A Hybrid Observer-based Controller for a Non-uniformly Observable System

Mohamed Maghenem, William Pasillas-Lépine, Antonio Loria, Missie Aguado-Rojas

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M. Maghenem W. Pasillas-Lépine A. Loría M. Aguado-Rojas

***Abstract*— For systems that are not observable at an equilibrium to be stabilized, output-feedback stabilization is considerably challenging. This paper solves this control problem for the case-study of a second-order system that is bilinear and affine, both in the input and the output, but it is unobservable at the target equilibrium. The case-study defines a class of non-uniformly observable systems and stems from automotive-control applications. Our main contribution is a novel observer-based hybrid controller that achieves asymptotic stabilization semiglobally. The controller relies on a switched observer that estimates the state, provided that the latter is 'kept away' from the singular equilibrium. To achieve both competing tasks, stabilization and estimation, the controller relies on the keen construction of a piecewise-constant, converging, output reference. Our main results are illustrated via numerical simulations on a meaningful example.**

***Index Terms*— Non-uniformly observable systems; observers; hybrid control; ABS.**

I. Introduction

"Stabilizing a system and estimating the state are two competing processes that need to happen simultaneously in order to stabilize a partially measured system in closed loop" [1]. This statement is particularly meaningful when the control goal is to stabilize an equilibrium, at which, observability is lost. The study of such systems, for about three decades, has yielded an ever-growing literature—see, *e.g.*, [1]–[5]. Beyond the mathematical difficulty imposed by designing an observer [6]–[8], the stabilization problem is well-motivated by concrete engineering applications, such as sensorless motor control [9], bioreactor systems [10], [11], electrical systems [12], and automotive applications [13].

Despite elegant solutions for particular instances, such as the one studied in [14], the asymptotic output-feedback stabilization of an unobservable equilibrium for nonlinear systems remains, in general, an open question. This is the case for the representative case-study, of interest here, given by the equations

$$\dot{z}_1 = -az_1z_2 + u \tag{1a}$$
$$\dot{z}_2 = (cz_2 + d)z_1, \quad z_1,\ z_2 \in \mathbb{R}, \tag{1b}$$

where a, c, and $d > 0$, u is the control input, and $y = Cz = z_1$ is the measured output.

M. Maghenem is with GIPSA-Lab, University of Grenoble Alpes, CNRS, France (e-mail: mohamed.maghenem@cnrs.fr). W. Pasillas-Lépine and A. Loría are with L2S, CNRS, Univ Paris-Saclay, CentraleSupélec, France (e-mails: wiliam.pasillas-lepine@centralesupelec.fr and antonio.loria@cnrs.fr). M. Aguado-Rojas is with Hitachi Astemo. The work of M. Maghenem and of A. Loría is supported by the French ANR via project HANDY, contract number ANR-18-CE40-0010.

This system belongs to the class of bilinear systems of the form $\dot{x} = A(y)x + Bu$—cf. [4], which are not observable at $\{y = 0\}$. Furthermore, up to a change of time-scale, (1) represents the dynamics of the so-called extended-braking stiffness system [13], [15], where the regulation of the unmeasured variable z_2 translates into maximizing the braking force in the antiblock braking system (ABS) [16]. This being said, stabilizing the origin $\{z_1 = z_2 = 0\}$ for (1), measuring only z_1 (which is the linear acceleration of the tire at the wheel-ground contact point, relative to the longitudinal acceleration of the vehicle) is the open problem that we solve here.

In this paper, we propose a novel hybrid controller that relies on the exponentially convergent observer previously proposed in [13], [15]. As explained in [17] and documented in the bilinear-systems literature [4], [18], such a state estimator relies on the output $y = z_1$ being persistently exciting, which is in clear conflict with the stabilization task. To circumvent this inadequacy, we propose a control strategy inspired from [15], [19] that consists in steering y to a vanishing piecewise-constant output reference z^*. This reference takes values in a discrete set that is symmetrical with respect to 0. The reference z^* converges to 0 at a controlled rate that maintain the output y sufficiently excited on specific windows of time. On each window, the estimation error decreases at a rate that depends on the level of excitation of y and, thus, on the magnitude of z^*. During each window, z_2 is maintained in a specific ball around 0, by switching z^* between a value and its opposite. In particular, the range of z_2 shrinks from one window to another as well as the size of z^*. Finally, we control the length of the different time windows by inspecting the decrease ratio of the estimation errors. This way, we guarantee convergence of the estimation errors as well as the system's variables to zero. Such a controller is hybrid in nature and successfully stabilizes the origin provided that an upper bound on the initial condition is known.

This paper is a natural follow up on long-standing work led by the second author [5], [13], [15], [17], which are exclusively devoted to the observer-design problem.

II. Switched-observer design

We revisit the switched observer for system (1) originally introduced in [13] and we refine the main result therein by providing an explicit estimation of the convergence rate.

