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# Particles, Quasi-particles, Elementary and Topological excitations in condensed matter

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Condensed Matter (CM) contains a broad spectrum of ingredients such as Particles, Quasi-particles and Elementary excitations that are different from their counterparts in vacuum, in air or other gases because of the smaller distances between the electrons, atoms and ions constituting the CM medium or because of specific structural (geometrical) symmetries or non-geometrical symmetries such as Time-reversal symmetry, particle-hole symmetry and chiral symmetry as in topological materials. In these interesting emergent materials, topological excitations have already been observed and potential paradigm shift applications are expected in the near future.

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## I. INTRODUCTION

Solids are made of atoms or molecules containing electrons and nuclei containing protons and neutrons. Free electrons, protons and neutrons are considered as particles but when they are inside a particular environment such as a solid or a nucleus... their behavior is strongly different from their free non-interacting counterparts.

Quasi-particles are particles dressed by their environment or by interactions among themselves. For instance, an electron in a solid is subject to a potential  $V(\mathbf{r})$  created by atoms, ions, impurities, dislocations... present inside the solid.

The idea of quasi-particles originated in Lev Landau's theory of many-body Fermi liquids, which was originally created phenomenologically for understanding liquid Helium-3. Landau [1, 2] assumed that the energy of the system is described by a simple parameter, the spatial density  $\rho(\mathbf{r})$  of quasi-particles and that these preserve the basic characteristics of the Fermi-liquid system such as charge, spin and Fermi-Dirac statistics implying that we are mapping the many-body system to an ensemble of weakly interacting fermions.

Landau [3] main hypothesis is that the energy is a functional  $F[\rho(\mathbf{r})]$  and not a simple function, enabling him to define specific conditions that  $F$  should satisfy in order to qualify as the right energy. This idea led, later on, to band-structure calculations of many systems with the DFT (Density Functional Theory) and afterwards to SDFT (Spin Density Functional Theory) pertaining to magnetic systems.

However, it does not apply to Luttinger-Tomonaga [4] 1D fermions, superconducting systems in any dimension or any strongly interacting fermion system.

Elementary excitations are quanta describing excited states of a solid or a liquid. They are different from both particles and quasi-particles. While particles and quasi-particles possess mass, charge and spin, elementary excitations do not have those characteristics and are rather described by their dispersion relations  $\omega(\mathbf{k})$ .

The excited states are obtained when parts of the solid or liquid are driven away from their equilibrium states. The description is akin to a pendulum that exhibits periodic harmonic oscillations with small amplitude about equilibrium when slightly disturbed. However, when it is excited with large perturbations, it will exhibit non-linear, non-harmonic and even chaotic motion.

A quantum many-body system such as a solid contains a large number of interacting particles or quasi-particles on the order of Avogadro number ( $N_A = 6.023 \times 10^{23}$ ) for a mole of material. In principle, Schrödinger equation predicts exactly how this system will behave, however it is a daunting task to solve systems of  $3N_A$  Schrödinger PDE (Partial Differential Equations) in 3D.

Quasi-particles and elementary excitations are types of low-lying excited state. A phonon is a quantum of crystal vibration called an elementary excitation, whereas an electron in a crystal is a quasi-particle.

Several distinctions exist between free particles in vacuum and particles in a crystal. In vacuum, momentum  $\mathbf{p}$  or wavevector  $\mathbf{k} = \mathbf{p}/\hbar$  are continuous as well as the corresponding energy  $\hbar\omega(\mathbf{k})$ .

In a crystal, momentum, wavevector and energy are discrete and called quasi-momentum and quasi-wavevector because translational symmetry is adapted to a discrete lattice and no longer to a continuum.

Translational symmetry restricts momentum and wavevector space to a finite Brillouin zone (BZ) describing physical phenomena in Fourier space much like a unit cell is sufficient to describe physical phenomena in real crystal space. For instance, a simple way to understand the transformation from a free electron kinetic energy  $E_K = \frac{\hbar^2 k^2}{2m_e}$  into a dispersion relation  $\hbar\omega_{\lambda,\sigma}(\mathbf{k})$  is to perform back-folding of  $E_K$  in the BZ (see e.g. Kittel [5]).

Dispersion relation  $\hbar\omega_{\lambda,\sigma}(\mathbf{k})$  is labeled with a band index  $\lambda$  that might take finite (vibration case) or infinite values (electronic excitations) depending on the underlying physics and  $\sigma$  the spin index that describe particles and quasi-particles but not elementary excitations.

Therefore, using quasi-particles/elementary excitations, instead of analyzing a large number ( $3N_A$  particles), boils down to handling somewhat-independent excitations. It is, therefore, an effective approach to simplify the many-body problem from the quantum statistical mechanics point of view.

This approach is not useful for strongly correlated materials where emergent quasi-particles are encountered throughout the discovery of high- $T_c$  superconductivity (HTS), the Fractional Quantum Hall Effect (FQHE) and Topological materials (Insulators, Superconductors...).

Many of the properties of these emergent materials can be explained by the Berry connection, the  $\mathbf{k}$ -space analog of the magnetic vector potential and the Berry curvature, the  $\mathbf{k}$ -space analog of the magnetic field, their peculiarity being revealed by integration over the Brillouin Zone (BZ) yielding a non-zero integer number (Chern) while, in ordinary materials, the integration yields zero.

Topological properties of these materials can be revealed by other special symmetries, sometimes not present in standard materials such as particle-hole symmetry, chiral symmetry, time-reversal symmetry and the number of topologically distinct ground states [6].

Additionally, strong correlations produce separation of charge and spin degrees of freedom [7]. Exceptions exist, however, in e.g. Graphene while not a strongly correlated material, displays topological order with dissipation-less transport along with fractional charge and statistics.

Thus, Topological properties of emergent materials are fascinating as they pave the way to a plethora of novel properties such as spinless or spin edge states in insulating (Chern or Topological), metallic (Weyl, Dirac, Majorana...), superconducting materials ... leading to novel applications and paradigms in condensed matter.

## II. STATISTICAL DISTRIBUTIONS

Generally, an ensemble of spatially distant non-interacting particles with small De Broglie wavelength  $\lambda_{dB}$  are considered as classical whereas in the case of large  $\lambda_{dB}$  the particles are considered as quantum.

In order to precisely qualify the notion of small and large  $\lambda_{dB}$ , one must consider the typical interparticle distance  $d_{int}$ . Thus if we have  $d_{int} \gg \lambda_{dB}$  the ensemble is considered as a classical gas whereas when  $\lambda_{dB} \gg d_{int}$  the ensemble is considered as a quantum gas.

In fact, there is a semi-quantum regime that occurs when  $\lambda_{dB}$  is large without overlap between nearest neighbors. Thus a purely classical regime exists when  $\lambda_{dB}$  is very small and simultaneously  $d_{int}$  is large.

### A. The 3D ideal classical gas

In the Microcanonical ensemble [8], the chemical potential  $\mu = -T \left( \frac{\partial S}{\partial N} \right)_{V,E}$  in terms of  $(E, V, N)$  canonical variables [9] with entropy  $S = S(E, V, N)$  (see Appendix A) related to  $\Omega(E, V, N)$  via Boltzmann relation  $S = k_B \ln \Omega(E, V, N)$  where  $\Omega$  is given by [10]:

$$\Omega(E, V, N) = \int \cdots \int \delta(E - \mathcal{H}) d\Gamma_N \quad (1)$$

where  $d\Gamma_N = \frac{1}{N! h^{3N}} \prod_{i=1}^N d\mathbf{r}_i d\mathbf{p}_i$  and  $\mathcal{H} = \sum_{i=1}^{3N} p_i^2/2m$  is the Hamiltonian for a system of  $N$  free particles.

Note that in the Microcanonical ensemble [8] the total energy is called  $E$  and not  $U$ .

Since  $\mathcal{H}$  is  $\mathbf{r}$ -independent and spherically symmetric with respect to momenta  $\mathbf{p}_i$ , we may write in 3D eq. 1 as:

$$\Omega(E, V, N) = \frac{V^N}{N! h^{3N}} \frac{2\pi^{3N/2}}{\Gamma(3N/2)} \times \int_0^\infty dP P^{3N-1} \delta(E - P^2/2m) \quad (2)$$

with a change of variable  $P \equiv \sum_{i=1}^{3N} p_i^2$  and hyper-volume (see Appendix B) element in  $3N$  dimensions with momenta  $p_i$ . Thus:

$$\Omega(E, V, N) = \frac{1}{N!} \frac{V^N}{h^{3N}} (2\pi m)^{3N/2} \frac{E^{3N/2-1}}{(3N/2-1)!} \approx \frac{1}{N!} \frac{V^N}{h^{3N}} (2\pi m)^{3N/2} \frac{E^{3N/2}}{(3N/2)!} \quad (3)$$

Using Stirling approximation [11], one obtains for the entropy  $S$ :

$$S = k_B N \left\{ \ln \frac{V}{N} + \frac{3}{2} \ln \left[ \frac{mE}{3\pi\hbar^2 N} \right] + \frac{5}{2} \right\}. \quad (4)$$

Substituting the above into the thermodynamic (see Appendix A) relation  $\mu = -T \left( \frac{\partial S}{\partial N} \right)_{V,E}$  leads to the ideal classical gas chemical potential:

$$\mu_C = -k_B T \ln \left[ \frac{V}{N} \left( \frac{mk_B T}{2\pi\hbar^2} \right)^{3/2} \right], \quad (5)$$

where the relation  $E = \frac{3}{2} N k_B T$  has been used.

### B. The 3D ideal quantum gas

Particles, quasi-particles and elementary excitations, are characterized by their Mass, Charge, Spin, Polarizations, Statistical properties and dispersion relations.

They can be statistically distributed along either Bose-Einstein or Fermi-Dirac depending on the value of their spin. In sharp contrast, with non-existing spin value, elementary excitations can be classified as Bosons or Fermions rather from the commutation relations related to their creation-annihilation operators.

In the Bose-Einstein case, the mean occupation number  $f_B(\epsilon) = \langle n \rangle$  of energy level  $\epsilon$  is:

$$f_B(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)} - 1} \quad (6)$$

whereas in the Fermi-Dirac case:

$$f_F(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)} + 1} \quad (7)$$

with  $\beta = 1/k_B T$  and  $\mu$  the chemical potential that allows to determine the number of particles in the system. The chemical potential is a Lagrange multiplier corresponding to the conservation of particles when one determines the statistical distribution of a given system of particles [1]. It is zero in the non-conserved case and negative in the Bose-Einstein case, dropping to zero at the condensation temperature.

