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# NULL CONTROLLABILITY OF STRONGLY DEGENERATE PARABOLIC EQUATIONS

ANTOINE BENOIT, ROMAIN LOYER, AND LIONEL ROSIER

ABSTRACT. We consider linear one-dimensional strongly degenerate parabolic equations with measurable coefficients that may be degenerate or singular. Taking 0 as the point where the strong degeneracy occurs, we assume that the coefficient  $a = a(x)$  in the principal part of the parabolic equation is such that the function  $x \rightarrow x/a(x)$  is in  $L^p(0, 1)$  for some  $p > 1$ . After establishing some spectral estimates for the corresponding elliptic problem, we prove that the parabolic equation is null controllable in the energy space by using one boundary control.

## 1. INTRODUCTION

We continue our investigation of the controllability of parabolic equations with measurable coefficients [24] (see also [2]) by studying the case of a *strongly degenerate* equation of the type

$$(a(x)u_x)_x + q(x)u = \rho(x)u_t, \quad x \in (0, 1), \quad t \in (0, T),$$

where the nonnegative function  $a$  may vanish *strongly* at  $x = 0$ , and the potential  $q$  may be singular at  $x = 0$ . Only *weakly degenerate* (i.e.  $1/a \in L^1(0, 1)$ ) parabolic equations were covered by the theory developed in [24].

The null controllability of (weakly or strongly) degenerate parabolic equations was considered in e.g. [1, 4, 5, 9, 10, 11, 13, 14, 15, 26]. Most of the papers were concerned with a parabolic equation with  $a(x) = x^{2-\varepsilon}$ , which is strongly (resp. weakly) degenerate for  $\varepsilon \in (0, 1]$  (resp.  $\varepsilon \in (1, 2)$ ). More general choices for the coefficient  $a$  were considered in e.g. [14]. However, several technical assumptions (e.g.  $x \rightarrow a(x)/x^\gamma$  nondecreasing for some exponent  $\gamma$  and  $a \in W^{1,\infty}(0, 1)$  in [14]) were required in order to derive some Carleman estimate to prove the null controllability of the parabolic equation. The purpose of this paper is to remove these technical assumptions in the derivation of the null controllability of the parabolic equation.

More precisely, we propose a general method based on the flatness approach to deal with quite general parabolic equations, displaying both a strong degeneracy of  $a$  and a singularity of the potential  $q$  at the same point, with measurable coefficients, and without any monotony assumption about  $a$ . Roughly, the main assumption is that the function  $x \rightarrow x/a(x)$  is in  $L^p(0, 1)$  for some  $p > 1$ . That assumption is slightly stronger (by Hölder inequality) than Trudinger assumption  $1/\sqrt{a} \in L^1(0, 1)$  (see e.g. [12, 28]) which was made in order to investigate

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*Key words and phrases.* strongly degenerate parabolic equations; null controllability; measurable and singular coefficient; spectral estimates; flatness.

the degenerate elliptic system

$$\begin{cases} -(au_x)_x = f, & x \in (0, 1), \\ (au_x)(0) = 0, \\ u(1) = 0. \end{cases}$$

Our method is based on the flatness approach, introduced in [20] and developed since then in [22, 24] for the heat equation, in [21] for the Korteweg-de Vries equation, and in [25] for Schrödinger equation. (See also [23] for a recent study of the reachable states of the heat equation and [19] for the exact controllability of semilinear heat equations.) In [26], the flatness approach is used to derive for  $\varepsilon \in (0, 1)$  the null controllability of the control system

$$\begin{cases} u_t - (x^{2-\varepsilon}u_x)_x = 0, & x \in (0, 1), t \in (0, T), \\ (x^{2-\varepsilon}u_x)(0, t) = 0, & t \in (0, T), \\ u(1, t) = h(t), & t \in (0, T), \\ u(x, \cdot) = u_0(x), & x \in (0, 1). \end{cases}$$

For the corresponding elliptic problem

$$\begin{cases} -(x^{2-\varepsilon}u_x)_x = f, & x \in (0, 1), \\ (x^{2-\varepsilon}u_x)(0) = 0, \\ u(1) = 0, \end{cases}$$

the eigenfunctions and eigenvalues can be expressed in terms of Bessel functions, and the asymptotic behaviour of the eigenvalues is perfectly known [26]. For a more general function  $a$ , however, Bessel functions cannot be used and, to the best knowledge of the authors, nothing is known about the sharp asymptotic behaviour of the eigenvalues. (See [3, 17, 18] for some results in that direction.) For the application of the flatness approach, what is needed is not a spectral gap, but merely that the eigenvalues tend to infinity faster than some power of the index of the eigenvalue.

To be more precise, we are concerned with the null controllability of the system

$$(a(x)u_x)_x + q(x)u = \rho(x)u_t, \quad x \in (0, 1), t \in (0, T), \quad (1.1)$$

$$(au_x)(0, t) = 0, \quad t \in (0, T), \quad (1.2)$$

$$\alpha u(1, t) + \beta(au_x)(1, t) = h(t), \quad t \in (0, T), \quad (1.3)$$

$$u(x, 0) = u_0(x), \quad x \in (0, 1) \quad (1.4)$$

where  $(\alpha, \beta) \in \mathbb{R}_+^2 \setminus \{(0, 0)\}$ ,  $u_0 \in L^2(0, 1)$  is the initial state, and  $h \in L^2(0, T)$  is the control input.

The given functions  $a, q, \rho$  are assumed to satisfy the following conditions:

$$a(x) > 0 \text{ and } \rho(x) > 0 \text{ for a.e. } x \in (0, 1), \quad (1.5)$$

$$a \in L^1_{loc}(0, 1), \quad \left(x \rightarrow \frac{x}{a(x)}\right) \in L^p(0, 1), \quad (1.6)$$

$$\rho \in L^r(0, 1), \quad \limsup_{x \rightarrow 0^+} \rho(x) < \infty, \quad (1.7)$$

$$\lim_{x \rightarrow 0^+} a(x)^{-1} \left( \int_x^1 \frac{ds}{a(s)} \right)^{-2} = +\infty, \quad (1.8)$$

$$\exists v \in W^{1,1}(0, 1) \text{ s.t. } \begin{cases} v(x) > 0 \text{ for all } x \in [0, 1], \\ (av_x)_x + qv = 0 \text{ in } (0, 1), \\ (av_x)(0) = 0, \end{cases} \quad (1.9)$$

for some numbers  $p, r$  with

$$p \in (1, +\infty], \quad r \in (p', +\infty] \quad (1.10)$$

where  $p' := \frac{p}{p-1}$ . As the functions  $a$  and  $\rho$  are defined a.e., the limits in (1.7) and (1.8) should be taken after modifying them on a zero measure set, if needed. Note that (1.6) and (1.8) are satisfied by any measurable function  $a : (0, 1) \rightarrow \mathbb{R}$  fulfilling the condition

$$C_1 x^{2-\varepsilon_1} \leq a(x) \leq C_2 x^{2-\varepsilon_2} \quad \forall x \in (0, 1) \quad (1.11)$$

for some positive constants  $C_1, C_2, \varepsilon_1, \varepsilon_2$  with  $\varepsilon_1 \leq \varepsilon_2 < 2\varepsilon_1 \leq 4$ . Such a function needs not be monotonous nor smooth. (Note that a continuous function  $a : (0, 1) \rightarrow \mathbb{R}_+$  satisfying both (1.6) and (1.8) and vanishing on a sequence  $x_n \searrow 0$  could also be constructed, so that (1.1) can be *strongly degenerate* at 0 and also *weakly degenerate* at each  $x_n$ .) A typical example displaying both a (possibly strong) degeneracy for  $a$  and a singularity for  $q$  at  $x = 0$  is the parabolic equation

$$(x^{2-\varepsilon} u_x)_x + \mu \frac{u}{x^\varepsilon} = u_t \quad (1.12)$$

for  $0 < \varepsilon < 2$  and  $\mu < \frac{1}{4}(1 - \varepsilon)^2$ .

Let us introduce some notations. For any  $t_1 < t_2$  and  $s \geq 0$ , we denote by  $G^s([t_1, t_2])$  the space of (Gevrey) functions  $h \in C^\infty([t_1, t_2])$  for which there exist some positive constants  $M, R$  such that

$$|h^{(p)}(t)| \leq M \frac{p!^s}{R^p} \quad \forall t \in [t_1, t_2], \quad \forall p \in \mathbb{N}.$$

Let  $L^2_\rho$  denote the space of (classes of) measurable functions  $f : (0, 1) \rightarrow \mathbb{R}$  such that

$$\|f\|_{L^2_\rho} := \left( \int_0^1 f(x)^2 \rho(x) dx \right)^{\frac{1}{2}} < \infty.$$

The main result in this paper is the following

**Theorem 1.1.** *Let the functions  $a, q, \rho, v : (0, 1) \rightarrow \mathbb{R}$  satisfy assumptions (1.5)-(1.9) for some numbers  $p$  and  $r$  as in (1.10). Let  $(\alpha, \beta) \in \mathbb{R}_+^2 \setminus \{(0, 0)\}$  and  $T > 0$ . Pick any  $u_0 \in L^2_\rho$  and*

any  $s \in (1, 1 + \frac{1}{p'} - \frac{1}{r})$ . Then there exists a function  $h \in G^s([0, T])$  such that the solution  $u$  of (1.1)-(1.4) satisfies  $u \in C^0([0, T], L^2_\rho) \cap C^1((0, T], W^{1,1}(0, 1))$ ,  $au_x \in C^1((0, T], W^{1,r}(0, 1))$ , and

$$u(x, T) = 0 \quad \text{for all } x \in [0, 1].$$

**Remark 1.2.** A null controllability result with a distributed control can be derived from Theorem 1.1 by using a partition of unity.

Clearly, assumptions (1.5)-(1.8) are easy to test, while assumption (1.9) is not obvious to check at first glance. We shall provide in the following propositions two classes of coefficients  $(a, q, \rho)$  satisfying (1.9).

**Proposition 1.3.** Let the numbers  $p, p', r$  be as in (1.10) and let the functions  $a, q, \rho$  satisfy (1.5)-(1.8). Assume in addition that  $q \in L^{p'}(0, 1)$  and that either

$$\int_0^1 \frac{1}{a(x)} \left( \int_0^x |q(s)| ds \right) dx < 1, \quad (1.13)$$

or

$$q(x) \leq 0 \quad \text{for a.e. } x \in (0, 1). \quad (1.14)$$

Then (1.9) holds for some function  $v \in W^{1,1}(0, 1)$  with  $av_x \in W^{1,r}(0, 1)$ , and the conclusion of Theorem 1.1 is valid for any  $u_0 \in L^2_\rho$ . Finally, if (1.14) is replaced by the condition

$$\exists K \in \mathbb{R}_+ \text{ such that } q(x) \leq K\rho(x) \text{ for a.e. } x \in (0, 1), \quad (1.15)$$

then the conclusion of Theorem 1.1 is still valid for any  $u_0 \in L^2_\rho$ .

**Proposition 1.4.** Let  $a(x) = x^{2-\varepsilon}$ ,  $q(x) = \mu x^{-\varepsilon}$  and  $\rho(x) = 1$  for  $x \in (0, 1)$ , where  $0 < \varepsilon < 1$  and

$$\frac{(1-\varepsilon)^2 - 1}{4} < \mu \leq 0. \quad (1.16)$$

Pick  $v(x) := x^\delta$  with  $\delta := (\varepsilon - 1 + \sqrt{(1-\varepsilon)^2 - 4\mu})/2$ . Then the function  $v : (0, 1) \rightarrow (0, \infty)$  fulfills

$$(av_x)_x + qv = 0 \quad \text{in } (0, 1), \quad (1.17)$$

$$(av_x)(0) = 0, \quad (1.18)$$

and (1.5)-(1.8) hold for some numbers  $p$  and  $r$  as in (1.10). Furthermore the conclusion of Theorem 1.1 is valid for (1.12) supplemented with (1.2)-(1.4) for any initial data  $u_0 : (0, 1) \rightarrow \mathbb{R}$  with  $u_0 \in L^2(0, 1)$ .

**Remark 1.5.** (1) Note that the main result in [26] corresponds to the case  $\mu = 0$ .

(2) Note that Proposition 1.4 is not a consequence of Proposition 1.3, since we cannot find any  $p$  in  $(1, +\infty]$  with both  $(x \rightarrow x^{\varepsilon-1}) \in L^p(0, 1)$  and  $(x \rightarrow x^{-\varepsilon}) \in L^{p'}(0, 1)$ .

(3) Our computations suggest that for  $a(x) = x^{2-\varepsilon}$  and  $q(x) = \mu x^{-\kappa}$ ,  $\kappa$  should be at most  $\varepsilon$ . It would be interesting to see whether it is a necessary condition, or merely a technical assumption.

Let us say a few words about the proof of the main result. In a first step, we show that we can get rid of the term  $q(x)u$  in (1.1) by a change of variables, using assumption (1.9). Therefore we can restrict to the simplified parabolic equation

$$(au_x)_x = \rho u_t.$$

We first prove that the boundary-value problem

$$\begin{cases} -(au_x)_x = f, \\ (au_x)(0) = 0, \\ \alpha u(1) + \beta(au_x)(1) = 0 \end{cases}$$

possesses a unique solution in some weighted Sobolev space. Next, we pay some attention to the spectral properties of this boundary-value problem. We show that the eigenvalue  $\lambda_n$  grows *at least* as a power of  $n$ , and that the  $L^\infty$ -norm of the corresponding eigenfunction  $e_n$  grows *at most* as a power of  $\lambda_n$ . This is done by using a modified Prüfer method (see [6, 29]) introducing a phase  $\theta_n$  associated with  $\lambda_n$ . However, since  $1/a \notin L^1(0,1)$  in the interesting situation of a strong degeneracy, the classical argument relating  $\lambda_n$  to the variation of the phase  $\theta_n$  has to be refined in using a splitting of the interval  $(0,1)$  involving the frequency  $\lambda_n$ . Roughly, we split  $(0,1)$  into  $(0, A_n) \cup [A_n, 1)$  with  $A_n := (C\lambda_n)^{-\frac{p'r}{r-p'}}$ ,  $C$  denoting some positive constant. We show that  $e_n(x)$  remains close to  $e_n(0)$  for  $x \in (0, A_n)$ , so that the (bad) integral term  $\int_0^{A_n} \frac{d\theta_n}{dx} dx$  does not contribute too much in the variation of the phase  $\theta_n(1) - \theta_n(0)$ .

With these spectral estimates at hand, we can prove that the eigenfunctions  $e_n$ ,  $n \geq 0$ , can be expressed in terms of the generating functions  $g_i$ ,  $i \geq 0$ , defined by  $g_0(x) = 1$  and the relation

$$g_i(x) = \int_0^x \frac{1}{a(s)} \left( \int_0^s \rho(\sigma) g_{i-1}(\sigma) d\sigma \right) ds, \quad i \geq 1.$$

Finally, the trajectories of the control problem (1.1)-(1.4) can be expanded in the form

$$u(x, t) = \sum_{i=0}^{\infty} y^{(i)}(t) g_i(x)$$

for some function  $y \in G^s([0, T])$  (as in [24]), the series being convergent thanks to the spectral estimates.

The paper is organized as follows. Section 2 is devoted to the study of the corresponding elliptic problem. We introduce the appropriate weighted Sobolev space, derive some generalized Hardy inequality and obtain some estimates for both the eigenfunctions and the eigenvalues. In Section 3, we define and investigate the generating functions. The proof of the main results are given in Section 4. Finally, in some appendix we prove that the conditions (1.6) and (1.8) are independent, and we provide a class of functions for which (1.8) holds.