Let $\hat{z} := (\hat{z}_1, \hat{z}_2)$ denote the estimate of $z := (z_1, z_2)$ and consider the observer

$$\dot{\hat{z}}_1 = -az_1\hat{z}_2 + u + k_1(z_1)z_1(z_1 - \hat{z}_1)$$
$$\dot{\hat{z}}_2 = cz_1\hat{z}_2 + dz_1 + k_2(z_1)z_1(z_1 - \hat{z}_1),$$

where

$$k_i(z_1) := \begin{cases} k_i^+ & \text{if } z_1 > 0 \\ k_i^- & \text{if } z_1 < 0 \\ 0 & \text{if } z_1 = 0, \end{cases} \qquad i \in \{1, 2\}.$$

and, following [15], we set $k_1^+ > c$, $k_2^+ < -\frac{c}{a}k_1^+$, $k_2^- < -\frac{c}{a}k_1^-$, $k_1^- = 2c - k_1^+$, $ck_1^+ + ak_2^+ = ck_1^- + ak_2^-$.

Then, the estimation error $\tilde{z}_i := \hat{z}_i - z_i$ is governed by

$$\begin{bmatrix} \dot{\tilde{z}}_1 \\ \dot{\tilde{z}}_2 \end{bmatrix} = z_1(t) \begin{bmatrix} -k_1(z_1(t)) & -a \\ -k_2(z_1(t)) & c \end{bmatrix} \begin{bmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{bmatrix}, \tag{3}$$

and depends on the measurable output trajectory $t \mapsto z_1(t)$.

Now, we consider the following hypothesis on the measured output z_1.

Assumption 1: There exist positive constants τ_d, τ_s, $\underline{z}$, and $\bar{z}$, and an infinite union of disjoint intervals, denoted I_d, such that:

- $|z_1(t)| \geq \underline{z}$ for all $t \in I_d$,
- $|z_1(t)| \leq \bar{z}$ for all $t \in \mathbb{R}_{\geq 0} \backslash I_d$,
- the length of each connected interval in I_d is no smaller than τ_d, and
- the length of each connected interval in $\mathbb{R}_{\geq 0} \backslash I_d$ is smaller than τ_s.

Assumption 1 relaxes the hypotheses imposed in [5], where it is additionally required that $t \mapsto z_1(t)$ passes by 0 only at isolated instants. As it is showed in the next section, Assumption 1 holds by design (of a particular output reference) on specific windows of time.

Next, for the purpose of analysis, we introduce the new time scale—cf. [5],

$$\tau := \int_{t_o}^{t} |z_1(s)| ds =: f_{z_1}(t), \tag{4}$$

which is well defined under Assumption 1, in the sense that, f_{z_1} is continuous, nondecreasing, and $\lim_{t \to \infty} f_{z_1}(t) = \infty$. Moreover, we note that Assumption 1 implies the existence of μ and $T > 0$ such that

$$\int_{t_o}^{t} |z_1(s)| ds \geq \mu(t - t_o) \qquad \forall t \geq t_o + T, \quad \forall t_o \geq 0. \tag{5}$$

In the new time-scale, system (3) is equivalent to

$$\tilde{w}' := \frac{d\tilde{w}}{d\tau} = A(w_1(\tau))\tilde{w} \qquad \forall \tau \in \mathrm{Im}(f_{z_1}), \tag{6}$$

where

$$A(w_1) := \begin{cases} A_+ & \text{if } w_1 > 0 \\ A_- & \text{if } w_1 < 0 \\ 0 & \text{if } w_1 = 0, \end{cases}$$

$\mathrm{Im}(f_{z_1}) := f_{z_1}([t_o, \infty))$ denotes the image of f_{z_1}, $w_1 : \mathrm{Im}(f_{z_1}) \to \mathbb{R}$ and $\tilde{w} : \mathrm{Im}(f_{z_1}) \to \mathbb{R}^2$ are given by

$$w_1(\tau) := \begin{cases} z_1(f_{z_1}^{-1}(\tau)) & \text{if } \mathrm{card}(f_{z_1}^{-1}(\tau)) = 1 \\ 0 & \text{otherwise,} \end{cases} \tag{7}$$

$$\tilde{w}(\tau) := \begin{cases} \tilde{z}(f_{z_1}^{-1}(\tau)) & \text{if } \mathrm{card}(f_{z_1}^{-1}(\tau)) = 1 \\ \tilde{z}(\min\{f_{z_1}^{-1}(\tau)\}) & \text{otherwise,} \end{cases} \tag{8}$$

where $\mathrm{card}(f_{z_1}^{-1}(\tau))$ is the number of elements in

$$f_{z_1}^{-1}(\tau) := \{t \in [t_o, +\infty) : f_{z_1}(t) = \tau\},$$

and A_+ and A_- correspond, respectively, to the matrix in (3) with $k_i = k_i^+$ and $k_i = k_i^-$.