For free non-interacting bosons in 3D, the conservation of particles condition is written by expressing their number  $N$  as an integral over energy levels  $\epsilon$  using the density of states  $g(\epsilon) = \frac{m^{2/3}}{\sqrt{2\pi^2\hbar^3}}\epsilon^{1/2}$ :

$$N = \frac{Vm^{3/2}}{\sqrt{2\pi^2\hbar^3}} \int_0^\infty \frac{\epsilon^{1/2}d\epsilon}{e^{\beta(\epsilon-\mu)} - 1} \quad (8)$$

Instead of using specific integration methods proper to Fermi integrals (see Appendix C), we use the Polylogarithm function by noting that:

$$\int_0^\infty \frac{\epsilon^{1/2}d\epsilon}{e^{\beta(\epsilon-\mu)} - 1} = \frac{1}{\beta^{3/2}} \int_0^\infty \frac{x^{1/2}dx}{\zeta e^x - 1} \quad (9)$$

where  $\zeta$  is defined as  $\zeta \equiv e^{-\beta\mu}$ . The Polylogarithm function  $Li_s(\zeta)$  defined as:

$$Li_s(\zeta) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}dt}{\zeta^{-1}e^t - 1}. \quad (10)$$

can be used to express the result in a compact way.

For  $s = 3/2$  and  $\zeta \rightarrow \zeta^{-1}$  we have:

$$Li_{3/2}\left(\frac{1}{\zeta}\right) = \frac{1}{\Gamma(3/2)} \int_0^\infty \frac{t^{1/2}dt}{\zeta e^t - 1} \quad (11)$$

which enable us to rewrite the last integral above as:

$$\frac{1}{\beta^{3/2}} \int_0^\infty \frac{x^{1/2}dx}{\zeta e^x - 1} = \frac{1}{\beta^{3/2}} Li_{3/2}(e^{\beta\mu}). \quad (12)$$

This allows the chemical potential  $\mu$  to be written as an implicit function of temperature  $T$  and particle density  $N/V$  as:

$$\frac{N}{V} = \left(\frac{mk_B T}{2\pi\hbar^2}\right)^{3/2} Li_{3/2}(e^{\beta\mu}). \quad (13)$$

The above corresponds to the equation of state  $\mu = \mu(N, V, T)$  for an ideal Bose gas.

Let us find explicitly the equation of state  $\mu = \mu(N, V, T)$  for an ideal Fermi gas. Proceeding as in the Bose case, we use the density of states  $g(\epsilon) = \frac{m^{2/3}}{\sqrt{2\pi^2\hbar^3}}\epsilon^{1/2}$  to get:

$$N = \sum_s \langle n_s \rangle = \sum_k \frac{1}{e^{\beta(\epsilon_k - \mu)} + 1} \quad (14)$$

This yields:

$$N = \frac{V}{\Gamma(3/2)} \left( \frac{m}{2\pi\hbar^2} \right)^{3/2} \int_0^\infty \frac{\epsilon^{1/2} d\epsilon}{e^{\beta(\epsilon-\mu)} + 1}, \quad (15)$$

Using particle density  $N/V$  and the polylogarithm function  $Li_s(z)$  we write:

$$\frac{N}{V} = - \left( \frac{mk_B T}{2\pi\hbar^2} \right)^{3/2} Li_{3/2}(-e^{\beta\mu}), \quad (16)$$

Thus, the equation of state  $\mu(N, V, T)$  is obtained with  $\mu(N, V, T = 0)$  and Fermi energy  $E_F = k_B T_F = \hbar^2 k_F^2 / 2m$  given by:

$$E_F = \frac{\hbar^2}{2m} \left( 6\pi^2 \frac{N}{V} \right)^{2/3}. \quad (17)$$

The Fermi-Dirac equivalent  $T_{FD}$  of the Bose-Einstein condensation temperature  $T_{BE}$  is given by:

$$\frac{N}{V} = - \left( \frac{mk_B T_{FD}}{2\pi\hbar^2} \right)^{3/2} Li_{3/2}(-1) = \left( \frac{mk_B T_{FD}}{2\pi\hbar^2} \right)^{3/2} \quad (18)$$

This enables us to check the temperature at which the chemical potential of a Fermi gas is zero (condensation forbidden by Pauli exclusion principle) as given by:

$$T_{FD} = \frac{2\pi\hbar^2}{mk_B} \left( \frac{N}{V} \right)^{2/3} \quad (19)$$

In order to evaluate numerically the polylogarithmic [12] function  $Li_s(z)$ , we express them with the Lerch transcendent  $\Phi$  as [13]:

$$Li_s(z) = z\Phi(s, z, 1) \quad Re(s) > 1, |z| \leq 1. \quad (20)$$

$\Phi(s, z, c)$  defined by  $\Phi(s, z, c) = \sum_{n=0}^{\infty} \frac{z^n}{(n+c)^s}$  is a fast converging sum. The Lerch transcendent  $\Phi(s, z, c)$  is obtained from the Lerch zeta function  $\zeta(s, a, c) = \sum_{n=0}^{\infty} e^{2i\pi na} (n+c)^{-s}$  under change of variable  $z = e^{2i\pi a}$  and sometimes called the *Lerch zeta function* (cf. Oberhettinger [14]) despite the fact  $\zeta(s, a, c)$  was originally studied in 1887 by the Czech mathematician Mathias Lerch [15, 16]. In the  $|z| > 1$  case, an analytical continuation [17, 18] is performed in order to calculate  $Li_s(z)$ .

### C. Spin, statistics and associated fields

If the spin of a P/Q/E (P: Particle, Q: Quasi-particle, E: Elementary excitation) is zero, the field associated with it, is Scalar.

If the spin of a P/Q/E is half, the field associated with it, is Spinorial [20]. Mathematically, a spinor reverses its sign when the polar angle is rotated by  $2\pi$  since  $|\psi(\theta + 2\pi, \phi)\rangle = \hat{R}_z(\theta + 2\pi, \phi) |\psi(\theta, \phi)\rangle = -|\psi(\theta, \phi)\rangle$

If the spin of a P/Q/E is one, the field associated with it, is Vectorial.

If the spin of a P/Q/E is a half-integer  $> 1/2$ , the field associated with it, is Spinorial of higher order [20].

If the spin of a P/Q/E is an integer  $> 1$ , the field associated with it, is Tensorial [21].

Intermediate statistics or para-statistics (also called Gentile) occur, for instance, in gas adsorption problems where several gas molecules can fill empty sites of a surface contrasting with the Langmuir (Fermi-Dirac) model where a site is empty or filled at most by a single gas molecule.

Para-statistics and intermediate or fractional statistics can be described by three methods at least:

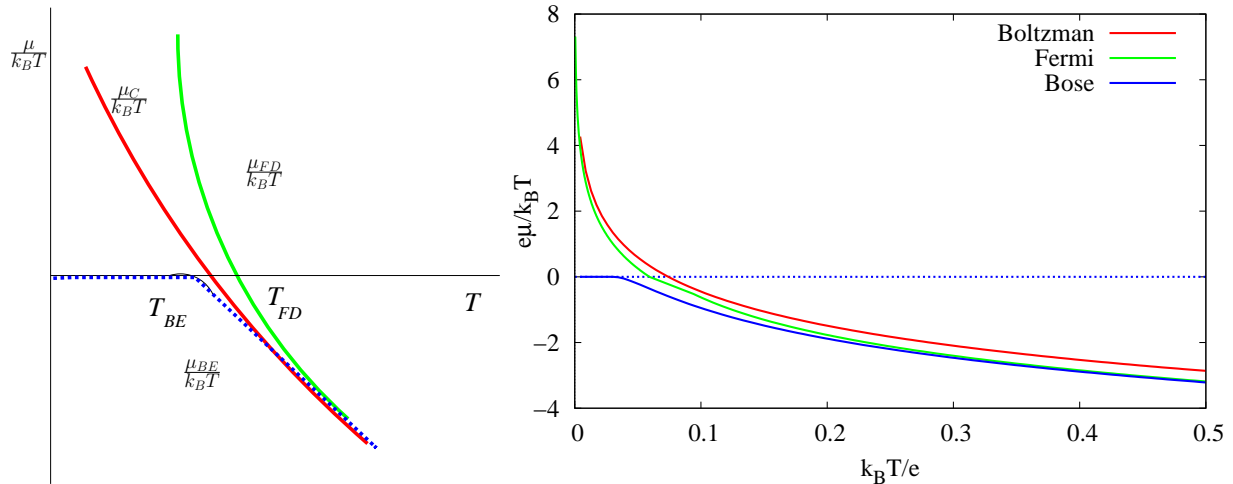


FIG. 1: (Color on-line) Variation of the chemical potential potential for a classical particle [19], a boson and a fermion. At left is Fetter *et al.* theoretical behavior and at right are our own results obtained from reverting the polylogarithmic [12] function  $Li_{3/2}(z)$ . The chemical potential of a boson is always negative reaching zero at the condensation temperature. The classical limit is when the Bose-Einstein factor  $\langle n \rangle = \frac{1}{e^{\beta(\hbar\omega-\mu)} - 1} \ll 1$  and the Fermi-Dirac factor  $\langle n \rangle = \frac{1}{e^{\beta(\hbar\omega-\mu)} + 1} \ll 1$  or when mean particle distance is much larger than thermal wavelength [1, 20]. When  $\beta\mu \rightarrow -\infty$ , we recover Boltzmann distribution  $\langle n \rangle \approx e^{-\beta(\hbar\omega-\mu)}$  with the merging of all chemical potentials: Bose-Einstein, Fermi-Dirac and Maxwell-Boltzmann along the classical limiting line:  $\beta\mu_C \rightarrow \ln \frac{N}{V} \left( \frac{2\pi\hbar^2}{mk_B T} \right)^{\frac{3}{2}}$ .

1. Interpolation method: A Fermion can fill a single quantum level according to Pauli Exclusion Principle whereas Bosons can occupy a single quantum level with no bounds (Condensation). Describing level occupation index with a number  $p$ , we infer that  $p = 1$  for Fermions, whereas  $p = \infty$  for Bosons, thus we can interpolate between these two extreme values, to finally obtain a mean occupation  $f_p(\epsilon)$  of energy level  $\epsilon$  as:

$$f_p(\epsilon) = \frac{1}{e^{\beta(\epsilon+\mu)} - 1} - \frac{p+1}{e^{(p+1)\beta(\epsilon+\mu)} - 1} \quad (21)$$

2. Operator method: Fermions obey anti-commutator relations  $\{a, a^\dagger\} = aa^\dagger + a^\dagger a = 1$  whereas Bosons obey commutator  $[a, a^\dagger] = aa^\dagger - a^\dagger a = 1$  relations. We thus introduce  $q$ -deformed commutation relations in the following way:  $[a, a^\dagger]_q = aa^\dagger - qa^\dagger a$  where  $q = 1$  for bosons and  $q = -1$  for fermions.