## 2. STUDY OF THE ELLIPTIC PROBLEM

Through the paper, we denote  $\|u\|_{L^p}$  for  $\|u\|_{L^p(0,1)}$  ( $1 \leq p \leq \infty$ ), and  $\|u\|_{L^p(x_1, x_2)}$  for the  $L^p$  norm of  $u$  on an interval  $(x_1, x_2) \neq (0, 1)$ .

In this section, we investigate the elliptic problem

$$-(au')' = \rho f \quad \text{in } (0, 1), \quad (2.1)$$

$$(au')(0) = 0, \quad (2.2)$$

$$\alpha u(1) + \beta(au')(1) = 0, \quad (2.3)$$

where  $' = d/dx$ , the functions  $a$  and  $\rho$  satisfy (1.5)-(1.8) for some numbers  $p, p'$ , and  $r$  as in (1.10),  $(\alpha, \beta) \in \mathbb{R}_+^2 \setminus \{(0, 0)\}$ .

Let us introduce the spaces

$$H_a := \{u \in W_{loc}^{1,1}(0, 1); \sqrt{a}u' \in L^2(0, 1) \text{ and } u(1) = 0\}, \quad (2.4)$$

$$H_{a,\rho} := \{u \in W_{loc}^{1,1}(0, 1); \sqrt{a}u' \in L^2(0, 1) \text{ and } \sqrt{\rho}u \in L^2(0, 1)\} \quad (2.5)$$

endowed respectively with the norms

$$\|u\|_{H_a} := \left( \int_0^1 a(x)u'(x)^2 dx \right)^{\frac{1}{2}}, \quad \|u\|_{H_{a,\rho}} := \left( \int_0^1 [a(x)u'(x)^2 + \rho(x)u(x)^2] dx \right)^{\frac{1}{2}}.$$

Then the following result holds.

**Proposition 2.1.** *The spaces  $H_a$  and  $H_{a,\rho}$  are complete. Furthermore, we have  $H_a \subset L^2(0, 1)$ ,  $H_{a,\rho} \subset L^2(0, 1)$  and  $H_{a,\rho} \subset L_\rho^2$  with continuous and compact embeddings.*

*Proof.* We first investigate the space  $H_a$ . By Hölder inequality, we have for all  $\varepsilon \in (0, 1)$

$$\int_\varepsilon^1 \frac{dx}{a(x)} \leq \left( \int_\varepsilon^1 \left( \frac{x}{a(x)} \right)^p dx \right)^{\frac{1}{p}} \left( \int_\varepsilon^1 x^{-p'} dx \right)^{\frac{1}{p'}} < \infty,$$

so that  $a^{-1} \in L^1(\varepsilon, 1)$ . Thus, by Cauchy-Schwarz inequality,

$$\int_\varepsilon^1 |u'(x)| dx \leq \left( \int_\varepsilon^1 a(x)u'(x)^2 dx \right)^{\frac{1}{2}} \left( \int_\varepsilon^1 \frac{dx}{a(x)} \right)^{\frac{1}{2}} < \infty \quad (2.6)$$

if  $\sqrt{a}u' \in L^2(0, 1)$ , so that  $u \in W^{1,1}(\varepsilon, 1) \subset C^0([\varepsilon, 1])$ . Therefore, the condition  $u(1) = 0$  is meaningful whenever  $\sqrt{a}u' \in L^2(0, 1)$ , and if  $u \in H_a$  satisfies  $\|u\|_{H_a} = 0$ , then  $\sqrt{a}u' = 0$  a.e.,  $u$  is constant and  $u = 0$  since  $u(1) = 0$ . Thus  $\|\cdot\|_{H_a}$  is a norm on  $H_a$ , which is clearly Hilbertian.

If  $(u_n)$  is a Cauchy sequence in  $H_a$ , then by (2.6) and the fact that  $u_n(1) = 0$ ,  $(u_n)$  is a Cauchy sequence in  $W^{1,1}(\varepsilon, 1)$  for all  $\varepsilon > 0$ . Therefore, there exists  $u \in W_{loc}^{1,1}(0, 1)$  such that  $u_n \rightarrow u$  in  $W^{1,1}(\varepsilon, 1)$  for all  $\varepsilon > 0$ , hence in  $\mathcal{D}'(0, 1)$ . There is also some  $v \in L_a^2$  such that  $u'_n \rightarrow v$  in  $L_a^2$ . But for any  $\varphi \in \mathcal{D}(0, 1) := C_c^\infty((0, 1))$ ,

$$\left| \int_0^1 (u'_n - v)\varphi dx \right| \leq \left( \int_0^1 (u'_n - v)^2 a dx \right)^{\frac{1}{2}} \left( \int_0^1 \frac{\varphi^2}{a} dx \right)^{\frac{1}{2}} \rightarrow 0,$$

that is,  $u'_n \rightarrow v$  in  $\mathcal{D}'(0, 1)$ . We infer that  $u' = v \in L_a^2$ , and hence, with  $u(1) = 0$  (since  $u_n \rightarrow u$  in  $W^{1,1}(\varepsilon, 1)$ ),  $u \in H_a$  and  $u_n \rightarrow u$  in  $H_a$ . Therefore  $H_a$  is complete.

Let us now show that  $H_{a,\rho}$  is complete. We first need the following

**Lemma 2.2.** *Let  $\delta \in (0, 1)$ . Then for any  $\varepsilon > 0$ , there exists some number  $C_{\delta, \varepsilon} > 0$  such that*

$$\int_{\delta}^1 |u| dx \leq \varepsilon \int_{\delta}^1 |u'| dx + C_{\delta, \varepsilon} \left( \int_{\delta}^1 \rho u^2 dx \right)^{\frac{1}{2}} \quad \forall u \in H_{a, \rho}. \quad (2.7)$$

*Proof of Lemma 2.2:* If (2.7) does not hold, there exists  $\varepsilon > 0$  and a sequence  $(u_n)$  in  $H_{a, \rho}$  such that for all  $n \geq 1$

$$1 = \int_{\delta}^1 |u_n| dx > \varepsilon \int_{\delta}^1 |u'_n| dx + n \left( \int_{\delta}^1 \rho u_n^2 dx \right)^{\frac{1}{2}}. \quad (2.8)$$

Recall that  $\rho \in L^r(0, 1)$  for some  $r \in (p', \infty]$ , by (1.7) and (1.10). Since  $(u_n)$  is bounded in  $W^{1,1}(\delta, 1)$ , which is compactly embedded in  $L^{2r'}(\delta, 1)$  ( $2 \leq 2r' < 2p \leq \infty$ ), there is a subsequence  $(u_{n_k})$  such that  $u_{n_k} \rightarrow u$  in  $L^{2r'}(\delta, 1)$ . Then, by Hölder inequality,

$$\int_{\delta}^1 \rho (u_{n_k} - u)^2 dx \leq \left( \int_{\delta}^1 \rho^r dx \right)^{\frac{1}{r}} \left( \int_{\delta}^1 |u_{n_k} - u|^{2r'} dx \right)^{\frac{1}{r'}} \rightarrow 0,$$

and hence  $\int_{\delta}^1 \rho u_{n_k}^2 dx \rightarrow \int_{\delta}^1 \rho u^2 dx$ . But  $\int_{\delta}^1 \rho u_n^2 dx \rightarrow 0$  by (2.8), and hence  $\int_{\delta}^1 \rho u^2 dx = 0$ . It follows that  $u = 0$  a.e. in  $(\delta, 1)$ . But this contradicts the fact that  $1 = \int_{\delta}^1 |u_{n_k}| dx \rightarrow \int_{\delta}^1 |u| dx$ . Lemma 2.2 is proved.  $\square$

Combining (2.6) and (2.7), we obtain that for all  $\delta \in (0, 1)$ , there is some  $C_{\delta} > 0$  such that

$$\|u\|_{W^{1,1}(\delta, 1)} \leq C_{\delta} \|a\|_{H_{a, \rho}} \quad \forall u \in H_{a, \rho}. \quad (2.9)$$

Proceeding as above and using the fact that  $W^{1,1}(\varepsilon, 1) \subset C^0([\varepsilon, 1])$  continuously for any  $\varepsilon \in (0, 1)$ , one can prove that  $H_{a, \rho}$  is a Hilbert space.

The next result is concerned with the density of spaces of smooth functions.

**Lemma 2.3.** *The space  $\mathcal{D}(0, 1)$  is dense in  $H_a$  if  $a^{-1} \notin L^1(0, 1)$ , while the space  $\{\varphi \in C^{\infty}([0, 1]); \varphi(1) = 0\}$  is dense in  $H_a$  if  $a^{-1} \in L^1(0, 1)$ .*

*Proof.* 1. Assume first that

$$a^{-1} \notin L^1(0, 1). \quad (2.10)$$

Let  $u \in H_a$  with  $\int_0^1 au'\varphi' dx = 0$  for all  $\varphi \in \mathcal{D}(0, 1)$ . Note that  $au' \in L^1_{loc}(0, 1) \subset \mathcal{D}'(0, 1)$ , for

$$\int_{\delta}^{\varepsilon} |au'| dx \leq \left( \int_{\delta}^{\varepsilon} au'^2 dx \right)^{\frac{1}{2}} \left( \int_{\delta}^{\varepsilon} \frac{dx}{a} \right)^{\frac{1}{2}} < \infty \quad \text{if } 0 < \delta < \varepsilon < 1.$$

Thus  $\langle (au')', \varphi \rangle_{\mathcal{D}', \mathcal{D}} = -\langle au', \varphi' \rangle_{\mathcal{D}', \mathcal{D}} = 0$  for all  $\varphi \in \mathcal{D}(0, 1)$  and  $(au')' = 0$  in  $\mathcal{D}'(0, 1)$ . Thus there is some number  $K \in \mathbb{R}$  such that  $au' = K$  a.e. in  $(0, 1)$ . But  $\frac{K}{\sqrt{a}} = \sqrt{a}u' \in L^2(0, 1)$ , hence  $\int_0^1 \frac{K^2}{a} dx < \infty$  and  $K = 0$ , by (2.10). Thus  $u = 0$  and  $\mathcal{D}(0, 1)$  is dense in  $H_a$ .

2. Assume now that

$$a^{-1} \in L^1(0, 1). \quad (2.11)$$

Pick any  $u \in H_a$  with  $\int_0^1 au'\varphi' dx = 0$  for all  $\varphi \in C^{\infty}([0, 1])$  with  $\varphi(1) = 0$ . (Such functions  $\varphi$  belong to  $H_a$ , for  $\int_0^1 a\varphi'^2 dx < \infty$ .) In particular, as  $\int_0^1 au'\varphi' dx = 0$  for all  $\varphi \in \mathcal{D}(0, 1)$ , we infer



that  $au' = K$  a.e. in  $(0, 1)$  for some  $K \in \mathbb{R}$ . Thus

$$0 = \int_0^1 au'\varphi' dx = K \int_0^1 \varphi' dx = -K\varphi(0)$$

for all  $\varphi \in C^\infty([0, 1])$  with  $\varphi(1) = 0$ . This yields again  $K = 0$ ,  $u = 0$ , and the density of the space  $\{\varphi \in C^\infty([0, 1]); \varphi(1) = 0\}$  in  $H_a$ .  $\square$

The following result gives a generalized Hardy inequality (see [7, 16, 27]).

**Lemma 2.4.** *Let  $a : (0, 1) \rightarrow \mathbb{R}$  be as in (1.5) and (1.8). Extend  $a$  to  $(0, \infty)$  by setting*

$$a(x) = x^2 \quad \text{for } x \geq 1, \quad (2.12)$$

and let

$$b(x) = a(x)^{-1} \left( \int_x^\infty \frac{ds}{a(s)} \right)^{-2}, \quad x \in (0, \infty). \quad (2.13)$$

Then

$$\lim_{x \rightarrow 0^+} b(x) = +\infty \quad (2.14)$$

and

$$\int_0^1 b(x)u(x)^2 dx \leq 4 \int_0^1 a(x)u'(x)^2 dx \quad \forall u \in H_a. \quad (2.15)$$

*Proof.* First, we note that (2.14) follows from (1.8) and the fact that

$$\lim_{x \rightarrow 0^+} \frac{\int_x^1 a(s)^{-1} ds}{\int_x^\infty a(s)^{-1} ds} = \begin{cases} \frac{\int_0^1 a(s)^{-1} ds}{\int_0^\infty a(s)^{-1} ds} \in (0, \infty) & \text{if } \int_0^1 a(s)^{-1} ds < \infty; \\ 1 & \text{if } \int_0^1 a(s)^{-1} ds = \infty. \end{cases}$$

From [7, 27], if  $\alpha, \beta, f$  are nonnegative measurable functions defined on  $\mathbb{R}_+$ , if

$$K := \sup_{r>0} \left[ \left( \int_0^r \beta(x)^2 dx \right)^{\frac{1}{2}} \left( \int_r^\infty \alpha(x)^{-2} dx \right)^{\frac{1}{2}} \right] < \infty \quad (2.16)$$

then

$$\left( \int_0^\infty [\beta(x) \int_x^\infty f(t) dt]^2 dx \right)^{\frac{1}{2}} \leq 2K \left( \int_0^\infty [\alpha(x) f(x)]^2 dx \right)^{\frac{1}{2}}. \quad (2.17)$$

Pick  $\alpha(x) := \sqrt{a(x)}$  and  $\beta(x) := \sqrt{b(x)}$ . Let us check that condition (2.16) is satisfied. For  $0 < \varepsilon < r$ , we have that

$$\begin{aligned} \int_\varepsilon^r \beta(x)^2 dx &= \int_\varepsilon^r a(x)^{-1} \left( \int_x^\infty a(s)^{-1} ds \right)^{-2} dx = \int_\varepsilon^r \frac{d}{dx} \left( \int_x^\infty a(s)^{-1} ds \right)^{-1} dx \\ &= \left( \int_r^\infty a(s)^{-1} ds \right)^{-1} - \left( \int_\varepsilon^\infty a(s)^{-1} ds \right)^{-1}. \end{aligned}$$

Note that by (2.12),

$$l := \lim_{\varepsilon \rightarrow 0^+} \left( \int_\varepsilon^\infty \frac{ds}{a(s)} \right)^{-1} \in [0, 1)$$

always exists. It follows that

$$\int_0^1 \beta(x)^2 dx = 1 - l < \infty.$$

Thus

$$\int_0^r \beta(x)^2 dx \int_r^\infty \alpha(x)^{-2} dx = \left( \left( \int_r^\infty \frac{ds}{a(s)} \right)^{-1} - l \right) \int_r^\infty \frac{ds}{a(s)} \leq 1 \quad \forall r \in (0, +\infty)$$

and (2.16) is indeed satisfied with  $K \leq 1$ .

Pick now any  $u \in H_a$ . Extend  $u$  by 0 for  $x \geq 1$ . (Note that  $u \in W^{1,1}(\varepsilon, +\infty)$  for all  $\varepsilon \in (0, 1)$ .) Pick  $f(x) = |u'(x)|$  for  $x \in (0, +\infty)$ . Then for  $x \in (0, 1)$

$$|u(x)| = \left| \int_x^\infty u'(t) dt \right| \leq \int_x^\infty f(t) dt$$

and (2.15) follows from (2.17).  $\square$

By (2.14), one may pick  $x_0 \in (0, 1)$  such that  $b(x) \geq 1$  for  $0 < x < x_0$ . Then (2.15) yields

$$\int_0^{x_0} u(x)^2 dx \leq 4 \|u\|_{H_a}^2 \quad \forall u \in H_a.$$

Combined with (2.6) and the fact that  $u(1) = 0$ , we infer the existence of some constant  $C > 0$  such that

$$\int_0^1 u(x)^2 dx \leq C \|u\|_{H_a}^2.$$

Thus  $H_a \subset L^2(0, 1)$  continuously. Actually, the embedding is also compact.