Note that, by definition, $w_1(\tau) = 0$ only for τ in a null-measure set. Hence,

$$A(w_1(\tau)) \in \{A_1, A_2\} \quad \text{for almost all} \quad \tau \in \mathrm{Im}(f_{z_1}).$$

Furthermore, under Assumption 1, $\mathrm{Im}(f_{z_1}) = \mathbb{R}_{\geq 0}$ and $\tau \mapsto w_1(\tau)$ satisfies the following property:

Property 1: There exist an infinite union of disjoint intervals, denoted $\bar{I}_d$, such that

- $|w_1(\tau)| > 0$ for all $t \in \bar{I}_d$,
- the length of each connected interval in $\bar{I}_d$ is no smaller than $(\tau_d \underline{z})$,
- the length of each connected interval in $\mathbb{R}_{\geq 0} \backslash \bar{I}_d$ is smaller than $(\tau_s \bar{z})$.

Lemma 1: Consider the non-autonomous system (6) such that $w_1 : \mathrm{Im}(f_{z_1}) \to \mathbb{R}$ satisfies Property 1. Then,

$$|\tilde{w}(\tau)| \leq \kappa_1 |\tilde{w}(0)| \exp^{-\kappa_2 \tau} \quad \forall \tau \in \mathrm{Im}(f_{z_1}),$$

where, given $\lambda > 0$, $\kappa_1 := \gamma \frac{\bar{k} + 2\bar{c}^2 \gamma}{\rho(1 + \bar{k})}$, $\kappa_2 := -\ln(\rho)/L$,

$$\gamma := \lambda_{\max}(P)/\lambda_{\min}(P), \quad \bar{c} := \exp^{\max\{|A_1|, |A_2|\} \bar{z} \tau_s},$$
$$k := \max\{|K_1|, |K_2|\}, \quad \bar{k} := \lambda_{\max}(P) \bar{c}^2 k^2 / \lambda,$$

L is such that

$$\rho := \frac{2\gamma \bar{c}^2 \exp^{-2\lambda L} + \bar{k}}{1 + \bar{k}} < 1,$$

and K_1, $K_2 \in \mathbb{R}^2$ are such that, for each $i \in \{1, 2\}$, we have

$$\left| \exp^{(A_i + K_i C)\tau} \right| \leq \frac{1}{\bar{c}} \exp^{-2\lambda(\tau - \frac{\tau_d \underline{z}}{2})} \quad \forall t \geq \frac{\tau_d \underline{z}}{2}.$$

Consequently, under Assumption 1—see (5),

$$\begin{aligned} |\tilde{z}(t)| &\leq \kappa_1 |\tilde{z}(t_o)| \exp^{-\kappa_2 \int_{t_o}^{t} |z_1(s)| ds} \\ &\leq \kappa_1 |\tilde{z}(t_o)| \exp^{-\kappa_2 \mu (t - t_o)} \ \forall t \geq t_o + T, \ t_o \geq 0. \end{aligned} \tag{9}$$

III. Observer-based Hybrid Control Algorithm

As explained in the Introduction, the difficulty in stabilizing the origin for (1) via output feedback resides in that, roughly speaking, the observer (2) relies on the output $z_1(t)$ remaining away from the origin, which is in clear conflict with the control objective. To circumvent this paradox we propose an observer-based hybrid controller, whose design starts with the simple control law

$$u := a z_1 \hat{z}_2 - k(z_1 - z^*), \tag{10}$$

in which, z^* is a piece-wise constant and vanishing reference trajectory, that is carefully designed in function of the bound on the estimation errors $|\tilde{z}(t)|$. More precisely, the design of z^* is driven by the analysis of the closed-loop of (1a) using the controller in (10), driven by the estimation error $\tilde{z}_2$.

Defining the tracking error $z_{1e} := z_1 - z^*$, this corresponds to

$$\dot{z}_{1e} = -(k + a\tilde{z}_2) z_{1e} + a z^* \tilde{z}_2. \tag{11}$$

Note that the latter constitutes a linear time-invariant system with a bounded perturbation $\tilde{z}_2$. Indeed, after [5], one can

compute a positive definite matrix $P \in \mathbb{R}^{2\times 2}$ such that the time derivative of

$$V_{obs}(\tilde{z}) := \tilde{z}^\top P \tilde{z}, \tag{12}$$

along the solutions to (3), verifies

$$\dot{V}_{obs}(\tilde{z}(t)) \leq 0 \qquad \forall\, t \geq 0. \tag{13}$$

Let $R > 0$ be arbitrarily fixed. Then, for any initial condition $z(0)$ such that $|z(0)| \leq R$, we may fix $\hat{z}(0)$ such that $|\tilde{z}(0)| \leq \tilde{R}$, where $\tilde{R} > 0$ is known. Therefore, from (13) we have

$$V_{obs}(\tilde{z}(0)) \leq \lambda_{\max}(P)\tilde{R}^2,$$

which implies that

$$\tilde{z}_2(t)^2 \leq \gamma^2 \tilde{z}(0)^2 \leq \gamma^2 \tilde{R}^2 \qquad \forall t \geq 0, \tag{14}$$

with $\gamma := \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}$.

