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Marc Cabanes, Britta Späth. On semisimple classes and component groups in type D. 2022. hal-03789899

HAL Id: hal-03789899

https://hal.science/hal-03789899v1

Preprint submitted on 27 Sep 2022

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On semisimple classes and component groups in type D

Dedicated to Pham Huu Tiep on the occasion of his 60th birthday

Marc Cabanes Britta Späth *

September 27, 2022

Abstract

In adjoint reductive groups $\mathbf{H}$ of type D we show that for every semisimple element s, its centralizer splits over its connected component, i.e. $\mathrm{C}_{\mathbf{H}}(s) = \mathrm{C}_{\mathbf{H}}(s)^\circ \rtimes \breve{A}$ for some complement $\breve{A}$ with strong stability properties. We derive several consequences about the action of automorphisms on semisimple conjugacy classes. This helps to parametrize characters of the finite groups $\mathsf{D}_{l,\mathrm{sc}}(q)$ and ${}^2\mathsf{D}_{l,\mathrm{sc}}(q)$ and describe the action of automorphisms on them. It is also a contribution to the final proof of the McKay conjecture for the prime 3, see [S22].

Introduction

One of the many outstanding ideas in Deligne-Lusztig theory was to point out the relevance of a dual group $\mathbf{G}^*$ to the study of representations of a finite group of Lie type $G = \mathbf{G}^F$ (see for instance [DM, Ch. 11, 12]). In particular the set $\mathrm{Irr}(G)$ of irreducible (complex) characters is partitioned according to semisimple elements of $G^* = \mathbf{G}^{*F}$ up to G^*-conjugacy ("rational series") or $\mathbf{G}^*$-conjugacy ("geometric series"). This explains how the study of $\mathrm{Irr}(G)$, for instance as an $\mathrm{Out}(G)$-set, may lead to purely group-theoretic questions about G^* and its semisimple classes.

Together with Lusztig's Jordan decomposition of characters, rational series allow to study characters with the help of unipotent characters of the centralizer of semisimple elements $\mathrm{C}_{G^*}(s)$. It is a well known fact that the rational classes in a given geometric class are parametrized via the component group $A(s) := \mathrm{C}_{\mathbf{G}^*}(s)/\mathrm{C}^\circ_{\mathbf{G}^*}(s)$ of the algebraic centralizer $\mathrm{C}_{\mathbf{G}^*}(s)$. The various lifts in $\mathrm{C}_{\mathbf{G}^*}(s)$ of a fixed element of $A(s)$ provide different

*Some of this research was conducted in the framework of the research training group *GRK 2240: Algebro-Geometric Methods in Algebra, Arithmetic and Topology*, funded by the DFG. The authors would like to thank the Isaac Newton Institute for Mathematical Sciences for support during the programme *Groups, Representations and Applications*, when work on this paper was undertaken. The Institute is supported by EPSRC Grant Number EP/R014604/1.

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Mathematics Subject Classification (2010): 20C20 (20C33 20C34)

rational structures on the centralizer $\mathrm{C}_{\mathbf{G}^*}(s)$ and different finite groups whose unipotent characters are then in correspondence with the characters of a given rational series. The results presented here are applied in [S22] to establish a Jordan decomposition compatible with automorphisms.

The study of $\mathrm{Irr}(\mathbf{G}^F)$ is probably most difficult when $\mathbf{G}$ is a simply connected simple group and its type is D. The difficulties come from the graph automorphism and a possibly non-cyclic fundamental group. However the dual group $\mathbf{G}^*$ in that type, denoted here as $\mathbf{H}$, has a couple of remarkable features with regard to semisimple classes and their centralizers. In particular we single out, for s a semisimple element of $\mathbf{H}$, a semidirect decomposition

$$\mathrm{C}_{\mathbf{H}}(s) = \mathrm{C}^\circ_{\mathbf{H}}(s) \rtimes \check{A}$$

where the supplement $\check{A}$ has a strong stability with regard to Frobenius endomorphisms and graph automorphims. In [S22], the elements of $\check{A}$ are crucial to parametrize the different rational structures on $\mathrm{C}_{\mathbf{G}^*}(s)$ and the action of $\mathrm{Out}(\mathbf{G}^F)$ on rational series of characters. This then also contributes to the verification of McKay's conjecture for the prime 3 in [S22].

In type different from D our questions in representation theory have been solved by simpler methods not requiring this study of adjoint groups. It could be interesting however to prove results similar to Theorem A below in other types, at least classical. See [E08, 3.2.1] for a property a bit weaker than the semidirect product structure. Theorem A also verifies that Condition (*) introduced in [DM21, Def. 2] is satisfied for the groups $\mathrm{C}_{\mathbf{G}^*}(s)$ and all semisimple elements s, whenever $\mathbf{G}$ is of type D, thus establishing the first (minor) part of [DM21, Conjecture 7].

1 Notations and main theorems

Let $\mathbf{H}_0$ be a reductive group over the algebraically closed field $\mathbb{F}$ of positive characteristic p. Assume $\mathbf{H}_0$ is simple simply connected and let $\mathbf{H} := \mathbf{H}_0/\mathrm{Z}(\mathbf{H}_0)$ be its adjoint group with the surjective map $\pi\colon \mathbf{H}_0 \to \mathbf{H}$. Let $\mathbf{B}_0 \geqslant \mathbf{T}_0$ some Borel subgroup and maximal torus of $\mathbf{H}_0$. Let us denote by $X(\mathbf{T}_0) \supseteq \Phi \supseteq \Delta$ the associated roots and basis of the root system. Let $x_\alpha\colon \mathbb{F} \to \mathbf{H}_0$ be the unipotent 1-parameter subgroup associated with $\alpha \in \Phi$. Let $F_p\colon \mathbf{H}_0 \to \mathbf{H}_0$ be defined by $F_p(x_\alpha(t)) = x_\alpha(t^p)$ for any $\alpha \in \Phi$ and $t \in \mathbb{F}$. Let Γ be the group of algebraic automorphisms of $\mathbf{H}_0$ that satisfy $x_{\epsilon\delta}(t) \mapsto x_{\epsilon\delta'}(t)$ for any $t \in \mathbb{F}$, $\epsilon \in \{1, -1\}$, $\delta \in \Delta$ and where $\delta \mapsto \delta'$ is an automorphism of the Dynkin diagram. Note that Γ is cyclic of order 2 whenever $\mathbf{H}$ is of type A_l ($l \geqslant 2$), D_l ($l \geqslant 5$) or E_6, dihedral of order 6 for type D_4 and trivial for other types. Let $E := \langle F_p \rangle \Gamma$ be the group of abstract automorphisms of $\mathbf{H}_0$ generated by Γ and F_p. It also clearly acts on $\mathbf{H}$ and we can form the semidirect product $\mathbf{H} \rtimes E$.

