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# The defocusing NLS equation : quasi-rational and rational solutions

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## Abstract

Quasi-rational solutions to the defocusing nonlinear Schrödinger equation (dNLS) in terms of wronskians and Fredholm determinants of order  $2N$  depending on  $2N - 2$  real parameters are given. We get families of quasi-rational solutions to the dNLS equation expressed as a quotient of two polynomials of degree  $N(N + 1)$  in the variables  $x$  and  $t$ .

We present also rational solutions as a quotient of determinants involving certain particular polynomials.

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**Keywords :** Fredholm determinants, defocusing NLS equation, wronskians.

## 1 Introduction

We consider the one dimensional defocusing nonlinear Schrödinger equation (dNLS) which can be written in the form

$$iv_t + v_{xx} - 2|v|^2v = 0. \quad (1)$$

Nakemura and Hirota presented solutions to this equation in terms of wronskians in 1985 [1] using bilinear method. They constructed rational solutions by using a connection with a Bäcklund transformation for the classical Boussinesq system (*BS*)

$$\begin{cases} u_t = ((1 + u)v + a^2v_{xx})_x \\ v_t = (u + \frac{1}{2}v^2)_x. \end{cases} \quad (2)$$

Hone [2] constructed rational solutions in terms of determinant by using Crum dressing method in 1997. In 1999, Barran and Kovalyov presented slowly oscillatory decaying solutions in terms of determinants [3].

Clarkson presented rational solutions and rational-oscillatory solutions expressed in terms of special polynomials associated with rational solutions of the fourth Painlevé equation in [4]. Lenells considered in 2015 solutions of the dNLS equation on the halfline [5] whose Dirichlet and Neumann boundary values become

periodic for sufficiently large  $t$ . In the same year, Prinari et al. [6] derived novel dark-bright soliton solutions with nonzero boundary conditions obtained within the framework of the inverse scattering transform.

Here we present solutions to the defocusing nonlinear Schrödinger equation (dNLS) of order  $N$  depending on  $2N - 2$  real parameters in terms of wronskians and Fredholm determinants. Families of quasi-rational solutions to the dNLS equation are obtained. These quasi-rational solutions can be expressed as a quotient of two polynomials of degree  $N(N + 1)$  in the variables  $x$  and  $t$ . We present also rational solutions as a quotient of determinants using certain particular polynomials.

## 2 Different representations of quasi-rational solutions to the dNLS equation

### 2.1 Quasi-rational solutions of the dNLS equation in terms of Fredholm determinant

We have to define the following notations.

The terms  $\kappa_\nu, \delta_\nu, \gamma_\nu$  and  $x_{r,\nu}$  are functions of the parameters  $\lambda_\nu, 1 \leq \nu \leq 2N$ ; they are defined by the formulas:

$$\begin{aligned} \kappa_\nu &= 2\sqrt{1 - \lambda_\nu^2}, & \delta_\nu &= \kappa_\nu \lambda_\nu, & \gamma_\nu &= \sqrt{\frac{1 - \lambda_\nu}{1 + \lambda_\nu}}, \\ x_{r,\nu} &= (r - 1) \ln \frac{\gamma_\nu - i}{\gamma_\nu + i}, & r &= 1, 3. \end{aligned} \quad (3)$$

The parameters  $-1 < \lambda_\nu < 1, \nu = 1, \dots, 2N$ , are real numbers such that

$$\begin{aligned} -1 < \lambda_{N+1} < \lambda_{N+2} < \dots < \lambda_{2N} < 0 < \lambda_N < \lambda_{N-1} < \dots < \lambda_1 < 1, \\ \lambda_{N+j} &= -\lambda_j, \quad j = 1, \dots, N. \end{aligned} \quad (4)$$

The condition (4) implies that

$$\kappa_{j+N} = \kappa_j, \quad \delta_{j+N} = -\delta_{j+N}, \quad \gamma_{j+N} = \gamma_j^{-1}, \quad x_{r,j+N} = x_{r,j}, \quad j = 1, \dots, N. \quad (5)$$

Complex numbers  $e_\nu, 1 \leq \nu \leq 2N$  are defined in the following way:

$$\begin{aligned} e_j &= i \sum_{l=1}^{N-1} a_l(j\epsilon)^{2l+1} - \sum_{l=1}^{N-1} b_l(j\epsilon)^{2l+1}, \\ e_{j+N} &= i \sum_{l=1}^{N-1} a_l(j\epsilon)^{2l+1} + \sum_{l=1}^{N-1} b_l(j\epsilon)^{2l+1}, \\ 1 &\leq j \leq N - 1. \end{aligned} \quad (6)$$

$\epsilon, a_\nu, b_\nu, \nu = 1 \dots 2N$  are arbitrary real numbers.

Let  $I$  be the unit matrix, and

$$\epsilon_j = j \quad 1 \leq j \leq N, \quad \epsilon_j = N + j, \quad N + 1 \leq j \leq 2N. \quad (7)$$

Let's consider the matrix  $D_r = (d_{jk}^{(r)})_{1 \leq j, k \leq 2N}$  defined by:

$$d_{\nu\mu}^{(r)} = (-1)^{\epsilon_\nu} \prod_{\eta \neq \mu} \left| \frac{\gamma_\eta + \gamma_\nu}{\gamma_\eta - \gamma_\mu} \right| \exp(i\kappa_\nu x - 2\delta_\nu t + x_{r,\nu} + e_\nu). \quad (8)$$

Using all the previous notations, the solution to the dNLS equation can be written as

**Theorem 2.1** *The function  $v$  defined by*

$$v(\tilde{x}, \tilde{t}) = \frac{\det(I + D_3(x, t))}{\det(I + D_1(x, t))} e^{2it - i\varphi} \Big|_{\{x=i\tilde{x}, t=-\tilde{t}\}}, \quad (9)$$

*is a solution to the defocusing dNLS equation depending on  $2N - 1$  real parameters  $a_j, b_j, \epsilon, 1 \leq j \leq N - 1$  with the matrix  $D_r = (d_{jk}^{(r)})_{1 \leq j, k \leq 2N}$  defined by*

$$d_{\nu\mu}^{(r)} = (-1)^{\epsilon_\nu} \prod_{\eta \neq \mu} \left| \frac{\gamma_\eta + \gamma_\nu}{\gamma_\eta - \gamma_\mu} \right| \exp(i\kappa_\nu x - 2\delta_\nu t + x_{r,\nu} + e_\nu).$$

where  $\kappa_\nu, \delta_\nu, x_{r,\nu}, \gamma_\nu, e_\nu$  being defined in(3), (4) and (6).