This is useful to estimate an ultimate bound on the tracking errors z_{1e}. Indeed, using the function $V(z_{1e}) := (1/2)z_{1e}^2$ and setting

$$k := \gamma a \tilde{R} + k', \quad k' > 0, \tag{15}$$

we see that, in view of (14), the derivative of V along the trajectories of (11) satisfies

$$\dot{V}(z_{1e}) \leq -k' z_{1e}^2 + a|z^*|_\infty |\tilde{z}_2|_\infty |z_{1e}|, \tag{16}$$

where $|\phi|_\infty := \operatorname{ess\,sup}_{t\geq 0}|\phi(t)|$.

Roughly speaking, we have that the tracking errors converge to a ball of radius that depend on the bounds on $|z^*(t)|$ and $|\tilde{z}_2(t)|$. Therefore, the achievement of the control goal resides, as mentioned above, on the appropriate design of a reference z^* that, on the one hand, converges to zero and, on the other hand, it does so at a rate sufficiently "slow" that Assumption 1 holds on specific windows of time. As a result, the estimation errors at the end of each window diminish the size they had at the beginning of the window at a specific rate.

To that end, we design the reference z^* as a piecewise-constant function taking values, sequentially, in the set

$$S^* := \bigcup_{i=0}^{\infty} \left\{ -\frac{z_{in}^*}{2^i}, \frac{z_{in}^*}{2^i} \right\}, \tag{17}$$

where $z_{in}^* > 0$ is an initialization value and i denotes an increasing cycle index. More precisely, z^* changes value according to the following algorithm:

Initialization step: We initially set $z^*(t) = z_{in}^*$ for all $t \in [0, t_1]$, where t_1 is to be defined, and $\hat{z}(0)$ such that $|\tilde{z}(0)| \leq \tilde{R}$. Then,

$$|\tilde{z}_2(t)| \leq \gamma \tilde{R} \qquad \forall t \geq 0.$$

From (14) and (16), we conclude the existence of $T > 0$ such that

$$|z_{1e}(t)| \leq \frac{2a\tilde{R}}{\sqrt{k'}}|z^*(t)| \qquad \forall t \geq T,$$

so, by setting $k' \geq 16a^2\tilde{R}^2$, it follows that

$$|z_{1e}(t)| \leq z_{in}^*/2 \qquad \forall t \geq T,$$

so $z_1(t) \in [\frac{z_{in}^*}{2}, z_{in}^*]$ and, consequently, Assumption 1 holds. On the other hand, there exist κ_{1o}, $\kappa_{2o} > 0$ such that—see (9),

$$|\tilde{z}(t)| \leq |\tilde{z}(0)|\kappa_{1o} \exp\big(-\frac{\kappa_{2o} z_{in}^* t}{2}\big) \qquad \forall t \geq 0,$$

so, for any $\varepsilon > 0$, there exists $T_o \geq T > 0$ such that

$$|\tilde{z}(t)| \leq g(0)(\varepsilon/\gamma) \qquad \forall t \geq T_o, \quad g(0) := 1,$$

and, from the first inequality in (14), we have $|\tilde{z}_2(t)| \leq \varepsilon$ for all $t \geq T_o$. Next, we define $t_1 := T_o$ and generate the first cycle.

Remark 1: Note that on the interval $[0, T_o]$, z_2 remains bounded since z_1 is bounded and the dynamics of z_2 is linear in z_2. Furthermore, z_2 remains away from the singular point $-\frac{d}{c}$ provided that $z_2(0) > -\frac{d}{c}$. The latter property always holds for real-world ABS systems.

First cycle: From $t_1 := T_o$, we set z^* to satisfy $|z^*| = \frac{z_{in}^*}{2}$, moreover, the tracking error z_{1e} satisfies (16) with $|\tilde{z}_2| \leq \varepsilon$. As a result, a limit cycle is generated by switching z^* between $-z_{in}^*/2$ and $z_{in}^*/2$ each time $\hat{z}_2(t)$ reaches $d/2c$ or $-d/2c$, as described below:

1) If $\hat{z}_2(t_1) \leq 0$, $z^*(t_1)$ is set to $z_{in}^*/2$. Then, at $t_1' \geq t_1$ such that $\hat{z}_2(t_1') = \frac{d}{2c}$, which means that $z_2(t_1') \in [\frac{d}{2c} - \varepsilon, \frac{d}{2c} + \varepsilon]$, the reference z^* is set to $z^*(t_1') = -\frac{z_{in}^*}{2}$. Then, at $t_1'' \geq t_1'$ such that $\hat{z}_2(t_1'') = -\frac{d}{2c}$, which means that $z_2(t_1'') \in [-\varepsilon - \frac{d}{2c}, \varepsilon - \frac{d}{2c}]$, the reference z^* is set back to $\frac{z_{in}^*}{2}$.
2) If $\hat{z}_2(t_1) \geq 0$, the reference is set to $z^*(t_1) = -z_{in}^*/2$ and the same switching rules as above apply *mutatis mutandis*.