Proceeding exactly as done with the standard harmonic oscillator ladder operators, we get:  $a^\dagger |n\rangle = \sqrt{[n+1]_q} |n+1\rangle$  and  $a |n\rangle = \sqrt{[n]_q} |n-1\rangle$  where  $q$ -numbers are defined as  $[n]_q = \frac{q^n - 1}{q - 1}$ .

Orthonormalized set of eigenstates is obtained by repetitive application of the creation operator  $a^\dagger$  onto vacuum state  $|0\rangle$  defined by  $a|0\rangle = 0$  to obtain:

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{[n]_q!}} |0\rangle \quad n = 0, 1, \dots \quad (22)$$

where the  $q$ -factorial  $[n]_q!$  is defined from  $q$ -numbers such that  $[n]_q! = [n]_q [n-1]_q \dots [1]_q$  with  $[0]_q = 1$ .

Finally, we get the mean occupation of energy level  $\epsilon$  as:

$$f_q(\epsilon) = \frac{1}{\ln q} \ln \left( \frac{e^{\beta(\epsilon-\mu)} - 1}{e^{\beta(\epsilon-\mu)} - q} \right) \quad (23)$$

3. Exchange operations: In Pauli spin-statistics theorem, a many-particle wave-function has the following property:

$$\Psi(\mathbf{r}_1, \mathbf{r}_2 \dots \mathbf{r}_j, \dots \mathbf{r}_i \dots) = e^{i\theta} \Psi(\mathbf{r}_1, \mathbf{r}_2 \dots \mathbf{r}_i, \dots \mathbf{r}_j \dots) \quad \theta = 0, \pi \quad (24)$$

The generalization of this theorem proceeds through consideration of an arbitrary value of  $\theta$ . In addition, there is a peculiarity of 2D spatial dimension from a topological point of view.

Firstly, the occurrence of surface states (called edge states) and secondly the permutation group describing spatial exchange of particles is transformed into a Braid group [22] leading to Non-Abelian statistics in sharp contrast to the permutation group that is Abelian in any dimension  $D > 2$ .

This has consequences on the phase factor affecting the many-particle wave-function during exchange of particles:

$$\psi_\alpha(\mathbf{r}_1, \mathbf{r}_2 \dots \mathbf{r}_i, \mathbf{r}_j \dots) \rightarrow e^{i\theta_p T_{\alpha\beta}^p} \psi_\beta(\mathbf{r}_1, \mathbf{r}_2 \dots \mathbf{r}_j, \mathbf{r}_i \dots) \quad (25)$$

where the former wave-function  $\Psi$  has been replaced by a multi-component  $\psi_\alpha$  and the phase has acquired several values  $\theta_p$  that are altered by a  $n \times n$  transformation matrix  $T^p$  where  $n$  is the corresponding Lie Group size [6] (see Appendix D).

On the statistics side we get either bosons or fermions for  $D > 2$  according to Pauli Spin-Statistics [21] theorem, however arbitrary statistics are allowed in 2D paving the way to novel exotic physical properties such as the FQHE to be discussed further below.

### III. QUASI-PARTICLES AND EMERGENCE OF TOPOLOGICAL EXCITATIONS

In solids, an electron quasi-particle is an electron affected by other forces and interactions in the solid. The electron quasi-particle has the same charge and spin as a free electron. However, its mass can differ substantially from that of a free electron. Its electric field is also modified, as a result of screening produced by a large number of surrounding electrons. In some metals under ordinary conditions, these so-called Landau quasi-particles are similar to free electrons.

A hole is a positively charged quasi-particle consisting of the absence of an electron in a state; it is used to describe empty states of a semiconductor valence band.

A photon in vacuum has zero rest mass and travels at the speed of light  $c$ , thus its energy is  $pc$  with  $p$  its momentum. Note that the photon number not being conserved,  $\mu = 0$  for all temperatures, implying that the photon character is always quantum and the "classical" limit  $k_B T \gg \hbar\omega$  invoked by Planck [23, 24] bypasses the quantum nature of the photon.

In sharp contrast, a photon inside a photonic crystal has an energy  $\hbar\omega(\mathbf{k})$ , a mass and a velocity depending on the dispersion law  $\omega(\mathbf{k})$ .

A phonon [5] is a elementary excitation associated with the vibration of atoms in a crystal. It is a quantum of a sound wave.

A magnon [5] is an elementary excitation associated with the electronic spin structure in a crystal lattice. It is a spin wave quantum.

A plasmon [5] is an elementary excitation, a quantum of plasma (ionized medium) oscillation with electrons moving as a whole with respect to ions.

A polaron [5] is a quasi-particle consisting of an electron interacting with the polarization of its surrounding ions.

An exciton [5] is an electron and hole bound together by Coulomb interaction.

A roton [2, 3] is an elementary excitation (vortex quantum) associated with the rotation of a fluid or a superfluid.

According to Kittel [5], the number of polarizations arise in the density of states. This function is  $g(\omega)$  in  $d$  dimensions for a system of typical linear length  $L$  given by  $g(\omega) = \left(\frac{L}{2\pi}\right)^d \int \frac{dS_\omega}{v_g}$  with  $\mathbf{k}$  integration performed such

Name	Nature	Mass	Charge	Spin	$N_p$	Statistics	Chemical Potential	Dispersion relation
Free Boson P	P	$m$	$q$	$S$	$2S + 1$	Bose-Einstein	$\mu(T, P)$	$\frac{\hbar^2 k^2}{2m}$
Free Fermion P	P	$m$	$q$	$S$	$2S + 1$	Fermi-Dirac	$\mu(T, P)$	$\frac{\hbar^2 k^2}{2m}$
Free Electron	P	$m_e$	$-e$	$1/2$	2	Fermi-Dirac	$\mu(T, P)$	$\frac{\hbar^2 k^2}{2m_e}$
Electron in crystal	Q	$m_e^*$	$-e$	$1/2$	2	Fermi-Dirac	$\mu(T, P)$	$\hbar\omega_{\lambda,\sigma}(\mathbf{k})$
Hole in crystal	Q	$m_h^*$	$+e$	$1/2$	2	Fermi-Dirac	$\mu(T, P)$	$\hbar\omega_{\lambda,\sigma}(\mathbf{k})$
Free Photon	P	0	0	1	2	Bose-Einstein	0	$pc$
Photon in Photonic crystal	P	0	0	1	2	Bose-Einstein	0	$\hbar\omega_{\lambda,\sigma}(\mathbf{k})$
Phonon	E	No	No	No	1	Bose-Einstein	0	$\hbar\omega_{\lambda}(\mathbf{k})$
Magnon	E	No	No	No	1	Bose-Einstein	0	$\hbar\omega_{\lambda}(\mathbf{k})$
Graviton in Vacuum	P	0	0	2	2	Bose-Einstein	$\mu(T, P)$	$pc$

TABLE I: Examples of Particles (P), Quasi-Particles (Q) and Elementary excitations (E). While P and Q possess mass, spin and number of polarization states  $N_p$ , E do not have neither mass, nor charge, nor spin nor polarization states (however we take, by default,  $N_p = 1$ ). If a P is massive and possesses a spin  $S$ , it has  $N_p = 2S + 1$  polarization states, whereas a zero-mass P with a spin  $S$  has only two polarization states  $\pm S$  ( $N_p = 2$ ). Particles, quasi-particles and elementary excitations are characterized by their dispersion relation  $\omega(\mathbf{k})$  with  $\mathbf{k}$  continuous for free particles, discrete and restricted to the BZ in crystals for quasi-particles and elementary excitations along with the corresponding dispersion relations  $\omega_{\lambda,\sigma}(\mathbf{k})$  and  $\omega_{\lambda}(\mathbf{k})$  since spin is inexistent in the case of elementary excitations. When we have conservation of the number of particles or quasi-particles we should have a defined chemical potential  $\mu(T, P)$  depending on system temperature and pressure ( $T, P$ ). In the non-conserved case,  $\mu(T, P) = 0, \forall T, P$ .  $\lambda, \sigma$  are band index and spin. For electrons, holes and photons in crystals  $\lambda$  is unbounded whereas for phonons it is bounded by the total number of vibrational modes.  $c$  is vacuum speed of light.

that  $\omega < \omega(\mathbf{k}) < \omega + d\omega$ .  $v_g$  is the group velocity modulus of the elementary excitations [5]:  $v_g = |\nabla_{\mathbf{k}}\omega(\mathbf{k})|$  and  $dS_{\omega}$  is the differential area element on the constant dispersion surface  $\omega(\mathbf{k}) = \omega$ . This formula is generalized into  $g(\omega) = N_p \left(\frac{L}{2\pi}\right)^d \int \frac{dS_{\omega}}{v_g}$  where  $N_p$  is the number of polarizations.

When the excitations are real particles (photons, electrons...) and possess a spin  $S$ ,  $N_p = 2S + 1$  when the particles have non-zero mass (electrons) and  $N_p = 2$  for zero-mass particles (such as photons). Elementary excitations (phonons, plasmons, magnons, excitons...) possess  $N_p = 1$  regardless of statistics. E. Wigner [25] showed in 1939 with Lorentz invariance, that a photon (or any other massless particle with spin  $S$ ) moves with the velocity of light  $c$ . Exploiting rotational symmetry around  $c$  direction provides two polarizations: left or right circular corresponding to  $m_S = \pm S$  spin eigenstates.

### A. Berry phase and the Quantum adiabatic theorem

In classical physics, rotating an object by an integral number  $m$  of full revolutions about some axis returns the object to its initial state. This means the rotation operation by angle  $2m\pi$  radians, with integer  $m$ , is equivalent to the identity operation producing no observable consequences. In Quantum Mechanics, the situation is different with the adiabatic theorem allowing the wave function to alter its phase after its control variables are cycled around a circuit.

Berry [22] first showed that by recalling the adiabatic theorem extension of quantum mechanics, the wave function of a quantum system does not necessarily return to its original phase after circulating around a closed path.

The phase possesses a topological character determined by underlying atomic symmetries that include both inversion and time-reversal symmetry.