**Proof:** It is a consequence of the previous works of the author [12, 13, 14] with the change of variables defined by  $\{x = i\tilde{x}, t = -\tilde{t}\}$ .  $\square$

## 2.2 Wronskian representation

For this, we need to define the following notations :

$$\phi_{r,\nu} = \sin \Theta_{r,\nu}, \quad 1 \leq \nu \leq N, \quad \phi_{r,\nu} = \cos \Theta_{r,\nu}, \quad N + 1 \leq \nu \leq 2N, \quad r = 1, 3, \quad (10)$$

with the arguments

$$\Theta_{r,\nu} = \kappa_\nu x/2 + i\delta_\nu t - ix_{r,\nu}/2 + \gamma_\nu y - ie_\nu/2, \quad 1 \leq \nu \leq 2N. \quad (11)$$

The functions  $\phi_{r,\nu}$  are defined by

$$\phi_{r,\nu} = \sin \Theta_{r,\nu}, \quad 1 \leq \nu \leq N, \quad \phi_{r,\nu} = \cos \Theta_{r,\nu}, \quad N + 1 \leq \nu \leq 2N, \quad r = 1, 3. \quad (12)$$

We denote  $W_r(y)$  the wronskian of the functions  $\phi_{r,1}, \dots, \phi_{r,2N}$  defined by

$$W_r(y) = \det[(\partial_y^{\mu-1} \phi_{r,\nu})_{\nu, \mu \in [1, \dots, 2N]}]. \quad (13)$$

We consider the matrix  $D_r = (d_{\nu\mu})_{\nu, \mu \in [1, \dots, 2N]}$  defined in (8). Then we have the following statement [13]:

**Theorem 2.2**

$$\det(I + D_r) = k_r(0) \times W_r(\phi_{r,1}, \dots, \phi_{r,2N})(0), \quad (14)$$

where

$$k_r(y) = \frac{2^{2N} \exp(i \sum_{\nu=1}^{2N} \Theta_{r,\nu})}{\prod_{\nu=2}^{2N} \prod_{\mu=1}^{\nu-1} (\gamma_\nu - \gamma_\mu)}.$$

With these notations, we have the following result

**Theorem 2.3** *The function  $v$  defined by*

$$v(\tilde{x}, \tilde{t}) = \frac{W_3(\phi_{3,1}, \dots, \phi_{3,2N})(0)}{W_1(\phi_{1,1}, \dots, \phi_{1,2N})(0)} e^{2it - i\varphi} \Big|_{\{x=i\tilde{x}, t=-\tilde{t}\}}.$$

is a solution to the defocusing dNLS equation depending on  $2N - 1$  real parameters  $a_j, b_j, \epsilon, 1 \leq j \leq N - 1$  with  $\phi_\nu^r$  defined in (12)

$$\begin{aligned} \phi_{r,\nu} &= \sin(\kappa_\nu x/2 + i\delta_\nu t - ix_{r,\nu}/2 + \gamma_\nu y - ie_\nu/2), & 1 \leq \nu \leq N, \\ \phi_{r,\nu} &= \cos(\kappa_\nu x/2 + i\delta_\nu t - ix_{r,\nu}/2 + \gamma_\nu y - ie_\nu/2), & N + 1 \leq \nu \leq 2N, \quad r = 1, 3. \end{aligned}$$

$\kappa_\nu, \delta_\nu, x_{r,\nu}, \gamma_\nu, e_\nu$  being defined in(3), (4) and (6).

**Proof:** It is a consequence of [13] with the change of variables defined by  $\{x = i\tilde{x}, t = -\tilde{t}\}$ .  $\square$

We can give another representation of the solutions to the dNLS equation depending only on terms  $\gamma_\nu, 1 \leq \nu \leq 2N$ . From the relations (3), we can express the terms  $\kappa_\nu, \delta_\nu$  and  $x_{r,\nu}$  in function of  $\gamma_\nu$ , for  $1 \leq \nu \leq 2N$  and we obtain:

$$\begin{aligned} \kappa_j &= \frac{4\gamma_j}{(1 + \gamma_j^2)}, & \delta_j &= \frac{4\gamma_j(1 - \gamma_j^2)}{(1 + \gamma_j^2)^2}, & x_{r,j} &= (r - 1) \ln \frac{\gamma_j - i}{\gamma_j + i}, & 1 \leq j \leq N, \\ \kappa_j &= \frac{4\gamma_j}{(1 + \gamma_j^2)}, & \delta_j &= -\frac{4\gamma_j(1 - \gamma_j^2)}{(1 + \gamma_j^2)^2}, & x_{r,j} &= (r - 1) \ln \frac{\gamma_j + i}{\gamma_j - i}, & N + 1 \leq j \leq 2N. \end{aligned} \quad (15)$$

We have the following new representation

**Theorem 2.4** *The function  $v$  defined by*

$$v(\tilde{x}, \tilde{t}) = \frac{\det[(\partial_y^{\mu-1} \tilde{\phi}_{3,\nu}(0))_{\nu, \mu \in [1, \dots, 2N]}]}{\det[(\partial_y^{\mu-1} \tilde{\phi}_{1,\nu}(0))_{\nu, \mu \in [1, \dots, 2N]}]} e^{2it - i\varphi} \Big|_{\{x=i\tilde{x}, t=-\tilde{t}\}} \quad (16)$$

is a solution to the defocusing dNLS equation (1) depending on  $2N - 1$  real

parameters  $a_j, b_j, \epsilon, 1 \leq j \leq N-1$ . The functions  $\tilde{\phi}_{r,\nu}$  are defined by

$$\begin{aligned}\tilde{\phi}_{r,j}(y) &= \sin \left( \frac{2\gamma_j}{(1+\gamma_j^2)}x + i \frac{4\gamma_j(1-\gamma_j^2)}{(1+\gamma_j^2)^2}t - i \frac{(r-1)}{2} \ln \frac{\gamma_j-i}{\gamma_j+i} + \gamma_j y - i e_j \right), \\ \tilde{\phi}_{r,N+j}(y) &= \cos \left( \frac{2\gamma_j}{(1+\gamma_j^2)}x - i \frac{4\gamma_j(1-\gamma_j^2)}{(1+\gamma_j^2)^2}t + i \frac{(r-1)}{2} \ln \frac{\gamma_j-i}{\gamma_j+i} + \frac{1}{\gamma_j}y - i e_{N+j} \right),\end{aligned}\tag{17}$$

where  $\gamma_j = \sqrt{\frac{1-\lambda_j}{1+\lambda_j}}, 1 \leq j \leq N$ .

$\lambda_j$  is an arbitrary real parameter such that  $0 < \lambda_j < 1, \lambda_{N+j} = -\lambda_j, 1 \leq j \leq N$ .