**Lemma 2.5.** *The embedding  $H_a \subset L^2(0, 1)$  is compact.*

*Proof of Lemma 2.5:* Let  $(u_n)$  be a sequence in  $H_a$  and let  $u \in H_a$  be such that  $u_n \rightarrow u$  weakly in  $H_a$ . We have to show that  $u_n \rightarrow u$  strongly in  $L^2(0, 1)$ . Since for  $\delta \in (0, 1)$  the embedding  $W^{1,1}(\delta, 1) \subset L^2(\delta, 1)$  is compact, the map  $v \in H_a \rightarrow v|_{(\delta, 1)} \in L^2(\delta, 1)$  is compact for any  $\delta \in (0, 1)$ , and hence  $u_n \rightarrow u$  in  $L^2(\delta, 1)$ . Let  $\varepsilon > 0$  be given. By (2.14), there exists some  $\delta \in (0, 1)$  such that

$$b(x) \geq B := \frac{8}{\varepsilon^2} \left( 1 + 4 \sup_{n \geq 0} \|u_n\|_{H_a}^2 \right) \quad \forall x \in (0, \delta).$$

Using the fact that  $\|u\|_{H_a} \leq \sup_{n \geq 0} \|u_n\|_{H_a}$  and (2.15), we obtain

$$\int_0^\delta |u_n(x) - u(x)|^2 dx \leq B^{-1} \int_0^\delta b(x) |u_n(x) - u(x)|^2 dx \leq 4B^{-1} \|u_n - u\|_{H_a}^2 \leq \frac{\varepsilon^2}{2}.$$

Since  $u_n \rightarrow u$  in  $L^2(\delta, 1)$ , we have that  $\int_\delta^1 |u_n(x) - u(x)|^2 dx \leq \varepsilon^2/2$  for  $n \geq n_0$  for some  $n_0 \in \mathbb{N}$ . Thus  $\|u_n - u\|_{L^2(0,1)} \leq \varepsilon$  for  $n \geq n_0$ , and  $u_n \rightarrow u$  in  $L^2(0, 1)$ .  $\square$

**Lemma 2.6.** *The embeddings  $H_{a,\rho} \subset L^2(0, 1)$  and  $H_{a,\rho} \subset L_\rho^2$  are compact. The space  $C^\infty([0, 1])$  is dense in  $H_{a,\rho}$ .*

*Proof of Lemma 2.6:* By (1.7) one may pick some numbers  $\delta \in (0, \frac{1}{2})$  and  $C > 0$  such that

$$0 \leq \rho(x) \leq C \quad \forall x \in (0, 2\delta). \quad (2.18)$$

Let  $\theta \in C^\infty([0, 1])$  be such that

$$\theta(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq \delta, \\ 0 & \text{if } 2\delta \leq x \leq 1. \end{cases}$$

Let  $u \in H_{a,\rho}$ . Then  $\theta u \in H_a$ , for  $(\theta u)(1) = 0$  and

$$\sqrt{a}(\theta u)' = (\sqrt{a}u')\theta + \sqrt{a}\theta'u \in L^2(0, 1).$$

Indeed,  $\sqrt{a}u' \in L^2(0, 1)$ ,  $\text{Supp } \theta' \subset [\delta, 2\delta]$ ,  $\sqrt{a} \in L^2(\delta, 2\delta)$ , and  $u \in L^\infty(\delta, 2\delta)$ . Furthermore,

$$\|(\theta u)'\|_{L_a^2} \leq C\|u\|_{H_{a,\rho}},$$

so that the map  $u \in H_{a,\rho} \rightarrow \theta u \in H_a$  is continuous. The embedding  $H_a \subset L^2(0, 1)$  being compact, the map  $u \in H_{a,\rho} \rightarrow \theta u \in L^2(0, 1)$  is compact. Clearly, the map  $u \in H_{a,\rho} \rightarrow (1-\theta)u \in W^{1,1}(0, 1)$  is continuous. The embedding  $W^{1,1}(0, 1) \subset L^2(0, 1)$  being compact, we infer that the map  $u \in H_{a,\rho} \rightarrow (1-\theta)u \in L^2(0, 1)$  is compact. Thus the embedding  $H_{a,\rho} \subset L^2(0, 1)$  is compact.

The fact that  $H_{a,\rho} \subset L_\rho^2$  continuously comes from the definition of the spaces  $H_{a,\rho}$  and  $L_\rho^2$  and of their norms. Using (4.2) and the lines above, we infer that the map  $u \in H_{a,\rho} \rightarrow \theta u \in L_\rho^2$  is compact. On the other hand, the embedding  $W^{1,1}(0, 1) \subset L^{2r'}(0, 1)$  is compact, and by Hölder inequality

$$\int_0^1 \rho(1-\theta)^2 u^2 dx \leq C \left( \int_0^1 \rho^r dx \right)^{\frac{1}{r}} \left( \int_0^1 u^{2r'} dx \right)^{\frac{1}{r'}}.$$

It follows that the map  $u \in H_{a,\rho} \rightarrow (1-\theta)u \in L_\rho^2$  is compact. Thus the embedding  $H_{a,\rho} \subset L_\rho^2$  is compact.

Let us prove that  $C^\infty([0, 1])$  is dense in  $H_{a,\rho}$ . Pick any  $u \in H_{a,\rho}$ . If we set

$$a(x) = x, \quad \rho(x) = 1, \quad u(x) = (2-x)u(1) \quad \forall x \in (1, 2),$$

then  $u \in H_a(0, 2) := \{u \in L_{loc}^1(0, 2); \sqrt{a}u' \in L^2(0, 2) \text{ and } u(2) = 0\}$ . As in Lemma 2.3,  $C^\infty([0, 2])$  is dense in  $H_a(0, 2)$ , so that we can pick a sequence  $(\varphi_n)$  in  $C^\infty([0, 2])$  with  $\varphi_n \rightarrow u$  in  $H_a(0, 2)$ , and also in  $L^2(0, 2)$ . This gives

$$\int_0^1 [a(\varphi_n' - u')^2 + (\varphi_n - u)^2] dx \rightarrow 0.$$

By (4.2), we have  $\int_0^{2\delta} \rho(\varphi_n - u)^2 dx \rightarrow 0$ . Since  $\varphi_n \rightarrow u$  in  $W^{1,1}(2\delta, 1)$  and in  $L^{2r'}(2\delta, 1)$ , we also have that

$$\int_{2\delta}^1 \rho(\varphi_n - u)^2 dx \rightarrow 0.$$

We conclude that  $\varphi_n \rightarrow u$  in  $H_{a,\rho}$ . □

The proof of Proposition 2.1 is complete. □

Next, we investigate the elliptic problem (2.1)-(2.3). Introduce the symmetric bilinear form

$$a(u, v) = \int_0^1 au'v' dx + a_b(u, v)$$

where

$$a_b(u, v) = \begin{cases} \frac{\alpha}{\beta}u(1)v(1) & \text{if } \beta \neq 0, \\ 0 & \text{if } \beta = 0. \end{cases}$$

Let

$$H = \begin{cases} H_{a,\rho} & \text{if } \alpha\beta \neq 0, \\ \{u \in H_{a,\rho}; u(1) = 0\} & \text{if } \beta = 0, \\ \{u \in H_{a,\rho}; \int_0^1 u\rho dx = 0\} & \text{if } \alpha = 0, \end{cases}$$

be endowed with the norm  $\|\cdot\|_{H_{a,\rho}}$ . By (2.9),  $H$  is a closed subspace of  $H_{a,\rho}$ , and the bilinear form  $a$  is continuous on  $H \times H$ . To prove that the bilinear form is coercive, we need the following lemma.

**Lemma 2.7.** *There exist a constant  $C > 0$  such that*

$$\int_0^1 |u|^2 \rho dx \leq C \left( \int_0^1 |u'|^2 a dx + u(1)^2 \right) \quad \forall u \in H_{a,\rho}, \quad (2.19)$$

$$\int_0^1 |u|^2 \rho dx \leq C \left( \int_0^1 |u'|^2 a dx + \left( \int_0^1 u\rho dx \right)^2 \right) \quad \forall u \in H_{a,\rho}. \quad (2.20)$$

*Proof of Lemma 2.7:* We prove (2.19) only, the proof of (2.20) being similar. If (2.19) is false, one can pick a sequence  $(u_n)$  in  $H_{a,\rho}$  such that

$$1 = \int_0^1 |u_n|^2 \rho dx > n \left( \int_0^1 |u_n'|^2 a dx + u_n(1)^2 \right). \quad (2.21)$$

As the sequence  $(u_n)$  is bounded in  $H_{a,\rho}$ , one can extract a subsequence  $(u_{n_k})$  such that  $u_{n_k} \rightharpoonup u$  weakly in  $H_{a,\rho}$  for some  $u \in H_{a,\rho}$ . By Lemma 2.6,  $u_{n_k} \rightarrow u$  strongly in  $L^2_\rho$ . As  $u_{n_k}' \rightarrow 0$  strongly in  $L^2_a$  by (2.21), we infer that  $(u_{n_k})$  is a Cauchy sequence in  $H_{a,\rho}$ , and hence  $u_{n_k} \rightarrow u$  strongly in  $H_{a,\rho}$ . Letting  $n \rightarrow \infty$  in (2.21), we obtain

$$\int_0^1 |u'|^2 a dx = 0, \quad u(1) = 0.$$

Thus  $u = 0$ , but this contradicts the condition  $\int_0^1 |u|^2 \rho dx = 1$ . The proof of Lemma 2.7 is complete.  $\square$

We have to prove that

$$a(u, u) \geq K \|u\|_{H_{a,\rho}}^2 \quad \forall u \in H \quad (2.22)$$

for some constant  $K > 0$ . If  $\beta = 0$  (resp.  $\alpha = 0$ ), then (2.19) (resp. (2.20)) yields  $\|u\|_{L^2_\rho}^2 \leq C \|u'\|_{L^2_a}^2$  for  $u \in H$ , which gives (2.22).

If  $\alpha\beta > 0$ , then (2.19) yields

$$\int_0^1 |u|^2 \rho dx \leq C' a(u, u)$$

for some  $C' > 0$ , which gives again (2.22). Thus the bilinear form  $a$  is coercive.

Let  $f \in L^2_\rho$  be given (with also  $\int_0^1 f \rho dx = 0$  if  $\alpha = 0$ ). Since the linear form  $L(v) = \int_0^1 f v \rho dx$  is continuous on  $H$ , it follows from Lax-Milgram theorem that there exists a unique function  $u \in H$  such that

$$a(u, v) = L(v) \quad \forall v \in H. \quad (2.23)$$

Assume first  $\alpha\beta \neq 0$ . Clearly  $\mathcal{D}(0, 1) \subset H$ . Picking  $v \in \mathcal{D}(0, 1)$  in (2.23), we obtain (2.1) in the sense of distributions. Multiplying in (2.1) by  $v \in C^\infty([0, 1]) \subset H$ , integrating by parts and comparing with (2.23), we infer that

$$(au')(1)v(1) - (au')(0)v(0) + \frac{\alpha}{\beta}u(1)v(1) = 0.$$

Picking  $v(x) = 1 - x$  (resp.  $v(x) = 1$ ) yields (2.2) (resp. (2.3)).

If  $\beta = 0$ , we have again  $\mathcal{D}(0, 1) \subset H$ , and picking  $v \in \mathcal{D}(0, 1)$  in (2.23) yields (2.1) in the sense of distributions. Multiplying in (2.1) by  $v(x) = 1 - x$  and integrating by parts yields (2.2). Finally (2.3) comes from the definition of  $H$ .

If  $\alpha = 0$ , then for any given  $\theta \in \mathcal{D}(0, 1)$  the function  $\varphi(x) = \theta(x) - \int_0^1 \theta \rho ds$  belongs to  $H$ . Picking such a function in (2.23) and using the fact that  $\int_0^1 f \rho dx = 0$ , we obtain  $\int_0^1 au'\theta' dx = \int_0^1 f\theta\rho dx$ , so that (2.1) holds in the sense of distributions. Multiplying in (2.1) by  $v(x) = 1$  (resp. by  $v(x) = 1 - 2x$ ), we obtain (2.2)-(2.3).

Finally, using

$$\int_0^1 |\rho f| dx \leq \left( \int_0^1 \rho f^2 dx \right)^{\frac{1}{2}} \left( \int_0^1 \rho dx \right)^{\frac{1}{2}} < \infty,$$

we infer from (2.1) that  $au' \in W^{1,1}(0, 1)$  for any value of  $(\alpha, \beta)$ .

We are in a position to study the spectral problem associated with (2.1)-(2.3).

**Theorem 2.8.** *Let  $a, \rho$  and  $(\alpha, \beta)$  be as above. Then there are a sequence  $(e_n)_{n \geq 0}$  in  $L^2_\rho$  and a nondecreasing sequence  $(\lambda_n)_{n \geq 0}$  in  $(0, +\infty)$  such that*

(i)  $(e_n)_{n \geq 0}$  is an orthonormal basis in  $L^2_\rho$ ;

(ii) for all  $n \geq 0$ ,  $e_n \in H_{a,\rho}$ ,  $ae'_n \in W^{1, \min(2,r)}(0, 1)$ , and  $e_n$  solves

$$-(ae'_n)' = \lambda_n \rho e_n \quad \text{in } (0, 1), \quad (2.24)$$

$$(ae'_n)(0) = 0, \quad (2.25)$$

$$\alpha e_n(1) + \beta (ae'_n)(1) = 0. \quad (2.26)$$

*Proof.* Assume first that  $\alpha \neq 0$ . For  $f \in L^2_\rho$ , let  $T(f)$  denote the unique solution  $u \in H$  of (2.23). The operator  $T : f \in L^2_\rho \rightarrow u = T(f) \in L^2_\rho$  is continuous, compact, and selfadjoint by (2.22), (2.23), and Proposition 2.1. It is also positive definite, for

$$K\|u\|_{H_{a,\rho}}^2 \leq a(u, u) = (f, u)_{L^2_\rho}.$$

By the spectral theorem, there are an orthonormal basis  $(e_n)_{n \geq 0}$  in  $L^2_\rho$  and a sequence  $(\mu_n)_{n \geq 0}$  in  $(0, +\infty)$  with  $\mu_n \searrow 0$  such that  $T(e_n) = \mu_n e_n$  for all  $n \geq 0$ . Then (2.24)-(2.26) hold with  $\lambda_n := \mu_n^{-1} > 0$ .

Assume now that  $\alpha = 0$ , and let  $V := \{f \in L^2_\rho; \int_0^1 f \rho dx = 0\}$ . For  $f \in V$ , let  $T(f)$  denote the unique solution  $u \in H$  of (2.23). Again, the operator  $T : f \in V \rightarrow u = T(f) \in V$  is continuous, compact, selfadjoint and positive definite. Therefore there are an orthonormal basis  $(e_n)_{n \geq 1}$  in

$V$  and a sequence  $(\mu_n)_{n \geq 1}$  in  $(0, +\infty)$  with  $\mu_n \searrow 0$  such that  $T(e_n) = \mu_n e_n$  for all  $n \geq 1$ . Let  $e_0 := 1$ . Noticing that  $V = \{u \in L^2_\rho; (u, e_0)_{L^2_\rho} = 0\}$ , we see that  $(e_n)_{n \geq 0}$  is an orthonormal basis of  $L^2_\rho$  and that (2.24)-(2.26) hold with  $\lambda_0 := 0$  and  $\lambda_n := \mu_n^{-1} > 0$  for  $n \geq 1$ .