Along this cycle, Assumption 1 holds on $[t_1, +\infty)$; thus, there exist positive constants $(\kappa_{11}, \kappa_{21})$ such that

$$|\tilde{z}(t)| \leq \kappa_{11}|\tilde{z}(t_1)| \exp^{-\kappa_{21}\int_{t_1}^t |z_1(s)|ds} \qquad \forall t \geq t_1.$$

The first cycle is ended at $t_2 := T_o + T_1 > 0$, where T_1 is chosen so that

$$|\hat{z}_2(t)| \leq d/2c \quad \text{and} \quad |\tilde{z}(t)| \leq (\varepsilon/\gamma)g(1) \qquad \forall t \geq t_2,$$

where $g(1) \in (0, 1)$. Thus, from t_2, we start a new cycle. Moreover, the latter is repeated successively by defining $t_i := T_o + T_1 + \cdots + T_{i-1}$ and the ith cycle as follows.

ith cycle: From $t_i = T_o + T_1 + \cdots + T_{i-1}$, the reference z^* is set to satisfy $|z^*| = z_{in}^*/2^i$, moreover, z_{1e} satisfies (16) with $|\tilde{z}_2| \leq \varepsilon g(i-1)$ for some $g(i-1) \in (0, 1)$. As a result, a limit cycle is created by switching z^* between $-z_{in}^*/2^i$ and $z_{in}^*/2^i$ each time $\hat{z}_2(t)$ equals to $d/2^i c$ or to $-d/2^i c$, as described below:

1) If $\hat{z}_2(t_i) \leq 0$, $z^*(t_i)$ is set to $z_{in}^*/2^i$. Then, at $t_i' \geq t_i$ such that $\hat{z}_2(t_i') = \frac{d}{2^i c}$, which means that $z_2(t_i') \in [\frac{d}{2^i c} - \varepsilon, \frac{d}{2^i c} + \varepsilon]$, the reference z^* is set to $z^*(t_i') = -\frac{z_{in}^*}{2^i}$. Then, at $t_i'' \geq t_i'$ such that $\hat{z}_2(t_i'') = -\frac{d}{2^i c}$, which means that $z_2(t_i'') \in [-\frac{d}{2^i c} - \varepsilon, -\frac{d}{2^i c} + \varepsilon]$, the reference z^* is set back to $\frac{z_{in}^*}{2^i}$.
2) Similarly, if $\hat{z}_2(t_i) \geq 0$, the reference is set to $z^*(t_i) = -z_{in}^*/2^i$ and the same switching rules as in the previous item are applied.

During the ith cycle, Assumption 1 holds on $[t_i, +\infty)$, so there exist positive constants $(\kappa_{1i}, \kappa_{2i})$ such that

$$|\tilde{z}(t)| \leq \kappa_{1i}|\tilde{z}(t_i)| \exp^{-\kappa_{2i}\int_{t_i}^{t}|z_1(s)|ds} \quad \forall t \geq t_i.$$

The cycle ends at $t_{i+1} := T_o + T_1 + ... + T_i > 0$, where T_i is chosen so that

$$|\hat{z}_2(t)| \leq d/2^i c \quad \text{and} \quad |\tilde{z}(t)| \leq (\varepsilon/\gamma)g(i) \qquad \forall t \geq t_{i+1},$$

where $g(i) \in (0,1)$. Hence, a new cycle starts over and the algorithm carries on.

During each cycle, the measured output z_1 satisfies Assumption 1 and the observer is "excited" enough to make $|\tilde{z}|$ decrease. On the other hand, the certainty-equivalence controller (10) makes $|z_{1e}|$ decrease too. The design of the switched reference z^* ensures that $|\tilde{z}|$, at the beginning of the $(i+1)$th cycle, shrinks the size it has at the beginning of the ith cycle, provided that the sequence g is well designed. In addition to switching, the reference decreases to zero so that z_1 also converge.

IV. Main Statement

The ABS system (1), under the previously described control algorithm may be implemented and formally analyzed as a hybrid system. To make a precise statement on asymptotic stability of the origin we rely on the framework of definition of hybrid systems, as laid in [20]. Following the latter, we first show that the closed-loop composed of the plant (1), the controller (10), and the observer (2), may be expressed as a hybrid system that consists in the combination of a constrained differential and a constrained difference equations,

$$\mathcal{H}: \begin{cases} \dot{x} &= F(x) \quad x \in C \\ x^+ &= G(x) \quad x \in D, \end{cases} \tag{18}$$

where the state variable $x \in \mathcal{X} \subset \mathbb{R}^n$ has a continuous evolution while in the flow set $C \subset \mathcal{X}$ and it is allowed to jump if in the jump set $D \subset \mathcal{X}$. The continuous- and the discrete-time evolution of x are governed by the flow and the jump maps $F : C \to \mathbb{R}_{\geq 0} \times \{0\} \times \mathbb{R}^2 \times \mathbb{R}^2 \times \{0\}$ and $G : D \to \mathcal{X}$, respectively. Furthermore, the closed-loop state is defined as

$$x := (\tau, i, z, \tilde{z}, z^*) \in \mathcal{X},$$
$$\mathcal{X} := \mathbb{R}_{\geq 0} \times \mathbb{N} \times \mathbb{R} \times \left(-\frac{d}{c}, +\infty\right) \times \mathbb{R}^2 \times S^*.$$