The terms  $e_\nu$  are defined by (6), where  $a_j$  and  $b_j$  are arbitrary real numbers,  $1 \leq j \leq N-1$ .

**Proof:** We have to make the following change of variables defined by  $\{x = i\tilde{x}, t = -\tilde{t}\}$  in the previous works [13, 29, 30, 36].  $\square$

**Remark 2.1** In the formula (16), the determinants  $\det[(\partial_y^{\mu-1} f_\nu(0))_{\nu, \mu \in [1, \dots, 2N]}]$  are the wronskians of the functions  $f_1, \dots, f_{2N}$  evaluated in  $y = 0$ . In particular  $\partial_y^0 f_\nu$  means  $f_\nu$ .

### 2.3 Families of quasi-rational solutions of dNLS equation in terms of a quotient of two determinants

The following notations are used:

$$X_\nu = \kappa_\nu x/2 + i\delta_\nu t - ix_{3,\nu}/2 - ie_\nu/2,$$

$$Y_\nu = \kappa_\nu x/2 + i\delta_\nu t - ix_{1,\nu}/2 - ie_\nu/2,$$

for  $1 \leq \nu \leq 2N$ , with  $\kappa_\nu, \delta_\nu, x_{r,\nu}$  defined in (3).

Parameters  $e_\nu$  are defined by (6).

Below the following functions are used :

$$\begin{aligned}\varphi_{4j+1,k} &= \gamma_k^{4j-1} \sin X_k, & \varphi_{4j+2,k} &= \gamma_k^{4j} \cos X_k, \\ \varphi_{4j+3,k} &= -\gamma_k^{4j+1} \sin X_k, & \varphi_{4j+4,k} &= -\gamma_k^{4j+2} \cos X_k,\end{aligned}\tag{18}$$

for  $1 \leq k \leq N$ , and

$$\begin{aligned}\varphi_{4j+1,N+k} &= \gamma_k^{2N-4j-2} \cos X_{N+k}, & \varphi_{4j+2,N+k} &= -\gamma_k^{2N-4j-3} \sin X_{N+k}, \\ \varphi_{4j+3,N+k} &= -\gamma_k^{2N-4j-4} \cos X_{N+k}, & \varphi_{4j+4,N+k} &= \gamma_k^{2N-4j-5} \sin X_{N+k},\end{aligned}\tag{19}$$

for  $1 \leq k \leq N$ .

We define the functions  $\psi_{j,k}$  for  $1 \leq j \leq 2N, 1 \leq k \leq 2N$  in the same way, the term  $X_k$  is only replaced by  $Y_k$ .

$$\begin{aligned}\psi_{4j+1,k} &= \gamma_k^{4j-1} \sin Y_k, & \psi_{4j+2,k} &= \gamma_k^{4j} \cos Y_k, \\ \psi_{4j+3,k} &= -\gamma_k^{4j+1} \sin Y_k, & \psi_{4j+4,k} &= -\gamma_k^{4j+2} \cos Y_k,\end{aligned}\tag{20}$$

for  $1 \leq k \leq N$ , and

$$\begin{aligned}\psi_{4j+1,N+k} &= \gamma_k^{2N-4j-2} \cos Y_{N+k}, & \psi_{4j+2,N+k} &= -\gamma_k^{2N-4j-3} \sin Y_{N+k}, \\ \psi_{4j+3,N+k} &= -\gamma_k^{2N-4j-4} \cos Y_{N+k}, & \psi_{4j+4,N+k} &= \gamma_k^{2N-4j-5} \sin Y_{N+k},\end{aligned}\quad (21)$$

for  $1 \leq k \leq N$ .

Then we get the following result

**Theorem 2.5** *The function  $v$  defined by*

$$v(\tilde{x}, \tilde{t}) = \frac{\det((n_{jk})_{j,k \in [1,2N]})}{\det((d_{jk})_{j,k \in [1,2N]})} e^{2it - i\varphi} \Big|_{\{x=i\tilde{x}, t=-\tilde{t}\}} \quad (22)$$

is a quasi-rational solution of the defocusing dNLS equation (1) depending on  $2N - 2$  real parameters  $a_j, b_j, 1 \leq j \leq N - 1$ , where

$$\begin{aligned}n_{j1} &= \varphi_{j,1}(x, t, 0), \quad 1 \leq j \leq 2N & n_{jk} &= \frac{\partial^{2k-2} \varphi_{j,1}}{\partial \epsilon^{2k-2}}(x, t, 0), \\ n_{jN+1} &= \varphi_{j,N+1}(x, t, 0), \quad 1 \leq j \leq 2N & n_{jN+k} &= \frac{\partial^{2k-2} \varphi_{j,N+1}}{\partial \epsilon^{2k-2}}(x, t, 0), \\ d_{j1} &= \psi_{j,1}(x, t, 0), \quad 1 \leq j \leq 2N & d_{jk} &= \frac{\partial^{2k-2} \psi_{j,1}}{\partial \epsilon^{2k-2}}(x, t, 0), \\ d_{jN+1} &= \psi_{j,N+1}(x, t, 0), \quad 1 \leq j \leq 2N & d_{jN+k} &= \frac{\partial^{2k-2} \psi_{j,N+1}}{\partial \epsilon^{2k-2}}(x, t, 0), \\ & & & 2 \leq k \leq N, \quad 1 \leq j \leq 2N.\end{aligned}$$

The functions  $\varphi$  and  $\psi$  are defined in (18), (19), (20), (21).

**Proof:** It is also a consequence of the previous work [29] with the following change of variables defined by  $\{x = i\tilde{x}, t = -\tilde{t}\}$ .  $\square$

We don't give examples of solutions in terms of Fredholm determinants, wronskians or quasi-rational solutions because these types of solutions have been already explicitly constructed by the author until order 13 in the case of the focusing equation and it is easy to deduce these in the defocusing case. These results can be found from the previous published works. We do not give all the references; for the first orders in [18], until last orders (13) in [38].