Let s be a semisimple element of $\mathbf{H}$, then $\mathrm{C}_{\mathbf{H}}(s)$ is reductive but may not be connected. The component group $A_{\mathbf{H}}(s) := \mathrm{C}_{\mathbf{H}}(s)/\mathrm{C}^\circ_{\mathbf{H}}(s)$ is abelian and can be seen as a subgroup of $Z(\mathbf{H}_0)$ via the map

$$\omega_s\colon \mathrm{C}_{\mathbf{H}}(s) \to \mathrm{Z}(\mathbf{H}_0) \text{ defined by } g \mapsto [\dot{g}, \dot{s}] \text{ for } \dot{g} \in \pi^{-1}(g),\ \dot{s} \in \pi^{-1}(s)$$

(see [B06, 8.2]). In this paper we study the question of finding subgroups $\check{A} \leqslant \mathrm{C}_{\mathbf{H}}(s)$ such that $\mathrm{C}_{\mathbf{H}}(s) = \mathrm{C}^\circ_{\mathbf{H}}(s)\check{A}$ and the decomposition has a strong compatibility with the action of E.

In type D, we denote by $\underline{E} = \langle F_p, \gamma \rangle$ the group generated by F_p and some $\gamma \in \Gamma$ of order 2 (essentially in the case of type D_4 a choice is made and the graph automorphism of order 3 is left out).

Note that $\underline{E}$ acts on both $\mathbf{H}_0$ and $\mathbf{H}$. For $\sigma \in \mathbf{H}\underline{E}\backslash\mathbf{H}$ and $h \in \mathbf{H}$ we write $\sigma(h)$ for $\sigma h \sigma^{-1} \in \mathbf{H}$. We also define the submonoid

$$\underline{E}^+ := \{F_p^i \circ \gamma^j \mid i \geqslant 1,\ j \in \{0,1\}\}$$

whose elements can be considered as Frobenius endomorphims $F\colon \mathbf{H} \to \mathbf{H}$ with fixed points forming the finite group $\mathbf{H}^F := \{h \in \mathbf{H} \mid F(h) = h\}$. For $x \in \mathbf{H}^F$ ($F \in \underline{E}^+$) we denote by $[x]_{\mathbf{H}^F}$ its conjugacy class in $\mathbf{H}^F$.

In the following theorems, $\mathbf{H} = \mathsf{D}_{l,\mathrm{ad}}(\mathbb{F})$ and $\mathbf{H}_0 = \mathsf{D}_{l,\mathrm{sc}}(\mathbb{F})$ are groups of rank $l \geqslant 4$ over a field $\mathbb{F}$ of characteristic $p \neq 2$.

Theorem A *Let $F_0 \in \underline{E}^+$, $s \in \mathbf{H}^{F_0}$ a semisimple element. Then there is a semidirect product decomposition*

$$\mathrm{C}_{\mathbf{H}}(s) = \mathrm{C}^\circ_{\mathbf{H}}(s) \rtimes \breve{A}$$

with both sides F_0-stable.

Moreover, if $\sigma' \in \underline{E}$, $F := F_0^k$ for some $k \geqslant 1$ and $\sigma'(s) \in [s]_{\mathbf{H}^F}$, then there exists $\sigma \in \mathbf{H}^F\sigma' \cap \mathrm{C}_{\mathbf{H}\underline{E}}(s)$ such that $\sigma(\breve{A}) = \breve{A}$ and $[F_0, \sigma] \in \breve{A}$.

The considerations to prove the following are more classical, using only the covering $\pi\colon \mathbf{H}_0 \to \mathbf{H}$.

Theorem B *Let $\mathcal{C}$ be the $\mathbf{H}$-conjugacy class of a semisimple element $t \in \mathbf{H}$ such that $|A_{\mathbf{H}}(t)| = 4$. Let $F_0 \in \underline{E}^+$ and assume $F_0(\mathcal{C}) = \gamma(\mathcal{C}) = \mathcal{C}$. Then $\pi(\mathbf{H}_0^{F_0}) \cap \mathcal{C} \neq \varnothing$ and there exists $s \in \pi(\mathbf{H}_0^{F_0}) \cap \mathcal{C}$ such that $\gamma(s) \in [s]_{\mathbf{H}^{F_0}}$.*

More technical corollaries will be given in our last section.

2 Lifting of component groups and automorphisms

We continue with the same notations about $\mathbf{H}$, an adjoint group of type D_l ($l \geqslant 4$) over an algebraically closed field $\mathbb{F}$ of odd characteristic, its maximal torus $\mathbf{T} = \pi(\mathbf{T}_0)$ and the abstract automorphism group $\underline{E}$.

Proposition 2.1 *There exists an $\underline{E}$-stable subgroup $\breve{W} \leqslant \mathbf{H}$ such that $\mathrm{N}_{\mathbf{H}}(\mathbf{T}) = \mathbf{T} \rtimes \breve{W}$.*

Proof. The statement is probably known to experts familiar with adjoint groups (see [AH17] and its references). We recall an elementary construction of $\breve{W}$ and emphasize the stability property. For $d \geqslant 1$ let $J_d \in \mathrm{GL}_d(\mathbb{F})$ be the permutation matrix corresponding to the product of transpositions $(1,d)(2,d-1)\cdots \in \mathcal{S}_d$.

Let $\mathbf{G} := \mathrm{SO}_{2l}(\mathbb{F}) = \mathrm{O}_{2l}(\mathbb{F}) \cap \mathrm{SL}_{2l}(\mathbb{F})$, where $\mathrm{O}_{2l}(\mathbb{F}) := \{x \in \mathrm{GL}_{2l}(\mathbb{F}) \mid {}^t x J_{2l} x = J_{2l}\}$. Then $\mathbf{G}$ is connected simple of type D_l (see for instance [DM, 1.5.5]) with center $\{\mathrm{Id}_{2l}, -\mathrm{Id}_{2l}\}$ and $\mathbf{H} = \mathbf{G}/\{\mathrm{Id}_{2l}, -\mathrm{Id}_{2l}\}$. The maximal torus of $\mathbf{G}$ is the group $\overline{\mathbf{T}}$ of diagonal matrices with diagonal $(t_1, \ldots, t_l, t_l^{-1}, \ldots, t_1^{-1})$ with $t_i \in \mathbb{F}^\times$. Numbering the basis in dimension $2l$ as $(1, 2, \ldots, l, -l, -l+1, \ldots, -2, -1)$ the normalizer of $\mathbf{T}$ in $\mathrm{GL}_{2l}(\mathbb{F})$

is generated by permutation matrices corresponding to $(l,-l)$ and the various involutions $(i,j)(-i,-j)$ $(1 \leqslant i,j \leqslant l)$. All those permutation matrices clearly belong to $\mathrm{O}_{2l}(\mathbb{F})$ and generate a subgroup $\overline{V} \leqslant \mathrm{O}_{2l}(\mathbb{F})$ with $\overline{V} \cong \mathbb{Z}/2\mathbb{Z} \wr \mathcal{S}_l$. The intersection $\overline{W} := \overline{V} \cap \mathrm{SL}_{2l}(\mathbb{F})$ corresponds to elements of the wreath product whose number of non-trivial coordinates in the base group is even, thus forming a Coxeter group of type D_l.