Then, the jump and flow sets are defined as follows. The flow set $C := \mathrm{cl}\,(\mathcal{X} \backslash D)$, where $\mathrm{cl}(\,\cdot\,)$ denotes *closure relative to* $\mathcal{X}$ and the jump set $D := D_c \cup D_{nc}$. The set D_c, which determines the jump conditions *within* the ith cycle, is given by

$$D_c := \left\{ x \in \mathcal{X} : |\hat{z}_2| \geq \frac{d|z^*|}{cz^*_{in}},\ \hat{z}_2 z^* \geq 0 \right\} \tag{19}$$

and the set D_{nc}, which determines the jump condition *from* the ith to the $(i+1)$th cycle, is given by

$$D_{nc} := \Bigg\{ x \in \mathcal{X} :\ |\hat{z}_2| \leq \frac{d|z^*|}{cz^*_{in}},\ \hat{z}_2 z^* \leq 0, \\ |\Phi_i(\tau,0)^\top P \Phi_i(\tau,0)|^{\frac{1}{2}} \leq \lambda_{\min}(P)^{\frac{1}{2}} h(i) \Bigg\}, \tag{20}$$

where Φ_i is the transition matrix corresponding to the system

$$\frac{d\tilde{w}}{d\tau} = A(w_1(\tau + \tau_i))\tilde{w} \qquad \tau \geq 0,$$

and $\tau_i := \int_0^{t_i} |z_1(s)| ds$. Moreover $h(0) := \varepsilon/(\gamma \tilde{R})$, and $h(i) := g(i)/g(i-1) \in (0,1)$ for all $i \geq 1$.

The definition of the jump sets D_c and D_{nc} follows the rationale developed in the previous section, but certain technical aspects are also considered in order to cast the analysis in the framework of [20]. The respective first inequalities in D_c and D_{nc} correspond to the switch conditions explained in Section III. The constraint $\hat{z}_2 z^* \leq 0$, which requires that the signs of $\hat{z}_2$ and z^* be different, is imposed in the definition of D_{nc}, while the opposite is used to define D_c, to render the intersection of these sets empty (the apparent intersection $\{\hat{z}_2 = z^* = 0\}$ is void since $z^* \neq 0$ by design). A somewhat more natural manner to define the jump sets D_{nc} and D_c, which is in more strict concordance with the algorithm loosely described in the previous section, would not include such condition and would simply impose a strict inequality in either set. However, such definition would lead to the hybrid system being not well-posed [20].

The third inequality in the definition of D_{nc},

$$|\Phi_i(\tau,0)^\top P \Phi_i(\tau,0)| \leq \lambda_{\min}(P) h(i)^2, \tag{21}$$

is a conservative, yet verifiable, condition that essentially tests the size of the otherwise non-measurable estimation errors $\tilde{z}(t) \equiv \tilde{w}(\tau)$. To better see this, consider the function V_{obs} in (12). Its total derivative along the solutions to (6) satisfies $\dot{V}_{obs}(\tilde{w}(\tau)) \leq 0$, so $V_{obs}(\tilde{w}(\tau)) \leq V_{obs}(\tilde{w}(0))$ for all $\tau \geq 0$. Hence, equivalently,

$$\tilde{w}(\tau)^\top P \tilde{w}(\tau) \leq \tilde{w}(0)^\top P \tilde{w}(0).$$

Therefore, using $\tilde{w}(\tau) = \Phi_i(\tau,0)\tilde{w}(0)$, we see that (21) implies that, for any $\tilde{w}(0) \in \mathbb{R}^2$,

$$\tilde{w}(\tau)^\top P \tilde{w}(\tau) \leq \lambda_{\min}(P) h(i)^2 \tilde{w}(0)^2,$$

that is, $|\tilde{w}(\tau)|^2 \leq h(i)^2 |\tilde{w}(0)|^2$.

Then, with these definitions of the flow and jump sets, we introduce the flow map

$$F(x) := \begin{bmatrix} |z_1| \\ 0 \\ \begin{bmatrix} -(k + a\tilde{z}_2)z_{1e} + az^*\tilde{z}_2 \\ (cz_2 + d)z_1 \end{bmatrix} \\ z_1 \begin{bmatrix} -k_1(z_1) & -a \\ -k_2(z_1) & c \end{bmatrix} \tilde{z} \\ 0 \end{bmatrix}. \tag{22}$$

Note that in the definition of F, the dynamics of the discrete variables (i, z^*) is null, the dynamics of τ corresponds to (4). Finally, the dynamics of z and $\tilde{z}$ are simply repeated from (11) and (3), respectively.