### 3 Structure of the multi-parametric quasi-rational solutions to the dNLS equation

Here we present a result which states the structure of the quasi-rational solutions of the dNLS equation. In this section we use the notations defined in the previous sections. The functions  $\varphi$  and  $\psi$  are defined in (18), (19), (20), (21). The structure of the quasi rational solutions to the dNLS equation is given by the following theorem

**Theorem 3.1** *The function  $v$  defined by*

$$v(\tilde{x}, \tilde{t}) = \frac{\det((n_{jk})_{j,k \in [1, 2N]})}{\det((d_{jk})_{j,k \in [1, 2N]})} e^{2it - i\varphi} \{x = i\tilde{x}, t = -\tilde{t}\} \quad (23)$$

*is a quasi-rational solution of the defocusing dNLS equation (1) quotient of two polynomials of degrees  $N(N+1)$  in  $x$  and  $t$  depending on  $2N-2$  real parameters  $a_j$  and  $b_j$ ,  $1 \leq j \leq N-1$ .*

**Proof:** It is sufficient to realize the following change of variables defined by  $\{x = i\tilde{x}, t = -\tilde{t}\}$  in [30, 36].  $\square$

## 4 Rational solutions of order $k$ to the dNLS equation

### 4.1 Expression of the rational solutions of order $k$

We consider the polynomials  $p_n(x, t)$  defined by

$$\begin{cases} p_n(x, t) = \sum_{k=0}^n \frac{(-x)^k t^{\binom{n-k}{2}}}{k! \binom{n-k}{2}!} \left(1 - (n-k) + 2 \left\lfloor \frac{n-k}{2} \right\rfloor\right), & n \geq 0, \\ p_n(x, t) = 0, & n < 0, \end{cases} \quad (24)$$

where  $\lfloor x \rfloor$  is the greater integer less or equal to  $x$ .

We denote  $W_{n,k}(x, t)$  the following determinants

$$W_{n,k}(x, t) = \begin{vmatrix} p_n & p_{n-1} & \cdots & p_k \\ -p_{n-1} & -p_{n-2} & \cdots & -p_{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{n-k} p_k & (-1)^{n-k} p_{k-1} & \cdots & (-1)^{n-k} p_{2k-n} \end{vmatrix} \quad (25)$$

We define the function  $v_k$  by

$$v_k(x, t) = \frac{W_{2k+1,k}(x, t)}{W_{2k+1,k+1}(x, t)}.$$

We will call this function a function of order  $k$  and with these notations we have the following result

**Theorem 4.1** *The function  $v_k(x, t)$  defined by*

$$v_k(x, t) = \frac{W_{2k+1,k}(x, t)}{W_{2k+1,k+1}(x, t)} \quad (26)$$

*is a rational to the (dNLS) equation*

$$iv_t + v_{xx} - 2|v|^2 v = 0.$$



**Proof:** It is well known that  $v = \frac{G}{F}$ , where  $F$  and  $G$  are polynomials, is a solution to the dNLS equation if  $G$  and  $F$  verify the two following equations:

$$(iD_t + D_x^2)G \cdot F = 0 \quad (27)$$

$$D_x^2 F \cdot F + 2\bar{G}G = 0, \quad (28)$$

where  $D$  is the bilinear differential Hirota operator.

We have to verify (27) for  $G = W_{2k+1,k}(x, t)$  and  $F = W_{2k+1,k+1}(x, t)$ . We denote  $C_l$  and  $\tilde{C}_l$  the following columns :

$$C_l = \begin{pmatrix} p_l \\ -p_{l-1} \\ \vdots \\ (-1)^{k+1} p_{l-k-1} \end{pmatrix}, \quad \tilde{C}_l = \begin{pmatrix} p_l \\ -p_{l-1} \\ \vdots \\ (-1)^k p_{l-k} \end{pmatrix}. \quad (29)$$

With these notations,  $W_{2k+1,k}(x, t)$  and  $W_{2k+1,k+1}(x, t)$  can be written as

$$W_{2k+1,k}(x, t) = |C_{2k+1}, \dots, C_k| \quad \text{and} \quad W_{2k+1,k+1}(x, t) = |\tilde{C}_{2k+1}, \dots, \tilde{C}_{k+1}|.$$

We denote  $A$  the expression  $A = (iD_t + D_x^2)W_{2k+1,k}(x, t) \cdot W_{2k+1,k+1}(x, t)$ . We have to evaluate  $A$ .

The polynomials  $p_k$  verify  $\partial_x(p_k) = -p_{k-1}$  and  $\partial_t(p_k) = ip_{k-2}$ .

So  $A$  can be written as

$$\begin{aligned} A &= |C_{2k+1}, \dots, C_{k+2}, C_k, C_{k-1}| \times |\tilde{C}_{2k+1}, \dots, \tilde{C}_{k+1}| \\ &- |C_{2k+1}, \dots, C_{k+1}, C_{k-2}| \times |\tilde{C}_{2k+1}, \dots, \tilde{C}_{k+1}| \\ &- |C_{2k+1}, \dots, C_k| \times |\tilde{C}_{2k+1}, \dots, \tilde{C}_{k+3}, \tilde{C}_{k+1}, \tilde{C}_k| \\ &+ |C_{2k+1}, \dots, C_k| \times |\tilde{C}_{2k+1}, \dots, \tilde{C}_{k+2}, \tilde{C}_{k-1}| \\ &+ |C_{2k+1}, \dots, C_{k+2}, C_k, C_{k-1}| \times |\tilde{C}_{2k+1}, \dots, \tilde{C}_{k+1}| \\ &+ |C_{2k+1}, \dots, C_{k+1}, C_{k-2}| \times |\tilde{C}_{2k+1}, \dots, \tilde{C}_{k+1}| \\ &- 2|C_{2k+1}, \dots, C_{k+1}, C_{k-1}| \times |\tilde{C}_{2k+1}, \dots, \tilde{C}_{k+3}, \tilde{C}_k| \\ &+ |C_{2k+1}, \dots, C_k| \times |\tilde{C}_{2k+1}, \dots, \tilde{C}_{k+3}, \tilde{C}_{k+1}, \tilde{C}_k| \\ &+ |C_{2k+1}, \dots, C_k| \times |\tilde{C}_{2k+1}, \dots, \tilde{C}_{k+2}, \tilde{C}_{k-1}|. \end{aligned}$$

$A$  can be reduced to

$$\begin{aligned} A &= 2(|C_{2k+1}, \dots, C_{k+2}, C_k, C_{k-1}| \times |\tilde{C}_{2k+1}, \dots, \tilde{C}_{k+1}| \\ &+ |C_{2k+1}, \dots, C_k| \times |\tilde{C}_{2k+1}, \dots, \tilde{C}_{k+2}, \tilde{C}_{k-1}| \\ &- |C_{2k+1}, \dots, C_{k+1}, C_{k-1}| \times |\tilde{C}_{2k+1}, \dots, \tilde{C}_{k+2}, \tilde{C}_k|). \end{aligned}$$

$A$  can be rewritten as the following determinant of order  $2k + 3$

$$A = \begin{vmatrix} C_{2k+1} & \dots & C_{k+2} & C_{k+1} & C_k & 0 & \dots & 0 & -C_{k-1} \\ 0 & \dots & 0 & -C_{k+1} & -\tilde{C}_k & C_{2k+1} & \dots & C_{k+2} & C_{k-1} \end{vmatrix}. \quad (30)$$

We denote by  $\mathcal{L}$  the rows and by  $\mathcal{C}$  the columns of this determinant of order  $2k + 3$ .