In $\mathbf{G} = \mathrm{SO}_{2l}(\mathbb{F})$ we indeed get $\mathrm{N}_{\mathbf{G}}(\overline{\mathbf{T}}) = \overline{\mathbf{T}} \rtimes \overline{W}$. The elements of $\overline{V}$ are clearly fixed by F_p which here is just the raising of matrix entries to the p-th power. On the other hand, conjugation by the permutation matrix associated to $(l,-l)$ induces the graph automorphism of $\mathbf{G}$ of order 2 associated to the upper triangular Borel subgroup containing $\overline{\mathbf{T}}$ and swapping the fundamental reflections corresponding to $(l-1,l)(-l+1,-l)$ and $(l-1,-l)(-l+1,l)$ (see [DM, 4.3.6]). This too preserves $\overline{W}$ since the permutation matrix associated to $(l,-l)$ belongs to $\overline{V}$. We get our Proposition by taking for $\breve{W}$ the image of $\overline{W}$ in $\mathbf{H} = \mathbf{G}/\{\mathrm{Id}_{2l}, -\mathrm{Id}_{2l}\}$. □

Remark 2.2. In type B_l the same statement is proved similarly in $\mathbf{H} = \mathrm{SO}_{2l+1}(\mathbb{F})$ seen as $\mathrm{O}_{2l+1}(\mathbb{F})/\{\mathrm{Id}_{2l+1}, -\mathrm{Id}_{2l+1}\}$. In type A_{l-1} with $\mathbf{H} = \mathrm{GL}_l(\mathbb{F})/\{\lambda\,\mathrm{Id}_l \mid \lambda \in \mathbb{F}^\times\}$, the permutation matrices also provide a supplement whose elements are fixed under F_p. Note that the non-trivial element of Γ sends our group of permutation matrices to a distinct $\mathrm{N}_{\mathbf{H}}(\mathbf{T})$-conjugate in odd characteristic. Note also that permutation matrices have a somewhat cumbersome expression in terms of Chevalley generators. For instance the permutation matrix for the transposition $(1,2)$ has a class mod $\mathrm{Z}(\mathrm{GL}_l(\mathbb{F}))$ corresponding to $n_{\alpha_1}(1)h_{\alpha_1}(-\omega)h_{\alpha_2}(-\omega^2)\dots h_{\alpha_{l-1}}(-\omega^{l-1}) \in \mathrm{SL}_l(\mathbb{F})$ for some ω with $\omega^l = -1$ in the notations of [GLS, 1.12.1], a similarly complex formula being necessary in type D.

In type C_l, or exceptional types $\neq \mathsf{G}_2$ it is known that $\mathbf{T}$ has no supplement in $\mathrm{N}_{\mathbf{H}}(\mathbf{T})$ (see [AH17, 4.11]).

We now consider the centralizer of some $x \in \mathbf{T}$, where $\mathbf{T}$ is as before a maximal torus of an adjoint group that is split for any Frobenius endomorphism belonging to $\underline{E}^+$.

Recall that for any $x \in \mathbf{T}$ the connected centralizer $\mathrm{C}^\circ_{\mathbf{H}}(x)$ is a reductive group containing $\mathbf{T}$ as maximal torus (see for instance [C, 3.5.4]).

Proposition 2.3 *Let $\mathbf{T}$ and $\breve{W}$ be as in Proposition 2.1, noting that $\breve{W}$ acts on $\mathbf{T}$ hence naturally on the set Φ of $\mathbf{T}$-roots of $\mathbf{H}$. Let $x \in \mathbf{T}$, $\Phi_x \subseteq X(\mathbf{T})$ the root system of $\mathrm{C}^\circ_{\mathbf{H}}(x)$ with respect to $\mathbf{T}$ and Δ_x a basis of Φ_x. We set $\breve{B} := \mathrm{C}_{\breve{W}}(x)_{\Delta_x} = \{w \in \breve{W} \mid {}^w x = x,\ w(\Delta_x) = \Delta_x\}$ and $\breve{B}_{\underline{E}} := \mathrm{C}_{\breve{W}\rtimes\underline{E}}(x)_{\Delta_x}$ similarly defined. Then*

(a) $\mathrm{C}_{\mathbf{H}}(x) = \mathrm{C}^\circ_{\mathbf{H}}(x) \rtimes \breve{B}$, $\mathrm{C}_{\mathbf{H}\underline{E}}(x) = \mathrm{C}^\circ_{\mathbf{H}}(x) \rtimes \breve{B}_{\underline{E}}$,

(b) $\breve{B} \lhd \breve{B}_{\underline{E}}$ *with abelian quotient.*

Proof. Note first that $\underline{E}$ stabilizes $\mathbf{T}$ and sends maximal tori of $\mathbf{H}$ to maximal tori since $\sigma({}^g\mathbf{T}) = {}^{\sigma(g)}\mathbf{T}$ for any $g \in \mathbf{H}$ and $\sigma \in \underline{E}$. So $\mathbf{H}\underline{E}$ permutes the maximal tori of $\mathbf{H}$. The connected centralizer $\mathbf{C} := \mathrm{C}^\circ_{\mathbf{H}}(x)$ is the reductive group containing $\mathbf{T}$ with root system Φ_x, so

$$\begin{aligned}
\mathrm{N}_{\mathbf{H}\underline{E}}(\mathbf{C}) &= \mathbf{C}.\mathrm{N}_{\mathbf{H}\underline{E}}(\mathbf{T},\mathbf{C}) && \text{by conjugacy of maximal tori in } \mathbf{C}\\
&= \mathbf{C}.\mathrm{N}_{\breve{W}\underline{E}}(\mathbf{T},\mathbf{C}) && \text{by Proposition 2.1}\\
&= \mathbf{C} \rtimes (\breve{W}\underline{E})_{\Delta_x} && \text{by the regular action of } \mathrm{N}_{\mathbf{C}}(\mathbf{T})/\mathbf{T} \text{ on the bases of } \Phi_x.
\end{aligned}$$

On the other hand $\mathbf{C}$ is obviously normal in $\mathrm{C}_{\mathbf{H}\underline{E}}(x)$, so $\mathrm{C}_{\mathbf{H}\underline{E}}(x)$ is the centralizer of x in the above $\mathbf{C} \rtimes (\breve{W}\underline{E})_{\Delta_x}$. This gives the second claim of (a) with $\breve{B}_{\underline{E}} := (\breve{W}\underline{E})_{\Delta_x}$. The first is clear by taking the intersection with $\mathbf{H}$, which also gives (b). The quotient $\breve{B}_{\underline{E}}/\breve{B}$ is abelian since it injects into $\underline{E}$. □

Remark 2.4. Using the simply connected covering $\mathbf{H}_0$ we can embed $\mathrm{C}_{\mathbf{H}\underline{E}}(x)/\mathrm{C}^\circ_{\mathbf{H}}(x)$, or equivalently $\breve{B}_{\underline{E}}$, into $\mathbb{Z} \times V \times \mathbb{Z}/2\mathbb{Z}$ where V is the dihedral group of order 8. It could be interesting to classify those groups $\mathrm{C}_{\mathbf{H}\underline{E}}(x)/\mathrm{C}^\circ_{\mathbf{H}}(x)$.

