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# SPANS OF TRANSLATES IN WEIGHTED $\ell^p$ SPACES

K. KELLY\*, F. LE MANACH, AND M. ZARRABI

ABSTRACT. We study the cyclic vectors and the spanning set of the circle for the  $\ell_\beta^p(\mathbb{Z})$  spaces of all sequences  $u = (u_n)_{n \in \mathbb{Z}}$  such that  $(u_n(1 + |n|)^\beta)_{n \in \mathbb{Z}} \in \ell^p(\mathbb{Z})$  with  $p > 1$  and  $\beta > 0$ . By duality the spanning set is the uniqueness set of the distribution on the circle whose Fourier coefficients are in  $\ell_{-\beta}^q(\mathbb{Z})$  where  $q$  is the conjugate of  $p$ . Our characterizations are given in terms of the Hausdorff dimension and capacity.

## 1. INTRODUCTION AND MAIN RESULTS

The cyclic vectors are, amongst others, an important tool in the study of invariant subspaces and their characterization [1, 21, 26]. For the shift operator, the problem of cyclic vectors in the space of sequences  $\ell^p(\mathbb{Z})$  goes back to the works of Wiener [27] for  $p = 1$  and  $p = 2$ , Beurling [3], Salem [25] for  $1 < p < 2$ , and Newman [18] for  $p > 1$ . This problem is still far from being resolved.

A vector  $u \in \ell^p(\mathbb{Z})$  is called *cyclic* in  $\ell^p(\mathbb{Z})$  if the linear span of its translates,

$$\{(u_{n+k})_{n \in \mathbb{Z}}, k \in \mathbb{Z}\}$$

is dense in  $\ell^p(\mathbb{Z})$ . The Fourier transform of  $u \in \ell^p(\mathbb{Z})$  is given by  $\hat{u}(t) = \sum_{n \in \mathbb{Z}} u_n e^{int}$  where the trigonometric series is to be interpreted as a distribution on the circle group  $\mathbb{T} = \mathbb{R} \setminus 2\pi\mathbb{Z}$ . For  $u \in \ell^p(\mathbb{Z})$  with  $1 \leq p \leq 2$ ,  $\hat{u}$  becomes a function. We denote by  $\mathcal{Z}(\hat{u})$  the zero set of  $\hat{u}$  in  $\mathbb{T}$ . Notice that for  $u \in \ell^1(\mathbb{Z})$ , the set  $\mathcal{Z}(\hat{u})$  is well-defined, since  $\hat{u}$  is continuous. The cyclicity can be viewed as an approximation problem or a uniqueness/removable singularities problem. Following Newman [18], a closed subset  $E \subset \mathbb{T}$  is called *p-spanning* if every  $u \in \ell^1(\mathbb{Z})$  with  $\mathcal{Z}(\hat{u}) \subset E$ , is cyclic in  $\ell^p(\mathbb{Z})$ . On the other hand,  $E$  is called a *q-uniqueness* set if  $E$  does not support any non-vanishing distribution  $\sum_{n \in \mathbb{Z}} c_n e^{int}$  with  $(c_n)_{n \in \mathbb{Z}}$  in  $\ell^q(\mathbb{Z})$ . It is well known that  $E$  is *p-spanning* if and only if  $E$  is *q-uniqueness* where  $q$  is the conjugate of  $p$ .

Wiener characterized the cyclic vectors in  $\ell^1(\mathbb{Z})$  and in  $\ell^2(\mathbb{Z})$  [27]. Further, Beurling, Salem, and Newman [3, 25, 18] provided either necessary or sufficient conditions for  $u$  to be cyclic in  $\ell^p(\mathbb{Z})$  for  $p > 1$ . Their characterization of the cyclic vectors was given in terms of the size (capacity and Hausdorff dimension) of the zero set of the Fourier transform

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$\widehat{u}$ . However, Lev and Olevskii [14, 15, 16] showed that for  $1 < p < 2$ , the problem of cyclicity in  $\ell^p(\mathbb{Z})$  is more complicated even for sequences in  $\ell^1(\mathbb{Z})$  : we cannot characterize the cyclicity of  $u$  in  $\ell^p(\mathbb{Z})$  in terms of  $\mathcal{Z}(\widehat{u})$  alone, which contradicts Wiener's conjecture.

We summarize the results of the previous works cited above. We denote by  $q$  the Hölder conjugate of  $p \geq 1$  with  $1/p + 1/q = 1$ , and  $\dim(E)$  the Hausdorff dimension of a subset  $E \subset \mathbb{T}$ .

- (1) WIENER:  $u$  is cyclic in  $\ell^1(\mathbb{Z})$  if and only if  $\widehat{u}$  has no zeros on  $\mathbb{T}$ .
- (2) WIENER:  $u$  is cyclic in  $\ell^2(\mathbb{Z})$  if and only if  $\widehat{u}$  is non-zero almost everywhere.
- (3) BEURLING: Let  $1 \leq p \leq 2$ . If  $\dim(E) < 2/q$  then  $E$  is a  $p$ -spanning.
- (4) SALEM: Let  $1 \leq p \leq 2$ . For  $2/q < \alpha \leq 1$ , there exists  $E \subset \mathbb{T}$  such that  $\dim(E) = \alpha$  and  $E$  is not  $p$ -spanning.
- (5) NEWMAN: There exists a  $p$ -spanning set  $E$  for all  $1 < p < 2$  such that  $\dim(E) = 1$ .
- (6) LEV & OLEVSKII: If  $1 < p < 2$ , there exist  $u$  and  $v$  in  $\ell^1(\mathbb{Z})$  such that  $\mathcal{Z}(\widehat{u}) = \mathcal{Z}(\widehat{v})$ ,  $u$  is not cyclic in  $\ell^p(\mathbb{Z})$ , and  $v$  is cyclic in  $\ell^p(\mathbb{Z})$ .

In this paper, we shall focus on the cyclic vectors on weighted  $\ell^p(\mathbb{Z})$  spaces, namely  $\ell_\beta^p(\mathbb{Z})$ , the space of sequences  $u = (u_n)_{n \in \mathbb{Z}}$  for which  $(u_n(1 + |n|)^\beta) \in \ell^p(\mathbb{Z})$  for  $p \geq 1$  and  $\beta > 0$ . A vector  $u \in \ell_\beta^p(\mathbb{Z})$  is called *cyclic* in  $\ell_\beta^p(\mathbb{Z})$  if the linear span of  $\{(u_{n+k})_{n \in \mathbb{Z}}, k \in \mathbb{Z}\}$  is dense in  $\ell_\beta^p(\mathbb{Z})$ . For every closed subset  $E$  of  $\mathbb{T}$ ,  $E$  is called  $(p, \beta)$ -spanning if every  $u \in \ell^1(\mathbb{Z})$  such that  $\mathcal{Z}(\widehat{u}) \subset E$  is cyclic in  $\ell_\beta^p(\mathbb{Z})$ , and  $E$  is called  $(q, \beta)$ -uniqueness if  $E$  does not support any non-zero distribution  $\sum_{n \in \mathbb{Z}} c_n e^{int}$  with  $(c_n)_{n \in \mathbb{Z}}$  in  $\ell_{-\beta}^q(\mathbb{Z})$ . As before,  $E$  is  $(p, \beta)$ -spanning if and only if  $E$  is  $(q, \beta)$ -uniqueness where  $1/p + 1/q = 1$ .

Observe that the shift operator does not act as an isometry on  $\ell_\beta^p(\mathbb{Z})$  unlike on  $\ell^p(\mathbb{Z})$  spaces which presents a difficulty for the study of cyclic vectors in  $\ell_\beta^p(\mathbb{Z})$ .

Notice that  $\ell_\beta^p(\mathbb{Z})$  is a Banach algebra if and only if  $\beta q > 1$  (see [6]). Hence, in this case we have an analogue of (1) in Wiener's theorem : a vector  $u \in \ell_\beta^p(\mathbb{Z})$  is cyclic if and only if  $\widehat{u}$  has no zeros on  $\mathbb{T}$ . Thus in the sequel of the paper we will only be interested in pairs  $(p, \beta)$  such that  $\beta q < 1$ .

Richter, Ross and Sundberg [23] gave a complete characterization of hyperinvariant subspaces of the weighted harmonic Dirichlet spaces  $\ell_\beta^2(\mathbb{Z})$ ,  $0 < \beta \leq \frac{1}{2}$ . Their characterization and a relation between capacity and Hausdorff dimension led to the result that  $u \in \ell_\beta^1(\mathbb{Z})$  is cyclic in  $\ell_\beta^2(\mathbb{Z})$  if and only if  $\dim(\mathcal{Z}(\widehat{u})) \leq 1 - 2\beta$ . Their result may be considered as an analog of Wiener's theorem about the cyclic vector of  $\ell^2$ . Hence, we study the case of  $p \neq 1$  and  $p \neq 2$ . Our main result for  $1 < p < 2$  is the following (see Theorem 3.3).

**Theorem A.** *Let  $1 < p < 2$ ,  $\beta > 0$  such that  $\beta q \leq 1$  and let  $E$  be a closed subset of  $\mathbb{T}$ .*

- (1) *If  $\dim(E) < \frac{2}{q}(1 - \beta q)$ , then  $E$  is  $(p, \beta)$ -spanning.*
- (2) *If  $\dim(E) > 1 - \beta q$ , then  $E$  is not  $(p, \beta)$ -spanning.*

- (3) For  $\frac{2}{q}(1 - \beta q) \leq \alpha \leq 1$ , there exists a closed subset  $E \subset \mathbb{T}$  such that  $\dim(E) = \alpha$  and  $E$  is not  $(p, \beta)$ -spanning.
- (4) If  $p = 2k/(2k - 1)$  for some  $k \in \mathbb{N} \setminus \{0\}$  there exists a  $(p, \beta)$ -spanning  $E \subset \mathbb{T}$  such that  $\dim(E) = 1 - \beta q$ .

The property (4) shows that the constant  $1 - \beta q$  obtained in (2) is sharp. Indeed, on one hand, there is no cyclic vector  $u$  such that  $\dim(\mathcal{Z}(\hat{u})) > 1 - \beta q$ , and on the other hand, we can find some cyclic vector  $u$  with  $\dim(\mathcal{Z}(\hat{u})) = 1 - \beta q$ . However, this is only proved when  $p = 2k/(2k - 1)$  for some positive integer  $k$ . The proof is based on the construction of a closed subset  $E$  of  $\mathbb{T}$  whose  $k$ -sums  $E + \dots + E$  are of zero capacity and of given Hausdorff dimension (see Lemma 3.2). The arithmetic structure of  $E$  allows us to reach the best constant  $1 - \beta q$  only for  $p = 2k/(2k - 1)$ .