As an example, in a material with a band structure defined with Bloch cell periodic eigenstates  $|u_{n\mathbf{k}}\rangle$ , altering adiabatically the wave-vector from  $\mathbf{k}$  to  $\mathbf{k}'$ , we define the Berry connection (see Appendix E) by  $\mathbf{A}_n = -i \langle u_{n\mathbf{k}} | \frac{\partial}{\partial \mathbf{k}} | u_{n\mathbf{k}} \rangle$ .

The Berry phase is given by:  $\Phi_B(\mathbf{k}, \mathbf{k}') = \int_{\mathbf{k}}^{\mathbf{k}'} d\mathbf{k} \cdot \mathbf{A}_n = -i \int_{\mathbf{k}}^{\mathbf{k}'} d\mathbf{k} \cdot \langle u_{n\mathbf{k}} | \frac{\partial}{\partial \mathbf{k}} | u_{n\mathbf{k}} \rangle$ .

For a closed circuit, we get:  $\Phi_B = \oint d\mathbf{k} \cdot \mathbf{A}_n = 2\pi N$  with  $N$  a non-zero integer.

Consequently, we infer that the Berry connection is simply the  $\mathbf{k}$ -space analog of the magnetic vector potential.

Additionally, the Berry curvature, the  $\mathbf{k}$ -space analog of the magnetic field,  $F_n = (\nabla \times \mathbf{A}_n)_z$  yields the Chern number as an integral over the Brillouin Zone (BZ) summed over filled states:

$$\mathcal{C} = \sum_n^{\text{filled}} \frac{1}{2\pi} \int_{BZ} dk_x dk_y F_n = 0, \pm 1, \pm 2, \pm 3 \dots \quad (26)$$

$\mathcal{C} = 0$  corresponds to a standard material whereas  $\mathcal{C} = \pm 1, \pm 2, \pm 3 \dots$  corresponds to a Chern insulator material. The Hall conductivity [26], in a Chern insulator, is given by:  $\sigma_{xy} = \mathcal{C} \frac{e^2}{h}$  and the edge state is a charge current.

The  $\mathbb{Z}_2$  invariant is calculated over half the Brillouin Zone (BZ/2) :

$$\mathbb{Z}_2 = \sum_{n=1}^{\text{filled}} \frac{1}{2\pi} \left( \oint_{BZ/2} \mathbf{A}_n \cdot d\mathbf{k} - \int_{BZ/2} dk_x dk_y F_n \right) = 0, 1 \pmod{2} \quad (27)$$

$\mathbb{Z}_2$  corresponds to a standard material whereas  $\mathbb{Z}_2 = 1$  corresponds to a topological insulator material [22] possessing time-reversal symmetry and in which the edge state is a spin current.

In addition there are other indicators such as surface states (or edge states in 2D) and excitations that possess fractional charge or statistics.

They can be characterized by topological entropy that measures the complexity of their order. One example would be the complexity of the ground state wave-function [27].

Topological materials address several issues such as those mentioned by Thouless [28]:

1. Why does current flow without loss in a superconductor, although the current-carrying state does not have low free energy?
2. Why is a solid rigid?
3. Why can a poorly defined Josephson junction provide the world's best voltage measurements?
4. Why can the quantum Hall effect provide the best standards of electrical resistance?
5. How many different types of line defects are there in various liquid crystal phases?

Moreover, following Wen [27], another set of questions can be addressed in Topological materials along the following lines:

1. Topological insulators and superconductors (*i.e.* with  $T^2 = (-1)^{N_F}$  the fermion number parity operator [22] for electron systems time-reversal symmetry and weak interactions) has no topological order.

Topological insulators are insulating in bulk but their surface might be insulating or conducting (gapless) depending on a topological property. For example, a (2+1)  $D$  (2 spatial + 1 time) topological insulator is characterized by a non-zero term called  $\theta$  and time-reversal symmetric flux [29, 30]. Most materials have  $\theta = 0$  whereas a topological insulator has  $\theta = \pi$ .  $\theta$  is a topological quantity that controls charges in the topological material, for instance when  $\theta = 2\pi$  we get  $e$  the electron charge [29, 30].

For a 3D topological insulator gapless states might occur on its faces, along its sides or some directions or even at special points [31] as displayed in Fig. 2.

Note this is in sharp contrast with respect to (non-topological) amorphous Selenium films (used in photocopier photoconducting drums) that are conducting in the bulk and insulating on their surface [32] so that they retain charges reproducing the latent image for some required time (imaged document) before later processing.

2. (3+1) $D$  (3 spatial + 1 time)  $s$ -wave superconductors have no topological order, while (3+1) $D$  real-life  $s$ -wave superconductors have a  $\mathbb{Z}_2$ -topological order.
3. (2+1) $D$  fermion paired state and Integer Quantum Hall (IQH) states do not have any fractional topological excitations. They are considered either as long or short-range entangled states.
4. What are the difference between a Chern insulator, quantum anomalous Hall state, and IQH?
5. In non-Abelian statistics, should a Majorana fermion obey Fermi statistics?

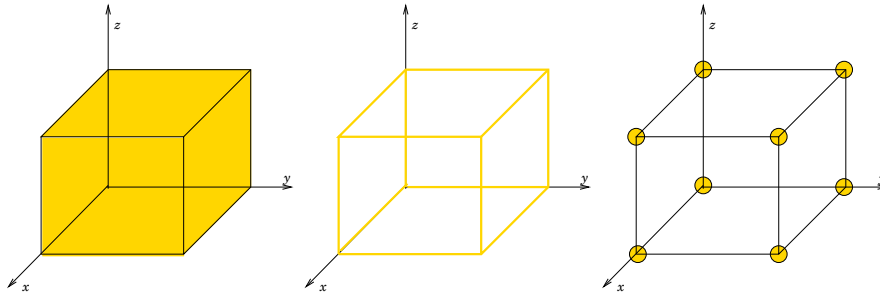


FIG. 2: (Color on-line) Cubic 3D topological insulator with gapless (conducting) states occurring on its 2D faces (left), along 1D sides (center) or at 0D apices (right).

6. Is the Bogoliubov [33] quasiparticle in a superconductor a Majorana fermion, knowing that a Bogoliubov quasiparticle is rather inspired from superfluidity than superconductivity?

We would add that it brings another look at exchange of identical particles based on the essence of topology i.e. continuity. Thus exchange might be viewed as an adiabatic, continuous transformation with the particles changing place via continuous paths.

Moreover there are indications from Chern-Simons theory based on physics belonging to a new type of gauge theory, completely different from Maxwell theory, in  $(2+1)D$  (see Appendix D) implying the charge density is locally proportional to the magnetic field meaning that in Chern-Simons theory magnetic flux is tied to electric charge through a coupling parameter  $\kappa$ .

This implies a coupling of charge and magnetic flux that is responsible for the notion of composite particle carrying an electric charge and a magnetic flux as in the FQHE.

**Note:** Let us insist there is a clear distinction between the 3D case and the 2D case where topological and fractional properties have been observed in many systems pertaining to spin behavior in 2D.

For instance, composite fermions arise in a 2D system subject to a large magnetic field, displaying the FQHE. One observes a fraction of the electron charge  $e$  with filling fraction values:

- $\nu = \frac{m}{2m\pm 1}$  for integer  $m \geq 1$  (principal series),
- $\nu = \frac{m}{2pm\pm 1}$  for integer  $m, p \geq 1$  (Jain series),
- $\nu = \frac{m}{m+2}$  for integer  $m \geq 2$  (Moore-Read-Rezayi series),
- $\nu = \frac{m}{2(m\pm 1)}$  for integer odd  $m \geq 3$  (Even denominator series) and finally
- $\nu = \frac{m}{(3m-2)}$  for integer  $m \geq 2$ .

#### IV. PARTICLES, QUASI-PARTICLES AND ELEMENTARY EXCITATIONS IN EMERGENT MATERIALS

Emergent and topological materials [34] are the source of novel Particles, Quasi-Particles and Elementary excitations.

While "standard" Particles (P) and Quasi-Particles (Q) are characterized by their Mass, Charge, Spin, Polarizations and dispersion relations, topological excitations are characterized by symmetries such as Time-reversal or  $\mathcal{T}$  symmetry  $\Theta$ , particle-hole symmetry  $\Xi$  and chiral symmetry  $\Pi = \Xi\Theta$ . Their charge, spin and statistics are fractional meaning they behave distinctly from both Bosons and Fermions.

In the case of Elementary excitations (E) Mass, Charge, Spin and Polarizations are not relevant but only the dispersion relations. Same is true for Topological Elementary excitations as well as symmetry properties.

Topological excitations were considered after three major paradigm shifts [35] in condensed matter physics:

1. Discovery of High Temperature Superconductivity (HTSC).

2. Discovery of the Integer (IQHE) and Fractional Quantum Hall Effect (FQHE).
3. Discovery of Bose-Einstein condensation in dilute atomic gases such as vapors of Rubidium, Sodium, and Lithium.

Topological symmetry [22] is higher than geometrical (spatial) symmetry in the sense it does not care about specific angles or lengths as in standard crystallography. Landau argument [1], defining a phase transition as broken symmetry that is a change from higher (at higher temperature) to lower geometrical symmetry (at lower temperature) has to be modified in the case of Topological symmetry since it is not about geometrical or spatial symmetry.

In Landau theory [1] the main ingredient is a local order parameter which is zero in the higher symmetry (higher temperature) phase and non-zero in the lower symmetry (lower temperature) phase like magnetization in magnetic materials which is zero in the paramagnetic phase and non-zero in the ferromagnetic phase.

Topological symmetry endows the corresponding material, such an insulator having an energy gap with "protection" meaning that the gap value cannot change under any deformation of the Hamiltonian, does not cause it to decrease and does not trigger, for instance, an Insulator-Metal (first-order) phase transition. This protection implies also that any disorder considered as a deformation to the Hamiltonian cannot affect certain physical quantities belonging to a given topological symmetry.

We recall that an insulating state might occur in one-body band structure (weakly interacting Fermi gas) from the appearance of a large (several eV) energy gap (ordinary insulator) or in many-body systems (strongly interacting Fermi gas) from Coulomb interactions (Mott insulator) or from disorder (Anderson insulator). We will show below that in topological materials we have occurrence of two new types of insulating materials, Chern and topological materials who have both special edge (surface) states.

Topological materials display non-local effects and the corresponding topological order parameter (TOP) cannot be described by a local Landau theory [1] but by special numbers and integrals over the entire material also called topological numbers or invariants [27] characterizing their nature such as charge or density...

Examples of invariants are Chern numbers that are equivalent to Berry flux evaluated through integration over the Brillouin Zone (or its half in the case of Time-Reversal symmetry) and  $\mathbb{Z}_2$ . Both can be evaluated directly from the Bloch cell periodic eigenstates  $|u_{n\mathbf{k}}\rangle$  or other mathematical quantities associated with the state of interest [27].