Our applications to centralizers of semisimple elements and semisimple conjugacy classes will stem from the omnibus lemma below. Note that it deals with the action of three elements of $\underline{E}$.

Lemma 2.5 *Let $F_0 \in \underline{E}^+$. Let $s \in \mathbf{H}^{F_0}$ be a semisimple element and let $x \in \mathbf{T}$ be an $\mathbf{H}$-conjugate of s. Let $\mathrm{C}_{\mathbf{H}\underline{E}}(x) = \mathrm{C}^\circ_{\mathbf{H}}(x) \rtimes \breve{B}_{\underline{E}}$ the decomposition of Proposition 2.3. Then there is an inner automorphism $\iota\colon \mathbf{H}\underline{E} \to \mathbf{H}\underline{E}$ induced by an element of $\mathbf{H}$ such that*

(a) $\iota(s) = x$,
(b) $\iota(F_0) \in \breve{B}_{\underline{E}}$ and
(c) if $F = F_0^i$ for some $i \geqslant 1$, $\tau \in \underline{E}$ and $h \in \mathbf{H}^F$ are such that $\tau(s) = s^h$, then

$$\iota(h\tau) \in \mathrm{C}^\circ_{\mathbf{H}}(x)\mathrm{C}_{\breve{B}_{\underline{E}}}(\iota(F)).$$

Proof. We denote $x := s^g$ for some $g \in \mathbf{H}$ and $\iota' : \mathbf{H}\underline{E} \longrightarrow \mathbf{H}\underline{E}$ given by $y \mapsto y^g$. Note that ι' preserves cosets $\mathbf{H}e$ ($e \in \mathbf{H}\underline{E}$). We have $\iota'(F_0) \in \mathbf{H}F_0 \cap \mathrm{C}_{\mathbf{H}\underline{E}}(x)$. Recalling $\breve{B}_{\underline{E}}$ the group associated to x from Proposition 2.3 and the decomposition $\mathrm{C}_{\mathbf{H}\underline{E}}(x) = \mathrm{C}^\circ_{\mathbf{H}}(x)\breve{B}_{\underline{E}}$ we can write $\iota'(F_0)$ as cb with $c \in \mathrm{C}^\circ_{\mathbf{H}}(x)$ and $b \in \breve{B}_{\underline{E}}$. Lang's theorem (see for instance [G, 4.1.2]) applied to $\mathrm{C}^\circ_{\mathbf{H}}(x)$ and $\iota'(F_0)$ allows to write $c^{-1} = c'^{-1}\iota'(F_0)c'\iota'(F_0)^{-1}$ in $\mathbf{H}\underline{E}$ for some $c' \in \mathrm{C}^\circ_{\mathbf{H}}(x)$. Calling now $\iota : \mathbf{H} \rtimes \underline{E} \longrightarrow \mathbf{H} \rtimes \underline{E}$ the isomorphism given by $y \mapsto y^{gc'}$, we get

$$b = c^{-1}\iota'(F_0) = c'^{-1}\iota'(F_0)c' = \iota'(F_0)^{c'} = \iota(F_0)$$

and $\iota(s) = s^{gc'} = x^{c'} = x$.

It remains to check (c). We have $\tau \in \underline{E}$ and $h \in \mathbf{H}^F$ such that $h\tau \in \mathrm{C}_{\mathbf{H}\underline{E}}(s)$. Then $\iota(h\tau) \in \mathrm{C}_{\mathbf{H}\underline{E}}(\iota(s)) = \mathrm{C}_{\mathbf{H}\underline{E}}(x) = \mathrm{C}^\circ_{\mathbf{H}}(x) \rtimes \breve{B}_{\underline{E}}$ by Proposition 2.3(a). So

$$\iota(h\tau) \in \mathrm{C}^\circ_{\mathbf{H}}(x)b'$$

for some $b' \in \breve{B}_{\underline{E}}$. Note that $h\tau$ commutes with F and accordingly $\iota(h\tau)$ commutes with $\iota(F)$. Because of the semidirect product structure $\mathrm{C}_{\mathbf{H}\underline{E}}(x) = \mathrm{C}^\circ_{\mathbf{H}}(x) \rtimes \breve{B}_{\underline{E}}$ and $\iota(F) = \iota(F_0)^i \in \breve{B}_{\underline{E}}$ thanks to (b), we see that b' commutes with $\iota(F)$ as well. This gives our point (c). □

Proof of Theorem A. By Lemma 2.5, we have an inner automorphism ι of $\mathbf{H}\underline{E}$ such that $x := \iota(s) \in \mathbf{T}$, $\mathrm{C}_{\mathbf{H}\underline{E}}(x) = \mathrm{C}^\circ_{\mathbf{H}}(x) \rtimes \breve{B}_{\underline{E}}$ with $\iota(F_0) \in \breve{B}_{\underline{E}}$. We define $\breve{A} := \iota^{-1}(\breve{B}_{\underline{E}} \cap \mathbf{H})$. It is F_0-stable because $\breve{B}_{\underline{E}} \cap \mathbf{H}$ is stable under $\iota(F_0)$ since the latter belongs to $\breve{B}_{\underline{E}}$.