Next we will deal with the case  $p > 2$ . Newman in [18] showed that for all  $\varepsilon > 0$ , there exists a  $p$ -spanning set  $E \subset \mathbb{T}$  which has a Lebesgue measure  $|E| > 2\pi - \varepsilon$ . The existence of  $q$ -uniqueness sets of arbitrary large measure for the spaces  $\ell^q(\mathbb{Z})$ ,  $1 < q < 2$ , was established also independently, by Katznelson [10] (see also [11, IV. 2.5 Theorem]). Extensions of their result to a more general setting were given in [12] where they studied the uniqueness set of  $\ell_{-\beta}^q(\mathbb{Z})$ . We have the following result.

**Theorem B.** *Let  $p > 2$ ,  $\beta > 0$  such that  $\beta q \leq 1$ .*

- (1) *If  $\beta > \frac{1}{2} - \frac{1}{p}$ , then every a closed subset  $E$  of  $\mathbb{T}$  of positive Lebesgue measure, is not  $(p, \beta)$ -spanning.*
- (2) *If  $\beta < \frac{1}{2} - \frac{1}{p}$  then for every  $\varepsilon > 0$ , there exists a  $(p, \beta)$ -spanning set  $E \subset \mathbb{T}$  such that  $|E| > 2\pi - \varepsilon$ .*

Nikolski in [20, Corollary 6] considered the weighted space

$$\ell_{\omega}^p(\mathbb{Z}) = \{(u_n)_{n \in \mathbb{Z}} : (u_n \omega_n)_{n \in \mathbb{Z}} \in \ell^p(\mathbb{Z})\}$$

where  $\omega_n = \log(e + |n|)^{\gamma}$ ,  $\gamma > 0$ . He showed that if  $p > 2/(1 - \gamma)$ ,  $0 < \gamma < 1$ , then there exists  $E \subset \mathbb{T}$  with large Lebesgue measure which is a uniqueness set for the dual of  $\ell_{\omega}^p(\mathbb{Z})$ , which is equivalent to the cyclicity in  $\ell_{\omega}^p(\mathbb{Z})$  of every  $u \in \ell_{\omega}^1(\mathbb{Z})$  satisfying  $\mathcal{Z}(\hat{u}) \subset E$ . As a by-product of Theorem B, we show in Corollary 5.1 that the result of Nikolski remains valid for all  $p > 2$  and  $\gamma > 0$ .

This paper is organized as follows. In the next section, we present the background and recall some properties of distribution spaces. Section 3 is devoted to the proof of Theorem A. We construct in Lemma 3.2 a Cantor type set of zero capacity whose  $k^{\text{th}}$  sum remains of zero capacity. Section 4 provides the proof of Theorem B based on the estimation of power sums of unimodular complex numbers (see Lemma 4.1). Finally, Section 5 is dedicated to some results on the  $\ell^p$  spaces with logarithmic weights.

## 2. NOTATIONS AND PRELIMINARIES

**2.1. Background on  $\ell^p$  weighted spaces.** Let  $1 \leq p < \infty$  and  $\beta \in \mathbb{R}$ . We denote by  $\mathcal{D}'(\mathbb{T})$  the set of distributions on  $\mathbb{T}$  and  $\mathcal{M}(\mathbb{T})$  the set of measures on  $\mathbb{T}$ . For  $S \in \mathcal{D}'(\mathbb{T})$ , we denote by  $\widehat{S} = (\widehat{S}(n))_{n \in \mathbb{Z}}$  the sequence of Fourier coefficients of  $S$  and, we write  $S = \sum_n \widehat{S}(n) e_n$ , where  $e_n(t) = e^{int}$ . Notice that we use the same notation  $\widehat{u}$  and  $\widehat{S}$  to denote respectively the Fourier transform of  $u \in \ell^p$  and of  $S \in \mathcal{D}'(\mathbb{T})$ . The space  $A_\beta^p(\mathbb{T})$  will be the set of all distributions  $S \in \mathcal{D}'(\mathbb{T})$  such that  $\widehat{S}$  belongs to  $\ell_\beta^p(\mathbb{Z})$ . We endow  $A_\beta^p(\mathbb{T})$  with the norm

$$\|S\|_{A_\beta^p(\mathbb{T})} = \|\widehat{S}\|_{\ell_\beta^p} = \left( \sum_{n \in \mathbb{Z}} |\widehat{S}(n)|^p (1 + |n|)^{\beta p} \right)^{1/p}.$$

We will write  $A^p(\mathbb{T})$  for the space  $A_0^p(\mathbb{T})$ . By construction the Fourier transform  $u \rightarrow \widehat{u}$  is an isometric isomorphism between  $\ell_\beta^p(\mathbb{Z})$  and  $A_\beta^p(\mathbb{T})$ . We prefer to work with  $A_\beta^p(\mathbb{T})$  rather than  $\ell_\beta^p(\mathbb{Z})$ . In this section, we establish some properties of  $A_\beta^p(\mathbb{T})$  which will be needed to prove Theorem A and Theorem B. For  $1 \leq p < \infty$  and  $\beta \geq 0$  we define the product of  $f \in A_\beta^1(\mathbb{T})$  and  $S \in A_\beta^p(\mathbb{T})$  by

$$fS = \sum_{n \in \mathbb{Z}} (\widehat{f} * \widehat{S})(n) e_n = \sum_{n \in \mathbb{Z}} \left( \sum_{k \in \mathbb{Z}} \widehat{f}(k) \widehat{S}(n - k) \right) e_n,$$

and we see that  $\|fS\|_{A_\beta^p(\mathbb{T})} \leq \|f\|_{A_\beta^1(\mathbb{T})} \|S\|_{A_\beta^p(\mathbb{T})}$ . Note that if  $S \in A_{-\beta}^p(\mathbb{T})$ , we can also define the product  $fS \in A_{-\beta}^p(\mathbb{T})$  by the same formula and obtain a similar inequality:  $\|fS\|_{A_{-\beta}^p(\mathbb{T})} \leq \|f\|_{A_\beta^1(\mathbb{T})} \|S\|_{A_{-\beta}^p(\mathbb{T})}$ .

For  $p \neq 1$ , the dual space of  $A_\beta^p(\mathbb{T})$  can be identified with  $A_{-\beta}^q(\mathbb{T})$ ,  $1/p + 1/q = 1$ , by the formula

$$\langle S, T \rangle = \sum_{n \in \mathbb{Z}} \widehat{S}(n) \widehat{T}(-n), \quad S \in A_\beta^p(\mathbb{T}), T \in A_{-\beta}^q(\mathbb{T}).$$

We need the following lemmas which gives us different inclusions between the  $A_\beta^p(\mathbb{T})$  spaces.

**Lemma 2.1.** *Let  $1 \leq r, s < \infty$  and  $\beta, \gamma \in \mathbb{R}$ .*

- (1) *If  $r \leq s$  then  $A_\beta^r(\mathbb{T}) \subset A_\gamma^s(\mathbb{T})$  if and only if  $\gamma \leq \beta$ .*
- (2) *If  $r > s$  then  $A_\beta^r(\mathbb{T}) \subset A_\gamma^s(\mathbb{T})$  if and only if  $\beta - \gamma > \frac{1}{s} - \frac{1}{r}$ .*

*Proof.* (1) : Suppose that  $r \leq s$ . If  $\gamma \leq \beta$ , then  $A_\beta^s(\mathbb{T}) \subset A_\gamma^s(\mathbb{T})$ . Since  $\|\cdot\|_{\ell^s} \leq \|\cdot\|_{\ell^r}$ , we get  $A_\beta^r(\mathbb{T}) \subset A_\gamma^s(\mathbb{T})$ . Now suppose  $\gamma > \beta$ . Let  $S \in \mathcal{D}'(\mathbb{T})$  such that  $\widehat{S}(n)(1+|n|)^\beta = (1+m)^{-2/r}$  if  $|n| = 2^m$  and  $\widehat{S}(n) = 0$  otherwise. Then we have  $S \in A_\beta^r(\mathbb{T}) \setminus A_\gamma^s(\mathbb{T})$ .

(2) : Suppose that  $r > s$ . If  $\beta - \gamma > \frac{1}{s} - \frac{1}{r}$ , then by Hölder's inequality, we obtain  $A_\beta^r(\mathbb{T}) \subset A_\gamma^s(\mathbb{T})$ . Now suppose that  $\beta - \gamma < \frac{1}{s} - \frac{1}{r}$ . Let  $\varepsilon > 0$  such that  $\beta - \gamma + \varepsilon < \frac{1}{s} - \frac{1}{r}$ ,  $\alpha = -\frac{1}{s} - \gamma + \varepsilon$  and let  $S \in \mathcal{D}'(\mathbb{T})$  be such that  $\widehat{S}(n) = n^\alpha$ . We have  $S \in A_\beta^r(\mathbb{T}) \setminus A_\gamma^s(\mathbb{T})$ .