Topological insulators and superconductors [22] properties are determined by specifying the symmetry class and the dimensionality. The symmetry class depends on the presence or absence of time-reversal  $\mathcal{T}$  symmetry (see Table II), with  $\Theta^2 = \pm 1$  and/or particle-hole symmetry (see Table II) with  $\Xi^2 = \pm 1$ .

In topological insulators and superconductors there are ten symmetry classes [21] specified by presence or absence of  $\mathcal{T}$  symmetry  $\Theta$ , particle-hole symmetry  $\Xi$  and chiral symmetry  $\Pi = \Xi\Theta$ .

The symmetry classes are closely related to the classification of random matrices [21].

The similarity between insulators and superconductors originates from the presence of an energy gap. In an insulator a (direct or indirect) gap occurs in its one-body band structure whereas a gap in a superconductor originates from a many-body interaction and separates the ground state from the first excited state.

Some examples of emergent fermionic quasi-particles are:

A Dirac fermion is a relativistic electron observed in materials such as Graphene.

A Majorana fermion is a uncharged particle which is its own antiparticle, and can emerge as a quasi-particle in certain superconductors, or in a quantum spin liquid.

A Weyl fermion is a massless fermion.

Non-linear excitations are solitons as the Su-Schrieffer-Heeger (SSH) in Polyacetylene or magnetic monopoles, vortices, domain walls and skyrmions (topological spin structures, see Appendix F)...

### A. Bose-Einstein condensation in dilute atomic gases and Quantum Gas Microscopes

Bose-Einstein condensation (BEC) in dilute atomic gases such as vapors of Rubidium, Sodium, and Lithium constitutes the Third Quantum Revolution after the First one signaling the birth of Quantum Mechanics and the Second exploiting Bell inequalities to distinguish Quantum Theory as clearly characterized by its non-locality. The third quantum revolution opens the doors of Quantum Physics to macroscopic applications and Quantum

System	Symmetry				$d$		
	Cartan	$\Theta$	$\Xi$	$\Pi$	1	2	3
Wigner-Dyson	A	0	0	0	0	$\mathbb{Z}$	0
	AI	1	0	0	0	0	0
	AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
Chiral	AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$
	BDI	1	1	1	$\mathbb{Z}$	0	0
	CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$
BdG	D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0
	C	0	-1	0	0	$\mathbb{Z}$	0
	DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0
	CI	1	-1	1	0	0	$\mathbb{Z}$

TABLE II: Periodic table of topological insulators and superconductors. The 10 symmetry classes are labeled using Cartan labeling [34] and are specified by presence or absence of  $\mathcal{T}$  symmetry  $\Theta$ , particle-hole symmetry  $\Xi$  and chiral symmetry  $\Pi = \Xi\Theta$ .  $\mathbb{Z}_2$  means two topologically distinct ground states whereas  $\mathbb{Z}$  means that the topologically distinct ground states can be labeled by the set of integers [6]. Wigner-Dyson refers to traditional Quantum Field Theory whereas BdG refers to Bogoliubov-De-Gennes [33] superconductivity theory.

Emergent field	Basic component	Control parameter	Bottleneck/key experiment	Target applications
Topological Electronics	Berry phase	Band structure design Spin texture	RT Zero-field edge current Skyrmionic circuit (see Appendix F)	Information technology Sustainable energy
Magneto-Electronics or Spintronics	Spin-orbit interaction	Broken symmetries in space and time	RT E-field switching Ultrafast photo-switching	Low Dissipation circuits Information technology Sustainable energy
MIPT Electronics	Mott Transistor Phase Transition	Band-filling Bandwidth	RT E-field switching Above-RT superconductor	Low Dissipation circuits Sustainable energy
Quantum Computing	Quantum coherence	Nanomaterials design Topological protection	Qubit/photon interface Quantum simulator	Quantum computer Information security Ultra-fast decryption
Large scale Computing	Quantum Gas microscope	Bose-Einstein condensation (BEC)	Optical Lattice	Many-body problem solving: Hubbard, Bose-Hubbard...

TABLE III: Emergent applications of Quantum/Topological materials and Bose-Einstein condensation. Adapted from Tokura *et al.* [36]. **Note:** MIPT is Metal-Insulator Phase Transition. RT is Room Temperature.

Technologies such as Quantum Information, Communications and Computing.

Dilute quantum gases differ from ordinary gases, liquids and solids in a number of ways, The particle density at the centre of a Bose-Einstein condensed atomic cloud is typically  $10^{13}$ - $10^{15}/\text{cm}^3$ . By contrast, the density of molecules in air at room temperature and atmospheric pressure is about  $10^{19}/\text{cm}^3$ . In liquids and solids the density of atoms is of order  $10^{22}/\text{cm}^3$ , while the density of nucleons in atomic nuclei is about  $10^{38}/\text{cm}^3$ .

BEC has been observed in Excitons and Photons under specific conditions. This led to many scientific advances such as the reaching of extremely low temperatures (nano-K and pico-K ranges). Moreover, BEC of Magnons and Photons have been both observed surprisingly at room temperature.

Another application of BEC is the development of Quantum Gas Microscopes allowing to study in a straightforward way the properties of many-body systems...

Putting atoms into an optical lattice, paves the way to study of many-body systems that are realizations of models used in condensed matter physics such as the Hubbard model for fermions or Bose-Hubbard model for bosons.

Although the gases are very dilute, the atoms can be made to interact strongly, thus providing new challenges for the description of strongly correlated many-body systems. In a period of less than ten years, the study of dilute quantum gases has changed from an unusual topic to an integral part of contemporary physics, with strong ties to molecular, atomic, subatomic and condensed matter physics.

### APPENDIX A: CLASSICAL THERMODYNAMICS REFRESHER

The Jacobian method allows one to derive easily any thermodynamic identity as long as one respects the canonical variables associated with some potential.

This is provided with the Guggenheim table below where each potential Gibbs, Enthalpy, Energy and Free energy respectively  $G, H, U$  and  $F$  is flanked on each side by the appropriate canonical variables.

$P$	$H$	$S$
$G$		$U$
$T$	$F$	$V$

FIG. 3: (Color on-line) The table is constructed with a simple sentence: Great Physicists Have Studied Under Very Fine Teachers starting with  $G$  at left of middle row and proceeding clockwise. The corner highlighted variables make Maxwell relation table.

If we want to derive the enthalpy differential  $dH$  from the table, we write  $dH = adP + bdS$  and  $a, b$  are found by inspecting the element in box across the diagonal from the box containing the canonical variable. We have a (+) sign if we are moving down from the canonical variable to reach out for the diagonally opposite corner box and a (-) sign in the opposite case. Thus  $a = V$  and  $b = T$  yielding  $dH = VdP + TdS$ . In the energy differential case  $dU$  we have  $dU = adS + bdV$  with  $a = T, b = -P$  yielding  $dU = TdS - PdV$ . In the case of free energy we have  $dF(T, V) = -SdT - PdV$  whereas for Gibbs potential,  $dG(P, T) = VdP - SdT$ .

Maxwell identities are constructed as well from the table corners. Thus we have:  $(\frac{\partial P}{\partial T})_V = (\frac{\partial S}{\partial V})_T$  and each time we exchange variables diagonally we pick up a sign change as in the case  $P \leftrightarrow V$ :  $(\frac{\partial V}{\partial T})_P = -(\frac{\partial S}{\partial P})_T$ .

Note that if we want to enlarge the number of thermodynamic variables to the Grand Canonical ensemble case that includes  $N, \mu$  we write:  $dU(S, V, N) = TdS - PdV + \mu dN$ ,  $dH(P, S, N) = VdP + TdS + \mu dN$ ,  $dF(T, V, N) = -SdT - PdV + \mu dN$  and  $dG(P, T, N) = VdP - SdT + \mu dN$ . From  $dU(S, V, N)$  we obtain  $\mu = -T(\frac{\partial U}{\partial N})_{S, V}$  or  $\mu = -T(\frac{\partial S}{\partial N})_{V, U}$  by taking into  $dU(S, V, N) = TdS - PdV + \mu dN$   $V, U$  as constants implying  $dV = 0, dU = 0$ . Same can be done with the other thermodynamic potentials  $H, F, G$ .

In order to describe the use of Jacobians in Thermodynamics, let us recall some of their properties [1, 9]:

- $\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$
- $\frac{\partial(u, v)}{\partial(x, y)} = -\frac{\partial(v, u)}{\partial(x, y)}$
- $\frac{\partial(u, y)}{\partial(x, y)} = (\frac{\partial u}{\partial x})_y$
- $\frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial(u, v)}{\partial(s, t)} \frac{\partial(s, t)}{\partial(x, y)}$
- $\frac{\partial(u, v)}{\partial(x, y)} = 1 / \frac{\partial(x, y)}{\partial(u, v)}$

This means that any term such as  $\partial(x, y)$  behaves as if it were an elementary algebraic coefficient simplifying and speeding up enormously mathematical operations.

Nonetheless, there is a limit to express any physical quantity: It should depend on three experimentally measurable quantities [9]:

1. Dilation coefficient  $\alpha_P = \frac{1}{V} \left( \frac{\partial V}{\partial T} \right)_P$
2. Compressibility  $\kappa_T = -\frac{1}{V} \left( \frac{\partial V}{\partial P} \right)_T$
3. Heat capacity  $C_V = T \left( \frac{\partial S}{\partial T} \right)_V$  or  $C_P = T \left( \frac{\partial S}{\partial T} \right)_P$

As an example, let us find the expression of  $C_P - C_V$ . We start by writing:  $C_V = T \left( \frac{\partial S}{\partial T} \right)_V = T \frac{\partial(S, V)}{\partial(T, V)} = \frac{\partial(S, V)}{\partial(T, P)} \frac{\partial(T, P)}{\partial(T, V)}$  which gives:

$$C_P - C_V = -T \left( \frac{\partial V}{\partial T} \right)_P^2 / \left( \frac{\partial V}{\partial P} \right)_T = TV \alpha_P^2 / \kappa_T.$$

In the ideal gas case we have the state equation  $PV \propto T$  resulting in  $\alpha_P = 1/T$  and  $\kappa_T = 1/P$ . Thus  $C_P - C_V = PV/T = Nk_B$  where  $N$  is the number of particles.