By (1.7), one can pick some numbers  $C > 0$  and  $\delta \in (0, 1)$  such that  $0 \leq \rho(x) \leq C$  for  $0 < x < \delta$ , so that  $(ae'_n)' = -\lambda_n \rho e_n \in L^2(0, \delta)$ . Next,  $e_n \in W^{1,1}(\delta, 1) \subset L^\infty(\delta, 1)$ , and  $(ae'_n)' \in L^r(\delta, 1)$ . Thus  $(ae'_n)' \in L^{\min(2,r)}(0, 1)$  and  $ae'_n \in W^{1, \min(2,r)}(0, 1)$ .  $\square$

We are now interested in the asymptotic behavior of the eigenvalues  $\lambda_n$ ,  $n \geq 0$ . Indeed, to apply the flatness approach, we need to prove that  $\lambda_n \geq Cn^\kappa$  for some  $C, \kappa > 0$  and all  $n \geq 0$ . The estimate we shall derive is likely not sharp, but it is sufficient for the sequel.

**Theorem 2.9.** *Let  $a, \rho, (\alpha, \beta)$  and the sequences  $(e_n)_{n \geq 0}$ ,  $(\lambda_n)_{n \geq 0}$  be as in Theorem 2.8. Then*

- (i)  $e_n \in W^{1,1}(0, 1)$  and  $ae'_n \in W^{1,r}(0, 1)$  for all  $n \geq 0$ ;
- (ii) there exists some constant  $C_1 > 0$  such that

$$\|e_n\|_{L^\infty(0,1)} \leq C_1 \lambda_n^{\frac{3}{4}(1+\frac{p'r}{r-p})} \quad \text{if } \lambda_n > 0; \quad (2.27)$$

(iii) let  $\kappa := [\frac{1}{2} + \frac{1}{p}(\frac{p'r}{r-p})]^{-1} > 0$  if  $p < \infty$  and pick any  $\kappa < 2$  if  $p = \infty$ . Then there exists some constant  $C_2 > 0$  such that

$$\lambda_n \geq C_2 n^\kappa \quad \forall n \geq 0. \quad (2.28)$$

*Proof.* (i) We need several lemmas.

**Lemma 2.10.** *Let  $p, p'$  be as in (1.10), let  $a$  be as in (1.5)-(1.6), and let  $q \in L^{p'}(0, 1)$ . Then there exists a unique function  $v \in W^{1,1}(0, 1)$  with  $av' \in W^{1,p'}(0, 1)$  and such that*

$$(av')' = -qv \quad \text{a.e. in } (0, 1), \quad (2.29)$$

$$(av')(0) = 0, \quad (2.30)$$

$$v(0) = 1. \quad (2.31)$$

*Proof of Lemma 2.10.* If a function  $v$  as in Lemma 2.10 does exist, then  $v \in C^0([0, 1])$  and we obtain by successive integrations

$$(av')(y) = - \int_0^y (qv)(s) ds \quad \forall y \in [0, 1], \quad (2.32)$$

and

$$v(x) = 1 - \int_0^x \frac{dy}{a(y)} \int_0^y (qv)(s) ds \quad \forall x \in [0, 1]. \quad (2.33)$$

Using Hardy inequality

$$\int_0^1 \left[ \frac{1}{y} \int_0^y |q(s)| ds \right]^{p'} dy \leq C \int_0^1 |q(s)|^{p'} ds,$$

Hölder inequality and (1.6), we infer that

$$\int_0^1 \frac{dy}{a(y)} \int_0^y |q(s)| ds < \infty. \quad (2.34)$$

Therefore, if  $v \in C^0([0, 1])$  satisfies (2.33), then  $v \in W^{1,1}(0, 1)$  by (2.34) and  $av' \in W^{1,p'}(0, 1)$  by (2.32). Therefore, it is sufficient to prove the existence and uniqueness of a solution  $v \in C^0([0, 1])$  of (2.33).

First, one may pick some  $\delta \in (0, 1)$  such that for all  $x_1, x_2 \in [0, 1]$  with  $0 \leq x_2 - x_1 \leq \delta$ , we have

$$\int_{x_1}^{x_2} \frac{dy}{a(y)} \int_0^y |q(s)| ds \leq \frac{1}{2}.$$

For  $0 \leq x_1 < x_2 \leq 1$  and  $\gamma_1, \gamma_2 \in \mathbb{R}$  with  $\gamma_1 = 0$  if  $x_1 = 0$ , we consider the map  $\Gamma : C^0([x_1, x_2]) \rightarrow C^0([x_1, x_2])$  defined by

$$\Gamma(v)(x) = \gamma_2 - \int_{x_1}^x \frac{dy}{a(y)} \left( \gamma_1 + \int_{x_1}^y (qv)(s) ds \right). \quad (2.35)$$

By (1.6) and (2.34), the map  $\Gamma$  is well defined. Let us prove that it is a contraction provided that  $|x_2 - x_1|$  is “small enough”. For  $v, w \in C^0([x_1, x_2])$ , we have

$$\|\Gamma(v) - \Gamma(w)\|_{L^\infty(x_1, x_2)} \leq \int_{x_1}^{x_2} \frac{dy}{a(y)} \int_{x_1}^y |q(s)(v-w)(s)| ds.$$

The map  $\Gamma$  is a contraction in  $C^0([0, \delta])$  for  $x_1 = 0, x_2 = \delta, \gamma_1 = 0$  and  $\gamma_2 = 1$ , since

$$\|\Gamma(v) - \Gamma(w)\|_{L^\infty(0, \delta)} \leq \frac{1}{2} \|v - w\|_{L^\infty(0, \delta)}.$$

This gives a solution  $v$  of (2.33) on  $[0, \delta]$  by the contraction principle. Note that

$$v(\delta) = 1 - \int_0^\delta \frac{dy}{a(y)} \int_0^y (qv)(s) ds. \quad (2.36)$$

If  $\delta \leq x_1 \leq x_2 \leq 1$  and  $x_2 - x_1 \leq \delta$ , then

$$\begin{aligned} \|\Gamma(v) - \Gamma(w)\|_{L^\infty(x_1, x_2)} &\leq \left( \int_{x_1}^{x_2} \frac{dy}{a(y)} \int_{x_1}^y |q(s)| ds \right) \|v - w\|_{L^\infty(x_1, x_2)} \\ &\leq \frac{1}{2} \|v - w\|_{L^\infty(x_1, x_2)}. \end{aligned}$$

Then  $\Gamma$  is a contraction in  $C^0([x_1, x_2])$  if  $\delta \leq x_1 \leq x_2 \leq 1$  and  $x_2 - x_1 \leq \delta$ . Picking  $x_1 = \delta, x_2 = x_1 + \delta, \gamma_1 = \int_0^\delta (qv)(s) ds$ , and  $\gamma_2 = v(\delta)$ , we obtain a solution of (2.33) on  $[0, 2\delta]$ . We can proceed in a similar way to extend  $v$  to  $[0, 3\delta], [0, 4\delta], \dots$  and finally to  $[0, 1]$ .  $\square$

**Corollary 2.11.** *Let  $\rho$  and  $r$  be as in (1.7) and (1.10), and let  $\lambda \in \mathbb{R}$ . Then there exists a unique function  $u \in W^{1,1}(0, 1)$  with  $au' \in W^{1,r}(0, 1)$  such that*

$$(au')' = -\lambda\rho u \quad \text{a.e. in } (0, 1), \quad (2.37)$$

$$(au')(0) = 0, \quad (2.38)$$

$$u(0) = 1. \quad (2.39)$$

Furthermore, the map  $\lambda \rightarrow (u, au')$  from  $\mathbb{R}$  to  $W^{1,1}(0, 1) \times W^{1,r}(0, 1)$  is continuous.

*Proof.* Since  $\rho \in L^r(0,1) \subset L^{p'}(0,1)$ , the existence and uniqueness of  $u$  follows from Lemma 2.10. The continuity of the map  $\lambda \in \mathbb{R} \rightarrow u \in C^0([0,\delta])$  (or  $C^0([\delta,2\delta])$ , etc.) follows from the version of the contraction principle with a parameter. Using (2.32) and (2.33) gives the last sentence in the statement.  $\square$

**Lemma 2.12.** *Let  $\lambda \in \mathbb{R}$ . If  $0 < x_0 - \delta < x_0 + \delta < 1$  and  $u \in W^{1,1}(x_0 - \delta, x_0 + \delta)$  is such that  $au' \in W^{1,1}(x_0 - \delta, x_0 + \delta)$  and*

$$\begin{cases} (au')' = -\lambda\rho u & \text{in } (x_0 - \delta, x_0 + \delta), \\ u(x_0) = 0, \\ (au')(x_0) = 0, \end{cases}$$

then  $u = 0$  in  $(x_0 - \delta, x_0 + \delta)$ .

The proof is similar to those of Lemma 2.10 by applying the contraction principle to the map  $\Gamma$  from  $C^0([x_0 - \delta, x_0 + \delta])$  into itself defined by

$$\Gamma(u)(x) = -\lambda \int_{x_0}^x \frac{dy}{a(y)} \int_{x_0}^y (\rho u)(s) ds$$

when  $\delta > 0$  is small enough, and by propagating the uniqueness up to  $[x_0 - \delta, x_0 + \delta]$  when  $\delta$  is as in the statement of the lemma.  $\square$

**Lemma 2.13.** *Let  $n \geq 0$ , and let  $e_n$  and  $\lambda_n$  be as in Theorem 2.8. If  $u$  is the function associated with  $\lambda = \lambda_n$  in Corollary 2.11, then*

$$e_n(x) = e_n(0)u(x) \quad \forall x \in (0,1), \quad (2.40)$$

so that  $e_n \in W^{1,1}(0,1)$  and  $ae'_n \in W^{1,r}(0,1)$ .

*Proof of Lemma 2.13:* As  $u(0) = 1$  and  $u \in C^0([0,1])$ , we can pick some  $\varepsilon \in (0, 1/2)$  such that

$$u(x) \geq \frac{1}{2} \quad \forall x \in [0, 2\varepsilon].$$

Let

$$v(x) := u(x) \int_{\varepsilon}^x \frac{ds}{u(s)^2 a(s)} \quad x \in (0, 2\varepsilon].$$

Since  $a^{-1} \in L^1_{loc}(0,1)$ ,  $v \in W^{1,1}_{loc}(0, 2\varepsilon)$  and

$$v'(x) = u'(x) \int_{\varepsilon}^x \frac{ds}{u(s)^2 a(s)} + \frac{1}{u(x)a(x)} \quad \text{a.e. in } (0, 2\varepsilon). \quad (2.41)$$

Since  $u, au' \in W^{1,1}(0,1)$ , it follows that  $av' \in W^{1,1}_{loc}(0, 2\varepsilon)$  and that

$$\begin{aligned} (av')'(x) &= (au')'(x) \int_{\varepsilon}^x \frac{ds}{u(s)^2 a(s)} + \frac{(au')(x)}{u(x)^2 a(x)} - \frac{u'(x)}{u(x)^2} \\ &= -\lambda(\rho u)(x) \int_{\varepsilon}^x \frac{ds}{u(s)^2 a(s)} \\ &= -\lambda(\rho v)(x) \quad \text{a.e. in } (0, 2\varepsilon). \end{aligned}$$



Note that the function  $u$  satisfies (2.37) for  $\lambda = \lambda_n$  in  $(0, 2\varepsilon)$  and that it is not proportional to  $v$  (otherwise we would have  $[u(s)^2 a(s)]^{-1} = 0$  a.e.). Note also that  $v(\varepsilon) = 0$  and  $(av')(\varepsilon) = 1/u(\varepsilon) > 0$ . We may therefore find some number  $\mu \in \mathbb{R}$  so that the function  $U(x) := e_n(x) - \frac{e_n(\varepsilon)}{u(\varepsilon)}u(x) - \mu v(x)$  satisfies the assumptions of Lemma 2.12 for  $0 < \delta < x_0 = \varepsilon$ . We infer from Lemma 2.12 that  $U = 0$  on  $(0, 2\varepsilon)$ , that is

$$e_n(x) = \frac{e_n(\varepsilon)}{u(\varepsilon)}u(x) + \mu v(x), \quad x \in (0, 2\varepsilon). \quad (2.42)$$

If  $\mu \neq 0$ , then by (2.25), (2.38) and (2.42), we obtain  $(av')(0) = 0$  which, combined to (2.39) and (2.41), yields

$$[au'(x) \int_{\varepsilon}^x \frac{ds}{u(s)^2 a(s)}](0^+) = -1.$$

But this is impossible, since  $\int_{\varepsilon}^x \frac{ds}{u(s)^2 a(s)} < 0$  and  $(au')(x) \leq 0$  for  $0 < x < \varepsilon$ . Indeed,  $\lambda_n \geq 0$  yields

$$(au')(x) = - \int_0^x \lambda_n \rho(x) u(x) dx \leq 0, \quad 0 < x < \varepsilon.$$

Thus  $\mu = 0$  and  $e_n(x) = \frac{e_n(\varepsilon)}{u(\varepsilon)}u(x)$  for  $x \in (0, 2\varepsilon)$ .

Using Lemma 2.12 several times, we infer that

$$e_n(x) = \frac{e_n(\varepsilon)}{u(\varepsilon)}u(x) \quad \forall x \in (0, 1).$$

Thus, by Corollary 2.11,  $e_n \in W^{1,1}(0, 1)$ ,  $ae'_n \in W^{1,r}(0, 1)$ , and letting  $x \rightarrow 0$  yields  $e_n(0) = e_n(\varepsilon)/u(\varepsilon)$ , so that

$$e_n(x) = e_n(0)u(x) \quad \forall x \in (0, 1).$$

The proof of Lemma 2.13 is complete. This ends the proof of (i) in Theorem 2.9.

(ii) In what follows, we fix some  $n \geq 0$  for which  $\lambda_n > 0$  and denote  $e = e_n$  and  $\lambda = \lambda_n$  to simplify the writing. The letter  $C$  will denote a constant that may change from line to line.