If  $\beta - \gamma = \frac{1}{s} - \frac{1}{r}$ , then let  $S \in \mathcal{D}'(\mathbb{T})$  such that  $\widehat{S}(n)^r (1+|n|)^{\beta r} = 1/(1+|n|) \ln(1+|n|)^{1+\varepsilon}$  with  $\varepsilon = \frac{r}{s} - 1 > 0$ . We have  $S \in A_\beta^r(\mathbb{T}) \setminus A_\gamma^s(\mathbb{T})$  which proves that  $A_\beta^r(\mathbb{T}) \not\subset A_\gamma^s(\mathbb{T})$ .  $\square$

**2.2. Cyclicity in  $A_\beta^p(\mathbb{T})$ .** We denote by  $\mathcal{P}(\mathbb{T})$  the set of trigonometric polynomials on  $\mathbb{T}$ . We say that  $S \in A_\beta^p(\mathbb{T})$  is a cyclic vector in  $A_\beta^p(\mathbb{T})$  if the set  $\{PS, P \in \mathcal{P}(\mathbb{T})\}$  is dense in  $A_\beta^p(\mathbb{T})$ . It is clear that the cyclicity of  $S$  in  $A_\beta^p(\mathbb{T})$  is equivalent to the cyclicity of the sequence  $\widehat{S}$  in  $\ell_\beta^p(\mathbb{Z})$ . Moreover for  $1 \leq p < \infty$  and  $\beta \geq 0$ ,  $S$  is cyclic in  $A_\beta^p(\mathbb{T})$  if and only if there exists a sequence  $(P_n)$  of trigonometric polynomials such that

$$\lim_{n \rightarrow \infty} \|1 - P_n S\|_{A_\beta^p(\mathbb{T})} = 0. \quad (2.1)$$

We obtain the first cyclicity results for the spaces  $A_\beta^p(\mathbb{T})$  when  $A_\beta^p(\mathbb{T})$  is a Banach algebra. More precisely, we have (see [6]):

**Lemma 2.2.** *Let  $1 \leq p < \infty$  and  $\beta > 0$ . Then  $A_\beta^p(\mathbb{T})$  is a Banach algebra if and only if  $\beta q > 1$ . Moreover, when  $\beta q > 1$ , a vector  $f \in A_\beta^p(\mathbb{T})$  is cyclic in  $A_\beta^p(\mathbb{T})$  if and only if  $f$  has no zeros on  $\mathbb{T}$ .*

Let  $f \in A_\beta^1(\mathbb{T})$  and  $S \in \mathcal{D}'(\mathbb{T})$ . We denote by  $\mathcal{Z}(f)$  the zero set of the function  $f$

$$\mathcal{Z}(f) = \{\zeta \in \mathbb{T} : f(\zeta) = 0\}.$$

**Lemma 2.3.** *Let  $1 \leq p < \infty$  and  $0 \leq \beta < 1/2$ . Let  $f \in A_\beta^1(\mathbb{T})$  and  $S \in A_{-\beta}^p(\mathbb{T})$ . If for all  $n \in \mathbb{Z}$ ,  $\langle S, e_n f \rangle = 0$  then  $\text{supp}(S) \subset \mathcal{Z}(f)$ .*

*Proof.* Recall that  $e_n(t) = e^{int}$ . We have

$$\langle S, e_n f \rangle = \langle f S, e_n \rangle = 0.$$

Hence  $f S = 0$ . Let  $\varphi \in C^\infty(\mathbb{T})$  such that  $\text{supp}(\varphi) \subset \mathbb{T} \setminus \mathcal{Z}(f)$ . We claim that  $\frac{\varphi}{f} \in A_\beta^1(\mathbb{T}) \subset A_\beta^q(\mathbb{T})$  where  $1/p + 1/q = 1$ . So we obtain

$$\langle S, \varphi \rangle = \langle f S, \frac{\varphi}{f} \rangle = 0$$

which proves that  $\text{supp}(S) \subset \mathcal{Z}(f)$ .

Now we prove the claim. Let  $\varepsilon = \min\{|f(t)|, t \in \text{supp}(\varphi)\} > 0$  and  $P \in \mathcal{P}(\mathbb{T})$  such that  $\|f - P\|_{A_\beta^1(\mathbb{T})} \leq \varepsilon/3$ .

By the Cauchy-Schwarz and Parseval inequalities, for every  $g \in C^1(\mathbb{T})$ , we get

$$\|g\|_{A_\beta^1(\mathbb{T})} \leq \|g\|_\infty + 2^{1+\beta} \sqrt{\frac{1-\beta}{1-2\beta}} \|g'\|_\infty. \quad (2.2)$$

As in [19], by applying (2.2) to  $\frac{\varphi}{P^n}$  we see that

$$\frac{\varphi}{f} = \sum_{n \geq 1} \varphi \frac{(P - f)^{n-1}}{P^n} \in A_\beta^1(\mathbb{T}),$$

which finishes the proof. □

**Lemma 2.4.** *Let  $1 < p < \infty$  and  $f \in A_\beta^1(\mathbb{T})$  with  $\beta \geq 0$ . We have*

- (1) If  $f$  is not cyclic in  $A_\beta^p(\mathbb{T})$ , then there exists  $S \in A_{-\beta}^q(\mathbb{T}) \setminus \{0\}$  such that  $\text{supp}(S) \subset \mathcal{Z}(f)$ .
- (2) If there exists a nonzero measure  $\mu \in A_{-\beta}^q(\mathbb{T})$  such that  $\text{supp}(\mu) \subset \mathcal{Z}(f)$ , then  $f$  is not cyclic in  $A_\beta^p(\mathbb{T})$ .

*Proof.* (1) If  $f$  is not cyclic in  $A_\beta^p(\mathbb{T})$ , by duality there exists  $S \in A_{-\beta}^q(\mathbb{T}) \setminus \{0\}$  such that

$$\langle S, e_n f \rangle = 0, \quad \forall n \in \mathbb{Z}.$$

Thus, by Lemma 2.3, we have  $\text{supp}(S) \subset \mathcal{Z}(f)$ .

(2) Let  $\mu \in A_{-\beta}^q(\mathbb{T}) \cap \mathcal{M}(\mathbb{T}) \setminus \{0\}$  such that  $\text{supp}(\mu) \subset \mathcal{Z}(f)$ . Since  $\mu$  is a measure on  $\mathbb{T}$  we have  $\langle \mu, e_n f \rangle = 0$ , for all  $n \in \mathbb{Z}$ . So  $f$  is not cyclic in  $A_\beta^p(\mathbb{T})$ .  $\square$

Recall that  $A_\beta^1(\mathbb{T})$  is a Banach algebra. Let  $I$  be a closed ideal in  $A_\beta^1(\mathbb{T})$ . We denote by  $\mathcal{Z}_I$  the set of common zeros of the functions of  $I$  given by

$$\mathcal{Z}_I = \bigcap_{f \in I} \mathcal{Z}(f).$$

We have the following result about spectral synthesis in  $A_\beta^1(\mathbb{T})$ , for the case  $\beta = 0$ , also called Beurling-Pollard Technique, see [8, pp. 121-123].

**Lemma 2.5.** *Let  $0 \leq \beta < 1/2$ . Let  $I$  be a closed ideal in  $A_\beta^1(\mathbb{T})$ . If  $g$  is a Lipschitz function which vanishes on  $\mathcal{Z}_I$ , then  $g \in I$ .*

*Proof.* Notice first that since  $g$  is Lipschitz function, Bernstein's Theorem [7, p.13] gives that  $g \in A_\beta^1(\mathbb{T})$ . Let  $I^\perp$  be the set of all  $S$  in the dual space of  $A_\beta^1(\mathbb{T})$  satisfying  $\langle S, f \rangle = 0$  for all  $f \in I$ . Hence,  $S \in I^\perp$  and  $\text{supp}(S) \subset \mathcal{Z}_I$  see [4, Remarque 1.3]. For  $h > 0$ , we set  $S_h = S * \Delta_h$  where  $\Delta_h : t \mapsto \frac{-|t|}{h^2} + \frac{1}{h}$  if  $t \in [-h, h]$  and 0 otherwise. We have

$$\widehat{\Delta}_h(0) = 1/2\pi, \quad \widehat{\Delta}_h(n) = \frac{1}{2\pi} \frac{4 \sin(nh/2)^2}{(nh)^2}, \quad n \neq 0.$$

Since  $S$  is in the dual of  $A_\beta^1(\mathbb{T})$ ,  $S_h \in A^1(\mathbb{T})$ . Moreover, we have  $\text{supp}(S_h) \subset \text{supp}(S) + \text{supp}(\Delta_h) \subset \mathcal{Z}_I^h := \mathcal{Z}_I + [-h, h]$ . Let  $g$  be a Lipschitz function which vanishes on  $\mathcal{Z}_I$ , We have

$$\begin{aligned} |\langle S_h, g \rangle|^2 &= \left| \int_{\mathcal{Z}_I^h \setminus \mathcal{Z}(g)} S_h(x) g(x) dx \right|^2 \\ &\leq \left( \sum_{n \in \mathbb{Z}} |\widehat{S}(n) \widehat{\Delta}_h(n)|^2 \right) \left( \int_{\mathcal{Z}_I^h \setminus \mathcal{Z}(g)} |g(x)|^2 dx \right) \end{aligned}$$

Since  $\mathcal{Z}_I \subset \mathcal{Z}_g$ , for every  $x \in \mathcal{Z}_I^h$ ,  $|g(x)| \leq ch$  for some positive constant  $c$ . Thus

$$|\langle S_h, g \rangle|^2 \leq c^2 \left( \sum_{n \in \mathbb{Z}} \frac{\widehat{S}(n)^2}{1+n^2} \right) (|\mathcal{Z}_I^h \setminus \mathcal{Z}(g)|)$$

Hence  $\lim_{h \rightarrow 0} \langle S_h, g \rangle = 0$ . By the dominated convergence theorem, we obtain that

$$\lim_{h \rightarrow 0} \langle S_h, g \rangle = \lim_{h \rightarrow 0} \sum_{n \in \mathbb{Z}} \widehat{S}_h(n) \widehat{g}(-n) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \widehat{S}(n) \widehat{g}(-n) = \frac{1}{2\pi} \langle S, g \rangle.$$

So  $\langle S, g \rangle = 0$ . Therefore,  $g \in I$ . □

We also need the following lemma which is a consequence of Lemma 2.5. Newman gave a proof of this when  $\beta = 0$  (see [18, Lemma 2]).

**Lemma 2.6.** *Let  $0 \leq \beta < 1/2$  and a closed set  $E \subset \mathbb{T}$ . There exists a sequence of Lipschitz functions  $(f_n)$  which vanish on  $E$  such that*

$$\lim_{n \rightarrow \infty} \|f_n - 1\|_{A_\beta^p(\mathbb{T})} = 0$$

*if and only if every  $f \in A_\beta^1(\mathbb{T})$  satisfying  $\mathcal{Z}(f) = E$  is cyclic in  $A_\beta^p(\mathbb{T})$ .*

We finish this subsection by the following result of Newman (see the proof of Theorem 5 in [18]):

**Lemma 2.7.** *Let  $p > 2$ . Assume that for every  $\varepsilon > 0$ , there exists a Lipschitz function  $f$  such that  $|\mathcal{Z}(f)| > 2\pi - \varepsilon$  and*

$$\|f - 1\|_{A_\beta^p(\mathbb{T})} \leq \varepsilon.$$

*Then for every  $\varepsilon$  there exists a  $(p, \beta)$ -spanning closed set  $E \subset \mathbb{T}$  with Lebesgue measure  $|E| > 2\pi - \varepsilon$ .*

**2.3. Generalized Cantor set and capacity.** Given  $E \subset \mathbb{T}$  and a non-decreasing continuous function  $h$  such that  $h(0) = 0$ , we define the  $h$ -measure of  $E$  by

$$H_h(E) = \liminf_{\delta \rightarrow 0} \left\{ \sum_{i=0}^{\infty} h(|U_i|), E \subset \bigcup_{i=0}^{\infty} U_i, |U_i| \leq \delta \right\}$$

where each  $U_i$  is an open interval inside  $\mathbb{T}$  and  $|U_i|$  denotes its length.