## APPENDIX B: SURFACE AND VOLUME OF A HYPERSPHERE IN $d$ DIMENSIONS

Surface and volume of a 3D sphere is done through rationalisation of coordinate expressions via the following transformations:

$$\begin{aligned} (x, y, z) &\rightarrow (x_1, x_2, x_3) \\ (r, \theta, \phi) &\rightarrow (r, \theta_1, \theta_2); \theta_1 \in [0, \pi], \theta_2 \in [0, 2\pi] \end{aligned} \quad (\text{B1})$$

Cartesian coordinates use  $(r, \theta_1, \theta_2)$  in the rationalized case as:

$$\begin{aligned} x_1 &= r \sin \theta_1 \sin \theta_2 \\ x_2 &= r \sin \theta_1 \cos \theta_2 \\ x_3 &= r \cos \theta_1 \end{aligned} \quad (\text{B2})$$

**Note:** Angles  $(\theta_1, \theta_2)$  are different from spherical angles  $(\theta, \phi)$  since  $\theta_1 = (\hat{z}, \mathbf{r})$ ,  $\theta_2 = (\hat{y}, \mathbf{r})$  whereas  $\theta = (\hat{z}, \mathbf{r})$ ,  $\phi = (\hat{x}, \mathbf{r})$ . Unit vectors  $\hat{y}, \hat{z}$  are along  $y, z$  axes respectively.

Jacobian  $J$  of transformation  $(x_1, x_2, x_3) \rightarrow (r, \theta_1, \theta_2)$  links volume elements  $dx_1 dx_2 dx_3 = |J| dr d\theta_1 d\theta_2$  in both systems.

Jacobian modulus is:

$$|J| = \left| \frac{\partial(x_1, x_2, x_3)}{\partial(r, \theta_1, \theta_2)} \right| = \left\| \begin{array}{ccc} \frac{\partial x_1}{\partial r} & \frac{\partial x_1}{\partial \theta_1} & \frac{\partial x_1}{\partial \theta_2} \\ \frac{\partial x_2}{\partial r} & \frac{\partial x_2}{\partial \theta_1} & \frac{\partial x_2}{\partial \theta_2} \\ \frac{\partial x_3}{\partial r} & \frac{\partial x_3}{\partial \theta_1} & \frac{\partial x_3}{\partial \theta_2} \end{array} \right\| = \left\| \begin{array}{ccc} \sin \theta_1 \sin \theta_2 & r \cos \theta_1 \sin \theta_2 & r \sin \theta_1 \cos \theta_2 \\ \sin \theta_1 \cos \theta_2 & r \cos \theta_1 \cos \theta_2 & -r \sin \theta_1 \sin \theta_2 \\ \cos \theta_1 & -r \sin \theta_1 & 0 \end{array} \right\| = r^2 \sin \theta_1, \quad (\text{B3})$$

yielding expressions for volume  $dV$  and surface  $dS$  elements:

$$dV = r^2 \sin \theta_1 dr d\theta_1 d\theta_2, \quad dS = r^2 \sin \theta_1 d\theta_1 d\theta_2 = r^2 d\Omega, \quad d\Omega = \sin \theta_1 d\theta_1 d\theta_2 \quad (\text{B4})$$

Solid angle element  $d\Omega$  is defined by direction  $(\theta_1, \theta_2)$  is simply elemental surface of unit radius sphere along direction  $(\theta_1, \theta_2)$ .

For a  $d$  dimension hypersphere its surface and volume are obtained with an extension from 3D to  $d$  dimensions via Cartesian and rationalized hyperspherical coordinates:

$$\begin{aligned} (x, y, z, u, v, \dots) &\rightarrow (x_1, x_2, x_3, x_4, x_5, \dots, x_d) \\ (r, \theta, \phi, \psi, \dots) &\rightarrow (r, \theta_1, \theta_2, \dots, \theta_{d-1}); \theta_1 \in [0, \pi], \theta_2 \in [0, 2\pi] \dots \end{aligned} \quad (\text{B5})$$

Exploiting an analogy with eq. B2 the coordinates are expressed as:

$$\begin{aligned}
x_1 &= r \sin \theta_1 \sin \theta_2 \dots \sin \theta_{d-2} \sin \theta_{d-1} \\
x_2 &= r \sin \theta_1 \sin \theta_2 \dots \sin \theta_{d-2} \cos \theta_{d-1} \\
x_3 &= r \sin \theta_1 \sin \theta_2 \dots \cos \theta_{d-2} \\
&\vdots \\
x_d &= r \cos \theta_1
\end{aligned} \tag{B6}$$

The respective domains pertaining to each variable are given by:

$$r \in ]0, \infty[, \theta_1 \in [0, \pi], \theta_2 \in [0, 2\pi] \dots \theta_{d-1} \in [0, 2\pi] \tag{B7}$$

Rationalized angles are given by:  $\theta_i = (\hat{x}_{d-i+1}, \mathbf{r})$  (cf. **Note** above).

One may generalize the Jacobian to  $d$  dimensions according to the following.

A volume element in  $d$  dimensions can be obtained from Jacobian  $J_d$  with coordinate transformation  $dx_1 dx_2 dx_3 dx_4 dx_5 \dots dx_d = |J_d| dr d\theta_1 d\theta_2 \dots d\theta_{d-1}$ , to obtain:

$$|J_d| = \left| \frac{\partial(x_1, x_2, x_3, x_4, x_5 \dots x_d)}{\partial(r, \theta_1, \theta_2, \dots \theta_{d-1})} \right| = r^{d-1} \sin \theta_1 \sin \theta_2 \sin \theta_3 \dots \sin \theta_{d-2}, \tag{B8}$$

proceeding along B3, obtaining the volume element as:

$$dV = r^{d-1} \sin \theta_1 \sin \theta_2 \sin \theta_3 \dots \sin \theta_{d-2} dr d\theta_1 d\theta_2 \dots d\theta_{d-1} \tag{B9}$$

For the surface element we have:

$$\begin{aligned}
dS &= r^{d-1} \sin \theta_1 \sin \theta_2 \sin \theta_3 \dots \sin \theta_{d-2} d\theta_1 d\theta_2 \dots d\theta_{d-1} \equiv r^{d-1} d\Omega_d \\
d\Omega_d &= \sin \theta_1 \sin \theta_2 \sin \theta_3 \dots \sin \theta_{d-2} d\theta_1 d\theta_2 \dots d\theta_{d-1}
\end{aligned} \tag{B10}$$

$d\Omega_d$  is a hyper solid angle given by directions  $(\theta_1, \theta_2, \dots \theta_{d-1})$ . It is in fact an area element on the unit sphere contained between directions  $(\theta_1, \theta_2, \dots \theta_{d-1})$ .

Thus we are able to evaluate volumes and surfaces in  $d$  dimensions by direct integration.

### 1. Laurent Schwartz method

Another elegant method attributed to Mathematician Laurent Schwartz (creator of Distribution Theory) consists of using relations based on Eulerian  $\Gamma$  function.

Since:

$$\int_{-\infty}^{\infty} dx \exp(-x^2) = \sqrt{\pi} \tag{B11}$$

Taking this identity to  $d$ -th power yields:

$$I_d = \left[ \int_{-\infty}^{\infty} dx \exp(-x^2) \right]^d = [\sqrt{\pi}]^d \tag{B12}$$

This is equivalent to turning  $d$  1D integrals into a single integral in  $d$  dimensions:

$$I_d = \int_{-\infty}^{\infty} dx \exp(-x^2) \int_{-\infty}^{\infty} dy \exp(-y^2) \int_{-\infty}^{\infty} dz \exp(-z^2) \dots = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \exp -(x^2 + y^2 + z^2 \dots) dx dy dz \dots \tag{B13}$$

Following the above path, we are able to express this  $d$  dimensional integral as follows:

$$I_d = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \exp(-(x^2 + y^2 + z^2 \dots)) dx dy dz \dots = \int_0^{\infty} \int_{\mathcal{D}_\theta} \exp(-r^2) r^{d-1} dr d\Omega_d \quad (\text{B14})$$

where angular domain  $\mathcal{D}_\theta = [0, \pi] \times [0, 2\pi] \dots \times [0, 2\pi]$  corresponds to respective intervals :  $\theta_1 \in [0, \pi], \theta_2 \in [0, 2\pi] \dots \theta_{d-1} \in [0, 2\pi]$ .

The hypersphere surface is given by:  $S_d = \int_{\mathcal{D}_\theta} d\Omega_d$  whereas the radial integral  $I_r = \int_0^{\infty} \exp(-r^2) r^{d-1} dr$  and consequently  $I_d$  is finally reduced to:  $I_d = \pi^{d/2} = I_r S_d$ .

Using Gamma function definition  $\Gamma(x) = \int_0^{\infty} \exp(-t) t^{x-1} dt$  after change of variable  $u = r^2$  yields  $I_r = \Gamma(d/2)/2$ .

Thus  $S_d = 2\pi^{d/2}/\Gamma(d/2)$  for a unit radius hypersphere. For a radius  $R$  hypersphere, its surface is  $S_R = 2R^{d-1}\pi^{d/2}/\Gamma(d/2)$  and its volume is  $V_R = 2R^d\pi^{d/2}/[d\Gamma(d/2)]$ .

### APPENDIX C: FERMI INTEGRALS

A Fermi integral is defined by:

$$I = \int_0^{+\infty} \frac{G(\epsilon) d\epsilon}{e^{(\epsilon-\mu)/k_B T} + 1} \quad (\text{C1})$$

with  $G(\epsilon)$  an arbitrary function of energy  $\epsilon$ .