We combine the lines of the previous determinant in the following way:

we replace  $\mathcal{L}_{k+2+j}$  by  $\mathcal{L}_{k+2+j} + \mathcal{L}_j$  for  $1 \leq j \leq k+1$ , then we obtain the following determinant

$$A = \begin{vmatrix} p_{2k+1} & p_{2k} & \cdots & p_{k+1} & p_k & 0 & \cdots & 0 & -p_{k-1} \\ -p_{2k} & -p_{2k-1} & \cdots & -p_k & -p_{k-1} & 0 & \cdots & 0 & p_{k-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (-1)^{k+1}p_k & (-1)^{k+1}p_{k-1} & \cdots & \cdots & 0 & 0 & \cdots & 0 & 0 \\ p_{2k+1} & p_{2k} & \cdots & 0 & 0 & p_{2k+1} & \cdots & p_{k+2} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (-1)^k p_{k+1} & (-1)^k p_k & \cdots & 0 & 0 & (-1)^k p_{k+1} & \cdots & (-1)^k p_2 & 0 \end{vmatrix}. \quad (31)$$

Then replacing  $C_j$  by  $C_j - C_{k+2+j}$  for  $1 \leq j \leq k+1$ , when we obtain the following determinant

$$A = \begin{vmatrix} p_{2k+1} & p_{2k} & \cdots & p_{k+1} & p_k & 0 & \cdots & 0 & -p_{k-1} \\ -p_{2k} & -p_{2k-1} & \cdots & -p_k & -p_{k-1} & 0 & \cdots & 0 & p_{k-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (-1)^{k+1}p_k & (-1)^{k+1}p_{k-1} & \cdots & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & p_{2k+1} & \cdots & p_{k+2} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & p_k & \cdots & p_2 & 0 \end{vmatrix}. \quad (32)$$

This last determinant is clearly equal to 0, which proves that:

$$A = (iD_t + D_x^2)W_{2k+1,k}(x,t) \cdot W_{2k+1,k+1}(x,t) = 0.$$

The relation (28) can be proven with the same type of arguments.

We give a sketch of the proof.

We denote  $B$  the expression  $B = D_x^2 F \cdot F + 2\overline{G}G$ . We have to evaluate  $B$ .

The polynomials  $p_k$  verify  $\partial_x(p_k) = -p_{k-1}$ .

So  $B$  can be written as

$$\begin{aligned} B &= 2(|\tilde{C}_{2k+1}, \dots, \tilde{C}_{k+3}, \tilde{C}_{k+1}, \tilde{C}_k| \times |\tilde{C}_{2k+1}, \dots, \tilde{C}_{k+1}| \\ &+ |\tilde{C}_{2k+1}, \dots, \tilde{C}_{k+2}, \tilde{C}_{k-1}| \times |\tilde{C}_{2k+1}, \dots, \tilde{C}_{k+1}| \\ &- |\tilde{C}_{2k+1}, \dots, \tilde{C}_{k+2}, \tilde{C}_k| \times |\tilde{C}_{2k+1}, \dots, \tilde{C}_{k+2}, \tilde{C}_k| \\ &+ |\tilde{C}_{2k+1}, \dots, \tilde{C}_k| \times |\overline{C}_{2k+1}, \dots, \overline{C}_k|). \end{aligned}$$

The determinant  $\overline{G} = |\overline{C}_{2k+1}, \dots, \overline{C}_k|$  is equal to  $|C_{2k+1}^*, \dots, C_{k+2}^*|$ , where  $C_l^*$  is defined by:

$$C_l^* = \begin{pmatrix} p_l \\ -p_{l-1} \\ \vdots \\ (-1)^{k-1}p_{1-k+1} \end{pmatrix}. \quad (33)$$

The product  $G \times \overline{G}$  can be written as  $G \times (G[k+1, k+1])[k+2, k+2]$ , where  $G[i, j]$  means that  $G[i, j]$  is obtained from  $G$  by deleting the row  $i$  and the

column  $j$ .

We denote  $\widehat{C}_l$

$$\widehat{C}_l = \begin{pmatrix} p_l \\ -p_{1-l} \\ \vdots \\ (-1)^{k-1} p_{1-k+1} \\ (-1)^{k+1} p_{1-k-1} \end{pmatrix}. \quad (34)$$

Using the Jacobi identity, we can write  $G \times \overline{G}$  as  $G \times (G[k+1, k+1])G[k+2, k+2] - G[k+1, k+2]G[k+2, k+1] = |\widehat{C}_{2k+1}, \dots, \widehat{C}_{k+2}, \widehat{C}_k| \times |\widetilde{C}_{2k+1}, \dots, \widetilde{C}_{k+1}| - |\widehat{C}_{2k+1}, \dots, \widehat{C}_{k+1}| \times |\widetilde{C}_{2k+1}, \dots, \widetilde{C}_{k+2}, \widetilde{C}_k|$ .

So,  $B$  can be rewritten as the sum

$$B = 2|\widetilde{C}_{2k+1}, \dots, \widetilde{C}_{k+1}| \times (|\widetilde{C}_{2k+1}, \dots, \widetilde{C}_{k+3}, \widetilde{C}_{k+1}, \widetilde{C}_k| + |\widetilde{C}_{2k+1}, \dots, \widetilde{C}_{k+3}, \widetilde{C}_{k+2}, \widetilde{C}_{k-1}| + |\widehat{C}_{2k+1}, \dots, \widehat{C}_{k+2}, \widehat{C}_k|) - 2|\widetilde{C}_{2k+1}, \dots, \widetilde{C}_{k+2}, \widetilde{C}_k| \times (|\widetilde{C}_{2k+1}, \dots, \widetilde{C}_{k+2}, \widetilde{C}_k| + |\widehat{C}_{2k+1}, \dots, \widehat{C}_{k+2}, \widehat{C}_{k+1}|).$$

But the sums  $|\widetilde{C}_{2k+1}, \dots, \widetilde{C}_{k+3}, \widetilde{C}_{k+1}, \widetilde{C}_k| + |\widetilde{C}_{2k+1}, \dots, \widetilde{C}_{k+3}, \widetilde{C}_{k+2}, \widetilde{C}_{k-1}| + |\widehat{C}_{2k+1}, \dots, \widehat{C}_{k+2}, \widehat{C}_k|$  and  $(|\widetilde{C}_{2k+1}, \dots, \widetilde{C}_{k+2}, \widetilde{C}_k| + |\widehat{C}_{2k+1}, \dots, \widehat{C}_{k+2}, \widehat{C}_{k+1}|)$  are equal to 0 which proves that  $B = 0$ .