We also define the *Hausdorff dimension* of a subset  $E \subset \mathbb{T}$ , given by

$$\dim(E) = \inf\{\alpha \in (0, 1), H_\alpha(E) = 0\} = \sup\{\alpha \in (0, 1), H_\alpha(E) = \infty\},$$

where  $H_\alpha = H_h$  for  $h(t) = t^\alpha$  (see [8, pp. 23-30]).

Let  $\mu$  be a probability measure on  $\mathbb{T}$  and  $\alpha \in (0, 1)$ . We define its  $\alpha$ -energy by

$$I_\alpha(\mu) := \iint \frac{d\mu(t)d\mu(s)}{|t-s|^\alpha}.$$



Note that  $I_\alpha(\mu) \in [0, +\infty]$ . Simple calculations shows that

$$I_\alpha(\mu) \asymp \sum_{n \geq 1} \frac{|\widehat{\mu}(n)|^2}{(1 + |n|)^{1-\alpha}}.$$

The  $\alpha$ -capacity of a Borel set  $E$  is given by

$$C_\alpha(E) = 1/\{\inf\{I_\alpha(\mu), \mu \in \mathcal{M}_{\mathcal{P}}(E)\},$$

where  $\mathcal{M}_{\mathcal{P}}(E)$  is the set of all probability measures on  $\mathbb{T}$  which are supported on a compact subset of  $E$ .

An important property which connects capacity and Hausdorff dimension is that (see [8, p. 34])

$$\dim(E) = \inf\{\alpha \in (0, 1), C_\alpha(E) = 0\} = \sup\{\alpha \in (0, 1), C_\alpha(E) > 0\}. \quad (2.3)$$

For  $E \subset \mathbb{T}$ , we denote by  $A_\beta^p(E)$  the collection of  $S \in A_\beta^p(\mathbb{T})$  such that  $\text{supp}(S) \subset E$ , where  $\text{supp}(S)$  denotes the support of the distribution  $S$ . The following lemma is a direct consequence of the definition of capacity and the inclusion  $A_{-\beta}^q(\mathbb{T}) \subset A_{\frac{\alpha-1}{2}}^2(\mathbb{T})$  when  $q \geq 2$  and  $0 \leq \alpha < \frac{2}{q}(1 - \beta q)$ .

**Lemma 2.8.** *Let  $E$  be a Borel set,  $\beta \geq 0$  and  $q \geq 2$ . If there exists  $\alpha \in \mathbb{R}$  with  $0 \leq \alpha < \frac{2}{q}(1 - \beta q)$  such that  $C_\alpha(E) = 0$ , then  $A_{-\beta}^q(E) = \{0\}$ .*

Let us recall Salem's Theorem (see [25] and [8, p. 106-110]).

**Theorem 2.9.** *Let  $0 < \alpha < 1$  and  $q > \frac{2}{\alpha}$ . There exists a compact set  $E \subset \mathbb{T}$  which satisfies  $\dim(E) = \alpha$  and there exists a positive measure  $\mu \in A^q(\mathbb{T}) \setminus \{0\}$  such that  $\text{supp}(\mu) \subset E$ .*

The following theorem is due to Körner (see [13, Theorem 1.2]).

**Theorem 2.10.** *Let  $h : [0, \infty) \rightarrow [0, \infty)$  be an increasing continuous function with  $h(0) = 0$  and let  $\phi : [0, \infty) \rightarrow [0, \infty)$  be a decreasing function. Suppose that*

- (1)  $\int_1^\infty \phi(x)^2 dx = \infty$ ;
- (2) there exist  $K_1, K_2 > 1$  such that for all  $1 \leq x \leq y \leq 2x$ ,  $K_1\phi(2x) \leq \phi(x) \leq K_2\phi(y)$ ;
- (3) there exists  $\gamma > 0$  such that  $\lim_{x \rightarrow \infty} x^{1-\gamma}\phi(x) = \infty$ ;
- (4) there exist  $0 < K_3 < K_4 < 1$  such that for all  $t > 0$ ,  $K_3h(2t) \leq h(t) \leq K_4h(2t)$ .

*Then there exists a probability measure  $\mu$  with support of Hausdorff  $h$ -measure zero such that*

$$|\widehat{\mu}(n)| \leq \phi\left(\frac{1}{h(|n|^{-1})}\right) \left(\ln\left(\frac{1}{h(|n|^{-1})}\right)\right)^{1/2}, \quad \forall n \neq 0.$$

We finish this section by describing the construction of the generalized Cantor set. Let  $(k_j)_{j \geq 0}$  be integers and let  $(l_j)_{j \geq 0}$  be a sequence of positive number such that  $k_0 = 1$  and

$$k_j \geq 2, \quad \text{and} \quad k_j l_j < l_{j-1}, \quad j \geq 1.$$

Let  $E_0 = [0, l_0]$ . We dissect the interval  $E_0 = [0, l_0]$  in  $2k_1 - 1$  intervals of lengths respectively  $l_1$  and  $d_1 = (l_0 - k_1 l_1)/(k_1 - 1)$ :

$$\begin{aligned} & [0, l_1]; \quad ]l_1, d_1 + l_1[; \\ & \quad \vdots \\ & [md_1 + ml_1, md_1 + (m+1)l_1]; \quad ]md_1 + (m+1)l_1, (m+1)d_1 + (m+1)l_1[; \\ & \quad \vdots \\ & ]l_0 - l_1 - d_1, l_0 - l_1[; \quad [l_0 - l_1, l_0]. \end{aligned}$$

We delete the  $k_1 - 1$  open interval of length  $d_1$  and we keep the  $k_1$  equidistant closed intervals of length  $l_1$ . We set

$$E_1 = \bigcup_{m=0}^{k_1-1} [md_1 + ml_1, md_1 + (m+1)l_1].$$

Suppose that the set  $E_{n-1}$ ,  $n \geq 1$ , has already been constructed and this set consists of  $p_{n-1}$  closed intervals of length  $l_{n-1}$ :

$$E_{n-1} = \bigcup_{j=1}^{p_{n-1}} [a_j, a_j + l_{n-1}].$$

We operate the same dissection on each of the intervals  $[a_j, a_j + l_{n-1}]$  with the parameters  $(l_n, k_n)$  instead of  $(l_1, k_1)$ , thus we obtain

$$E_n = \bigcup_{j=1}^{p_n} \bigcup_{s=0}^{k_n-1} [a_j + s(l_n + d_n), a_j + s(l_n + d_n) + l_n]$$

where  $d_n = (l_{n-1} - k_n l_n)/(k_n - 1)$ . The compact set

$$E = \bigcap_{n \geq 0} E_n$$

is called the generalized Cantor set. Ohtsuka [22] obtained a criterion for vanishing  $C_\alpha(E)$ , see also [5]:

**Theorem 2.11.** *Let  $E$  be a generalized Cantor set. Then.*

$$C_\alpha(E) = 0 \iff \sum_{n \geq 0} \frac{1}{(k_0 k_1 \dots k_n) l_n^\alpha} = \infty.$$

### 3. PROOF OF THEOREM A

**3.1. Cyclicity and the set all sum of  $k$  elements from  $\mathcal{Z}(f)$ .** For  $k \in \mathbb{N}$  and  $E \subset \mathbb{T}$ , let  $k \times E$  denote the set all sum of  $k$  elements from  $E$ ,

$$k \times E = E + E + \dots + E = \left\{ \sum_{n=1}^k x_n, x_n \in E \right\}.$$

We have the following result, the case  $\beta = 0$  was considered by Newman in [18].

**Lemma 3.1.** *Let  $1 < p < 2$  and  $\beta > 0$  such that  $\beta q \leq 1$ , and let  $f \in A_\beta^1(\mathbb{T})$ .*

- (a) *Let  $k \in \mathbb{N} \setminus \{0\}$  be such that  $k \leq q/2$ . If  $C_\alpha(k \times \mathcal{Z}(f)) = 0$  for some  $\alpha < \frac{2}{q}(1 - \beta q)k$ , then  $f$  is cyclic in  $A_\beta^p(\mathbb{T})$ .*
- (b) *Let  $k \in \mathbb{N} \setminus \{0\}$  be such that  $q/2 \leq k \leq 1/(2\beta)$ . If  $C_\alpha(k \times \mathcal{Z}(f)) = 0$  where  $\alpha = 1 - 2k\beta$ , then  $f$  is cyclic in  $A_\beta^p(\mathbb{T})$ .*

*Proof.* Let  $k \in \mathbb{N} \setminus \{0\}$ . Suppose that  $f$  is not cyclic in  $A_\beta^p(\mathbb{T})$ . Then there exists  $L \in A_{-\beta}^q(\mathbb{T})$ , the dual of  $A_\beta^p(\mathbb{T})$ , such that  $L(1) = 1$  and  $L(Pf) = 0$ , for all  $P \in \mathcal{P}(\mathbb{T})$ .

Since  $\beta < \frac{1}{2}$ , by (2.2), we get  $C^1(\mathbb{T}) \subset A_\beta^1(\mathbb{T}) \subset A_\beta^p(\mathbb{T})$ . Moreover, by [17] (see also [18, Lemma 5]), there exists  $\phi \in L^2(\mathbb{T})$  such that

$$L(g) = \int_{\mathbb{T}} (g'(x)\phi(x) + g(x)) \, dx, \quad g \in C^1(\mathbb{T}).$$

Since  $L \in A_{-\beta}^q(\mathbb{T})$  which implies  $(L(e_n))_{n \in \mathbb{Z}} \in \ell_{-\beta}^q(\mathbb{Z})$ , we obtain

$$\sum_{n \in \mathbb{Z}} |n\widehat{\phi}(n)|^q (1 + |n|)^{-\beta q} < \infty. \quad (3.1)$$

Moreover we have,

$$\int_{\mathbb{T}} ((e_n f)'(x)\phi(x) + (e_n f)(x)) \, dx = 0, \quad n \in \mathbb{Z},$$

hence,  $\langle \phi' - 1, e_n f \rangle = 0$  where  $\phi'$  is defined in terms of distribution. By (3.1),  $\phi' - 1 \in A_{-\beta}^q(\mathbb{T})$ , by Lemma 2.3, we get  $\text{supp}(\phi' - 1) \subset \mathcal{Z}(f)$ .