We now consider some $\sigma' \in \underline{E}$ such that $\sigma'(s) \in [s]_{\mathbf{H}^F}$ for some $F = F_0^k$, $k \geqslant 1$. Let $h \in \mathbf{H}^F$ such that $\sigma'(s) = s^h$. By Lemma 2.5(c) applied to $\tau := \sigma'$, we get some $b \in \mathrm{C}_{\check{B}_{\underline{E}}}(\iota(F))$ such that $\iota(h\tau) \in \mathrm{C}^\circ_{\mathbf{H}}(x)b$. We show that $\sigma := \iota^{-1}(b)$ satisfies our claims. First σ centralizes s since b centralizes x. Also σ belongs to $\mathbf{H}\tau$ since ι and its inverse, being conjugations, are the identity on $\mathbf{H}\underline{E}/\mathbf{H}$ and therefore $\mathbf{H}\sigma = \mathbf{H}b = \mathbf{H}\tau$. Moreover σ commutes with F since b commutes with $\iota(F)$, so $\sigma \in (\mathbf{H}\tau)^F = \mathbf{H}^F\tau$. The group $\check{A}$ is σ-stable by the same argument used for F_0, namely $b \in \check{B}_{\underline{E}}$ and therefore $\iota^{-1}(b)$ normalizes $\check{A} = \iota^{-1}(\check{B}_{\underline{E}} \cap \mathbf{H})$. That $b \in \check{B}_{\underline{E}}$ also implies that $[\iota(F_0), b] \in \check{B}$ since $\check{B}_{\underline{E}}/\check{B}$ is abelian by Proposition 2.3(b). This in turn tells us that $[F_0, \sigma] \in \check{A}$, thus completing the proof of Theorem A. □

In the next section, Theorem A will be applied mostly with σ' being a graph automorphism. But here is a case reminsicent of Shintani descent with two commuting Frobenius endomorphisms.

Corollary 2.6 *Let $F, F_0 \in \underline{E}^+$, let $\mathcal{C}$ be a semisimple conjugacy class of $\mathbf{H}$ and assume we have $s_0 \in \mathcal{C}^{F_0}$ such that $F(s_0) \in [s_0]_{\mathbf{H}^{F_0}}$. Applying Theorem A with (F_0, s, σ', k) being here $(F_0, s_0, F, 1)$, we get a group $\check{A}$ and some $\sigma \in \mathbf{H}\underline{E}$ that we denote as $\check{A}(s_0)$ and F'. Then there exist $s \in \mathcal{C}^F$, $F_0' \in \mathbf{H}^F F_0$ centralizing s, and a group $\check{A}(s)$ such that*

(i) $\mathrm{C}_{\mathbf{H}}(s) = \mathrm{C}^\circ_{\mathbf{H}}(s) \rtimes \check{A}(s)$;

(ii) *$\check{A}(s)$ is $\langle F, F_0'\rangle$-stable; and there exists an inner isomorphism $\iota_{s_0,s} : \mathbf{H}\underline{E} \to \mathbf{H}\underline{E}$ such that*
- $\iota_{s_0,s}(\check{A}(s_0)) = \check{A}(s)$;
- $\iota_{s_0,s}(s_0) = s$;
- $\iota_{s_0,s}(F_0) = F_0'$ *and* $\iota_{s_0,s}(F') = F$.

Proof. Let us recall how Theorem A is proved by application of Lemma 2.5. Choosing $x \in \mathbf{T} \cap \mathcal{C}$, we have $\mathrm{C}_{\mathbf{H}}(x) = \mathrm{C}^\circ_{\mathbf{H}}(x) \rtimes \check{B}$ from Proposition 2.3. Lemma 2.5 gives us an inner automorphism $\iota\colon \mathbf{H}\underline{E} \to \mathbf{H}\underline{E}$ with $\iota(s_0) = x$, $\iota(F_0) := f_0 \in \check{B}$, and we define $\check{A}(s_0) := \iota^{-1}(\check{B})$.

Having $F(s_0) \in [s_0]_{\mathbf{H}^{F_0}}$ implies that $[hF, s_0] = 1$ for some $h \in \mathbf{H}^{F_0}$, so that $\iota(hF) \in \mathrm{C}^\circ_{\mathbf{H}}(x)f$ with $f \in \mathrm{C}_{\check{B}_{\underline{E}}}(f_0)$ thanks to Lemma 2.5(c) for $i = 1$ and $\tau = F$. Following the proof of Theorem A, we define $F' := \iota^{-1}(f) \in \mathrm{C}_{\mathbf{H}^{F_0}F}(s_0)$ and get $F'(\check{A}(s_0)) = \check{A}(s_0)$.

Recalling that ι is an inner automorphism and $\mathbf{H}\underline{E}/\mathbf{H}$ is abelian, we have $\mathbf{H}f = \mathbf{H}F' = \mathbf{H}F$ and Lang's theorem ensures the existence of some inner automorphism ι' of $\mathbf{H}\underline{E}$ induced by an element of $\mathbf{H}$ such that $\iota'(f) = F$. We define $\iota_{s_0,s} := \iota'\iota$, $F_0' := \iota_{s_0,s}(F_0) = \iota'(f_0)$, $s := \iota_{s_0,s}(s) = \iota'(x)$. The latter belongs to $\mathcal{C}^F$ since ι and ι' are induced by elements of $\mathbf{H}$ and $[F, s] = \iota_{s_0,s}([\iota^{-1}(f), s_0]) = 1$. We have $\iota_{s_0,s}(F') = \iota'(f) = F$. We get $\mathrm{C}_{\mathbf{H}}(s) = \mathrm{C}^\circ_{\mathbf{H}}(s) \rtimes \check{A}(s)$ for $\check{A}(s) := \iota_{s_0,s}(\check{A}(s_0)) = \iota'(\check{B})$ as a consequence of the same property of $\check{A}(s_0)$ for s_0. Moreover $\check{A}(s)$ is $\langle F, F_0'\rangle$-stable since $\check{A}(s_0)$ has been seen to be F_0-stable and $\check{B}$ is f-stable since $f \in \check{B}_{\underline{E}}$. We also have $F_0' \in \mathbf{H}F_0$ while $[F_0', F] = \iota'([f_0, f]) = 1$ as seen above. So $F_0' \in \mathbf{H}^F F_0$ as claimed. We now have all our claims. □

3 Proof of Theorem B and some corollaries

We prove Theorem B but also some more technical statements, that can be seen as strengthenings of Theorem A, about semisimple classes of $\mathbf{H}$ and the action of elements of $\underline{E}^+$.

In the following $F_0 \in \underline{E}^+$ and $\mathcal{C}$ is the $\mathbf{H}$-conjugacy class of a semisimple element.

Corollary 3.1 *Assume $F_0(\mathcal{C}) = \gamma(\mathcal{C}) = \mathcal{C}$ and there is some $s \in \mathcal{C}^{F_0}$ such that $\gamma(s) \in [s]_{\mathbf{H}^{F_0}}$. Then there is $\gamma' \in \mathbf{H}^{F_0}\gamma \cap \mathrm{C}_{\mathbf{H}\underline{E}}(s)$ and a $\langle F_0, \gamma'\rangle$-invariant subgroup $\breve{A}$ with $\mathrm{C}_{\mathbf{H}}(s) = \mathrm{C}^{\circ}_{\mathbf{H}}(s) \rtimes \breve{A}$.*

Recall that $\pi\colon \mathbf{H}_0 \to \mathbf{H}$ is a composition $\mathbf{H}_0 \to \mathrm{SO}_{2l}(\mathbb{F}) \to \mathbf{H}$. The kernel of the second map is $\{\pm \mathrm{Id}_{2l}\}$ while the kernel of the first is $\{1, h_0\}$ where $h_0 \in \mathrm{Z}(\mathbf{H}_0)$ is of order 2.

