetilde{D_n^1 \cap Q_1^n}$$

is $\mathfrak{T}_q$-quasi open. By Lemma 5.17 there exists an open set D_n^2 such that

$$D_n^2 \subset \widetilde{D}_n^2 \subset D_n^1,$$

and by induction we construct a sequence of open sets D_n^j such tht

$$D_n^{j+1} \subset \widetilde{D}_n^{j+1} \subset D_n^j \quad \text{and } cap_{\frac{2}{q},q'}(D_n^j) \leq c2^{-n}.$$

By Lemma 5.17 the set

$$Q_j := \bigcup_n = 1^{\infty} D_n^j \cap Q_j^n$$

is open and the set

$$\bigcup_n = 1^{\infty} \widetilde{D_n^j \cap Q_j^n}$$

is $\mathfrak{T}_q$-quasi closed. For any $n \in \mathbb{N}_*$ we have

$$D_n^j \cap Q_j^n \subset \widetilde{D_n^j \cap Q_j^n} \subset \widetilde{D}_n^j \cap \widetilde{Q}_j^n \subset D_n^{j-1} \cap Q_{j-1}^n.$$

Therefore

$$Q_j \subset \widetilde{Q}_j \subset \bigcup_{n=1}^{\infty} \widetilde{D_n^j \cap Q_j^n} \subset \bigcup_{n=1}^{\infty} D_n^{j-1} \cap Q_{j-1}^n \subset D_j.$$

Since the set $\bigcup_{n=1}^{\infty} \widetilde{D_n^j \cap Q_j^n}$ is $\mathfrak{T}_q$-quasi closed, we have

$$Q_j \subset \widetilde{Q}_j \subset Q_{j-1}.$$

Finally,

$$F' \subset Q_j \subset F'^{2^{-j}} \Longrightarrow F' = \bigcap_{n=1}^{\infty} Q_j.$$

Because we have

$$cap_{\frac{2}{q},q'}(F) \leq \lim_{j \to \infty} cap_{\frac{2}{q},q'}(Q_j) \leq \lim_{j \to \infty} cap_{\frac{2}{q},q'}(D_j) = cap_{\frac{2}{q},q'}(F),$$

the assertion follows. □

The next results are classical in the framework of the $\mathfrak{T}_q$-topology.

Proposition 5.19 *I- Any family $\mathcal{D}$ of $\mathfrak{T}_q$-open sets contains a countable subfamily $\mathcal{D}'$ whose union differs from the union of the sets of the whole family $\mathcal{D}$ by a set with zero $cap_{\frac{2}{q},q'}$-capacity.*
II- Let F be a bounded $\mathfrak{T}_q$-open set and let $\mathcal{D}$ be a covering of F consisting of $\mathfrak{T}_q$-open sets. Then, for every $\epsilon > 0$ there exists an open subset $\mathcal{O}_\epsilon$ of F such that $cap_{\frac{2}{q},q'}(\mathcal{O}_\epsilon) < \epsilon$ and $F \cap \mathcal{O}_\epsilon^c$ is covered by a finite subfamily of $\mathcal{D}$.
III- Let F be a $\mathfrak{T}_q$-open set. Then for any $\xi \in F$ there exists a $\mathfrak{T}_q$-open set Q_ξ such that

$$\xi \in Q_\xi \subset \widetilde{Q}_\xi \subset F.$$

Proof. Assertion I is the quasi-Lindelöf property, see [1, 6.5.11]. The second assertion is a consequence of the quasi-Lindelöf property and is proved in [46, Lemma 2.5] and the last assertion is a consequence of the fact that any point in F is a $\mathfrak{T}_q$-thin point of F^c and is proved in [46, Lemma 2.7] using the definition. □

5.2.2 Approximations in Besov spaces

Lemma 5.20 *Let $U \subset \mathbb{R}^N$ be a $\mathfrak{T}_q$-open set and $z \in U$. Then there exists a function f in $B^{\frac{2}{q},q'}(\mathbb{R}^N)$ with support in U such that $f(z) > 0$. In particular, there exists a bounded $\mathfrak{T}_q$-open set V such that $\overline{V} \subset U$.*

Proof. The result is clear if z is an interior point of U with respect to the Euclidean topology. Thus we assume that it is not the case. Since U is $\mathfrak{T}_q$-open, U^c is thin at z. By the assumption we have that $z \in \overline{U}^c \setminus \overset{\circ}{U}$. By [1, p. 174] there exists an open set W such that $z \in \overline{W} \cap W^c$ and W is thin at z. We recall (see [1, Theorems 2.2.7, 2.5.6]) that for a Borel set E with positive $cap_{\frac{2}{q},q'}$-capacity, we define the *Besov nonlinear potential* of the capacitary measure μ_E by

$$F_E := \mathcal{V}^{\mu_E} = G_{\frac{1}{q}} * (G_{\frac{1}{q}} * \mu_E)^{\frac{1}{q-1}},$$

where $G_{\frac{1}{q}}$ is the Bessel kernel in $\mathbb{R}^N$. By [1, Theorem 6.3.9] there holds

$$\mathcal{V}^{\mu_E}(z) < \frac{1}{2},$$

if we take for E the set $B_r(z) \cap W$ for $r > 0$ small enough. By [1, Theorem 6.3.9] we have $\mathcal{V}^{\mu_E} \geq 1$-$\mathfrak{T}_q$-q.e. on $B_r(z) \cap W$, and by [1, Theorem 2.6.7] $\mathcal{V}^{\mu_E} \geq 1$ everywhere on $B_r(z) \cap W$. Therefore

$$\mathcal{V}^{\mu_E}(z) < \frac{1}{2} < 1 \leq \mathcal{V}^{\mu_E}(x) \quad \text{for all } x \in B_r(z) \cap W.$$

This implies that for $r > 0$ small enough there holds

$$\mathcal{V}^{\mu_E}(z) < \frac{1}{2} < 1 \leq \inf\{\mathcal{V}^{\mu_E}(x) : x \in B_r(z) \cap W\}.$$

Now let H be a smooth nondecreasing function defined on $\mathbb{R}$, such that $H(t) = 0$ for $t \leq 0$ and $H(t) = t$ for $t > \frac{1}{4}$. If $\eta \in C_c^\infty(\mathbb{R}^N)$ satisfies $0 \leq \eta \leq 1$, $supp(\eta) \subset B_r(z)$ and $\eta(z) = 1$, then the function

$$f := \eta H \circ (1 - \mathcal{V}^{\mu_E})$$

satisfies the requirements of the Lemma. □

Lemma 5.21 *Let U be a $\mathfrak{T}_q$- open set and $z \in U$. Then there exists a $\mathfrak{T}_q$-open set V, such that $z \in V \subset U$, and a function $\psi \in B^{\frac{2}{q},q}(\mathbb{R}^N)$ such that $0 \leq \psi \leq 1$, $\psi = 1$ q-a.e. on V and $\psi = 0$ in U^c.*

Proof. We keep the notations of Lemma 5.20 and assume that z is not interior to U. Let μ be the capacitary measure of $B_r(z) \cap U$ with (up to changing r),

$$\mathcal{V}^\mu(z) < \frac{1}{4} \quad \text{and} \quad \mathcal{V}^\mu(x) = 1 \quad \text{for all } x \in B_r(z) \cap U^c.$$

By [1, Proposition 6.3.10] $\mathcal{V}^\mu$ is quasi continuous, hence there exists a $\mathfrak{T}_q$- open set W which contains z such that

$$\mathcal{V}^\mu \leq \frac{1}{4} \quad \text{q. a.e. on } W.$$

Let $\eta \in C_c^\infty(\mathbb{R}^N)$ such that $0 \leq \eta \leq 1$, $supp(\eta) \subset B_r(z)$ and $\eta(x) = 1$ for all $x \in B_r(z)$. We set

$$f(x) = 2\eta(x) H \circ \left(1 - H \circ \left(\frac{1}{2} - \mathcal{V}^\mu(x)\right) - \mathcal{V}^\mu(x)\right).$$

Then $f \in B^{\frac{2}{q},q'}(\mathbb{R}^N)$, $0 \leq f \leq 1$ on $B_r(z) \cap U^c$, $f = 1$ on $B_{\frac{r}{2}}(z) \cap W$ and $f = 0$ outside of $B_r(z) \cap U$. □

Definition 5.22 *If ζ is a function defined in $\mathbb{R}^N$ we denote by $\mathfrak{T}_q$-supp(ζ) the closure in the $\mathfrak{T}_q$-topology of the set $\{x \in \mathbb{R}^N : |\zeta(x)| > 0\}$.*

Lemma 5.23 *Assume $q \geq 2$. Let K be a compact set and U a $\mathfrak{T}_q$-open set containing K. Let $\{U_j\}$ be a sequence of $\mathfrak{T}_q$- open subsets of U covering U up to a set of zero Z of zero $cap_{\frac{2}{q},q}$-capacity.*

1- If there exists a nonnegative function $u \in B^{\frac{2}{q},q'}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $\mathfrak{T}_q$-supp(u) included in K, then for any $k \in \mathbb{N}_*$ there exist an integer $m(k)$ and nonnegative functions $u_{k,j} \in B^{\frac{2}{q},q'}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $\mathfrak{T}_q$-supp$(u_{k,j})$ included in U_j such that

$$\sum_{j=1}^{m(k)} u_{k,j} \leq u \quad \text{in } \mathbb{R}^N, \tag{5.8}$$

and

$$\left\| u - \sum_{j=1}^{m(k)} u_{k,j} \right\|_{B^{\frac{2}{q},q}} \to 0 \quad \text{as } k \to \infty. \tag{5.9}$$

2- If u is a signed function, and since $q \geq 2$, $u^{\pm}$ belongs to $B^{\frac{2}{q},q'}(\mathbb{R}^N)$. The existence of the $\{u_{k,j}\}$ is replaced by existence of $\{u_{k,j,\pm}\}$. Estimate (5.8) is replaced by

$$\sum_{j=1}^{m(k)} u_{k,j,\pm} \leq u^{\pm} \quad in\ \mathbb{R}^N, \tag{5.10}$$

estimate (5.9) remains valid with $u_{k,j}$ replaced by $u_{k,j,+} - u_{k,j,-}$.

Proof. We can assume that U and U_j are bounded. For any j, k there exists an open set $G_{k,j}$ such that $cap_{\frac{2}{q},q'}(G_{k,j}) \leq 2^{-k-j}$ for $j \geq 1$, $K \subset G_{k,0}$, and for $j \geq 1$, the sets $U_j \cup G_{k,j}$ are open. Furthermore the sets

$$\bigcup_{j=0}^{\infty} G_{k,j} \quad \text{and} \quad \bigcup_{j=0}^{\infty} G_k \bigcup_j U_j$$

are open, and clearly $cap_{\frac{2}{q},q'}(G_k) \to 0$ when $k \to \infty$. Since G_k is open, its Besov potential $\mathcal{V}^{\mu_{G_k}} := F^{G_k}$ is larger or equal to 1 on G_k [1, Theorems 2.5.6, 2.6.7] and there holds

$$\|\mathcal{V}^{\mu_{G_k}}\|^{q'}_{B^{\frac{2}{q},q'}} \leq \widetilde{C} cap_{\frac{2}{q},q'}(G_k),$$

for some $C = C(N, q) > 0$. Let H be a smooth nondecreasing defined on $\mathbb{R}_+$ function such that $H(t) = 1$ for $t \geq 1$ and $H(t) = t$ for $0 \leq t \leq \frac{1}{2}$. Then the function $\phi_k = H \circ \mathcal{V}^{\mu_{G_k}}$ belongs to $B^{\frac{2}{q},q'}(\mathbb{R}^N)$, satisfies $0 \leq \phi_k \leq 1$, $\phi_k = 1$ on G_k and there exists $C' = C'(N, q) > 0$ such that

$$\|\phi_k\|^{q'}_{B^{\frac{2}{q},q'}} \leq C' cap_{\frac{2}{q},q'}(G_k),$$

We set $\psi_k = 1 - \phi_k$. Then

$$\|u - \psi_k u\|_{B^{\frac{2}{q},q'}} \to 0 \quad \text{as } k \to \infty. \tag{5.11}$$

For $k \in \mathbb{N}_*$ fixed, there exist open balls $B_{k,j,i}$ such that

$$B_{k,j,i} \subset U_j \bigcup G_k \quad \text{and} \bigcup_j \Big(G_k \bigcup U_j\Big) = G_k \bigcup \left(\bigcup_j U_j\right) = \bigcup_{i,j=1}^{\infty} B_{k,j,i}.$$

Since K is compact there exists $m(k) \in \mathbb{N}_*$ such that

$$K \subset \bigcup_{i,j=1}^{m(k)} B_{k,j,i}.$$

Now we consider functions $w_{k,j,i} \in C_c^{\infty}(\mathbb{R}^N)$ such that

$$B_{k,j,i} = \{x \in \mathbb{R}^N : w_{k,j,i}(x) > 0\},$$

and we set

$$u_{k,j} = u\psi_k \frac{\sum_{i=1}^{m(k)} w_{k,j,i}}{\sum_{j,i=1}^{m(k)} w_{k,j,i}}.$$

Then $u_{k,j} \in L^\infty(\mathbb{R}^N) \cap B^{\frac{2}{q},q'}(\mathbb{R}^N)$ and

$$\mathfrak{T}_q\text{-}supp\, u_{k,j} \subset \left(K \bigcap G_k^c\right) \bigcap \left(\bigcup_i B_{k,j,i}\right) \subset U_j.$$

which ends the proof. □

Remark. The construction can be made also in the case $1 < q < 2$, but the proof of (5.11) is still pending.

5.3 Regular sets

5.3.1 The regular initial set

In order to define the precise trace we recall that for any Borel set $U \subset \mathbb{R}^N$, $\mathbf{1}_U$ denotes the characteristic function of U and

$$\mathbb{H}[\mathbf{1}_U](x,t) = \frac{1}{(4\pi t)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}} \mathbf{1}_U(y) dy.$$

If $u \in \mathcal{U}_+(Q_T)$ (i.e. a positive solution of (3.13) in Q_T), the following dichotomy occurs for any $\xi \in \mathbb{R}^N$:
(i) either there exists a bounded $\mathfrak{T}_q$-open neighbourhood $U = U_\xi$ of ξ such the

$$\int_0^T \int_{\mathbb{R}^N} u^q \left(\mathbb{H}[\mathbf{1}_U]\right)^{2q'} dxdt < \infty, \tag{5.12}$$

(ii) or for any $\mathfrak{T}_q$-open neighbourhood U of ξ there holds

$$\int_0^T \int_{\mathbb{R}^N} u^q \left(\mathbb{H}[\mathbf{1}_U]\right)^{2q'} dxdt = \infty. \tag{5.13}$$

Definition 5.24 *Let $u \in \mathcal{U}_+(Q_T)$. The set of $\xi \in \mathbb{R}^N$ such that (i) occurs is $\mathfrak{T}_q$-open and denoted by $\mathcal{R}_q(u)$. It is called the q-regular set of u. The set $\mathcal{S}_q(u) := \mathbb{R}^N \setminus \mathcal{R}_q(u)$ is $\mathfrak{T}_q$-closed and called the q-singular set of u.*

Proposition 5.25 *Let $\eta \in B^{\frac{2}{q},q'}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $\mathfrak{T}_q$-supp(η) in a bounded $\mathfrak{T}_q$-open set U, and let $u \in \mathcal{U}_+(Q_T)$ satisfy*

$$M_U = \int_0^T \int_{\mathbb{R}^N} u^q \left(\mathbb{H}[\mathbf{1}_U]\right)^{2q'} dxdt < \infty. \tag{5.14}$$

Then there exists the following limit

$$\ell(\eta) = \lim_{t\to 0} \int_0^T \int_{\mathbb{R}^N} u^q \left(\mathbb{H}[\eta]_+\right)^{2q'}(x,t)dxdt. \tag{5.15}$$

Furthermore there exists $C = C(M_U, q, N) > 0$ *such that*

$$|\ell(\eta)| \le C\left(\|\eta\|^{2q'}_{B^{\frac{2}{q},q'}} + \|\eta\|^{2q'}_{L^\infty}\right). \tag{5.16}$$

Proof. Set $h = \mathbb{H}[\eta]$ and $\phi(r) = r_+^{2q'}$. Since $|\eta| \le \|\eta\|_{L^\infty} \mathbf{1}_U]$, there holds

$$\left|\int_0^T \int_{\mathbb{R}^N} u^q \phi(h) dxdt\right| \le \|\eta\|^{2q'}_{L^\infty} \int_0^T \int_{\mathbb{R}^N} u^q \left(\mathbb{H}[\mathbf{1}_U]\right)^{2q'} dxdt = \|\eta\|^{2q'}_{L^\infty} M_U < \infty. \tag{5.17}$$

Note that for $0 < s < t < T$,

$$\int_s^t \int_{\mathbb{R}^N} \left(-u\left(\partial_t\phi(h) + \Delta\phi(h)\right) + u^q\phi(h)\right) dxd\tau = \int_{\mathbb{R}^N} u\phi(h)(.,s)dx - \int_{\mathbb{R}^N} u\phi(h)(.,t)dx. \tag{5.18}$$

But

$$\partial_t\phi(h) + \Delta\phi(h) = 2q'\phi(h)h_+^{-2}\left(2h_+\partial_t h + (2q'-1)|\nabla h|^2\right).$$

By Hölder's inequality,

$$\left|\int_s^t \int_{\mathbb{R}^N} u\left(\partial_t\phi(h) + \Delta\phi(h)\right) dxd\tau\right|$$

$$\hat{E} \le \left(\int_s^t \int_{\mathbb{R}^N} u^q\phi(h)dxd\tau\right)^{\frac{1}{q}} \left(\int_s^t \int_{\mathbb{R}^N} (\phi(h))^{-\frac{q'}{q}} \left|\partial_t\phi(h) + \Delta\phi(h)\right|^{q'} dxd\tau\right)^{\frac{1}{q'}}$$

$$\hat{E} \le 4q' \left(\int_s^t \int_{\mathbb{R}^N} u^q\phi(h)dxd\tau\right)^{\frac{1}{q}} \left(\int_s^t \int_{\mathbb{R}^N} \left(h_+|\partial_t h| + |\nabla h|^2\right)^{q'} dxd\tau\right)^{\frac{1}{q'}}.$$

Since

$$\int_s^t \int_{\mathbb{R}^N} |\partial_t h|^{q'} dxd\tau \le \int_0^T \int_{\mathbb{R}^N} |\partial_t h|^{q'} dxd\tau \le c\|\eta\|^{q'}_{B^{\frac{2}{q},q'}}$$

and

$$\int_s^t \int_{\mathbb{R}^N} |\nabla h|^{2q'} dxd\tau \le \int_0^T \int_{\mathbb{R}^N} |\nabla h|^{2q'} dxd\tau \le C\|\eta\|^{q'}_{L^\infty} \|\Delta\eta\|^{q'}_{L^{q'}} = C\|\eta\|^{q'}_{L^\infty} \|\partial\eta\|^{q'}_{L^{q'}}$$

by Gagliardo-Nirenberg's inequality and the maximum principle, we obtain

$$\left|\int_s^t \int_{\mathbb{R}^N} u\left(\partial_t\phi(h) + \Delta\phi(h)\right) dxd\tau\right| \le C\left(\int_s^t \int_{\mathbb{R}^N} u^q\phi(h)dxd\tau\right)^{\frac{1}{q}} \|\eta\|_{L^\infty} \|\eta\|_{B^{\frac{2}{q},q'}}. \tag{5.19}$$

As a consequence of (5.18) and (5.19), we infer the two following inequalities

$$\begin{aligned}\int_s^t \int_{\mathbb{R}^N} u^q \phi(h) dx d\tau + C \left(\int_s^t \int_{\mathbb{R}^N} u^q \phi(h) dx d\tau \right)^{\frac{1}{q}} \|\eta\|_{L^\infty} \|\eta\|_{B^{\frac{2}{q},q'}} \\ \geq \left| \int_{\mathbb{R}^N} u\phi(h)(.,s) dx - \int_{\mathbb{R}^N} u\phi(h)(.,t) dx \right|,\end{aligned} \tag{5.20}$$

and

$$\begin{aligned}\int_s^t \int_{\mathbb{R}^N} u^q \phi(h) dx d\tau - C \left(\int_s^t \int_{\mathbb{R}^N} u^q \phi(h) dx d\tau \right)^{\frac{1}{q}} \|\eta\|_{L^\infty} \|\eta\|_{B^{\frac{2}{q},q'}} \\ \leq \int_{\mathbb{R}^N} u\phi(h)(.,s) dx - \int_{\mathbb{R}^N} u\phi(h)(.,t) dx.\end{aligned} \tag{5.21}$$

Under the assumption (5.14) the left-hand side of (5.20) tends to zero when $s, t \to 0$, therefore, we deduce from (5.18) that the function

$$t \mapsto \int_{\mathbb{R}^N} u\phi(h)(.,t) dx$$

admits a limit that is denoted by $\ell(\eta)$ when $t \to 0$. Using again (5.18) we get

$$\int_0^T \int_{\mathbb{R}^N} \left(-u \left(\partial_t \phi(h) + \Delta \phi(h) \right) + u^q \phi(h) \right) dx d\tau + \int_{\mathbb{R}^N} u\phi(h)(.,T) dx = \ell(\eta). \tag{5.22}$$

Since

$$\left| \int_{\mathbb{R}^N} u\phi(h)(.,T) dx \right| \leq C(T) \|\eta\|_{L^\infty}^{2q'}, \tag{5.23}$$

we infer from (5.19)

$$\ell(\eta) \leq C_1 \|\eta\|_{L^\infty}^{2q'} + C_2 \|\eta\|_{L^\infty}^{q'} \|\eta\|_{B^{\frac{2}{q},q'}}^{q'} \leq C \left(\|\eta\|_{L^\infty} + \|\eta\|_{B^{\frac{2}{q},q'}} \right)^{2q'}. \tag{5.24}$$

□

This estimate can be improve in order to show that the initial trace holds in the usual sense.

Proposition 5.26 *Let the assumptions of Proposition 5.25 be satisfied, then*

$$\lim_{t \to 0} \int_{\mathbb{R}^N} u(x,t) \eta^{2q'}(x) dx = \ell(\eta). \tag{5.25}$$

Proof. Using (5.18) wit $t = T$ and replacing $h(x)$ by $h_s(x,t) = \mathbb{H}[\eta](x, t-s)$ we have

$$\begin{aligned}\int_s^T \int_{\mathbb{R}^N} \left(-u \left(\partial_t \phi(h_s) + \Delta \phi(h_s) \right) + u^q \phi(h_s) \right) dx d\tau \\ + \int_{\mathbb{R}^N} u\phi(h_s)(.,T) dx = \int_{\mathbb{R}^N} u\phi(h_s)(.,s) dx.\end{aligned} \tag{5.26}$$

When $s \to 0$, one has by the Lebesgue dominated convergence theorem

$$\int_{\mathbb{R}^N} u\phi(h_s)(.,T)dx \to \int_{\mathbb{R}^N} u\phi(h)(.,T)dx$$

and

$$\int_s^T \int_{\mathbb{R}^N} u^q \phi(h_s)dxd\tau \to \int_s^T \int_{\mathbb{R}^N} u^q \phi(h)dxd\tau.$$

Furthermore

$$\left| \int_0^{T-s} \int_{\mathbb{R}^N} \left(u(x,t+s) - u(x,t)\right) \left(\partial_t \phi(h) + \Delta\phi(h)\right) dxd\tau \right|$$
$$\leq C \left(\int_0^{T-s} \int_{\mathbb{R}^N} |u(x,t+s) - u(x,t)|^q \, h_+^{2q'} \right)^{\frac{1}{q}} \|\eta\|_{L^\infty}^{q'} \|\eta\|_{B^{\frac{2}{q},q'}}^{q'}.$$

By Proposition 5.25, the right-hand side of the above inequality tends to 0 when $s \to 0$. Clearly

$$\lim_{s\to 0} \int_{T-s}^T u^q \phi(h)dxd\tau = 0.$$

Combining (5.18) and (5.26) we obtain

$$\lim_{s\to 0} \int_{\mathbb{R}^N} u(x,s) \left(\phi(h)(x,s) - \phi(\eta)\right) dx = 0, \tag{5.27}$$

which ends the proof. □

Combining Proposition 5.25 and Proposition 5.26 one obtain

Corollary 5.27 *Assume $U \subset \mathbb{R}^N$ is a bounded $\mathfrak{T}_q$-open set such that*

$$\lim_{s\to 0} \int_{\mathbb{R}^N} u(x,s)\eta^{2q'}(x)dx = \infty, \tag{5.28}$$

for some nonnegative $\eta \in L^\infty(\mathbb{R}^N) \cap B^{\frac{2}{q},q'}(\mathbb{R}^N)$ with $\mathfrak{T}_q$-supp$(\eta) \subset U$. Then

$$\int_0^T \int_{\mathbb{R}^N} u^q \left(\mathbb{H}[\eta]\right)^{2q'} dxdt = \infty. \tag{5.29}$$

The next result shows that the q-singular set of u inherits the main properties of the singular set $\mathcal{S}(u)$ of the rough trace of u

Proposition 5.28 *Let $\xi \in \mathcal{S}_q(u)$. Then for any $\mathfrak{T}_q$-open set G containing ξ, there holds*

$$\lim_{t\to 0} \int_G u(x,t)dx = \infty, \tag{5.30}$$

Proof. If $\xi \in \mathcal{S}_q(u)$ and if G is a $\mathfrak{T}_q$-open set containing ξ, then by Lemma 5.21 there exists $\eta \in L^\infty(\mathbb{R}^N) \cap B^{\frac{2}{q},q'}(\mathbb{R}^N)$ such that $0 \le \eta \le 1$, and a $\mathfrak{T}_q$-open set $D \subset G$ such that $\eta = 1$ on D and $\eta = 0$ in G^c. Therefore

$$\lim_{s\to 0} \int_s^T \int_{\mathbb{R}^N} u^q \left(\mathbb{H}[\eta]\right)^{2q'} dxdt \ge \lim_{s\to 0} \int_s^T \int_{\mathbb{R}^N} u^q \left(\mathbb{H}[\mathbf{1}_{\mathbf{D}}]\right)^{2q'} dxdt = \infty.$$

This implies that the left-hand side of (5.21) tends to ∞ when $s \to 0$. Using again (5.23) we obtain

$$\lim_{s\to 0} \int_{\mathbb{R}^N} u \left(\mathbb{H}[\eta]\right)^{2q'} (x,s)dx = \infty,$$

which implies

$$\lim_{s\to 0} \int_{\mathbb{R}^N} u\eta^{2q'}(x,s)dx = \infty.$$

Since $\eta = 1$ on D the result follows. □

5.3.2 Moderate solutions

We recall that a solution u of (3.13) in Q_T is called *moderate* if $u \in L^q(K)$ for any compact set $K \subset \mathbb{R}^N \times [0,T)$. Then there exists a Radon measure μ on $\mathbb{R}^N$ such that

$$\lim_{t\to 0} \int_{\mathbb{R}^N} u(x,t)\zeta(x)dx = \int_{\mathbb{R}^N} \zeta d\mu(x) \quad \text{for all } \zeta \in C_c(\mathbb{R}^N). \tag{5.31}$$

Equivalently, for any $\phi \in C_c^{1,2}(\mathbb{R}^N \times [0,T))$, there holds

$$\int_0^T \int_{\mathbb{R}^N} \left(-u\left(\partial_t\phi + \Delta\phi\right) + |u|^{q-1}u\phi\right) dxdt = \int_{\mathbb{R}^N} \phi(x,0)d\mu(x). \tag{5.32}$$

It is proved in [9] that the measure μ vanishes on Borel subsets of $\mathbb{R}^N$ with $cap_{\frac{2}{q},q'}$-capacity zero.