For  $m \in \mathbb{N}$ , we denote by  $\phi^{*m}$  the result obtained from convolving  $\phi$  with itself  $m$  times. Using the fact that  $S' * T = S * T'$  and  $1 * S' = 0$  for any distributions  $S$  and  $T$ , we have

$$(\phi' - 1) * \left( (\phi^{*(m-1)})^{(m-1)} + (-1)^{m-1} \right) = (\phi^{*m})^{(m)} + (-1)^m.$$

By induction and by the formula  $\text{supp}(T * S) \subset \text{supp}(T) + \text{supp}(S)$  that

$$\text{supp} \left( (\phi^{*m})^{(m)} + (-1)^m \right) \subset m \times \mathcal{Z}(f), \quad m \geq 1. \quad (3.2)$$

Note that  $\widehat{(\phi^{*k})^{(k)}}(n) = i^k n^k \widehat{\phi}(n)^k$  for  $k \geq 1$  and  $n \in \mathbb{Z}$ .

(a) Suppose that  $0 < k \leq q/2$  and  $C_\alpha(k \times \mathcal{Z}(f)) = 0$  for some  $\alpha < \frac{2}{q}(1 - \beta q)k$ . We rewrite (3.1) as

$$\sum_{n \in \mathbb{Z}} \left( |n\widehat{\phi}(n)|^k \right)^{\frac{q}{k}} (1 + |n|)^{-\frac{q}{k}\beta k} < \infty.$$

If we set  $q' = \frac{q}{k} \geq 2$  and  $\beta' = \beta k$ , we have  $(\phi^{*k})^{(k)} \in A_{-\beta'}^{q'}(\mathbb{T})$ . By (3.2) and by Lemma 2.8 we obtain that  $(\phi^{*k})^{(k)} = (-1)^{k-1}$ . This contradicts the fact that  $\widehat{(\phi^{*k})^{(k)}}(0) = 0$ .

(b) Now suppose that  $k \geq q/2$  and  $C_\alpha(k \times \mathcal{Z}(f)) = 0$  where  $\alpha = 1 - 2k\beta$ . Since  $q \leq 2k$ , we have by (3.1),

$$\sum_{n \in \mathbb{Z}} |n\widehat{\phi}(n)|^{2k} (1 + |n|)^{-2k\beta} < \infty.$$

Then  $(\phi^{*k})^{(k)} \in A_{-k\beta}^2(\mathbb{T})$  and  $(\phi^{*k})^{(k)} = (-1)^{k-1}$ , which contradicts  $(\widehat{\phi^{*k}})^{(k)}(0) = 0$ .  $\square$

**3.2. Construction of generalized Cantor set.** We need to compute the capacity of the Minkowski sum of some Cantor type subset of  $\mathbb{T}$ . We denote by  $[x]$  the integer part of  $x \in \mathbb{R}$ . For  $\lambda \in [0, 1]$  and  $k \in \mathbb{N} \setminus \{0\}$ , we define

$$K_\lambda^k = \{m \in \mathbb{N}, \exists j \in \mathbb{N}, m \in [2^j, 2^j(1 + \lambda + 1/j) - k + 1]\}$$

and we set in  $\mathbb{R}/\mathbb{Z} \simeq [0, 1[$ ,

$$S_\lambda^k = \left\{ x = \sum_{i=0}^{\infty} \frac{x_i}{2^{i+1}}, (x_i) \in \{0, 1\}^{\mathbb{N}} \text{ such that } i \in K_\lambda^k \Rightarrow x_i = 0 \right\}.$$

We denote  $K_\lambda = K_\lambda^1$  and  $S_\lambda = S_\lambda^1$ . We the following lemma.

**Lemma 3.2.** *For all  $k \geq 1$ , we have*

- (1)  $k \times S_\lambda \subset S_\lambda^k$ ;
- (2)  $C_\alpha(S_\lambda^k) = 0$  if and only if  $\alpha \geq \frac{1 - \lambda}{1 + \lambda}$ ;
- (3)  $\dim(k \times S_\lambda) = \frac{1 - \lambda}{1 + \lambda}$  and  $C_{\frac{1-\lambda}{1+\lambda}}(k \times S_\lambda) = 0$ .

*Proof.* (1) We prove this by induction. If  $k = 1$  we have  $S_\lambda = S_\lambda^1$ . We suppose the result true for  $k - 1$  for some  $k \geq 2$ , and we will show that  $k \times S_\lambda \subset S_\lambda^k$ . Observe that we have

$$k \times S_\lambda \subset (k - 1) \times S_\lambda + S_\lambda \subset S_\lambda^{k-1} + S_\lambda.$$

Let  $x \in S_\lambda^{k-1}$ ,  $y \in S_\lambda$  and  $z = x + y$ . Denote by  $(x_i)$ ,  $(y_i)$  and  $(z_i)$  their binary decomposition. Let  $m \in K_\lambda^k$ , then there exists  $j \in \mathbb{N}$  such that  $m \in [2^j, 2^j(1 + \lambda + 1/j) - k + 1]$ . Since  $m \in K_\lambda^k$  and  $m, m + 1 \in K_\lambda^{k-1} \subset K_\lambda$ , we get  $x_m = y_m = x_{m+1} = y_{m+1} = 0$ . Therefore, we write

$$z = x + y = \sum_{i=0}^{m-1} \frac{x_i + y_i}{2^{i+1}} + \sum_{i=m+2}^{\infty} \frac{x_i + y_i}{2^{i+1}}.$$

Note that for infinitely many  $i \geq m + 2$ , we have  $x_i + y_i < 2$ , we see that

$$\sum_{i=m+2}^{\infty} \frac{x_i + y_i}{2^{i+1}} < \frac{1}{2^{m+1}}.$$

We denote by  $[s]$  the integer part of  $s$ , we have

$$[2^{m+1}z] = 2[2^m z] = 2^{m+1} \sum_{i=0}^{m-1} \frac{x_i + y_i}{2^{i+1}}.$$

Therefore, we obtain by uniqueness of the decomposition that

$$z_m = [2^{m+1}z] - 2[2^m z] = 0.$$

This proves that  $z = x + y \in S_\lambda^k$  and  $k \times S_\lambda \subset S_\lambda^k$ .

(2) We will first show that the set  $S_\lambda^k$  is a generalized Cantor set. Let

$$\nu_j = [2^j(1 + \lambda + 1/j) - k + 1] + 1$$

and  $N_0$ , depending only on  $k$  and  $\lambda$ , such that for all  $j \geq N_0$ ,  $2^j < \nu_j < 2^{j+1}$ . We set for  $N \geq N_0$ ,

$$l_N = \sum_{j=N}^{\infty} \left( \frac{1}{2^{\nu_j}} - \frac{1}{2^{2^{j+1}}} \right).$$

Since

$$2^j(1 + \lambda + 1/j) - k + 1 < \nu_j \leq 2^j(1 + \lambda + 1/j) - k + 2,$$

we have

$$\sum_{j=N}^{\infty} \frac{1}{2^{2^j(1+\lambda+\frac{1}{j})}} \left( \frac{1}{2^{2-k}} - \frac{1}{2^{2^j(1-\lambda-\frac{1}{j})}} \right) \leq l_N \leq \sum_{j=N}^{\infty} \frac{1}{2^{2^j(1+\lambda+\frac{1}{j})}} \left( \frac{1}{2^{1-k}} - \frac{1}{2^{2^j(1-\lambda-\frac{1}{j})}} \right).$$

There exists  $C \geq 1$  such that for all  $j \geq N$ ,

$$\frac{1}{C} \leq \frac{1}{2^{2-k}} - \frac{1}{2^{2^j(1-\lambda-\frac{1}{j})}} \leq \frac{1}{2^{1-k}} - \frac{1}{2^{2^j(1-\lambda-\frac{1}{j})}} \leq C.$$

And for  $N \geq N_0$ ,

$$\begin{aligned} \frac{1}{2^{2^N(1+\lambda+\frac{1}{N})}} &\leq \sum_{j=N}^{\infty} \frac{1}{2^{2^j(1+\lambda+\frac{1}{j})}} \\ &\leq \frac{1}{2^{2^N(1+\lambda+\frac{1}{N})}} + \sum_{j=0}^{\infty} \left( \frac{1}{2^{2^{N+1}(1+\lambda)}} \right)^{2^j} \\ &\leq \frac{1}{2^{2^N(1+\lambda+\frac{1}{N})}} + \sum_{j=0}^{\infty} \left( \frac{1}{2^{2^{N+1}(1+\lambda)}} \right)^{j+1} \\ &\leq \frac{1}{2^{2^N(1+\lambda+\frac{1}{N})}} + \frac{2}{2^{2^{N+1}(1+\lambda)}} \\ &\leq \frac{3}{2^{2^N(1+\lambda+\frac{1}{N})}}. \end{aligned}$$

Hence we obtain that  $l_N$  is comparable to  $2^{-2^N(1+\lambda+1/N)}$ , that is,

$$\frac{1}{C 2^{2^N(1+\lambda+\frac{1}{N})}} \leq l_N \leq \frac{3C}{2^{2^N(1+\lambda+\frac{1}{N})}}. \quad (3.3)$$

Moreover, we have

$$\begin{aligned}
 l_N &= \frac{1}{2^{\nu_N}} - \sum_{j=N+1}^{\infty} \left( \frac{1}{2^{2^j}} - \frac{1}{2^{\nu_j}} \right) \\
 &< \frac{1}{2^{\nu_N}} \\
 &\leq \frac{1}{2^{2^N}}.
 \end{aligned} \tag{3.4}$$

For  $N \geq N_0$ , we set

$$E_N = \left\{ \sum_{i=0}^{2^N-1} \frac{x_i}{2^{i+1}} + l_N z, z \in [0, 1], x_i \in \{0, 1\}, i \in K_\lambda^k \Rightarrow x_i = 0 \right\}.$$