• Assume first that  $\alpha \neq 0$ . We first check that

$$|e(1)| \leq \sqrt{\frac{\beta}{\alpha}} \sqrt{\lambda} \quad (2.43)$$

The estimate (2.43) is obvious if  $\beta = 0$ , for  $e(1) = 0$ . If  $\beta \neq 0$ , we infer from (2.23) with  $u = v = e$  that

$$\int_0^1 a|e'|^2 dx + \frac{\alpha}{\beta} e(1)^2 = \lambda \int_0^1 e^2 \rho dx.$$

This yields

$$|e(1)|^2 \leq \frac{\beta}{\alpha} \lambda \int_0^1 e^2 \rho dx = \frac{\beta}{\alpha} \lambda.$$

Note also that

$$\int_0^1 a|e'|^2 dx \leq \lambda \int_0^1 e^2 \rho dx = \lambda.$$

But for  $0 < x < 1$  we have that

$$|e(x) - e(1)| \leq \int_x^1 |e'(s)| ds \leq \left( \int_x^1 \frac{ds}{a(s)} \right)^{\frac{1}{2}} \left( \int_x^1 a e'^2 ds \right)^{\frac{1}{2}} \leq \sqrt{\lambda} \left( \int_x^1 \frac{ds}{a(s)} \right)^{\frac{1}{2}}$$

and

$$\int_x^1 \frac{ds}{a(s)} \leq \frac{1}{x} \int_x^1 \frac{s}{a(s)} ds \leq \frac{1}{x} \left\| \frac{s}{a(s)} \right\|_{L^1} \quad (2.44)$$

where  $\left\| \frac{s}{a(s)} \right\|_{L^1}$  denotes  $\int_0^1 \frac{s}{a(s)} ds$ . It follows that

$$|e(x)| \leq \left( \frac{\beta}{\alpha} \lambda \right)^{\frac{1}{2}} + \left( \left\| \frac{s}{a(s)} \right\|_{L^1} \frac{\lambda}{x} \right)^{\frac{1}{2}}, \quad 0 < x < 1. \quad (2.45)$$

On the other hand, for  $0 < x < 1$ , by Hölder inequality

$$\begin{aligned} \int_0^x \frac{1}{a(s)} \left( \int_0^s \rho(t) dt \right) ds &\leq \|\rho\|_{L^r} \int_0^x \frac{s^{1-\frac{1}{r}}}{a(s)} ds \\ &\leq \|\rho\|_{L^r} \left\| \frac{s}{a(s)} \right\|_{L^p} \left( \int_0^x s^{-\frac{p'}{r}} ds \right)^{\frac{1}{p'}} \\ &\leq C x^{\frac{1}{p'} - \frac{1}{r}} \end{aligned} \quad (2.46)$$

with  $C := \|\rho\|_{L^r} \left\| \frac{s}{a(s)} \right\|_{L^p} (1 - \frac{p'}{r})^{-\frac{1}{p'}}$ . The function  $u(x)$  still denoting the solution to (2.37)-(2.39), we have that

$$u'(x) = -\frac{\lambda}{a(x)} \int_0^x (\rho u)(s) ds \leq 0$$

if  $u \geq 0$  on  $[0, x]$ , and hence for such  $x$

$$0 \leq 1 - u(x) \leq \int_0^x |u'(s)| ds \leq \lambda \|u\|_{L^\infty(0,x)} \int_0^x \frac{1}{a(s)} \left( \int_0^s \rho(t) dt \right) ds \leq C \lambda x^{\frac{1}{p'} - \frac{1}{r}}.$$

It follows that

$$\frac{1}{2} \leq u(x) \leq 1 \quad \text{for } x \in [0, (2C\lambda)^{\frac{p'r}{p'-r}}].$$

(Note that  $\lambda^{\frac{p'r}{p'-r}} \rightarrow 0$  as  $\lambda \rightarrow \infty$ , for  $r > p'$ .) Replacing  $e$  by  $-e$  if needed, we can assume that  $e(0) > 0$ . Using Lemma 2.13, we infer that

$$\frac{e(0)}{2} \leq e(x) \leq e(0) \quad \text{for } x \in [0, (2C\lambda)^{\frac{p'r}{p'-r}}]. \quad (2.47)$$

It follows from (2.24) that

$$\begin{aligned} e(0)^2 - e(1)^2 &= -2 \int_0^1 e e' dx = 2\lambda \int_0^1 \frac{e(x)}{a(x)} \left( \int_0^x \rho e ds \right) dx \\ &= 2\lambda \left( \int_0^{(4C\lambda)^{\frac{p'r}{p'-r}}} \cdots dx + \int_{(4C\lambda)^{\frac{p'r}{p'-r}}}^1 \cdots dx \right) =: 2\lambda(I_1 + I_2). \end{aligned} \quad (2.48)$$

Then, by (2.46) and (2.47),

$$|I_1| \leq e(0)^2 \int_0^{(4C\lambda)^{\frac{p'r}{p'-r}}} \frac{1}{a(x)} \left( \int_0^x \rho(s) ds \right) dx \leq \frac{e(0)^2}{4\lambda}. \quad (2.49)$$

On the other hand, by (2.45), we have for  $x \in ((4C\lambda)^{\frac{p'r}{p'-r}}, 1)$

$$\begin{aligned} |e(x)| &\leq \left(\frac{\beta}{\alpha}\lambda\right)^{\frac{1}{2}} + \left\| \frac{s}{a(s)} \right\|_{L^1} (4C)^{-\frac{p'r}{p'-r}} \frac{1}{2} \lambda^{\frac{1}{2}(1-\frac{p'r}{p'-r})} \\ &\leq C' \lambda^{\frac{1}{2}(1+\frac{p'r}{r-p'})} \end{aligned} \quad (2.50)$$

for  $\lambda \geq 1$ , where  $C' := \sqrt{\frac{\beta}{\alpha}} + \left\| \frac{s}{a(s)} \right\|_{L^1}^{\frac{1}{2}} (4C)^{-\frac{p'r}{2(p'-r)}}$ . Since by Cauchy-Schwarz inequality

$$\int_0^x \rho |e| ds \leq \left( \int_0^1 \rho e^2 dx \right)^{\frac{1}{2}} \left( \int_0^1 \rho dx \right)^{\frac{1}{2}} = \|\rho\|_{L^1}^{\frac{1}{2}}$$

we obtain with (2.44) that

$$|I_2| \leq \int_{(4C\lambda)^{\frac{p'r}{p'-r}}}^1 \frac{|e(x)|}{a(x)} \left( \int_0^x \rho(s) |e(s)| ds \right) dx \leq C' \frac{\lambda^{\frac{1}{2}(1+\frac{p'r}{r-p'})}}{(4C\lambda)^{\frac{p'r}{p'-r}}} \left\| \frac{s}{a(s)} \right\|_{L^1} \|\rho\|_{L^1}^{\frac{1}{2}}. \quad (2.51)$$

Gathering together (2.43) and (2.48)-(2.51), we infer that

$$\frac{1}{2} e(0)^2 \leq \frac{\beta}{\alpha} \lambda + \frac{2C'}{(4C)^{\frac{p'r}{p'-r}}} \left\| \frac{s}{a(s)} \right\|_{L^1} \|\rho\|_{L^1}^{\frac{1}{2}} \lambda^{\frac{3}{2}(1+\frac{p'r}{r-p'})}.$$

It follows that

$$|e(0)| \leq C'' \lambda^{\frac{3}{4}(1+\frac{p'r}{r-p'})}$$

for  $\lambda \geq 1$ , where  $C'' = C''(p, r, \|\rho\|_{L^1}, \left\| \frac{s}{a(s)} \right\|_{L^1}) > 0$ . As

$$e(y)^2 - e(1)^2 = 2\lambda \int_y^1 \frac{e(x)}{a(x)} \left( \int_0^x \rho e ds \right) dx, \quad \forall y \in (0, 1),$$

and  $|e(1)| \leq \sqrt{\beta\lambda/\alpha}$ , the same calculations as above yield

$$|e(y)| \leq C'' \lambda^{\frac{3}{4}(1+\frac{p'r}{r-p'})} \quad \forall y \in (0, 1) \quad (2.52)$$

for  $\lambda \geq 1$ .

• Assume now that  $\alpha = 0$ . Then  $\int_0^1 \rho e dx = 0$  and hence there exists  $\bar{x} \in ((2C\lambda)^{\frac{p'r}{p'-r}}, 1]$  such that  $e(\bar{x}) = 0$ . For any  $y \in (0, 1)$ , we have

$$\begin{aligned} e(y)^2 &= 2\lambda \int_y^{\bar{x}} \frac{e(x)}{a(x)} \left( \int_0^x \rho e ds \right) dx \\ &\leq 2\lambda \left( \int_0^{(4C\lambda)^{\frac{p'r}{p'-r}}} \cdots dx + \int_{(4C\lambda)^{\frac{p'r}{p'-r}}}^1 \cdots dx \right) =: 2\lambda(I_1 + I_2). \end{aligned}$$

The estimate (2.49) for  $I_1$  is still valid without any change. For  $I_2$ , we first notice that for  $x > (4C\lambda)^{\frac{p'r}{p'-r}}$ ,

$$|e(x)| \leq \left| \int_x^{\bar{x}} e'(s) ds \right| \leq \sqrt{\lambda} \left( \int_{\min(x, \bar{x})}^1 \frac{ds}{a(s)} \right)^{\frac{1}{2}} \leq \left( \left\| \frac{s}{a(s)} \right\|_{L^1} \frac{\lambda}{(4C\lambda)^{\frac{p'r}{p'-r}}} \right)^{\frac{1}{2}},$$

so that (2.50) holds again. Therefore (2.51) and (2.52) hold.

• Since  $\lambda_n > 0$  for  $n \geq 1$  and since the number of  $\lambda_n$ 's in  $(0, 1)$  is finite, we infer that (2.52) is still valid for all the eigenvalues  $\lambda_n > 0$  by replacing the constant  $C''$  by a larger one denoted  $C_1$ . The proof of (ii) is complete.

(iii) Let  $\lambda \in [0, \infty)$ , let  $u$  given by Corollary 2.11 and let  $e(x) := e(0)u(x)$  with  $e(0) > 0$ . We use a modified Prüfer substitution (see e.g. [6, 29]). We set

$$ae' = \lambda^{\frac{1}{4}} R \cos \theta, \quad (2.53)$$

$$e = \lambda^{-\frac{1}{4}} R \sin \theta, \quad (2.54)$$

so that

$$\begin{aligned} R &= (\lambda^{-\frac{1}{2}}(ae')^2 + \lambda^{\frac{1}{2}}e^2)^{\frac{1}{2}}, \\ \cot \theta &= \frac{\cos \theta}{\sin \theta} = \lambda^{-\frac{1}{2}} \frac{ae'}{e}. \end{aligned}$$

We can impose that  $\theta(0) = \frac{\pi}{2}$ , for  $(ae')(0) = 0$  and  $e(0) > 0$ .

We note that  $R \in W^{1,1}(0, 1)$ , since

$$\begin{aligned} R &\leq \lambda^{-\frac{1}{4}} |ae'| + \lambda^{\frac{1}{4}} |e|, \\ |R'| &\leq R^{-1} |\lambda^{-\frac{1}{2}}(ae')(ae')' + \lambda^{\frac{1}{2}} ee'| \\ &\leq (\lambda^{-\frac{1}{2}} |(ae')'|^2 + \lambda^{\frac{1}{2}} |e'|^2)^{\frac{1}{2}} \\ &\leq \lambda^{-\frac{1}{4}} |(ae')'| + \lambda^{\frac{1}{4}} |e'| \end{aligned}$$

and  $e, ae' \in W^{1,1}(0, 1)$ .

Since  $R \in C^0([0, 1])$  and  $R(x) > 0$  for all  $x \in [0, 1]$  by Lemma 2.12, we infer that  $\inf_{x \in [0, 1]} R(x) > 0$  and  $R^{-1} \in W^{1,1}(0, 1)$ . Thus  $\cos \theta = \lambda^{-\frac{1}{4}} \frac{ae'}{R}$  and  $\sin \theta = \lambda^{\frac{1}{4}} \frac{e}{R}$  are both in  $W^{1,1}(0, 1)$ . The functions arcsin and arccos being both of class  $C^1$  on  $(-1, 1)$ , we infer that  $\theta \in W^{1,1}(0, 1)$ .

We obtain by straightforward computations that the pair  $(R, \theta)$  solves the following Cauchy problem

$$R' = \lambda^{\frac{1}{2}} R \left( \frac{1}{a} - \rho \right) \cos \theta \sin \theta, \quad (2.55)$$

$$\theta' = \lambda^{\frac{1}{2}} \left( \rho \sin^2 \theta + \frac{1}{a} \cos^2 \theta \right), \quad (2.56)$$

$$R(0) = \lambda^{\frac{1}{4}} e(0), \quad (2.57)$$

$$\theta(0) = \frac{\pi}{2} \quad (2.58)$$

on  $(0, 1)$ . Conversely, if  $R, \theta \in W^{1,1}(0, 1)$  satisfy (2.55)-(2.58), then the function  $e$ , defined in (2.54), is in  $W^{1,1}(0, 1)$ , (2.53) holds,  $(ae)'' = -\lambda \rho e$  a.e. in  $(0, 1)$ , and  $e(x) = e(0)u(x)$ , where  $u$  is as given in Corollary 2.11.

**Lemma 2.14.** *The map  $\lambda \rightarrow \theta(x, \lambda)$  is continuous and strictly increasing for all  $x \in (0, 1]$ .*

*Proof of Lemma 2.14:* The continuity of the map  $\lambda \rightarrow \theta(x, \lambda)$  follows from those of the maps  $\lambda \in \mathbb{R}_+ \rightarrow u \in W^{1,1}(0, 1)$  and  $\lambda \in \mathbb{R}_+ \rightarrow au' \in W^{1,r}(0, 1)$  and from the definition of  $\theta$ .

Let us show that the map  $\lambda \rightarrow \theta(x, \lambda)$  is strictly increasing for all  $x \in (0, 1]$ . Assume  $\lambda_1 < \lambda_2$ , and let  $\theta_1$  and  $\theta_2$  be associated with  $\lambda_1$  and  $\lambda_2$ , respectively. Let  $w := \theta_2 - \theta_1$ . Then  $w(0) = 0$ , and we have a.e. in  $(0, 1)$

$$w' = \left( \lambda_2^{\frac{1}{2}} - \lambda_1^{\frac{1}{2}} \right) \left[ \rho \sin^2 \theta_2 + \frac{1}{a} \cos^2 \theta_2 \right] + \lambda_1^{\frac{1}{2}} \left[ \rho (\sin^2 \theta_2 - \sin^2 \theta_1) + \frac{1}{a} (\cos^2 \theta_2 - \cos^2 \theta_1) \right] =: J_1 + J_2.$$

Then  $J_1 > 0$  and  $|J_2| \leq 2\lambda_1^{\frac{1}{2}} \left( \rho + \frac{1}{a} \right) |w|$  a.e. in  $(0, 1)$ , where we used the mean value theorem. It follows that

$$w' > -2\lambda_1^{\frac{1}{2}} \left( \rho + \frac{1}{a} \right) |w| \quad \text{a.e. in } (0, 1).$$

Assume that there exists  $d \in (0, 1]$  with  $w(d) < 0$ . If  $[c, d]$  denotes the largest segment to the left of  $d$  where  $w \leq 0$ , then

$$w' > 2\lambda_1^{\frac{1}{2}} \left( \rho + \frac{1}{a} \right) w \quad \text{a.e. in } (c, d)$$

and

$$\left( w e^{-2\lambda_1^{\frac{1}{2}} \int_d^x (\rho + a^{-1}) ds} \right)' = \left[ w' - 2\lambda_1^{\frac{1}{2}} \left( \rho + \frac{1}{a} \right) w \right] e^{-2\lambda_1^{\frac{1}{2}} \int_d^x (\rho + a^{-1}) ds} > 0 \quad \text{a.e. in } (c, d),$$

so that the function  $x \rightarrow w(x) e^{-2\lambda_1^{\frac{1}{2}} \int_d^x (\rho + a^{-1}) ds}$  is strictly increasing in  $[c, d]$ , and thus negative in  $[c, d]$ . This yields  $c = 0$  and  $w(0) < 0$ , which is a contradiction. We infer that  $w(x) \geq 0$  for all  $x \in [0, 1]$ . If  $w(x) = 0$  for all  $x \in [c, d]$  for some  $c, d$  with  $0 \leq c < d \leq 1$ , then

$$w' = \left( \lambda_2^{\frac{1}{2}} - \lambda_1^{\frac{1}{2}} \right) \left[ \rho \sin^2 \theta_2 + \frac{1}{a} \cos^2 \theta_2 \right] > 0 \quad \text{a.e. in } (c, d)$$

which is a contradiction. Thus for any  $d \in (0, 1)$  one can pick some  $c \in (0, d)$  with  $w(c) > 0$ . Then

$$\left( w e^{2\lambda_1^{\frac{1}{2}} \int_d^x (\rho + a^{-1}) ds} \right)' = \left[ w' + 2\lambda_1^{\frac{1}{2}} \left( \rho + \frac{1}{a} \right) w \right] e^{2\lambda_1^{\frac{1}{2}} \int_d^x (\rho + a^{-1}) ds} > 0 \quad \text{a.e. in } (0, 1),$$

and hence  $w(x) > 0$  on  $[c, 1]$ . We conclude that  $w(x) > 0$  for all  $x \in (0, 1]$ .  $\square$

Note that the map  $x \rightarrow \theta(x, \lambda)$  is also strictly increasing for all  $\lambda > 0$ , for the r.h.s. of (2.56) is positive a.e. on  $(0, 1)$ , while  $\theta(\cdot, 0) \equiv \frac{\pi}{2}$ .