Lemma 5.29 *Let u be a nonnegative moderate solution of u of (3.13) in Q_T with initial trace $\mu \in \mathfrak{M}_+(\mathbb{R}^N)$. Then for any $\mathfrak{T}_q$-open bounded set $\mathcal{O}$ one has*

$$\int_0^T \int_{\mathbb{R}^N} u^q(x,t) \left(\mathbb{H}[\mathbf{1}_{\mathcal{O}}]\right)^{2q'} dxdt < \infty. \tag{5.33}$$

Proof. Let $\eta \in C_c^\infty(\mathbb{R}^N)$ be a nonnegative function with value 1 on $\mathcal{O}$. We put $h(x,t) = \mathbb{H}[\eta](x,t)$ and for $0 \le s \le t \le 1$, $h_s(x,t) = \mathbb{H}[\eta](x,t-s)$. We also set $\phi(r) = |r|^{2q'}$. Using again the identities in Proposition 5.25 we have that

$$\int_0^T \int_{\mathbb{R}^N} u^q\phi(h_s)dxd\tau + \int_{\mathbb{R}^N} (u\phi(h_s))(.,T)dx \\ \le C\left(\int_{\mathbb{R}^N} u(x,s)\phi(\eta)dx + \|\eta\|_{L^\infty}^{q'} \|\eta\|_{B^{\frac{2}{q},q'}}^{q'}\right).$$

Because for any Borel set E, one has

$$\limsup_{s\to 0}\int_E u(x,s)dx<\infty,$$

we obtain (5.33) by Fatou's lemma. □

Definition 5.30 *A Radon measure μ in $\mathbb{R}^N$ is regular with respect to the $\mathfrak{T}_q$-topology if for any Borel set E one has*

$$\mu(E)=\inf\{\mu(D): D\supset E,\ D\ \mathfrak{T}_q\text{-}open\}=\sup\{\mu(K): K\subset E,\ K\ compact\}. \tag{5.34}$$

Theorem 5.31 *Let u be a nonnegative solution of (3.13) in Q_T with initial data μ. Then*
(i) The measure μ is a regular measure with respect to the $\mathfrak{T}_q$-topology.
(ii) For any quasi continuous function $\phi\in L^\infty(\mathbb{R}^N)$ with bounded $\mathfrak{T}_q$-support in $\mathbb{R}^N$, we have

$$\lim_{t\to 0}\int_{\mathbb{R}^N}u(x,t)\phi(x)dx=\int_{\mathbb{R}^N}\phi d\mu(x).$$

Proof. (i) We recall that a Radon measure is regular with respect to the standard topology. Moreover, if $E\subset\mathbb{R}^N$ is a Borel set and D is open and contains D, then D is open for the $\mathfrak{T}_q$-topology, hence

$$\mu(E)\le\inf\{\mu(D): D\supset E,\ D\ \mathfrak{T}_q\text{-open}\}\le\inf\{\mu(D): D\supset E,\ D\ \text{ open}\}=\mu(E).$$

The assertion on compact sets is unchanged and the statement (i) follows.
(ii) The measure $\mu_t:=u(t,.)dx$ converges to μ in the weak-$*$ topology. Hence we have

$$\begin{array}{ll}\limsup_{t\to 0}\mu_t(E)\le\mu(E) & \text{for any compact set } E\subset\mathbb{R}^N\\ \liminf_{t\to 0}\mu_t(A)\ge\mu(A) & \text{for any open set } A\subset\mathbb{R}^N.\end{array} \tag{5.35}$$

If E is a $\mathfrak{T}_q$-closed set, there exists an increasing sequence of closed sets $\{K_m\}$ such that $cap_{\frac{2}{q},q'}(E\cap K_m^c)\to 0$ when $m\to\infty$. Then, for any open $\mathcal{O}$ containing E, one has

$$\limsup_{t\to 0}\mu_t(E)\le\limsup_{t\to 0}\mu_t(K_m)+\limsup_{t\to 0}\mu_t(E\cap K_m^c)\le\mu_t(\mathcal{O})+\limsup_{t\to 0}\mu_t(E\cap K_m^c).$$

We will prove by contradiction that

$$\lim_{m\to\infty}\limsup_{t\to 0}\mu_t(E\cap K_m^c)=0. \tag{5.36}$$

Assume that (5.36) does not hold and let $\epsilon>0$ be the value of the above limit. For fixed $m\in\mathbb{N}$, let $\{t_{n,m}\}$ be a decreasing sequence converging to 0 such that

$$\lim_{t_{n,m}\to 0}\mu_{t_{n,m}}(K_m)=\limsup_{t\to 0}\mu_t(K_m)=\epsilon_m.$$

The sequence $\{\epsilon_m\}$ is decreasing with limit ϵ when $m \to \infty$. Let $u_{n,m}$ be the sequence of solutions of (3.13) in Q_∞ such that $u_{n,m}(.,0) = \mathbf{1}_{E\cap K_m^c}\mu_{t_{n,m}}$. Clearly

$$u_{n,m}(x,t) \le u(x,t+t_{n,m}) \quad \text{for all } (x,t) \in Q_T,$$

and

$$u_{n,m}(x,t) \le V_{\widetilde{E\cap K_m^c}} \quad \text{for all } (x,t) \in Q_T,$$

where $V_{\widetilde{E\cap K_m^c}}$ is the maximal σ-moderate solution of (3.13) in Q_∞ with initial data ν where $\nu \in \mathfrak{M}_+(\mathbb{R}^N)$ vanishes in $E \cap K_m^c$ and is q-admissible (this notion is developped in the next section). Because

$$cap_{\frac{2}{q},q'}(\widetilde{E \cap K_m^c}) \le \widetilde{C} cap_{\frac{2}{q},q'}(E \cap K_m^c) \to 0 \quad \text{as } m \to \infty,$$

it follows from Proposition 5.39 that

$$V_{\widetilde{E\cap K_m^c}} \to 0 \quad \text{as } m \to \infty.$$

This is a contradiction. Hence (5.36) holds. Thus the proof is complete if E is a $\mathfrak{T}_q$-closed. If E is $\mathfrak{T}_q$-open, then

$$\mu(E) = \mu(\widetilde{E}),$$

since μ is q-admissible and the proof follows.
Let ϕ be a quasi continuous function. Without loss of generality, we can suppose that it is nonnegative since $\phi = \phi_+ - \phi_-$ and bounded above by 1. If $k \in \mathbb{N}$ and $m = 2^k - 1, 2^k - 2, ..., 0$, we denote by $a_{m,k}$ a real number in the interval $(m2^{-k}, (m+1)2^{-k}$ such that

$$\mu\left(\phi^{-1}(\{a_{m,k}\})\right) = 0.$$

Set

$$A_{m,k} = \phi^{-1}\left((a_{m,k}, a_{m+1,k}]\right) \text{ for } m = 1, 2, ..., 2^k - 1 \text{ and } A_{0,k} = \phi^{-1}\left((a_{0,k}, a_{1,k}]\right).$$

Since ϕ has compact support, all the above sets are bounded and

$$\lim_{t\to 0} \mu_t(A_{m,k}) = \mu(A_{m,k}). \tag{5.37}$$

If we denote by ϕ_k the step function

$$\phi_k = \sum_{m=0}^{2^k-1} m2^{-k}\mathbf{1}_{A_{m,k}}.$$

Then $\phi_k \uparrow \phi$ uniformly, and by (5.37),

$$\lim_{t\to 0} \int_{\mathbb{R}^N} u(x,t)\phi_k(x) = \int_{\mathbb{R}^N} \phi_k d\mu.$$

This implies that (ii) holds. □

5.4 Localization

5.4.1 Vanishing properties

Definition 5.32 *A continuous function $u \in \mathcal{U}_+(Q_T)$ vanishes on a $\mathfrak{T}_q$-open set $G \subset \mathbb{R}^N$, if for any $\eta \in B^{\frac{2}{q},q'}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $\mathfrak{T}_q$-supp$(\eta) \subset^q G$, there holds*

$$\lim_{t\to 0} \int_{\mathbb{R}^N} u(x,t)\eta_+^{2q'}(x)dx = 0. \tag{5.38}$$

We write $u \approx_G 0$. We denote by $\mathcal{U}_G(Q_T)$ the subset of $u \in \mathcal{U}_+(Q_T)$ which vanish in the previous sense on G.

The following result is obvious.

Proposition 5.33 *Let $A \subset \mathbb{R}^N$ be a $\mathfrak{T}_q$-open set, and $u_1, u_2 \in \mathcal{U}_+(Q_T)$. If $u_1 \approx_A 0$ and $0 \le u_2 \le u_1$, then $u_2 \approx_A 0$.*

Proposition 5.34 *Let $G, G' \subset \mathbb{R}^N$ be $\mathfrak{T}_q$-open set such that $G \sim^q G'$. If $u \in \mathcal{U}_G(Q_T)$, then $u \in \mathcal{U}_{G'}(Q_T)$.*

Proof. If $\eta \in B^{\frac{2}{q},q'}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $\mathfrak{T}_q$-supp$(\eta) \subset^q G$, then $\mathfrak{T}_q$-supp$(\eta) \subset^q G'$. Since $|G' \cap G^c| = |G \cap G'^c|$, the result follows. □

If G is an open subset, this notion coincides with the usual definition of vanishing, since we can take a test function $\eta \in C_c^\infty(G)$. In that case $u \in C(Q_T \cup (G \times \{0\})$.

Lemma 5.35 *Assume that $u \in \mathcal{U}_G(Q_T)$. Then for any $\eta \in B^{\frac{2}{q},q'}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $\mathfrak{T}_q$-supp$(\eta) \subset^q G$, there holds*

$$\int_0^T \int_{\mathbb{R}^N} u^q \left(\mathbb{H}[\eta]_+\right)^{2q'} dxdt + \int_{\mathbb{R}^N} u(x,T) \left(\mathbb{H}[\eta]_+\right)^{2q'} dx \le C \|\eta\|_{L^\infty}^{q'} \|\eta\|_{B^{\frac{2}{q},q'}}^{q'}. \tag{5.39}$$

Proof. Let u and η be as in the statement of the lemma, $h = \mathbb{H}[\eta]$ and $\phi(r) : r_+^{2q'}$. Then

$$\int_0^T \int_{\mathbb{R}^N} \left(-u\left(\partial_t\phi(h) + \Delta\phi(h)\right) + u^q\phi(h)\right) dxdt + \int_{\mathbb{R}^N} u(x,T)\phi(h)dx = 0. \tag{5.40}$$

Inequality (5.39) is a consequence of (5.19). □

Lemma 5.36 *Let $G \subset \mathbb{R}^N$ be a $\mathfrak{T}_q$-open set. Then there exists a nondecreasing sequence $\{u_n\} \subset \mathcal{U}_G(Q_T)$ which converges to $\sup\{v : v \in \mathcal{U}_G(Q_T)\}$. In addition, the function $u := \sup\{v : v \in \mathcal{U}_G(Q_T)\}$ belongs to $\mathcal{U}_G(Q_T)$.*

Proof. We recall that by definition, $u = \sup\{v : v \in \mathcal{U}_G(Q_T)\}$ is defined by

$$u(x,t) = \sup\{v : v \in \mathcal{U}_G(Q_T)\}(x,t) := \sup\{v(x,t) : v \in \mathcal{U}_G(Q_T)\} \quad \text{for all } (x,t) \in Q_T.$$

If u_1 and u_2 belong to $\mathcal{U}_G(Q_T)$, then u_1+u_2 is a supersolution of (3.13) which vanishes on G. Hence $u_1 \vee u_2$ is a solution smaller than u_1+u_2, hence $u_1 \vee u_2 \in \mathcal{U}_G(Q_T)$. By Proposition 5.6, there exists an increasing sequence $\{u_n\} \subset \mathcal{U}_G(Q_T)$ which converges to u. Then

$$\int_0^T \int_{\mathbb{R}^N} \left(-u_n \left(\partial_t \phi(h) + \Delta \phi(h)\right) + u_n^q \phi(h)\right) dxdt + \int_{\mathbb{R}^N} u_n(x,T)\phi(h)dx = 0. \tag{5.41}$$

As in (5.21) $\{u_n^q \phi(h)\}$ and $\{u_n(x,T)\phi(h)\}$ are uniformly bounded in $L^1(Q_T)$ and $L^1(\mathbb{R}^N)$ respectively, and by Fatou's theorem $u_n^q \phi(h) \uparrow u^q \phi(h)$ in $L^1(Q_T)$ and $u_n(x,T)\phi(h) \uparrow u(x,T)\phi(h)$ in $L^1(\mathbb{R}^N)$. Furthermore, if E is any Borel subset of Q_T, we have from (5.19) and Hölder's inequality

$$\left| \int_0^T \int_E u_n \left(\partial_t \phi(h) + \Delta \phi(h)\right) dxdt \right| \leq C(T) \left(\int_0^T \int_E u_n^q \phi(h) dxdt \right)^{\frac{1}{q}} \|\eta\|_{L^\infty} \|\eta\|_{B^{\frac{2}{q},q'}},$$

and the right-hand side tends to 0 as $|E| \to 0$ since $u_n^q \phi(h) \leq u^q \phi(h) \in L^1(Q_T)$. By Vitali's convergence theorem, we infer that

$$\int_0^T \int_{\mathbb{R}^N} \left(-u \left(\partial_t \phi(h) + \Delta \phi(h)\right) + u^q \phi(h)\right) dxdt + \int_{\mathbb{R}^N} u(x,T)\phi(h)dx = 0. \tag{5.42}$$

Thus $u \in \mathcal{U}_G(Q_T)$. □

Definition 5.37 *(i) Let $u \in \mathcal{U}_+(Q_T)$ and let A denote the union of all $\mathfrak{T}_q$-open sets on which u vanishes. Then $u \in \mathcal{U}_A(Q_T)$ and A^c is called the precise initial support of u, denoted by $\mathfrak{T}_q$-supp(u).*
(ii) Let $F \subset \mathbb{R}^N$ be a Borel set, we denote by U_F the maximal element of $\mathcal{U}_{\widetilde{F^c}}(Q_T)$.

Note that by definition

$$U_F = U_{\widetilde{F}}. \tag{5.43}$$

5.4.2 Maximal solutions

If μ is a q-admissible measure, i.e. μ is absolutely continuous with respect to $cap_{\frac{2}{q},q'}$, u_μ denote the solution of (3.13) in Q_∞ with initial data μ.

Definition 5.38 *If E is a Borel set with positive $cap_{\frac{2}{q},q'}$-capacity, we set*

$$\mathcal{V}_{mod}(E) = \left\{ u_\mu : \mu \in \mathfrak{M}_+^b(\mathbb{R}^N) \cap B^{-\frac{2}{q},q}(\mathbb{R}^N)\}, \mu(E^c) = 0 \right\}$$

and

$$V_E = \sup\{u_\mu : u_\mu \in \mathcal{V}_{mod}(E)\} = \sup \mathcal{V}_{mod}(E).$$

We recall that we have proved in Section 3 the following result due to Marcus and Véron [47]. *If $F \subset \mathbb{R}^N$ is a closed set $U_F = V_F$.*

Proposition 5.39 *If $\{A_n\}$ is a collection of Borel sets such that $cap_{\frac{2}{q},q'}(A_n) \to 0$ as $n \to \infty$, then $U_{A_n} \to 0$.*

Proof. Let O_n be an open set such that $A_n \subset O_n$ and $cap_{\frac{2}{q},q'}(O_n) \leq cap_{\frac{2}{q},q'}(A_n) + \frac{1}{n}$. By the Kellogg's result in Proposition 5.9,

$$cap_{\frac{2}{q},q'}(\overline{O}_n) = cap_{\frac{2}{q},q'}(\widetilde{O}_n) \leq \widetilde{C} cap_{\frac{2}{q},q'}(O_n).$$

Therefore $cap_{\frac{2}{q},q'}(\overline{O}_n) \to 0$ when $n \to \infty$. Since

$$U_{A_n} \leq U_{\overline{O}_n},$$

and the result follows. □

Corollary 5.40 *If $E \subset \mathbb{R}^N$ is a Borel set such that $cap_{\frac{2}{q},q'}(E) = 0$ then $\mathcal{U}_{\widetilde{E^c}} = \{0\}$.*

Proposition 5.41 *Let E and F be Borel sets.*
(i) If E and F are $\mathfrak{T}_q$-closed, then $U_E \wedge U_F = U_{E \cap F}$.
(ii) If E and F are $\mathfrak{T}_q$-closed, then

$$\begin{aligned} &U_E < U_F \Longleftrightarrow E \subset^q F \text{ and } cap_{\frac{2}{q},q'}(F \setminus E) > 0. \\ &U_E = U_F \Longleftrightarrow E \sim^q F. \end{aligned} \tag{5.44}$$

(iii) If $\{F_n\}$ is a decreasing sequence of $\mathfrak{T}_q$-closed subset of $\mathbb{R}^N$, then

$$\lim_{n \to \infty} U_{F_n} = U_{\cap F_n}.$$

(iv) Let A be a $\mathfrak{T}_q$-closed subset of $\mathbb{R}^N$ and $u \in \mathcal{U}_+(Q_T)$. Assume that for any $\sigma \in A$ there exists a $\mathfrak{T}_q$-open subset A_σ of $\mathbb{R}^N$ containing σ and contained in A such that

$$u \approx_{A_\sigma} 0.$$

Then u vanishes on A. In particular any $u \in \mathcal{U}_+(Q_T)$ vanishes on the complement of the $\mathfrak{T}_q$ support of u.

Proof. (i) $U_E \wedge U_F$ is the largest solution below $\inf\{U_E, U_F\}$. Hence it is the largest solution which vanishes outside $E \cap F$.
(ii) By construction, $E \sim^q F$ implies $U_E = U_F$, and $U_E < U_F \Longleftrightarrow E \subset^q F$. Furthermore, if $cap_{\frac{2}{q},q'}(F \setminus E) > 0$ there exists a compact set $K \subset F \setminus E$ with $cap_{\frac{2}{q},q'}(K) > 0$. Hence $0 < U_K \leq U_F$. Consequently, $u_E = U_F$ implies $E \sim^q F$.
(iii) Let $V = \lim_{n \to \infty} U_{F_n}$. Since $F \subset F_n$, we have $F \subset^q F$, hence $U_F \leq U_{F_n}$ which implies $U_F \leq V$. But the $\mathfrak{T}_q$-support of V is included in F_n, therefore is is alsoe included in $F = \cap_n F_n$, which implies $v \leq U_F$, and finally $V = U_F$.
(iv) First we assume that $A = \cup_n A_n$ where A_n is $\mathfrak{T}_q$-open and $u \approx_{A_n} 0$ for every n. Then, for every $k \in \mathbb{N}_*$, u vanishes on $\cup_{n=0}^k A_k$, and we can assume that the sequence $\{A_k\}$ is

increasing. Set $F_n = A_n^c$. Then $u \le U_{F_n^c}$ and by (iii) $U_{F_n^c} \downarrow U_F$, thus $u \le U_F$. Equivalently $u \approx_A 0$.
In the general case, we use the quasi-Lindelöf property which is satisfied by the $\mathfrak{T}_q$-topology. From the covering of A by the family of $\mathfrak{T}_q$-open subsets of A indexed by the $\sigma \in A$, we can extract a countable subcovering A_{σ_n} such that $cap_{\frac{2}{q},q'}(A \setminus \cup_n A_{\sigma_n}) = 0$. Since $u \approx_{\cup_n A_{\sigma_n}} 0$, the claim follows. □

Proposition 5.42 *(i) Let E be a $\mathfrak{T}_q$-closed set. Then*

$$U_E = \inf\{U_D : E \subset D, D \text{ open }\} = \sup\{U_K : K \subset E, K \text{ closed }\} \tag{5.45}$$

(ii) Let E, F be Borel sets. Then

$$U_E = U_{F\cap E} \oplus U_{F\cap E^c}.$$

(iii) Let $E, \{F_n\}$ be a countable family of Borel sets. Assume either $cap_{\frac{2}{q},q'}(E\Delta F_n) \to 0$, or $\widetilde{F}_n \downarrow \widetilde{E}$. Then

$$U_{F_n} \to U_E \quad \text{as } n \to \infty.$$

Proof. (i) Let $\{D_j\}$ be the decreasing sequence of open sets containing E already used in Lemma 5.18 and satisfying

$$\cap_j D_j = \cap_j \widetilde{D}_j = E' \sim^q E.$$

Then, by Proposition 5.41, there holds $U_{D_j} \to U_E$, which implies the first equality in (i). For the second equality, let $\{F_n\}$ be a nondecreasing sequence of compact subsets of E such that $cap_{\frac{2}{q},q'}(E \setminus F_n) \to 0$. If $\{D_j\}$ is the decreasing sequence used above, then $cap_{\frac{2}{q},q'}(D_j \setminus E) \to 0$. Because $E \subset F_n \cup (D_n \cap F_n^c)$ we have

$$U_{F_n} \le U_E \le U_{F_n} + U_{D_n\setminus F_n}.$$

But

$$cap_{\frac{2}{q},q'}(D_n \setminus F_n) \le cap_{\frac{2}{q},q'}(E \setminus F_n) + cap_{\frac{2}{q},q'}(D_n \setminus E) \to 0 \quad \text{as } n \to \infty.$$

By Proposition 5.39 $U_{D_n\setminus F_n} \to 0$. This implies the claim.
(ii) Using (5.45) we have

$$U_E \le U_{E\cap F} + U_{E\cap F^c} \text{ hence } U_E \le U_{E\cap F} \oplus U_{E\cap F^c}.$$

Since $U_{E\cap F}$ and $U_{E\cap F^c}$ vanish outside $\widetilde{E}$, it follows that $U_{E\cap F} \oplus U_{E\cap F^c}$ vanishes outside $\widetilde{E}$, hence

$$U_E = U_{\widetilde{E}} \ge U_{E\cap F} \oplus U_{E\cap F^c}.$$

which is the claim.
(iii) Using (ii) we have

$$U_E \le U_{E\cap F_n^c} + U_{E\cap F_n} \text{ and } U_{F_n} \le U_{F_n\cap E^c} + U_{F_n\cap E}.$$

If $cap_{\frac{2}{q},q'}(E\Delta F_n) \to 0$, then $U_{E\Delta F_n} \to 0$ by Proposition 5.39.
If $\widetilde{F}_n \downarrow E$, the result follows by (iii). □

Theorem 5.43 *If E is a $\mathfrak{T}_q$-closed set, then V_E and U_E satisfy the same capacitary estimates as if E were a closed set. Hence $V_E = U_E$ and therefore U_E is σ-moderate.*

Proof. The proof follows [46, Theorem 3.10] If $\{E_k\}$ is a proper q-stratification of E and μ is a bounded nonnegative measure belonging to $B^{-\frac{2}{q},q}(\mathbb{R}^N)\}$ and satisfying $\mu(E^c) = 0$, then

$$u_\mu = \sup\{u_{\mu_k} : \mu_k = \mathbf{1}_{E_k}\mu\}.$$

Therefore $V_E = \sup_n V_{E_n}$. By Marcus-Véron's theorem (Section 3), $V_{E_k} = U_{E_k}$, and by Proposition 5.42-(iii), $U_{E_k} \to U_E$. Hence $U_E = V_E$.
Note also that if W_{E_k} is the capacitary potential defined by (4.6) with F replaced by E_k. Hence

$$cap_{\frac{2}{q},q'}\left(\frac{E_k \cap F_n(x,t)}{\sqrt{t(n+1)}}\right) \to cap_{\frac{2}{q},q'}\left(\frac{E \cap F_n(x,t)}{\sqrt{t(n+1)}}\right) \quad \text{as } k \to \infty.$$

Then by the Lebesgue convergence theorem (applied to series) $W_{E_n}(x,t) \to W_F(x,t)$. Hence if E is just $\mathfrak{T}_q$-closed set U_E satisfies the same capacitary quasi-representation as if it were closed and given in Theorem 4.8 and Theorem 4.21. □

5.4.3 The local restrictions

The local restrictions are key processes compatible with the supercritical range. They roughly consist in truncating a solution u of (3.13) outside a Borel set A^c. More precisely,

Definition 5.44 *Let A be a Borel subset of $\mathbb{R}^N$. We denote by $[u]_A$ the supremum of the $v \in \mathcal{U}_+(Q_T)$ which are smaller than u and vanish on $\widetilde{A}^c$. Equivalently $[u]_A = u \wedge U_A$, that is the largest solution smaller than the subsolution* $\inf\{u, U_A\}$.

The following result is an immediate consequence of the fact that $U_A = U_{\widetilde{A}}$ and $[u]_A = u \wedge U_A$.

Lemma 5.45 *For any Borel set $A \subset \mathbb{R}^N$, $[u]_A = [u]_{\widetilde{A}}$.*

Lemma 5.46 *If $G \subset \mathbb{R}^N$ is a $\mathfrak{T}_q$-open set and $u \in \mathcal{U}_G(Q_T)$, then*

$$u = \sup\left\{v \in \mathcal{U}_G(Q_T) : v \le u,\ v \text{ vanishes in some open neighbourhood of } \widetilde{G}\right\}. \quad (5.46)$$

Proof. Set $A = G^c$ and $\{A_n\}$ be a nondecreasing sequence of closed subsets of A such that $cap_{\frac{2}{q},q'}(A \cap A_n^c) \to 0$ as $n \to \infty$. By Proposition 5.42, there holds

$$U_A \le U_{A_n} + U_{A \cap A_n^c}.$$

Hence

$$u = u \wedge U_A \le u \wedge U_{A_n} + u \wedge U_{A \cap A_n^c}.$$

By Proposition 5.39, $U_{A \cap A_n^c} \to 0$ as $n \to \infty$. Therefore $u \wedge U_{A \cap A_n^c}$ converges also to 0, and

$$u = \lim_{n \to \infty} u \wedge U_{A_n},$$

which implies the claim. □

In the next result we analyse the regularity of the correspondence $E \mapsto [u]_E$.

Proposition 5.47 *Let $u \in \mathcal{U}_+(Q_T)$.*
(i) If E is $\mathfrak{T}_q$-closed, then,

$$[u]_E = \inf\{[u]_D : E \subset D,\ D \text{ open}\} = \inf\{[u]_F : F \subset E,\ F \text{ closed}\}. \tag{5.47}$$

(ii) If E and F are two $\mathfrak{T}_q$-closed sets then

$$[u]_E \le [u]_{F\cap E} + [u]_{E\cap F^c}, \tag{5.48}$$

and

$$[[u]_E]_F = [[u]_F]_E = [u]_{F\cap E}. \tag{5.49}$$

(iii) Let E and F_n, $n = 1, 2, ...$ be Borel sets. If either $cap_{\frac{2}{q},q'}(E\Delta F_n) \to 0$, or $\widetilde{F}_n \downarrow \widetilde{E}$, then

$$[u]_{F_n} \to [u]_E.$$

Proof. Mutatis mutandis the arguments we use are very similar to the ones in [46], but we keep them for the sake of completeness.
(i) Let $\mathcal{D} = \{D\}$ be the family of all open sets containing E as in (5.47). Using the first equality of (5.45), we have

$$\inf\{u, U_E\} = \inf\left\{u, \inf_{D\in\mathcal{D}} U_D\right\} = \inf \inf_{D\in\mathcal{D}} \{u, U_D\} \ge \inf_{D\in\mathcal{D}} [u]_D. \tag{5.50}$$

Clearly

$$[u]_{D_1} \wedge [u]_{D_2} \ge [u]_{D_1\cap D_2},$$

then it is a consequence of Proposition 5.6 that $v = \inf_{D\in\mathcal{D}} [u]_D$ is a solution of (3.13). It follows from (5.50) that $[u]_E \ge v$. The reverse inequality is clear.
For the second equality, let $\{F_n\}$ be a nondecreasing sequence of compact subsets of E such that $cap_{\frac{2}{q},q'}(E \setminus F_n) \to 0$. If $\{D_j\}$ is the decreasing sequence used above, then $cap_{\frac{2}{q},q'}(D_j \setminus E) \to 0$. Because $E \subset F_n \cup (D_n \cap F_n^c)$ we have

$$U_{F_n} \le U_E \le U_{F_n} + U_{D_n\setminus F_n}.$$

But

$$cap_{\frac{2}{q},q'}(D_n \setminus F_n) \le cap_{\frac{2}{q},q'}(E \setminus F_n) + cap_{\frac{2}{q},q'}(D_n \setminus E) \to 0 \quad \text{as } n \to \infty.$$

By Proposition 5.39 $U_{D_n\setminus F_n} \to 0$. This implies the claim.
(ii) Let $v \in \mathcal{U}_+(Q_T)$, dominated by u with $\mathfrak{T}_q$ support in E and let D and D' be open sets such that $\widetilde{E\cap F} \subset D$ and $\widetilde{E\cap F^c} \subset D'$. For any integer $j > \frac{1}{T}$, let v_j^1 be the solution of (3.13) in $\mathbb{R}^N \times (\frac{1}{j}, T)$ satisfying $v_j^1(., \frac{1}{j}) = v(., \frac{1}{j})\mathbf{1}_D$. We also denote by v_j^2 and v_j^3 the solution of (3.13) in $\mathbb{R}^N \times (\frac{1}{j}, T)$ with respective initial data $v_j^2(., \frac{1}{j}) = v(., \frac{1}{j})\mathbf{1}_{D'}$ and

$v_j^3(.,\frac{1}{j}) = v(.,\frac{1}{j})\mathbf{1}_{(D\cup D')^c}$. Since v vanishes outside E, it vanishes in $(D\cup D')^c$, consequently $v(.,\frac{1}{j})\mathbf{1}_{(D\cup D')^c} \to 0$ when $j \to \infty$, which implies v_j^3 when $j \to \infty$. Therefore

$$v \le \liminf_{j\to\infty}(v_j^1 + v_j^2) \le [u]_D + [u]_{D'}.$$

Since $\widetilde{E\cap F} \subset D$ and $\widetilde{E\cap F^c} \subset D'$, it follows from (5.47)

$$v \le [u]_{\widetilde{E\cap F}} + [u]_{\widetilde{E\cap F^c}} = [u]_{E\cap F} + [u]_{E\cap F^c}.$$

This implies (5.48).
For proving (5.49), we just have to notice that

$$[[u]_E]_F = [u]_E \vee U_F = (u \vee U_E) \vee U_F = [\max\{u, U_E, U_F\}]_\dagger = [[u]_F]_E.$$

(iii) By (5.48) there holds

$$[u]_E \le [u]_{F_n\cap E} + [u]_{E\cap F_n^c} \quad \text{and } [u]_{F_n} \le [u]_{F_n\cap E} + [u]_{F_n\cap E^c}.$$

if $cap_{\frac{2}{q},q'}(E\Delta F_n) \to 0$, then by Proposition 5.39 $U_{E\Delta F_n} \to 0$. Since

$$\max\{[u]_{E\cap F_n^c}, [u]_{F_n\cap E^c}\} \le U_{E\Delta F_n},$$

we have that $U_{E\Delta F_n} \to 0$ when $n \to \infty$.
If $\widetilde{F}_n \downarrow \widetilde{E}$, then $U_{E_n} \to U_E$ by Proposition 5.39. Therefore

$$[u]_E \le \lim_{n\to\infty} U_{E_n} = \lim_{n\to\infty} u \vee U_{E_n} \le \lim_{n\to\infty}\{u, U_{E_n}\} \le \inf\{u, U_E\}.$$

Since $[u]_E$ is the largest solution dominated by $\inf\{u, U_E\}$ and the function $v = \lim_{n\to\infty}[u]_{F_n}$ is a solution, there holds $U_E \le v$. Thus (iii) follows. □

Definition 5.48 *Let μ be a nonnegative Radon measure which is absolutely continuous with respect to the $cap_{\frac{2}{q},q'}$-capacity.*
(i) The $\mathfrak{T}_q$-support of μ, denoted by $\mathfrak{T}_q$-supp(μ) is the intersection of all the $\mathfrak{T}_q$-closed sets F such that $\mu(F^c) = 0$.
(ii) We say that μ is concentrated on a Borel set E if $\mu(E^c) = 0$.