Observe that we can write  $E_N$  as a union of disjoint intervals given by

$$E_N = \bigcup_{\substack{(x_i) \in \{0,1\}^{2^N} \\ i \in K_\lambda^k \Rightarrow x_i = 0}} E_N^{(x_i)}$$

where

$$E_N^{(x_i)} = \sum_{i=0}^{2^N-1} \frac{x_i}{2^{i+1}} + l_N [0, 1].$$

Since by (3.4),  $l_N < 1/2^{2^N}$ , the intervals  $E_N^{(x_i)}$  are disjoint

$$E_N^{(x_i)} \cap E_N^{(x'_i)} = \emptyset, \quad (x_i) \neq (x'_i).$$

For fixed  $N \geq N_0$ , let  $(x_i)_{0 \leq i \leq 2^N-1} \in \{0, 1\}^{2^N}$  and  $(y_i)_{0 \leq i \leq 2^{N+1}-1} \in \{0, 1\}^{2^{N+1}}$ . We claim that:

$$E_{N+1}^{(y_i)} \subset E_N^{(x_i)} \text{ if and only if } x_i = y_i \text{ for all } 0 \leq i < 2^N \text{ and } y_i = 0 \text{ for all } 2^N \leq i < \nu_N.$$

Indeed, suppose that  $E_{N+1}^{(y_i)} \subset E_N^{(x_i)}$  and let  $u \in E_{N+1}^{(y_i)}$ . We have

$$\begin{aligned}
 u &= \sum_{i=0}^{2^{N+1}-1} \frac{y_i}{2^{i+1}} + l_{N+1} z_2 \\
 &= \sum_{i=0}^{2^N-1} \frac{x_i}{2^{i+1}} + l_N z_1,
 \end{aligned}$$

where  $z_1, z_2 \in [0, 1]$ . By (3.4),  $l_N < 1/2^{\nu_N}$ , and using the uniqueness of the binary representation, we obtain  $x_i = y_i$  for all  $0 \leq i < 2^N$  and  $y_i = 0$  for all  $2^N \leq i < \nu_N$ . Now suppose  $x_i = y_i$  for all  $0 \leq i < 2^N$  and  $y_i = 0$  for all  $2^N \leq i < \nu_N$ . Let  $u \in E_{N+1}^{(y_i)}$ . We write

$$u = \sum_{i=0}^{2^N-1} \frac{x_i}{2^{i+1}} + \sum_{i=\nu_N}^{2^{N+1}-1} \frac{y_i}{2^{i+1}} + l_{N+1} z, \quad z \in [0, 1].$$

Since

$$\sum_{i=\nu_N}^{2^{N+1}-1} \frac{1}{2^{i+1}} + l_{N+1} = \frac{1}{2^{\nu_N}} - \frac{1}{2^{2^{N+1}}} + l_{N+1} = l_N,$$

we get

$$\begin{aligned} \sum_{i=0}^{2^N-1} \frac{x_i}{2^{i+1}} &\leq \sum_{i=0}^{2^N-1} \frac{x_i}{2^{i+1}} + \sum_{i=Z_N}^{2^{N+1}-1} \frac{y_i}{2^{i+1}} + l_{N+1}z \\ &\leq \sum_{i=0}^{2^N-1} \frac{x_i}{2^{i+1}} + l_N, \end{aligned}$$

and  $u \in E_N^{(x_i)}$ . This concludes the proof of the claim.

By the claim, for fixed  $(x_i)$  and for  $N \geq N_0$ , we have the following properties:

(i) the interval  $E_N^{(x_i)}$  contains precisely

$$k_{N+1} = \#\{(y_i)_{\nu_N \leq i \leq 2^{N+1}-1} : y_i \in \{0, 1\}\} = 2^{2^{N+1}-\nu_N}$$

intervals of the form  $E_{N+1}^{(y_i)}$ ,

(ii) the intervals of the form  $E_{N+1}^{(y_i)}$  contained in  $E_N^{(x_i)}$  are equidistant intervals of length  $l_{N+1}$ ; the distance of two consecutive intervals of the form  $E_{N+1}^{(y_i)}$  is equal to  $\frac{1}{2^{2^{N+1}-l_{N+1}}}$ ,

(iii) writing  $E_N^{(x_i)} = [a, b]$ , there exist  $(y_i)$  and  $(z_i)$  such that  $E_{N+1}^{(y_i)} = [a, a + l_{N+1}]$  and  $E_{N+1}^{(z_i)} = [b - l_{N+1}, b]$ .

Finally we can write  $S_\lambda^k$  as

$$S_\lambda^k = \bigcap_{N \geq N_0} E_N.$$

This shows that  $S_\lambda^k$  is a generalized Cantor set. By Theorem 2.11, we have for  $0 < \alpha < 1$  that  $C_\alpha(S_\lambda^k) = 0$  if and only if

$$\sum_{N=N_0}^{\infty} \frac{1}{(k_{N_0} \cdots k_{N-1})l_N^\alpha} = \infty.$$

where  $k_{N_0} = 1$ . Since

$$2^{(k-2)(N-N_0)+(2^N-2^{N_0})(1-\lambda)-\sigma_N} \leq k_{N_0} \cdots k_{N-1} \leq 2^{(k-1)(N-N_0)+(2^N-2^{N_0})(1-\lambda)-\sigma_N}$$

where

$$\sigma_N = \sum_{j=N_0}^{N-1} \frac{2^j}{j}$$

we have, by (3.3),  $C_\alpha(S_\lambda^k) = 0$  if and only if

$$\sum_{N=N_0}^{\infty} 2^{2^N(\alpha(1+\lambda)-(1-\lambda))+\alpha 2^N/N+\sigma_N-(k-1)(N-N_0)+2^{N_0}(1-\lambda)} = \infty.$$

Therefore,  $C_\alpha(S_\lambda^k) = 0$  if and only if  $\alpha \geq \frac{1-\lambda}{1+\lambda}$ .

Finally, (3) follows from (1) and (2) by the capacity property.  $\square$

**3.3. Proof of Theorem A.** We are now ready to prove Theorem A. It follows immediately from the following theorem stated in  $A_\beta^p(\mathbb{T})$  spaces.

**Theorem 3.3.** *Let  $1 < p < 2$ ,  $\beta > 0$  be such that  $\beta q \leq 1$ .*

- (1) *If  $f \in A_\beta^1(\mathbb{T})$  and  $\dim(\mathcal{Z}(f)) < \frac{2}{q}(1 - \beta q)$  then  $f$  is cyclic in  $A_\beta^p(\mathbb{T})$ .*
- (2) *If  $f \in A_\beta^1(\mathbb{T})$  and  $C_{1-\beta q}(\mathcal{Z}(f)) > 0$  then  $f$  is not cyclic in  $A_\beta^p(\mathbb{T})$ .*
- (3) *For  $\frac{2}{q}(1 - \beta q) \leq \alpha \leq 1$ , there exists a closed set  $E \subset \mathbb{T}$  such that  $\dim(E) = \alpha$  and every  $f \in A_\beta^1(\mathbb{T})$  satisfying  $\mathcal{Z}(f) = E$  is not cyclic in  $A_\beta^p(\mathbb{T})$ .*
- (4) *Let  $k = [q/2]$ . For all  $\varepsilon > 0$ , there exists a closed set  $E \subset \mathbb{T}$  such that*

$$\dim(E) \geq \max\left(\frac{2}{q}(1 - \beta q)k - \varepsilon, 1 - 2(k + 1)\beta\right) \quad (3.5)$$

*and every  $f \in A_\beta^1(\mathbb{T})$  satisfying  $\mathcal{Z}(f) = E$  is cyclic in  $A_\beta^p(\mathbb{T})$ . Furthermore, if  $p = 2k/(2k - 1)$  for some  $k \in \mathbb{N} \setminus \{0\}$ ,  $E$  can be chosen such that  $\dim(E) = 1 - \beta q$ .*

*Proof.* (1) Note that, by (2.3),  $\dim(\mathcal{Z}(f)) < \frac{2}{q}(1 - \beta q)$  if and only if there exists  $\alpha < \frac{2}{q}(1 - \beta q)$  such that  $C_\alpha(\mathcal{Z}(f)) = 0$ . If  $C_\alpha(\mathcal{Z}(f)) = 0$ , by Lemma 2.8, there is no  $S \in A_{-\beta}^q(\mathbb{T}) \setminus \{0\}$  such that  $\text{supp}(S) \subset \mathcal{Z}(f)$ . So, by Lemma 2.4 (1),  $f$  is cyclic in  $A_\beta^p(\mathbb{T})$ .

(2) Suppose that  $C_{1-\beta q}(\mathcal{Z}(f)) > 0$ . There exists a probability measure  $\mu$  of energy  $I_{1-\beta q}(\mu) < \infty$ , such that  $\text{supp}(\mu) \subset \mathcal{Z}(f)$ . So  $\mu \in A_{-\beta q/2}^2(\mathbb{T}) \setminus \{0\}$ . Since  $|\widehat{\mu}(n)| \leq 1$  for all  $n \in \mathbb{Z}$  and  $q \geq 2$ , we have  $\mu \in A_{-\beta}^q(\mathbb{T})$ . By Lemma 2.4 (2),  $f$  is not cyclic in  $A_\beta^p(\mathbb{T})$ .

(3) Suppose that  $\frac{2}{q}(1 - \beta q) < \alpha \leq 1$ . There exists  $\varepsilon > 0$  such that  $\frac{2}{q}(1 - \beta q) + \varepsilon < \alpha$ . Let  $q'$  such that  $\frac{2}{q} - 2\beta + \varepsilon = \frac{2}{q'}$ . Since  $\beta > \frac{1}{q} - \frac{1}{q'}$ , by Lemma 2.1,  $A^{q'}(\mathbb{T}) \subset A_{-\beta}^q(\mathbb{T})$ . By Theorem 2.9, as  $q'$  satisfies  $q' > \frac{2}{\alpha}$ , there exists a closed subset  $E \subset \mathbb{T}$  such that  $\dim(E) = \alpha$  and a non-zero positive measure  $\mu \in A^{q'}(\mathbb{T}) \subset A_{-\beta}^q(\mathbb{T})$  such that  $\text{supp}(\mu) \subset E$ . Now (3) follows from Lemma 2.4 (2).