On the other hand, the jump map is given by

$$G(x) := \begin{bmatrix} \begin{cases} 0 & \text{if} \quad x \in D_{nc} \\ \tau & \text{if} \quad x \in D_c \end{cases} \\ \begin{cases} i+1 & \text{if} \quad x \in D_{nc} \\ i & \text{if} \quad x \in D_c \end{cases} \\ z \\ \tilde{z} \\ \begin{cases} z^*/2 & \text{if} \quad x \in D_{nc} \\ -z^* & \text{if} \quad x \in D_c \end{cases} \end{bmatrix}. \tag{23}$$

The map G is designed to reset the value of τ to 0 each time a new cycle starts and updates the cycle index i. The variables z and $\tilde{z}$ are continuous variables, so they do not change their values during jumps. According to the algorithm previously explained, the variable z^* halves its size in absolute value whenever a jump to a new cycle occurs, otherwise (while switching within a cycle), z^* only alternates sign. It is important to note that since $D_c \cap D_{nc} = \emptyset$, then the map G is continuous on D. This is important for the system to be well-posed [20].

In addition, the initial state $x_o := (\tau_o, i_o, z_o, \tilde{z}_o, z_o^*) \in \mathcal{X}$ is defined as follows. By assumption, a number R is known such that $|z_o| \leq R$. Then, the estimation errors $\tilde{z}_o$ are set so that $|\tilde{z}_o| \leq \tilde{R}$ for a given $\tilde{R} > 0$. Hence, when a reliable estimate of $|z_o|$ is available, the Initialization step described on p. 3 may be skipped by defining the initial cycle index as $i_o := \max\{0, \kappa_1(\tilde{R})\}$, where

$$\kappa_1(\tilde{R}) := \max\left\{ i \in \mathbb{Z} : \tilde{R} \leq \frac{\varepsilon g(i-1)}{\gamma} \right\}.$$

Furthermore, according to (4), $\tau_o = 0$. Finally, the vanishing reference trajectory z^* is initialized to

$$z_o^* := \begin{cases} \left\{ \begin{cases} \frac{z_{in}^*}{2^{i_o}} & \text{if } \hat{z}_{2o} < 0 \\ -\frac{z_{in}^*}{2^{i_o}} & \text{otherwise} \end{cases} \right\} & \text{if} \quad i_o \geq 1 \\ z_{in}^* & \text{if} \quad i_o = 0. \end{cases}$$

Our main statement establishes semi-global attractivity of the set $\mathcal{A} := \{x \in \mathcal{X} : z = \tilde{z} = 0\}$ for the closed-loop system. That is, that for any ball of initial conditions of radius R, there exists a control gain $k(R)$, as defined in (15), such that all trajectories converge to the set $\mathcal{A}$. In particular, the domain of attraction may be enlarged by increasing the control gain.

Theorem 1: Consider the closed-loop hybrid system $\mathcal{H} = (C, F, D, G)$ defined by (18)–(20), (22), and (23). Let R, $\tilde{R} > 0$ be such that $|z_o| \leq R$ and $|\tilde{z}_o| \leq \tilde{R}$, and let (i_o, z_o^*, τ_o) be defined as above. Then, for each R, there exists $k > 0$ such that:

(i) each solution to[1] $\mathcal{H}$ satisfies

$$\lim_{(t+j)\to+\infty} |x(t,j)|_{\mathcal{A}} = 0,$$

provided that $\lim_{i\to\infty} g(i) = 0$;

(ii) the set $\mathcal{A}$ is locally stable in the sense that there exist a sufficiently small constant $\delta > 0$ and a class-$\mathcal{K}$ function[2] κ such that $|(z, \tilde{z})|_\infty \leq \kappa(|z_o| + \delta)$;

(iii) the system $\mathcal{H}$ is well posed—see [20], and its solutions are uniformly non-Zeno, that is, there exist $T > 0$ and $J \in \mathbb{N}$ such that, on any time period of length T, at most J jumps can occur.

In a general setting, the assumption in item (ii) is restrictive. However, this is not the case for commercial ABS systems, for which the initial condition z_{1o} is usually known and there exists a small δ such that $z_{2o} \in [-\delta, \delta]$. Note that, in this case, items (i) and (ii) imply semi-global asymptotic stability in the hybrid sense [20]. This, together with item (iii) goes well beyond that of mere convergence of estimation errors $\tilde{z}$ and the state variable z since it guarantees certain robustness—see the latter reference for details.

V. A NUMERICAL EXAMPLE

To illustrate our theoretical findings, we performed some numerical simulations on a model of extended braking stiffness dynamics, described in detail in [15]. The switching logic is determined by the conditions in (19) and (20), with $a = 375$, $c = 24$, and $d = 12.5$. The scenario defined by these parameters corresponding to that of a hard braking on a dry road. We took, moreover, a gain $k = 500$ for the controller and gains $k_1^+ = 40$ and $k_2^+ = -3$ for the observer. The initial wheel acceleration reference z_o^* was set to 75 and the sequence h was chosen as

$$h(i) := \begin{cases} 1/(1+\frac{1}{4^i}) & \text{if} \quad i \in \{1, 2, ..., 8\}, \\ 1/2 & \text{if} \quad i \in \{9, 10, ...\}. \end{cases}$$

This choice of h results from extensive numerical tests, which showed that the natural choice $h(i) \equiv 1/2$, which satisfies the conditions of Theorem 1, leads to relatively slow convergence of z, while $h(i) = 1/(1+\frac{1}{4^i})$ for all $i \in \mathbb{N}$ violates the condition t $\lim_{i\to\infty} g(i) = 0$, with $g(i) = \prod_1^i h(j)$, in Item (i) of Theorem 1.