Using change of variable  $z = (\epsilon - \mu)/k_B T$   $I$  yields:

$$I = k_B T \int_{-\mu/k_B T}^{+\infty} \frac{G(\mu + k_B T z) dz}{e^z + 1} = k_B T \int_{-\mu/k_B T}^0 \frac{G(\mu + k_B T z) dz}{e^z + 1} + k_B T \int_0^{+\infty} \frac{G(\mu + k_B T z) dz}{e^z + 1} \quad (\text{C2})$$

Performing  $z \rightarrow -z$  in the first integral and using:

$$\frac{1}{e^{-z} + 1} = 1 - \frac{1}{e^z + 1} \quad (\text{C3})$$

we get:

$$I = \int_0^{\mu} G(z) dz - k_B T \int_0^{\mu/k_B T} \frac{G(\mu - k_B T z) dz}{e^z + 1} + k_B T \int_0^{+\infty} \frac{G(\mu + k_B T z) dz}{e^z + 1} \quad (\text{C4})$$

Using the approximation  $\mu/k_B T \approx \infty$ , we get:

$$I = \int_0^{\mu} G(z) dz + k_B T \int_0^{\infty} \frac{[G(\mu + k_B T z) - G(\mu - k_B T z)] dz}{e^z + 1} \quad (\text{C5})$$

Expanding  $G(z)$  around  $\mu$ :

$$G(\mu + x) - G(\mu - x) \approx G(\mu) + xG'(\mu) + \frac{x^2}{2!}G''(\mu) \dots - [G(\mu) - xG'(\mu) + \frac{x^2}{2!}G''(\mu) \dots] \quad (\text{C6})$$

we get:

$$I \approx \int_0^{\mu} G(z) dz + 2(k_B T)^2 G'(\mu) \int_0^{\infty} \frac{z dz}{e^z + 1} \dots \quad (\text{C7})$$

Using the identity  $\int_0^{\infty} \frac{z dz}{e^z + 1} = \frac{\pi^2}{12}$ , we finally get:

$$I \approx \int_0^{\mu} G(z) dz + \frac{\pi^2}{6} (k_B T)^2 G'(\mu) \dots \quad (\text{C8})$$

## APPENDIX D: ABELIAN AND NON-ABELIAN GAUGE THEORY WITH IMPLICATIONS TO STATISTICS

Most of groups of interest in physics can be formulated as groups of matrices. For instance, the Lorentz group, consists of a set of  $4 \times 4$  matrices.

In elementary particle physics, a Lie group is a continuous group, with an infinite number of elements such as rotation angles, taking on a continuum of values. For example, a general 3D rotation matrix has three parameters (Euler angles) and nine elements. Lie groups are Non-Abelian as we recall that rotations (in 3D) do not commute and thus constitute examples of Non-Abelian groups.

$U(n)$  is made of all unitary  $n \times n$  matrices. A unitary matrix is such that its inverse is its adjoint i.e.  $U^{-1} = U^\dagger$ .  $SU(n)$  is made of unitary matrices with unit determinant,  $S$  stands for special meaning that the determinant=1. Real unitary matrices make the group  $O(n)$  where  $O$  stands for orthogonal; an orthogonal matrix inverse is equal to its transpose:  $O^{-1} = O^T$ . Finally, the group of real, orthogonal,  $n \times n$  unity determinant matrices is  $SO(n)$  that may be thought of as the group of all rotations in  $n$  dimensional space.

Let us introduce  $T^p$  Lie group generators where the index  $p = 1, 2, \dots, d_G$  with  $d_G$  the Group dimension [6]. For instance,  $d_G = n^2 - 1$  for  $SU(n)$  and  $d_G = \frac{1}{2}n(n - 1)$  for  $SO(n)$ .

The generators  $T^p$  define the group  $SU(n)$  with the commutation relations  $[T^a, T^b] = if^{abc}T^c$  where the numbers  $f^{abc}$  are called the group [37] structure constants (for example, in the  $SU(2)$  case [37], we have  $f^{abc} = \epsilon^{abc}$  where  $\epsilon^{abc}$  is the 3D Levi-Civita antisymmetric tensor).

We move on below to a  $(2 + 1)D$  Abelian Chern-Simons gauge theory which is different from standard Maxwell electromagnetism with a field Lagrangian [38] given by:

$$\mathcal{L}_M = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_\mu J^\mu \quad (D1)$$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$  is expressed with a 4-potential  $A_\mu = (A_0, \vec{A})$ ,  $A_0$  being the scalar potential and  $\vec{A}$  the 3D vector potential whereas the 4-current  $J_\mu = (\rho, \vec{j})$  in a Minkowski space with signature  $(-1,1,1,1)$ . We assume Einstein summation rule over repeated indices for the above Lagrangian as well as the rest of this section.

A global gauge transformation is given by:

$$\psi \rightarrow e^{-ie\alpha}\psi \quad (D2)$$

whereas a local gauge transformation is given by:

$$\psi \rightarrow e^{-ie\alpha(x)}\psi \quad (D3)$$

implying that

$$A_\mu \rightarrow A_\mu - \partial_\mu\alpha(x) \quad (D4)$$

The Abelian Chern-Simons (CS)  $(2 + 1)D$  Lagrangian uses  $\kappa$ , a coupling parameter such that:

$$\mathcal{L}_{CS} = \frac{\kappa}{2}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho - A_\mu j^\mu \quad (D5)$$

where  $\epsilon^{\mu\nu\rho}$  is the 3D Levi-Civita [37] antisymmetric tensor.

In order to understand the significance of coupling a matter current  $J^\mu$  to a Chern-Simons gauge field  $A_\mu$ , consider the electric charge density  $\rho$  and current components  $J_i$  :

$$\rho = \kappa B, \quad J^i = \kappa\epsilon^{ij}E_j \quad (D6)$$

where  $\epsilon^{ij}$  is the dielectric tensor.

These equations indicate the charge density is locally proportional to the magnetic field meaning that in Chern-Simons theory magnetic flux is tied to electric charge through parameter  $\kappa$ .

This implies a coupling of charge and magnetic flux that is responsible for the notion of composite particle carrying an electric charge and a magnetic flux as in the FQHE and explaining many other aspects of FQHE not explainable by Maxwell theory.

Regarding statistics, Pauli spin-statistics theorem, a many-particle wave-function has the following property:

$$\Psi(\mathbf{r}_1, \mathbf{r}_2 \dots \mathbf{r}_j, \dots \mathbf{r}_i \dots) = e^{i\theta} \Psi(\mathbf{r}_1, \mathbf{r}_2 \dots \mathbf{r}_i, \dots \mathbf{r}_j \dots) \quad \theta = 0, \pi \quad (\text{D7})$$

The generalization of this theorem is an arbitrary value for  $\theta$ . In addition, there is a peculiarity of  $2D$  spatial dimension from a topological point of view.

Since the permutation group describes spatial exchange of particles in order to describe the effect of exchange on statistics we recall that the general permutation group is Abelian for any dimension  $D > 2$ . It is thus remarkable that  $2D$  is special since the permutation group becomes a Braid group [39] characterized by Non-Abelian statistics.

In Non-Abelian gauge theory where 4-potentials  $A_\mu$  develop non-commuting components  $A_\mu^\alpha$  with same in the case of the many-particle wave-function leading to a phase factor affecting each wave-function component  $\psi_\alpha$  during exchange of particles:

$$\psi_\alpha(\mathbf{r}_1, \mathbf{r}_2 \dots \mathbf{r}_i, \mathbf{r}_j \dots) \rightarrow e^{i\theta_p T_{\alpha\beta}^p} \psi_\beta(\mathbf{r}_1, \mathbf{r}_2 \dots \mathbf{r}_j, \mathbf{r}_i \dots) \quad (\text{D8})$$

Note that  $T^p$  components are indexed with  $\alpha, \beta$  (Einstein summation rule being assumed over repeated indices  $p, \beta$ ) and the former wave-function  $\Psi$  has been replaced by a multi-component  $\psi_\alpha$  and the phase has acquired several values  $\theta_p$  that are altered by the matrix  $T^p$  whose size is  $n \times n$  with  $n$  the size of the corresponding Lie group.

## APPENDIX E: TOPOLOGY, DIFFERENTIAL GEOMETRY AND BAND STRUCTURE

Topology is a branch of mathematics concerned with continuity and geometrical properties of objects that are insensitive to smooth deformations [40].

Popular examples of topology is that a sphere cannot be smoothly deformed into the surface of a torus and that a torus can be smoothly deformed into a cup with a handle.

The genus,  $g$ , is an integer consisting of the number of holes in the object and obviously is 0 for the sphere and 1 for the torus. The genus is an example of a topological invariant.

Surfaces that can be deformed into one another are topologically equivalent, and those with different genus that cannot be deformed into one another, are said to be topologically distinct.

Moreover, topological invariants can be determined for surfaces with the help of Differential Geometry [41]. With the Gauss-Bonnet theorem stating that the integral of the Gaussian curvature [41],  $K$  over a surface, one obtains an integer number called the Euler characteristic. More precisely, if  $S$  is an orientable [41] compact surface of class  $C^2$  (twice derivable), then:

$$\iint_S K dS = 2\pi\chi(s) \quad (\text{E1})$$

The number  $\chi(s)$  is the Euler characteristic of  $S$  and  $2\pi\chi(s)$  is called the total curvature.

For instance, for a sphere of radius  $R$ ,  $K = 1/R^2$  and consequently  $\chi(s) = 2$  whereas  $\chi(s) = 0$  for a torus meaning zero total curvature.

If  $S$  has handles like a cup of coffee then  $\chi(s) = 2(1 - g)$  where  $g$  is the genus equivalent to the number of handles.

In order to relate topology to band structure we recall Bloch theorem [5] stating the eigenstates of an electron in a crystal is given by  $e^{i\mathbf{k}\cdot\mathbf{r}} |u_{n\mathbf{k}}\rangle$  that is a plane wave multiplied by a cell periodic eigenstate of the Bloch Hamiltonian  $\mathcal{H}(\mathbf{k}) = e^{i\mathbf{k}\cdot\mathbf{r}} \mathcal{H} e^{-i\mathbf{k}\cdot\mathbf{r}}$  the  $\mathbf{k}$  transform of the real space Hamiltonian  $\mathcal{H}$ .

The Bloch states are invariant under the transformation

$$|u_{n\mathbf{k}}\rangle \rightarrow e^{i\phi(\mathbf{k})} |u_{n\mathbf{k}}\rangle \quad (\text{E2})$$

and if we define the Berry connection [40] as  $\mathbf{A}_n = -i \langle u_{n\mathbf{k}} | \frac{\partial}{\partial \mathbf{k}} | u_{n\mathbf{k}} \rangle$  it transforms as  $\mathbf{A}_n \rightarrow \mathbf{A}_n + \nabla_{\mathbf{k}} \phi(\mathbf{k})$  thus it behaves as an electromagnetic vector potential with  $\phi(\mathbf{k})$  playing the role of a gauge.

Pushing the analogy further, we define the Berry phase [40]  $\Phi_B$  as the line integral over  $C$  a closed loop in  $\mathbf{k}$  space, we get  $\Phi_B = \oint_C \mathbf{A}_n \cdot d\mathbf{k}$ . This yields  $\Phi_B = 2\pi N$  with  $N$  a non-zero integer.

Transforming the line integral into a surface integral yields  $\Phi_B = \iint_S F_n d^2\mathbf{k}$  where  $F_n = (\nabla \times \mathbf{A}_n)_z$  is the Berry curvature [40], the analog of the magnetic field. Thus the link between topology and band structure is established.