Proposition 5.49 *Let μ be a Radon measure as in Definition 5.48. Then*

$$\mathfrak{T}_q\text{-}supp\,(\mu) \sim^q \mathfrak{T}_q\text{-}supp\,(u_\mu).$$

Proof. Set $F = \mathfrak{T}_q$-supp(u_μ). By Proposition 5.41-(iv), u_μ vanishes on F^c, and by Lemma 5.46 there exists an increasing sequence of positive solutions $\{u_n\}$ vanishing outside a closed subset F_n and converging to u. Set $S_n := \mathfrak{T}_q$-supp (u_n). Then $S_n \subset F_n$ and $S_n \subset S_{n+1}$. Thus $\{\overline{S}_n\}$ is an increasing sequence of closed subsets of F. If we set

$\mu_n = \mathbf{1}_{\overline{S}_n}\mu$, we have that $u_n \le u_{\mu_n} \le u_\mu$. Hence the increasing sequence $\{u_{\mu_n}\}$ converges to u_μ as $n \to \infty$. Consequently

$$\mu_n \uparrow \mu \quad \text{and } \mathfrak{T}_q\text{-supp}\,(\mu) \subset^q \widetilde{\bigcup_{n=1}^{\infty}} \overline{S}_n \subset F.$$

If D is open and $\mu(D) = 0$, then u_μ vanishes in D. Therefore u_{μ_n} vanishes outside $\overline{S}_n$ and consequently it vanishes outside $\mathfrak{T}_q$-supp (μ). Hence u_μ vanishes outside $\mathfrak{T}_q$-supp (μ). This means $F \subset^q \mathfrak{T}_q$-supp (μ). □

Definition 5.50 *Let $u \in \mathcal{U}_+(Q_T)$ and A be a Borel set. Then*

$$[u]^A := \sup\{[u]_F : F \subset^q A,\ F\ \mathfrak{T}_q\text{-closed}\}.$$

Remark. Note that since $[u]_E = [u]_{\widetilde{E}}$, if A is $\mathfrak{T}_q$-closed, we have $[u]_A = [u]^A$. In the general case, we have only $[u]^A \le [u]_A$.

Definition 5.51 *Let $\beta > 0$ and $u \in C(Q_T)$, $u \ge 0$. For any Borel set $A \subset \mathbb{R}^N$, we denote by u_β^A the solution of*

$$\begin{aligned} \partial_t v - \Delta v + |v|^{q-1}v = 0 \qquad & \text{in } \mathbb{R}^N \times (\beta, \infty) \\ v(.,\beta) = \mathbf{1}_A u(.,\beta) \quad & \text{in } \mathbb{R}^N. \end{aligned}$$

Proposition 5.52 *Let $u \in \mathcal{U}_+(Q_T)$ and E be $\mathfrak{T}_q$-supp (u).*
(i) If D is a $\mathfrak{T}_q$-open set such that $E \subset^q D$, then

$$[u]^D = \lim_{\beta \to 0} u_\beta^D = [u]_D = u. \tag{5.51}$$

(ii) If A is a $\mathfrak{T}_q$-open set, then

$$u \approx_A 0 \Longleftrightarrow u^Q = \lim_{\beta \to 0} u_\beta^Q = 0 \quad \text{for all } \mathfrak{T}_q\text{-open set s.t. } \widetilde{Q} \subset^q A. \tag{5.52}$$

(iii) Finally,

$$u \approx_A 0 \Longleftrightarrow [u]^A = 0. \tag{5.53}$$

Proof. Case 1: Assume first that E is closed. Since u vanishes on E^c and is continuous in $Q_T \cup E^c \times \{0\}$, we have that $u = 0$ on E^c hence $u \in C(Q_T \cup E^c \times \{0\}$. If D is an open neighbourhood of E, then for all $\phi \in C_c(E^c)$ there holds

$$\lim_{t \to 0} \int_{E^c} u(x,t)\phi(x)dx = 0.$$

Therefore

$$\lim_{\beta \to 0} u_\beta^{E^c} = 0.$$

But

$$u_\beta^D(x,t) \le u(x,t) \le u_\beta^D(x,t) + u_\beta^{D^c}(x,t) \quad \text{for all } (x,t)) \in \mathbb{R}^N \times [\beta, T).$$

From this relation we deduce that

$$u = \lim_{\beta \to 0} u_\beta^D. \tag{5.54}$$

If we assume now that D is $\mathfrak{T}_q$-open and $E \subset^q D$, then for avery $\epsilon > 0$ there exists an open set $\mathcal{O}_\epsilon$ such that $D \subset \mathcal{O}_\epsilon$, $E \subset \mathcal{O}_\epsilon$ and $cap_{\frac{2}{q},q'}(\mathcal{O}_\epsilon \cap D^c) < \epsilon$. Therefore

$$u_\beta^{\mathcal{O}_\epsilon}(x,t) - u_\beta^D(x,t) \le U_{\mathcal{O}'_\epsilon}(x, t-\beta) \quad \text{for all } t \ge \beta,$$

where $\mathcal{O}'_\epsilon = \mathcal{O}_\epsilon \cap D^c$. We observe that $\lim_{\epsilon \to 0} U_{\mathcal{O}'_\epsilon}(x, t-\beta) = 0$ uniformly w.r. to β. Since $\lim_{\beta \to 0} u_\beta^{\mathcal{O}_\epsilon}(x,t) = u(x,t)$ for all $(x,t) \in Q_T$, it follows that $\lim_{\beta \to 0} u_\beta^D(x,t) = u(x,t)$. The same argument shows that $\lim_{\beta \to 0} u_\beta^{D^c}(x,t) = 0$ for all $(x,t) \in Q_T$. Combining all these results we obtain

$$\lim_{\beta \to 0} u_\beta^D \le [u]_D \le u,$$

hence $[u]_D = u$. By Proposition 5.19 there exists a $\mathfrak{T}_q$-open set Q such that $E \subset^q \subset Q \subset \widetilde{Q} \subset D$, therefore $u = [u]_Q \le [u]^D$, hence $u = [u]^D$.

In addition there holds $E \subset^q A^c \subset^q \widetilde{Q}^c$. If we replace D by $\widetilde{Q}^c$ in the above argument, we have that $u \approx_A 0$ which implies $u^Q = \lim_{\beta \to 0} u_\beta^Q = 0$. For the opposite implication in equivalence (5.52) we use the fact that for any $\xi \in A$ there exists a $\mathfrak{T}_q$-open neighbourhood $\mathcal{O}_\xi$ of ξ such that $\widetilde{\mathcal{O}}_\xi \subset^q A$. By (i) we have that $\lim_{\beta \to 0} u_\beta^{\widetilde{\mathcal{O}}_\xi^c}$. Finally, since $u_\beta^{\widetilde{\mathcal{O}}_\xi^c} \approx_{\mathcal{O}_\xi} 0$ for all $\beta > 0$, we deduce that $u \approx_{\mathcal{O}_\xi} 0$ by Proposition 5.39. Using Proposition 5.41 (iv) we deduce (5.52) in the case where E is closed.

Case 2: Assume next that E is $\mathfrak{T}_q$-closed. Let $\{E_n\}$ be a $\mathfrak{T}_q$-stratification of E such that $cap_{\frac{2}{q},q'}(E \cap E_n^c) \to 0$ as $n \to \infty$. If D is a $\mathfrak{T}_q$-open set such that $E \subset^q D$, then by Case 1,

$$\lim_{\beta \to 0} ([u]_{E_n})_\beta^D = [u]_{E_n}. \tag{5.55}$$

By Proposition 5.47-(ii), using the definition of u_β^D and the fact that $[u]_E = u$,

$$u_\beta^D = ([u]_E)_\beta^D \le ([u]_{E \cap E_n})_\beta^D + ([u]_{E \cap E_n^c})_\beta^D = ([u]_{E_n})_\beta^D + ([u]_{E \cap E_n^c})_\beta^D. \tag{5.56}$$

Let $\{\beta_k\}$ be a sequence decreasing to 0 such that there exists

$$w := \lim_{\beta_k \to 0} u_{\beta_k}^D \quad \text{and } w_n := \lim_{\beta_k \to 0} ([u]_{E \cap E_n^c})_{\beta_k}^D \text{ for } n = 1, 2, ...$$

Then, using the two previous inequalities

$$[u]_{E_n} \le w \le [u]_{E_n} + w_n \le [u]_{E_n} + U_{E \cap E_n^c}.$$

Using (5.46) and the fact that $U_{E\cap E_n^c} \to 0$ and $U_{E_n} \to U_E$, we deduce that $w = u$. This implies (i).

In order to prove (ii), we apply (5.56) with D replaced by Q and get

$$([u]_E)^Q_\beta \leq ([u]_{E_n})^Q_\beta + ([u]_{E\cap E_n^c})^Q_\beta.$$

From Case 1 we have already proved that

$$\lim_{\beta\to 0}([u]_{E_n})^Q_\beta = 0.$$

There exists a decreasing sequence $\{\beta_k\}$ such that $\{u^Q_{\beta_k}\}$ and $\{([u]_{E\cap E_n^c})^Q_{\beta_k}\}$ admit a limit when $\beta_k \to 0$, for any $n = 1, 2, ...$ Therefore

$$\lim_{\beta_k\to 0} u^Q_{\beta_k} \leq \lim_{\beta_k\to 0}([u]_{E\cap E_n^c})^Q_{\beta_k} \leq U_{E\cap E_n}.$$

Since $U_{E\cap E_n} \to 0$ as $n \to \infty$, this implies the implication $\Longrightarrow$ in (5.52). The implication $\Longleftarrow$ in (5.52) is proved as in Case 1.

Proof of (iii). We assume first that $u \approx_A 0$. If F is a $\mathfrak{T}_q$-closed set such that $F \subset^q A$, then, by Lemma 5.17 there exists a $\mathfrak{T}_q$-open set Q such that $F \subset^q Q \subset \widetilde{Q} \subset^q A$. Applying (5.51) to $v := [u]_F$ and using (5.52) we obtain

$$v = \lim_{\beta\to 0} v^Q_\beta \leq \lim_{\beta\to 0} u^Q_\beta = 0.$$

It is thus a consequence of the definition of $[u]^A$ that $[u]^A = 0$.
If $[u]^A = 0$, then for any $\mathfrak{T}_q$-open set $Q \subset \widetilde{Q} \subset^q A$, there holds $[u]_Q = 0$. Because $\mathfrak{T}_q$-supp$(u^Q_\beta) \subset^q \widetilde{Q}$, there exists a subsequence β_k decreasing to 0 such that

$$\lim_{\beta_k\to 0} u^Q_{\beta_k} \leq [u]_Q = 0.$$

Therefore $u \approx_Q 0$ by (5.52). Applying again Lemma 5.17 and Proposition 5.41-(iv), we infer that $u \approx_A 0$. □

Definition 5.53 *Let $u, v \in \mathcal{U}_+(Q_T)$ and A be a $\mathfrak{T}_q$-open set. We say that $u = v$ on A if both $u \ominus v$ and $v \ominus u$ vanish on A. This relation is denoted by $u \approx_A v$.*

Proposition 5.54 *Let $u, v \in \mathcal{U}_+(Q_T)$ and A be a $\mathfrak{T}_q$-open set. Then,*
(i)

$$u \approx_A v \Longleftrightarrow \lim_{\beta\to 0} |u - v|^Q_\beta = 0, \tag{5.57}$$

for every $\mathfrak{T}_q$-open set Q such that $\widetilde{Q} \subset^q A$.
(ii)

$$u \approx_A v \Longleftrightarrow [u]_F = [v]_F, \tag{5.58}$$

for every $\mathfrak{T}_q$-closed set F such that $F \subset^q A$.

Proof. The idea of the proof is the adaptation to the parabolic framework of the construction in the elliptic case performed in [46]. If $u \approx_A v$, then $u \ominus v \approx_A 0$ and $v \ominus u \approx_A 0$. Hence, by (5.52), we have that $w_\beta := (u \ominus v)^Q_\beta \to 0$ as $\beta \to 0$. We set $f_\beta = ((u-v)_+)^Q_\beta$ and consider the truncated problem in $B_j \times (\beta, \infty)$ for $j = 1, 2, ...,$

$$\begin{aligned} \partial_t w - \Delta w + |w|^{q-1} w = 0 \qquad & \text{in } B_j \times (\beta, \infty) \\ w = 0 \qquad & \text{on } \partial B_j \times (\beta, \infty) \\ w(., \beta) = \phi \qquad & \text{in } B_j, \end{aligned}$$

and denote by w_j and f_j respectively the solutions with initial data $\mathbf{1}_Q(u \ominus v)(., \beta)$ and $\mathbf{1}_Q(u-v)_+(., \beta)$. By the maximum principle, the sequences $\{w_j\}$ and $\{f_j\}$ are increasing. Since $u \ominus v$ is the smallest solution which dominates the subsolution $(u-v)_+$, we have $w_j \geq f_j$ for all $j \in \mathbb{N}_*$. When $j \to \infty$, $w_i \to w_\beta$ and $f_i \to f_\beta$. Then $w_\beta \geq f_\beta$. This implies

$$((u-v)_+)^Q_\beta \quad \text{as } \beta \to 0.$$

Similarly

$$((v-u)_+)^Q_\beta \quad \text{as } \beta \to 0.$$

This yields the implication $\Longrightarrow$ in (5.57).
For the reverse implication we introduce the problem

$$\begin{aligned} \partial_t w - \Delta w + |w|^{q-1} w = 0 \qquad & \text{in } B_j \times (\beta, \infty) \\ w = h \qquad & \text{on } \partial B_j \times (\beta, \infty) \\ w(., \beta) = \phi \qquad & \text{in } B_j. \end{aligned}$$

Let $Q \subset \widetilde{Q} \subset^q A$ be a $\mathfrak{T}_q$-open. Denote by w_j the solution of the above problem with $h = \mathbf{1}_Q |u-v|\lfloor_{\partial B_j \times (\beta,\infty)}$ and $\phi = \mathbf{1}_Q |u-v|$, and f_j the solution with $h = \mathbf{1}_{Q^c} |u-v|\lfloor_{\partial B_j \times (\beta,\infty)}$ and $\phi = \mathbf{1}_{Q^c} |u-v|$. Then

$$|u-v| \leq w_j + f_j.$$

Up to some subsequence, w_j and f_j converge respectively to w and f which are solutions of (3.13) in $\mathbb{R}^N \times (\beta, \infty)$ with respective initial data $w(., \beta) = \mathbf{1}_Q |u-v|(., \beta)$ and $f(., \beta) = \mathbf{1}_{Q^c} |u-v|(., \beta)$. because of uniqueness and the definition Definition 5.51 , $w = |u-v|^Q_\beta$ and $f = |u-v|^{Q^c}_\beta$. When $\beta \to 0$ we have by assumption

$$\lim_{\beta \to 0} |u-v|^Q_\beta = 0.$$

Let $\{\beta_k\}$ be a subsequence decreasing to 0 such that there exists $\lim_{\beta_k \to 0} |u-v|^{Q^c}_\beta$. Then

$$|u-v| \leq \lim_{\beta_k \to 0} |u-v|^{Q^c}_{\beta_k}.$$

But $|u-v|^{Q^c}_{\beta_k} \approx_Q 0$, hence $\lim_{\beta_k \to 0} |u-v|^{Q^c}_{\beta_k} \approx_Q 0$. Since $u \ominus v$ is the smallest solution which dominates the subsolution $(u-v)_+$ there holds

$$\max\{u \ominus v, v \ominus u\} \leq \lim_{\beta_k \to 0} |u-v|^{Q^c}_{\beta_k}.$$

The proof follows from Lemma 5.46 and Proposition 5.41-(i).
(ii) Let us assume that $u \approx_A v$, then

$$u + (u - v)_+ \leq v + (u - v)_+ \leq v + u \ominus v. \tag{5.59}$$

If F is a $\mathfrak{T}_q$-closed set and Q a $\mathfrak{T}_q$-open set such that $F \subset^q Q$, we claim that

$$[u]_F \leq [v]_Q + [u \ominus v]_Q. \tag{5.60}$$

This can be proved as follows: we first have

$$u = [u]_{\mathbb{R}^N} \leq [u]_Q + [u]_{Q^c}$$

by (5.48). Using (5.59) it infers

$$[u]_F \leq [u]_{\mathbb{R}^N} \leq v + u \ominus v \leq [v]_Q + [v]_{Q^c} + [u \ominus v]_Q + [u \ominus v]_{Q^c}.$$

The subsolution $w := ([u]_F - ([v]_Q + [u \ominus v]_Q))_+$ is dominated by $[u \ominus v]_{Q^c} + [v]_{Q^c}$ which is a supersolution. From the definition we have

$$w \leq [w]_\dagger \leq [u \ominus v]_{Q^c} \oplus [v]_{Q^c} \leq [u \ominus v]_{Q^c} + [v]_{Q^c}.$$

Therefore $[w]_\dagger \approx_Q 0$. Since $w \leq [u]_F$ we deduce $[w]_\dagger \leq [u]_F$, which means $\mathfrak{T}_q\text{-supp}([w]_\dagger) \subset^q F \subset^q Q$. As $[w]_\dagger \approx_Q 0$ we obtain that $w = [w]_\dagger = 0$ and (5.60) follows.

Let Q be a $\mathfrak{T}_q$-open set such that $F \subset^q Q \subset \widetilde{Q} \subset^q A$, and because $u \ominus v \approx_A 0$ implies $[u \ominus v]_F = 0$ by (5.53) and (5.60), we deduce that

$$[u]_F \leq [v]_Q.$$

By Lemma 5.18-I, there exists a decreasing sequence $\{Q_j\}$ of open sets such that $\cap_j Q_j \sim^q F$. Then by Proposition 5.42-(iii) there holds

$$[u]_F \leq \lim_{j \to \infty} [v]_{Q_j} \leq [v]_F.$$

Similarly $[v]_F \leq [u]_F$.

To prove the reverse implication, we assume that $[v]_F = [u]_F$ for any $\mathfrak{T}_q$-closed set $F \subset^q A$. If Q is a $\mathfrak{T}_q$-open set such that $F \subset^q Q \subset^q \widetilde{Q} \subset A$, we notice that

$$u \ominus v \leq [u]_Q \oplus [u]_{Q^c} \ominus [v]_Q,$$

since (for the last inequality)

$$u = [u]_{\mathbb{R}^N} \leq [u]_Q + [u]_{Q^c} \Longrightarrow u \leq [u]_Q \oplus [u]_{Q^c} \leq [u]_Q + [u]_{Q^c}.$$

Because $([u]_Q \oplus [u]_{Q^c}) \ominus [v]_Q$ is the smallest solution dominating $(([u]_Q \oplus [u]_{Q^c}) \ominus [v]_Q)_+$, we have, using the assumption that $[u]_Q = [v]_Q$,

$$(([u]_Q \oplus [u]_{Q^c}) \ominus [v]_Q)_+ \leq (([u]_Q + [u]_{Q^c}) \ominus [v]_Q)_+ = [u]_Q + [u]_{Q^c} - [v]_Q = [u]_{Q^c}.$$

Therefore

$$[u \ominus v]_F \le u \ominus v \le [u]_{Q^c}.$$

Hence $\mathfrak{T}_q$-supp$([u \ominus v]_F) \subset^q F$ and $[u \ominus v]_F \approx Q0$. This in turn implies that $[u \ominus v]_F = 0$. Using (5.53) in Proposition 5.52 we obtain $u \ominus v \approx_A 0$. Similarly $v \ominus u \approx_A 0$. □

As an immediate consequence of (5.57), we have

Corollary 5.55 *If $A \subset \mathbb{R}^N$ is a $\mathfrak{T}_q$-open set, the relation $\approx_A$ is an equivalence relation in $\mathcal{U}_+(Q_T)$.*

5.5 The regular initial trace

5.5.1 The local test

Lemma 5.56 *Let $u \in \mathcal{U}_+(Q_T)$ and Q be a $\mathfrak{T}_q$-open set. Then for any $\eta \in B^{\frac{2}{q},q'}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $\mathfrak{T}_q$-support in $\widetilde{Q}^c$, we have*

$$\int_0^T \int_{\mathbb{R}^N} (u \wedge U_Q)^q \left(\mathbb{H}[\eta]_+\right)^{2q'}(t,x)dxdt < \infty. \tag{5.61}$$

Proof. By Proposition 5.33, there holds

$$\lim_{t\to 0} \int_Q (u \wedge U_Q)(x,t)\eta_+(x)dx = 0,$$

and the result follows by estimate (5.39) in Lemma 5.35. □

Proposition 5.57 *Let $u \in \mathcal{U}_+(Q_T)$ and Q be a $\mathfrak{T}_q$-open set such that $u \wedge U_Q$ is a moderate solution with initial data μ. Then for any $\xi \in Q$ there exists a $\mathfrak{T}_q$-open set $\mathcal{O}_\xi \subset Q$ such that*

$$\int_0^T \int_{\mathbb{R}^N} u^q \left(\mathbb{H}[\mathbf{1}_{Q_\xi}]\right)^{2q'}(x,t)dxdt < \infty. \tag{5.62}$$

Furthermore, for any $\eta \in B^{\frac{2}{q},q'}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $\mathfrak{T}_q$-support in Q, we have

$$\lim_{t\to 0} \int_Q u(x,t)\eta_+^{2q'}dx = \int_Q \eta^{2q'}d\mu. \tag{5.63}$$

Proof. If η is as above, the function $\eta_+^{2q'}$ is quasi-continuous and there holds by Theorem 5.31,

$$\lim_{t\to 0} \int_Q u \wedge U_Q(x,t)\eta_+^{2q'}(x)dx = \int_Q \eta_+^{2q'}d\mu,$$

and, by the properties of U_{Q^c},

$$\lim_{t\to 0} \int_Q u \wedge U_{Q^c}(x,t)\eta_+^{2q'}(x)dx = 0.$$

Since $u \wedge U_Q \le u \le u \wedge U_Q + u \wedge U_{Q^c}$, we get

$$\int_Q u \wedge U_Q(x,t)\eta_+^{2q'}dx \le \int_Q u(x,t)\eta_+^{2q'}dx \le \int_Q u \wedge U_Q(x,t)\eta_+^{2q'}dx + \int_Q u \wedge U_{Q^c}(x,t)\eta_+^{2q'}dx.$$

This implies

$$\lim_{t\to 0}\int_Q u(x,t)\eta_+^{2q'}dx = \int_Q \eta_+^{2q'}d\mu_Q.$$

By Proposition 5.25 and Proposition 5.26,

$$\int_0^T\int_{\mathbb{R}^N} (u \wedge U_Q)^q \left(\mathbb{H}[\eta]_+\right)^{2q'}(t,x)dxdt < \infty, \tag{5.64}$$

for any $\eta \in B^{\frac{2}{q},q'}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $\mathfrak{T}_q$-support in $\widetilde{Q}$. By Lemma 5.21, we can assume that the above function η has its values in $[0,1]$, with $\mathfrak{T}_q$-support in Q and value 1 on a $\mathfrak{T}_q$-open neighbourhood $\mathcal{O}_\xi$ of ξ. Then (5.64) implies

$$\int_0^T\int_{\mathbb{R}^N} (u \wedge U_Q)^q \left(\mathbb{H}[\mathbf{1}_{\mathcal{O}_\xi}]\right)^{2q'}(t,x)dxdt < \infty. \tag{5.65}$$

□

Definition 5.58 *If $E \subset \mathbb{R}^N$) is a Borel set, we denote by $B^{\frac{2}{q},q'}(E)$ the closure for the norm of $B^{\frac{2}{q},q'}$ of the set of C^∞ functions with compact support in E.*

If E is an open set, $B^{\frac{2}{q},q'}(E)$ coincides with $B_0^{\frac{2}{q},q'}(E)$.

Proposition 5.59 *Let $u \in \mathcal{U}_+(Q_T)$ and Q be a bounded $\mathfrak{T}_q$-open set such that*

$$\int_0^T\int_{\mathbb{R}^N} u^q \left(\mathbb{H}[\mathbf{1}_Q]\right)^{2q'}(x,t)dxdt < \infty. \tag{5.66}$$

(i) There exists an increasing sequence of $\mathfrak{T}_q$-open set $\{Q_n\}$ satisfying $Q_n \subset Q$, $\widetilde{Q}_n \subset^q Q_{n+1}$ and $Q_0 := \cup_{n=1}^\infty Q_n \sim^q \widetilde{Q}$ such that the solution $v_n := u \wedge U_{Q_n}$ is moderate, $v_n \uparrow [u]_Q$ and there exists a nonnegative measure μ_Q on Q such that $tr\,(v_n) \to \mu_Q$ as $n \to \infty$.
(ii) For any $\eta \in B^{\frac{2}{q},q'}(Q) \cap L^\infty(Q)$, we have

$$\lim_{t\to 0}\int_Q u(x,t)\eta_+^{2q'}(x)dx = \int_Q \eta_+^{2q'}d\mu_Q. \tag{5.67}$$

Proof. Let $z \in Q$. By Lemma 5.21 there exist a $\mathfrak{T}_q$-open set V such that $z \in V \subset \widetilde{V} \subset Q$ and a function $\psi \in B^{\frac{2}{q},q'}(\mathbb{R}^N)$ such that $\psi = 1$ q.a.e. on V, $\psi = 0$ outside Q and $0 \le \psi \le 1$. By Lemma 5.17 there exists a $\mathfrak{T}_q$-open neighbourhood $\mathcal{O}_z$ of z such that $\mathcal{O}_z \subset \widetilde{\mathcal{O}}_z \subset V$.

We claim that the function

$$v_z = u \wedge U_{\mathcal{O}_z}, \tag{5.68}$$

is a moderate solution. Actually, let $R>0$ such that $Q\subset\bar{Q}\subset B_R$ and let $\eta\in C_c^\infty(B_{2R})$ with value 1 on B_R and $0\le\eta\le 1$. Then the function $\zeta=(1-\psi)\eta$ belongs to $B^{\frac{2}{q},q'}(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ and has compact support in $B_{2R}\cap\widetilde{V}^c$. Therefore

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^N} v_z^q\left(\mathbb{H}[\mathbf{1}_{B_R}]\right)^{2q'}(x,t)dxdt &\le \int_0^T\int_{\mathbb{R}^N} v_z^q\left(\mathbb{H}[\psi]\right)^{2q'}(x,t)dxdt\\ &\qquad+\int_0^T\int_{\mathbb{R}^N} v_z^q\left(\mathbb{H}[1-\psi]\right)^{2q'}(x,t)dxdt\\ &\le \int_0^T\int_{\mathbb{R}^N} v_z^q\left(\mathbb{H}[\psi]\right)^{2q'}(x,t)dxdt\\ &\qquad+\int_0^T\int_{\mathbb{R}^N} v_z^q\left(\mathbb{H}[\zeta]\right)^{2q'}(x,t)dxdt<\infty,\end{aligned}$$

because the first integral in the last inequality is finite by assumption and the second integral is finite by Lemma 5.56. As R is arbitrary, $u\wedge U_{\mathcal{O}_z}$ is a moderate solution.