Then we get the relation (28).

So we get the result.  $\square$

## 4.2 Some examples of rational solutions to the dNLS equation

In this section we will give some explicit examples of rational solutions to the dNLS equation. We recall that  $k$  means the order of the solution defined by

$$v_k(x, t) = \frac{W_{2k+1, k}(x, t)}{W_{2k+1, k+1}(x, t)}.$$

### 4.2.1 Rational solutions of order 1 to the dNLS equation

**Example 4.1** *The function  $v_k(x, t)$  defined by*

$$v_k(x, t) = -2 \frac{x(-x^2 + 6it)}{-x^4 + 12t^2} \quad (35)$$

*is a rational solution to the (dNLS) equation.*

### 4.2.2 Rational solutions of order 2 to the dNLS equation

**Example 4.2** *The function  $v_k(x, t)$  defined by*

$$v_k(x, t) = -3 \frac{-x^8 + 16ix^6t + 120x^4t^2 - 720t^4}{x(-x^8 + 72x^4t^2 + 2160t^4)} \quad (36)$$

*is a rational solution to the (dNLS) equation.*

### 4.2.3 Rational solutions of order 3 to the dNLS equation

**Example 4.3** The function  $v_k(x, t)$  defined by

$$v_k(x, t) = \frac{n(x, t)}{d(x, t)} \quad (37)$$

with

$$n(x, t) = 4(-x^{14} + 30itx^{12} + 540t^2x^{10} - 4200it^3x^8 - 10800t^4x^6 + 151200it^5x^4 - 504000t^6x^2 + 3024000it^7)x$$

and

$$d(x, t) = x^{16} - 240t^2x^{12} - 7200t^4x^8 - 2016000t^6x^4 + 6048000t^8$$

is a rational solution to the (dNLS) equation.

### 4.2.4 Rational solutions of order 4 to the dNLS equation

**Example 4.4** The function  $v_k(x, t)$  defined by

$$v_k(x, t) = \frac{n(x, t)}{d(x, t)} \quad (38)$$

with

$$n(x, t) = -5x^{24} + 240itx^{22} + 7560t^2x^{20} - 134400it^3x^{18} - 1436400t^4x^{16} + 12096000it^5x^{14} + 98784000t^6x^{12} + 677376000it^7x^{10} - 1905120000t^8x^8 + 71124480000it^9x^6 + 533433600000t^{10}x^4 - 1066867200000t^{12}$$

and

$$d(x, t) = (x^{24} - 600t^2x^{20} + 25200t^4x^{16} - 14112000t^6x^{12} - 4021920000t^8x^8 + 106686720000t^{10}x^4 + 1066867200000t^{12})x$$

is a rational solution to the (dNLS) equation.

### 4.2.5 Rational solutions of order 5 to the dNLS equation

**Example 4.5** The function  $v_k(x, t)$  defined by

$$v_k(x, t) = \frac{n(x, t)}{d(x, t)} \quad (39)$$

with

$$n(x, t) = -6(-x^{34} + 70itx^{32} + 3360t^2x^{30} - 100800ix^{28}t^3 - 2116800t^4x^{26} + 33022080it^5x^{24} + 423360000t^6x^{22} - 3217536000ix^{20}t^7 - 1778112000t^8x^{18} + 522764928000it^9x^{16} + 2782389657600t^{10}x^{14} + 39431411712000it^{11}x^{12} + 1552163751936000t^{12}x^{10} - 11435109396480000ix^8t^{13} - 14195308216320000t^{14}x^6 + 198734315028480000it^{15}x^4 - 248417893785600000t^{16}x^2 + 1490507362713600000it^{17})x$$

and

$$d(x, t) = -x^{36} + 1260t^2x^{32} - 302400t^4x^{28} + 76204800t^6x^{24} + 30939148800t^8x^{20} + 12943232870400t^{10}x^{16} - 162385722776000t^{12}x^{12} - 21292962324480000t^{14}x^8 - 2235761044070400000t^{16}x^4 + 2981014725427200000t^{18}$$

is a rational solution to the (dNLS) equation.

#### 4.2.6 Rational solutions of order 6 to the dNLS equation

**Example 4.6** The function  $v_k(x, t)$  defined by

$$v_k(x, t) = \frac{n(x, t)}{d(x, t)} \quad (40)$$

with

$$\begin{aligned} n(x, t) = & 7x^{48} - 672itx^{46} - 45360t^2x^{44} + 2016000it^3x^{42} + 67102560t^4x^{40} - \\ & 1717148160it^5x^{38} - 35611349760t^6x^{36} + 580375756800it^7x^{34} + 6847687123200t^8x^{32} - \\ & 82242658713600it^9x^{30} - 1292786998272000t^{10}x^{28} - 2839061643264000it^{11}x^{26} - \\ & 158869157787648000t^{12}x^{24} + 2004377520144384000it^{13}x^{22} - 104275895095443456000t^{14}x^{20} \\ & + 2511365792151896064000it^{15}x^{18} + 22959387883325988864000t^{16}x^{16} \\ & - 142130012484808212480000it^{17}x^{14} - 1281924569969568645120000t^{18}x^{12} \\ & - 3781319401921409187840000it^{19}x^{10} - 4253984327161585336320000t^{20}x^8 \\ & - 158815414880699185889280000it^{21}x^6 - 1191115611605243894169600000t^{22}x^4 \\ & + 1191115611605243894169600000t^{24} \end{aligned}$$

and

$$\begin{aligned} d(x, t) = & (-x^{48} + 2352t^2x^{44} - 1481760t^4x^{40} + 516499200t^6x^{36} + 79481606400t^8x^{32} + \\ & 125617211596800t^{10}x^{28} + 52451663859302400t^{12}x^{24} - 25764484412620800000t^{14}x^{20} + \\ & 427924663835074560000t^{16}x^{16} - 154800517473763983360000t^{18}x^{12} \\ & - 17488602233886517493760000t^{20}x^8 + 238223122321048778833920000t^{22}x^4 \\ & + 1191115611605243894169600000t^{24})x \end{aligned}$$

is a rational solution to the (dNLS) equation.