We keep $\mathcal{C}$ a semisimple conjugacy class of $\mathbf{H}$ and $F_0 \in \underline{E}^+$. We describe the situation complementary to the one of Corollary 3.1, but while the latter is a direct consequence of Theorem A, the following will require a bit more work.

Corollary 3.2 *Assume that $F_0(\mathcal{C}) = \gamma(\mathcal{C}) = \mathcal{C}$ but $\gamma(s) \notin [s]_{\mathbf{H}^{F_0}}$ for every $s \in \mathcal{C}^{F_0}$.*
Let $F := F_0^{2k}$ for some $k \geqslant 1$ and $s \in \mathcal{C}^F$.
Then there exist $F_0' \in \mathrm{C}_{\mathbf{H}^F F_0}(s)$, $\gamma' \in \mathrm{C}_{\mathbf{H}^F\gamma}(s)$ and an $\langle F, F_0', \gamma'\rangle$-stable group $\breve{A}$ such that
(i) $\mathrm{C}_{\mathbf{H}}(s) = \mathrm{C}^{\circ}_{\mathbf{H}}(s) \rtimes \breve{A}$;
(ii) $\omega_s(\breve{A}) = \langle h_0\rangle$ and $|\mathrm{A}_{\mathbf{H}}(s)| = |\breve{A}| = 2$;
(iii) $[s_0, F_0'] \in \mathrm{Z}(\mathbf{H}_0)\backslash\langle h_0\rangle$ for every $s_0 \in \pi^{-1}(s)$ and
(iv) $[F_0', \gamma'] = \breve{a}$, where $\breve{a}$ is the generator of $\breve{A}$.

Let $\mathcal{C}$ be a semisimple conjugacy class of $\mathbf{H}$ and $F' \in \mathbf{H}\underline{E}^+$ (seen as an endomorphism of $\mathbf{H}$) such that $F'(\mathcal{C}) = \mathcal{C}$. Thanks to Lang's Theorem we have $\mathcal{C}^{F'} \neq \varnothing$.

Taking $s \in \mathcal{C}^{F'}$, we can combine the parametrization of $\mathbf{H}^{F'}$-conjugacy classes in $\mathcal{C}^{F'}$ by $A_{\mathbf{H}}(s)_{F'} := A_{\mathbf{H}}(s)/[A_{\mathbf{H}}(s), F']$ (see [G, 4.3.6]) and the injection $A_{\mathbf{H}}(s) \to \mathrm{Z}(\mathbf{H}_0)$ induced by the map ω_s from our introduction.

$$\begin{array}{ccccc} \mathcal{C}^{F'}/\mathbf{H}^{F'}\text{-conj} & \overset{\sim}{\longleftrightarrow} & A_{\mathbf{H}}(s)_{F'} & & \\ & & A_{\mathbf{H}}(s) & \hookrightarrow & \mathrm{Z}(\mathbf{H}_0) \end{array}$$

An easy calculation then yields that the resulting map

$$\begin{array}{rcll} \Omega_{F'} : \mathcal{C}^{F'}/\mathbf{H}^{F'}\text{-conj} & \hookrightarrow & \mathrm{Z}(\mathbf{H}_0)_{F'} & \text{is injective and defined by} \\ [x]_{\mathbf{H}^{F'}} & \mapsto & x_0^{-1}F'(x_0)[\mathrm{Z}(\mathbf{H}_0), F'] & \text{for any } x_0 \in \pi^{-1}(x). \end{array}$$

Note that, in contrast to the other two maps, $\Omega_{F'}$ is independent of the choice of s in $\mathcal{C}^{F'}$. Note also that if moreover $F' \in \underline{E}^+$, then $\Omega_{F'}$ is $\underline{E}_{\mathcal{C}}$-equivariant by its definition (since $\underline{E}$ commutes with F').

Let us show Theorem B. We consider the above with $F' = F_0 \in \underline{E}$ and $|A_{\mathbf{H}}(s)| = 4 = |\mathrm{Z}(\mathbf{H}_0)|$. The injection $A_{\mathbf{H}}(s) \hookrightarrow \mathrm{Z}(\mathbf{H}_0)$ is then an $\langle F_0\rangle$-equivariant isomorphism and therefore Ω_{F_0} is onto. So $1 = x_0^{-1}F_0(x_0)[\mathrm{Z}(\mathbf{H}_0), F_0]$ for some $x_0 \in \mathbf{H}_0$ with $\pi(x_0) \in \mathcal{C}^{F_0}$.

Making the adequate translate we get some $x_0' \in \mathbf{H}_0$ with $F_0(x_0') = x_0'$ and $\pi(x_0') = \pi(x_0) \in \mathcal{C}$. Note also that the equivariance and injectivity of Ω_{F_0} imply that $[\pi(x_0)]_{\mathbf{H}^{F_0}}$ is γ-stable. This shows all claims of Theorem B.

We now prove the corollaries. If there is some $s \in \mathcal{C}^{F_0}$ such that $[s]_{\mathbf{H}^{F_0}}$ is γ-stable then Theorem A with $k = 1$ clearly gives the claims of Corollary 3.1.

Proof of Corollary 3.2. The action of $\underline{E}$ on $\mathrm{Z}(\mathbf{H}_0)$ is easy to deduce from the description of $\mathrm{Z}(\mathbf{H}_0)$ (see [GLS, Table 1.12.6], [S21, 2.9]). In particular $\mathrm{C}_{\mathrm{Z}(\mathbf{H}_0)}(\gamma) = \langle h_0 \rangle$, a subgroup of $\mathrm{Z}(\mathbf{H}_0)$ of order 2.

Since no $\mathbf{H}^{F_0}$-class contained in $\mathcal{C}^{F_0}$ is γ-stable, the $\mathbf{H}^{F_0}$-conjugacy classes contained in $\mathcal{C}$ are mapped via Ω_{F_0} to elements in $\mathrm{Z}(\mathbf{H}_0)_{F_0}$ not fixed by γ. Then $\mathrm{Z}(\mathbf{H}_0)_{F_0}$ can't have order 1 or 2, so it implies the following

Lemma 3.3 *F_0, and therefore also F, act trivially on $\mathrm{Z}(\mathbf{H}_0)$.*

We also get that the image of Ω_{F_0}, being γ-stable, is indeed equal to the whole $\mathrm{Z}(\mathbf{H}_0)\backslash\langle h_0 \rangle$. This is of order 2, so $|A_{\mathbf{H}}(s)_{F_0}| = |A_{\mathbf{H}}(s)| = 2$ for any $s \in \mathcal{C}^{F_0}$ by the diagram defining the map Ω from ω_s. The equality $|A_{\mathbf{H}}(s)| = 2$ of course holds for any $s \in \mathcal{C}$. We have $\gamma\omega_s\gamma = \omega_{\gamma(s)}$ and the image of ω_s depends only on the conjugacy class of s (by the same formula for inner automorphisms), so $\gamma(\mathcal{C}) = \mathcal{C}$ implies that $\omega_s(A_{\mathbf{H}}(s))$ is γ-stable of order 2 for any $s \in \mathcal{C}$. Then indeed $\omega_s(A_{\mathbf{H}}(s)) = \langle h_0 \rangle$, which is part (ii).