By the quasi-Lindelöf property there exists a non decreasing sequence of $\mathfrak{T}_q$-open sets $\{\mathcal{O}_n\}$ such that $\cup_n\mathcal{O}_n\sim^q Q$ and, using the construction above, the solution $u\wedge U_{\mathcal{O}_n}$ is moderate for any $n\in\mathbb{N}_*$. By Proposition 5.42-(II), for any n there exists a sequence $\{A_{n,j}\}$ of $\mathfrak{T}_q$-open sets such that $\widetilde{A}_{n,j}\subset^q A_{n,j+1}\subset^q E_n$ and $\cup_{j=1}^\infty A_{n,j}\sim^q E_n$. Put

$$Q_n=\bigcup_{k+j=n}A_{k,j}.$$

Then

$$\widetilde{Q}_n\subset\bigcup_{k+j=n}\widetilde{A}_{k,j}\subset^q\bigcup_{k+j=n}\widetilde{A}_{k,j+1}=Q_{n+1}.$$

Therefore

$$Q_0:=\bigcup_n Q_n\sim^q Q.$$

Next we prove that $v_n=u\wedge U_{Q_n}\to u\wedge U_Q$. By Proposition 5.42-(ii),

$$v_n=u\wedge U_{Q_n}\le u\wedge U_Q\le u\wedge U_{Q_n}+u\wedge U_{Q\cap Q_n^c}.$$

Since $Q\cap Q_n^c\downarrow F$ and $cap_{\frac{2}{q},q'}(F)=0$, we infer from Proposition 5.42-(iii)

$$u\wedge U_{Q\cap Q_n^c}\to 0\quad\text{as } n\to\infty.$$

Hence $v_n\uparrow u\wedge U_Q$. Again, by Proposition 5.42-(ii), $v_n=[v_{n+k}]_{Q_n}$. Therefore, with $\mu_n=\mathbf{1}_{Q_n}\mu_Q$,

$$\mu_n(Q_n)=\mu_{n+k}(Q_n)=\mu_Q(Q_n)<\infty\Longrightarrow tr\,(v_n)\to\mu_Q. \tag{5.69}$$

(iii) We assume at first that the function $\eta\in B^{\frac{2}{q},q'}(Q)\cap L^\infty(Q)$ is nonnegative (which is not a restriction) and has compact support in Q. By Lemma 5.23 there exists a function η_k with $\mathfrak{T}_q$ support included in Q_k such that $0\le\eta_k\le\eta$, $\eta_{k+1}\le\eta_k$,

$$\|\eta-\eta_k\|_{B^{\frac{2}{q},q'}}\le\frac{1}{k}, \tag{5.70}$$

and, for k large enough,

$$\int_0^T \int_{\mathbb{R}^N} u^q \left(\mathbb{H}[\eta - \eta_k]\right)^{2q'} (x,t) dx dt \leq \left(\frac{1}{k}\right)^{2q'}.$$

Since

$$\lim_{t\to 0} \int_Q u(x,t)\eta_k dx = \int_Q \eta_k^{2q'} d\mu_Q \quad \text{and} \quad \lim_{k\to\infty} \int_Q \eta_k^{2q'} d\mu_Q = \int_Q \eta^{2q'} d\mu_Q,$$

by a standard limit theorem

$$\begin{aligned} \liminf_{t\to 0} \lim_{k\to\infty} \int_Q u(x,t)\eta_k^{2q'} dx &= \liminf_{t\to 0} \int_Q u(x,t)\eta^{2q'} dx \\ &\geq \lim_{k\to\infty} \lim_{t\to 0} \int_Q u(x,t)\eta_k^{2q'} dx = \int_Q \eta^{2q'} d\mu_Q \end{aligned} \tag{5.71}$$

By (5.69) and Proposition 5.57 and Hölder's inequality

$$\begin{aligned} \left(\int_Q u(x,t)\eta^{2q'} dx\right)^{\frac{1}{2q'}} &\leq \left(\int_Q u(x,t)(\eta-\eta_k)^{2q'} dx\right)^{\frac{1}{2q'}} + \left(\int_Q u(x,t)\eta_k^{2q'} dx\right)^{\frac{1}{2q'}} \\ &\leq \left(\int_Q u(x,t)\eta_k^{2q'} dx\right)^{\frac{1}{2q'}} + C\left\|\eta-\eta_k\right\|_{B^{\frac{2}{q},q'}} \left\|\eta-\eta_k\right\|_{L^\infty} \\ &\qquad + C\left(\int_0^T \int_{\mathbb{R}^N} u^q(x,t)\left(\mathbb{H}[\eta-\eta_k]\right)^{2q'}\right)^{\frac{1}{2q'}} \\ &\leq \left(\int_Q u(x,t)\eta_k^{2q'} dx\right)^{\frac{1}{2q'}} + \frac{C'}{k}. \end{aligned} \tag{5.72}$$

Hence

$$\limsup_{t\to 0} \left(\int_Q u(x,t)\eta^{2q'} dx\right)^{\frac{1}{2q'}} \leq \left(\int_Q \eta_k^{2q'} d\mu_Q\right)^{\frac{1}{2q'}} + \frac{C'}{k},$$

which implies, by letting $k \to \infty$,

$$\limsup_{t\to 0} \left(\int_Q u(x,t)\eta^{2q'} dx\right)^{\frac{1}{2q'}} \leq \left(\int_Q \eta^{2q'} d\mu_Q\right)^{\frac{1}{2q'}}. \tag{5.73}$$

Combining (5.71) and (5.73) we obtain (5.67).

In the general case, by Netrusov's approximation theorem [1, Theorem 10.1.1] there exists a function η_k with compact support in Q such that $0 \leq \eta_k \leq \eta$ and (5.70) holds. The end of the proof is as above. □

The Proposition 5.59 admits the following easy extension to the case where the set Q is non-necessarily bounded. An overview of the proof is given in Proposition 5.64.

Corollary 5.60 *Let $\mathcal{Q}$ be $\mathfrak{T}_q$-open set and $u \in \mathcal{U}_+(Q_T)$ satisfying* (5.66) *for any $\mathfrak{T}_q$-open and bounded subset of $\mathcal{Q}$.*
(i) There exists an increasing sequence of $\mathfrak{T}_q$-open set $\{Q_n\}$ satisfying $Q_n \subset Q$, $\widetilde{Q}_n \subset^q Q_{n+1}$ and $\mathcal{Q}_0 := \cup_{n=1}^{\infty} Q_n \sim^q \widetilde{\mathcal{Q}}$ such that the solution $v_n := u \wedge U_{Q_n}$ is moderate, $v_n \uparrow [u]_Q$ and there exists a nonnegative measure $\mu_{\mathcal{Q}}$ on $\mathcal{Q}$ such that $tr\,(v_n) \to \mu_{\mathcal{Q}}$ as $n \to \infty$.
(ii) For any $\eta \in B^{\frac{2}{q},q'}(Q_n) \cap L^{\infty}(Q_)$ for some $n \in \mathbb{N}$, we have

$$\lim_{t\to 0} \int_{\mathcal{Q}} u(x,t)\eta_+^{2q'}(x)dx = \int_{\mathcal{Q}} \eta_+^{2q'} d\mu_{\mathcal{Q}}. \tag{5.74}$$

Proposition 5.61 *Let Q be a $\mathfrak{T}_q$-open set and $u \in \mathcal{U}_+(Q_T)$ satisfying* (5.66)*. Then*
(i)

$$[u]_Q = \sup\{[u]_F : F \subset^q Q, F \ \mathfrak{T}_q\text{-closed}\}. \tag{5.75}$$

(ii) For every $\mathfrak{T}_q$-open set $\mathcal{O} \subset \widetilde{\mathcal{O}} \subset^q Q$ such that $[u]_{\mathcal{O}}$ is a moderate solution, we have

$$\mathbf{1}_{\widetilde{\mathcal{O}}}\mu_Q = tr\left(([[u]_Q]_{\mathcal{O}}\right). \tag{5.76}$$

Furthermore the measure μ_Q defined in Proposition 5.59-(i) is $\mathfrak{T}_q$-locally finite on Q and σ-finite on $Q' := \cup_n Q_n$ where the sets Q_n form an increasing sequence $\{Q_n\}$ of $\mathfrak{T}_q$-open subsets of Q satisfying $\widetilde{Q}_n \subset^q Q_{n+1}$ and $Q' \sim^q Q$ as in Proposition 5.59-(i).
(iii) If $\{w_n\} \subset \mathcal{U}_+(Q_T)$ is a nondecreasing sequence of moderate solutions of (3.13) such that $\mathfrak{T}_q$-supp.$(w_n) \subset^q Q$ and $w_n \uparrow [u]_Q$, then $tr\,(w_n) \uparrow \mu_Q$.

Proof. (i) Let u^* denote the right-hand side of (5.75). By Proposition 5.6 there exists a nonndecreasing sequence $\{[u]_{F_n}\}$ such that F_n is $\mathfrak{T}_q$-closed and $[u]_{F_n} \uparrow u^*$. By Proposition 5.47 we have

$$[u]_{F_n} \le [u]_{F_n \cap Q_m} + [u]_{F_n \cap Q_m^c}.$$

Notice that $F_n \cap Q_m^c$ is $\mathfrak{T}_q$-closed and $\cap_{m=1}^{\infty} F_n \cap Q_m^c = A_n$ and $cap_{\frac{2}{q},q'}(A_n) = 0$. Therefore, by Proposition 5.42 we have that $U_{F_n \cap Q_m^c} \to 0$ as $m \to \infty$, hence $[u]_{F_n \cap Q_m^c} \to 0$ as $m \to \infty$. Therefore $[u]_{F_n} \le \lim_{m\to\infty} [u]_{F_m} = u_Q$. Letting $n \to \infty$ we infer $u^* \le u_Q$. By the definition of u^* we have $u_Q \le u^*$.
(ii) Set $\mu_{\mathcal{O}} = tr\,([u]_{\mathcal{O}})$. If F is $\mathfrak{T}_q$-closed such that $F \subset^q \mathcal{O}$, then by Proposition 5.47-(ii),

$$tr\,([u]_F) = tr\left([[u]_{\widetilde{\mathcal{O}}}]_F\right) = \mathbf{1}_F\,\mu_{\mathcal{O}} = \mathbf{1}_F\,\mu_{\mathcal{O}}. \tag{5.77}$$

If $\mathcal{O}' \subset \widetilde{\mathcal{O}}' \subset^q Q$ is $\mathfrak{T}_q$-open such that $[u]_{\mathcal{O}'}$ is a moderate solution, then clearly

$$\mu_{\mathcal{O}\cap\mathcal{O}'} = \mathbf{1}_{\widetilde{\mathcal{O}}\cap\widetilde{\mathcal{O}}'}\mu_{\mathcal{O}} = \mathbf{1}_{\widetilde{\mathcal{O}}\cap\widetilde{\mathcal{O}}'}\mu_{\mathcal{O}'}. \tag{5.78}$$

Since $[u]_F$ is moderate,

$$[[u]_{Q_n}]_F = [u]_{Q_n\cap F} \uparrow [u]_F \quad \text{as } n \to \infty. \tag{5.79}$$

In addition, $[u_Q]_F \ge \lim_{n\to\infty} [[u]_{Q_n}]_F = [u]_F$, jointly with $u_Q \le u$, leads to

$$u_F = [u_Q]_F. \tag{5.80}$$

By (5.77) and (5.79), if F is a $\mathfrak{T}_q$-closed subset of $\mathcal{R}_q(u)$, and $[u]_F$ is moderate

$$tr\left([u]_F\right) = \lim_{n\to\infty} tr\left([[u]_{Q_n}]_F\right) = \mathbf{1}_F \mu_{\mathcal{R}_q(u)}, \tag{5.81}$$

and (5.76) follows.

Since $Q' = \cup_n Q_n$ and $\mu_Q(Q_n) < \infty$, μ_Q is σ-finite on $Q' \sim^q Q$. Since for any $\xi \in Q$ there exists a $\mathfrak{T}_q$-open neighbourhood $\mathcal{O}_\xi$ of ξ included in Q with the property that $\mu_Q(\mathcal{O}_\xi) < \infty$, μ_Q is $\mathfrak{T}_q$-locally finite on Q.

(iii) If w is a moderate solution dominated by u_Q, with $\mathfrak{T}_q$-supp $(w) \subset^q Q$, then $tr(w) \leq \mu_Q$ since

$$[w]_{Q_n} \leq [u]_{Q_n} \quad \text{and} \quad [w]_{Q_n} \uparrow w \Longrightarrow tr([w]_{Q_n}) \uparrow tr(w) \leq \lim_{n\to\infty} tr\left([u]_{Q_n}\right) = \mu_Q.$$

Let $\{w_n\}$ be an increasing sequence of moderate solutions with $F_n := \mathfrak{T}_q$-supp $(w) \subset^q Q$ and $w_n \uparrow u_Q$. We claim that if $\nu_n := tr(w_n)$, then

$$\nu := \lim_{n\to\infty} \nu_n = \mu_Q. \tag{5.82}$$

Clearly $\nu \leq \mu_Q$. To prove the reverse inequality, let D be a $\mathfrak{T}_q$-open set such that $[u]_D$ is moderate and $K \subset D$ a compact set such that $cap_{\frac{2}{q},q'}(K) > 0$. Then

$$w_n \leq [w_n]_D + [w_n]_{D^c} \longrightarrow u_Q = \lim_{n\to\infty} w_n \leq \lim_{n\to\infty} [w_n]_D + U_{D^c}.$$

Since $[u]_D$ is moderate, the sequence $\{[w_n]_D\}$ which is dominated by $[u]_D$ has an initial trace $tr\left([w_n]_D\right) := \mathbf{1}_{\widetilde{D}}\nu_n$ which increases and converges to $\mathbf{1}_{\widetilde{D}}\nu$. Hence, $\mathbf{1}_{\widetilde{D}}\nu$ is a Radon measure which vanishes on sets with zero $cap_{\frac{2}{q},q'}$-capacity. Hence

$$[w_n]_D \uparrow u_{\mathbf{1}_{\widetilde{D}}\nu}$$

where $u_{\mathbf{1}_{\widetilde{D}}\nu}$ is the moderate solution with initial trace $\mathbf{1}_{\widetilde{D}}\nu$. Therefore

$$u_Q = \lim_{n\to\infty} w_n \leq u_{\mathbf{1}_{\widetilde{D}}\nu} + U_{D^c}.$$

This implies

$$\left([u_Q]_K - u_{\mathbf{1}_{\widetilde{D}}\nu}\right)_+ \leq \inf\left\{U_{D^c}, U_K\right\}.$$

Notice that the left-hand side of the above inequality is a subsolution while the right-hand side is a supersolution. This implies

$$\left([u_Q]_K - u_{\mathbf{1}_{\widetilde{D}}\nu}\right)_+ \leq U_{D^c} \wedge U_K = [[U]_{D^c}]_K = 0.$$

Therefore $u_Q]_K \leq u_{\mathbf{1}_{\widetilde{D}}\nu}$ which implies $\mathbf{1}_K\mu_Q \leq \mathbf{1}_{\widetilde{D}}\nu$. Moreover, if $\mathcal{O}$ is a $\mathfrak{T}_q$-open set such that $\widetilde{\mathcal{O}} \subset^q D$, then, using the fact that

$$\mathbf{1}_{\widetilde{\mathcal{O}}}\mu_Q = \sup\left\{\mathbf{1}_K\mu_Q : K \subset \mathcal{O},\, K \text{ compact}\right\},$$

we obtain

$$\mathbf{1}_{\widetilde{O}}\mu_Q \leq \mathbf{1}_{\widetilde{D}}\nu. \tag{5.83}$$

Applying this series of inequalities to the sets $Q_m, Q_{m+1}, ...$, we infer

$$\mathbf{1}_{Q_m}\mu_Q \leq \mathbf{1}_{\widetilde{Q}_{m+1}}\nu \leq \mathbf{1}_{Q_{m+2}}\nu.$$

Letting $m \to \infty$ we deduce that $\mu_Q = \nu$. □

5.5.2 $\mathfrak{T}_q$-perfect measures

Definition 5.62 *Let μ be a positive Borel measure on $\mathbb{R}^N$.*
(i) We say that μ is essentially absolutely continuous with respect to the $cap_{\frac{2}{q},q'}$-capacity if the following condition holds:
If Q is a $\mathfrak{T}_q$-open set and A a Borel subset such that $cap_{\frac{2}{q},q'}(A) = 0$, then

$$\mu(Q) = \mu(Q \cap A^c).$$

This relation is denoted by

$$\mu \prec\prec cap_{\frac{2}{q},q'}.$$

(ii) We say that μ is regular with respect to the $\mathfrak{T}_q$-topology if, for every Borel set E, there holds

$$\mu(E) = \inf\{\mu(D) : E \subset D,\ D\ \mathfrak{T}_q\text{-open}\} = \sup\{\mu(K) : K \subset E,\ K \text{ compact}\}, \tag{5.84}$$

and μ is outer regular with respect to the $\mathfrak{T}_q$-topology if there only holds

$$\mu(E) = \inf\{\mu(D) : E \subset D,\ D\ \mathfrak{T}_q\text{-open}\}. \tag{5.85}$$

(iii) A positive Borel measure is called $\mathfrak{T}_q$-perfect if it is essentially absolutely continuous with respect to the $cap_{\frac{2}{q},q'}$ and outer regular with respect to the $\mathfrak{T}_q$-topology. The space of $\mathfrak{T}_q$-perfect Borel measures is denoted by $\mathfrak{M}_q(\mathbb{R}^N)$.

Proposition 5.63 *If $\mu \in \mathfrak{M}_q(\mathbb{R}^N)$ and A is a non-empty Borel subset of $\mathbb{R}^N$ such that $cap_{\frac{2}{q},q'}(A) = 0$. Then*

$$\mu = \begin{cases} \infty & \text{if } \mu(Q \cap A^c) = \infty \quad \text{for all } \mathfrak{T}_q\text{-open neighbourhood } Q \text{ of } A \\ 0 & \text{otherwise.} \end{cases} \tag{5.86}$$

If μ_0 is an essentially absolutely continuous positive measure in $\mathbb{R}^N$ and Q is a $\mathfrak{T}_q$-open set such that $\mu_0(Q) < \infty$, then $\mu_0\lfloor_Q$ is absolutely continuous with respect to the $cap_{\frac{2}{q},q'}$-capacity in the strong sense, that is for any sequence of Borel subsets $\{A_n\}$ of $\mathbb{R}^N$,

$$cap_{\frac{2}{q},q'}(A_n) \to 0 \Longrightarrow \mu_0(Q \cap A) \to 0 \quad \text{as } n \to \infty.$$

If μ_0 is an essentially absolutely continuous positive Borel measure on $\mathbb{R}^N$ and if for every Borel subset of $\mathbb{R}^N$ we denote

$$\mu(E) = \inf\{\mu_0(D) : E \subset D \quad D\ \mathfrak{T}_q\text{-open}\}, \tag{5.87}$$

then μ is a Borel measure and

$$\begin{aligned} &(i) \qquad \mu_0 \le \mu, \quad \mu_0(Q) = \mu(Q) \text{ for all } \mathfrak{T}_q\text{-open set } Q \\ &(ii) \qquad \mu\lfloor_Q = \mu_0\lfloor_Q \text{ for all } \mathfrak{T}_q\text{-open set } Q \text{ s.t. } \mu_0(Q) < \infty. \end{aligned} \tag{5.88}$$

Proof. The first assertion follows from the definition of $\mathfrak{M}_q(\mathbb{R}^N)$. Next, if μ_0 is essentially absolutely continuous and $\mu_0(Q) < \infty$ where Q is $\mathfrak{T}_q$-open, then $\mathbf{1}_Q\mu_0$ is a bounded Borel measure which vanishes on Borel sets with zero $cap_{\frac{2}{q},q'}$-capacity. If $\{A_n\}$ is a sequence of Borel sets that we can assume to be decreasing, such that $cap_{\frac{2}{q},q'}(A_n) \to 0$ when $n \to \infty$, and $\mu_n = \mathbf{1}_{Q\cap A_n}\mu_0$, then by [43, Lemma 2.8] there exists a unique moderate solution u_{μ_n} with initial trace μ_n. There holds $u_{\mu_n} \le U_{Q\cap A_n}$. Since $cap_{\frac{2}{q},q'}(Q \cap A_n) \to 0$, $U_{Q\cap A_n}$ converges to 0 when $n \to \infty$, and so does u_{μ_n}. Then $\mu_n \to 0$ in the weak topology of Radon measures, which implies that $u_{\mu_n} \to 0$ locally uniformly in Q_T. Therefore $\mu(Q \cap A_n) \to 0$ which implies that $\mu_0\lfloor_Q$ is absolutely continuous with respect to the $cap_{\frac{2}{q},q'}$-capacity in the usual sense.

Assertion (5.88)-(i) follows from the definition (5.87). If Q is a $\mathfrak{T}_q$-open set such that $\mu_0(Q) < \infty$ then $\mu(Q) < \infty$. Since $\mu_0\lfloor_Q$ and $\mu\lfloor_Q$ are regular Borel measures which coincide on open sets, they coincide on all Borel sets. This implies (5.88)-(ii).

At end, if A is a Borel set such that $cap_{\frac{2}{q},q'}(A) = 0$, then $cap_{\frac{2}{q},q'}(\widetilde{A}) = 0$. If Q is a $\mathfrak{T}_q$-open set, then $Q \cap \widetilde{A}^c$ is $\mathfrak{T}_q$-open. Therefore

$$\mu(Q) = \mu_0(Q) = \mu_0(Q \cap \widetilde{A}^c) = \mu(Q \cap \widetilde{A}^c).$$

Hence μ is essentially absolutely continuous. Using (5.88)-(i) and the definition of μ we infer that μ is outer regular with respect to the capacity $cap_{\frac{2}{q},q'}$. Hence $\mu \in \mathfrak{M}_q(\mathbb{R}^N)$. □

5.5.3 The initial trace on the regular set

In the next propositions we study the initial trace of a positive solution u of (3.13) on the regular initial set $\mathcal{R}_q(u)$ given in Definition 5.58 and we study the properties of the measure $\mu_{\mathcal{R}_q(u)}$ constructed by Proposition 5.59, Corollary 5.60 and Proposition 5.61.

Proposition 5.64 *Let $u \in \mathcal{U}_+(Q_T)$.*
(i) There exists an increasing sequence of $\mathfrak{T}_q$-open sets $\{Q_n\}$ with the following properties: $Q_n \subset \mathcal{R}_q(u)$, $\widetilde{Q}_n \subset^q Q_{n+1}$ and $\mathcal{R}_{q,0}(u) := \cup_{n=0}^{\infty} Q_n \sim^q \mathcal{R}_q(u)$, such

$$v_n := u \wedge U_{Q_n} \text{ is moderate } \ v_n \uparrow v_{\mathcal{R}_q(u)} \text{ and } \ tr\,(v_n) \uparrow \mu_{\mathcal{R}_q(u)}. \tag{5.89}$$

(ii) There holds

$$v_{\mathcal{R}_q(u)} := \sup\left\{[u]_F : F \subset^q \mathcal{R}_q(u),\ F\ \mathfrak{T}_q\text{ -closed}\right\}. \tag{5.90}$$

Thus $v_{\mathcal{R}_q(u)}$ is σ-moderate.
(iii) If $[u]_F$ is moderate and $F \subset^q \mathcal{R}_q(u)$, there exists a $\mathfrak{T}_q$-open set $Q \subset \mathcal{R}_q(u)$ such that $F \subset^q Q$, $[u]_Q$ is moderate.
(iv) For every $\mathfrak{T}_q$-open set Q such that $[u]_Q$ is a moderate solution, we have

$$\mathbf{1}_{\widetilde{Q}}\mu_{\mathcal{R}_q(u)} = tr\left([u]_Q\right) = tr\left(\left[v_{\mathcal{R}_q(u)}\right]\right). \tag{5.91}$$

Finally $\mu_{\mathcal{R}_q(u)}$ is $\mathfrak{T}_q$-locally finite on $\mathcal{R}_q(u)$ and σ-finite on $\mathcal{R}_{q,0}(u)$.
(v) If $\{w_n\}$ is a sequence of moderate solutions such that $w_n \uparrow u_{\mathcal{R}_q(u)}$, then

$$\mu_{\mathcal{R}_q(u)} = \lim_{n\to\infty} tr\left(w_n\right) = \lim_{n\to\infty} tr(v_n). \tag{5.92}$$

(vi) The regularised measure $\bar{\mu}_{\mathcal{R}_q(u)}$ defined for Borel sets $E \subset \mathbb{R}^N$ by

$$\bar{\mu}_{\mathcal{R}_q(u)}(E) = \inf\left\{\mu_{\mathcal{R}_q(u)}(Q) : E \subset Q,\ Q\,\mathfrak{T}_q\text{ -open}\right\} \tag{5.93}$$

is $\mathfrak{T}_q$-perfect.
(vii) There holds

$$u \approx_{\mathcal{R}_q(u)} v_{\mathcal{R}_q(u)}.$$

(viii) For every $\mathfrak{T}_q$-closed set $F \subset^q \mathcal{R}_q(u)$,

$$[u]_F = \left[v_{\mathcal{R}_q(u)}\right]_F. \tag{5.94}$$

Furthermore, if $\mu_{\mathcal{R}_q(u)}(F \cap K) < \infty$ for every compact subset $K \subset \mathbb{R}^N$, then $[u]_F$ is moderate and

$$tr\left([u]_F\right) = \mathbf{1}_F \mu_{\mathcal{R}_q(u)}. \tag{5.95}$$

(ix) If F is a $\mathfrak{T}_q$-closed set with positive $cap_{\frac{2}{q},q'}$-capacity, whe have

$$\mu_{\mathcal{R}_q(u)}(F \cap K) < \infty \quad \textit{for all compact set } K \subset \mathbb{R}^N \Longleftrightarrow [u]_F \quad \textit{is moderate.} \tag{5.96}$$

Proof. (i) For every $z \in \mathcal{R}_q(u)$ there exists a $\mathfrak{T}_q$-bounded open set $\mathcal{O}_z \subset \mathcal{R}_q(u)$ such that $[u]_{\mathcal{O}_z}$ is moderate. With the previous notations and the construction of the sequence $\{Q_n\}$ in Proposition 5.59, we recall that $v_n = [u]_{Q_n} = u \wedge U_{Q_n}$ satisfies $v_n = [v_{n+k}]_{Q_n}$ for any $k \in \mathbb{N}$ and

$$\mu_n(Q_n) = \mu_{n+k}(Q_n) = \mu_{\mathcal{R}_q(u)}(Q_n). \tag{5.97}$$

(ii) The proof has already been made in Proposition 5.61.
(iii) We assume firstly that F is bounded. Using the definition and (i), every point in $\mathcal{R}_q(u)$ possesses a $\mathfrak{T}_q$-open neighbourhood A such that $[u]_A$ is moderate. By Proposition 5.19-(II), for any $\epsilon > 0$ there exists a $\mathfrak{T}_q$-open set Q_ϵ such that $cap_{\frac{2}{q},q'}(F \cap Q_\epsilon^c) < \epsilon$ and $[u]_{Q_\epsilon}$ is moderate. Since F is bounded, we can assume that so is Q_ϵ. Let $\mathcal{O}_\epsilon$ be an open set

containing $F \cap Q_\epsilon^c$ and such that $cap_{\frac{2}{q},q'}(\mathcal{O}_\epsilon) < 2\epsilon$. We define a $\mathfrak{T}_q$-closed set F_ϵ included in Q_ϵ by

$$F_\epsilon = F \cap \mathcal{O}_\epsilon, \tag{5.98}$$

and $F_\epsilon \subset F$ with $cap_{\frac{2}{q},q'}(F \cap F_\epsilon^c) < 2\epsilon$.

Claim 1: Let E be a $\mathfrak{T}_q$-closed set, D a $\mathfrak{T}_q$-open set such that $[u]_D$ is moderate and $E \subset^q D$. There exists a decreasing sequence of $\mathfrak{T}_q$-open sets $\{G_n\}$ such that

$$E \subset^q G_{n+1} \subset \widetilde{G}_{n+1} \subset^q G_n \subset^q D, \tag{5.99}$$

and

$$[u]_{G_\epsilon} \to [u]_E \quad \text{in } L^q(K) \text{ for every compact set } K \subset \overline{Q}_T. \tag{5.100}$$

By Lemma 5.18 and Proposition 5.25-(iii), there exists a decreasing sequence of $\mathfrak{T}_q$-open sets $\{G_n\}$ satisfying (5.99) and such that $[u]_{G_n} \downarrow [u]_E$ locally uniformly in Q_T. Since $[u]_{G_n} \leq [u]_D$ which is a moderate solution, we deduce (5.100).