Now if  $\alpha = \frac{2}{q}(1 - \beta q)$  and  $\gamma > \frac{2}{q}$ . Then by Theorem 2.10 with  $\phi(t) = (t \ln(et))^{-1/2}$  for  $t \geq 1$  and  $h(t) = \frac{t^\alpha}{\ln(e/t)^\gamma}$  for  $t \in [0, \infty)$ , there exists a probability measure  $\mu$  with support of Hausdorff  $h$ -measure zero such that

$$|\widehat{\mu}(n)| \leq \phi\left(\frac{1}{h(|n|^{-1})}\right) \left(\ln\left(\frac{1}{h(|n|^{-1})}\right)\right)^{1/2} \leq (|n|^\alpha \ln(e|n|)^\gamma)^{-1/2},$$



for  $n \neq 0$ . So

$$\begin{aligned} \sum_{n \neq 0} |\widehat{\mu}(n)|^q (1 + |n|)^{-\beta q} &\leq C \sum_{n \neq 0} |n|^{-\alpha q/2 - \beta q} \ln(e|n|)^{-\gamma q/2} \\ &\leq C \sum_{n \neq 0} \frac{1}{|n| \ln(e|n|)^{\gamma q/2}} < \infty \end{aligned}$$

with  $C$  a positive constant. Hence,  $\mu \in A_{-\beta}^q(\mathbb{T})$ . We set  $E = \text{supp}(\mu)$ . By Lemma 2.4 the result is proved.

(4) Let  $k = [q/2]$ . Suppose first  $\frac{2}{q}(1 - \beta q)k > 1 - 2(k + 1)\beta$  and let  $0 < \varepsilon' < \varepsilon$  satisfy  $1 - 2(k + 1)\beta \leq \frac{2}{q}(1 - \beta q)k - \varepsilon'$ . Consider the set  $S_\lambda$  where  $\lambda$  satisfies

$$\frac{2}{q}(1 - \beta q)k - \varepsilon' < \frac{1 - \lambda}{1 + \lambda} < \frac{2}{q}(1 - \beta q)k.$$

By Lemma 3.2 (3) we have  $\dim(S_\lambda) = \frac{1 - \lambda}{1 + \lambda}$  and  $C_{\frac{1 - \lambda}{1 + \lambda}}(k \times S_\lambda) = 0$ . Therefore, by Lemma 3.1 (a), every  $f \in A_\beta^1(\mathbb{T})$  such that  $\mathcal{Z}(f) = S_\lambda$  is cyclic in  $A_\beta^p(\mathbb{T})$ .

Now, suppose  $\frac{2}{q}(1 - \beta q)k \leq 1 - 2(k + 1)\beta$ . We consider  $S_\lambda$  where

$$\frac{1 - \lambda}{1 + \lambda} = 1 - 2(k + 1)\beta.$$

By Lemma 3.2 (3) we have

$$\dim(S_\lambda) = \frac{1 - \lambda}{1 + \lambda} = 1 - 2(k + 1)\beta \quad \text{and} \quad C_{\frac{1 - \lambda}{1 + \lambda}}((k + 1) \times S_\lambda) = 0.$$

Thus, by Lemma 3.1 (b), every  $f \in A_\beta^1(\mathbb{T})$  such that  $\mathcal{Z}(f) = S_\lambda$  is cyclic in  $A_\beta^p(\mathbb{T})$ .

Suppose now that  $p = \frac{2k}{2k - 1}$  for some  $k \in \mathbb{N} \setminus \{0\}$ . As before, we consider  $S_\lambda$  where

$$\frac{1 - \lambda}{1 + \lambda} = 1 - 2k\beta = 1 - \beta q.$$

Again by Lemma 3.1 (b), every  $f \in A_\beta^1(\mathbb{T})$  such that  $\mathcal{Z}(f) = S_\lambda$  is cyclic in  $A_\beta^p(\mathbb{T})$ .

Note that the set  $E$  which was considered in Theorem 3.3 (4) satisfies  $C_\alpha(E) = 0$  where

$$\alpha \geq \max \left( \frac{2}{q}(1 - \beta q)k - \varepsilon, 1 - 2(k + 1)\beta \right).$$

□

## 4. PROOF OF THEOREM B

4.1. **Some power sum.** To prove Theorem B, we need the following lemmas.

**Lemma 4.1.** *Let  $R$  be a prime power and  $m$  a positive integer. We set  $k = R^{m+1}$  and  $N = (R - 1)(R^{m+1} + 1)$ . Then there exists  $N^{\text{th}}$  roots of unity  $z_1, \dots, z_k$  such that*

$$\left| \sum_{j=1}^k z_j^r \right| \leq \sqrt{k}, \quad r = 1, \dots, N - 1.$$

*Proof.* The proof is inspired from a result by Andersson [2, Lemma 1]. Let  $F = \{x_j, 1 \leq j \leq k\}$  be a finite field of order  $k$  and let  $E$  be an extension field of  $F$  of order  $k^2$ . Let  $\omega$  be an element that generates the multiplicative group  $E^*$  and let  $\chi$  be a multiplicative character on  $E$  of order  $k^2 - 1$ . We set

$$z_j = \chi^d(\omega + x_j), \quad 1 \leq j \leq k,$$

where  $d = \sum_{j=0}^m R^j$ . Since  $Nd = k^2 - 1$  the  $z_j$  are  $N^{\text{th}}$  roots of unity. For  $1 \leq r \leq N - 1$ , the characters  $\chi^{rd}$  are non-trivial on  $E$ , thus by [11, Theorem 1] we get

$$\left| \sum_{j=1}^k z_j^r \right| = \left| \sum_{j=1}^k \chi^{rd}(\omega + x_j) \right| \leq \sqrt{k}, \quad r = 1, \dots, N - 1.$$

□

**Lemma 4.2.** *With the notation of Lemma 4.1, we set*

$$c_n = \frac{\left( \sum_{j=1}^k z_j^n \right)}{k} \left( \frac{\sin(\pi n/N)}{\pi n/N} \right)^2, \quad n \in \mathbb{Z}.$$

Then

$$\sum_{n \in \mathbb{Z} \setminus \{0\}} |c_n|^p (1 + |n|)^{\beta p} \leq \frac{N^{1+\beta p}}{k^{\frac{p}{2}}}$$

*Proof.* We have

$$\begin{aligned} c_{r+N\ell} &= \frac{\left( \sum_{j=1}^k z_j^r \right)}{k} \left( \frac{\sin(\pi r/N)}{\frac{\pi r}{N} + \pi \ell} \right)^2 \\ \sum_{n \in \mathbb{Z} \setminus \{0\}} |c_n|^p (1 + |n|)^{\beta p} &= \sum_{r=1}^{N-1} \sum_{\ell \in \mathbb{Z}} |c_{r+N\ell}|^p (1 + |N\ell + r|)^{\beta p} \\ &= \sum_{r=1}^{N-1} \frac{\left| \sum_{j=1}^k z_j^r \right|^p}{k^p} \sum_{\ell \in \mathbb{Z}} \frac{|\sin(\pi r/N)|^{2p}}{\pi^{2p} \left| \frac{r}{N} + \ell \right|^{2p}} (1 + |N\ell + r|)^{\beta p}. \end{aligned}$$

To estimate

$$\frac{|\sin(\pi r/N)|^{2p}}{\pi^{2p} \left| \frac{r}{N} + \ell \right|^{2p}} (1 + |N\ell + r|)^{\beta p}$$

we will consider two cases:

**Case 1:**  $N \geq 2r$ .

$$\begin{aligned} \frac{\sin^2((r/N)\pi)}{\pi^2(r/N + \ell)^2} (1 + |r + \ell N|)^\beta &\leq \frac{(r/N)^{2-\beta}}{(|\ell| - 1/2)^2} (r/N + r^2/N + |\ell|r)^\beta \\ &\leq \frac{1}{2^{2-\beta}} \frac{1}{(|\ell| - 1/2)^2} (1/2 + r/2 + |\ell|r)^\beta \\ &\leq \frac{1}{2^{2-\beta}} \frac{(|\ell| + 1/2)^\beta}{(|\ell| - 1/2)^2} (1 + r)^\beta. \end{aligned} \quad (4.1)$$

Therefore

$$\sum_{\ell \in \mathbb{Z}} \frac{|\sin(\pi r/N)|^{2p}}{\pi^{2p} |\frac{r}{N} + \ell|^{2p}} (1 + |N\ell + r|)^{\beta p} \leq \frac{1}{2^{(2-\beta)p}} \sum_{\ell \in \mathbb{Z}} \frac{(|\ell| + 1/2)^{\beta p}}{(|\ell| - 1/2)^{2p}} (1 + r)^{\beta p}. \quad (4.2)$$

**Case 2:**  $N \leq 2r$ . For  $|r + \ell N| < N$ ,

$$\frac{\sin^2((r/N)\pi)}{\pi^2(r/N + \ell)^2} (1 + |r + \ell N|)^\beta \leq (1 + N)^\beta \leq 2^\beta (1 + r)^\beta.$$

We remark that there are at most two integers of the form  $r + \ell N$  with  $|r + \ell N| < N$ . Thus

$$\sum_{\ell : |r + \ell N| < N} \frac{|\sin(\pi r/N)|^{2p}}{\pi^{2p} |\frac{r}{N} + \ell|^{2p}} (1 + |N\ell + r|)^{\beta p} \leq 2^{1+\beta p} (1 + r)^{\beta p}. \quad (4.3)$$

Assume now that  $|r + \ell N| \geq N$  and  $\ell \in \mathbb{Z}$ . We note that in this case  $|\frac{r}{N} + \ell| = \frac{|r + \ell N|}{N} \geq 1$ . We have

$$\begin{aligned} \frac{\sin^2(r\pi/N)}{\pi^2(r/N + \ell)^2} (1 + |r + \ell N|)^\beta &\leq \frac{1}{(r/N + \ell)^2 \pi^2} N^\beta (1/N + |r/N + \ell|)^\beta \\ &\leq \frac{2^{2\beta}}{\pi^2 |r/N + \ell|^{2-\beta}} r^\beta. \end{aligned}$$

Then we get

$$\begin{aligned} \sum_{\ell \in \mathbb{Z}: |r + \ell N| \geq N} \frac{|\sin(\pi r/N)|^{2p}}{\pi^{2p} |\frac{r}{N} + \ell|^{2p}} (1 + |N\ell + r|)^{\beta p} &\leq \sum_{\ell \in \mathbb{Z}: |r/N + \ell| \geq 1} \frac{2^{2\beta p}}{\pi^{2p} |r/N + \ell|^{(2-\beta)p}} r^{\beta p} \\ &\leq \frac{2^{2\beta p+1}}{\pi^{2p}} \sum_{\ell \geq 1} \frac{1}{\ell^{(2-\beta)p}} r^{\beta p}. \end{aligned} \quad (4.4)$$