We claim that

$$\lim_{\lambda \rightarrow +\infty} \theta(1, \lambda) = +\infty.$$

Indeed

$$\theta(1, \lambda) - \theta\left(\frac{1}{2}, \lambda\right) = \lambda^{\frac{1}{2}} \int_{\frac{1}{2}}^1 \left(\rho \sin^2 \theta + \frac{1}{a} \cos^2 \theta\right) dx \geq \lambda^{\frac{1}{2}} \underbrace{\int_{\frac{1}{2}}^1 \min\left(\rho, \frac{1}{a}\right) dx}_{>0}.$$

It follows that

$$\theta(1, [0, +\infty)) = \left[\frac{\pi}{2}, +\infty\right).$$

Let  $\Lambda_n := \theta(1, \cdot)^{-1}(n\pi)$  for all  $n \in \mathbb{N}^*$ . We claim that for all  $n \in \mathbb{N}^*$  there exists a unique  $\lambda \in [\Lambda_n, \Lambda_{n+1})$  such that the function  $u$  associated with  $\lambda$  in Corollary 2.11 satisfies

$$\alpha u(1) + \beta (au')(1) = 0. \quad (2.59)$$

Indeed, if  $\beta = 0$ , then the condition  $u(1) = 0$  is equivalent to  $\sin \theta(1, \lambda) = 0$  and  $\lambda = \Lambda_n$  is the only solution in  $[\Lambda_n, \Lambda_{n+1})$ . If  $\beta \neq 0$ , then (2.59) can be written

$$\frac{\alpha}{\beta} \lambda^{-\frac{1}{2}} \sin \theta(1, \lambda) + \cos \theta(1, \lambda) = 0.$$

If  $\alpha = 0$ ,  $\cos \theta(1, \lambda) = 0$  gives  $\theta(1, \lambda) = n\pi + \frac{\pi}{2}$  and  $\lambda = \theta(1, \cdot)^{-1}(n\pi + \frac{\pi}{2})$ . If  $\alpha \neq 0$ , then both  $\cos \theta(1, \lambda)$  and  $\sin \theta(1, \lambda)$  have to be different from 0 and

$$h(\lambda) := \frac{\alpha}{\beta} \lambda^{-\frac{1}{2}} + \cot \theta(1, \lambda) = 0.$$

But the function  $h$  is continuous and strictly decreasing in  $(\Lambda_n, \Lambda_{n+1})$  with  $h(\Lambda_n^+) = +\infty$  and  $h(\Lambda_{n+1}^-) = -\infty$ . It follows that there exists a unique  $\lambda \in (\Lambda_n, \Lambda_{n+1})$  such that (2.59) holds.

Consider now the possible solutions  $\lambda$  of (2.59) in  $[0, \Lambda_1)$ . If  $\beta = 0$ , (2.59) cannot hold, for  $\sin \theta(1, \lambda) > 0$  (since  $\pi/2 \leq \theta(1, \lambda) < \pi$ ). If  $\alpha = 0$ , (2.59) holds only if  $\lambda = 0$ . If  $\alpha \neq 0$  and  $\beta \neq 0$ , then  $\alpha/\beta > 0$  and  $h(0^+) = +\infty$ ,  $h(\Lambda_1^-) = -\infty$ , so that there exists a unique  $\lambda \in (0, \Lambda_1)$  with  $h(\lambda) = 0$ .

We conclude that the eigenvalues  $\lambda_n$ ,  $n \in \mathbb{N}$ , which are all simple by Lemma 2.13, fulfill the following property:

$$\text{if } \beta = 0, \lambda_n = \Lambda_{n+1} \text{ for all } n \in \mathbb{N}; \quad (2.60)$$

$$\text{if } \alpha = 0, \lambda_0 = 0 \text{ and } \lambda_n = \theta(1, \cdot)^{-1}(n\pi + \frac{\pi}{2}) \text{ for all } n \in \mathbb{N}^*; \quad (2.61)$$

$$\text{if } \alpha\beta \neq 0, \lambda_0 \in (0, \Lambda_1) \text{ and } \lambda_n \in (\Lambda_n, \Lambda_{n+1}) \text{ for all } n \in \mathbb{N}^*. \quad (2.62)$$

Since  $\lambda \rightarrow \theta(1, \lambda)$  is strictly increasing and  $\theta(1, \Lambda_n) = n\pi$ , we infer that

$$\text{if } \beta = 0, \theta(1, \lambda_n) = (n+1)\pi \text{ for all } n \in \mathbb{N}; \quad (2.63)$$

$$\text{if } \alpha = 0, \theta(1, \lambda_n) = n\pi + \frac{\pi}{2} \text{ for all } n \in \mathbb{N}; \quad (2.64)$$

$$\text{if } \alpha\beta \neq 0, n\pi < \theta(1, \lambda_n) < (n+1)\pi \text{ for all } n \in \mathbb{N}^*. \quad (2.65)$$

Let  $n \in \mathbb{N}$  and let  $(\lambda_n, e_n)$  be as in Theorem 2.8. Assume that  $\lambda_n \neq 0$ . Let  $(R_n, \theta_n)$  denote the pair associated with  $(\lambda_n, e_n)$ . Since  $\theta_n = \theta(\cdot, \lambda_n) \in W^{1,1}(0,1)$ , we can integrate in (2.56) along  $(0,1)$  to obtain

$$\theta_n(1) - \frac{\pi}{2} = \lambda_n^{\frac{1}{2}} \int_0^1 \rho \sin^2 \theta_n dx + \lambda_n^{\frac{1}{2}} \int_0^1 \frac{\cos^2 \theta_n}{a} dx. \quad (2.66)$$

It follows from (2.63)-(2.65) that

$$\theta_n(1) - \frac{\pi}{2} \geq n\pi - \frac{\pi}{2}. \quad (2.67)$$

The first term in the r.h.s. of (2.66) is easily estimated:

$$\lambda_n^{\frac{1}{2}} \int_0^1 \rho \sin^2 \theta_n dx \leq \lambda_n^{\frac{1}{2}} \|\rho\|_{L^r}.$$

To estimate the second term in the r.h.s. of (2.66), we split the integral into two terms as in (ii), namely

$$\lambda_n^{\frac{1}{2}} \int_0^1 \frac{\cos^2 \theta_n}{a} dx = \lambda_n^{\frac{1}{2}} \int_0^{(2C\lambda_n)^{\frac{p'r}{p'-r}}} \frac{\cos^2 \theta_n}{a} dx + \lambda_n^{\frac{1}{2}} \int_{(2C\lambda_n)^{\frac{p'r}{p'-r}}}^1 \frac{\cos^2 \theta_n}{a} dx =: I_1 + I_2. \quad (2.68)$$

• Assume that  $p < \infty$  (so  $p' > 1$ ). Since for all  $y \in (0,1)$

$$\int_y^1 \frac{dx}{a(x)} \leq \left( \int_y^1 \left( \frac{x}{a(x)} \right)^p dx \right)^{\frac{1}{p}} \left( \int_y^1 x^{-p'} dx \right)^{\frac{1}{p'}} \leq \left\| \frac{x}{a(x)} \right\|_{L^p} \left( \frac{y^{1-p'} - 1}{p' - 1} \right)^{\frac{1}{p'}},$$

we infer that

$$\begin{aligned} I_2 &\leq \lambda_n^{\frac{1}{2}} \int_{(2C\lambda_n)^{\frac{p'r}{p'-r}}}^1 \frac{dx}{a(x)} \\ &\leq \lambda_n^{\frac{1}{2}} \frac{\left\| \frac{x}{a(x)} \right\|_{L^p}}{(p' - 1)^{\frac{1}{p'}}} (2C\lambda_n)^{\left(\frac{1}{p'} - 1\right) \frac{p'r}{p'-r}} \\ &\leq C'_2 \lambda_n^{\frac{1}{2} + \frac{1}{p} \frac{p'r}{r-p'}} \end{aligned} \quad (2.69)$$

for some constant  $C'_2 = C'_2(p, r, \left\| \frac{x}{a(x)} \right\|_{L^p}, C) > 0$ .

If  $p = \infty$ , then  $p' = 1$ ,  $\int_y^1 \frac{dx}{a(x)} \leq \left\| \frac{x}{a(x)} \right\|_{L^\infty} |\ln y|$ , and

$$I_2 \leq C'_2 \lambda_n^{\frac{1}{2}} |\ln \lambda_n| \quad (2.70)$$

for some constant  $C'_2 = C'_2(r, \left\| \frac{x}{a(x)} \right\|_{L^\infty}, C) > 0$ .

• Let us proceed to estimate  $I_1$ . First, recall that

$$0 \leq \frac{e_n(0)}{2} \leq e_n(x) \leq e_n(0) \quad \text{for } 0 < x < (2C\lambda_n)^{\frac{p'r}{p'-r}}.$$

Using (2.24)-(2.25) and (2.53), we obtain

$$I_1 = \lambda_n^{\frac{1}{2}} \int_0^{(2C\lambda_n)^{\frac{p'r}{p'-r}}} \frac{(ae'_n)^2}{a(\lambda_n^{\frac{1}{4}}R_n)^2} dx = \lambda_n^2 \int_0^{(2C\lambda_n)^{\frac{p'r}{p'-r}}} \frac{(\int_0^x (\rho e_n)(s) ds)^2}{a(x)R_n(x)^2} dx.$$

But for  $0 < x < (2C\lambda_n)^{\frac{p'r}{p'-r}}$ ,

$$\begin{aligned} \left| \int_0^x (\rho e_n)(s) ds \right| &\leq \left( \int_0^x \rho^r ds \right)^{\frac{1}{r}} \left( \int_0^x e_n^{r'} ds \right)^{\frac{1}{r'}} \\ &\leq \|\rho\|_{L^r} e_n(0) x^{\frac{1}{r'}} \end{aligned}$$

while

$$|R_n(x)|^2 \geq \lambda_n^{\frac{1}{2}} e_n(x)^2 \geq \frac{\lambda_n^{\frac{1}{2}}}{4} e_n(0)^2.$$

It follows that

$$\begin{aligned} I_1 &\leq 4\|\rho\|_{L^r}^2 \lambda_n^{\frac{3}{2}} \int_0^{(2C\lambda_n)^{\frac{p'r}{p'-r}}} \frac{x^{\frac{2}{r'}}}{a(x)} dx \\ &\leq 4\|\rho\|_{L^r}^2 \lambda_n^{\frac{3}{2}} \left\| \frac{x}{a(x)} \right\|_{L^p} \left( \int_0^{(2C\lambda_n)^{\frac{p'r}{p'-r}}} x^{(\frac{2}{r'}-1)p'} dx \right)^{\frac{1}{p'}}. \end{aligned}$$

Note that  $(\frac{2}{r'} - 1)p' = (\frac{1}{r'} - \frac{1}{r})p' > -p'/r > -1$  by (1.10). It follows that

$$\int_0^{(2C\lambda_n)^{\frac{p'r}{p'-r}}} x^{(\frac{2}{r'}-1)p'} dx = \frac{(2C\lambda_n)^{[(\frac{2}{r'}-1)p'+1]\frac{p'r}{p'-r}}}{(\frac{2}{r'}-1)p'+1} < \infty.$$

Thus

$$I_1 \leq C_2'' \lambda_n^{\frac{3}{2} - [\frac{2}{r'} - 1 + \frac{1}{p'}]\frac{p'r}{r-p'}} \quad (2.71)$$

for some constant  $C_2'' = C_2''(p, r, \|\frac{x}{a(x)}\|_{L^p}, \|\rho\|_{L^r}, C) > 0$ .

It remains to compare the exponents of  $\lambda_n$  in (2.69) and (2.71). We have

$$\frac{1}{2} + \frac{1}{p} \frac{p'r}{r-p'} > \frac{3}{2} - \left[ \frac{2}{r'} - 1 + \frac{1}{p'} \right] \frac{p'r}{r-p'}. \quad (2.72)$$

Indeed,

$$\left[ \frac{1}{p} + \frac{2}{r'} - 1 + \frac{1}{p'} \right] p'r - (r-p') = 2 \frac{p'r}{r'} - r + p' = (2p' - 1)r - p' \geq r - p' > 0.$$

If  $p < \infty$ , we infer from (2.67)-(2.71) and (2.72) that if  $\lambda_n \geq 1$ ,

$$n\pi - \frac{\pi}{2} \leq C_2''' \lambda_n^{\frac{1}{2} + \frac{1}{p} \frac{p'r}{r-p'}}$$



for some constant  $C_2''' > 0$ . Then (2.28) follows.

If  $p = \infty$  and  $\lambda_n > 1$ , we obtain for given  $\kappa < 2$

$$n\pi - \frac{\pi}{2} \leq C_2''' \lambda_n^{\frac{1}{2}} |\ln \lambda_n| \leq C_2'''' \lambda_n^{\kappa-1}$$

for some constants  $C_2''', C_2'''' > 0$ . Then (2.28) follows. The proof of Theorem 2.9 is complete.  $\square$

### 3. INTRODUCTION OF THE GENERATING FUNCTIONS

We shall see later that the zero-order term  $q(x)u$  in (1.1) can be removed thanks to a change of variables. Consider the simplified system

$$(au_x)_x = \rho u_t \quad (3.1)$$

$$(au_x)(0, t) = 0 \quad (3.2)$$

and search for a solution of it in the form

$$u(x, t) = \sum_{i=0}^{\infty} y^{(i)}(t) g_i(x) \quad (3.3)$$

where  $y$  is the *flat output* and the  $g_i$ 's are the *generating functions*.