Next we assume that F is a non-necessarily bounded $\mathfrak{T}_q$-closed set. If $x \in F$ we set $B_n = B_n(x) \cap F$, $n \in \mathbb{N}_*$ and

$$E_n = \bigcup_{m=1}^{n} (F \cap B_n)_{2^{-m}},$$

where $(F \cap B_n)_{2^{-m}}$ is the set defined in (5.98) with F replaced by $F \cap B_n$ and ϵ replaced by 2^{-m}. We can also assume that the sequence $\{E_n\}$ is increasing. We set $Q^n_{m^{-1}} = (F \cap B_n)_{m^{-1}}$ and

$$Q_n = \bigcup_{m=1}^{n} Q^n_{m^{-1}},$$

and as for $\{F_n\}$ we can assume that the sequence $\{Q_n\}$ is increasing. Therefore, we have that $E_n \subset E$, Q_n is $\mathfrak{T}_q$-open, $[u]_{Q_n}$ is moderate and $E_n \subset^q Q_n$. Furthermore $\cup_n E_n = E' \sim^q F$ since for any $n \in \mathbb{N}_*$ there holds

$$\begin{aligned} cap_{\frac{2}{q},q'}\left(F \setminus \bigcup_{j=1}^{\infty} E_j\right) &\leq \sum_{k=1}^{n} cap_{\frac{2}{q},q'}\left((F \cap B_k) \setminus \bigcup_{j=1}^{\infty} E_j\right) + \sum_{k=n+1}^{\infty} cap_{\frac{2}{q},q'}\left((F \cap B_k) \setminus E_k\right) \\ &\leq \frac{1}{2^n} + \sum_{k=n+1}^{\infty} \frac{1}{2^k} = \frac{1}{2^{n-1}}. \end{aligned}$$

Thus, by Assertion 1, we can choose a sequence of $\mathfrak{T}_q$-open sets $\{V_n\}$ such that

$$E_n \subset^q V_n \subset \widetilde{V}_n \subset^q Q_n \quad \text{and} \quad \|[u]_{V_n} - [u]_{E_n}\|_{L^q(B_n(0)\times(0,T))} \leq 2^{-n}. \tag{5.101}$$

Notice that since E_n and Q_n are bounded sets, the functions $[u]_{V_n}$ and $[u]_{E_n}$ which are moderate belong to $L^q(\mathbb{R}^N \times (0,T))$.

Because $[u]_F$ is moderate, there exists a Radon measure $\mu_F = tr\,([u]_F)$ and $[u]_F = [u]_{E'}$ since $F \sim^q E'$. At end, using (5.49) and the fact that $E_n \subset^q F$, we have

$$[u]_{E_n} = [u]_{F \cap E_n} = [[u]_{E_n}]_F.$$

Because $[u]_F$ is moderate we have $tr\,([u]_{E_n}) = \mathbf{1}_{E_n}\mu_F$. Since $E_n \uparrow E' \sim^q F$ we deduce that $[u]_{E_n} \uparrow [u]_F$ in $L^q_{loc}(\overline{Q}_T)$. Hence, we have from (5.101) that $[u]_{V_n} \to [u]_F$ in $L^q_{loc}(\overline{Q}_T)$ as $n \to \infty$.

Let $\{V_{n_k}\}$ be a subsequence such that

$$\left(\int_0^T \int_{B_k} \left|[u]_{V_{n_k}} - [u]_F\right|^q\right)^{\frac{1}{q}} \leq 2^{-k}. \tag{5.102}$$

If $K \subset \mathbb{R}^N$ is compact, it is included in B_k for $k \geq k_0$. We set $W = \cup_{k=1}^{\infty} V_{n_k}$, then

$$[u]_W \leq \sum_{k=1}^{\infty} [u]_{V_{n_k}}.$$

Therefore

$$\begin{aligned}\left(\int_0^T \int_K |[u]_W - [u]_F|^q\right)^{\frac{1}{q}} &\leq \sum_{k=1}^{k_0} \left(\int_0^T \int_{B_k} \left|[u]_{V_{n_k}} - [u]_F\right|^q\right)^{\frac{1}{q}} \\ &\qquad + \sum_{k=k_0+1}^{\infty} \left(\int_0^T \int_K \left|[u]_{V_{n_k}} - [u]_F\right|^q\right)^{\frac{1}{q}} \\ &\leq \sum_{k=1}^{k_0} \left(\int_0^T \int_{B_k} \left|[u]_{V_{n_k}} - [u]_F\right|^q\right)^{\frac{1}{q}} + \sum_{k=k_0+1}^{\infty} 2^{-k} \\ &< \infty.\end{aligned}$$

Because $F \subset^q W$, W is $\mathfrak{T}_q$-open, $[u]_F$ is moderate and K is arbitrary it follows from the above inequality that $[u]_W$ is moderate, therefore $W \subset \mathcal{R}_q(u)$ by Proposition 5.57.

(iv) Let Q be a $\mathfrak{T}_q$-open set such that $[u]_Q$ is a moderate solution, and $\mu_Q = tr\,([u]_Q)$. If F is a $\mathfrak{T}_q$-closed set such that $F \subset^q Q$, then by Proposition 5.25-(ii),

$$tr([u]_F) = tr\left([[u]_Q]_F\right) = \mathbf{1}_F \mu_Q. \tag{5.103}$$

In particular, if Q and Q' are regular sets in the sense of Definition 5.24, then

$$\mu_{Q\cap Q'} = \mathbf{1}_{\widetilde{Q}\cap\widetilde{Q}'}\mu_Q = \mathbf{1}_{\widetilde{Q}\cap\widetilde{Q}'}\mu_{Q'}. \tag{5.104}$$

Using the notations of (i), we have $[v_{n+k}]_{Q_k} = v_k$ and hence $\mathbf{1}_{\widetilde{Q}_k}\mu_{n+k} = \mu_k$ for every integer k.

Let F be a $\mathfrak{T}_q$-closed regular subset of $\mathcal{R}_q(u)$. Since $[u]_F$ is moderate we have by (5.104)

$$[v_n]_F = [u]_{F\cap\widetilde{Q}_n} \uparrow [u]_F. \tag{5.105}$$

Furthermore, since we have

$$\left[v_{\mathcal{R}_q(u)}\right]_F \geq \lim_{n\to\infty} [v_n]_F = [u]_F,$$

and $v_{\mathcal{R}_q(u)} \leq u$, we infer

$$[u]_F = \left[v_{\mathcal{R}_q(u)}\right]_F. \tag{5.106}$$

It follows from (5.103) and (5.105) that if F is a $\mathfrak{T}_q$-closed subset of $\mathcal{R}_q(u)$ and $[u]_F$ is moderate that

$$tr\left([u]_F\right) = \lim_{n\to\infty} tr\left([v_n]_F\right) = \lim_{n\to\infty} \mathbf{1}_F \mu_n = \mathbf{1}_F \mu_{\mathcal{R}_q(u)}, \tag{5.107}$$

which yields (5.91).

Finally, since $\mathcal{R}_q(u)$ has a regular decomposition, $\mu_{\mathcal{R}_q(u)}$ is σ-finite on $\mathcal{R}_{q,?^c}(u)$. As for the claim that $\mu_{\mathcal{R}_q(u)}$ is $\mathfrak{T}_q$-locally finite on $\mathcal{R}_q(u)$ it is a consequence of the fact that every point $\xi \in \mathcal{R}_q(u)$ is contained in a $\mathfrak{T}_q$-open set $\mathcal{O}_\xi \subset \mathcal{R}_q(u)$ such that $[u]_{\mathcal{O}_\xi}$ is moderate and thus $\mu_{\mathcal{R}_q(u)}(\mathcal{O}_\xi) < \infty$.

(v) If w is a moderate solution dominated by $v_{\mathcal{R}_q(u)}$ and the $\mathfrak{T}_q$-supp$(w) \subset^q \mathcal{R}_q(u)$ then $\tau := tr(w) \leq \mu_{\mathcal{R}_q(u)}$.

Now, let $\{w_n\}$ be an increasing sequence of moderate solutions such that $F_n := \mathfrak{T}_q$-supp$(w_n) \subset^q \mathcal{R}_q(u)$ and $w_n \uparrow v_{\mathcal{R}_q(u)}$. *We claim that*

$$\nu := \lim_{n\to\infty} \nu_n := \lim_{n\to\infty} tr(w_n) = \mu_{\mathcal{R}_q(u)}. \tag{5.108}$$

By the previous argument, $\nu \leq \mu_{\mathcal{R}_q(u)}$. In order to prove the opposite inequality, we proceed as follows: Let D be a $\mathfrak{T}_q$-open set such that $[u]_D$ is moderate and let K be a compact subset of D with positive $cap_{\frac{2}{q},q'}$-capacity. Then

$$w_n \leq [w_n]_D + [w_n]_{D^c} \Longleftarrow v_{\mathcal{R}_q(u)} := \lim_{n\to\infty} w_n \leq \lim_{n\to\infty} [w_n]_D + U_{D^c}.$$

The sequence $\{[w_n]_D\}$ is dominated by the moderate solution $[v_{\mathcal{R}_q(u)}]_D$. In addition $tr\left([w_n]_D\right) = \mathbf{1}_{\widetilde{D}}\nu_n \uparrow \mathbf{1}_{\widetilde{D}}\nu \leq \mathbf{1}_{\widetilde{D}}\nu_{\mathcal{R}_q(u)}$. Hence $\mathbf{1}_{\widetilde{D}}\nu$ is a Radon measure which vanishes onBorel sets with zero $cap_{\frac{2}{q},q'}$-capacity. Also $[w_n]_D \uparrow u_{\mathbf{1}_{\widetilde{D}}\nu}$, with the usual notation. Consequently

$$v_{\mathcal{R}_q(u)} := \lim_{n\to\infty} w_n \leq u_{\mathbf{1}_{\widetilde{D}}\nu} + U_{D^c}.$$

This implies

$$\left([v_{\mathcal{R}_q(u)}]_K - u_{\mathbf{1}_{\widetilde{D}}\nu}\right)_+ \leq \inf\{U_{D^c}, U_K\}.$$

By the same argument as in the proof of Proposition 5.61-(ii) this yields

$$\left([v_{\mathcal{R}_q(u)}]_K - u_{\mathbf{1}_{\widetilde{D}}\nu}\right)_+ \leq [[U_{D^c}]_K\} = 0.$$

Hence $[v_{\mathcal{R}_q(u)}]_K \leq u_{\mathbf{1}_{\widetilde{D}}\nu}$ and hence $\mathbf{1}_K \mu_{\mathcal{R}_q(u)} \leq\leq \mathbf{1}_{\widetilde{D}}\nu$. Next, if Q is a $\mathfrak{T}_q$-open set such that $\widetilde{Q} \subset^q D$, we use the fact that

$$\sup\{\mathbf{1}_K \mu_{\mathcal{R}_q(u)} : K \subset Q,\, K \text{ compact}\} = \mathbf{1}_Q \mu_{\mathcal{R}_q(u)}$$

to obtain that

$$\mathbf{1}_Q \mu_{\mathcal{R}_q(u)} \leq \mathbf{1}_{\widetilde{D}}\nu. \tag{5.109}$$

Applying this inequality to the couple of sets (Q_m, Q_{m+1}) we deduce that

$$\mathbf{1}_{Q_m}\mu_{\mathcal{R}_q(u)} \leq \mathbf{1}_{\widetilde{Q}_{m+1}}\nu \leq \mathbf{1}_{Q_{m+2}}\nu.$$

Letting $m \to \infty$ implies $\mu_{\mathcal{R}_q(u)} \leq \nu$. This completes the proof of the claim (5.109) and assertion (v).

(vi) Since the measure $\mu_{\mathcal{R}_q(u)}$ is essentially absolutely continuous with respect to the $cap_{\frac{2}{q},q'}$-capacity, the claim follows from Proposition 5.63.

(vii) For any $n \in \mathbb{N}$, we have

$$u \leq [u]_{Q_n} + [u]_{Q_n^c}.$$

Since Q_n^c is $\mathfrak{T}_q$-closed and $\cap_n Q_n^c = \mathcal{R}_{q,0}^c(u)$, we have by Proposition 5.47-(iii)

$$[u]_{Q_n^c} \downarrow [u]_{\mathcal{R}_{q,0}^c(u)}.$$

Therefore

$$\lim_{n\to\infty}(u - [u]_{Q_n} = u - v_{\mathcal{R}_q(u)} \leq [u]_{\mathcal{R}_{q,0}^c(u)}.$$

It follows that $u \ominus v_{\mathcal{R}_q(u)} \approx_{\mathcal{R}_{q,0}(u)} 0$. Because $v_{\mathcal{R}_q(u)} \leq u$, this is equivalent to the statement $u \approx_{\mathcal{R}_{q,0}(u)} v_{\mathcal{R}_q(u)}$.

(viii) The fact that $[u]_F = [v_{\mathcal{R}_q(u)}]_F$ for every $\mathfrak{T}_q$-closed subset $F \subset^q \mathcal{R}_q(u)$ follows from assertion (vii). Next we assume that $\mu_{\mathcal{R}_q(u)}(F \cap K) < \infty$ for any compact set K and we set $F_n = F \cap \widetilde{Q}_n$. By relation (5.48) we have

$$[u]_{F_n} \leq [u]_F \leq [u]_{F_n} + [u]_{F\cap F_n^c} = [u]_{F_n} + [u]_{F\cap \widetilde{Q}_n^c} \leq [u]_{F_n} + [u]_{F\cap Q_n^c}.$$

Since $F \cap Q_n^c$ is $\mathfrak{T}_q$-closed and $\cap_n F \cap Q_n^c = G$ with $cap_{\frac{2}{q},q'}(G) = 0$, we deduce from Proposition 5.47-(iii) that $[u]_{F\cap Q_n^c} \to [u]_G = 0$ as $n \to \infty$. Hence $[u]_{F_n} \uparrow [u]_F$ and

$$tr\,([u]_{F_n}) = \mathbf{1}_{F_n}\mu_{\mathcal{R}_q(u)} \uparrow \mathbf{1}_{F_0}\mu_{\mathcal{R}_q(u)} = \mathbf{1}_F\mu_{\mathcal{R}_q(u)} \quad \text{as } n \to \infty,$$

since $\mathbf{1}_{F_0}\mu_{\mathcal{R}_q(u)} = \mathbf{1}_F\mu_{\mathcal{R}_q(u)}$ if $F_0 = \cap_n F_n$. Because $\mathbf{1}_F\mu_{\mathcal{R}_q(u)}$ is a Radon measure essentially absolutely continuous with respect to the $cap_{\frac{2}{q},q'}$-capacity, $[u]_F$ is moderate and (5.95) is verified.

(ix) If $\mu_{\mathcal{R}_q(u)}(F \cap K) < \infty$ for any compact set K, then by (viii) $[u]_F$ is moderate. Conversely, if $[u]_F$ is moderate, then by (iv), there holds $\mu_{\mathcal{R}_q(u)}(F \cap K) < \infty$ for any compact set $K \subset \mathbb{R}^N$. □

Example There exist functions $u \in \mathcal{U}_+(Q_T)$ such that $\mathcal{R}_q(u) = \mathbb{R}^N$ which are not moderate solutions. We construct one of them as follows. Let $\eta : [0,\infty) \mapsto [0,\infty)$ be a smooth function which is positive on $(0,\infty)$, $\eta^k(0) = 0$ for all $k \in \mathbb{N}$ (e.g. $\eta(r) = e^{-r^{-2}}$). We define the closed set $K \subset \mathbb{R}^N$ by

$$K = \{x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}_+ : |x'| \leq \eta(x_N)\}.$$

Then K is $\mathfrak{T}_q$-thin at 0 for the capacity $cap_{\frac{2}{q},q'}$. We set

$$f(x) = \begin{cases} \eta^{-N}(x_N) & \text{if } x \in K \\ 0 & \text{if } x \in \mathbb{R}^N \setminus K, \end{cases}$$

and define the measure

$$\mu = f dx.$$

Then the following properties hold:
1- μ is $\mathfrak{T}_q$-locally finite.
2- $\mu(Q_n) < \infty$ if $Q_n = B_{2n} \setminus \overline{B}_{\frac{1}{n}}$ and $\cup_n Q_n \sim^q \mathbb{R}^N$.
3- $\mu(F) = 0$ for any Borel set F such that $cap_{\frac{2}{q},q'}(F) = 0$.
4- There exists a non-decreasing sequence of bounded nonnegative Radon measures $\{\mu_n\}$ absolutely continuous with respect to the $cap_{\frac{2}{q},q'}$-capacity such that
(i) $\mathfrak{T}_q$-supp $(\mu_n) \subset \widetilde{Q}_n$, $\mu_n(A) = \mu_{n+k}(A)$ for any $A \subset \widetilde{Q}_n$ and any $n, k \in \mathbb{N}_*$.
(ii) $\lim_{n\to\infty} \mu_n = \mu$.
5- We can construct a solution u of (3.13) such that $\mathcal{R}_q(u) = \mathbb{R}^N$ and $\mu_{\mathcal{R}_q(u)} = \mu$.

We will prove later on that this solution is actually the unique one towhich has this initial trace since it is σ-moderate.

Lemma 5.65 *Let μ satisfy the conditions 1-4 above.Then there exists an open set $\mathcal{R}_q \sim^q \mathbb{R}^N$ such that the measure μ is a Radon measure on $\mathcal{R}_q$.*

Proof. By [46, Lemma 2.5] for any $R > 1$ and $\epsilon > 0$ there exist a sequence of open sets $\{\mathcal{O}_m\}$ and $n(m) \in \mathbb{N}_*$ such that $cap_{\frac{2}{q},q'}(\mathcal{O}_m) < \epsilon 2^{-m}$ and

$$\overline{B}_R \setminus \mathcal{O}_m \subset \bigcup_{j=1}^{n(m)} Q_j \quad \text{where } Q_j = B_{2j} \setminus \overline{B}_{\frac{1}{j}}. \tag{5.110}$$

Since $\overline{\mathcal{O}}_m \subset \widetilde{\mathcal{O}}$

$$cap_{\frac{2}{q},q'}\left(\overline{\mathcal{O}}_m\right) \leq cap_{\frac{2}{q},q'}\left(\widetilde{\mathcal{O}}\right) \leq \widetilde{C} cap_{\frac{2}{q},q'}\left(\mathcal{O}_m\right) \leq c\epsilon 2^{-m}.$$

If $x \in B_R \setminus \bigcap_{m=1}^{\infty} \overline{\mathcal{O}}_m$ there exist $r_x > 0$ and $k \in \mathbb{N}$ such that

$$B_{r_x}(x) \subset B_R \setminus \bigcap_{m=1}^{k} \overline{\mathcal{O}}_m.$$

Jointly with (5.110) it implies that

$$\mu(B_{r_x}(x)) < \infty.$$

We set

$$\mathcal{R}_q = \left\{x \in \mathbb{R}^N : \exists r_x > 0 \text{ such that } \mu(B_{r_x}(x)) < \infty\right\}.$$

The set $\mathcal{R}_q$ is open and by letting $R \to \infty$ and $\epsilon \to 0$, we obtain that $\mathcal{R}_q \sim^q \mathbb{R}^N$. By the definition of $\mathcal{R}_q$, for any compact set $K \subset \mathcal{R}_q$ there holds $\mu(K) < \infty$. Hence μ is a Radon measure in $\mathcal{R}_q$. □

5.6 The precise initial trace

5.6.1 Definition and first properties

We can now define the *precise initial trace* of an element of $\mathcal{U}_+(Q_T)$ in the supercritical case.

Definition 5.66 *Let $q \geq q_c$ and $u \in \mathcal{U}_+(Q_T)$.*
1- The function $v_{\mathcal{R}_q(u)}$ defined in (5.90) is called the regular component of u and will be denoted by u_{reg}.
2- Let $\{v_n\}$ be an increasing sequence of moderate solutions satisfying condition (5.89) and put $\mu_{\mathcal{R}_q(u)} := \lim_{n\to\infty} tr(v_n)$. Then, the regularised measure $\overline{\mu}_{\mathcal{R}_q(u)}$, defined by (5.93), is called the regular initial trace of u. It will be denoted by $tr_{\mathcal{R}_q(u)}(u)$.
3- The couple $(tr_{\mathcal{R}_q(u)}(u), \mathcal{S}_q(u))$ is called the precise initial trace of u and will be denoted by $tr^c(u)$.
4- Let ν be the Borel measure on $\mathbb{R}^N$ given by

$$\nu(E) = \begin{cases} tr_{\mathcal{R}_q(u)}(E) & \text{if } E \subset \mathcal{R}_q(u) \\ \infty & \text{if } E \cap \mathcal{S}_q(u) \neq \emptyset \end{cases} \tag{5.111}$$

for every Borel set $E \subset \mathbb{R}^N$. Then ν is the measure representation of the precise trace of u and it is denoted by $tr(u)$.

Remark. In the definitions of $tr^c(u)$ and $tr(u)$, the exponent c stands for*couple*, but the two objects are the same in their respective classes. Thanks to Proposition 5.64 the measure $\mu_{\mathcal{R}_q(u)}$ is independent of the choice of the sequence $\{v_n\}$.

The next fundamental result is the parabolic version of the construction given in [46].

Theorem 5.67 *Assume that $u \in \mathcal{U}_+(Q_T)$ is a σ-moderate solution, and more precisely that there exists an increasing sequence $\{u_n\}$ of positive moderate solutions such that $u_n \uparrow u$ and $tr(u_n) = \mu_n$. Set $\mu_0 = \lim_{n\to\infty} \mu_n$ and define μ on Borel sets $E \subset \mathbb{R}^N$ by*

$$\mu(E) = \inf \{\mu_0(Q) : E \subset Q,\, Q \ \mathfrak{T}_q\text{-open}\}. \tag{5.112}$$

Then:
(i) μ is the precise initial trace of u and μ is $\mathfrak{T}_q$-perfect. In particular μ is independent of the sequence $\{u_n\}$ which appears in its definition.
(ii) If A is a Borel set such that $\mu(A) < \infty$, then $\mu(A) = \mu_0(A)$.
(iii) A solution $u \in \mathcal{U}_+(Q_T)$ is σ-moderate if and only if

$$u = \sup \{v \in \mathcal{U}_+(Q_T),\, v \leq u,\, v \text{ moderate}\}. \tag{5.113}$$

This statement is equivalent to

$$u = \sup \left\{u_\tau \in \mathcal{U}_+(Q_T) : \tau \in B^{-\frac{2}{q},q}(\mathbb{R}^N) \cap \mathfrak{M}^b_+(\mathbb{R}^N),\, \tau \leq tr(u)\right\}. \tag{5.114}$$

(iv) If u and w are σ-moderate solutions,

$$tr(w) \leq tr(u) \Longleftrightarrow w \leq u. \tag{5.115}$$

Proof. (i) Since the μ_n are Radon measures absolutely continuous with respect to $cap_{\frac{2}{q},q'}$, μ_0 which is the limit of the μ_n shares this property. By Proposition 5.63, μ is $\mathfrak{T}_q$-perfect.

Let $\{Q_n\}$ be the family of $\mathfrak{T}_q$-open sets of Proposition 5.64-(i). Set $Q'_n = \mathcal{R}_q(u) \setminus Q_n$. Since $\cup_n Q_n = \lim_{n\to\infty} Q_n \sim^q \mathcal{R}_q(u)$, then $Q'_n \downarrow E$ and $cap_{\frac{2}{q},q'}(E) = 0$. Consequently, for any $n \in \mathbb{N}$, we have

$$\lim_{m\to\infty} u_{\mathbf{1}_{Q'_m}\mu_n} = 0.$$

Therefore, there exists a subsequence still denoted by $\{Q'_n\}$ such that

$$\left(\int_0^T \int_{\mathbb{R}^N} u^q_{\mathbf{1}_{Q'_m}\mu_n} dxdt\right)^{\frac{1}{q}} \leq \frac{1}{2^n}.$$

Since

$$\mathbf{1}_{\mathcal{R}_q(u)}\mu_n = \mathbf{1}_{Q_n}\mu_n + \mathbf{1}_{Q_n^c}\mu_n,$$

it follows that

$$\lim_{n\to\infty} |u_{\mathbf{1}_{\mathcal{R}_q(u)}\mu_n} - u_{\mathbf{1}_{Q_n}\mu_n}| = \lim_{n\to\infty} u_{\mathbf{1}_{Q_n^c}\mu_n} = 0.$$

Since we have also

$$u_n = u_{\mu_n} \leq u_{\mathbf{1}_{\mathcal{R}_q(u)}\mu_n} + u_{\mathbf{1}_{\mathcal{S}_q(u)}\mu_n} \leq u_{\mathbf{1}_{\mathcal{R}_q(u)}\mu_n} + [u]_{\mathcal{S}_q(u)},$$

we infer

$$0 \leq u - [u]_{\mathcal{S}_q(u)} \leq w := \lim_{n\to\infty} u_{\mathbf{1}_{\mathcal{R}_q(u)}\mu_n} = \lim_{n\to\infty} u_{\mathbf{1}_{D_n}\mu_n} \leq u_{reg}.$$

This implies $u \ominus [u]_{\mathcal{S}_q(u)} \leq u_{reg}$ and $u \leq u_{reg} \oplus [u]_{\mathcal{S}_q(u)}$. For the opposite inequality, we have by Proposition 5.64-(iv)

$$[u]_{D_n} \uparrow u_{reg}.$$

By relation (5.60) in Proposition 5.54, using the fact that $\widetilde{D}_n \subset^q D_{n+1} \subset \widetilde{D}_{n+1} \subset^q \mathcal{R}_q(u)$, we have that $cap_{\frac{2}{q},q'}\left(\widetilde{D}_{n+1} \cap \mathcal{S}_q(u)\right) = 0$ and

$$[u]_{D_n} \leq [[u]_{\mathcal{S}_q(u)}]_{D_{n+1}} + [u \ominus [u]_{\mathcal{S}_q(u)}]_{D_{n+1}} = [u \ominus [u]_{\mathcal{S}_q(u)}]_{D_{n+1}} \leq u \ominus [u]_{\mathcal{S}_q(u)}.$$

Letting $n \to \infty$, we derive $u_{reg} \leq u \ominus [u]_{\mathcal{S}_q(u)}$. Therefore $\lim_{n\to\infty} u_{\mathbf{1}_{D_n}\mu_n} = u_{reg}$. Therefore the sequence $\left\{u_{\mathbf{1}_{D_n}\mu_n}\right\}$ satisfies condition (5.89) and by Proposition 5.64-(iv) and Definition 5.66 we obtain

$$\lim_{n\to\infty} \mathbf{1}_{D_n}\mu_n = \mu_{\mathcal{R}_q(u)} \quad \text{and} \quad tr_{\mathcal{R}_q(u)}(u) = \overline{\mu}_{\mathcal{R}_q(u)}. \tag{5.116}$$

Next we show that the q-singular set $\mathcal{S}_q(u)$ is singular for the sequences of measures $\{\mu_n\}$ in the sense that if $\xi \in \mathcal{S}_q(u)$, then for every $\mathfrak{T}_q$-open neighbourhood Q of ξ, $\mu_n(\widetilde{Q}) \to \infty$ when $n \to \infty$. Indeed, we can assume that Q is bounded and we consider a nonnegative function $\eta \in B^{\frac{2}{q},q'}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $\mathfrak{T}_q$-support included in Q. We put $h = \mathbb{H}\eta]$

and $\phi(r) = r_+^{2q'}$. Then, using Theorem 5.31, Proposition 5.26 and the computations in Proposition 5.25, we have

$$\int_0^T \int_{\mathbb{R}^N} \left(-u_n \left(\partial_t \phi(h) + \Delta \phi(h)\right) + u_n^q \phi(h)\right) dx d\tau + \int_{\mathbb{R}^N} (\phi(h) u_n)(x,T) dx = \int_{\mathbb{R}^N} \eta^{2q'} d\mu_n,$$

and

$$\int_0^T \int_{\mathbb{R}^N} u_n^q \phi(h) dx d\tau \leq C(q) \left(\|\eta\|^{2q'}_{B^{\frac{2}{q},q'}} + \|\eta\|_{L^\infty} + \int_{\mathbb{R}^N} \eta^{2q'} d\mu_n \right).$$

We can assume that the function η has value 1 in some $\mathfrak{T}_q$-open set $D \subset Q$ and $0 \leq \eta \leq 1$ (see Lemma 5.21). If we let $n \to \infty$, we obtain from the above relations

$$\lim_{n\to\infty} \int_0^T \int_{\mathbb{R}^N} u_n^q \left(\mathbb{H}[\mathbf{1}_D]\right)^{2q'} dx d\tau \leq C(q) \left(\|\eta\|^{2q'}_{B^{\frac{2}{q},q'}} + \|\eta\|_{L^\infty} + \lim_{n\to\infty} \int_{\mathbb{R}^N} \eta^{2q'} d\mu_n \right).$$

Then the assertion follows from Proposition 5.28.

In conclusion, we have proved that if $\xi \in \mathcal{S}_q(u)$ and Q is a $\mathfrak{T}_q$-open neighbourhood of ξ, then $\mu_0(\widetilde{Q}) = \infty$. By the outer regularity of μ with respect to the $\mathfrak{T}_q$-topology, it means that $\mu(\xi) = \infty$. Combined with (5.116) this implies that μ is the precise trace of u.

(ii) If $\mu(A) < \infty$, then A is contained in a $\mathfrak{T}_q$-open set D such that $\mu_0(D) < \infty$. By Proposition 5.63 we have that $\mu(A) = \mu_0(A)$.