In the following we prove the other statements (i), (iii) and (iv) in several steps first for some $s \in \mathcal{C}^{F_0}$ and then for $t \in \mathcal{C}^F$ via an application of Theorem A and Lemma 2.5.

For the following we fix some $x \in \mathcal{C} \cap \mathbf{T}$ and the groups $\check{B}_{\underline{E}} \leqslant \mathbf{H}\underline{E}$ and $\check{B} := \check{B}_{\underline{E}} \cap \mathbf{H}$ with $\mathrm{C}_{\mathbf{H}\underline{E}}(x) = \mathrm{C}^\circ_{\mathbf{H}}(x) \rtimes \check{B}_{\underline{E}}$ from Proposition 2.3.

Let us take $s \in \mathcal{C}^{F_0}$. Let $s_0 \in \pi^{-1}(s)$ and $h' := s_0^{-1}F_0(s_0)$. Note that $\Omega_{F_0}([s]_{\mathbf{H}^{F_0}}) = h' \in \mathrm{Z}(\mathbf{H}_0)\backslash\langle h_0 \rangle$ by the above. This ensures $[s_0, F_0] \in \mathrm{Z}(\mathbf{H}_0)\backslash\langle h_0 \rangle$ for every $s_0 \in \pi^{-1}(s)$. Defining $F_0' := F_0$ in that case, we get part (iii) for that s.

We turn to the question of actually finding γ' in $\mathrm{C}_{\mathbf{H}\underline{E}}(s)^F$ and proving (i) and (iv). In order to apply Theorem A with $\sigma' = \gamma$, we need to check that $[s]_{\mathbf{H}^F}$ is γ-stable.

Since $F = F_0^{2k}$ and $F_0(s_0) = s_0h'$ with $h' \in \mathrm{Z}(\mathbf{H}_0)$, we have

$$\Omega_F([s]_{\mathbf{H}^F}) = s_0^{-1}F_0^{2k}(s) = s_0^{-1}F_0(s)F_0(s_0^{-1}F_0(s))\dots F_0^{2k-1}(s_0^{-1}F_0(s)) = (h')^{2k}$$

thanks to Lemma 3.3. Squares in $\mathrm{Z}(\mathbf{H}_0)$ are $\underline{E}$-fixed since they form a stable subgroup of order 1 or 2. So $\Omega_F([s]_{\mathbf{H}^F})$ is γ-fixed and by equivariance of Ω_F, this implies that $[s]_{\mathbf{H}^F}$ is γ-stable. Note that it is also F_0-stable since $F_0(s) = s$. On the other hand $\mathcal{C}^F/\mathbf{H}^F$-conj is of cardinality $|A(s)_F| = |A(s)| = 2$ as seen before and using Lemma 3.3 again. Since $\mathcal{C}^F$ is $\langle F_0, \gamma \rangle$-stable and $[s]_{\mathbf{H}^F}$ is $\langle F_0, \gamma \rangle$-stable it implies the following

Lemma 3.4 *$\langle F_0, \gamma \rangle$ acts trivially on $\mathcal{C}^F/\mathbf{H}^F-$conj.*

We can now apply Theorem A to s with $i = 2k$, $\sigma' = \gamma$. Taking also $F_0' = F_0$, this gives us a σ that we call γ' and a decomposition $\mathrm{C}_{\mathbf{H}}(s) = \mathrm{C}^\circ_{\mathbf{H}}(s) \rtimes \check{A}(s)$ satisfying our part (i) and $[F_0, \gamma'] \in \check{A}(s)$.

Let us check that $[F_0, \gamma'] \neq 1$. Remembering that $\mathbf{H}\underline{E}$ acts on $\mathbf{H}_0$, it suffices to show that $[\gamma', F_0](s_0) \neq s_0$. Recall that $F_0(s_0) = s_0h'$ for some $h' \in \mathrm{Z}(\mathbf{H}_0)\backslash\langle h_0 \rangle$ so that

$\gamma'(h') = \gamma'^{-1}(h') = \gamma(h') = h_0h'$ since $\mathrm{Z}(\mathbf{H}_0)$ is a group of order 4 in which the centralizer of γ is the subgroup of order 2 generated by h_0. Let $z \in \mathrm{Z}(\mathbf{H}_0)$ such that $\gamma'(s_0) = s_0z$. Note that z exists since $\gamma'(s) = s$. Using also Lemma 3.3 we get

$$[\gamma', F_0](s_0) = \gamma' F_0 \gamma'^{-1}(s_0h'^{-1}) = \gamma' F_0(s_0\gamma'^{-1}(z^{-1})h_0h'^{-1}) =$$
$$= \gamma'(s_0\gamma'^{-1}(z^{-1})h_0) = s_0h_0 \neq s_0.$$

This implies our claim that $[F_0, \gamma'] \neq 1$. (Another proof is possible with the three subgroups Lemma.) Letting $\breve{a}_s$ be the generator of $\breve{A}(s)$, a group of order 2, we have $[F_0, \gamma'] = \breve{a}_s$ as $[F_0, \gamma'] \in \breve{A}(s)$. This gives (iv) in that case.

Let us recall however how Theorem A is deduced from Lemma 2.5 in our case (with σ' being γ). We first get an inner automorphism of $\mathbf{H}\underline{E}$ induced by an element of $\mathbf{H}$, namely $\iota_s\colon \mathbf{H}\underline{E} \to \mathbf{H}\underline{E}$, such that $\iota_s(s) = x$, $f_s := \iota_s(F_0) \in \breve{B}_{\underline{E}}$ and then $\breve{A}(s) := \iota_s^{-1}(\breve{B})$ satisfies $\mathrm{C}_{\mathbf{H}}(s) = \mathrm{C}^\circ_{\mathbf{H}}(s) \rtimes \breve{A}(s)$. Finally we take some $g_s \in \breve{B}_{\underline{E}} \cap \mathbf{H}^F\gamma$ and define $\gamma' := \iota_s^{-1}(g_s)$.