The numerical results are illustrated in Figs. 1 and 2. The NE plot in Fig. 1 shows (in black) the trajectories of the measured output, z_1, and the corresponding estimate generated by the observer (in blue). The yellow (apparently) solid lines correspond to the piece-wise constant reference z^* taking values in the discrete set S^* defined in (17), hence the "staircase-type" graph. It is noticeable that, *e.g.*, approximately $\forall\, t \in (0, 8)$, $|z^*(t)| \approx 75$, while $\forall\, t \in (15, 28)$ $|z^*(t)| \approx 2.3$. Other values are discernible over other intervals. These correspond to the duration of the different cycles triggered as explained in Section III. Within each cycle, the reference keeps switching between the positive and negative value of the same constant. In the zoomed windows, one can appreciate that, even if the system's output does not attain the imposed reference (because the latter switches) the observer keeps tracking the plant's states; the trajectories of z_2 and $\hat{z}_2$ are depicted in the NW plot. It is the "persistent excitation" induced by the switchings that the observer keeps

[1] Note that $(t, j) \mapsto x(t, j)$ are defined as absolutely continuous functions mapping their hybrid domain, dom $x \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$, into $\mathbb{R}^2$. See [20] for details.

[2] i.e. κ is continuous, strictly increasing, and $\kappa(0) = 0$.

up with a close estimate of the states. Then, as described in the algorithm, successive cycles, at the beginning of which the reference z^* is halved, are generated. Thus, the reference, the plant's states and the observer errors, all tend to zero asymptotically. In Fig. 2 one may appreciate the stabilization mechanism by looking at the response on the phase plane.

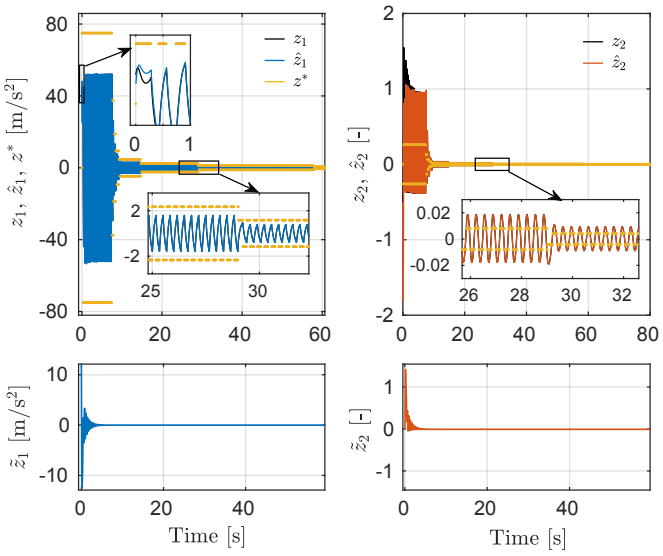

Fig. 1. System's and observer's response

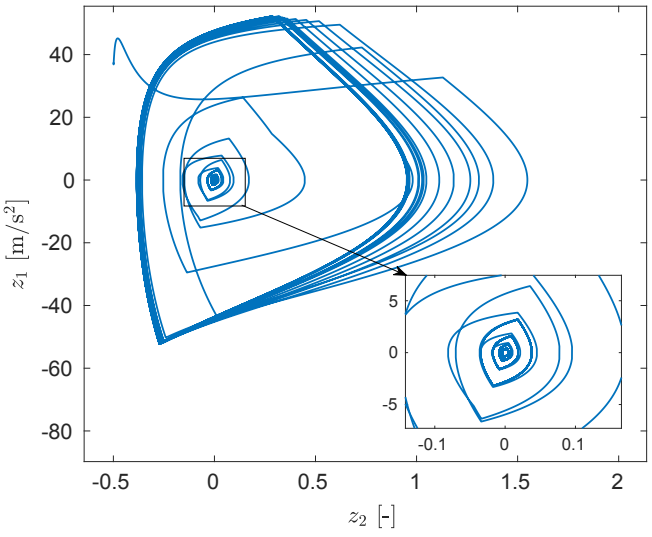

Fig. 2. System's response on the phase-plane

VI. Conclusion and Future Work

Simultaneous estimation and stabilization at an equilibrium where observability is lost is a challenging, but not impossible task. Our main statement focuses on a particular bilinear system, but of great interest in control practice. It sets the basis for future work oriented towards broadening the applicability of our switching-observer-based hybrid controller to other bilinear and, more generally, non-uniformly observable systems. Beyond these remaining open theoretical questions, a deeper study regarding control implementation is required to determine different cycle-jump conditions that deliver good performance while satisfying the technical conditions imposed by the analysis.

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