#### 4.2.7 Rational solutions of order 7 to the dNLS equation

**Example 4.7** The function  $v_k(x, t)$  defined by

$$v_k(x, t) = \frac{n(x, t)}{d(x, t)} \quad (41)$$

with

$$\begin{aligned} n(x, t) = & 8(-x^{62} + 126itx^{60} + 11340t^2x^{58} - 693000ix^{56}t^3 - 32916240t^4x^{54} + \\ & 1236422880it^5x^{52} + 38294182080t^6x^{50} - 981414403200it^7x^{48} - 20719094457600t^8x^{46} + \\ & 373342708569600it^9x^{44} + 6234317431372800t^{10}x^{42} - 78116020651468800ix^{40}t^{11} \\ & - 380937010696704000t^{12}x^{38} + 7441864983641088000it^{13}x^{36} + 234509627737921536000t^{14}x^{34} - \\ & 10491528929367822336000it^{15}x^{32} + 28872638199346765824000t^{16}x^{30} \\ & - 6740728931108306534400000it^{17}x^{28} - 169474893181970199183360000t^{18}x^{26} \\ & + 2400831552640985128304640000it^{19}x^{24} + 29802589444030637579304960000t^{20}x^{22} - \\ & 228707538154566157550223360000it^{21}x^{20} + 230292480111126109028352000000t^{22}x^{18} \\ & + 11716238554181895101585817600000it^{23}x^{16} + 91133252091673859628623462400000t^{24}x^{14} \\ & - 235200751536287015684171366400000it^{25}x^{12} + 13016193545913179761829058969600000t^{26}x^{10} \\ & - 93948903546617439225800294400000000it^{27}x^8 - 67643210553564556242576211968000000t^{28}x^6 \\ & + 947004947749903787396066967552000000it^{29}x^4 - 631336631833269191597377978368000000t^{30}x^2 \\ & + 3788019790999615149584267870208000000it^{31})x \end{aligned}$$

and

$$d(x, t) = x^{64} - 4032t^2x^{60} + 5201280t^4x^{56} - 3353011200t^6x^{52} + 613601049600t^8x^{48} -$$

$737286926745600 t^{10} x^{44} - 609647929867468800 t^{12} x^{40} - 164001698405056512000 t^{14} x^{36} +$   
 $436233208152604262400000 t^{16} x^{32} - 57117582890101985771520000 t^{18} x^{28}$   
 $+ 10315477913333460605337600000 t^{20} x^{24} + 1960527679738492460256460800000 t^{22} x^{20} +$   
 $327241269096776028738551808000000 t^{24} x^{16} - 24597531110387111360936804352000000 t^{26} x^{12} -$   
 $180381894809505483313536565248000000 t^{28} x^8 - 10101386109332307065558047653888000000 t^{30} x^4 +$   
 $7576039581999230299168535740416000000 t^{32}$   
*is a rational solution to the (dNLS) equation.*

#### 4.2.8 Rational solutions of order 8 to the dNLS equation

**Example 4.8** *The function  $v_k(x, t)$  defined by*

$$v_k(x, t) = \frac{n(x, t)}{d(x, t)} \quad (42)$$

*with*

$$\begin{aligned}
n(x, t) = & -9 x^{80} + 1440 itx^{78} + 166320 t^2 x^{76} - 13305600 it^3 x^{74} - 846568800 t^4 x^{72} + \\
& 43445445120 it^5 x^{70} + 1870141996800 t^6 x^{68} - 68163843686400 it^7 x^{66} - 2125554631200000 t^8 x^{64} + \\
& 57307521503232000 it^9 x^{62} + 1363297604917248000 t^{10} x^{60} - 28253298553159680000 it^{11} x^{58} - \\
& 484086300466728960000 t^{12} x^{56} + 7566415631881666560000 it^{13} x^{54} + 133152327700676444160000 t^{14} x^{52} - \\
& 2058963324486458277888000 it^{15} x^{50} - 22021666720077485260800000 t^{16} x^{48} \\
& - 1398578306925676894617600000 it^{17} x^{46} - 30664550434512031285248000000 t^{18} x^{44} - \\
& 437328109580302016210534400000 it^{19} x^{42} - 46579987446459613360163389440000 t^{20} x^{40} + \\
& 1425504528307712192388739891200000 it^{21} x^{38} + 31263949644881871172712634777600000 t^{22} x^{36} - \\
& 509018296855765937142651420672000000 it^{23} x^{34} - 4932515559735507065518869184512000000 t^{24} x^{32} + \\
& 39396796525811762450559075247718400000 it^{25} x^{30} \\
& + 684884899293436572778179103555584000000 t^{26} x^{28} \\
& - 2294663852713834896871972008886272000000 it^{27} x^{26} \\
& + 14778780673321768689451169942077440000000 t^{28} x^{24} \\
& - 947201085451035429276698502821314560000000 it^{29} x^{22} \\
& + 3321490984253367200445697603493953536000000 t^{30} x^{20} \\
& - 198711070178640618136097905352890122240000000 it^{31} x^{18} \\
& - 1648749560903000743664085988421633310720000000 t^{32} x^{16} \\
& + 747729465714665280479917863089932861440000000 it^{33} x^{14} \\
& + 69816498810133160472172135327273372876800000000 t^{34} x^{12} \\
& + 117025750386508916600974245881905844060160000000 it^{35} x^{10} \\
& + 268184011302416267210565980146034225971200000000 t^{36} x^8 \\
& + 2730600842351874720689399070577803028070400000000 it^{37} x^6 \\
& + 20479506317639060405170493029333522710528000000000 t^{38} x^4 \\
& - 12287703790583436243102295817600113626316800000000 t^{40}
\end{aligned}$$

*and*

$$\begin{aligned}
d(x, t) = & (x^{80} - 6480 t^2 x^{76} + 14968800 t^4 x^{72} - 17603308800 t^6 x^{68} + 10318053715200 t^8 x^{64} - \\
& 6006932976844800 t^{10} x^{60} - 2425558108925952000 t^{12} x^{56} - 3568118188245811200000 t^{14} x^{52} + \\
& 1771127741654469918720000 t^{16} x^{48} + 7598918248410742916382720000 t^{18} x^{44} - \\
& 3390923311730068095298437120000 t^{20} x^{40} + 916481140720063998978215116800000 t^{22} x^{36} + \\
& 106611960768624409466532003840000000 t^{24} x^{32} + 64248472376758454748787784024064000000 t^{26} x^{28} \\
& + 9122821692517058573907961319522304000000 t^{28} x^{24}
\end{aligned}$$

$- 3638429895511987315358195445179351040000000 t^{30} x^{20}$   
 $+ 32250817833265306075499427215808921600000000 t^{32} x^{16}$   
 $- 7927880817267868038964601447419320729600000000 t^{34} x^{12}$   
 $- 495734081498405827268015902694184478310400000000 t^{36} x^8$   
 $+ 4095901263527812081034098605866704542105600000000 t^{38} x^4$   
 $+ 12287703790583436243102295817600113626316800000000 t^{40} x$   
*is a rational solution to the (dNLS) equation.*