A formal computation shows that

$$0 = (au_x)_x - \rho u_t = \sum_{i=0}^{\infty} y^{(i)}(t) (ag_{i,x})_x - \sum_{i=0}^{\infty} y^{(i+1)}(t) \rho(x) g_i(x).$$

It is thus natural to impose

$$(ag_{0,x})_x = 0 \quad (3.4)$$

$$(ag_{i,x})_x = \rho g_{i-1}, \quad i \geq 1, \quad (3.5)$$

together with the condition

$$(ag_{i,x})(0) = 0 \quad \forall i \geq 0. \quad (3.6)$$

We infer from (3.4) and (3.6) that  $g_{0,x} = 0$  a.e. We pick

$$g_0(x) := 1 \quad \forall x \in [0, 1]. \quad (3.7)$$

Integrating in (3.5) yields

$$g_{i,x}(x) = \frac{1}{a(x)} \int_0^x \rho(s) g_{i-1}(s) ds.$$

We pick

$$g_i(0) = 0, \quad i \geq 1 \quad (3.8)$$

to obtain a rapid decay of  $\|g_i\|_{L^\infty}$  as  $i \rightarrow +\infty$ , so that

$$g_i(x) := \int_0^x \frac{1}{a(s)} \left( \int_0^s \rho(\sigma) g_{i-1}(\sigma) d\sigma \right) ds, \quad i \geq 1.$$

This defines formally the sequence  $(g_i)_{i \geq 0}$  of generating functions. To obtain the estimate of  $\|g_i\|_{L^\infty}$  which ensures the convergence of the series in (3.3) for  $y \in G^s([0, T])$ , we need the following

**Proposition 3.1.** *There are some constants  $C, R > 0$  such that*

$$\|g_i\|_{W^{1,1}(0,1)} + \|ag_{i,x}\|_{W^{1,r}(0,1)} \leq \frac{C}{R^i(i!)^{1+\frac{1}{p'}-\frac{1}{r}}} \quad \forall i \in \mathbb{N}. \quad (3.9)$$

*Proof.* We need the following

**Lemma 3.2.** *Let  $f \in L^\infty(0,1)$  and  $g(x) = \int_0^x \frac{1}{a(s)} \left( \int_0^s \rho(\sigma)f(\sigma)d\sigma \right) ds$ . Then  $g \in W^{1,1}(0,1)$  and  $ag_x \in W^{1,r}(0,1)$ . If, in addition,*

$$|f(x)| \leq Cx^\delta \quad \text{for a.e. } x \in (0,1), \quad (3.10)$$

for some constants  $C, \delta \geq 0$ , then

$$|g(x)| \leq C \frac{\|\frac{s}{a(s)}\|_{L^p}\|\rho\|_{L^r}}{p'^{\frac{1}{p'}}} \frac{x^{\delta+\omega}}{(r'\delta+1)^{\frac{1}{r'}}(\delta+\omega)^{\frac{1}{p'}}} \quad \forall x \in [0,1], \quad (3.11)$$

where  $\omega := \frac{1}{r'} - 1 + \frac{1}{p'} = \frac{1}{p'} - \frac{1}{r} > 0$ .

*Proof of Lemma 3.2:* We have

$$\left| \int_0^s \rho(\sigma)f(\sigma)d\sigma \right| \leq \|f\|_{L^\infty} \left( \int_0^s \rho(\sigma)^r d\sigma \right)^{\frac{1}{r}} s^{\frac{1}{r'}}$$

and

$$\|s^{-1} \int_0^s \rho(\sigma)f(\sigma)d\sigma\|_{L^{p'}} \leq \|f\|_{L^\infty} \|\rho\|_{L^r} \left( \int_0^1 s^{(\frac{1}{r'}-1)p'} ds \right)^{\frac{1}{p'}}. \quad (3.12)$$

But  $(\frac{1}{r'} - 1)p' = -p'/r > -1$ , since  $r > p'$ . Thus  $(s \rightarrow s^{-1} \int_0^s \rho f d\sigma) \in L^{p'}(0,1)$  and  $(s \rightarrow a(s)^{-1} \int_0^s \rho f d\sigma) \in L^1(0,1)$  by Hölder inequality. Therefore  $g \in W^{1,1}(0,1)$  and  $(x \rightarrow (ag_x)(x) = \int_0^x (\rho f)(s) ds) \in W^{1,r}(0,1)$ . Assume now that (3.10) holds. Then

$$\left| \int_0^s \rho(\sigma)f(\sigma)d\sigma \right| \leq C \|\rho\|_{L^r} \left( \int_0^s \sigma^{r'\delta} d\sigma \right)^{\frac{1}{r'}}$$

and

$$\begin{aligned} |g(x)| &\leq \left\| \frac{s}{a(s)} \right\|_{L^p(0,x)} \|s^{-1} \int_0^s \rho f d\sigma\|_{L^{p'}(0,x)} \\ &\leq C \left\| \frac{s}{a(s)} \right\|_{L^p} \|\rho\|_{L^r} \frac{\left( \int_0^x s^{(\delta+\frac{1}{r'}-1)p'} \right)^{\frac{1}{p'}}}{(r'\delta+1)^{\frac{1}{r'}}} \\ &\leq C \left\| \frac{s}{a(s)} \right\|_{L^p} \|\rho\|_{L^r} \frac{x^{\delta+\frac{1}{r'}-1+\frac{1}{p'}}}{(r'\delta+1)^{\frac{1}{r'}} \left( (\delta+\frac{1}{r'}-1)p'+1 \right)^{\frac{1}{p'}}} \\ &\leq C \frac{\|\frac{s}{a(s)}\|_{L^p}\|\rho\|_{L^r}}{p'^{\frac{1}{p'}}} \frac{x^{\delta+\omega}}{(r'\delta+1)^{\frac{1}{r'}}(\delta+\omega)^{\frac{1}{p'}}} \quad \forall x \in [0,1]. \end{aligned}$$

The proof of Lemma 3.2 is complete.  $\square$

Using (3.7) and (3.11), we obtain by an easy induction that for all  $i \in \mathbb{N}$ ,

$$|g_i(x)| \leq \left( \frac{\| \frac{s}{a(s)} \|_{L^p} \| \rho \|_{L^r}}{p'^{\frac{1}{p'}}} \right)^i \frac{x^{i\omega}}{\left[ \prod_{j=1}^i (1 + (j-1)r'\omega) \right]^{\frac{1}{r'}} \left[ \prod_{j=1}^i (j\omega) \right]^{\frac{1}{p'}}} \quad \forall x \in [0, 1]. \quad (3.13)$$

Since

$$\left[ \prod_{j=1}^i (1 + (j-1)r'\omega) \right]^{\frac{1}{r'}} \left[ \prod_{j=1}^i (j\omega) \right]^{\frac{1}{p'}} \geq \frac{[(r'\omega)^{\frac{1}{r'}} \omega^{\frac{1}{p'}}]^i}{(ir'\omega)^{\frac{1}{r'}}} (i!)^{\frac{1}{r'} + \frac{1}{p'}}$$

with  $\frac{1}{r'} + \frac{1}{p'} = 1 + \frac{1}{p'} - \frac{1}{r} > 1$  and  $i \leq 2^i$  for all  $i \geq 0$ , we infer that

$$\|g_i\|_{L^\infty} \leq \frac{C}{R^i} \frac{1}{(i!)^{1 + \frac{1}{p'} - \frac{1}{r}}} \quad \forall i \geq 0 \quad (3.14)$$

for some positive constants  $C$  and  $R$ .

On the other hand  $(ag_{i,x})_x = \rho g_{i-1}$  and  $g_{i,x}(x) = \frac{1}{a(x)} \int_0^x \rho g_{i-1} ds$ , so that, with (3.12)

$$\|(ag_{i,x})_x\|_{L^r} \leq \|\rho\|_{L^r} \|g_{i-1}\|_{L^\infty}, \quad \|g_{i,x}\|_{L^1} \leq C \|\rho\|_{L^r} \left\| \frac{x}{a(x)} \right\|_{L^p} \|g_{i-1}\|_{L^\infty}. \quad (3.15)$$

Finally, (3.9) follows from (3.6), (3.8), (3.14) and (3.15).  $\square$

Next, we show that the eigenfunctions can be expressed in terms of the generating functions.

**Proposition 3.3.** *Let  $n \in \mathbb{N}$  and let  $e_n$  and  $\lambda_n$  be as in Theorem 2.8. Then*

$$e_n = e_n(0) \sum_{i \geq 0} (-\lambda_n)^i g_i \quad \text{in } W^{1,1}(0,1).$$

*Proof.* Fix some  $n \in \mathbb{N}$  and set

$$\tilde{e} = \sum_{i \geq 0} (-\lambda_n)^i g_i. \quad (3.16)$$

The series in (3.16) converges in  $W^{1,1}(0,1)$  by (3.9). It follows that

$$a\tilde{e}_x = \sum_{i \geq 0} (-\lambda_n)^i ag_{i,x} \quad \text{in } L^1(0,1).$$

Using (3.9) again, we deduce that  $a\tilde{e}_x \in W^{1,r}(0,1)$ , and that we have in  $L^r(0,1)$

$$\begin{aligned} (a\tilde{e}_x)_x &= \sum_{i \geq 0} (-\lambda_n)^i (ag_{i,x})_x \\ &= \sum_{i \geq 1} (-\lambda_n)^i \rho g_{i-1} \\ &= -\lambda_n \rho \tilde{e}. \end{aligned}$$

On the other hand, by (3.6), (3.7) and (3.8), we have that

$$\tilde{e}(0) = 1, \quad (a\tilde{e}_x)(0) = 0.$$

It follows from Corollary 2.11 that  $\tilde{e} = u$  for  $\lambda = \lambda_n$ , and from Lemma 2.13 that

$$e_n(x) = e_n(0) \sum_{i \geq 0} (-\lambda_n)^i g_i(x) \quad \forall x \in [0, 1].$$

□

We are in a position to prove the main results in the paper.

#### 4. PROOF OF THE MAIN RESULTS

##### 4.1. Proof of Theorem 1.1.

4.1.1. **Step 1: Reduction to the case  $q = 0$ .** Let  $\hat{u}(x, t) = u(x, t)/v(x)$ , where  $u$  satisfies (1.1) and (1.2) and  $v$  is as in (1.9). Then

$$v^2 a \hat{u}_x = v^2 a \left( \frac{u_x}{v} - \frac{uv_x}{v^2} \right) = a(u_x v - uv_x)$$

and

$$\begin{aligned} (v^2 a \hat{u}_x)_x &= (au_x)_x v + au_x v_x - ((av_x)_x u + au_x v_x) \\ &= [(au_x)_x + qu]v \\ &= \rho u_t v \\ &= \rho v^2 \hat{u}_t. \end{aligned}$$

Let  $\hat{a}(x) := v(x)^2 a(x)$  and  $\hat{\rho}(x) := v(x)^2 \rho(x)$ . Then  $\hat{u}$  satisfies

$$\begin{aligned} (\hat{a} \hat{u}_x)_x &= \hat{\rho} \hat{u}_t \quad x \in (0, 1), \quad t \in (0, T), \\ (\hat{a} \hat{u}_x)(0) &= 0, \quad t \in (0, T) \end{aligned}$$

with  $\hat{a}$  and  $\hat{\rho}$  satisfying (1.5)-(1.8). Indeed, using (1.9), one may pick some constants  $C_1$  and  $C_2$  such that

$$0 < C_1 \leq v(x) \leq C_2 \quad \forall x \in [0, 1].$$

We infer from  $u = \hat{u}v$  and (1.3) that

$$(\alpha v(1) + \beta(av_x)(1))\hat{u}(1, t) + \beta v(1)(a\hat{u}_x)(1, t) = h(t).$$

Setting  $\hat{\alpha} := \alpha v(1) + \beta(av_x)(1)$  and  $\hat{\beta} := \beta/v(1)$ , we arrive to

$$\hat{\alpha} \hat{u}(1, t) + \hat{\beta} (\hat{a} \hat{u}_x)(1, t) = h(t).$$

Let  $\hat{u}_0 := u_0/v$ . Then  $\hat{u}_0 \in L^2_{\hat{\rho}} \iff u_0 \in L^2_{\rho}$ .

**4.2. Step 2: Flatness approach.** We follow closely [24]. We assume that  $q = 0$  in (1.1). Let  $u_0 \in L^2_\rho$ . As  $(e_n)_{n \geq 0}$  is an orthonormal basis in  $L^2_\rho$ , we can expand  $u_0$  as a series of the  $e_n$ 's:

$$u_0 = \sum_{n=0}^{\infty} c_n e_n \quad \text{in } L^2_\rho,$$

where the sequence  $(c_n)_{n \geq 0}$  of real numbers satisfies  $\sum_{n=0}^{\infty} c_n^2 < \infty$ .

Using (2.27) and (2.28), we notice that the map  $z \rightarrow \sum_{n \geq 0} c_n e_n(0) e^{-\lambda_n z}$  is analytic in the set  $\{z = t + it'; t > 0, t' \in \mathbb{R}\}$ . It follows that the map  $t \rightarrow \sum_{n \geq 0} c_n e_n(0) e^{-\lambda_n t}$  is real analytic in  $(0, \infty)$ , and its restriction to  $[\epsilon, T]$  belongs to  $G^1([\epsilon, T]) \subset G^s([\epsilon, T])$  for all  $\epsilon \in (0, T)$  and all  $s \in (1, 2)$ . Pick  $s \in (1, 1 + \frac{1}{p'} - \frac{1}{r})$  and  $\varphi \in G^s([0, T])$  with

$$\varphi(t) = \begin{cases} 1 & \text{if } t \leq \frac{T}{3}, \\ 0 & \text{if } t \geq \frac{2T}{3}. \end{cases}$$

Let

$$y(t) := \varphi(t) \sum_{n=0}^{\infty} c_n e_n(0) e^{-\lambda_n t}, \quad \text{for } 0 < t < T,$$

$$u(x, t) := \begin{cases} u_0(x) & \text{if } t = 0, \\ \sum_{i=0}^{\infty} y^{(i)}(t) g_i(x) & \text{if } 0 < t \leq T. \end{cases}$$

Pick any  $\epsilon \in (0, T/3)$ . Then  $y \in G^s([\epsilon, T])$ , and there exist some positive numbers  $\tilde{M}, \tilde{R}$  such that

$$|y^{(i)}(t)| \leq \tilde{M} \frac{(i!)^s}{\tilde{R}^i} \quad \forall t \in [\epsilon, T].$$

Combined with (3.9), this yields

$$u \in C^1([\epsilon, T], W^{1,1}(0, 1)), \quad au_x \in C^1([\epsilon, T], W^{1,r}(0, 1)).$$

Furthermore,

$$\rho u_t = \sum_{i=0}^{\infty} y^{(i+1)}(t) \rho g_i(x) = \sum_{i=0}^{\infty} y^{(i+1)}(t) (a g_{i+1, x})_x = (a u_x)_x$$

and  $au_x(0, t) = 0$ . We notice that for  $0 < t \leq T/3$ , we have that  $y(t) = \sum_{n=0}^{\infty} c_n e_n(0) e^{-\lambda_n t}$  and

$$\begin{aligned} u(x, t) &= \sum_{i=0}^{\infty} \partial_t^i \left[ \sum_{n=0}^{\infty} c_n e_n(0) e^{-\lambda_n t} \right] g_i(x) \\ &= \sum_{i=0}^{\infty} \left[ \sum_{n=0}^{\infty} c_n e_n(0) (-\lambda_n)^i e^{-\lambda_n t} \right] g_i(x) \\ &= \sum_{n=0}^{\infty} c_n e^{-\lambda_n t} e_n(0) \sum_{i=0}^{\infty} (-\lambda_n)^i g_i(x) \\ &= \sum_{n=0}^{\infty} c_n e^{-\lambda_n t} e_n(x) \end{aligned} \tag{4.1}$$

for all  $x \in [0, 1]$ . The above computations are valid, since for  $0 < \delta \leq t \leq T$  and  $0 \leq x \leq 1$

$$\begin{aligned} \sum_{i \geq 0, n \geq 0} |c_n e_n(0) (-\lambda_n)^i e^{-\lambda_n t} g_i(x)| &\leq \sum_{i \geq 0, n \geq 0} |c_n e_n(0)| \frac{e^{\frac{\delta}{2} \lambda_n}}{(\frac{\delta}{2})^i} (i!) e^{-\delta \lambda_n} \frac{C}{R^i (i!)^{1 + \frac{1}{p'} - \frac{1}{r}}} \\ &\leq \left( |c_0| + C_1 \sum_{n; \lambda_n > 0}^{\infty} |c_n| \frac{\lambda_n^{\frac{3}{4}(1 + \frac{p'}{r-p'})}}{e^{\frac{\delta}{2} \lambda_n}} \right) \sum_{i=0}^{\infty} \frac{1}{(\frac{R\delta}{2})^i (i!)^{\frac{1}{p'} - \frac{1}{r}}} \\ &< \infty \end{aligned}$$

where we used the estimate  $x^i/i! \leq e^x$  for  $x = \delta \lambda_n/2 \geq 0$  and  $i \in \mathbb{N}$ .