Moreover, integrating over the Brillouin Zone (BZ)  $F_n = (\nabla \times \mathbf{A}_n)_z$  yields the Chern number by summing over filled states:

$$\mathcal{C} = \sum_n^{\text{filled}} \frac{1}{2\pi} \int_{BZ} dk_x dk_y F_n = 0, \pm 1, \pm 2, \pm 3 \dots \quad (\text{E3})$$

$\mathcal{C} = 0$  corresponds to a standard material whereas  $\mathcal{C} = \pm 1, \pm 2, \pm 3 \dots$  corresponds to a Chern insulator material. For instance, the Hall conductivity of a Chern insulator is:

$$\sigma_{xy} = \mathcal{C} \frac{e^2}{h} \quad (\text{E4})$$

Thus another link between topology and a transport coefficient like Hall conductivity [26] is also established.

Another link is related to the  $\mathbb{Z}_2$  invariant calculated over half the Brillouin Zone (BZ/2) (by Kramers theorem related to time-reversal symmetry and resulting in a symmetrical band structure such that  $\omega_n(\mathbf{k}) = \omega_n(-\mathbf{k})$ ):

$$\mathbb{Z}_2 = \sum_{n=1}^{\text{filled}} \frac{1}{2\pi} \left( \oint_{BZ/2} \mathbf{A}_n \cdot d\mathbf{k} - \int_{BZ/2} dk_x dk_y F_n \right) = 0, 1 \pmod{2} \quad (\text{E5})$$

$\mathbb{Z}_2$  corresponds to a standard material whereas  $\mathbb{Z}_2 = 1$  corresponds to a topological insulator material [22].

## APPENDIX F: EXAMPLE OF TOPOLOGICAL EXCITATIONS: SKYRMIONS

Skyrmions are spin textures [42] carrying charge and possessing topological properties described further below.

In fact, they are spatially localized soliton-like [43] structures with fixed rotational sense magnetic whirls possessing axial symmetry. They have been observed in non-centro-symmetric (with no inversion symmetry) bulk crystals in which a Dzyaloshinskii-Moriya [43] interaction (DMI) of the form  $D_{ij} \mathbf{m}_i \times \mathbf{m}_j$  exists between two neighboring moments  $\mathbf{m}_i, \mathbf{m}_j$ . DMI interaction is the vector counterpart of the scalar Heisenberg exchange  $J_{ij} \mathbf{m}_i \cdot \mathbf{m}_j$  interaction.

The interest in Magnetic Skyrmions originate from the fact they offer great potential as information carriers in future robust, high-density, and energy-efficient spintronic devices.

In addition to their protected topology and nanoscale size, they can easily be moved by lateral spin currents and created as well as deleted by vertical spin-current injection since they couple very efficiently to spin currents and respond sensitively to spin transfer torques [43]. They might make racetrack-type memory using spin-current densities being five to six orders of magnitude smaller than those needed to move magnetic domain walls [43].

Their name originates from nuclear physics: the Skyrme model [44] that considers three linearly dispersing (massless) spin wave excitations corresponding to three oscillation directions about the ordered one. These three massless modes represent three almost massless pions  $\pi^+, \pi^0, \pi^-$ . The nucleons (proton and neutron) are represented by Skyrmion spin textures. It can be shown that these pions are fermions despite the fact they are a coherent superposition of an infinite number of bosonic spin waves akin to the buildup of a quantum wavepacket from an

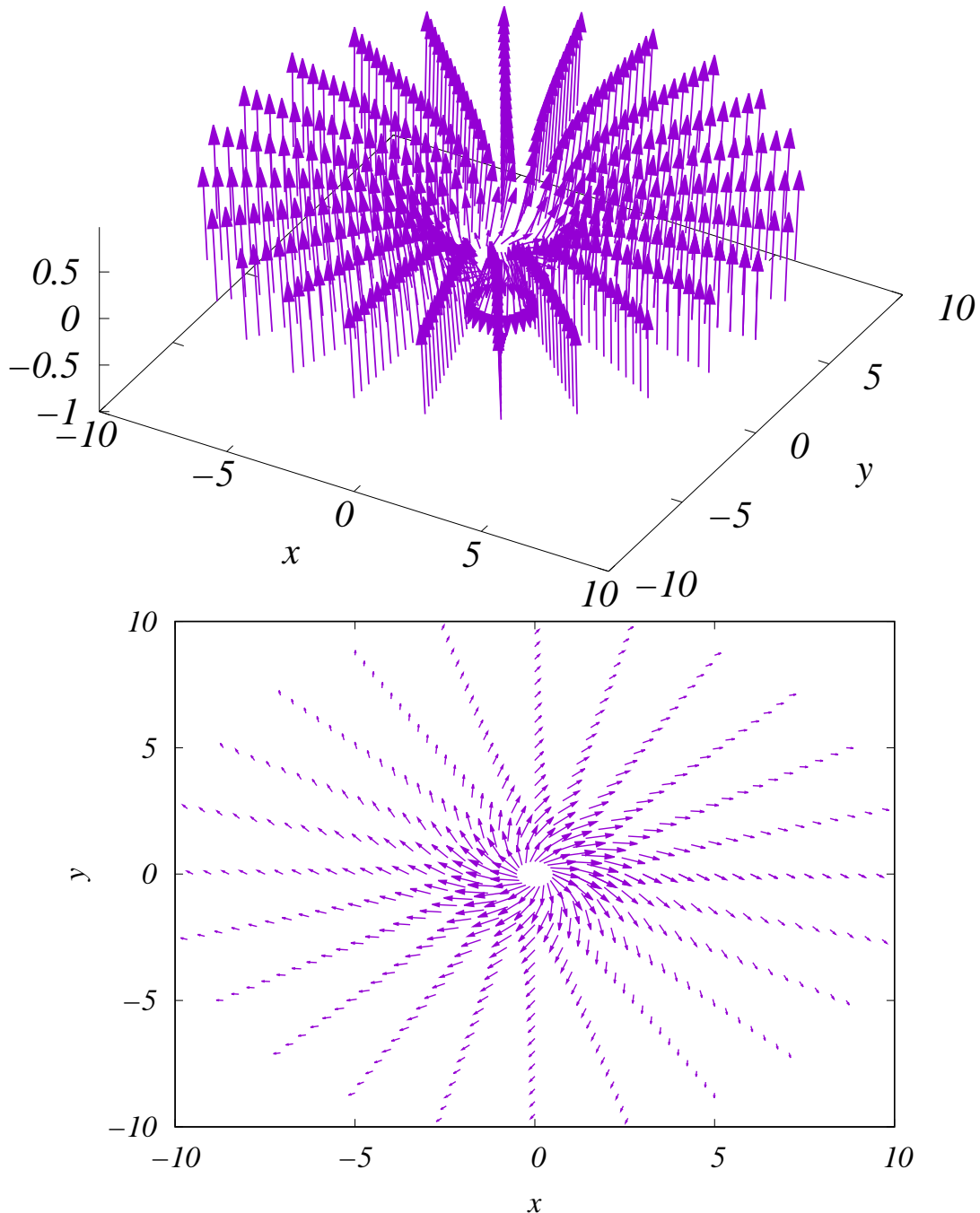


FIG. 4: (Color on-line) 3D view of a skyrmion spin texture (above) and same in 2D (below). The components are given by (see text)  $m_x(\mathbf{r}) = \frac{2\lambda r \cos(\theta-\varphi)}{\lambda^2+r^2}$ ,  $m_y(\mathbf{r}) = \frac{2\lambda r \sin(\theta-\varphi)}{\lambda^2+r^2}$ ,  $m_z(\mathbf{r}) = \frac{r^2-\lambda^2}{\lambda^2+r^2}$  with  $\lambda = 1$  and  $\phi = 0.6$ . The spins are down about the origin and gradually turn upwards for large distance  $r$ . At intermediate distances, the  $x - y$  components of the spin exhibit a vortex-like winding.

infinite superposition of plane waves.

Since magnetic spins tend to be locally parallel we obtain a smooth topological object as displayed in fig. 4.

While the spins point downwards about the origin and gradually turn upwards for large distance  $r$  their aspect at intermediate distances is a vortex-like configuration. However there is no singularity in the core region because the spins are able to rotate downwards out of the  $x - y$  plane as displayed in fig. 4.

Regarding the electric charge carried by a skyrmion, consider a ferromagnet with local spin orientation  $\mathbf{m}(\mathbf{r})$ . The

exchange field keeps the spin following the local orientation  $\mathbf{m}$  as each electron travels such that a charge density can be defined from the gradients of the local spin orientation. This results in a topological charge density [42] given by:

$$\rho_T = \frac{1}{8\pi} \epsilon^{\alpha\beta} \mathbf{m} \cdot \partial_\alpha \mathbf{m} \times \partial_\beta \mathbf{m} \quad (\text{F1})$$

where  $\epsilon^{\alpha\beta}$  is the 2D Levi-Civita antisymmetric tensor.

Performing 2D spatial integration, the Skyrmion has total topological charge:

$$Q_T \equiv \frac{1}{8\pi} \int d^2r \epsilon^{\alpha\beta} \mathbf{m} \cdot \partial_\alpha \mathbf{m} \times \partial_\beta \mathbf{m} \quad (\text{F2})$$

which is always an integer. In fact for any smooth spin texture in which the spins at infinity are all parallel,  $Q_T$  is always an integer.  $Q_T$  is a topological invariant meaning that it is stable against smooth continuous distortions of  $\mathbf{m}$ .

Since skyrmions carry charge, we can modify the spin configuration with an electric rather than a magnetic field.

One possible representation of the  $\mathbf{m}(\mathbf{r})$  skyrmion components is:

$$m_x(\mathbf{r}) = \frac{2\lambda r \cos(\theta - \varphi)}{\lambda^2 + r^2} \quad (\text{F3})$$

$$m_y(\mathbf{r}) = \frac{2\lambda r \sin(\theta - \varphi)}{\lambda^2 + r^2} \quad (\text{F4})$$

$$m_z(\mathbf{r}) = \frac{r^2 - \lambda^2}{\lambda^2 + r^2} \quad (\text{F5})$$

where  $(r, \theta)$  are the polar coordinates in the plane,  $\lambda$  is a constant that controls the Skyrmion size scale, and  $\varphi$  is a constant that controls the  $xy$  spin orientation. The Skyrmion spin texture shape arises from the fact  $m_z(\mathbf{r})$  component points downwards when  $|\mathbf{r}| \rightarrow 0$  and upwards when  $|\mathbf{r}| \rightarrow \infty$  for a positive  $\lambda$  while  $m_x(\mathbf{r}), m_y(\mathbf{r})$  components rotate about  $z$  axis as displayed in fig.4.

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