(iii) Let $u \in \mathcal{U}_+(Q_T)$ be σ-moderate and denote

$$u^* := \sup \{v : v \text{ moderate } v \leq u\}. \tag{5.117}$$

By expression (5.117) $u^* \leq u$. Since u is σ-moderate there exists an increasing sequence $\{u_n\}$ of moderate solutions which converges to u. For any n we have proved in the beginning of the Section on moderate solutions that given u_n there exists an increasing sequence $\{u_{n,m}\}_{m\in\mathbb{N}} = \{u_{\mu_{n,m}}\}_{m\in\mathbb{N}}$ of elements of $\mathcal{U}_+(Q_T)$ where $\mu_{n,m} \in B^{-\frac{2}{q},q}(\mathbb{R}^N) \cap \mathfrak{M}_+^b(\mathbb{R}^N)$ converging to u_n. Therefore

$$u_n \leq \sup \left\{ u_\tau : \tau \in B^{-\frac{2}{q},q}(\mathbb{R}^N) \cap \mathfrak{M}_+^b(\mathbb{R}^N),\, \tau \leq tr(u) \right\} = u^{**}.$$

By letting $n \to \infty$ we infer $u \leq u^{**}$.

However, if u is σ-moderate, $\tau \in B^{-\frac{2}{q},q}(\mathbb{R}^N) \cap \mathfrak{M}_+^b(\mathbb{R}^N)$ and $\tau \leq tr(u)$, then we have that $tr(u_\tau \ominus u_n) = (\tau - \mu_n)_+$ and the corresponding sequence decreases to 0 when $n \to \infty$. Therefore $u_\tau \ominus u_n \downarrow 0$ which implies $u_\tau \leq u$ and thus $u^{**} \leq u$. Consequently, (5.113) implies (5.114). This shows that the two identities which define σ-moderate solutions are equivalent.

(iv) The implication $\Longrightarrow$ follows from (5.114). For proving the opposite implication, it is sufficient to show that if u is σ-moderate, w is moderate and $w \leq u$, then $tr(w) \leq tr(u)$. For this task, we consider an increasing sequence of moderate solutions $\{u_n\}$ which converges to u. Then $u_n \wedge w \leq u$ and consequently $u_n \leq u_n \wedge w \uparrow u$.. This implies $tr(u_n \wedge w) \uparrow \mu' \leq tr(u)$. Hence $tr(w) \leq tr(u)$. □

This results extends Proposition 5.64 which deals with the regular initial trace.

Theorem 5.68 *Let $u \in \mathcal{U}_+(Q_T)$ and $\nu = tr(u)$.*
(i) u_{reg} is σ-moderate and $tr(u_{reg}) = tr_{\mathcal{R}_q(u)}(u)$.
(ii) If $v \in \mathcal{U}_+(Q_T)$, then

$$v \le u \Longrightarrow tr(v) \le tr(u), \tag{5.118}$$

and if F is a $\mathfrak{T}_q$-closed set, then

$$tr([u]_F) \le \mathbf{1}_F \nu. \tag{5.119}$$

(iii) A singular point of the trace can be characterized in terms of the measure ν as follows:

$$\xi \in \mathcal{S}_q(u) \Longrightarrow \nu(Q) = \infty \quad \text{for all } \mathfrak{T}_q\text{-open set containing } \xi. \tag{5.120}$$

(iv) If Q is a $\mathfrak{T}_q$-open set then:

$$[u]_Q \text{ moderate} \iff \exists \text{ Borel set } A \text{ s.t. } cap_{\frac{2}{q},q'}(A) = 0 \text{ and } \nu(A \cap \widetilde{Q} \setminus K) < \infty, \tag{5.121}$$

for any compact set $K \subset \mathbb{R}^N$.
(v) The singular set of u_{reg} may not be empty. Actually

$$\mathcal{S}_q(u) \setminus b_q(\mathcal{S}_q(u)) \subset \mathcal{S}_q(u_reg) \subset \mathcal{S}_q(u) \cap \widetilde{\mathcal{R}_q(u)}, \tag{5.122}$$

where $b_q(\mathcal{S}_q(u))$ is the set of thick points of $\mathcal{S}_q(u)$ for the $\mathfrak{T}_q$-topology.
(vi) Put

$$\mathcal{S}_{q,0}(u) := \left\{\xi \in \mathbb{R}^N : \nu(Q \setminus S_q(u)) = \infty \quad \text{for all } \mathfrak{T}_q\text{-neighbourhood of } \xi\right\}. \tag{5.123}$$

Then

$$\mathcal{S}_q(u_{reg}) \setminus b_q(\mathcal{S}_q(u)) \subset \mathcal{S}_{q,0}(u) \subset \mathcal{S}_q(u_{reg}) \bigcup b_q(\mathcal{S}_q(u)). \tag{5.124}$$

Remark. We will prove later on that any element of $\mathcal{U}_+(Q_T)$ is σ-moderate. Hence implication (5.118) is actually an equivalence.

Proof. The first part of assertion (i) is proved in Proposition 5.64-(i) and the fact that $u_{reg} = v_{\mathcal{R}_q(u)}$. The second part follows from Definition 5.66 and Theorem 5.67-(i).
(ii) If $v \le u$, then $\mathcal{R}_q(u) \subset \mathcal{R}_q(v)$ and by definition $v_{reg} \le u_{reg}$. By Theorem 5.67-(iv) we have $tr(v_{reg}) \le tr(u_{reg})$. This implies $tr(v) \le tr(u)$. Inequality (5.119) is a consequence of (5.118).
(iii) If ξ is a regular point, there exists a $\mathfrak{T}_q$-open neighbourhood of ξ, say Q, such that $[u]_Q$. Therefore $\nu(Q) = tr_{\mathcal{R}_q(u)}(u)(Q) < \infty$. Conversely, if $\xi \in \mathcal{S}_q(u)$, it follows from the definition of the precise trace that $\nu(Q) = \infty$ for all $\mathfrak{T}_q$-open neighbourhood Q of ξ.
(iv) If Q is $\mathfrak{T}_q$-open and $[u]_Q$ is moderate, then $Q \subset \mathcal{R}_q(u)$. By Proposition 5.64-(ix) we obtain the implication $\Longrightarrow$ in (5.121). Conversely,

$$\nu(\widetilde{Q} \cap K \setminus A) < \infty, \text{ for all compact set } K \subset \mathbb{R}^N \Longrightarrow \widetilde{Q} \subset^q \mathcal{R}_q(u),$$

and $\mu_{\mathcal{R}_q(u)}(\widetilde{Q}\cap K)=\mu_{\mathcal{R}_q(u)}(\widetilde{Q}\cap K\setminus A)<\infty$. It follows by Proposition 5.64-(ix) that $[u]_Q$ is moderate.
(v) Because $\mathfrak{T}_q$-supp $(u_{reg})\subset\widetilde{\mathcal{R}_q(u)}$ and $\mathcal{R}_q(u)\subset\mathcal{R}_q(u_{reg})$, we have

$$\mathcal{S}_q(u_{reg})\subset\mathcal{S}_q(u)\cap\widetilde{\mathcal{R}_q(u)}.$$

Next we prove that $\mathcal{S}_q(u)\setminus b_q(\mathcal{S}_q(u))\subset\mathcal{S}_q(u_{reg})$.
If $\xi\in\mathcal{S}_q(u)\setminus b_q(\mathcal{S}_q(u))$, then $\mathcal{R}_q(u)\cup\{\xi\}$ is a $\mathfrak{T}_q$-open neighbourhood of ξ. By (i) u_{reg} is σ-moderate and thus its trace is $\mathfrak{T}_q$-perfect (see Theorem 5.67)-(i)). Therefore if Q_0 is a $\mathfrak{T}_q$-open neighbourhood of ξ and $Q=Q_0\cap(\{\xi\}\cup\mathcal{R}_q(u))$, then

$$tr(u_{reg})(Q)=tr(u_{reg})(Q\setminus\{\xi\})=tr(u)(Q\setminus\{\xi\}),$$

where, it the last inequality, we have used the fact that $Q\setminus\{\xi\}\subset\mathcal{R}_q(u)$. Now, let D be a $\mathfrak{T}_q$-open set such that $\xi\in D\subset\widetilde{D}\subset Q$. If $tr(u)(\widetilde{Q}\setminus\{\xi\})<\infty$, then, by (iv), $[u]_D$ is moderate and $\xi\in\mathcal{R}_q(u$, contrary to our assumption. Therefore $tr(u)(\widetilde{Q}\setminus\{\xi\})=\infty$ which implies $tr(u_{reg})(Q_0\setminus\{\xi\})=\infty$ for every bounded $\mathfrak{T}_q$-open neighbourhood Q_0 of ξ, and consequently $\xi\in\mathcal{S}_q(u_{reg})$, which ends the proof of (v).
(vi) If $\xi\notin b_q(\mathcal{S}_q(u))$ there exists a $\mathfrak{T}_q$-open neighbourhood D of ξ such that $(D\setminus\{\xi\})\cap\mathcal{S}_q(u)=\emptyset$, and thus

$$tr(u_{reg})(D\setminus\{\xi\}=tr(u_{reg})(D\setminus\mathcal{S}_q(u))=tr(u)(D\setminus\mathcal{S}_q(u)).\tag{5.125}$$

Furthermore, if we assume that $\xi\in\mathcal{S}_{q,0}(u)$, then

$$tr(u)(D\setminus\mathcal{S}_q(u))=tr(u_{reg})(D\setminus\mathcal{S}_q(u))=\infty.$$

If Q is an arbitrary $\mathfrak{T}_q$-open neighbourhood of ξ, then the same relation holds if D is replaced by $D\cap Q$. Therefore $tr(u_{reg})(Q\setminus\{\xi\}=\infty$ for any such Q. This implies that $\xi\in\mathcal{S}_q(u_{reg})$ and $\mathcal{S}_{q,0}(u)\setminus b_q(\mathcal{S}_q(u))\subset\mathcal{S}_q(u_{reg})$.

On the other hand, if $\xi\in\mathcal{S}_q(u_{reg})\setminus b_q(\mathcal{S}_q(u))$, there exists a $\mathfrak{T}_q$-open neighbourhood D of ξ such that (5.125) holds and $tr(u_{reg})(D)=\infty$. Since u_{reg} is σ-moderate $tr(u_{reg})$ is $\mathfrak{T}_q$-perfect, which infers $tr(u_{reg})(D)=tr(u_{reg})(D\setminus\{\xi\})=\infty$. Using (5.125) we obtain that $tr(u)(D\setminus\mathcal{S}_q(u))=\infty$. At end, if Q is $\mathfrak{T}_q$-open neighbourhood of ξ, then D can be replaced by $D\cap Q$, which yields $tr(u)(Q\setminus\mathcal{S}_q(u))=\infty$. This proves that $\xi\in\mathcal{S}_{q,0}(u)$ and ends the proof of (5.125). □

Proposition 5.69 *Let F be a $\mathfrak{T}_q$-closed set. Then $\mathcal{S}_q(U_F)=b_q(F)$.*

Proof. Let $\xi\in\mathbb{R}^N$ such that F is $\mathfrak{T}_q$-thin at ξ. Let Q be a $\mathfrak{T}_q$-open neighbourhood of ξ such that $\widetilde{Q}\subset^q F^c$. Then $[U_F]_Q=U_{F\cap\widetilde{Q}}=0$. Then $\xi\in\mathcal{R}_q(u)$.

Conversely, if $\xi\in F\cap\mathcal{R}_q(U_F)$ there exists a $\mathfrak{T}_q$-open neighbourhood Q of ξ such that $[U_F]_Q$ is moderate. But the relation $[U_F]_Q=U_{F\cap\widetilde{Q}}$, combined with the previous assertion, implies that $cap_{\frac{2}{q},q'}(F\cap\widetilde{Q})=0$ and therefore $Q\subset\mathcal{R}_q(u)$. Since

$$cap_{\frac{2}{q},q'}(F)\leq cap_{\frac{2}{q},q'}(F\cap\widetilde{Q})+cap_{\frac{2}{q},q'}(Q^c),$$

we conclude that F is $\mathfrak{T}_q$-thin at ξ. □

5.6.2 The initial value problem

We introduce below some definitions and notations which will be useful in the sequel.

Definition 5.70 *I-* $\mathfrak{M}_+(\mathbb{R}^N)$ *is the space of positive outer regular Borel measure in* $\mathbb{R}^N$.
II- $\mathcal{C}_q(\mathbb{R}^N)$ *is the space of couples* (τ, F) *such that* F *is a* $\mathfrak{T}_q$*-closed subset of* $\mathbb{R}^N$, $\tau \in \mathfrak{M}_+(\mathbb{R}^N)$ *with* $\mathfrak{T}_q$*-supp* $(\tau) \subset \widetilde{F^c}$ *and* $\mathbf{1}_{F^c}\tau$ *is* $\mathfrak{T}_q$*-locally finite.*
III- $\mathbb{T}$ *denotes the mapping from* $\mathcal{C}_q(\mathbb{R}^N)$ *into* $\mathfrak{M}_+(\mathbb{R}^N)$ *defined by* $\nu = \mathbb{T}(\tau, F)$ *where* ν *is defined as in (5.111) with* $\mathcal{R}_q(u)$ *and* $\mathcal{S}_q(u)$ *replaced respectively by* F^c *and* F. *In this setting* ν *is the measure representation of the couple* (τ, F).
IV- If $(\tau, F) \in \mathcal{C}_q(\mathbb{R}^N)$, *the set*

$$F_\tau = \{\xi \in \mathbb{R}^N : \tau(Q \setminus F) = \infty \ \text{ for all } \mathfrak{T}_q\text{-open neighbourhood of } \xi\}, \tag{5.126}$$

is called the set of explosion points of τ.

Remark. Since $\mathbf{1}_{F^c}\tau$ is locally finite, $F_\tau \subset F$. If F_τ is not included in $\widetilde{F^c}$, there would exist a $\mathfrak{T}_q$-open neighbourhood Q of ξ with an empty intersection with F^c, hence included in F, thus $Q \setminus F = \emptyset$ and $\tau(Q \setminus F) = 0$, contradiction. Therefore $F_\tau \subset \widetilde{F^c}$ and consequently

$$F_\tau \subset \widetilde{F^c} \cap F = \left(F^c \bigcup b_q(F^c)\right) \cap F = b_q(F^c) \cap F. \tag{5.127}$$

This result has to be compared with Theorem 3.14 which deals with a necessary and sufficient conditions for the existence of a maximal solution u of (3.13) with a rough initial trace (S, μ).

The next result points out the crucial role of the set $\mathfrak{M}_q(\mathbb{R}^N)$ defined in Definition 5.62 for describing the link between $\mathcal{U}_+(Q_T)$ and $\mathcal{C}_q(\mathbb{R}^N)$.

Proposition 5.71 *Let* ν *be a positive Borel measure in* $\mathbb{R}^N$.
(i) The initial value problem

$$\begin{array}{rl} \partial_t u - \Delta u + |u|^{q-1}u = 0 & \text{in } Q_\infty \\ u \geq 0 & \text{in } Q_\infty \\ tr(u) = \nu & \end{array} \tag{5.128}$$

possesses a solution if and only if $\nu \in \mathfrak{M}_q(\mathbb{R}^N)$.
(ii) Let $(\tau, F) \in \mathcal{C}_q(\mathbb{R}^N)$ *and set* $\nu := \mathbb{T}(\tau, F)$. *Then* $\nu \in \mathfrak{M}_q(\mathbb{R}^N)$ *if and only if*

$$\tau \in \mathfrak{M}_q(\mathbb{R}^N) \ \text{ and } \ F = b_q(F) \bigcup F_\tau. \tag{5.129}$$

(iii) Let $\nu \in \mathfrak{M}_q(\mathbb{R}^N)$ *and set*

$$\begin{array}{l} \mathcal{E}_\nu := \{E : E \ \mathfrak{T}_q\text{-closed sets s.t. } \nu(E \cap K) < \infty \text{ for all compact } K \subset \mathbb{R}^N\} \\ \mathcal{D}_\nu := \{D : D \ \mathfrak{T}_q\text{-open sets s.t. } \widetilde{D} \sim^q E \text{ for some } E \in \mathcal{E}_\nu\}. \end{array} \tag{5.130}$$

Then a solution of (5.128) is given by $u = v \oplus U_F$ where

$$G := \bigcup_{D \in \mathcal{D}_\nu} D, \quad F := G^c, \quad \nu := \sup\{u_{\mathbf{1}_E \nu} : E \in \mathcal{E}_\nu\}. \tag{5.131}$$

(iv) The solution $u := v \oplus U_F$ is σ-moderate and it is the unique solution of problem (5.128) in the class of σ-moderate solutions. Furthermore u is the largest solution of this problem.

Remark. 1- We recall that if $E \in E_\nu$ then $\mathbf{1}_E \nu$ is a locally bounded Borel measure which does not charge sets of $cap_{\frac{2}{q},q'}$-capacity zero. Recall also that if μ is a positive measure possessing these properties, then u_μ denotes the moderate solution with initial trace μ.
2- We will see later on that $u := v \oplus U_F$ is the only solution to problem (5.128) since every solution happens to be σ-moderate.

Proof. (I) If $u \in \mathcal{U}_+(Q_T)$,

$$tr(u) = \nu \Longrightarrow \nu \in \mathfrak{M}_q(\mathbb{R}^N). \tag{5.132}$$

By Proposition 5.64, u_{reg} is σ-moderate and $u \approx_{\mathcal{R}_q(u)} u_{reg}$. Therefore

$$\mathbf{1}_{\mathcal{R}_q(u)} tr(u) = \mathbf{1}_{\mathcal{R}_q(u)} tr(u_{reg}).$$

By Theorem 5.67 $\widetilde{\mu}_{\mathcal{R}_q} = tr(u_{reg}) \in \mathfrak{M}_q(\mathbb{R}^N)$. If ν is defined by (5.130), then

$$v = \sup\{[u]_F : F\, \mathfrak{T}_q\text{-closed}, F \subset^q \mathcal{R}_q(u)\} = u_{reg}, \tag{5.133}$$

where the second equality holds by definition. Actually, by Theorem 5.68, for every $\mathfrak{T}_q$-open set Q $[u]_Q$ is moderate if and only if $\nu(K \cap \widetilde{Q} \setminus A) < \infty$ for some set A with $cap_{\frac{2}{q},q'}(A) = 0$ and for every compact set $K \subset \mathbb{R}^N$. Hence, by Proposition 5.64-(ix), $\widetilde{E}$ is regular, in the sense that there exists a $\mathfrak{T}_q$-open regular set such that $E \subset^q Q$. Hence $u_{\mathbf{1}_E \nu} \leq [u]_Q$. This implies that $v \leq u_{reg}$, which proves (5.133). Furthermore, if $E \cap \mathcal{S}_q(u) \neq \emptyset$, then $\nu(E) = \infty$ by Definition 5.66. Therefore ν is outer regular with respect to the $\mathfrak{T}_q$-topology.

Next we prove that ν is essentially absolutely continuous (cf. Definition 5.62-(iii)). Let Q be a $\mathfrak{T}_q$-open set and A a non-empty $\mathfrak{T}_q$-closed subset of Q such that $cap_{\frac{2}{q},q'}(A) = 0$. either $\nu(Q \setminus A) = \infty$ in which case $\nu(Q \setminus A) = \nu(Q) = \infty$, or $\nu(Q \setminus A) < \infty$. In that case $Q \setminus A \subset \mathcal{R}_q(u)$ and

$$\nu(Q \setminus A) = \widetilde{\mu}(Q \setminus A) = \widetilde{\mu}(Q)$$

since $cap_{\frac{2}{q},q'}(A) = 0 \Longrightarrow \widetilde{\mu}(A) = 0$.

Let $\xi \in A$ and D be a $\mathfrak{T}_q$-open subset of Q such that $\xi \in D \subset \widetilde{D} \subset^q Q$. Consider now a sequence $\{B_n\}$ of $\mathfrak{T}_q$-open neighbourhoods of $A \cap \widetilde{D}$ such that $B_n \subset^q D$ and $cap_{\frac{2}{q},q'}(B_n) < 2^{-n}$. If we set $E_n = \widetilde{D} \setminus B_n$, we have

$$[u]_D \leq [u]_{E_n} + [u]_{B_n}.$$

Since $\lim_{n\to\infty}[u]_{B_n}=0$, it follows that $[u]_D \le \liminf_{n\to\infty}[u]_{E_n}$. Because $E_n \subset \mathcal{R}_q(u)$, $\nu(E_n) \le \nu(Q\setminus A) < \infty$, we have by the definition of ν and Proposition 5.64-(ix) that $[u]_{E_n}$ is moderate. Using Lemma 5.18, Lemma 5.17-(ii) and [43] there holds

$$\int_0^T \int_K [u]_{E_n}^q dxdt \le C\nu(E_n) \le C\nu(Q\setminus A) < \infty,$$

for any compact set $K \subset \mathbb{R}^N$. Therefore

$$\int_0^T \int_K [u]_D^q dxdt \le C\nu(E_n) \le C\nu(Q\setminus A) < \infty \quad \text{for all compact set } K \subset \mathbb{R}^N.$$

This implies that $[u]_D$ is moderate and thus $D \subset \mathcal{R}_q(u)$. Therefore, since every point A has a neighbourhood D as above, we conclude that $A \subset \mathcal{R}_q(u)$ and hence $\nu(A) = tr_{\mathcal{R}_q(u)}(u)(A) = 0$. If A is any a non-empty Borel subset of Q such that $cap_{\frac{2}{q},q'}(A)=0$, we use the inequality $cap_{\frac{2}{q},q'}(\widetilde{A}) \le \widetilde{C} cap_{\frac{2}{q},q'}(A)$ to conclude that ν is absolutely continuous and hence $\nu \in \mathfrak{M}_q(\mathbb{R}^N)$.

Next we prove:
(II) *Suppose that $(\tau, F) \in \mathcal{C}_q(\mathbb{R}^N)$ satisfies (5.129) and put $\nu = \mathbb{T}(\tau, F)$. Then the solution $u := v \oplus U_F$ with ν as in (5.131) satisfies $tr(u) = \nu$. Notice that implies $\nu \in \mathfrak{M}_q(\mathbb{R}^N)$ by (5.131).*

The solution v is σ-moderate by construction. Since τ is locally $\mathfrak{T}_q$-finite in F^c and essentially absolutely continuous with respect to $cap_{\frac{2}{q},q'}$ we have that

$$G := F^c \subset \mathcal{R}_q(u) \quad \text{and } \mathbf{1}_G tr(v) = \tau_G. \tag{5.134}$$

Therefore, it follows from the definition of v that $F_\tau \subset S_q(v)$. By Proposition 5.69 and Theorem 5.68-(iv) we have that

$$F = b_q(F) \bigcup F_\tau \subset S_q(v) \bigcup S_q(U_F) \subset S_q(u) \subset F. \tag{5.135}$$

Hence $F = S_q(u)$, $v = u_{reg}$ and $\tau = tr(u_{reg})$. In turn, this implies $tr(u) = (\tau, F)$, which is equivalent to $\tau = tr(u)$.

Then we prove:
(III) *Suppose that $(\tau, F) \in \mathcal{C}_q(\mathbb{R}^N)$ and that there exists a solution u such that $tr^c(u) = (\tau, F)$ (see Definition 5.66 for the definition of tr^c). Then*

$$\tau = tr_{\mathcal{R}_q(u)}(u) = tr(u_{reg}) \quad \textit{and } F = S_q(u). \tag{5.136}$$

If $U := u_{reg} \oplus U_F$, then $tr(U) = tr(u)$ and $u \le U$. U is the only σ-moderate solution of (5.128) and (τ, F) satisfies (5.129). Assertion (5.136) follows by Proposition 5.64-(i). and Definition 5.66. Since u_{reg} is σ-moderate, we have that $\tau \in \mathfrak{M}_q(\mathbb{R}^N)$ by Theorem 5.67.

By Proposition 5.64-(vi) there holds $u \approx_{\mathcal{R}_q(u)} u_{reg}$. Therefore the function $w := u \ominus u_{reg}$ which vanishes on $\mathcal{R}_q(u)$ is dominated by U. Note that $u - u_{reg} \le w$ and therefore

$$u \le u_{reg} \oplus w \le U. \tag{5.137}$$

By defintion, $\mathcal{S}_{q,0}(u) = F_\tau$ and by Theorem 5.68(vi) and Proposition 5.69 we have

$$\begin{aligned}\mathcal{S}_q(u) &= \mathcal{S}_q(u_{reg})\bigcup \mathcal{S}_q(U_F) = \mathcal{S}_q(u_{reg})\bigcup b_q(U_F)\\ &= \mathcal{S}_{q,0}(u)\bigcup b_q(U_F) = F_\tau \bigcup b_q(U_F).\end{aligned} \tag{5.138}$$

On the other-hand $\mathcal{R}_q(U) \supset \mathcal{R}_q(u_{\mathcal{R}_q(u)}) = \mathcal{R}_q(u)$. As $u \le U$ we have $\mathcal{R}_q(U) \subset \mathcal{R}_q(u)$. Hence $\mathcal{R}_q(U) = \mathcal{R}_q(u)$ and $\mathcal{S}_q(U) = \mathcal{S}_q(u)$. Therefore, by (5.135), (5.137),

$$F = \mathcal{S}_q(U) = F_\tau \cup b_q(U_F).$$

This implies that (τ, F) satisfies (5.129) and $tr^c(u) = (\tau, F)$. That U is the maximal solution with this trace follows from (5.137).

The solution U is σ-moderate because u_{reg} and U_F are σ-moderate (see Theorem 5.43).

Finally we prove:
(IV)*If $\nu \in \mathfrak{M}_q(\mathbb{R}^N)$ then the couple (τ, F) defined by*

$$v := \sup\{u_{\mathbf{1}_E\nu} : E \in \mathcal{E}_\nu\}\ ,\ \tau = tr(v)\,,\ F = \mathcal{R}_q^c(v), \tag{5.139}$$

(see (5.130) for the definition of $\mathcal{E}_\nu$) satisfies (5.129). This is the only couple belonging to $\mathcal{C}_q(\mathbb{R}^N)$ satisfying $\nu = \mathbb{T}(\tau, F)$. The solution v is σ-moderate so that $\tau \in \mathfrak{M}_q(\mathbb{R}^N)$.

We first prove that $u := v \oplus U_F$ is a solution with initial trace $tr(u) = (\tau, F)$. Actually $u \ge v$, so that $\mathcal{R}_q(u) \subset \mathcal{R}_q(v)$. On the other hand, since τ is $\mathfrak{T}_q$-locally finite in $\mathcal{R}_q(v) = F^c$, it follows that $\mathcal{S}_q(u) \subset F$. Therefore $\mathcal{R}_q(v) \subset \mathcal{R}_q(u)$, and finally $\mathcal{R}_q(u) = \mathcal{R}_q(v)$ and $F = \mathcal{S}_q(u)$. This also implies $v = u_{reg}$.

At end

$$\mathcal{S}_q(u) = \mathcal{S}_q(v)\bigcup b_q(\mathcal{S}_q(U_F)) = b_q(F)\bigcup F,$$

which means that (5.129) holds.

That for $\nu \in \mathfrak{M}_q(\mathbb{R}^N)$ the couple (τ, F) defined by (5.129) is the only one couple belonging to $\mathcal{C}_q(\mathbb{R}^N)$ satisfying $\nu = \mathbb{T}(\tau, F)$ is a mere consequence of their expression in Definition 5.70.

Finally, statements (i)-(iv) follow from (I)-(IV). □

Remark. If $\nu \in \mathfrak{M}_q(\mathbb{R}^N)$ then G and v as defined by (5.131) have the following alternative representation:

$$G := \bigcup_{\mathcal{E}_\nu} E = \bigcup_{Q\in\mathcal{F}_\nu} Q, \qquad v := \sup\left\{u_{\mathbf{1}_Q\nu} : Q \in \mathcal{F}_\nu\right\}), \tag{5.140}$$

where

$$\mathcal{F}_\nu = \left\{Q :\, Q\ \mathfrak{T}_q\text{-open},\ \nu(Q\cap K) < \infty \text{ for all compact set } K \subset \mathbb{R}^N\right\} r. \tag{5.141}$$

In order to prove (5.140) we first observe that if A is a $\mathfrak{T}_q$-open set, then there exists an increasing sequence of $\mathfrak{T}_q$-quasi closed sets $\{E_n\}$ such that $A = \cup_1^\infty E_n$. This follows from Lemma 5.18-(II-i-ii) with $E_n = F_n \setminus L$ where $L = A' \setminus A$ and $cap_{\frac{2}{q},q'}(L) = 0$. Thus

$$\bigcup_{\mathcal{D}_\nu} D \subset \bigcup_{\mathcal{F}_\nu} Q \subset \bigcup_{\mathcal{E}_\nu} E = H.$$

On the other hand, if $E \in \mathcal{E}_\nu$, then $\mu_{\mathcal{R}_q(u)}(K \cap \widetilde{E}) = \mu_{\mathcal{R}_q(u)}(K \cap E) = \nu(E \cap K) < \infty$ for any compact set $K \subset \mathbb{R}^N$. By Proposition 5.64 (ix), $\widetilde{E}$ is regular in the sense that there exists a $\mathfrak{T}_q$-open regular set Q such that $E \subset^q Q$, therefore $H = \bigcup_{\mathcal{D}_\nu} D$.