It follows from (4.1) that  $u$  is the free evolution (i.e. with a null control) of the parabolic equation for  $0 < t \leq T/3$ . Therefore

$$\lim_{t \rightarrow 0^+} u(\cdot, t) = u_0 \quad \text{in } L^2_\rho.$$

We pick as control input

$$h(t) := \begin{cases} 0 & \text{for } t = 0, \\ \sum_{i=0}^{\infty} y^{(i)}(t) [\alpha g_i(1) + \beta (a g_{i,x})(1)] & \text{for } 0 < t \leq T. \end{cases}$$

It follows from (4.1) that  $h(t) = 0$  for  $0 < t \leq T/3$  and from (3.9) combined with the choice of  $s$  that  $h \in C^\infty([0, T])$  with  $h^{(j)}(t) = \sum_{i=0}^{\infty} y^{(i+j)}(t) [\alpha g_i(1) + \beta (a g_{i,x})(1)]$ . Let us check that  $h \in G^s([0, T])$ . We have for  $t \in [\epsilon, T]$

$$\begin{aligned} |h^{(j)}(t)| &\leq \sum_{i \geq 0} |y^{(i+j)}(t) [\alpha g_i(1) + \beta (a g_{i,x})(1)]| \\ &\leq C \sum_{i \geq 0} \frac{(i+j)!^s}{\tilde{R}^{i+j}} \frac{1}{R^i i!^{1 + \frac{1}{p'} - \frac{1}{r}}} \\ &\leq C \left( \frac{2^s}{\tilde{R}} \right)^j \left( \sum_{i \geq 0} \left( \frac{2^s}{R \tilde{R}} \right)^i \frac{1}{i!^{1 + \frac{1}{p'} - \frac{1}{r} - s}} \right) j!^s \end{aligned}$$

for some constant  $C$  which does not depend on  $j$  and  $t$ , where we used  $(i+j)! \leq 2^{i+j} i! j!$ . As  $h(t) = 0$  for  $0 \leq t \leq T/3$ , we conclude that  $h \in G^s([0, T])$ . Finally

$$u(x, t) = 0 \quad \forall (x, t) \in [0, 1] \times \left[ \frac{2T}{3}, T \right].$$

The proof of Theorem 1.1 is complete.

**Remark 4.1.** We stress that assumption (1.9) was used only in Step 1 to get rid of the term  $q(x)u$  in (1.1). If  $q \equiv 0$  in (1.1), then Theorem 1.1 is still valid with the assumptions (1.5)-(1.8).

**4.3. Proof of Proposition 1.3.** Since  $q \in L^{p'}(0, 1)$ , we infer from Lemma 2.10 the existence and uniqueness of a function  $v \in W^{1,1}(0, 1)$  with  $av_x \in W^{1,r}(0, 1)$  such that (2.29)-(2.31) hold. The only property still to establish is the fact that  $v(x) > 0$  for all  $x \in [0, 1]$ . We know that  $v$  satisfies the integral equation (2.33). Therefore, we have

$$v(x) \geq 1 - \|v\|_{L^\infty(0,x)} \int_0^x \frac{dy}{a(y)} \int_0^y |q(s)| ds \quad \forall x \in (0, 1).$$

If (1.13) holds, we claim that  $v(x) > 0$  for all  $x \in [0, 1]$ . Otherwise, there would exist some  $x_0 \in (0, 1]$  such that  $v(x_0) = 0$ , and we can assume that it is the least, so that  $\|v\|_{L^\infty(0, x_0)} = 1$ . But this yields

$$v(x_0) \geq 1 - \int_0^{x_0} \frac{dy}{a(y)} \int_0^y |q(s)| ds > 0,$$

a contradiction. If we assume that  $q(x) \leq 0$  for a.e.  $x \in (0, 1)$ , then we can prove in much the same way that  $v(x) \geq 1$  for all  $x \in [0, 1]$ . Finally, if (1.15) holds, the change of unknowns  $\tilde{u}(x, t) = e^{-Kt}u(x, t)$  transforms (1.5) into

$$(a(x)\tilde{u}_x)_x + [q(x) - K\rho(x)]\tilde{u} = \rho(x)\tilde{u}_t,$$

and the conclusion follows from the previous case with (1.14) satisfied.  $\square$

**4.4. Proof of Proposition 1.4.** Plugging  $v(x) = x^\delta$  in (1.17) results in  $[\delta(1+\delta-\varepsilon)+\mu]x^{\delta-\varepsilon} = 0$  for  $x \in (0, 1)$ , so that the choice  $\delta = \frac{1}{2}[\varepsilon - 1 + ((1-\varepsilon)^2 - 4\mu)^{\frac{1}{2}}]$  gives (1.17). We note that (1.18) holds, for

$$(av_x)(x) = \delta x^{\frac{1-\varepsilon+\sqrt{(1-\varepsilon)^2-4\mu}}{2}}$$

with  $1-\varepsilon+\sqrt{(1-\varepsilon)^2-4\mu} > 0$ . Using (1.16), we infer that  $\delta \geq 0$  and that  $\sqrt{(1-\varepsilon)^2-4\mu} < 1$ . Therefore, we can pick some  $p \in (1, \infty)$  so that

$$1 > \frac{1}{p} > \max(1-\varepsilon, \sqrt{(1-\varepsilon)^2-4\mu}). \quad (4.2)$$

Then the functions  $x \rightarrow x/a(x)$  and  $x \rightarrow x/(a(x)v(x)^2)$  are in  $L^p(0, 1)$ . Applying Lemma 4.2 (see below), we see that (1.8) and

$$\lim_{x \rightarrow 0^+} [a(x)v(x)^2]^{-1} \left( \int_x^1 \frac{ds}{a(s)v(s)^2} \right)^{-2} = +\infty$$

hold. As  $\delta \geq 0$ ,  $v \in C^0([0, 1])$  so that  $\rho v^2 \in L^r(0, 1)$  with  $r = \infty$  and  $\limsup_{x \rightarrow 0^+} [\rho(x)v(x)^2] < \infty$ .

For the application of Theorem 1.1, we do again the change of unknown  $\hat{u}(x, t) := u(x, t)/v(x)$ , and we set  $\hat{a} = av^2$ ,  $\hat{\rho} = \rho v^2$ , and  $\hat{u}_0 = u_0/v$ . From

$$\hat{a}\hat{u}_x = a(u_x v - uv_x) \quad (4.3)$$

we see readily that  $(au_x)(0, t) = 0$  implies  $(\hat{a}\hat{u}_x)(0, t) = 0$ . Let

$$\begin{aligned} \hat{\alpha} &= \alpha v(1) + \beta(av_x)(1) = \alpha + \beta\delta, \\ \hat{\beta} &= \frac{\beta}{v(1)} = \beta. \end{aligned}$$

An application of Theorem 1.1 to the simplified system

$$\begin{cases} (\hat{a}\hat{u}_x)_x = \hat{\rho}\hat{u}_t, \\ (\hat{a}\hat{u}_x)(0, t) = 0, \\ \hat{\alpha}\hat{u}(1, t) + \hat{\beta}(\hat{a}\hat{u}_x)(1, t) = h(t), \\ \hat{u}(x, 0) = \hat{u}_0(x) \end{cases} \quad (4.4)$$

yields the existence of a control input  $h \in G^s([0, T])$  such that the solution  $\hat{u}$  of (4.4) satisfies  $\hat{u}(\cdot, T) = 0$ .

Going back to the original dependent variable  $u(x, t) = \hat{u}(x, t)v(x)$ , we see that  $u$  satisfies (1.1), (1.3)-(1.4) and  $u(\cdot, T) = 0$ . It remains to show that (1.2) holds. Note that (1.2) is not a direct consequence of (4.3)-(4.4), for  $v(0) = 0$  if  $\delta > 0$ . We infer from (4.3) that

$$(au_x)(0^+, t) = \left(\frac{\hat{a}\hat{u}_x}{v}\right)(0^+, t) + \left(\frac{u}{v}av_x\right)(0^+, t).$$

The second term is easy to handle. Indeed,  $(av_x/v)(x) = \delta x^{1-\varepsilon}$ , so that  $(uav_x/v)(0^+, t) = 0$ . To deal with the first term, we write

$$\hat{u}(x, t) = \sum_{i \geq 0} y^{(i)}(t)g_i(x)$$

where the functions  $g_i$ 's and  $y$  are respectively the generating functions and the flat output for system (4.4). It follows that

$$\frac{\hat{a}(x)\hat{u}_x(x, t)}{v(x)} = \sum_{i \geq 0} y^{(i)}(t) \frac{\hat{a}(x)g_{i,x}(x)}{v(x)}.$$

But  $g_{0,x} = 0$  and for  $i \geq 1$

$$(\hat{a}g_{i,x})(x) = \int_0^x \hat{\rho}(s)g_{i-1}(s)ds = \int_0^x s^{2\delta}g_{i-1}(s)ds.$$

This yields

$$\left| \frac{(\hat{a}g_{i,x})(x)}{v(x)} \right| \leq \frac{x^{\delta+1}}{2\delta+1} \|g_{i-1}\|_{L^\infty} \leq \frac{C}{(2\delta+1)R^{i-1}[(i-1)!]^{1+\frac{1}{p'}-\frac{1}{r}}} x^{\delta+1}$$

where we used (3.9). It follows that  $(\hat{a}\hat{u}_x/v)(0^+, t) = 0$ . We conclude that  $(au_x)(0^+, t) = 0$  for  $0 < t < T$ .  $\square$

#### APPENDIX

**The conditions (1.6) and (1.8) are independent.** 1. Pick  $a(x) = x^2|\ln x|^\alpha$  for some  $\alpha > 0$ . Then (1.6) is never satisfied, since  $\int_0^1 x^{-p}|\ln x|^{-\alpha p}dx = +\infty$  for all  $p > 1$ . On the other hand, by integrating by parts, we have for any  $\varepsilon \in (0, 1)$

$$\int_x^{1-\varepsilon} \frac{ds}{s^2|\ln s|^\alpha} = \alpha \int_x^{1-\varepsilon} \frac{ds}{s^2|\ln s|^{\alpha+1}} + [-s^{-1}|\ln s|^{-\alpha}]_x^{1-\varepsilon}$$

and for  $1 - \varepsilon$  small enough and for  $x \in (0, 1 - \varepsilon)$

$$\left| \alpha \int_x^{1-\varepsilon} \frac{ds}{s^2|\ln s|^{\alpha+1}} \right| \leq \frac{1}{2} \int_x^{1-\varepsilon} \frac{ds}{s^2|\ln s|^\alpha}$$

so that for some positive constants  $C_1, C_2$  and  $x > 0$  small enough

$$C_1 x^{-1} |\ln x|^{-\alpha} \leq \int_x^{1-\varepsilon} \frac{ds}{s^2|\ln s|^\alpha} \leq C_2 x^{-1} |\ln x|^{-\alpha}.$$



This yields  $a(x)^{-1} \left( \int_x^1 \frac{ds}{a(s)} \right)^{-2} \geq C_2^{-2} |\ln x|^\alpha$  for  $0 < x \ll 1$ , and (1.8) is satisfied.

2. Pick  $\Omega = \bigcup_{n \geq 1} (2^{-n} - 10^{-n}, 2^{-n} + 10^{-n}) \subset (0, 1)$ , and  $a(x) = x^{2-\varepsilon} 1_{\Omega^c}(x) + 1_\Omega(x)$  for some  $\varepsilon \in (0, 1)$ . Then (1.6) is satisfied for some  $p > 1$ , since  $x/a(x) \leq x^{\varepsilon-1}$  for  $0 < x < 1$ . On the other hand,

$$\int_\Omega \frac{dx}{x^{2-\varepsilon}} \leq \sum_{n=1}^{\infty} \frac{2 \cdot 10^{-n}}{(2^{-n} - 10^{-n})^{2-\varepsilon}} < \infty = \int_0^1 \frac{dx}{x^{2-\varepsilon}},$$

which yields  $\int_{\Omega^c} \frac{dx}{x^{2-\varepsilon}} = +\infty$  and  $1/a \notin L^1(0, 1)$ . It follows that

$$\lim_{n \rightarrow +\infty} a(2^{-n}) \left( \int_{2^{-n}}^1 \frac{ds}{a(s)} \right)^2 = +\infty$$

and that (1.8) fails.

**Lemma 4.2.** *Assume that  $a(x) = [xg(x)]^2$  for some function  $g \in C^1((0, 1), (0, +\infty))$  satisfying for some  $\varepsilon \in (0, \frac{1}{2})$  and some  $x_0 \in (0, 1)$*

$$\begin{aligned} \left| \frac{g'(x)}{g(x)} \right| &\leq \frac{\varepsilon}{x} \quad \text{for } 0 < x < x_0, \\ g(0^+) &= +\infty, \\ (xg(x))(0^+) &= 0, \\ \int_{x_0}^1 \frac{dx}{g(x)^2} &< \infty. \end{aligned}$$

Then (1.8) is satisfied.

*Proof of Lemma 4.2:* We obtain by an integration by parts that

$$\begin{aligned} \int_x^{x_0} \frac{ds}{s^2 g(s)^2} &= [-s^{-1} g(s)^{-2}]_x^{x_0} - 2 \int_x^{x_0} \frac{g'(s)}{s g(s)^3} ds \\ &\leq \frac{1}{xg(x)^2} - \frac{1}{x_0 g(x_0)^2} + 2\varepsilon \int_x^{x_0} \frac{ds}{s^2 g(s)^2}, \end{aligned}$$

which yields

$$\sqrt{a(x)} \int_x^{x_0} \frac{ds}{a(s)} \leq (1 - 2\varepsilon)^{-1} \left( \frac{1}{g(x)} - \frac{\sqrt{a(x)}}{x_0 g(x_0)^2} \right) \rightarrow 0 \quad \text{as } x \rightarrow 0^+.$$

□

**Example 4.3.** (1) Let  $g(x) = x^{-\varepsilon}$  for some  $\varepsilon \in (0, \frac{1}{2})$ . Then  $|g'(x)/g(x)| = \varepsilon/x$  and Lemma 4.2 may be applied for any  $x_0 \in (0, 1)$ .

(2) Let  $g(x) = |\ln x|^\alpha$  for some  $\alpha > 0$ . Then  $|g'(x)/g(x)| = \alpha/(x |\ln x|)$  and Lemma 4.2 may be applied for e.g.  $\varepsilon = 1/4$  and  $x_0 \in (0, 1)$  small enough.

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