Combining (4.3) and (4.4) we obtain

$$\sum_{\ell \in \mathbb{Z}} \frac{|\sin(\pi r/N)|^{2p}}{\pi^{2p} |\frac{r}{N} + \ell|^{2p}} (1 + |N\ell + r|)^{\beta p} \leq \underbrace{\max \left( 2^{1+\beta p}, \frac{2^{2\beta p+1}}{\pi^{2p}} \sum_{k \geq 1} \frac{1}{k^{(2-\beta)p}} \right)}_{c_{\beta,p}} (1+r)^{\beta p}. \quad (4.5)$$

Therefore

$$\begin{aligned} \sum_{n \in \mathbb{Z} \setminus \{0\}} |c_n|^p (1 + |n|)^{\beta p} &= \sum_{r=1}^{N-1} \frac{|\sum_{j=1}^k z_j^r|^p}{k^p} \sum_{\ell \in \mathbb{Z}} \frac{|\sin(\pi r/N)|^{2p}}{\pi^{2p} |\frac{r}{N} + \ell|^{2p}} (1 + |N\ell + r|)^{\beta p} \\ &\leq c_{\beta,p} \sum_{r=1}^{N-1} \frac{|\sum_{j=1}^k z_j^r|^p}{k^p} (1+r)^{\beta p} \\ &\leq c_{\beta,p} \frac{N^{1+\beta p}}{k^{\frac{p}{2}}}. \end{aligned}$$

□

**4.2. Proof of Theorem B.** To prove (1), we suppose that  $p \geq 2$  and  $\beta > \frac{1}{2} - \frac{1}{p}$ . It suffices to check that the characteristic function of  $E$ ,  $\chi_E$  is in  $\ell_{-\beta}^q$ , the dual space of  $\ell_{\beta}^p$ . By Hölder's inequality

$$\sum_{n \in \mathbb{Z}} |\widehat{\chi}_E(n)|^q (1 + |n|)^{-\beta q} \leq \left( \sum_{n \in \mathbb{Z}} |\widehat{\chi}_E(n)|^2 \right)^{q/2} \left( \sum_{n \in \mathbb{Z}} (1 + |n|)^{-\frac{2\beta q}{2-q}} \right)^{\frac{2-q}{q}}.$$

The sums  $\sum_{n \in \mathbb{Z}} |\widehat{\chi}_E(n)|^2$  and  $\sum_{n \in \mathbb{Z}} (1 + |n|)^{-\frac{2\beta q}{2-q}}$  converge since  $\chi_E$  is in  $L^2(\mathbb{T})$  and  $\beta > \frac{1}{q} - \frac{1}{2}$ .

In order to prove (2), using notations from Lemmas 4.1 and 4.2, we first define

$$f(x) = \sum_{n \in \mathbb{Z}} c_n e^{inx}.$$

The function  $f$  is the sum of  $k$  triangles each with base  $4\pi/N$  and height  $N/k$ , then  $f$  is a Lipschitz function and its support has measure  $4\pi k/N$ .

We recall that  $k = R^{m+1}$  and  $N = (R-1)(R^{m+1} + 1)$ . Let  $0 \leq \beta < \frac{1}{2} - \frac{1}{p}$  and choose  $m$  such that  $\beta < \frac{1}{2} \frac{m+1}{m+2} - \frac{1}{p}$ . We have

$$\frac{N^{1+\beta p}}{k^{\frac{p}{2}}} \sim \frac{R^{(m+2)(1+\beta p)}}{R^{\frac{m+1}{2}p}} \sim R^{p(m+2)(\beta + \frac{1}{p} - \frac{1}{2} \frac{m+1}{m+2})}.$$

Since

$$p(m+2)\left(\beta + \frac{1}{p} - \frac{1}{2} \frac{m+1}{m+2}\right) < 0,$$

$$\frac{N^{1+\beta p}}{k^{\frac{p}{2}}} \rightarrow 0, \quad R \rightarrow \infty.$$

By Lemma 4.1 we have

$$\|f - 1\|_{A_\beta^p(\mathbb{T})}^p \leq \frac{N^{1+\beta p}}{k^{\frac{p}{2}}}.$$

On the other hand  $k/N \rightarrow 0$  as  $R \rightarrow \infty$ , hence for every  $\epsilon > 0$ , there exists a Lipschitz function  $f$  such that  $\|f - 1\|_{A_\beta^p(\mathbb{T})} < \epsilon$  and with support of measure less than  $\epsilon$ . Finally, Lemma 2.7 concludes the proof.

## 5. REMARKS

We say that  $(\omega_n) \in \mathbb{R}$  is a *weight* if there exists  $C > 0$  such that  $w_n \geq 1$  and  $\omega_{n+k} \leq C\omega_n\omega_k$  for all  $k, n \in \mathbb{Z}$ . For a weight  $\omega$  and  $1 \leq p < \infty$ , we set

$$A_\omega^p(\mathbb{T}) = \left\{ f \in \mathcal{C}(\mathbb{T}) : \|f\|_{A_\omega^p(\mathbb{T})}^p = \sum_{n \in \mathbb{Z}} |\widehat{f}(n)|^p \omega_n^p < \infty \right\}.$$

Note that  $\|fS\|_{A_\omega^p(\mathbb{T})} \leq \|f\|_{A_\omega^1(\mathbb{T})} \|S\|_{A_\omega^p(\mathbb{T})}$  for  $f \in A_\omega^1(\mathbb{T})$  and  $S \in A_\omega^p(\mathbb{T})$ . Hence we have the same result as (2.1) to characterize cyclicity in  $A_\omega^p(\mathbb{T})$  by norm.

When  $\omega_n = O((1 + |n|)^\epsilon)$  for all  $\epsilon > 0$ , for instance, by letting  $\omega_n = \ln(e + |n|)^\gamma$  where  $\gamma \geq 0$ , we can show the same result as Lemma 2.6. By noting that

$$A_\beta^p(\mathbb{T}) \subset A_\omega^p(\mathbb{T}) \subset A^p(\mathbb{T})$$

for all  $p \geq 1$  and  $\beta > 0$ , we obtain the following result by Theorem A and Theorem B :

**Corollary 5.1.** *Let  $\omega = (\omega_n)_{n \in \mathbb{Z}}$  be a weight such that  $\lim_{n \rightarrow +\infty} \frac{\log \omega_n}{\log n} = 0$ .*

- (1) Let  $1 < p < 2$ .
  - (a) If  $f \in A_\omega^1(\mathbb{T})$  and  $\dim(\mathcal{Z}(f)) < 2/q$  then  $f$  is cyclic in  $A_\omega^p(\mathbb{T})$ .
  - (b) For  $\frac{2}{q} < \alpha \leq 1$ , there exists a closed subset  $E \subset \mathbb{T}$  such that  $\dim(E) = \alpha$  and every  $f \in A_\omega^1(\mathbb{T})$  satisfying  $\mathcal{Z}(f) = E$  is not cyclic in  $A_\omega^p(\mathbb{T})$ .
  - (c) For all  $0 < \epsilon < 1$ , there exists a closed subset  $E \subset \mathbb{T}$  such that  $\dim(E) = 1 - \epsilon$  and every  $f \in A_\omega^1(\mathbb{T})$  satisfying  $\mathcal{Z}(f) = E$  is cyclic in  $A_\omega^p(\mathbb{T})$ .
- (2) Let  $p > 2$ . For every  $\epsilon > 0$ , there exists a closed subset  $E \subset \mathbb{T}$  such that  $|E| > 2\pi - \epsilon$  and every  $u \in A_\beta^1(\mathbb{Z})$  satisfying  $\mathcal{Z}(\widehat{u}) = E$  is cyclic in  $A_\beta^p(\mathbb{Z})$ .

*Proof.* (1) Suppose that  $1 < p < 2$ .

(a) Let  $f \in A_\omega^1(\mathbb{T})$  such that  $\dim(\mathcal{Z}(f)) < 2/q$ . Then there exists  $0 < \beta < 1/2$  such that  $\dim(\mathcal{Z}(f)) < \frac{2}{q}(1 - \beta q)$ . By Theorem 3.3 (1), every  $g \in A_\beta^1(\mathbb{T})$  satisfying  $\mathcal{Z}(g) = \mathcal{Z}(f)$  is cyclic in  $A_\beta^p(\mathbb{T})$ . Therefore, by Lemma 2.6, there exist a sequence of Lipschitz functions  $(f_n)$  which are zero on  $\mathcal{Z}(f)$  such that

$$\lim_{n \rightarrow \infty} \|f_n - 1\|_{A_\beta^p(\mathbb{T})} = 0.$$

Moreover,  $\omega_n = O((1 + |n|)^\beta)$ , therefore,

$$\lim_{n \rightarrow \infty} \|f_n - 1\|_{A_\omega^p(\mathbb{T})} = 0.$$

Again by Lemma 2.6 in  $A_\omega^p(\mathbb{T})$ , we obtain that  $f$  is cyclic in  $A_\omega^p(\mathbb{T})$ .

(b) By Theorem 2.9 there exists a closed set  $E \subset \mathbb{T}$  such that  $\dim(E) = \alpha$  and every  $f \in A^1(\mathbb{T})$  satisfying  $\mathcal{Z}(f) = E$  is not cyclic in  $A^p(\mathbb{T})$ . Let  $f \in A_\omega^1(\mathbb{T})$  such that  $\mathcal{Z}(f) = E$ . Since  $f \in A^1(\mathbb{T})$ ,  $f$  is not cyclic in  $A^p(\mathbb{T})$ . However,  $\|\cdot\|_{A^p(\mathbb{T})} \leq \|\cdot\|_{A_\omega^p(\mathbb{T})}$ , therefore  $f$  is not cyclic in  $A_\omega^p(\mathbb{T})$ .

(c) Let  $0 < \varepsilon < 1$  and  $\beta > 0$  such that  $1 - 2([q/2] + 1)\beta \geq 1 - \varepsilon$ . By Theorem 3.3 (4), there exists a closed set  $E \subset \mathbb{T}$  such that

$$\dim(E) \geq 1 - 2([q/2] + 1)\beta \geq 1 - \varepsilon$$

and every  $f \in A_\beta^1(\mathbb{T})$  satisfying  $\mathcal{Z}(f) = E$  is cyclic in  $A_\beta^p(\mathbb{T})$ . Since  $A_\beta^p(\mathbb{T}) \subset A_\omega^p(\mathbb{T})$ , by Lemma 2.6, by Lemma 2.6, we get our result.

(2) If  $p > 2$ , then the result immediately follows from Theorem B. □

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