If D is a $\mathfrak{T}_q$-open regular set, then $D = \cup_{n=1}^\infty E_n$ where $\{E_n\}$ is an increasing sequence of $\mathfrak{T}_q$-quasi closed sets. This implies

$$u_{\mathbf{1}_D \nu} = \lim_{n\to\infty} u_{\mathbf{1}_{E_n}\nu}.$$

Hence

$$\sup\left\{u_{\mathbf{1}_Q\nu} : Q \in \mathcal{D}_\nu\right\} \leq \sup\left\{u_{\mathbf{1}_Q\nu} : Q \in \mathcal{F}_\nu\right\} \leq \sup\left\{u_{\mathbf{1}_Q\nu} : Q \in \mathcal{E}_\nu\right\}.$$

However, if $E \in \mathcal{E}_\nu$ there exists a $\mathfrak{T}_q$-open regular set Q such that $E \subset^q Q$. This implies the inequality in (5.140).

5.7 Representation of positive solutions of $\partial_t u - \Delta u + Vu = 0$

In this section we prove a general representation theorem for positive solutions of

$$\partial_t u - \Delta u + Vu = 0 \qquad \text{in } Q_T := \mathbb{R}^N \times (0,T) \tag{5.142}$$

where $V : Q_T \to \mathbb{R}$ is a Borel function satisfying

$$0 \leq V(x,t) \leq \frac{C}{t} \qquad \text{for all } (x,t) \in Q_T, \tag{5.143}$$

for some positive constant. Our results are the parabolic counterpart of Ancona's results [3] concerning representation of positive solutions of

$$-\Delta u + V(x)u = 0 \qquad \text{in } \Omega, \tag{5.144}$$

by means of a Martin operator when $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain and V a Borel function defined in Ω satisfying

$$0 \leq V(x) \leq \frac{C}{(\rho(x))^2} \qquad \text{for almost all } x \in \Omega, \tag{5.145}$$

with $\rho(x) = \text{dist}\,(x, \partial\Omega)$ and $C > 0$.

We recall first some well-known facts concerning weak solutions of (5.142).

Definition 5.72 *Let $\mu \in \mathfrak{M}(\mathbb{R}^N)$. We say that u is a weak solution of*

$$\begin{array}{rl} \partial_t u - \Delta u + Vu = 0 & \text{in } Q_T \\ u(.,0) = \mu & \text{in } \mathbb{R}^N, \end{array} \tag{5.146}$$

if $u, Vu \in L^1_{loc}(\overline{Q}_T)$ and there holds

$$\int\int_{Q_T} u\left(-\partial_t\zeta - \Delta\zeta + V\zeta\right) dxdt = \int_{\mathbb{R}^N} \zeta(x,0) d\mu(x) \quad \text{for all } \zeta \in \mathbb{X}(Q_T), \tag{5.147}$$

where

$$\mathbb{X}(Q_T) = \left\{\zeta \in C_c(\mathbb{R}^N \times [0,T)) : \zeta + \Delta\zeta \in L^\infty_{loc}(\overline{Q}_T)\right\}.$$

Note that this definition implies that the function u admits the measure μ as an initial trace as described in Section 2. The next result is an easy adaptation of the techniques developed in Section 2.

Lemma 5.73 *Let $\mu \in \mathfrak{M}_+(\mathbb{R}^N)$ and assume that there exists a positive weak solution u of problem (5.146) where V satisfies (5.145). Then for any smooth bounded domain Ω there exists a unique positive weak solution $v = v_\Omega$ of problem*

$$\begin{array}{rl} \partial_t v - \Delta v + Vv = 0 & \text{in } Q^\Omega_T := \Omega \times (0,T) \\ v = 0 & \text{in } \partial_\ell Q^\Omega_T := \partial\Omega \times (0,T) \\ v(.,0) = \mathbf{1}_\Omega \mu & \text{in } \Omega. \end{array} \tag{5.148}$$

Furthermore $0 \le v_\Omega \le u$ and the mapping $\Omega \mapsto v_\Omega$ is nondecreasing.

Proof. Let ϵ_n be a sequence converging to 0 and v_n the solution of

$$\begin{array}{rl} \partial_t v - \Delta v + Vv = 0 & \text{in } \Omega \times (\epsilon_n, T) \\ v = 0 & \text{in } \partial\Omega \times (\epsilon_n, T) \\ v(.,\epsilon_n) = \mathbf{1}_\Omega u(.,\epsilon_n) & \text{in } \Omega. \end{array} \tag{5.149}$$

Such a solution exists since $u(.,\epsilon_n) \in L^1(\Omega)$ and it satisfies $0 \le v_n \le u$ in $\Omega \times (\epsilon_n, T)$. By classical parabolic regularity estimates we may assume that the sequence $\{v_n\}$ converges locally uniformly in Q^Ω_T to a nonnegative function v dominated by u. Let $\zeta \in C^{1,1;1}(\overline{Q}^T_\Omega)$ vanish on $\partial_\ell Q^\Omega_T$ and for $t \ge T - \delta$ for some $\delta > 0$. Set $\zeta_n(x,t) = \zeta(x, t-\epsilon_n)$, then from (5.148), and assuming that $\epsilon_n \le \delta$ we have

$$\int_{\epsilon_n}^T \int_\Omega v_n\left(-\partial_t\zeta_n - \Delta\zeta_n + V\zeta_n\right) dxdt = \int_\Omega \zeta(.,0) u(.,\epsilon_n) dx. \tag{5.150}$$

Because $v_n V \le uV \in L^1_{loc}(\overline{Q}_T)$ we deduce from the dominated convergence theorem that the left-hand side of (5.150) converges to $\displaystyle\int_0^T \int_\Omega v\left(-\partial_t\zeta - \Delta\zeta + V\zeta_n\right) dxdt$ while the right-hand side converges to $\displaystyle\int_\Omega \zeta(.,0) d\mu(x)$. The final assertion on the monotonicity of $\Omega \mapsto v_\Omega$ is a consequence of the maximum principle. This ends the proof. □

Lemma 5.74 *Let the assumptions on μ, V and u of Lemma 5.73 be satisfied and denote by $v_R := v_{B_R}$ the solution of (5.147) with $\Omega = B_R$. Then*

$$v_R \uparrow u \qquad \text{as } R \to \infty.$$

Furthermore this convergence is uniform on compact subsets of Q_T.

Proof. Since the mapping $R \mapsto v_R$ is increasing and v_R is dominated by u, there exists a function w such that

$$v_R \uparrow w \le u \qquad \text{as } R \to \infty,$$

and this convergence is locally uniformly in Q_T^Ω. Because for any $\zeta \in C_c(\overline{Q}_T)$,

$$\lim_{R\to\infty} \int_{B_R} \zeta(.,0) d\mu(x) = \int_{\mathbb{R}^N} \zeta(.,0) d\mu(x),$$

we infer that w is a weak solution of problem (5.146). Therefore the function $\widetilde{w} = u - w$ is nonnegative and satisfies

$$\begin{aligned} \partial_t \widetilde{w} - \Delta \widetilde{w} &\le 0 \qquad \text{in } Q_T \\ \widetilde{w} &\ge 0 \qquad \text{in } Q_T \\ \widetilde{w}(.,0) &= 0 \qquad \text{in } \mathbb{R}^N. \end{aligned} \tag{5.151}$$

Moreover $\widetilde{w}$ belongs to $L^1_{loc}(\overline{Q}_T)$. We extend it by 0 in $\mathbb{R}^N \times (-T, 0)$ and the resulting function $\widetilde{w}^*$ is a nonnegative sub-caloric function in $\mathbb{R}^N \times (-T, T)$ that we can suppose to be C^∞ by replacing it by $J_\epsilon * \widetilde{w}^*$ where J_ϵ is a sequence of mollifiers in $\mathbb{R}^{N+1}$. By the maximum principle $J_\epsilon * \widetilde{w}^* = 0$. Hence $\widetilde{w}^* = 0$ which yields $u = w$. □

The next result is the extension of the initial trace theorem for nonnegative caloric functions to nonnegative solutions of (5.142).

Lemma 5.75 *Let V satisfy (5.143) and $u \in C^{2;1}(Q_T)$ be a nonnegative function satisfying (5.142). Assume that for any $x \in \mathbb{R}^N$ there exists a bounded open set $U \subset \mathbb{R}^N$ such that*

$$\int_0^T \int_U u(y,t) V(y,t) dy dt < \infty.$$

Then $u \in L^1(\ \times (0,T))$ and there exists a nonnegative Radon measure μ in $\mathbb{R}^N$ such that

$$\lim_{t\to 0} \int_{\mathbb{R}^N} u(x,t)\zeta(x) dx = \int_{\mathbb{R}^N} \zeta d\mu \quad \text{for all } \zeta \in C_c^\infty(\mathbb{R}^N).$$

Proof. Without loss of generality we can assume that ∂U is smooth and since $Vu \in L^1(U \times (0,T))$ it is classical that there exists a solution v to the problem

$$\begin{aligned} \partial_t v - \Delta v &= Vu \qquad \text{in } Q_T^U \\ v &= 0 \qquad \text{in } \partial_\ell Q_T^U \\ v(.,0) &= 0 \qquad \text{in } U. \end{aligned}$$

The function v is nonnegative and $w = u + v$ is a positive solution of the heat equation. Hence w admits an initial trace on U which is a nonnegative Radon measure. This implies that u admits the same initial trace on U. We end the proof by using a partition of unity. □

Now we can prove our fundamental *Representation Theorem*.

We assume that V satisfies (5.143) and let $u \in C^{2;1}(Q_T)$ be a nonnegative solution of (5.142). If $\psi \in C^{2;1}(Q_T)$ we define $v \in C^{2;1}(Q_T)$ by $v(x,t) = e^{-\psi(x,t)}u(x,t)$. then

$$\partial_t v - \Delta v - 2\nabla v.\nabla\psi - |\nabla\psi|^2 v - 2v\Delta\psi + (\partial_t\psi + \Delta\psi + V)v = 0 \quad \text{in } Q_T. \tag{5.152}$$

We choose ψ to be the solution of

$$\begin{array}{rl} -\partial_t\psi - \Delta\psi = V & \text{in } Q_T \\ \psi(.,T) = 0 & \text{in } \mathbb{R}^N. \end{array} \tag{5.153}$$

Then

$$\psi(x,t) = \int_t^T \int_{\mathbb{R}^N} \frac{e^{-\frac{|x-y|^2}{4(s-t)}}}{(4\pi(s-t))^{\frac{N}{2}}} V(y,s)dyds. \tag{5.154}$$

Because of (5.143) the following estimates hold:

$$\begin{array}{ll} (i) & 0 \le \psi(x,t) \le C_0(T)\ln\left(\frac{T}{t}\right) \\ (ii) & |\nabla\psi(x,t)| \le C_1(T) + C_2(T)\ln\left(\frac{T}{t}\right). \end{array} \tag{5.155}$$

With this choice of ψ, equation (5.152) becomes

$$\partial_t v - \Delta v - 2\sum_{i=1}^n (v\psi_{x_i})_{x_i} - |\nabla\psi|^2 v = 0 \quad \text{in } Q_T. \tag{5.156}$$

Because $\ln t \in L^p(0,1)$ for all $p \in [1,\infty)$, it follows that for any $p \in [1,\infty)$ there exists $M_j := M_j(p) > 0$, $j = 1,2$ such that

$$\begin{array}{ll} (i) & \displaystyle\int_0^T \sup_{x\in\mathbb{R}^N} |\psi(x,t)|^p dt \le M_1 \\ (ii) & \displaystyle\int_0^T \sup_{x\in\mathbb{R}^N} |\nabla\psi(x,t)|^p dt \le M_2. \end{array} \tag{5.157}$$

Using Aronson's estimates [4] with $A_{ij} = \delta_{ij}$, $A_i = 2\psi_{x_i}$, $B_i = 0$, $C = |\nabla\psi|^2$ and $p = \infty$ with the notations of this article, then the condition H therein is satisfied. Therefore there exists a kernel $\Gamma(x,t;y,s)$ defined in $Q_T \times Q_T$ satisfying

$$C_1(T,N,M_2)\frac{e^{-a_1\frac{|x-y|^2}{4(s-t)}}}{(4\pi(s-t))^{\frac{N}{2}}} \le \Gamma(x,t;y,s) \le C_2(T,N,M_2)\frac{e^{-a_2\frac{|x-y|^2}{4(s-t)}}}{(4\pi(s-t))^{\frac{N}{2}}}, \tag{5.158}$$

for all $(x,t,y,s) \in Q_T \times Q_T$ with $a_1 \geq a_2 > 0$ depending on T, N and M_2, and such that v admits the following representation

$$v(x,t) = \int_{\mathbb{R}^N} \Gamma(x,t;y,0) d\mu(y), \tag{5.159}$$

where μ is the initial trace of u obtained in Lemma 5.75. Furthermore there holds

$$\lim_{t\to 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \Gamma(x,t;y,0)\zeta(x) d\mu(y) dx = \int_{\mathbb{R}^N} \zeta d\mu \quad \text{for all } \zeta \in C_c^\infty(\mathbb{R}^N).$$

Note that if the initial trace of u is a function u_0 such that $e^{-\gamma|x|^2} u_0(.) \in L^2(\mathbb{R}^N)$ for some $\gamma > 0$ and u_0 is continuous at some $x \in \mathbb{R}^N$ then

$$\lim_{t\to 0} \int_{\mathbb{R}^N} \Gamma(x,t;y,0) u_0(y) dy = u_0(x). \tag{5.160}$$

Finally, we have the representation

$$u(x,t) = e^{\psi(x,t)} \int_{\mathbb{R}^N} \Gamma(x,t;y,0) d\mu(y). \tag{5.161}$$

5.8 σ-moderate solutions

5.8.1 The Marcus approach

In this paragraph we adapt to the parabolic framework the construction in [41] used for characterising, by mean of their precise boundary trace, all the positive solutions of

$$-\Delta u + u^q = 0 \quad \text{in } \Omega \tag{5.162}$$

in a smooth bounded domain Ω.

Proposition 5.76 *Let $u \in \mathcal{U}_+(Q_T)$, then*

$$\max\left\{u_{\mathcal{R}_q(u)}, [u]_{\mathcal{S}_q(u)}\right\} \leq u \leq u_{\mathcal{R}_q(u)} + [u]_{\mathcal{S}_q(u)}. \tag{5.163}$$

Proof. From Proposition 5.64-(ii) the function $v = u \ominus u_{\mathcal{R}_q(u)}$ has it $\mathfrak{T}_q$-support included in $\mathcal{S}_q(u)$ since its vanishes on $\mathcal{R}_q(u)$. Furthermore $v \leq u$, hence $v \leq [u]_{\mathcal{S}_q(u)}$. Therefore $u - u_{\mathcal{R}_q(u)} \leq v$, which implies $u \leq u_{\mathcal{R}_q(u)} + [u]_{\mathcal{S}_q(u)}$. The left-hand side inequality in (5.163) follows by the construction of $u_{\mathcal{R}_q(u)}$ and the definition of $[u]_{\mathcal{S}_q(u)}$. □

Proposition 5.77 *Let $u \in \mathcal{U}_+(Q_T)$ and A, B be two disjoint $\mathfrak{T}_q$-closed Borel susbets of $\mathbb{R}^N$. If the $\mathfrak{T}_q$-support of u is included in $A \cup B$ and $[u]_A$ and $[u]_B$ are σ-moderate, then u is σ-moderate. Furthermore*

$$u = [u]_A \oplus [u]_B = [u]_A \vee [u]_B. \tag{5.164}$$

Proof. Because $[u]_A$ and $[u]_B$ are σ-moderate there exist two increasing sequences $\{\tau_n\}$ and $\{\tau'_n\}$ included in $B^{-\frac{2}{q},q'}(\mathbb{R}^N) \cap \mathfrak{M}^b_+(\mathbb{R}^N)$ such that

$$u_{\tau_n} \uparrow [u]_A \quad \text{and} \quad u_{\tau'_n} \uparrow [u]_B \quad \text{as} \quad n \to \infty,$$

and $\mathfrak{T}_q\text{-supp}\,(\tau_n) \subset^q A$ while $\mathfrak{T}_q\text{-supp}\,(\tau'_n) \subset^q B$ (see Proposition 5.49). Thus

$$cap_{\frac{2}{q},q'}\left(\mathfrak{T}_q\text{-supp}\,(\tau_n) \cap \mathfrak{T}_q\text{-supp}\,(\tau'_n)\right) = 0,$$

and

$$u_{\tau_n} \wedge u_{\tau'_n} = u_{\tau_n} \oplus u_{\tau'_n} = u_{\tau_n+\tau'_n}.$$

Moreover, by Proposition 5.42-(ii) and Definition 5.50,

$$\max\{[u]_A, [u]_B\} \le u \le [u]_A + [u]_B. \tag{5.165}$$

Therefore

$$\max\{u_{\tau_n}, u_{\tau'_n}\} \le u \Longrightarrow u_{\tau_n+\tau'_n} \le u.$$

On the other hand

$$u - u_{\tau_n+\tau'_n} \le [u]_A - u_{\tau_n} + [u]_B - u_{\tau'_n}.$$

Since the right-hand side tends to 0 as $n \to \infty$ we obtain

$$\lim_{n\to\infty} u_{\tau_n+\tau'_n} = u, \tag{5.166}$$

which means that u is σ-moderate.

By definition of the operations $\oplus$ and $\vee$, identity (5.164) admits the following equivalent formulation;
(a) u is the largest solution dominated by $[u]_A + [u]_B$,
(b) u is the smallest solution dominating by $\max\{[u]_A, [u]_B\}$.
Set $w := [u]_A \oplus [u]_B$, then

$$u \le w \le [u]_A + [u]_B.$$

Clearly $[u]_A \le [w]_A$. Since $[w]_A \le w \le [u]_A + [u]_B$ implies $[w]_A - [u]_A \le [u]_B$. This implies

$$v := [([w]_A - [u]_A)_+]_\dagger \le [u]_B ?\text{`} \;\; v \le [w]_A,$$

where $[([w]_A - [u]_A)_+]_\dagger$ is defined in the notations (e) in Section 4.1. This implies that

$$\mathfrak{T}_q\text{-supp}\,(\nu) \subset A \quad \text{and} \quad \mathfrak{T}_q\text{-supp}\,(\nu) \subset B.$$

Since $A \cap B = \emptyset$ we obtain $v = 0$ and $[w]_A \le [u]_A$. In a similar way $[w]_B \le [u]_B$. Using (5.165) and the fact that for any Borel set E we have $[u]_E \le [u]_{\widetilde{E}\cap A} + [u]_{\widetilde{E}\cap B}$ we infer

$$\mathcal{S}_q(u) = \mathcal{S}_q(w).$$

As a consequence any regular $\mathfrak{T}_q$-open regular subset $Q \subset \mathcal{R}_q(w)$ is included into $\mathcal{R}_q(u)$. Using now Proposition 5.42-(ii) and the fact that the $\mathfrak{T}_q$-support of w is included into $A \cap B$ we deduce

$$[w]_Q \leq [w]_{\widetilde{Q}\cap A} + [w]_{\widetilde{Q}\cap B} = [[w]_A]_{\widetilde{Q}} + [[w]_B]_{\widetilde{Q}} = [[u]_A]_{\widetilde{Q}} + [[u]_B]_{\widetilde{Q}w}.$$

Now $[w]_Q$ and $[u]_Q$ are moderate solutions. Because $A \cap B = \emptyset$ there also holds $[u]_{\widetilde{Q}\cap A} \oplus [u]_{\widetilde{Q}\cap B} \leq [u]_Q$, which implies in turn $[u]_Q = [w]_Q$. Therefore, by Proposition 5.64-(ii), $w_{\mathcal{R}_q(u)} = u_{\mathcal{R}_q(u)}$. Using Proposition 5.71 and the remark hereafter we derive

$$u \leq w \leq u_{\mathcal{R}_q(u)} + U_F,$$

where F is defined in (5.131). Since σ-moderate solutions are uniquely defined, w and u coincide. Hence the result follows from (5.165) and (5.166) by letting $n \to \infty$. □

5.8.2 Characterization of positive solutions of $\partial_t u - \Delta u + u^q = 0$

If $u \in \mathcal{U}_+(Q_T)$ we set

$$V(x,t) = u^{q-1}. \tag{5.167}$$

Then u is a solution of

$$\partial_t u - \Delta u + Vu = 0 \qquad \text{in } Q_T, \tag{5.168}$$

where V satisfies estimate (5.143) with $C = (q-1)^{-\frac{1}{q-1}}$. The function u belongs to $C^{2;1}(Q_T)$ and there exists a nonnegative Radon measure μ in $\mathbb{R}^N$ such that the following representation formula holds:

$$u(x,t) = e^{\psi(x,t)} \int_{\mathbb{R}^N} \Gamma(x,t;y,0) d\mu(y) \quad \text{for all } (x,t) \in Q_T, \tag{5.169}$$

where ψ is the solution of (5.153) expressed by (5.154). The measure μ is called the *extended initial trace of* u.

If $E \subset \mathbb{R}^N$ is a Borel set we put

$$\mu_E = \mathbf{1}_E \mu \quad \text{and} \quad (u)_E := e^{\psi(x,t)} \int_{\mathbb{R}^N} \Gamma(x,t;y,0) d\mu_E(y) \quad \text{in } Q_T.$$

The next result is fundamental and points out the importance of the function $(u)_E$.

Lemma 5.78 *Let $F \subset \mathbb{R}^N$ be compact, then*

$$(u)_E \leq [u]_E \qquad \text{in } Q_T. \tag{5.170}$$

Proof. Let $A \subset \mathbb{R}^N$ be a Borel set and $0 < \beta \leq \frac{T}{2}$. We denote by v_β^A the solution of

$$\begin{aligned} \partial_t v - \Delta v + Vv &= 0 && \text{in } \mathbb{R}^N \times (\beta, T) \\ v(.,\beta) &= \mathbf{1}_A u(.,\beta) && \text{in } \mathbb{R}^N, \end{aligned} \tag{5.171}$$

and by w^A_β the one of

$$\begin{array}{ll} \partial_t w - \Delta w + w^q = 0 & \text{in } \mathbb{R}^N \times (\beta, T) \\ v(.,\beta) = \mathbf{1}_A u(.,\beta) & \text{in } \mathbb{R}^N. \end{array} \tag{5.172}$$

Since $u^{q-1} \leq V$, there holds $0 \leq w^A_\beta \leq v^A_\beta \leq u$. For any sequence $\{\beta_k\}$ decreasing to 0 one can extract a subsequence still denoted by $\{\beta_k\}$ such that $\{v^A_{\beta_k}\}$ and $\{w^A_{\beta_k}\}$ converges locally uniformly to v^A and w^A respectively. Clearly $w^A \in \mathcal{U}_+(Q_T)$ while v^A is a solution of (5.168). Since the $\mathfrak{T}_q$- support of $w^A_\beta(.,\beta)$ is included into $\widetilde{Q}$ for any open set Q which contains A, we have

$$v^A \leq w^A \leq [u]_{\widetilde{Q}}. \tag{5.173}$$

Next we set $\widetilde{v}_k = e^{-\psi} v^A_{\beta_k}$, then $\widetilde{v}_k$ satisfies

$$\begin{array}{ll} \partial_t v - \Delta v - 2\nabla v.\nabla\psi - |\nabla\psi|^2 v - 2v\Delta\psi + (\partial_t\psi + \Delta\psi + V)v = 0 & \text{in } Q_T \\ v(.,\beta_k) = \mathbf{1}_A \displaystyle\int_{\mathbb{R}^N} \Gamma(.,\beta_k,y,0)d\mu(y) & \text{in } \mathbb{R}^N. \end{array} \tag{5.174}$$

Using Duhamel's formula (see [4] in a similar case), we have

$$\begin{aligned} \widetilde{v}_k(x,t) &= \int_{\mathbb{R}^N} \mathbf{1}_A(x)\Gamma(x,t-\beta_k;y,0)\left(\int_{\mathbb{R}^N}\Gamma(x,\beta_k;y,0)d\mu(y)\right)dx \\ &= \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\mathbf{1}_A(x)\Gamma(x,t-\beta_k;y,0)\Gamma(x,\beta_k;y,0)dx\right)d\mu(y) \\ &\leq \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\mathbf{1}_Q(x)\Gamma(x,t-\beta_k;y,0)\Gamma(x,\beta_k;y,0)dx\right)d\mu(y). \end{aligned}$$

Using the estimates on Γ (see (5.158)) the continuity and the property (5.160) we can let $k \to \infty$ and obtain by the dominated convergence theorem

$$\lim_{k\to\infty} \widetilde{v}_k(x,t) \leq \int_{\mathbb{R}^N} \Gamma(x,t_k;y,0)d\mu_Q(y).$$

This implies

$$v^A \leq (u)_{\widetilde{Q}}. \tag{5.175}$$

Then we can proceed in the same way with A^c. Extracting a subsequence from the previous subsequence (and denoting it still by $\{k\}$) we obtain limits v^{A^c} and w^{A^c} and they satisfy

$$v^{A^c} \leq w^{A^c} \leq [u]_{\widetilde{Q'^c}} \quad \text{for all open sets } Q' \supset A^c.$$

Since $v^A_{\beta_k} + v^{A^c}_{\beta_k}$ satisfies (3.13) in $(\beta_k, T) \times \mathbb{R}^N$ with initial data $u(.,\beta_k)$, we have

$$v^A + v^{A^c} = u\,, \quad v^A \leq (u)_{\widetilde{Q}}\,, \quad v^{A^c} \leq (u)_{\widetilde{Q'^c}}$$

from what we derive

$$v^A = u - v^{A^c} \geq (u)_{\widetilde{Q'^c}}. \tag{5.176}$$

Next, if F is a compact subset of $\mathbb{R}^N$, $\mathcal{O}$ an open set such that $F \subset \mathcal{O} \subset A$, we obtain from (5.176) with $Q' = A^c$ (and thus $A^c \cap F = Q' \cap F = \emptyset$),

$$v^A \geq (u)_{\mathcal{O}}.$$

By (5.175),

$$v\hat{A} \leq w\hat{A} \leq [u]_{\widetilde{Q}} \quad \text{for all open set } Q \supset A,$$

which implies

$$(u)_F \leq (u)_{\mathcal{O}} \leq [u]_{\widetilde{Q}}. \tag{5.177}$$

By Lemma 5.18 we can fix a sequence of open sets $\{Q_n\}$ such that $\cap_n \widetilde{Q}_n = E' \sim^q F$. This implies $[u]_{Q_n} \downarrow [u]_F$ (see Proposition 5.47-(iii)). The result follows from (5.177). □

In the next result we prove that the extended initial trace of a positive solution of (3.13) is absolutely continuous with respect to the $cap_{\frac{2}{q},q'}$ -capacity.

Proposition 5.79 *Let $u \in \mathcal{U}_+(Q_T)$ and μ be its extended initial trace as defined in (5.169). Then $\mu(E) = 0$ for any Borel set $E \subset \mathbb{R}^N$ such that $cap_{\frac{2}{q},q'}(E) = 0$.*

Proof. If K is a compact set satisfying $cap_{\frac{2}{q},q'}(K) = 0$, then $U_K = 0$ by Corollary 5.40. Therefore $[u]_K = u \vee U_K = 0$. Consequently, by Lemma 5.78 $(u)_K = 0$ and $\mu(K) = 0$. Since this holds for any such K, it also holds for E by outer regularity. □

We recall that for any $\nu \in B^{-\frac{2}{q},q}(\mathbb{R}^N) \cap \mathfrak{M}^b_+(\mathbb{R}^N)$ and any $T > 0$ $C(T) > 0$ such that

$$C^{-1} \left\|\nu\right\|_{B^{-\frac{2}{q},q}(\mathbb{R}^N)} \leq \left\|\mathbb{H}[\nu]\right\|_{L^q(Q_T)} \leq C \left\|\nu\right\|_{B^{-\frac{2}{q},q}(\mathbb{R}^N)}. \tag{5.178}$$

Proposition 5.80 *Let $u \in \mathcal{U}_+(Q_T)$, μ be its extended initial trace and $\nu \in B^{-\frac{2}{q},q}(\mathbb{R}^N) \cap \mathfrak{M}^b_+(\mathbb{R}^N)$. Suppose that there exists no positive solution of (3.13) dominated by $v = \inf\{u, \mathbb{H}[\nu]\}$. Then μ and ν are mutually singular, that we denote $\mu \perp \nu$.*

Proof. Set $V' = v^{q-1}$. Then v is a supersolution of

$$\partial_t w - \Delta w + V'w = 0 \quad \text{in } \ Q_T. \tag{5.179}$$

We first prove by contradiction that there exists no positive solution of (5.179) dominated by v. Indeed, if such a solution w of this equation does exist, there holds

$$\partial_t w - \Delta w + w^q \leq \partial_t w - \Delta w + V'w = 0.$$

Because of (5.178) the function w is a moderate solution of (3.13) dominated by v, contrary to the assumption. Next, we have a representation formula valid in Q_T where we use Aronson's estimates 5.157 and the constants a_2 and C from this inequality,

$$\begin{aligned}
\inf\{u, \mathbb{H}[\nu]\} &= \inf\left\{e^{\psi} \int_{\mathbb{R}^N} \Gamma(x,t;y,0) d\mu(y), \mathbb{H}[\nu]\right\} \\
&\geq \inf\left\{\int_{\mathbb{R}^N} \Gamma(x,t;y,0) d\mu(y), \mathbb{H}[\nu]\right\} \\
&\geq C \inf\left\{\mathbb{H}[\mu]\left(\tfrac{t}{a_2}, x\right), \mathbb{H}[\nu](t,x)\right\} \\
&\geq C \inf\left\{\mathbb{H}[\mu]\left(\tfrac{t}{\max\{a_2,1\}}, x\right), \mathbb{H}[\nu]\left(\tfrac{t}{\max\{a_2,1\}}, x\right)\right\}.
\end{aligned}$$

We notice that

$$(t,x)\mapsto \inf\left\{\mathbb{H}[\mu]\left(\tfrac{t}{\max\{a_2,1\}},x\right),\mathbb{H}[\nu]\left(\tfrac{t}{\max\{a_2,1\}},x\right)\right\},$$

is a supersolution of the equation $\partial_t w-\frac{1}{\max\{a_2,1\}}\Delta w=0$, therefore there exists a nonnegative Radon measure $\widetilde{\nu}$ in $\mathbb{R}^N$ such that

$$\lim_{t\to 0}\int_{\mathbb{R}^N}\phi(x)\inf\left\{\mathbb{H}[\mu]\left(\tfrac{t}{\max\{a_2,1\}},x\right),\mathbb{H}[\nu]\left(\tfrac{t}{\max\{a_2,1\}},x\right)\right\}dx=\int_{\mathbb{R}^N}\phi(x)d\widetilde{\nu}(x). \tag{5.180}$$

By Lemma 5.74 and Lemma 5.75 there exists a positive solution $\tilde{v}\le v$ of the initial value problem

$$\begin{aligned}\partial_t w-\Delta w+Vw&=0 \qquad \text{in } Q_T\\ w(.,0)&=\widetilde{\nu} \qquad \text{in } \mathbb{R}^N.\end{aligned}$$

By the first claim it yields $\widetilde{\nu}=0$.