Before turning to the general case of some $t \in \mathcal{C}^F$, let us apply now ι_s to F_0 and γ'. From what has been said about $[F_0, \gamma']$ we see that $[f_s, g_s] = \breve{b}$, the generator of $\breve{B}$. Since $|\breve{B}| = |\mathrm{A}_{\mathbf{H}}(s)| = 2$ and hence $\breve{B} \leqslant \mathrm{Z}(\breve{B}_{\underline{E}})$ this implies

$$[f, g] = \breve{b} \text{ for every } f \in \breve{B}_{\underline{E}} \cap \mathbf{H}F_0 \ (= \breve{B}f_s) \text{ and } g \in \breve{B}_{\underline{E}} \cap \mathbf{H}\gamma \ (= \breve{B}g_s). \tag{3.5}$$

This is a general property since such an s always exists.

Let us now look at the general case where $t \in \mathcal{C}^F \backslash \mathcal{C}^{F_0}$. Using again Lemma 2.5 but this time with $i = 1$ (so that the F_0 of that lemma is our F), we get a conjugation $\iota_t\colon \mathbf{H}\underline{E} \to \mathbf{H}\underline{E}$ such that $\iota_t(t) = x$ and some other properties.

We define $\breve{A}(t) := \iota_t^{-1}(\breve{B})$ which then satisfies $\mathrm{C}_{\mathbf{H}}(t) = \mathrm{C}^\circ_{\mathbf{H}}(t) \rtimes \breve{A}(t)$. Let $f := f_s = \iota_s(F_0) \in \breve{B}_{\underline{E}} \cap \mathbf{H}F_0$ (see above), and take any $g \in \breve{B}_{\underline{E}} \cap \mathbf{H}\gamma$. Let $F_0' := \iota_t^{-1}(f)$ and $\gamma_t' := \iota_t^{-1}(g)$. By this construction $\breve{A}(t)$ is $\langle F_0', \gamma_t' \rangle$-stable and $F_0'(t) = \gamma_t'(t) = t$.

In the next step we show that F centralizes in $\mathbf{H}\underline{E}$ both γ_t' and F_0'. In applying Lemma 2.5, we can take $\tau \in \{F_0, \gamma\}$ and $h \in \mathbf{H}^F$ such that $\tau(s) = s^h$ thanks to Lemma 3.4. Lemma 2.5(c) tells us that $\iota_t(h\tau) = cf'$ with $c \in \mathbf{H}$, $f' \in \breve{B}_{\underline{E}}$, and $[f', \iota_t(F)] = 1$. Since $\iota_t(h\tau) \in \mathbf{H}\tau$, this shows that $\iota_t(F)$ fixes an element of the set with two elements $\breve{B}_{\underline{E}} \cap \mathbf{H}\tau$. On the other hand $\iota_t(F) \in \breve{B}_{\underline{E}}$ by Lemma 2.5(b), so it stabilizes $\breve{B}_{\underline{E}} \cap \mathbf{H}\tau$ (remember again that cosets mod $\mathbf{H}$ are stabilized by inner automorphisms of $\mathbf{H}\underline{E}$) hence fixes any of its two elements. Taking the image by ι_t^{-1}, we get that F centralizes both $\iota_t^{-1}(\breve{B}_{\underline{E}} \cap \mathbf{H}F_0)$ and $\iota_t^{-1}(\breve{B}_{\underline{E}} \cap \mathbf{H}\gamma)$, hence both F_0' and γ_t'. So we get that $F_0' \in \mathrm{C}_{\mathbf{H}^F F_0}(s)$ and $\gamma_t' \in \mathrm{C}_{\mathbf{H}^F\gamma}(s)$. This ensures part (i) in that case.

Letting $t_0 \in \pi^{-1}(t)$, $x_0 \in \pi^{-1}(x)$, we have $[t_0, F_0'] = \iota_t^{-1}([x_0, f]) = \iota_t^{-1}\iota_s([s_0, F_0]) = [s_0, F_0]$, since ι_s and ι_t are conjugations by elements of $\mathbf{H}$ hence act trivially on $\mathrm{Z}(\mathbf{H}_0)$. Then $[t_0, F_0'] = [s_0, F_0] \in \mathrm{Z}(\mathbf{H}_0) \backslash \langle h_0 \rangle$ as seen before. We then get (iii) for t.

Let now $\breve{a}$ be the generator of $\breve{A}(t)$ and hence $\breve{a} = \iota_t^{-1}(\breve{b})$. By (3.5), $[f, g] = \breve{b}$ and hence $[F_0', \gamma_t'] = \iota_t^{-1}([f, g]) = \iota_t^{-1}(\breve{b}) = \breve{a}$. This ensures part (iv). $\square$

Remark 3.6. Both cases covered by our Corollaries are non empty. It's clear for the first.

In order to check the relevance of the second corollary we use the notations of [S21, Not. 2.3] to describe elements of the maximal torus $\mathbf{T}_0$ of $\mathbf{H}_0$. Let q be a power of p with $q \equiv 1 \mod 4$. Let F_q be the corresponding power of F_p. Let $\zeta, \varpi \in \mathbb{F}^\times$ with $\zeta^{q^2-1} = \varpi^2 = -1$. Recall that $\mathbf{h}_{\underline{l}}(\varpi) := \mathbf{h}_{e_1}(\varpi) \dots \mathbf{h}_{e_l}(\varpi)$ is an element of $\mathrm{Z}(\mathbf{H}_0) \backslash \langle h_0 \rangle$, not fixed by γ. Let $l = 4$, $s_0 = \mathbf{h}_{e_2}(\varpi)\mathbf{h}_{e_3}(\zeta)\mathbf{h}_{e_4}(\zeta^q\varpi) \in \mathbf{T}_0$. One has $\gamma(s_0) = s_0$. It is not difficult to see that with the equation $\zeta^{q^2}\varpi^q\varpi = \zeta$ one gets that $F_q(s_0)$ and $s_0\mathbf{h}_{\underline{4}}(\varpi)$ are $\mathrm{N}_{\mathbf{H}_0}(\mathbf{T}_0)$-conjugate, so we can write $F_q(s_0) = s_0^n\mathbf{h}_{\underline{4}}(\varpi)$ for some $n \in \mathrm{N}_{\mathbf{H}_0}(\mathbf{T}_0)$. Then $\Omega_{nF_q}([s]_{\mathbf{H}^{nF_q}}) = \mathbf{h}_{\underline{4}}(\varpi)$ for $s := \pi(s_0)$.

By Lang's Theorem we can take $\iota\colon \mathbf{H}\underline{E} \to \mathbf{H}\underline{E}$ an inner automorphism induced by an element of $\mathbf{H}$ such that $\iota(nF_q) = F_q$. Then $s' := \iota(s) \in \mathbf{H}^{F_q}$ and $\Omega_{F_q}([s']_{\mathbf{H}^{F_q}}) = \mathbf{h}_{\underline{4}}(\varpi)$ which is not γ-stable. Then $[s']_{\mathbf{H}^{F_q}}$ is not γ-stable while s and s' are in the same $\mathbf{H}$-class.

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