#### 4.2.9 Rational solutions of order 9 to the dNLS equation

**Example 4.9** *The function  $v_k(x, t)$  defined by*

$$v_k(x, t) = \frac{n(x, t)}{d(x, t)} \quad (43)$$

*with*

$$\begin{aligned}
n(x, t) = & -10(-x^{98} + 198 itx^{96} + 28512 t^2 x^{94} - 2882880 ix^{92} t^3 - 235414080 t^4 x^{90} + \\
& 15723227520 it^5 x^{88} + 892747215360 t^6 x^{86} - 43550538071040 it^7 x^{84} - 1851377348620800 t^8 x^{82} + \\
& 69126330181708800 it^9 x^{80} + 2290609252566835200 t^{10} x^{78} - 67587198729187737600 it^{11} x^{76} - \\
& 1769295226634333798400 t^{12} x^{74} + 41464108043171573760000 it^{13} x^{72} + 896769265742927216640000 t^{14} x^{70} - \\
& 17596890565184340393984000 it^{15} x^{68} - 294155011471335642980352000 t^{16} x^{66} + \\
& 3342933346282600754331648000 it^{17} x^{64} - 2450968429006146998108160000 t^{18} x^{62} - \\
& 2433287180223360105672867840000 it^{19} x^{60} - 130937651100257873779296829440000 t^{20} x^{58} + \\
& 3075516842912088157390236549120000 it^{21} x^{56} - 62774936194714235337761850654720000 t^{22} x^{54} + \\
& 6426139169515222305039698834227200000 it^{23} x^{52} \\
& + 271153781425549208557771420807987200000 t^{24} x^{50} \\
& - 7801408186146480945265464162071347200000 it^{25} x^{48} \\
& - 159344181851764568345772306837641625600000 t^{26} x^{46} \\
& + 2488514956058463917662084936610768486400000 it^{27} x^{44} \\
& + 3866255559865508062803692486882492416000000 t^{28} x^{42} \\
& - 503814497606994369286408442200694194176000000 it^{29} x^{40} \\
& - 2892799079812774780714673299868438495232000000 t^{30} x^{38} \\
& + 11449554864512161733828943380840391376896000000 it^{31} x^{36} \\
& - 196667249543515120711633358856754871402496000000 t^{32} x^{34} \\
& - 19007340765009740342410339347299858833735680000000 it^{33} x^{32} \\
& - 243813871772100040030824399553967097674465280000000 t^{34} x^{30} \\
& - 2039857021478025692575568254747153812066140160000000 it^{35} x^{28} \\
& - 92354753492736174581412317337491805743590932480000000 t^{36} x^{26} \\
& + 1178508838134387044632345359572365599835705835520000000 it^{37} x^{24} \\
& + 13978640714927931017608180739336927908177379328000000000 t^{38} x^{22} \\
& - 109578921093144586384883140031773893192623063040000000000 it^{39} x^{20} \\
& + 105103418022380273009575830246200385511960766054400000000 t^{40} x^{18} \\
& + 2409791909103468872738751319695802585131816635596800000000 it^{41} x^{16} \\
& + 18657374134884620337030400509550060264339394474803200000000 t^{42} x^{14} \\
& - 101005642479635415063356068952410491738385847156736000000000 it^{43} x^{12} \\
& + 1358944910391950715328384491014745524218195611746304000000000 t^{44} x^{10} \\
& - 9713302439679175155131997163624383620361280026574848000000000 it^{45} x^8
\end{aligned}$$

$-4541284257512341630970803868707504030298780272164864000000000 t^{46} x^6$   
 $+63577979605172782833591254161905056424182923810308096000000000 it^{47} x^4$   
 $-2649082483548865951399635590079377351007621825429504000000000 t^{48} x^2$   
 $+158944949012931957083978135404762641060457309525770240000000000 it^{49})x$   
*and*  
 $d(x, t) = -x^{100} + 9900 t^2 x^{96} - 37540800 t^4 x^{92} + 75243168000 t^6 x^{88} - 86477751360000 t^8 x^{84} +$   
 $69030822212352000 t^{10} x^{80} - 18841861512714240000 t^{12} x^{76} + 32133190371945062400000 t^{14} x^{72} +$   
 $8001282884188898304000000 t^{16} x^{68} - 71696063588052183920640000000 t^{18} x^{64} -$   
 $122350825624182265506299904000000 t^{20} x^{60} + 163239507932764545783559618560000000 t^{22} x^{56} -$   
 $88634453383495458565418231267328000000 t^{24} x^{52}$   
 $+ 5976929116686443410537623060480000000000 t^{26} x^{48}$   
 $- 9037618519244139622561816267613798400000000 t^{28} x^{44}$   
 $- 2978595180594090148758587450945867612160000000 t^{30} x^{40}$   
 $+ 182653836076595912222107290719280011673600000000 t^{32} x^{36}$   
 $+ 732385622391101433187551102767316006548275200000000 t^{34} x^{32}$   
 $- 59727919899994725237823441224569207950147584000000000 t^{36} x^{28}$   
 $+ 774336147693216625072155002615624980513161216000000000 t^{38} x^{24}$   
 $+ 872923144660058559307277530405151995358292738048000000000 t^{40} x^{20}$   
 $+ 81506554657420499600797701201590344042893237288960000000000 t^{42} x^{16}$   
 $- 4042431062558334406356586777069179723940202136207360000000000 t^{44} x^{12}$   
 $- 1892201773963475679571168278628126679291158446735360000000000 t^{46} x^8$   
 $- 66227062088721648784990889751984433775190545635737600000000000 t^{48} x^4$   
 $+ 317889898025863914167956270809525282120914619051540480000000000 t^{50}$   
*is a rational solution to the (dNLS) equation.*

We could go on and present more explicit rational solutions, but they become very complicated. For example, in the case of order 10 the numerator includes 60 terms and the denominator 31 terms with big coefficients. It will be relevant to study in detail the structure of these solutions.

## 5 Conclusion

Different representations of quasi-rational solutions to the defocusing nonlinear Schrödinger equation have been given. First quasi rational solutions in terms of wronskians of order  $2N$  depending on  $2N - 2$  real parameters have been presented. Another representation in terms of Fredholm determinants are given depending on  $2N - 2$  real parameters. These solutions give families of quasi-rational solutions to the dNLS equation expressed as a quotient of two polynomials of degree  $N(N + 1)$  in the variables  $x$  and  $t$  depending on  $2N - 2$  real parameters.

Rational solutions as a quotient of determinants have been also given using certain particular polynomials and some explicit expressions are given for some orders. It will be relevant to study the structure of these last solutions.



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