By the Radon-Nikodym theorem there exists a positive measure σ and a Borel function $\theta\in L^1(\mathbb{R}^N,\mu)$ such that $\sigma\perp\mu$ and $\nu=\theta\mu+\sigma$. Therefore if H is the heat kernel in Q_∞, we obtain

$$\begin{aligned}0&=\lim_{t\to 0}\int_{\mathbb{R}^N}\phi(x)\inf\left\{\mathbb{H}[\mu]\left(\tfrac{t}{\max\{a_2,1\}},x\right),\mathbb{H}[\nu]\left(\tfrac{t}{\max\{a_2,1\}},x\right)\right\}dx\\ &\ge\lim_{t\to 0}\int_{\mathbb{R}^N}\phi(x)H\left(\tfrac{t}{\max\{a_2,1\}},x,y\right)\min\{1,\theta(y)\}d\mu(y)dx\\ &\ge\lim_{t\to 0}\int_{\mathbb{R}^N}\phi(y)\min\{1,\theta(y)\}d\mu(y)\\ &=0.\end{aligned}$$

Hence $\theta=0$ and $\nu\perp\mu$. □

Lemma 5.81 *Let $u\in U_+(Q_T)$, μ its extended initial trace and suppose that for every $\nu\in\mathfrak{M}^b_+(\mathbb{R}^N)\cap B^{-\frac{2}{q},q'}(\mathbb{R}^N)$ there exists no positive solution of (3.13) dominated by $v=\inf\{u,\mathbb{H}[\nu]\}$. Then $u=0$.*

Proof. . As in the previous lemma, the proof is an adaptation to the parabolic framework of the construction in [41]. By the previous lemma,

$$\mu\perp\nu\quad\text{for all }\nu\in\mathfrak{M}^b_+(\mathbb{R}^N)\cap B^{-\frac{2}{q},q'}(\mathbb{R}^N).$$

Suppose now that $\mu\neq 0$, then by Lemma 5.78 μ vanishes on Borel sets $E\subset\mathbb{R}^N$ such that $cap_{\frac{2}{q},q'}(E)=0$. Therefore, there exists an increasing sequence $\{\nu_k\}\subset\mathfrak{M}^b_+(\mathbb{R}^N)\cap B^{-\frac{2}{q},q'}(\mathbb{R}^N)$ which converges to μ. Therefore $\mu\perp\nu_k$ and for every $k\in\mathbb{N}$ there exists a Borel set $A_k\subset\mathbb{R}^N$ such that

$$\mu(A_k)=0\quad\text{and}\quad\nu_k(A_k^c)=0.$$

If we denote $A = \cup_k A_k$, then

$$\mu(A) = 0 \quad \text{and for all integer } k,\ \nu_k(A^c) = 0.$$

But since $\nu_k \le \mu$ we have also $\nu_k(A) = 0$ and thus $\nu_k = 0$ for all k, contradiction. □

The next result is fundamental.

Proposition 5.82 *Let $u \in U_+(Q_T$, then $[u]_{\mathcal{S}_q(u)}$ is σ-moderate.*

Proof. We simplify the notations in setting $u_{\mathcal{S}} = [u]_{\mathcal{S}_q(u)}$ (there will be no ambiguity), and we denote $F = \mathfrak{T}_q$-supp $(u_{\mathcal{S}})$. Then $F \subset \mathcal{S}_q(u)$. We know that if $\mathcal{S}_q(u)$ is thin at ξ, then $\mathcal{S}_q(u)^c \cup \{\xi\}$ is $\mathfrak{T}_q$-open and $\mathcal{S}_q(u)^c \cup \{\xi\} \sim^q \mathcal{S}_q(u)^c$. Since F is the $\mathfrak{T}_q$-support of $u_{\mathcal{S}}$ we see that F consists exactly in the set of $cap_{\frac{2}{q},q'}$-thick points of $\mathcal{S}_q(u)$, and therefore $\mathcal{S}_q(u) \setminus F$ is contained in the singular set of $u_{\mathcal{R}_q(u)}$.

If $\nu \in \mathfrak{M}^b_+(\mathbb{R}^N) \cap B^{-\frac{2}{q},q'}(\mathbb{R}^N)$ and u_ν is the solution of (3.13) with initial trace ν we put

$$u^* := \sup\left\{u_\nu : \nu \in \mathfrak{M}^b_+(\mathbb{R}^N) \cap B^{-\frac{2}{q},q'}(\mathbb{R}^N),\ u_\nu \le u_{\mathcal{S}}\right\}. \tag{5.181}$$

By the previous lemma, u^* do exist since some elements u_ν of this family exist. Also u^* is σ-moderate by Theorem 5.67-(iii). Therefore u^* is the largest σ-moderate moderate solution of (3.13) dominated by $u_{\mathcal{S}}$. Let $\{\nu_k\} \subset \mathfrak{M}^b_+(\mathbb{R}^N) \cap B^{-\frac{2}{q},q'}(\mathbb{R}^N)$ be an increasing sequence such that $u_{\nu_k} \uparrow u^*$.

Let F^* be the $\mathfrak{T}_q$ support of u^*, then F^* is $\mathfrak{T}_q$-closed and included in F. Let us assume that

$$cap_{\frac{2}{q},q'}(F \setminus F^*) > 0,$$

then there exists a compact set $E \subset F \setminus F^*$ such that $cap_{\frac{2}{q},q'}(E) > 0$ and $(F^*)^c := Q^*$ is $\mathfrak{T}_q$-open and contains E. By Lemma 5.17 there exists a $\mathfrak{T}_q$-open set Q' such that $E \subset^q Q' \subset \widetilde{Q}' \subset^q Q^*$. Because $Q' \subset^q \mathfrak{T}_q$-supp $(u_{\mathcal{S}})$, $[U_{\mathcal{S}}]_{Q'} > 0$ and by Proposition 5.79 there exists a positive bounded measure $\tau \in B^{-\frac{2}{q},q'}(\mathbb{R}^N)$ with support in $\widetilde{Q}'$ such that $u_\tau \le u_{\mathcal{S}}$. As the $\mathfrak{T}_q$-support of τ is a $\mathfrak{T}_q$-closed set disjoint from F^*, the inequality $u^* \ge u_\tau$ cannot hold. However since $\tau \in \mathfrak{M}^b_+(\mathbb{R}^N) \cap B^{-\frac{2}{q},q'}(\mathbb{R}^N$ is such that $u_\tau \le u_{\mathcal{S}}$, it follows that $u \le u^*$, which is a contradiction. Hence

$$cap_{\frac{2}{q},q'}(F \setminus F^*) = 0.$$

Since $u_{\nu_k} \uparrow u^*$, the $\mathfrak{T}_q$ support of ν_k is contained into the $\mathfrak{T}_q$ support of u^* which is F^*. Therefore there exists a $\mathfrak{T}_q$-closed set F_0^* contained into F such that $\mathcal{S}_q(u^*) = F_0^*$ and $\mathcal{R}_q(u^*) = (F_0^*)^c$. Suppose now that

$$cap_{\frac{2}{q},q'}(F \setminus F_0^*) > 0,$$

and let Q' be a $\mathfrak{T}_q$-open set contained into $\mathcal{R}_q(u^*)$ such that $[u_{\mathcal{S}}]_{Q'}$ is a moderate solution of (3.13). Then $\widetilde{Q}' \subset^q \mathcal{R}_q(u^*)$ and $[u^*]_{\widetilde{Q}'}$ is a moderate solution too, thus

$$\int\int_{Q_T} [u^*]^q_{\widetilde{Q}'} \phi(x)dxdt < \infty \quad \text{for all } \phi \in C_0^\infty(\mathbb{R}^N),\ \phi \ge 0.$$

On the other hand Q' is a $\mathfrak{T}_q$-open subset of F which is the $\mathfrak{T}_q$ support of $u_{\mathcal{S}}$. Consequently the initial trace of $[u^*]_{\widetilde{Q}'}$ has no regular part, that is

$$\mathcal{R}_q([u^*]_{\widetilde{Q}'}) = \emptyset \quad \text{and } \mathcal{S}_q([u^*]_{\widetilde{Q}'}) = \mathfrak{T}_q\text{-support}([u^*]_{\widetilde{Q}'}).$$

In such a case we call $[u^*]_{\widetilde{Q}'}$ a *purely singular solution of (3.13)*. It implies that

$$v = \left[[u_{\mathcal{S}}]_{\widetilde{Q}'} - [u^*]_{\widetilde{Q}'}\right]_+$$

is a purely singular solution too.

Let v^* be defined as in expression (5.181) with $u_{\mathcal{S}}$ replaced by v. Then v^* is a singular σ-moderate solution of (3.13). As it is dominated by u and σ-moderate, it is smaller than u^*. Now, $\mathfrak{T}_q$-supp$(v^*) \subset^q \widetilde{Q}' \subset^q \mathcal{R}_q(u^*)$, therefore u^* cannot be larger or equal to v^*, hence $(v^* - u^*)_+$ is not identically zero. Since both u^* and v^* are σ-moderate, it follows that there exists a nonnegative bounded measure $\tau \in B^{-\frac{2}{q},q'}(\mathbb{R}^N)$ such that $u_\tau \le v^*$ and $(u_\tau - u^*)_+$ is not identically zero, and obviously that $u^* \le \max\{u_\tau, v^*\}$. The function $\max\{u^*, u_\tau\}$ is a nontrivial subsolution of (3.13) and there exists a smallest solution Z above it, which also strictly larger than u^*. However $u_\tau \le v^* \le u^*$ and thus $u^* = Z$, contradiction. As a consequence $cap_{\frac{2}{q},q'}(Q') = 0$ for any $\mathfrak{T}_q$-open set included in $\mathcal{R}_q(u^*)$ such that $[u^*]_{Q'}$ is a moderate solution. Hence

$$cap_{\frac{2}{q},q'}(F \setminus F_0^*) = 0. \tag{5.182}$$

In conclusion u^* is σ-moderate, $\mathfrak{T}_q$-supp$\,(u^*) \subset F$ and $F_0^* = \mathcal{S}_q(u^*) \sim^q F$. Therefore, by Proposition 5.71 and the remark which follows $u^* = U_F$. Since by definition (5.181) $u^* \le u_{\mathcal{S}} \le U_F$ it follows that $u^* = u_{\mathcal{S}}$ and thus $u_{\mathcal{S}}$ is σ-moderate. □

The following result is the *icing on the cake* of the precise trace theory.

Theorem 5.83 *Every positive solution of (3.13) is σ-moderate.*

Proof. Let $u \in \mathcal{U}_+(Q_T)$. By Proposition 5.64-(i), $\mathcal{R}_q(u)$ has a regular decomposition $\{Q_n\}$ and

$$v_n := [u]_{Q_n} \uparrow u_{\mathcal{R}_q(u)}.$$

Then $u_{\mathcal{R}_q(u)}$ is σ-moderate and

$$u \ominus u_{\mathcal{R}_q(u)} \le [u]_{\mathcal{S}_q(u)}.$$

Set

$$u_n = v_n \oplus [u]_{\mathcal{S}_q(u)}.$$

By Proposition 5.82 $[u]_{\mathcal{S}_q(u)}$ is σ-moderate. Using the fact that $\widetilde{Q}_n \cap \mathcal{S}_q(u)$, it follows by Proposition 5.77 that u_n is σ-moderate. The sequence $\{u_n\}$ is increasing and converges to some $\overline{u}$ of (3.13) which is σ-moderate too. Furthermore

$$v_n \vee [u]_{\mathcal{S}_q(u)} = u_n = v_n \oplus [u]_{\mathcal{S}_q(u)} \Longrightarrow \max\{u_{\mathcal{R}_q(u)}, [u]_{\mathcal{S}_q(u)}\} \le \overline{u} \le u_{\mathcal{R}_q(u)} \oplus [u]_{\mathcal{S}_q(u)}.$$

This implies that $\mathcal{S}_q(u) = \mathcal{S}_q(\overline{u})$. Now, by construction we have

$$v_n = [u]_{Q_n} \leq [\overline{u}]_{Q_n}.$$

then, letting $n \to \infty$ we obtain by Proposition 5.64

$$u_{\mathcal{R}_q(u)} \leq \overline{u}_{\mathcal{R}_q(u)} \Longrightarrow u_{\mathcal{R}_q(u)} = \overline{u}_{\mathcal{R}_q(u)},$$

therefore $tr(u) = tr(\overline{u})$. But since $\overline{u} \leq u$, it follows by Proposition 5.71 and the uniqueness of σ-moderate solutions that $\overline{u} = u$. □

5.9 Further studies and open problems

5.9.1 Lateral boundary trace

Let Ω be either a C^2 open subset or $\mathbb{R}^N_+$. The problem is to analyse the trace on the lateral boundary of $\Omega \times (0,T)$ of any positive solution of (3.13). It is proved in [42] that there exists a lateral trace in the class of outer regular Borel measures in $\partial_\ell \Omega \times (0,T) := \partial\Omega \times (0,T)$. The critical value for q is $\tilde{q}_c = \frac{N+3}{N+1}$ above this value the boundary isolated singularities are removable. The geometry of the cylindrical domain makes much more difficult the study of the supercritical case. A similar study was performed by Kuznetsov [36], [37] in the framework of superprocesses and with the restriction that $1 < q < 2$.

5.9.2 Full trace problem

It is an extension of the initial trace problem treated in this survey and the lateral boundary trace. In the paper [44] the initial trace is considered for solutions in a cylinder Q_T^Ω and the existence and uniqueness theorem in the subcritical case is proved provided the lateral boundary value is integrable in $\partial\Omega \times (0,T)$ and the initial measure has compact support in Ω, or at least is bounded near $\partial\Omega$. This is due to the fact that for general measure μ in $\mathfrak{M}_+(\Omega)$, the behaviour of μ near $\partial\Omega$ is fundamental. A general study of the trace of positive solutions of (3.13) on $\partial Q_T^\Omega := (\Omega \times \{0\}) \bigcup (\partial\Omega \times [0,T))$ would be a great interest. In this direction we can mention the work [33] dealing with the lateral boundary trace of positive solutions of

$$\frac{\partial u}{\partial t} - \Delta u - u^q = 0 \qquad \text{in } \mathbb{R}^N_+ \times (0,T). \tag{5.183}$$

Extension to general domain are expected.

5.9.3 Equations of general absorption-convection

$$\partial_t u - \Delta u + u^p |\nabla u|^q = 0. \tag{5.184}$$

Since this is an equation with absorption the construction of an initial trace should be tractable. To our knowledge the study of the self-similar solutions and isolated singularities has not yet been done. This study needs a preliminary analysis of the problem

$$\begin{aligned} \partial_t u - \Delta u + u^p |\nabla u|^q = 0 \quad &\text{in } Q_T \\ u(.,0) = \mu \quad &\text{in } \mathcal{D}'(\mathbb{R}^N), \end{aligned} \tag{5.185}$$

where μ is a nonnegative Radon measure.

5.9.4 Equations of Hamilton Jacobi type

$$\partial_t u - \Delta u + m|\nabla u|^q = 0 \quad \text{in } Q_T. \tag{5.186}$$

The subcritical case has been treated by Bidaut-Véron-Dao [13]. They prove the existence of a critical exponent $q^* = \frac{N+2}{N+1}$. When $1 < q < q^*$ they obtain the existence of solutions u with a Dirac mass as initial data and the existence and uniqueness of a positive very singular solution. When $q \geq q^*$ they prove that isolated singularities at $t = 0$ are removable. The detailed analysis of the initial trace in the supercritical case seems open.

5.9.5 Equations of mixed absorption-reaction-convection

$$\partial_t u - \Delta u + u^p - m|\nabla u|^q = 0, \tag{5.187}$$

or

$$\partial_t u - \Delta u + m|\nabla u|^q - u^p = 0. \tag{5.188}$$

For these two types of equations the existence of an initial trace seems open except in some specific cases. The study has to be put in parallel with the ones dealing with the boundary value problem and the boundary trace for the elliptic equations

$$-\Delta u + u^p - m|\nabla u|^q = 0 \qquad \text{in } \Omega, \tag{5.189}$$

obtained in [15] or

$$-\Delta u + m|\nabla u|^q - u^p = 0 \qquad \text{in } \Omega, \tag{5.190}$$

obtained in [16]. In these two papers, it is developed a method which associates some specific supersolutions and subsolutions namely

$$-\Delta u - m|\nabla u|^q = 0 \quad \text{in } \Omega \quad \text{and} \quad -\Delta u + u^p = 0 \quad \text{in } \Omega \tag{5.191}$$

for (5.189) in [15] and

$$-\Delta u - u^p = 0 \quad \text{in } \Omega \quad \text{and} \quad -\Delta u + m|\nabla u|^q = 0 \quad \text{in } \Omega \tag{5.192}$$

for (5.190) in [16]. It appears that this could be adapted to the study of (5.187) and (5.188). We also refer to the book of Quittner and Souplet [50] which contains an impressive quantity of results concerning semilinear heat equations with reaction terms of the type $-u^p$ or $-|\nabla u|^q$.

References

[1] D. R. Adams, L. I. Hedberg. *Function spaces and potential theory.* Grundlehren Math. Wissen. **145**, Springer (1967).

[2] H. Aikawa, A. A. Borichev. Quasiadditivity and measure property of capacity and the tangential boundary behavior of harmonic functions. *Trans. Amer. Math. Soc.* **348**, 1013-1030 (1996).

[3] A. Ancona. Negatively curved manifolds, elliptic operators and the Martin boundary. *Annals of Mathematics, Second Series* **125**, 495-536 (1987).

[4] D. G. Aronson. Non-negative solutions of linear parabolic equations. *Annali Scuola Normale Sup. Pisa, Classe di Sci.* **22**, 607-694 (1968).

[5] D. G. Aronson. Widder's inversion theorem and the initial distribution problems. *SIAM J. Math. Anal.* **12**, 639-651 (1981).

[6] D. Andreucci. Degenerate parabolic equations with initial data measures. *Trans. Amer. Math. Soc.* **349**, 3911-3923 (1997).

[7] C. Bandle, M. Marcus. Large solutions of semilinear elliptic equations : existence, uniqueness and asymptotic behaviour. *J. Anal. Math.* **58**, 9-24 (1992).

[8] P. Baras, M. Pierre. Singularités éliminables pour des équations semilinéaires. *Ann. Inst. Fourier* **34**, 185-206 (1984).

[9] P. Baras, M. Pierre. Problèmes paraboliques semi-linéaires avec données mesures. *Applicable Anal.* **18**, 111-149 (1984).

[10] P. Baras, M. Pierre. Critère d'existence de solutions positives pour des équations semi-linéaires non monotones. *Ann. Inst. H. Poincaré, Anal. Non Linéaire.* **2**, 185-212 (1985).

[11] H. Berens, P. Butzer. *Semigroups of Operators and Approximations.* Grundlehren Math. Wissen., **vol. 314**. Springer, New York (1996).

[12] M.F. Bidaut-Véron, E. Chasseigne, L. Véron. Initial trace of solutions of some quasilinear parabolic equations with absorption. *J. Funct. Anal.* **193**, 140-205 (2002).

[13] M. F. Bidaut-Véron, A.N. Dao. Initial trace of solutions of Hamilton-Jacobi parabolic equation with absorption. *Adv. Nonlinear. Stud.* **15**, 889-921 (2015).

[14] M.F. Bidaut-Véron, A.N. Dao. Isolated initial singularities for the viscous Hamilton-Jacobi equation. *Adv. Differential Equations* **17**, 903-934 (2012).

[15] M.F. Bidaut-Véron, M. Garcia-Huidobro, L. Véron. Boundary singular solutions of a class of equations with mixed absorption-reaction. *Calc. Var. Partial Differential Equations* **61**, Paper No. 2150051, 37 pp (2022).

[16] M.F. Bidaut-Véron, L. Véron. Trace and boundary singularities of solutions of a class of quasilinear equations. *Discr. Cont. Dyn. Systems*, **43**, 1026-1051 (2023).

[17] H. Brezis, Nonlinear elliptic equation in $\mathbb{R}^N$ without condition at infinity. *Appl. Math. Opt.* **12**, 271-282 (1985).

[18] H. Brezis, A. Friedman. Nonlinear parabolic equations involving measures as initial conditions. *J. Math. Pures Appl.* **62**, 73-97 (1983).

[19] H. Brezis, L. A. Peletier, D. Terman. A very singular solution of the heat equation with absorption. *Arch. Rational Mech. Anal.* **95**, 185-209 (1986).

[20] E. Chasseigne. Initial trace for a porous medium equation. II. The critical absorption term. *Asymptotic Anal.* **24**, 37-72 (2000).

[21] E. Chasseigne. Initial trace for a porous medium equation. I. The strong absorption case. *Ann. Mat. Pura Appl.* **169**, 413-458 (2001).

[22] E. Chasseigne, J. L. Vazquez. Theory of extended solutions for fast-diffusion equations in optimal classes of data. Radiation from singularities. *Arch. Rat. Mech. Anal.* **164**, 133-187 (2002).

[23] E. Chasseigne, J. L. Vazquez, Sets of admissible initial data for porous-medium equations with absorption. Proceedings of the 2001 Luminy Conference on Quasilinear Elliptic and Parabolic Equations and System. *electron. J. Differ. Equ. Conf.*, **8** 53-83 (2002)

[24] E. Di Nezza , G. Palatucci, E. Valdinoci. Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Amer. Math. Soc.* **136**, 521-573 (2012).

[25] E. B. Dynkin. *Superdiffusions and Partial Differential Equations*, American Mathematical Society Colloquium Publications **50**, American Mathematical Society, Providence, vi+236 pp (2002).

[26] E. B. Dynkin. *Superdiffusions and positive solutions of nonlinear partial differential equations*. University Lecture Series **34**. Amer. Math. Soc., Providence, vi+120 pp (2004).

[27] E. B. Dynkin, S. E. Kuznetsov. Superdiffusions and removable singularities for quasilinear partial differential equations. *Comm. Pure Appl. Math.* **49**, 125-176 (1996).

[28] E. B. Dynkin, S. E. Kuznetsov. Solutions of $Lu = u^\alpha$ dominated by harmonic functions. *J. Analyse Math.* **68**, 15-37 (1996).

[29] E. B. Dynkin, S. E. Kuznetsov. Fine topology and fine trace on the boundary associated with a class of quasilinear differential equations. *Comm. Pure Appl. Math.* **51**, 897-936 (1998).

[30] D. Feyel, A. de la Pradelle. Topologies fines et compactifications associées à certains espaces de Dirichlet, *Ann. Inst. Fourier Grenoble* **27**, 121-146 (1977).

[31] K. Gkikas, L. Véron. Initial value problems for diffusion equations with singular potential. *Contemporary Math.* **594**, 201-230 (2013).

[32] K. Gkikas, L. Véron. Complete classification of the positive solutions of heat equation with super critical absorption. *Adv. Nonlinear. Stud.* **14**, 47-113 (2014).

[33] K. Hisa, K. Ishige, J. Takahashi. Initial traces and solvability for a semilinear heat equation on a half space of $\mathbb{R}^N$. *Trans. amer. Math. Soc*, to appear.

[34] G. Grillo. Lower bounds for the Dirichlet heat kernel. *Quart. J. Math. Oxford Ser.* **48**, 203-211 (1997).

[35] P. Grisvard. Commutativité de deux foncteurs d'interpolation et applications. *J. Math. Pures et Appl.*, **45**, 143-290 (1966).

[36] S. Kuznetsov. On removable lateral singularities for quasilinear parabolic PDE'. *C. R. Acad. Sci. Paris*, **t. 325** , 627-632 (1997).

[37] S. Kuznetsov. Polar boundary sets for superdiffusions and removable lateral singularities for non-linear parabolic PDE's, *Comm. Pure Appl. Math.* **51**, 303-340 (1998).

[38] J. F. Le Gall. The Brownian snake and solutions of $\Delta u = u^2$ in a domain. *Probab. Th. Rel. Fields* **102**, 393-432 (1995).

[39] J.F. Le Gall. A probabilistic approach to the trace at the boundary for solutions of a semilinear parabolic partial differential equation. *Jl Appl. Math. Stochastic Anal.* **9**, 399-414 (1996).

[40] J.L. Lions, J. Petree. Sur une classe d'espaces d'interpolation. *Inst. Hautes Etudes Sci. Publ. Math.* **19**, 5-68 (1964).

[41] M. Marcus. Complete classification of the positive solutions of $-\Delta u + u^q = 0$. *J. Anal. Mathématique* **117**, 187-220 (2012).

[42] M. Marcus, L. Véron. Trace au bord lat éral des solutions positives d'équations paraboliques non-linéaires.*C. R. Acad. Sci. Paris*, **t. 324** , 783-788 (1997).

[43] M. Marcus, L. Véron. Initial trace of positive solutions of some nonlinear parabolic equations. *Comm. in P.D.E.* **24**, 1445-1499 (1999).

[44] M. Marcus, L. Véron. Initial trace of positive solutions to semilinear parabolic inequalities. *Adv. Nonlinear. Stud.* **4**, 395-436 (2002).

[45] M. Marcus, L. Véron. Capacitary estimates of positive solutions of semilinear elliptic equations with absorbtion. *J. Eur. Math. Soc.* **6**, 483-527 (2004).

[46] M. Marcus, L. Véron. The precise boundary trace of positive solutions of the equation $\Delta u = u^q$ in the supercritical case. Perspectives in nonlinear partial differential equations. *Contemp. Math.* **446**, 345-383, Amer. Math. Soc., Providence, RI (2007).

[47] M. Marcus, L. Véron. Capacitary estimates of positive solutions of semilinear parabolic equations with absorbtion. *Calc. Var. Partial Differential Equations* **48**, 131-183 (2013).

[48] M. Marcus, L. Véron. Semilinear parabolic equations with measure boundary data and isolated singularities. *J. Anal. Mathématique* **80**, 870-900 (2001).

[49] P.T. Nguyen, L. Véron. Initial trace of positive solutions of a class of degenerate heat equation with absorption *Discrete and Continuous Dyn. Syst.* **33**, 2033-2063 (2013).

[50] P. Quittner, Ph. Souplet. *Superlinear parabolic problems. Blow-up, global existence and steady states*, 2nd edition. Birkhäuser Advanced Texts, Birkhäuser Verlag, Basel, xvi+725 pp (2007).

[51] E. M. Stein. *Singular integrals and differentiability properties of functions*. Princeton Univ. Press **30** (1970).

[52] L. Tartar. Sur un lemme d'équivalence utilisé en analyse Numérique. *Calcolo* **24**, 129-140 (1987).

[53] L. Tartar. *Personal communication*, February 2012.

[54] H. Triebel. *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland Publishing Company, North-Holland Mathematical Library **18**. Amsterdam-New-york-Oxford (1978).

[55] F. B. Weissler. Existence and non-existence of global solutions for a semilinear heat equation. *Israel J. Math.***38**, 29-40 (1981).

[56] D. V. Widder, Positive temperature on an infinite rod, *Trans. Amer. Math. Soc.* **55**, 85-95 (1944).

[57] E. T. Whittaker, G. N. Watson. *A Course of Modern Analysis*. Cambridge University Press, 4th Ed. Chapter XXI (1927).